



Charles Bordenave · Arnab Sen · Bálint Virág

Mean quantum percolation

Received December 25, 2014

Abstract. We study the spectrum of adjacency matrices of random graphs. We develop two techniques to lower bound the mass of the continuous part of the spectral measure or the density of states. As an application, we prove that the spectral measure of bond percolation in the two-dimensional lattice contains a non-trivial continuous part in the supercritical regime. The same result holds for the limiting spectral measure of a supercritical Erdős–Rényi graph and for the spectral measure of a unimodular random tree with at least two ends. We give examples of random graphs with purely continuous spectrum.

Keywords. Expected spectral measure, continuous spectra, sparse random graphs, supercritical percolation, unimodular tree, Erdős–Rényi graph

1. Introduction

This work is devoted to the spectral analysis of adjacency matrices of deterministic and random graphs (the latter is traditionally called quantum percolation). The motivation comes from three distinct directions: random matrices, random trees, and random Schrödinger operators.

1.1. Random matrices and Erdős–Rényi graphs

Wigner introduced the study of random matrices to mathematical physics, and his first paper on the subject was on the density of states. He showed that the empirical distribution of the eigenvalues of a random symmetric matrix with entries picked from some fixed distribution with exponential moments, converges, after scaling, to the famous Wigner semicircle law.

C. Bordenave: Institut de Mathématiques de Toulouse, CNRS and University of Toulouse, 118 route de Narbonne, 31062 Toulouse Cedex 09, France;
e-mail: bordenave@math.univ-toulouse.fr, <http://www.math.univ-toulouse.fr/~bordenave>

A. Sen: School of Mathematics, University of Minnesota, 206 Church Street SE, Minneapolis, MN 55455, USA; e-mail: arnab@math.umn.edu, <http://math.umn.edu/~arnab>

B. Virág: Department of Mathematics and Statistics, University of Toronto, 40 St George Street, Toronto, ON, M5S 3G3, Canada;
e-mail: balint@math.toronto.edu, <http://www.math.toronto.edu/~balint>

Mathematics Subject Classification (2010): Primary 60B20; Secondary 60K35, 60D05

A particular example is the adjacency matrix of Erdős–Rényi random graphs on n vertices, where each edge is present with a fixed probability p . In this case, it is not hard to show that the semicircle limit holds as $np \rightarrow \infty$. At the forefront of current research is the sparse case, when $np \rightarrow c \in (0, \infty)$. In this case, the limit μ_c is quite unlike the Wigner semicircle law: it is supported on the entire real line and has a dense set of atoms [13]. This and related questions have been discussed in the physics literature—see notably Bauer and Golinelli [5, 4]. When $c \rightarrow \infty$, the measure μ_c rescaled by c converges to the Wigner semicircle law. This suggests (but by no means implies) that there may be a continuous part for large c , which has been an open problem.

Our first theorem establishes this in a strong sense.

Theorem 1.1. *The measure μ_c has a continuous part if and only if $c > 1$.*

This result is a corollary to our investigation of random trees.

1.2. Percolation on regular trees

The spectra of random trees is a very active area of random Schrödinger operators; it is in this setting that the celebrated extended states conjecture was first proved for a continuous perturbation of the regular tree.

A slightly different perturbation is given by Bernoulli bond percolation with high enough parameter p . Here each edge of the graph of the regular tree T_d is removed independently with probability $1 - p \in [0, 1]$.

First, a quick definition: the *spectral measure* of the adjacency matrix of a (possibly infinite) bounded degree rooted graph is the unique probability measure whose k th moments are given by the number of closed walks of length k . This can be extended to a more general setting (Section 1.5).

For the spectral measure (without taking expectation), for p close to 1, Keller [26] has shown that the component of the root has continuous spectrum with positive probability. It is an open problem when the continuous part appears. However, for the expected spectral measure μ , we can show

Theorem 1.2. *The critical value for the existence of continuous part in the expected spectral measure μ for percolation on d -regular trees is $1/d$.*

This can be generalized to Galton–Watson trees. In fact, we will show that a bounded degree unimodular tree has continuous spectrum as long as it has at least two ends (Section 1.6).

In contrast Bhamidi, Evans and Sen [8] have proved that the limiting spectral measures for various popular models of random trees, for example uniform random trees and trees generated by preferential attachment scheme, have a dense set of atoms (this does not rule out the existence of some continuous part). Lelarge, Salez and the first author [11] have given examples of Galton–Watson trees with arbitrarily high minimal degree and an atom at 0 for the expected spectral measure.

1.3. Percolation on Euclidean lattices

The study of the regularity of the density of states is of prime importance in the literature on random Schrödinger operators. The study of random Hamiltonians generated by percolation on \mathbb{Z}^d was initiated by de Gennes, Lafore and Millot [14, 15] in the 1950's under the name of quantum percolation. The study of the density of states is a preliminary step toward understanding the behavior of the eigenvectors—see e.g. Kirkpatrick and Egarter [30], Chayes et al. [13] or Veselić [40] and references therein. After more than half a century, it is still a very active field of research, and proving the existence of Anderson delocalization remains the main open challenge in the area. One of the issues of quantum percolation models is that the lack of regularity of percolation graphs does not allow one to use Wegner estimates.

In parallel, the study of the spectral properties of graphs and countable groups has a long history; see Mohar and Woess [34] for an early survey on the matter. Kesten [27] computed the spectral measure of the infinite d -regular tree (the Cayley graph of the free group with d generators). This spectral measure is absolutely continuous. This is not the case for every infinite Cayley graph. Indeed, Grigorchuk and Żuk [20] have proved that the spectral measure of the usual lamplighter group is purely atomic (see also Lehner, Neuhauser and Woess [31]). Hence, neither connectivity nor regularity are necessary to guarantee the regularity of the spectral measure.

For site or bond percolation on \mathbb{Z}^d the expected spectral measure μ can be defined through moments, spectral theory, or simply as the limit of the empirical eigenvalue distribution on finite boxes (Section 1.5).

When the percolation has only finite components, the expected spectral measure μ is a countable mixture of atomic measures, so it is purely atomic. On the other end, for $p = 1$ the percolation is simply \mathbb{Z}^d rooted at the origin and its spectral measure is absolutely continuous. In fact, it is the convolution of d arcsine distributions [34, Section 7.B].

Theorem 1.3. *For Bernoulli bond percolation on \mathbb{Z}^2 , the expected spectral measure has a continuous part if and only if $p > p_c$.*

In this paper, the focus is on the adjacency operator of a graph. The same study could be generalized to discrete Laplacian or combinatorial Laplacian or weighted graphs. In the next section, we discuss unimodular random graphs, a convenient framework which allows us to discuss all the above spectral questions. Our contribution to this theory is that we can define the expected spectral measure for unimodular random graphs in complete generality (Proposition 1.4).

1.4. Unimodular random graphs

We first briefly introduce the theory of local weak convergence of graph sequences and the notion of unimodularity. It was introduced by Benjamini and Schramm [7] and has become a popular topology for studying sparse graphs since then. Let us briefly introduce this topology; for further details we refer to Aldous and Lyons [2].

A graph $G = (V, E)$ is said to be *locally finite* if for all $v \in V$, the degree of v in G , $\deg_G(v)$, is finite. A *rooted graph* (G, o) is a locally finite and connected graph

$G = (V, E)$ with a distinguished vertex $o \in V$, the *root*. Two rooted graphs $(G_i, o_i) = (V_i, E_i, o_i)$, $i \in \{1, 2\}$, are *isomorphic* if there exists a bijection $\sigma : V_1 \rightarrow V_2$ such that $\sigma(o_1) = o_2$ and $\sigma(G_1) = G_2$, where σ acts on E_1 through $\sigma(\{u, v\}) = \{\sigma(u), \sigma(v)\}$. We will denote this equivalence relation by $(G_1, o_1) \simeq (G_2, o_2)$. In graph theory terminology, an equivalence class of rooted graphs is an *unlabeled rooted graph*. We denote by \mathcal{G}^* the set of unlabeled rooted graphs.

The *local topology* is the smallest topology such that for any $g \in \mathcal{G}^*$ and integer $t \geq 1$, the $\mathcal{G}^* \rightarrow \{0, 1\}$ function $f(G, o) = \mathbf{1}((G, o)_t \simeq g)$ is continuous, where $(G, o)_t$ is the induced rooted graph spanned by the vertices at graph distance at most t from o . This topology is metrizable and the space \mathcal{G}^* is separable and complete.

For a finite graph $G = (V, E)$ and $v \in V$, one writes $G(v)$ for the connected component of G at v . One defines the probability measure $U(G) \in \mathcal{P}(\mathcal{G}^*)$ as the law of the equivalence class of the rooted graph $(G(o), o)$ where the root o is sampled uniformly on V . If $(G_n)_{n \geq 1}$ is a sequence of finite graphs, we shall say that G_n has *local weak limit* $\rho \in \mathcal{P}(\mathcal{G}^*)$ if $U(G_n) \rightarrow \rho$ weakly in \mathcal{G}^* . A measure $\rho \in \mathcal{P}(\mathcal{G}^*)$ is called *sofic* if there exists a sequence $(G_n)_{n \geq 1}$ of finite graphs whose local weak limit is ρ .

The notion of unimodularity can be thought of as invariance under moving the root, but it requires some subtlety to get the definition right. Towards this end, we define locally finite connected graphs with two roots (G, o, o') and extend the notion of isomorphism to such structures. We define \mathcal{G}^{**} as the set of equivalence classes of graphs (G, o, o') with two roots and equip \mathcal{G}^{**} with its natural local topology. A function f on \mathcal{G}^{**} can be extended to a function on connected graphs with two roots (G, o, o') through the isomorphism classes. Then a measure $\rho \in \mathcal{P}(\mathcal{G}^*)$ is called *unimodular* if for any measurable function $f : \mathcal{G}^{**} \rightarrow \mathbb{R}_+$, we have

$$\mathbb{E}_\rho \sum_{v \in V} f(G, o, v) = \mathbb{E}_\rho \sum_{v \in V} f(G, v, o), \quad (1)$$

where under \mathbb{P}_ρ , (G, o) has law ρ . It is immediate to check that if G is finite then $U(G)$ is unimodular. More generally, all sofic measures are unimodular, and the converse is open; for a discussion see [2]. It is however known that all unimodular probability measures supported on rooted trees are sofic (see Elek [18], Bowen [12] and Benjamini, Lyons and Schramm [6]). We will denote by $\mathcal{P}_{\text{uni}}(\mathcal{G}^*)$ the set of unimodular measures. It is closed under the local weak topology.

Any Cayley graph G of a finitely generated group Γ is unimodular (more precisely, for any $v \in \Gamma$, the measure ρ which puts a Dirac mass at the equivalence class of (G, v) is unimodular) [2, Section 3].

With a slight abuse of language, we shall say that a random rooted graph (G, o) is unimodular if the law of its equivalence class in \mathcal{G}^* is unimodular.

1.5. The spectral measure of graphs

Let V be countable and $G = (V, E)$ be a locally finite graph. Its *adjacency operator*, denoted by A , is defined for vectors $\psi \in \ell^2(V)$ with finite support by the formula

$$A\psi(u) = \sum_{v: \{u,v\} \in E} \psi(v).$$

By construction A is symmetric. If it is assumed further that the degrees of vertices are bounded by an integer d , then we readily check that A has norm bounded by d . Hence, A is a self-adjoint operator. For any $\psi \in \ell^2(V)$ with $\|\psi\|_2^2 = 1$, we may thus define the *spectral measure with vector ψ* , denoted by μ_A^ψ , as the unique probability measure on \mathbb{R} such that for all integers $k \geq 1$,

$$\int x^k d\mu_A^\psi = \langle \psi, A^k \psi \rangle. \tag{2}$$

For example if $|V| = n$ is finite, then A is a symmetric matrix. If (v_1, \dots, v_n) is an orthonormal basis of eigenvectors associated to eigenvalues $(\lambda_1, \dots, \lambda_n)$, we find

$$\mu_A^\psi = \sum_{k=1}^n \langle v_k, \psi \rangle^2 \delta_{\lambda_k} \tag{3}$$

(if V is not finite, μ_A^ψ has a similar decomposition over the resolution of the identity of A but we shall not need this here).

Now, we denote by $e_v(u) = \mathbf{1}_{\{u=v\}}$ the coordinate vector associated to $v \in V$. Note that if two rooted graphs are isomorphic then the spectral measures associated to the coordinate vector for the root (simply called the *spectral measure at the root*) are equal. It thus makes sense to define $\mu_A^{e_o}$ for elements of \mathcal{G}^* . Then, if $\rho \in \mathcal{P}(\mathcal{G}^*)$ is supported on graphs with bounded degrees, we may consider the expected spectral measure of the root:

$$\mu_\rho = \mathbb{E}_\rho \mu_A^{e_o}. \tag{4}$$

In particular, if $|V| = n$ is finite, (3) implies

$$\mu_{U(G)} = \frac{1}{n} \sum_{k=1}^n \delta_{\lambda_k}.$$

It is the *empirical distribution of the eigenvalues* of the adjacency matrix.

It is not clear a priori how to extend this construction to random graphs without bounded degrees. For finitely supported $\psi \in \ell^2(V)$, the spectral measure μ_A^ψ is properly defined if A is essentially self-adjoint. More generally, if $\rho \in \mathcal{P}(\mathcal{G}^*)$ is such that ρ -a.s. A is essentially self-adjoint, we define $\mu_\rho = \mathbb{E}_\rho \mu_A^{e_o}$ as in (4). It can be difficult to check that adjacency operators are essentially self-adjoint (for a criterion of essential self-adjointness of the adjacency operator of trees see [11], and for a characterization see Salez [37, Theorem 2.2]). It turns out however that for unimodular measures, A is always ρ -a.s. essentially self-adjoint and $\mu_\rho = \mathbb{E}_\rho \mu_A^{e_o}$ is thus well-defined without any bounded degree assumption.

Proposition 1.4. *For any $\rho \in \mathcal{P}_{\text{uni}}(\mathcal{G}^*)$,*

- (i) *the adjacency operator A is ρ -a.s. essentially self-adjoint,*
- (ii) *if $\rho_n \in \mathcal{P}_{\text{uni}}(\mathcal{G}^*)$ and $\rho_n \rightarrow \rho$, then μ_{ρ_n} converges weakly to μ_ρ .*

If a sequence $(G_n)_{n \geq 1}$ of finite graphs has local weak limit ρ then the empirical distribution of the eigenvalues of their adjacency matrices converges weakly to μ_ρ . In this last case, if moreover $\deg_{G_n}(v) \leq \theta$ for some $\theta > 0$ and all $v \in V(G_n)$, then by using Lück’s

approximation [32, 39, 1], the convergence can even be strengthened to the pointwise convergence of all atoms.

1.6. Existence of continuous spectrum in unimodular trees

In this paper, we will develop two simple tools to prove the existence of a continuous part of the spectral measure of unimodular graphs. In addition to the examples given at the beginning of the introduction, we will present many cases where these two tools can be applied.

A *weighted graph* (G, ω) is a graph $G = (V, E)$ equipped with a weight function $\omega : V^2 \rightarrow \mathbb{Z}$ such that $\omega(u, v) = 0$ if $u \neq v$ and $\{u, v\} \notin E$. The weight function is *edge-symmetric* if $\omega(u, v) = \omega(v, u)$ and $\omega(u, u) = 0$. Note that, for edge-symmetric weight functions, the set of edges such that $\omega(e) = k$ spans a subgraph of G . A *line ensemble* of G is an edge-symmetric weight function $L : V^2 \rightarrow \{0, 1\}$ such that for all $v \in V$,

$$\sum_u L(u, v) \in \{0, 2\}.$$

We will think of L as a subgraph of G which consists of a union of vertex-disjoint copies of \mathbb{Z} .

It is straightforward to extend the local weak topology to weighted graphs. The definition of unimodularity carries over naturally to weighted graphs (see the definition of unimodular network in [2]). Now, consider a unimodular graph (G, o) . If, on an enlarged probability space, the weighted graph (G, L, o) is unimodular and L is a.s. a line ensemble then we shall say that L is an *invariant line ensemble* of (G, o) . We call $\mathbb{P}(o \in L)$ the *density* of L .

Theorem 1.5. *Let (T, o) be a unimodular tree with law ρ . If L is an invariant line ensemble of (T, o) then for each real λ ,*

$$\mu_\rho(\lambda) \leq \mathbb{P}(o \notin L)\mu_{\rho'}(\lambda)$$

where, if $\mathbb{P}(o \notin L) > 0$, ρ' is the law of the rooted tree $(T \setminus L, o)$ conditioned on the root $o \notin L$. In particular, the total mass of atoms of μ_ρ is bounded above by $\mathbb{P}(o \notin L)$.

We will check in Section 5.1 below that the measure ρ' is indeed unimodular. As a consequence, if (T, o) has an invariant line ensemble such that $\mathbb{P}(o \in L) = 1$ then μ_ρ is continuous. Our next result gives the existence of an invariant line ensemble for a large class of unimodular trees. We recall that for a rooted tree (T, o) , a *topological end* is just an infinite non-backtracking path in T starting from o .

Proposition 1.6. *Let (T, o) be a unimodular tree. If T has at least two topological ends with positive probability, then (T, o) has an invariant line ensemble L with positive density, $\mathbb{P}(o \in L) > 0$. Moreover, we have the following lower bounds.*

- (i) $\mathbb{P}(o \in L) \geq \frac{1}{6} \frac{(\mathbb{E} \deg(o) - 2)^2}{\mathbb{E} \deg(o)^2}$ as long as the denominator is finite.
- (ii) Let q be the probability that $T \setminus \{o\}$ has at most one infinite component. If $\deg(o) \leq d$ a.s., then $\mathbb{P}(o \in L) \geq \frac{1}{3}(\mathbb{E} \deg(o) - 2q)/d$.

One of the natural examples where the conditions of Proposition 1.6 are not satisfied is the infinite skeleton tree which consists of a semi-infinite line $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$ with i.i.d. critical Poisson Galton–Watson trees attached to each of the vertices of \mathbb{Z}_+ . Assign $0 \in \mathbb{Z}_+$ to be the root of this infinite skeleton tree. It is the local weak limit of the uniform trees on n labeled vertices.

Let $P \in \mathcal{P}(\mathbb{Z}_+)$ with positive and finite mean. The *unimodular Galton–Watson tree with degree distribution P* (commonly known as *size-biased Galton–Watson tree*) is the law of the random rooted tree obtained as follows. The root has a number d of children sampled according to P , and, given d , the subtrees of the children of the root are independent Galton–Watson trees with offspring distribution

$$\widehat{P}(k) = \frac{(k + 1)P(k + 1)}{\sum_{\ell} \ell P(\ell)}. \tag{5}$$

These unimodular trees appear naturally as a.s. local weak limits of random graphs with a given degree distribution [17, 16, 9]. It is also well known that the Erdős–Rényi $G(n, c/n)$ has a.s. local weak limit the Galton–Watson tree with offspring distribution $\text{Poi}(c)$. Note that if P is $\text{Poi}(c)$ then $\widehat{P} = P$. The percolation on the hypercube $\{0, 1\}^n$ with parameter c/n has the same a.s. local weak limit.

If P has first moment μ_1 and second moment μ_2 , then the first moment of \widehat{P} is $\widehat{\mu} = (\mu_2 - \mu_1)/\mu_1$. If $P \neq \delta_2$ and $\widehat{\mu} \leq 1$, then the unimodular Galton–Watson tree is a.s. finite. If $\widehat{\mu} > 1$ ($\widehat{\mu} = \infty$ is allowed), the tree is infinite with positive probability. Proposition 1.6 now implies that the following phase transition exists for the existence of a continuous part in the expected spectral measure.

Corollary 1.7. *Let ρ be a unimodular Galton–Watson tree with degree distribution $P \neq \delta_2$. Denote by $\widehat{\mu}$ the first moment of \widehat{P} . Then μ_ρ contains a non-trivial continuous part if and only if $\widehat{\mu} > 1$.*

Note that for some choices of P , it is false that the total mass of the atomic part of μ_ρ is equal to the probability of extinction of the tree—it is only a lower bound (see [11]).

Let us conclude the introduction with a few open questions.

1.7. Open questions

Question 1.8. *Consider a unimodular Galton–Watson tree with degree distribution P with finite support and $P(0) = P(1) = 0$. Does the expected spectral measure have only finitely many atoms?*

Theorem 1.3 naturally inspires the following question. We strongly believe that the answer is yes.

Question 1.9. *Does supercritical bond percolation on \mathbb{Z}^d have a continuous part in its expected spectral measure for every $d \geq 2$?*

In view of the result of Grigorchuk and Żuk [20] on the lamplighter group, the next problem has some subtlety.

Question 1.10. *Is there some monotonicity in the weights of the atoms of the spectral measure (for some non-trivial partial order on unimodular measures)?*

Our main results concern percolation on lattices and trees. This motivates the following question.

Question 1.11. *What can be said about the regularity of the spectral measure for other non-amenable/hyperbolic graphs and for other planar graphs (such as the uniform infinite planar triangulation in Angel and Schramm [3])?*

We have seen that regular trees with degree at least 2 contain invariant line ensembles with density 1. A quantitative version of this would be that if the degree is concentrated, then the density is close to 1. Based on the last part of Proposition 1.6, the following formulation is natural.

Question 1.12. *Is there a function f with $f(x) \rightarrow 1$ as $x \rightarrow 1$ such that every unimodular tree of maximal degree $d \geq 2$ contains an invariant line ensemble with density at least $f(\mathbb{E} \deg(o)/d)$?*

Two more open questions (Questions 5.8 and 5.9) can be found in Section 5.

2. The monotone labeling technique

In this section we will use a carefully chosen labeling of the vertices of a graph to prove regularity of its spectrum, the intuition being that a labeling gives an order on the vertices to solve the eigenvalue equation at each vertex.

Definition 2.1. Let $G = (V, E)$ be a graph. A map $\eta : V \rightarrow \mathbb{Z}$ is a *labeling* of the vertices of G with integers. We shall call a vertex v

- (i) *prodigy* if it has a neighbor w with $\eta(w) < \eta(v)$ and such that all other neighbors of w also have label less than $\eta(v)$,
- (ii) *level* if it is not prodigy and if all of its neighbors have the same or lower labels,
- (iii) *bad* if none of the above holds.

Finite graphs. We start with the simpler case of finite graphs.

Theorem 2.2. *Let G be a finite graph, and consider a labeling η of its vertices with integers. Let ℓ, b denote the numbers of level and of bad vertices, respectively. For any eigenvalue λ with multiplicity m , if ℓ_j is the multiplicity of the eigenvalue λ in the subgraph induced by level vertices with label j , then*

$$m \leq b + \sum_j \ell_j.$$

Consequently, for any multiplicities m_1, \dots, m_k of distinct eigenvalues we have

$$m_1 + \dots + m_k \leq kb + \ell.$$

Proof. Let S be the eigenspace for the eigenvalue λ of multiplicity m . Consider the set of bad vertices, and let B be the space of vectors which vanish on that set. For every integer j , let L_j denote the set of level vertices with label j and let A_j denote the eigenspace of λ in the induced subgraph of L_j . With the notation of the theorem, $\dim A_j = \ell_j$. We extend the vectors in A_j to the whole graph by setting them to zero outside L_j . Let A_j^\perp be the orthocomplement of A_j . Recall that for any vector spaces A, B we have $\dim(A \cap B) \geq \dim A - \text{codim } B$. Let $S' = S \cap B \cap \bigcap_j A_j^\perp$, and note that

$$\dim S' \geq \dim S - \text{codim } B - \sum_j \text{codim } A_j^\perp = m - b - \sum_j \dim A_j. \tag{6}$$

However, we claim that the subspace S' is trivial. Let $f \in S'$. We now prove, by induction on the label j of the vertices, low to high, that f vanishes on vertices with label j . Suppose that f vanishes on all vertices with label strictly below j . Clearly, f vanishes on all bad vertices since $f \in B$. Consider a prodigy vertex v with label j . Then, by induction hypothesis, v has a neighbor w such that f vanishes on all neighbors of w except perhaps at v . But the eigenvalue equation

$$\lambda f(w) = \sum_{u \sim w} f(u)$$

implies that f also vanishes at v . Now, observe that the outer vertex boundary of L_j (all vertices that have a neighbor in L_j but are not themselves in L_j) is contained in the union of the set of bad vertices, the set of level vertices with label strictly below j and the set of prodigy vertices with label j . Hence, we know that f vanishes on the outer vertex boundary of L_j . This means that the restriction of f to L_j has to satisfy the eigenvector equation. But since $f \in A_j^\perp$, we get $f(v) = 0$ for $v \in L_j$, and the induction is complete.

We have thus proved that S' is trivial. Now (6) implies that $m \leq b + \sum_j \dim A_j$. This gives the first statement of Theorem 2.2.

For the second statement, let $A_{i,j}$ denote the eigenspace of λ_i in the induced subgraph of L_j . Summing the above inequality over i , we get

$$m_1 + \dots + m_k \leq bk + \sum_j \sum_i \dim A_{i,j} \leq bk + \sum_j |L_j| = bk + \ell. \quad \square$$

Unimodular graphs. We now prove the same theorem for unimodular random graphs which may possibly be infinite. To make the above proof strategy work, we need a suitable notion of normalized dimension for infinite-dimensional subspaces of $\ell^2(V)$. This requires some basic concepts of operator algebras. First, as usual, if (G, o) is a unimodular random graph, we shall say that a labeling $\eta : V(G) \rightarrow \mathbb{Z}$ is *invariant* if on an enlarged probability space, the vertex-weighted rooted graph (G, η, o) is unimodular.

There is a natural von Neumann algebra associated to unimodular measures. More precisely, let \mathcal{G}^* denote the set of equivalence classes of locally finite connected (possibly weighted) graphs endowed with the local weak topology. There is a canonical way to represent an element $(G, o) \in \mathcal{G}^*$ as a rooted graph on the vertex set $V(G) = \{o, 1, 2, \dots, N\}$ with $N \in \mathbb{N} \cup \{\infty\}$ (see Aldous and Lyons [2]). We set $V = \{o, 1, 2, \dots\}$, $H = \ell^2(V)$ and define $\mathcal{B}(H)$ as the set of bounded linear operators on H . Now, for a fixed

$\rho \in \mathcal{P}_{\text{uni}}(\mathcal{G}^*)$, we consider the Hilbert space \mathcal{H} of ρ -measurable functions $\psi : \mathcal{G}^* \rightarrow H$ such that $\mathbb{E}_\rho \|\psi\|_2 < \infty$ with inner product $\mathbb{E}_\rho \langle \psi, \phi \rangle$. Let us denote by $L^\infty(\mathcal{G}^*, \mathcal{B}(H), \rho)$ the ρ -measurable maps $B : \mathcal{G}^* \rightarrow \mathcal{B}(H)$ with $\|B\| \in L^\infty(\mathcal{G}^*, \rho)$.

Now, for any bijection $\sigma : V \rightarrow V$, we consider the orthogonal operator λ_σ defined by $\lambda_\sigma(e_v) = e_{\sigma(v)}$ for all $v \in V$. We introduce the algebra \mathcal{M} of maps in $L^\infty(\mathcal{G}^*, \mathcal{B}(H), \rho)$ which commute with the operators λ_σ , i.e. for any σ, ρ -a.s. $B(G, o) = \lambda_\sigma^{-1} B(\sigma(G), o) \lambda_\sigma$. In particular, $B(G, o)$ does not depend on the root. It can be checked that \mathcal{M} is a von Neumann algebra of operators on the Hilbert space \mathcal{H} (see [2, §5] and Lyons [33] for details). Moreover, the linear map $\mathcal{M} \rightarrow \mathbb{C}$ defined by

$$\tau(B) = \mathbb{E}_\rho \langle e_o, B(G, o)e_o \rangle,$$

where $B : (G, o) \mapsto B(G, o) \in \mathcal{M}$ and under \mathbb{E}_ρ , G has distribution ρ , is a normalized faithful trace. Observe finally that $G = (V(G), E) \in \mathcal{G}^*$ can be extended to a graph on V (all vertices in $V \setminus V(G)$ are isolated). Then the adjacency operator $A : (G, o) \mapsto A(G)$ defines a densely defined operator affiliated to \mathcal{M} (see again [33] for details).

A closed vector space S of H such that P_S , the orthogonal projection to S , is an element of \mathcal{M} will be called an *invariant subspace*. Recall that the *von Neumann dimension* of such a vector space S is just

$$\dim S := \tau(P_S) = \mathbb{E}_\rho \langle e_o, P_S e_o \rangle.$$

We refer e.g. to Kadison and Ringrose [25].

Theorem 2.3. *Let (G, o) be a unimodular random graph with distribution ρ , and consider an invariant labeling η of its vertices with integers. Let ℓ, b denote the probabilities that the root is level or bad, respectively. For integer j and real λ , let ℓ_j be the von Neumann dimension of the eigenspace of λ in the subgraph spanned by level vertices with label j . The spectral measure μ_ρ satisfies*

$$\mu_\rho(\lambda) \leq b + \sum_j \ell_j.$$

Consequently, for any distinct real numbers $\lambda_1, \dots, \lambda_k$, we have

$$\mu_\rho(\lambda_1) + \dots + \mu_\rho(\lambda_k) \leq kb + \ell.$$

In particular, if $b = 0$, then the atomic part of μ_ρ has total weight at most ℓ .

Proof. We first assume that there are only finitely many labels. Let S be the eigenspace of λ , that is, the subspace of $f \in \ell^2(V)$ satisfying, for all $w \in V$,

$$\lambda f(w) = \sum_{u \sim w} f(u). \tag{7}$$

Consider the set of bad vertices, and let B be the space of vectors which vanish on that set. For every integer j let L_j denote the set of level vertices with label j . Let A_j denote the eigenspace of λ in the induced subgraph of L_j ; extend the vectors in A_j to the whole graph by setting them to zero outside L_j . Let A_j^\perp be the orthocomplement of A_j .

For any two invariant vector spaces R, Q we have

$$\dim(R \cap Q) \geq \dim R + \dim Q - 1$$

(see e.g. [23, exercise 8.7.31]). Setting $S' = S \cap B \cap \bigcap_j A_j^\perp$ leads to

$$\dim S' \geq \dim S + \dim B - 1 + \sum_j (\dim A_j^\perp - 1) = \mu_\rho(\lambda_i) - b - \sum_j \dim A_j.$$

However, we claim that the subspace V'_i is trivial. Let $f \in V'_i$. We now prove, by induction on the label j of the vertices, low to high, that f vanishes on vertices with label j . The argument is exactly similar to the case of finite graphs presented before. Suppose that f vanishes on all vertices with label strictly below j . Clearly, f vanishes on all bad vertices since $f \in B$. Consider a prodigy vertex v with label j . Then v has a neighbor w such that f vanishes on all neighbors of w except perhaps at v . But the eigenvalue equation (7) implies that f also vanishes at v . By now, we know that f vanishes on the outer vertex boundary of L_j . This means that the restriction of f to L_j has to satisfy the eigenvector equation. But since $f \in A_j^\perp$, we get $f(v) = 0$ for $v \in L_j$, and the induction is complete.

We have proved that $\mu_\rho(\lambda_i) \leq b + \sum_j \dim A_j$; it is the first statement of the theorem in the case of finitely many labels. When there are infinitely many labels, for every ε we can find n such that $\mathbb{P}(|\eta(o)| > n) \leq \varepsilon$. We can relabel all vertices with $|\eta(v)| > n$ by $-n - 1$; this may make them bad vertices, but will not make designation of vertices with other labels worse. The argument for finitely many labels gives

$$\mu_\rho(\lambda) \leq b + \varepsilon + \sum_{j=-n-1}^n \dim A_j \leq b + 2\varepsilon + \sum_{j=-n}^n \dim A_j \leq b + 2\varepsilon + \sum_j \ell_j,$$

and letting $\varepsilon \rightarrow 0$ completes the proof of the first statement.

For the second statement, let $A_{i,j}$ denote the eigenspace of λ_i in the induced subgraph of L_j . Summing the above inequality over i , we get

$$\mu_\rho(\lambda_1) + \dots + \mu_\rho(\lambda_k) \leq bk + \sum_j \sum_i \dim A_{i,j} \leq bk + \sum_j \mathbb{P}(o \in L_j) = bk + \ell. \quad \square$$

Vertical percolation. There are simple examples where we can apply Theorems 2.2–2.3. Consider the graph of \mathbb{Z}^2 . We perform a vertical percolation by removing some vertical edge $\{(x, y), (x, y + 1)\}$. We restrict to the $n \times n$ box $[0, n - 1]^2 \cap \mathbb{Z}^2$. We thus obtain a finite graph Λ_n on n^2 vertices. We consider the labeling $\eta((x, y)) = x$. It appears that all vertices with label different from 0 are prodigy. The vertices on the y -axis are bad and there are no level vertices. By Theorem 2.2, the multiplicity of any eigenvalue of the adjacency matrix of Λ_n is bounded by $n = o(n^2)$.

Similarly, let $p \in [0, 1]$. We remove each vertical edge $\{(x, y), (x, y + 1)\}$ independently with probability $1 - p$. We obtain a random graph $\Lambda(p)$ with vertex set \mathbb{Z}^2 . Now, we root this graph $\Lambda(p)$ at the origin and obtain a unimodular random graph. We claim that its expected spectral measure μ_ρ is continuous for any $p \in [0, 1]$. Indeed, let $k \geq 1$

be an integer and U be a random variable sampled uniformly on $\{0, \dots, k-1\}$. We consider the labeling $\eta((x, y)) = x + U \bmod(n)$. It is not hard to check that this labeling is invariant. Moreover, all vertices such that $\eta(x, y) \neq 0$ are prodigy while vertices such that $\eta(x, y) = 0$ are bad. It follows from Theorem 2.3 that the mass of any atom of μ_ρ is bounded by $1/k$. Since k is arbitrary, we deduce that μ_ρ is continuous.

The same holds on \mathbb{Z}^d , $d \geq 2$, in the percolation model where we remove edges of the form $\{u, u + e_k\}$ with $u \in \mathbb{Z}^d$ and $k \in \{2, \dots, d\}$.

3. The minimal path matching technique

In this section, we give a new tool to upper bound the multiplicities of eigenvalues.

Definition 3.1. Let $G = (V, E)$ be a finite graph, $I = \{i_1, \dots, i_b\}$ and $J = \{j_1, \dots, j_b\}$ be two disjoint subsets of V of equal cardinality. A *path matching* $\Pi = \{\pi_\ell\}_{1 \leq \ell \leq b}$ from I to J is a collection of self-avoiding paths $\pi_\ell = (u_{\ell,1}, \dots, u_{\ell,p_\ell})$ in G such that for some permutation σ on $\{1, \dots, b\}$ and all $1 \leq \ell \neq \ell' \leq b$,

- $\pi_{\ell'} \cap \pi_\ell = \emptyset$,
- $u_{\ell,1} = i_\ell$ and $u_{\ell,p_\ell} = j_{\sigma(i_\ell)}$.

We will call σ the *matching map* of Π . The length of Π is defined as the sum of the lengths of the paths,

$$|\Pi| = \sum_{\ell=1}^b |\pi_\ell| = \sum_{\ell=1}^b p_\ell.$$

(Note that by the length of a self-avoiding path π_ℓ , we mean the number of vertices on it, rather than the number of edges.) Finally, Π is a *minimal path matching* from I to J if its length is minimal among all possible path matchings.

Connections between multiplicities of eigenvalues and paths have already been known for a long time (see e.g. Godsil [19]). Kim and Shader [29, Theorem 8] provide a nice argument that connects the two notions in trees. This was the starting point for the proof of the following theorem.

Theorem 3.2. Let $G = (V, E)$ be a finite graph and $I, J \subset V$ be two subsets of cardinality b . Assume that the set of path matchings from I to J is not empty and that all minimal path matchings from I to J have the same matching map. Then if $|V| - \ell$ is the length of a minimal path matching and if m_1, \dots, m_r are the multiplicities of the distinct eigenvalues $\lambda_1, \dots, \lambda_r$ of the adjacency matrix of G , we have

$$\sum_{i=1}^r (m_i - b)_+ \leq \ell.$$

Consequently, for any $1 \leq k \leq r$,

$$m_1 + \dots + m_k \leq kb + \ell.$$

We will aim at applying Theorem 3.2 with b small and $|V| - \ell$ proportional to $|V|$. Observe that ℓ is the number of vertices not covered by the union of the paths involved

in a minimal path matching. Theorems 3.2 and 2.3 have the same flavor but they are not equivalent to each other. We note that, in contrast to Theorems 2.2 and 2.3, we do not have a version of Theorem 3.2 which holds for possibly infinite unimodular graphs. Unlike Theorem 2.2, we do not have either a version which bounds the multiplicity of an eigenvalue in terms of its multiplicities in subgraphs. On the other hand, Theorem 3.2 will be used to show the existence of non-trivial continuous part for the expected spectral measure of two-dimensional supercritical bond percolation. It is not clear how to apply Theorem 2.2 or Theorem 2.3 to get this result.

Following [29], the proof of Theorem 3.2 is based on the divisibility properties of characteristic polynomials of subgraphs. For $I, J \subset V$, we define $(A - x)_{I,J}$ as the matrix $A - x$ where the rows with indices in I and columns with indices in J have been removed. We define the polynomial associated to the (I, J) -minor as

$$P_{I,J}(A) : x \mapsto \det(A - x)_{I,J}.$$

We introduce the polynomial

$$\Delta_b(A) = \text{GCD}(P_{I,J}(A) : |I| = |J| = b),$$

where GCD is the (unique) monic polynomial g of highest degree such that all arguments are some polynomial multiples of g . Recall also that any polynomial divides 0. Observe that if $|I| = b$ then $P_{I,J}(A)$ is a polynomial of degree $|V| - b$. It follows that the degree of Δ_b is at most $|V| - b$. For the reader's convenience, let us record that Δ_b in this article corresponds to $\Delta_{|V|-b}$ in [29].

The next lemma is the key to relating multiplicities of eigenvalues and characteristic polynomial of subgraphs.

Lemma 3.3. *If A is the adjacency matrix of a finite graph and m_1, \dots, m_r are the multiplicities of its distinct eigenvalues $\lambda_1, \dots, \lambda_r$, then*

$$\Delta_b(A) = \prod_{i=1}^r (x - \lambda_i)^{(m_i - b)_+}.$$

Consequently,

$$\sum_{i=1}^r (m_i - b)_+ = \text{deg}(\Delta_b(A)).$$

Proof. We set $|V| = n$. If $B(x) \in \mathcal{M}_n(\mathbb{R}[x])$ is an $n \times n$ matrix with polynomial entries, we may define analogously $P_{I,J}(B(x)) = \det B(x)_{I,J}$ and $\Delta_b(B(x))$ (we retrieve our previous definition with $B(x) = A - x$). Let $B_1(x), \dots, B_n(x)$ be the columns of $B(x)$. A simple calculation gives, for $j \neq 1$,

$$\det(B_j(x), B_2(x), \dots, B_n(x))_{I,J} = \begin{cases} \det B(x)_{I,J} & \text{if } 1 \in J, \\ 0 & \text{if } 1 \notin J, j \notin J, \\ \pm \det B(x)_{I, (J \setminus \{j\}) \cup \{1\}} & \text{if } 1 \notin J, j \in J, \end{cases}$$

where the sign in the third case above is a function of j and J . Now the multi-linearity of the determinant implies that

$$\det(w_{11}B_1(x) + w_{21}B_2(x) + \dots + w_{n1}B_n(x), B_2(x), \dots, B_n(x))_{I,J}$$

is a weighted sum of determinants of minors of the form (I, J') where $|J'| = |J|$. It is thus divisible by $\Delta_b(B(x))$. The same holds for the rows of $B(x)$. We deduce that if $U, W \in \mathcal{M}_n(\mathbb{R})$, then $\Delta_b(B(x))$ divides $\Delta_b(UB(x)W)$. It follows that if U and W are invertible,

$$\Delta_b(UB(x)W) = \Delta_b(B(x)).$$

We may now come back to our matrix A . Since A is symmetric, the spectral theorem gives $A = UDU^*$ with U orthogonal and D diagonal with m_i entries equal to λ_i . We have $U(D - x)U^* = A - x$. Hence,

$$\Delta_b(A - x) = \Delta_b(D - x).$$

It is immediate to check that if $I \neq J$, then $P_{I,J}(D - x) = 0$ and

$$P_{I,I}(D - x) = \prod_{k \notin I} (D_{kk} - x) = \prod_{i=1}^r (\lambda_i - x)^{m_i - m_i(I)},$$

where $m_i(I) = \sum_{k \in I} \mathbf{1}(D_{kk} = \lambda_i)$. The lemma follows easily. □

Proof of Theorem 3.2. We set $|V| = n$. We can assume without loss of generality that the matching map of minimal length matchings is the identity. We consider the matrix $B \in \mathcal{M}_n(\mathbb{R})$ obtained from A by setting

$$Be_{j_\ell} = e_{i_\ell} \quad \text{for } 1 \leq \ell \leq b, \quad Be_j = \sum_{i \notin I} A_{ij}e_i \quad \text{for } j \notin J.$$

In graphical terms, B is the adjacency matrix of the oriented graph \bar{G} obtained from G as follows: (1) all edges adjacent to a vertex in J are oriented inwards, (2) all edges adjacent to a vertex in I are oriented outwards, (3) all other edges of G have both orientations, and (4) for each $1 \leq \ell \leq b$, an oriented edge from j_ℓ to i_ℓ is added. We define

$$B(x) = B - xD,$$

where D is the diagonal matrix with $D_{ii} = 1 - \mathbf{1}(i \in I \cup J)$. Expanding the determinant along the columns J , it is immediate to check that

$$\det B(x) = \det(A - x)_{I,J}.$$

We find

$$P_{I,J}(A) = \sum_{\tau} (-1)^\tau \prod_{v \in V} B(x)_{v, \tau(v)} = \sum_{\tau} (-1)^\tau Q_{\tau}(x),$$

where the sum is over all permutations of V . Consider a permutation such that $Q_{\tau} \neq 0$. We decompose τ into disjoint cycles. Observe that $Q_{\tau} \neq 0$ implies that any cycle of length at least 2 coincides with a cycle in the oriented graph \bar{G} . Hence, $Q_{\tau} = 0$ unless $\tau(j_\ell) = i_\ell$ and $(\tau^k(i_\ell), k \geq 0)$ is a path in \bar{G} . We define $\sigma(i_\ell) = \tau^{p_\ell}(i_\ell)$ as the first

element in J which is met in the path. We may decompose these paths into disjoint paths $\pi_\ell = (\tau^k(i_\ell), 0 \leq k \leq p_\ell)$ in G from i_ℓ to $j_{\sigma(\ell)}$. This defines a path matching $\Pi = \{\pi_1, \dots, \pi_b\}$. The contribution to Q_τ of any cycle of length at least 2 is 1 (since off-diagonal entries of A and B are 0 or 1). Also, the signature of a product of disjoint cycles is the product of their signatures. So finally, it follows that

$$P_{I,J}(A) = \sum_{\Pi} \varepsilon(\Pi) \det B(x)_{\Pi,\Pi} = \sum_{\Pi} \varepsilon(\Pi) \det (A - x)_{\Pi,\Pi}, \tag{8}$$

where the sum is over all path matchings from I to J and $\varepsilon(\Pi)$ is the signature of the permutation τ on Π defined as follows: if $\Pi = \{\pi_1, \dots, \pi_b\}$, $\pi_\ell = (i_{\ell,1}, \dots, i_{\ell,p_\ell})$ and σ is the matching map of Π , then $\tau(i_{\ell,k}) = i_{\ell,k+1}$ for $1 \leq k \leq p_\ell - 1$, and $\tau(i_{\ell,p_\ell}) = \tau(j_{\sigma(\ell)}) = i_{\sigma(\ell)}$.

Observe that $\det (A - x)_{\Pi,\Pi}$ is a polynomial of degree $n - |\Pi|$ and leading coefficient $(-1)^{n-|\Pi|}$. Recall also that the signature of a cycle of length k is $(-1)^{k+1}$. By assumption, if Π is a minimal path matching then its matching map is the identity; it follows that

$$\varepsilon(\Pi) = (-1)^{n-\ell+b}.$$

Hence, from (8), $P_{I,J}(A)$ is a polynomial of degree ℓ and leading coefficient $m(-1)^b$ where m is the number of minimal path matchings. By assumption $\Delta_b(A)$ divides $P_{I,J}(A)$, in particular $\deg(\Delta_b(A)) \leq \ell$. It remains to apply Lemma 3.3. \square

Vertical percolation (revisited). Let us revisit the example of vertical percolation on \mathbb{Z}^2 introduced in the previous paragraph. We consider the graph Λ_n on the vertex set $[0, n - 1]^2 \cap \mathbb{Z}^2$ where some vertical edges $\{(x, y), (x, y + 1)\}$ have been removed. We set $I = \{(0, 0), (0, 1), \dots, (0, n - 1)\}$ and $J = \{(n - 1, 0), (n - 1, 1), \dots, (n - 1, n - 1)\}$. Consider the path matchings from I to J . Since none of the horizontal edges of the graph of \mathbb{Z}^2 have been removed, the minimal path matching is unique: it matches $(0, k)$ to $(n - 1, k)$ along the path $((0, k), (1, k), \dots, (n - 1, k))$. In particular, the length of the minimal path matching is n^2 . We may thus apply Theorem 3.2; we find that the multiplicity of any eigenvalue is bounded by $n = o(n^2)$. By pointwise convergence of atoms, this implies that the limiting spectral measure is continuous. Note that Theorems 2.2 and 3.2 give the same bound on the multiplicities for this example.

Lamplighter group. The assumption that all minimal path matchings have the same matching map is important in the proof of Theorem 3.2. It is used to guarantee that the polynomial in (8) is not identically zero. Consider a Følner sequence B_n in the Cayley graph of the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ [20] where B_n consists of the vertices of the form $(v, k) \in \mathbb{Z}_2^{\mathbb{Z}} \times \mathbb{Z}$ with $v(i) = 0$ for $|i| > n$ and $|k| \leq n$. There is an obvious minimal matching in B_n covering all the vertices where each path is obtained by shifting the marker from $-n$ to n and keeping the configurations of the lamps unaltered along the way. But the condition on the unicity of the matching map is not fulfilled. In this case, it is not hard to check that there is a perfect cancellation on the right hand side of (8). This is consistent with the fact that the spectral measure of this lamplighter group is purely atomic.

4. Supercritical bond percolation on \mathbb{Z}^2

In this section, we will prove Theorem 1.3 by finding an explicit lower bound on the total mass of the continuous part of μ_ρ in terms of the speed of the point-to-point first passage percolation on \mathbb{Z}^2 . We fix $p > p_c(\mathbb{Z}^2) = 1/2$.

We will use a finite approximation of \mathbb{Z}^2 . Let $\Lambda_n(p)$ be the (random) subgraph of the lattice \mathbb{Z}^2 obtained by restricting the p -percolation on \mathbb{Z}^2 to the $(n+1) \times (n+1)$ box $[0, n]^2 \cap \mathbb{Z}^2$. We simply write Λ_n for $\Lambda_n(1)$. As mentioned in the introduction, $\text{perc}(\mathbb{Z}^2, p)$, the open connected component of the origin for the p -percolation on \mathbb{Z}^2 , is the local weak limit of $U(\Lambda_n(p))$, and hence by Proposition 1.4, $\mathbb{E}\mu_n^p$ converges weakly to μ_ρ as $n \rightarrow \infty$, where μ_n^p is the empirical eigenvalue distribution of $\Lambda_n(p)$ and the average \mathbb{E} is taken with respect to the randomness of $\Lambda_n(p)$.

Now, assume that, given a realization of the random graph $\Lambda_n(p)$, we can find two disjoint subsets of vertices U and V of $\Lambda_n(p)$ with $|U| = |V|$ and a minimal vertex-disjoint path matching M_n of $\Lambda_n(p)$ between U and V such that:

- (i) The vertices of U and V are uniquely paired up in any such minimal matching of $\Lambda_n(p)$ between U and V .
- (ii) $|U| = o(n^2)$.
- (iii) There exists a constant $c > 0$ such that the size of M_n is at least cn^2 , with probability converging to one.

If there exists such a matching satisfying (i)–(iii) above, then Theorem 3.2 says that for any finite subset $S \subset \mathbb{R}$,

$$\mathbb{P}(\mu_n^p(S) \leq 1 - c) = 1 - o(1),$$

and consequently $\mathbb{E}\mu_n^p(S) \leq (1 - c) + o(1)$. Then by Lück approximation (see [40, Corollary 2.5], [39, Theorem 3.5] or [1]), $\mu_\rho(S) = \lim_{n \rightarrow \infty} \mathbb{E}\mu_n^p(S) \leq 1 - c$ for any finite subset S , which implies that the total mass of the continuous part of μ_ρ is at least c . Hence, in order to prove Theorem 1.3, it is sufficient to prove the existence with high probability of such pairs of sets of disjoint vertices.

A natural way to construct this is to find $\Theta(n)$ vertex-disjoint paths in $\Lambda_n(p)$ between its left and right boundary. Suppose that there exists a collection of m disjoint left-to-right crossings of $\Lambda_n(p)$ that match the vertex $(0, u_i)$ on the left boundary to the vertex (n, v_i) on the right boundary for $1 \leq i \leq m$. Without loss of generality, we can assume $0 \leq u_1 < \dots < u_m \leq n$. Since two vertex-disjoint left-to-right crossings in \mathbb{Z}^2 can never cross each other, we always have $0 \leq v_1 < \dots < v_m \leq n$. Now we take $U = \{(0, u_i) : 1 \leq i \leq m\}$ and $V = \{(n, v_i) : 1 \leq i \leq m\}$. We consider all vertex-disjoint path matchings between U and V in $\Lambda_n(p)$ (there exists at least one such matching by our hypothesis) and take M_n to be a minimal matching between U and V . Clearly, (i) and (ii) above are satisfied. Since any left-to-right crossing contains at least $n+1$ vertices, the size of M_n is at least $(n+1)m$. Thus to prove (iii) we need to show that with high probability we can find at least cn vertex-disjoint left-to-right crossings in $\Lambda_n(p)$. This is a well-known fact in percolation theory (see, for example, [21, Lemma 11.22, Theorem 7.68]); we include a proof for completeness.

Towards this end, let ℓ_n denote the maximum number of vertex-disjoint paths in $\Lambda_n(p)$ between its left and right boundary. By Menger’s theorem, ℓ_n is also equal to the size of a minimum vertex cut of $\Lambda_n(p)$, that is, a set of vertices of smallest size that must be removed to disconnect the left and right boundary of $\Lambda_n(p)$. Note that to bound ℓ_n from below, it suffices to find a lower bound on the size of a minimum edge cut of $\Lambda_n(p)$, since the size of a minimum edge cut is always bounded above by 4 times the size of a minimum vertex cut—because deleting all the edges incident to the vertices in a minimum vertex cut gives an edge cut. The reason behind considering minimum edge cuts instead of minimum vertex cuts is that the size of the former can be related to a certain line-to-line first passage time in the dual graph of Λ_n , whose edges are weighted by i.i.d. $\text{Ber}(p)$. We describe this connection below.

Let Λ_n^* be the dual graph of Λ_n , i.e., the graph with vertex set $\{(x + 1/2, y + 1/2) : 0 \leq x \leq n - 1, -1 \leq y \leq n\}$, and edges connecting vertices in neighboring cells such that each crosses an edge of Λ_n . To each edge e of Λ_n^* , we assign a random weight of value 1 or 0 depending on whether the unique edge of Λ_n which e crosses is present or absent in the graph $\Lambda_n(p)$. Hence, the edge weights of Λ_n^* are i.i.d. $\text{Ber}(p)$. Now here is a crucial observation. The size of a minimum edge cut of $\Lambda_n(p)$, by duality, is the same as the minimum weight of a path from the top to the bottom boundary of Λ_n^* . Moreover, since the dual lattice of \mathbb{Z}^2 is isomorphic to \mathbb{Z}^2 , the minimum weight of a top-to-bottom crossing in Λ_n^* is equal in distribution to the line-to-line passage time $t_{n+1,n-1}(\text{Ber}(p))$ in \mathbb{Z}^2 , where

$$t_{n,m}(F) := \inf \left\{ \sum_{e \in \gamma} t(e) : \gamma \text{ is a path in } \mathbb{Z}^2 \text{ joining } (0, a), (n, b) \text{ for some } 0 \leq a, b \leq m \right. \\ \left. \text{and } \gamma \text{ is contained in } [0, n] \times [0, m] \right\},$$

and the $t(e)$ ’s, the weights of the edges e of \mathbb{Z}^2 , are i.i.d. with non-negative distribution F . By [22, Theorem 2.1(a)], for any non-negative distribution F we have

$$\liminf_{n \rightarrow \infty} \frac{1}{n} t_{n,n}(F) \geq v(F) \quad \text{a.s.}, \tag{9}$$

where $v(F) < \infty$ is called the *speed* (or *time-constant*) of the first passage percolation on \mathbb{Z}^2 with i.i.d. F edge weights, that is,

$$\frac{1}{n} a_{0,n}(F) \rightarrow v(F) \quad \text{in probability,}$$

where

$$a_{0,n}(F) := \inf \left\{ \sum_{e \in \gamma} t(e) : \gamma \text{ is a path in } \mathbb{Z}^2 \text{ joining } (0, 0) \text{ and } (n, 0) \right\}.$$

It is a classical fact due to Kesten [28] that $v(F) > 0$ if and only if $F(0) < p_c(\mathbb{Z}^2) = 1/2$. This ensures that $v(\text{Ber}(p)) > 0$ in the supercritical regime $p > 1/2$. Therefore, for any $\varepsilon > 0$, with probability tending to one,

$$t_{n+1,n-1}(\text{Ber}(p)) \geq t_{n+1,n+1}(\text{Ber}(p)) \geq (v(\text{Ber}(p)) - \varepsilon)(n + 1),$$

which implies that

$$\lim_{n \rightarrow \infty} \mathbb{P}(\ell_n \geq \frac{1}{4}(v(\text{Ber}(p)) - \varepsilon)n) = 1.$$

Hence (iii) is satisfied with $c = \frac{1}{4}(v(\text{Ber}(p)) - \varepsilon)$ for any $\varepsilon > 0$. Therefore, the total mass of the continuous part of μ_ρ is bounded below by $\frac{1}{4}v(\text{Ber}(p))$.

This concludes the proof of Theorem 1.3. \square

5. Spectrum of unimodular trees

5.1. Stability of unimodularity

We will repeatedly use the fact that unimodularity is stable by weight mappings, global conditioning and invariant percolation. More precisely, let (G, o) be a unimodular random weighted rooted graph with distribution ρ . Weights on G are denoted by $\omega : V^2 \rightarrow \mathbb{Z}$. The following trivially holds:

Weight mapping: Let $\psi : \mathcal{G}^* \rightarrow \mathbb{Z}$ and $\phi : \mathcal{G}^{**} \rightarrow \mathbb{Z}$ be measurable functions. We define \tilde{G} as the weighted graph with weights $\tilde{\omega}$, obtained from G by setting $\omega(u, u) = \psi(G, u)$ for $u \in V$, and $\omega(u, v) = \phi(G, u, v)$ for $u, v \in V^2$ with $\{u, v\} \in E(G)$. The random rooted weighted graph (\tilde{G}, o) is unimodular. Indeed, the $\mathcal{G}^* \rightarrow \mathcal{G}^*$ map $G \mapsto \tilde{G}$ is measurable and we can apply (1) to $f(G, u, v) = h(\tilde{G}, u, v)$ for any measurable $h : \mathcal{G}^{**} \rightarrow \mathbb{R}_+$.

Global conditioning: Let A be a measurable event on \mathcal{G}^* which is invariant by re-rooting, i.e. for any (G, o) and (G', o') in \mathcal{G}^* such that G and G' are isomorphic, we have $(G, o) \in A$ iff $(G', o') \in A$. Then, if $\rho(A) > 0$, the random rooted weighted graph (G, o) conditioned on $(G, o) \in A$ is also unimodular (apply (1) to $f(G, u, v) = \mathbf{1}((G, u) \in A)h(G, u, v)$ for any measurable $h : \mathcal{G}^{**} \rightarrow \mathbb{R}_+$).

Invariant percolation: Let $B \subset \mathbb{Z}$. We may define a random weighted graph \hat{G} with edge set $E(\hat{G}) \subset E(G)$ by putting the edge $\{u, v\} \in E(G)$ in $E(\hat{G})$ if both $\omega(u, v)$ and $\omega(v, u)$ are in B . We leave the remaining weights unchanged. Then the random weighted rooted graph $(\hat{G}(o), o)$ is also unimodular (apply (1) to $f(G, u, v) = h(\hat{G}(u), u, v)$ for any measurable $h : \mathcal{G}^{**} \rightarrow \mathbb{R}_+$).

As an application, the measure ρ' defined in the statement of Theorem 1.5 is unimodular. Indeed, consider the weight mapping defined by $\omega(v, v) = \mathbf{1}(v \in L)$ for $v \in V$, and $\omega(u, v) = \omega(v, u) = \mathbf{1}(\omega(u, u) = \omega(v, v))$ for $\{u, v\} \in E$. Then we perform an invariant percolation with $B = \{1\}$ and finally a global conditioning by $A = \{\text{all vertices in } G \text{ satisfying } \omega(v, v) = 0\}$.

5.2. Proof of Theorem 1.5

Consider the unimodular weighted tree (T, L, o) . Our main strategy will be to construct a suitable invariant labeling on T using the invariant line ensemble L and then apply Theorem 2.3.

We may identify L as a disjoint union of countable lines $(\ell_i)_i$. Each such line $\ell \subset L$ has two topological ends. We enlarge our probability space and associate to each line

an independent Bernoulli variable with parameter $1/2$. This allows us to orient each line $\ell \subset L$. This can be done by choosing the unique vertex on the line ℓ whose distance from the root o is minimum and then by picking one of its two neighbors on ℓ using the Bernoulli coin toss.

Let us denote by $(\vec{\ell}_i)_i$ the oriented lines. We thus obtain a unimodular weighted graph (T, ω, o) where $\omega(u, v) = 1$ if the oriented edge $(u, v) \in \vec{\ell}_i$ for some k , $\omega(u, v) = -1$ if $(v, u) \in \vec{\ell}_i$, and otherwise $\omega(u, v) = 0$.

Now, we fix some integer $k \geq 1$. There are exactly k functions $\eta : V \rightarrow \mathbb{Z}/k\mathbb{Z}$ such that the discrete gradient of η is equal to ω (i.e. $\eta(u) - \eta(v) = \omega(v, u) \pmod{k}$) for any $u, v \in V$ with $\{u, v\} \in E$ since given the gradient ω , the function η is completely determined by its value at the root. We may enlarge our probability space in order to sample, given (T, ω, o) , such a function η uniformly at random. Then the vertex-weighted random rooted graph (T, η, o) is unimodular.

In summary, we have obtained an invariant labeling η of (T, o) such that all vertices $v \in V$ outside L are level, all vertices in L such that $\eta(v) \neq 0$ are prodigy, and vertices in L such that $\eta(v) = 0$ are bad. By Theorem 2.3, we deduce that for any real λ ,

$$\mu_\rho(\lambda) \leq \mathbb{P}(o \text{ is bad}) + \sum_j \ell_j,$$

where $\ell_j = \mathbb{E}\langle e_o, P_j e_o \rangle$ and P_j is the projection operator of the eigenspace of λ in the adjacency operator A_j spanned by the vertices with label j . Now, observe that all level vertices with label j are at graph distance at least 2 from the set of level vertices with label $i \neq j$. This implies that the operators A_j commute and A' , the adjacency operator of $T' = T \setminus L$, can be decomposed as a direct sum of the operators A_j . It follows that if P' is the projection operator of the eigenspace of λ in A' then

$$\sum_j \ell_j = \mathbb{E}\langle e_o, P' e_o \rangle = \mathbb{P}(o \notin L) \mu_{\rho'}(\lambda).$$

Also, by construction, $\mathbb{P}(o \text{ is bad})$ is upper bounded by $1/k$. Since k is arbitrary, we find

$$\mu_\rho(\lambda) \leq \mathbb{P}(o \notin L) \mu_{\rho'}(\lambda).$$

This concludes the proof of Theorem 1.5. □

Remark 5.1. In the proof of Theorem 1.5, we have used Theorem 2.3. It is natural to ask if we could have used Theorem 3.2 together with some finite graph sequence (G_n) having local weak limit (T, o) instead. We could match the set of $v \in L$ such that $\eta(v) = 1$ to the set of $v \in L$ such that $\eta(v) = k - 1$ forbidding the set of $v \in L$ with $\eta(v) = 0$. Note however that if the weighted graph (G_n, η_n) has local weak limit (T, η, o) then the boundary of $\eta_n^{-1}(j)$ for $j \in \mathbb{Z}/k\mathbb{Z}$ has cardinality $(2/k + o(1))\mathbb{P}(o \in L)|V(G_n)|$. In particular, the sequence (G_n) must have a small Cheeger constant. This implies for example that we could not use the usual random graphs as finite approximations of infinite unimodular Galton–Watson trees with degree distribution P such that $P(0) = P(1) = 0$ since they have a Cheeger constant bounded away from 0 (see Durrett [17]).

5.3. Construction of invariant line ensembles on unimodular trees

We will say that a unimodular tree (T, o) is *Hamiltonian* if there exists an invariant line ensemble L that contains the root o with probability 1. As a first example, we show that infinite d -regular trees are Hamiltonian.

Lemma 5.2. *For any integer $d \geq 2$, the infinite d -regular tree is Hamiltonian.*

Proof. The case $d = 2$ is trivial: in this case $T = (V, E)$ itself is a line ensemble. Let us assume $d \geq 3$. On a probability space, we attach to each oriented edge (u, v) an independent variable, $\xi(u, v)$, uniformly distributed on $[0, 1]$. With probability one, for each $u \in V$, we may then order its d neighbors according to the value of $\xi(u, \cdot)$. This gives a weighted graph (T, ω, o) such that, for each $u \in V$ with $\{u, v\} \in E$, $\omega(u, v) \in \{1, \dots, d\}$ is the rank of the vertex v with respect to u . Note that $\omega(u, v)$ may be different from $\omega(v, u)$.

We now build a line ensemble as follows. The root picks its first two neighbors, say u_1, u_2 , and we set $L(u_1, o) = L(u_2, o) = 1$; for the other neighbors, we set $L(u, o) = 0$. To define further values of L , let us introduce some notation. For $u \neq v$, let T_u^v be the tree rooted at u spanned by the vertices whose shortest path in T to v meets u , and let $a^v(u)$ be the first visited vertex on the shortest path from u to v (see Figure 1). Then, we iteratively define $L(u, \cdot)$ as follows for a vertex u for which $L(a^o(u), \cdot)$ has already been defined: If $L(u, a^o(u)) = 1$, then we pick the first neighbor of u in T_u^o according to the ranking $\omega(u, \cdot)$, say v_1 , and set $L(u, v_1) = 1$. If, on the other hand, $L(u, a^o(u)) = 0$, then we pick the first two neighbors of u in T_u^o , say v_1, v_2 , and we set $L(u, v_1) = L(u, v_2) = 1$. In both cases, for any other neighbor w of u in T_u^o , we set $L(u, w) = 0$.

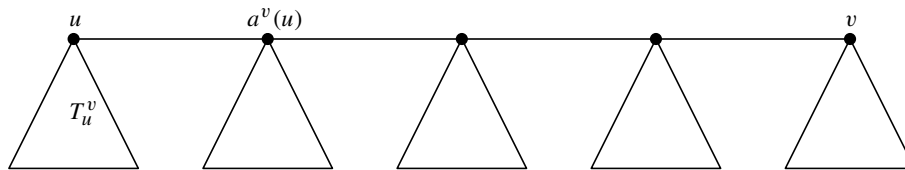


Fig. 1. Definition of $a^v(u)$ and T_u^v .

Iterating this procedure gives a line ensemble which covers all vertices. It is however not so clear that this line ensemble is indeed invariant since, in the construction, the root seems to play a special role. In order to verify (1), it is sufficient to restrict to functions $f(G, L, u, v)$ such that $f(G, L, u, v) = 0$ unless $\{u, v\} \in E$ (see [2, Proposition 2.2]). Letting v_1, \dots, v_d be the neighbors of the root, we have

$$\begin{aligned} \mathbb{E} \sum_{k=1}^d f(T, L, o, v_k) &= (d - 2)\mathbb{E}[f(T, L, o, v_1) \mid L(v_1, o) = 0] + 2\mathbb{E}[f(T, L, o, v_1) \mid L(v_1, o) = 1]. \end{aligned}$$

We notice that the rooted trees $T_u^v, u \neq v$, are isomorphic (T_u^v is a $(d - 1)$ -ary tree) and that, given the value of $L(u, v_1)$, the restrictions of L to $T_{v_1}^o$ and $T_o^{v_1}$ have the same law

(and are independent). Since $L(u, v) = L(v, u)$, it follows that, for $\varepsilon \in \{0, 1\}$,

$$\mathbb{E}[f(T, L, o, v_1) \mid L(v_1, o) = \varepsilon] = \mathbb{E}[f(T, L, v_1, o) \mid L(o, v_1) = \varepsilon].$$

We have thus checked that L is an invariant line ensemble. □

Lemma 5.3. *Let $k \geq 3$. Every unimodular tree with all degrees either 2 or k has an invariant line ensemble of density $\mathbb{E} \deg(o)/k$.*

Proof. Sample the unimodular random tree (T, o) . Consider the k -regular labeled tree T' that one gets by contracting each induced subgraph which is a path to a single edge labeled by the number of vertices. This tree has an invariant line ensemble L' with density 1; it corresponds to a line ensemble L in T . Since each edge in T' is contained in L' with probability $2/k$, it follows that each edge of T is contained in L with probability $2/k$. Thus the expected degree of L at the root of T given T is $(2/k) \deg(o)$. The claim follows after averaging over T . □

The following proves Proposition 1.6(ii) for the case $q = 0$ (i.e. when there are no “bushes”).

Proposition 5.4. *Let T be a unimodular tree with degrees in $\{2, 3, \dots, d\}$. Then T contains an invariant line ensemble with density at least $\frac{1}{3} \mathbb{E} \deg(o)/d$. In fact, when $d \geq 6$ the density is at least $\frac{1}{3} \mathbb{E} \deg(o)/(d - 4)$.*

A tree constructed of d -stars with paths of length m emanating shows that in some cases the optimal density can be arbitrarily close to $\mathbb{E} \deg(o)/d$. In this sense our bound is sharp up to a factor of $1/3$.

Proof of Proposition 5.4. If $d \geq 6$ we argue as follows. We may write uniquely the degree of a vertex as $3k + 2j$ where $j = 0, 1, 2$ and k is a non-negative integer. We then split this vertex into k groups of vertices of degree 3 and j groups of vertices of degree 2 (see Figure 2). We can perform this in a unimodular fashion by ordering the adjacent edges of a vertex uniformly at random (see the proof of Lemma 5.2). This way we obtain a countable collection $(T_n)_{n \geq 1}$ of trees where all degrees are either 2 or 3.

By Lemma 5.3 each of these trees contains invariant line ensembles with expected degree $\frac{2}{3} \mathbb{E} \deg_{T_n}(o)$. In particular, the expected degree of their union F_1 in T is $\frac{2}{3} \mathbb{E} \deg(o)$. We thus have found an invariant subforest F_1 of $F_0 = T$ with degrees in $\{0, 2, 4, \dots, 2k + 2j\}$ and expected degree $\frac{2}{3} \mathbb{E} \deg(o)$.

Iterating this construction i times we get a sequence of subforests F_i with expected degree $(\frac{2}{3})^i \mathbb{E} \deg(o)$. The maximal degree of F_i is bounded above by some d_i (with $d_0 = d$), which satisfy the following recursion: if $d_i = 3k + 2j$ with $j = 0, 1, 2$, then $d_{i+1} = 2k + 2j$. In particular, d_i is even for $i \geq 1$, and

$$d_{i+1} \leq \frac{2}{3}d_i + \frac{4}{3}. \tag{10}$$

Let k be the first value such that $d_k \leq 4$; by checking cases we see that $d_k = 4$, and either $d_{k-1} = 5$ or $d_{k-1} = 6$. Assuming $k > 1$ we also know that d_{k-1} is even, so

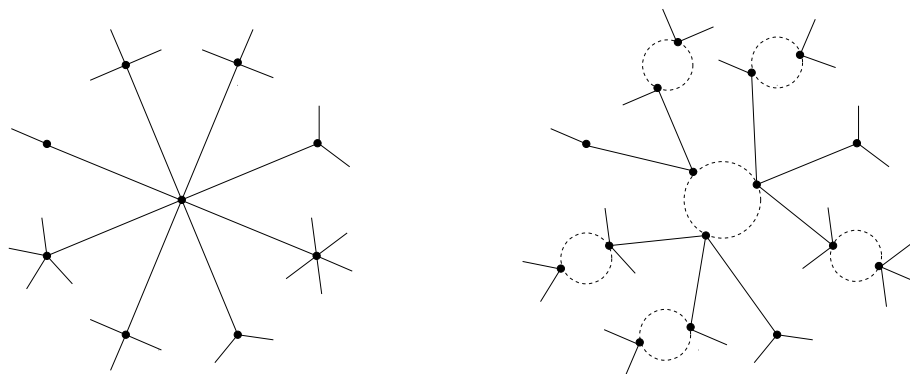


Fig. 2. Example of a splitting: the vertex of degree $8 = 3 \times 2 + 2 \times 1$ is split into two vertices of degree 3 and one vertex of degree 2, and similarly for its neighbors of degree larger than 3.

$d_{k-1} = 6$. Otherwise, $k = 1$ and then $d_0 = d$. However the assumption $d \geq 6$ yields $d_0 = d = 6$. Hence in any case $d_{k-1} = 6$. Now using (10) inductively we see that for $1 \leq i \leq k$ we have $d_{k-i} \geq \frac{4}{3} \left(\frac{3}{2}\right)^i + 4$. Setting $i = k$ and rearranging we get

$$\left(\frac{2}{3}\right)^k \geq \frac{4}{3} \frac{1}{d-4}.$$

The forest F_k has degrees in $\{0, 2, 4\}$. Another application of Lemma 5.3 (with $k = 4$ there) gives an invariant line ensemble with density

$$\frac{1}{4} \left(\frac{2}{3}\right)^k \mathbb{E} \deg(o) \geq \frac{1}{3} \frac{\mathbb{E} \deg(o)}{d-4}.$$

If $d = 5$, then $k = 1$, and the above argument gives an invariant line ensemble with density $\frac{1}{4} \left(\frac{2}{3}\right) \mathbb{E} \deg(o)$.

The only cases left are $d = 3, 4$. In the first case, just use Lemma 5.3 with $k = 3$. In the second, split each degree 4 vertex into two groups of degree 2 vertices as above. Then apply Lemma 5.3 with $k = 3$ to get a subforest with degrees in $\{0, 2, 4\}$. Then apply the lemma again with $k = 4$. The density lower bounds are given by $\frac{1}{3} \mathbb{E} \deg(o)$, $\frac{1}{6} \mathbb{E} \deg(o)$ respectively, and this proves the remaining cases. \square

Recall that the *core* C of a tree T is the induced subgraph of cut vertices. Here a *cut vertex* is a vertex whose removal breaks T into at least two infinite components. The following is a reformulation of part (ii) of Proposition 1.6.

Corollary 5.5 (Removing bushes). *Let (T, o) be an infinite unimodular tree with core C and maximal degree d . Then Proposition 5.4 holds with $\mathbb{E} \deg(o)$ replaced by $\mathbb{E} \deg(o) - 2\mathbb{P}(o \notin C)$.*

Proof. We clarify that $\text{deg}_C(o) = 0$ if $o \notin C$. It suffices to show that $\mathbb{E} \text{deg}_C(o) = \mathbb{E} \text{deg}(o) - 2\mathbb{P}(o \notin C)$. For this, let every vertex v with $\text{deg}_C(v) = 0$ send unit mass to the unique neighbor vertex closest to C (or closest to the single end of T in case C is empty). We have

$$\text{deg}_C(o) = \text{deg}(o) - r - \mathbf{1}(o \notin C)$$

where r is the amount of mass o receives. The claim now follows by mass transport: formula (1) applied to $f(G, o, v)$ equal to the amount of mass sent by o to v gives $\mathbb{P}(o \notin C) = \mathbb{E}r$. \square

We are now ready to prove the main assertion of Proposition 1.6, repeated here as follows.

Corollary 5.6. *Let (T, o) be a unimodular tree with at least two ends with positive probability. Then T contains an invariant line ensemble with positive density.*

Proof. We may decompose the measure according to whether T is finite or infinite and prove the claim separately. The finite case being trivial, we now assume that T is infinite.

Consider the core C of T . If T has more than one end, then C has the same ends as T , in particular it is not empty. Thus for the purposes of this corollary we may assume that $T = C$, or in other words all degrees of T are at least 2.

If $\mathbb{E} \text{deg}(o) = 2$, then T is a line almost surely and we are done. So next we consider the case $\mathbb{E} \text{deg}(o) > 2$.

Let F_d be a subforest where all edges incident to vertices of degree more than d are removed. Then $\text{deg}_{F_d}(o) \rightarrow \text{deg}_T(o)$ a.s. in a monotone way. Thus by the Monotone Convergence Theorem, $\mathbb{E} \text{deg}_{F_d}(o) \rightarrow \mathbb{E} \text{deg}_T(o) > 2$. Pick a d so that $\mathbb{E} \text{deg}_{F_d}(o) > 2$. Corollary 5.5 applied to the components of F_d now yields the claim. \square

Part (i) of Proposition 1.6 is restated here as follows.

Corollary 5.7. *Let T be a unimodular tree and assume that $\mathbb{E} \text{deg}(o)^2$ is finite. Then T contains an invariant line ensemble L with density*

$$\mathbb{P}(o \in L) \geq \frac{1}{6} \frac{(\mathbb{E} \text{deg}(o) - 2)_+^2}{\mathbb{E} \text{deg}(o)^2}.$$

Proof. Let $d \geq 1$ be an integer. For each vertex v we mark $(\text{deg}(v) - d)_+$ incident edges at random. To set up a mass transport argument, we let each vertex send mass 1 along each of its marked edges. The unmarked edges form a forest F_d with the same vertices as T and maximal degree d ; we now bound its expected degree. Note that the degree of the root in F_d is bounded below by the same in T minus the total amount of mass sent or received. These two quantities are equal in expectation, so we get

$$\mathbb{E} \text{deg}_{F_d}(o) \geq \mathbb{E} \text{deg}(o) - 2\mathbb{E}(\text{deg}(o) - d)_+.$$

By Proposition 5.4 applied to components of F_d , as long as $d \geq 6$ we get an invariant line ensemble L with density

$$\mathbb{P}(o \in L) \geq \frac{1}{3} \frac{1}{d - 4} (\mathbb{E} \text{deg}(o) - 2 - 2\mathbb{E}(\text{deg}(o) - d)_+).$$

To bound the last term, note that if we set $c = \deg(o) - d$, then the inequality $4(\deg(o) - d)_+ d \leq \deg(o)^2$ reduces to $4cd \leq (c + d)^2$, which certainly holds. Thus we can bound

$$\mathbb{P}(o \in L) \geq \frac{1}{3} \frac{1}{d-4} \left(\mathbb{E} \deg(o) - 2 - \frac{\mathbb{E} \deg(o)^2}{2d} \right).$$

Now set $d = \lceil \mathbb{E} \deg(o)^2 / (\eta - 2) \rceil \geq \eta^2 / (\eta - 2) \geq 8$, where $\eta = \mathbb{E} \deg(o)$ can be assumed to be more than 2. Using the bound $\lceil x \rceil - 4 \leq x$ we get the claim. \square

5.4. Maximal invariant line ensemble

Let (T, o) be a unimodular rooted tree with distribution ρ . In view of Theorem 1.5 and Proposition 1.6, we may wonder what is the value

$$\Sigma(\rho) = \sup \mathbb{P}(o \in L),$$

where the supremum is taken over all invariant line ensembles L of (T, o) . Recall that a line ensemble L of (T, o) is a weighted graph (T, L, o) with weights $L(u, v)$ in $\{0, 1\}$. By diagonal extraction, the set of $\{0, 1\}$ -weighted graphs of a given (locally finite) rooted graph $G = (G, o)$ is compact for the local topology. Hence, the set of probability measures on rooted $\{0, 1\}$ -weighted graphs such that the law of the corresponding unweighted rooted graph is fixed is a compact set for the local weak topology. Recall also that the set of unimodular measures is closed in the local weak topology. By compactness, it follows that there exists an invariant line ensemble, say L^* , such that

$$\Sigma(\rho) = \mathbb{P}(o \in L^*).$$

It is natural to call such an invariant line ensemble a maximal invariant line ensemble.

Question 5.8. *What is the value of $\Sigma(\rho)$ for ρ a unimodular Galton–Watson tree?*

Let L^* be a maximal invariant line ensemble and assume $\mathbb{P}(o \in L^*) < 1$. Then ρ' , the law of $(T \setminus L^*, o)$ conditioned on $o \notin L^*$, is unimodular. Assume for simplicity that ρ is supported on rooted trees with uniformly bounded degrees. Then, by Proposition 1.6 and the maximality of L^* , it follows that if (T', o) has law ρ' , then a.s. T' has either 0 or 1 topological end. Theorem 1.5 asserts that the atoms of μ_ρ are atoms of $\mu_{\rho'}$. We believe that the following is true.

Question 5.9. *Is it true that if ρ is a unimodular Galton–Watson tree then ρ' is supported on finite rooted trees?*

5.5. Two examples

With Theorem 1.5, we can give many examples of unimodular rooted trees (T, o) with continuous expected spectral measure. Indeed, by Theorem 1.5 all Hamiltonian trees have continuous spectrum.

Ring graphs. An example of a Hamiltonian unimodular tree is the *unimodular ring tree* obtained as follows. Let $P \in \mathcal{P}(\mathbb{Z}_+)$ with finite positive mean. We build a multi-type Galton–Watson tree with three types $\{o, a, b\}$. The root o has type o and has two type- a children and a number of type- b children sampled according P . Then, a type- b vertex has two type- a children and a number of type- b children sampled independently according to \widehat{P} given by (5). A type- a vertex has one type- a child and a number of type- b children sampled according to P . We then remove the types and obtain a rooted tree. By construction, it is Hamiltonian: the edges connecting type- a vertices to their genitor is a line ensemble covering all vertices. We can also check easily that it is unimodular.

If P has two finite moments, consider a graphic sequence $\underline{d}(n) = (d_1(n), \dots, d_n(n))$ such that the empirical distribution of $\underline{d}(n)$ converges weakly to P and whose second moment is uniformly integrable. Sample a graph G_n with vertex set $\mathbb{Z}/n\mathbb{Z}$ uniformly on graphs with degree sequence $\underline{d}(n)$ and, if they are not already present, add the edges $\{k, k + 1\}, k \in \mathbb{Z}/n\mathbb{Z}$. The a.s. weak limit of G_n is the above ring tree. This follows from the known result that the uniform graph with degree sequence $\underline{d}(n)$ has a.s. weak limit the unimodular Galton–Watson tree with degree distribution P [17, 16, 9].

Alternatively, consider a random graph G_n on $\mathbb{Z}/n\mathbb{Z}$ with the edges $\{k, k + 1\}, k \in \mathbb{Z}/n\mathbb{Z}$, and any other edge is present independently with probability c/n . Then the a.s. weak limit of G_n will be the unimodular ring tree with $P = \text{Poi}(c)$. Note that G_n is the Watts–Strogatz graph [41].

Stretched regular trees. Let us give another example of application of Theorem 1.5. Fix an integer $d \geq 3$. Consider a unimodular rooted tree (T, o) with only vertices of degree 2 and degree d . Denote its law by ρ . An example is a unimodular Galton–Watson tree with degree distribution $P = p\delta_2 + (1 - p)\delta_d, 0 < p < 1$. Then, arguing as in Proposition 1.6, a.s., all segments of degree 2 vertices are finite. Contracting these finite segments, we obtain a d -regular infinite tree. Hence, by Lemma 5.2, there exists an invariant line ensemble L of (T, o) such that a.s. all degree d vertices are covered. By Theorem 1.5, the atoms of μ_ρ are contained in the set of atoms in the expected spectral measure of rooted finite segments. Eigenvalues of finite segments of length n are of the form $\lambda_{k,n} = 2 \cos(\pi k/(n + 1)), 1 \leq k \leq n$. This proves that the atomic part of μ_ρ is contained in $\Lambda = \bigcup_{k,n} \{\lambda_{k,n}\} \subset (-2, 2)$.

On the other hand, if ρ is a unimodular Galton–Watson tree with degree distribution $P = p\delta_2 + (1 - p)\delta_d, 0 < p < 1$, then the support of μ_ρ is equal to $[-2\sqrt{d - 1}, 2\sqrt{d - 1}]$. Indeed, recall that $\mu_\rho = \mathbb{E}_\rho \mu_A^{e_o}$ and

$$\int x^{2k} \mu_A^{e_o} = \langle e_o, A^{2k} e_o \rangle$$

is equal to the number of paths in T of length $2k$ starting and ending at the root. An upper bound is certainly the number of such paths in the infinite d -regular tree. In particular, from Kesten [27],

$$\int x^{2k} \mu_A^{e_o} \leq (2\sqrt{d - 1} + o(1))^{2k}.$$

This implies that the convex hull of the support of μ_ρ lies in $[-2\sqrt{d-1}, 2\sqrt{d-1}]$. The other way around, recall first that if μ is the spectral measure of the infinite d -regular tree then $\mu(I) > 0$ if I is an open interval in $[-2\sqrt{d-1}, 2\sqrt{d-1}]$ (see [27]). Recall also that for the local topology on rooted graphs with degrees bounded by d , the map $G \mapsto \mu_{A(G)}^{e_o}$ is continuous in $\mathcal{P}(\mathbb{R})$ equipped with the weak topology (follows e.g. from Reed and Simon [36, Theorem VIII.25(a)]). Hence, there exists an integer $t \geq 1$ such that if $(T, o)_t$ is d -regular then $\mu_{A(T)}^{e_o}(I) > 0$. Observe finally that under ρ the probability that $(T, o)_t$ is d -regular is positive. Since $\mu_\rho = \mathbb{E}_\rho \mu_A^{e_o}$, this implies that $\mu_\rho(I) > 0$.

We have thus proved that for a unimodular Galton–Watson tree with degree distribution $P = p\delta_2 + (1-p)\delta_d$, μ_ρ restricted to the interval $[2, 2\sqrt{d-1}]$ is continuous.

6. Proof of Proposition 1.4

Restricted to sofic measures, the proof of this proposition is contained in [11], [10]. To bypass this limitation, we introduce some concepts of operator algebras. We consider the von Neumann algebra \mathcal{M} introduced in Section 2.

Statement (i) is a consequence of Nelson [35]. First, since $A : (G, o) \mapsto A(G)$ is affiliated to \mathcal{M} , from [24, Remark 5.6.3], \bar{A} , the closure of A , is also affiliated to \mathcal{M} . Moreover, from [35, Theorem 1], A^* is affiliated to \mathcal{M} (see [38, discussion below Theorem 2.2]). To prove statement (i) we should check that $\bar{A} = A^*$ (indeed, denoting by \mathcal{R} the range of an operator, if $\mathcal{R}(\bar{A} + iI) = \mathcal{H}$ then ρ -a.s. $\mathcal{R}(A(G) + iI) = \mathcal{H}$). Now, we introduce $V_n(G) = \{v \in V : \deg_G(v) \leq n, \deg_G(u) \leq n \text{ and for all } \{u, v\} \in E(G)\}$ and let $P_n \in \mathcal{M}$ be the projection onto $\mathcal{H}_n = \{\psi \in \mathcal{H} : \rho\text{-a.s. } \text{supp}(\psi(G, o)) \subset V_n(G)\}$. Observe that if $\psi \in \mathcal{H}_n$, then ρ -a.s.

$$\|AP_n\psi\|^2 = \sum_{v \in V} \mathbf{1}(\deg(v) \leq n) \left(\sum_{u: \{u,v\} \in E} \psi(u) \right)^2 \leq n \sum_{v \in V} \sum_{u: \{u,v\} \in E} \psi^2(u) \leq n^2 \|\psi\|^2.$$

Hence, AP_n is bounded and it follows that \mathcal{H}_n is in the domain of both \bar{A} and A^* . We deduce that \bar{A} and A^* coincide on \mathcal{H}_n . Moreover, since ρ is a probability measure on locally finite graphs,

$$\mathbb{P}_\rho(\deg(o) > n \text{ or } \exists v : \{v, o\} \in E, \deg(v) > n) = \varepsilon(n) \rightarrow 0. \tag{11}$$

Finally, since for any $B \in \mathcal{M}$, $\dim \ker(B) \geq \mathbb{P}_\rho(e_o \in \ker(B))$, we deduce that $\dim \mathcal{H}_n \geq 1 - \varepsilon(n)$. From [35, Theorem 3], \bar{A} and A^* are equal.

Let us prove statement (ii). Consider a sequence (ρ_n) converging to ρ in the local weak topology. From Skorokhod’s representation theorem one can define a common probability space such that the rooted graphs (G_n, o) converge for the local topology to (G, o) where (G_n, o) has distribution ρ_n and (G, o) has distribution ρ . Then, the following two facts hold true: (a) for any compactly supported $\psi \in \ell^2(V)$, for n large enough, $A_n\psi = A\psi$, where A_n and A are the adjacency operators of G_n and G ; and (b) if \mathbb{P} denotes the probability measure of the joint laws of (G_n, o) and (G, o) , from statement (i), \mathbb{P} -a.s. A and A_n are essentially self-adjoint with common core, the compactly supported

$\psi \in \ell^2(V)$. These last two facts imply the strong resolvent convergence [36, Theorem VIII.25(a)]. From (2), the Cauchy–Stieltjes transform of $\mu_A^{e_o}$ is a diagonal coefficient of the resolvent,

$$\langle e_o, (A - zI)^{-1} e_o \rangle = \int \frac{d\mu_A^{e_o}}{x - z}.$$

This implies that ρ -a.s. $\mu_{A_n}^{e_o}$ converges weakly to $\mu_A^{e_o}$ (recall that the pointwise convergence of the Cauchy–Stieltjes transform on \mathbb{C}_+ is equivalent to weak convergence). Taking expectation, we conclude that $\mu_{\rho_n} = \mathbb{E}_\rho \mu_{A_n}^{e_o}$ converges weakly to $\mu_\rho = \mathbb{E}_\rho \mu_A^{e_o}$. \square

Acknowledgments. C.B. and A.S. thank for hospitality of the University of Toronto where most of this work was done.

Research of C.B. was partially supported by ANR-11-JS02-005-01. Research of A.S. was partially supported by DMS 1406247. Research of B.V. was partially supported by the Canada Research Chair program and the NSERC Discovery Accelerator Supplement.

References

- [1] Abért, M., Thom, A., Virág, B.: Benjamini–Schramm convergence and pointwise convergence of the spectral measure. Preprint, <http://www.math.uni-leipzig.de/MI/thom/> (2013)
- [2] Aldous, D., Lyons, R.: Processes on unimodular random networks. *Electron. J. Probab.* **12**, no. 54, 1454–1508 (2007) [Zbl 1131.60003](#) [MR 2354165](#)
- [3] Angel, O., Schramm, O.: Uniform infinite planar triangulations. *Comm. Math. Phys.* **241**, 191–213 (2003) [Zbl 1098.60010](#) [MR 2013797](#)
- [4] Bauer, M., Golinelli, O.: Exactly solvable model with two conductor-insulator transitions driven by impurities. *Phys. Rev. Lett.* **86**, 2621–2624 (2001)
- [5] Bauer, M., Golinelli, O.: Random incidence matrices: moments of the spectral density. *J. Statist. Phys.* **103**, 301–337 (2001) [Zbl 0999.82035](#) [MR 1828732](#)
- [6] Benjamini, I., Lyons, R., Schramm, O.: Unimodular random trees. *Ergodic Theory Dynam. Systems* **35**, 359–373 (2015) [Zbl 1328.05166](#) [MR 3316916](#)
- [7] Benjamini, I., Schramm, O.: Recurrence of distributional limits of finite planar graphs. *Electron. J. Probab.* **6**, no. 23, 13 pp. (2001) [Zbl 1010.82021](#) [MR 1873300](#)
- [8] Bhamidi, S., Evans, S. N., Sen, A.: Spectra of large random trees. *J. Theoret. Probab.* **25**, 613–654 (2012) [Zbl 1255.05114](#) [MR 2956206](#)
- [9] Bordenave, C.: Notes on random graphs and combinatorial optimization. <http://www.math.univ-toulouse.fr/~bordenave/>.
- [10] Bordenave, C., Lelarge, M.: Resolvent of large random graphs. *Random Structures Algorithms* **37**, 332–352 (2010) [Zbl 1209.05222](#) [MR 2724665](#)
- [11] Bordenave, C., Lelarge, M., Salez, J.: The rank of diluted random graphs. *Ann. Probab.* **39**, 1097–1121 (2011) [Zbl 1298.05283](#) [MR 2789584](#)
- [12] Bowen, L.: Periodicity and circle packings of the hyperbolic plane. *Geom. Dedicata* **102**, 213–236 (2003) [Zbl 1074.52007](#) [MR 2026846](#)
- [13] Chayes, J. T., Chayes, L., Franz, J. R., Sethna, J. P., Trugman, S. A.: On the density of states for the quantum percolation problem. *J. Phys. A* **19**, L1173–L1177 (1986) [MR 0869300](#)
- [14] de Gennes, P.-G., Lafore, P., Millot, J.: Amas accidentels dans les solutions solides désordonnées. *J. Phys. Chem. Solids* **11**, 105–110 (1959)
- [15] de Gennes, P.-G., Lafore, P., Millot, J.: Sur un phénomène de propagation dans un milieu désordonné. *J. Phys. Rad.* **20**, 624 (1959)

- [16] Dembo, A., Montanari, A.: Gibbs measures and phase transitions on sparse random graphs. *Braz. J. Probab. Statist.* **24**, 137–211 (2010) [Zbl 1205.05209](#) [MR 2643563](#)
- [17] Durrett, R.: *Random Graph Dynamics*. Cambridge Ser. Statist. Probab. Math., Cambridge Univ. Press, Cambridge (2010) [Zbl 1223.05002](#) [MR 2656427](#)
- [18] Elek, G.: On the limit of large girth graph sequences. *Combinatorica* **30**, 553–563 (2010) [Zbl 1231.05259](#) [MR 2776719](#)
- [19] Godsil, C. D.: Spectra of trees. In: *Convexity and Graph Theory* (Jerusalem, 1981), North-Holland Math. Stud. 87, North-Holland, Amsterdam, 151–159 (1984) [Zbl 0559.05040](#) [MR 0791025](#)
- [20] Grigorchuk, R. I., Żuk, A.: The lamplighter group as a group generated by a 2-state automaton, and its spectrum. *Geom. Dedicata* **87**, 209–244 (2001) [Zbl 0990.60049](#) [MR 1866850](#)
- [21] Grimmett, G.: *Percolation*. 2nd ed., Grundlehren Math. Wiss. 321, Springer, Berlin (1999) [Zbl 0926.60004](#) [MR 1707339](#)
- [22] Grimmett, G., Kesten, H.: First-passage percolation, network flows and electrical resistances. *Z. Wahrsch. Verw. Gebiete* **66**, 335–366 (1984) [Zbl 0525.60098](#) [MR 0751574](#)
- [23] Kadison, R. V., Ringrose, J. R.: *Fundamentals of the Theory of Operator Algebras*. Vol. IV. Special Topics. Advanced Theory—an Exercise Approach. Birkhäuser Boston, Boston, MA (1992) [Zbl 0869.46029](#) [MR 1170351](#)
- [24] Kadison, R. V., Ringrose, J. R.: *Fundamentals of the Theory of Operator Algebras*. Vol. I. Elementary Theory, Grad. Stud. Math. 15, Amer. Math. Soc., Providence, RI (1997) [Zbl 0888.46039](#) [MR 1468229](#)
- [25] Kadison, R. V., Ringrose, J. R.: *Fundamentals of the Theory of Operator Algebras*. Vol. II. Advanced Theory, Grad. Stud. Math. 16, Amer. Math. Soc., Providence, RI (1997) [Zbl 0991.46031](#) [MR 1468230](#)
- [26] Keller, M.: Absolutely continuous spectrum for multi-type Galton Watson trees. *Ann. Henri Poincaré* **13**, 1745–1766 (2012) [Zbl 1255.05046](#) [MR 2994759](#)
- [27] Kesten, H.: Symmetric random walks on groups. *Trans. Amer. Math. Soc.* **92**, 336–354 (1959) [Zbl 0092.33503](#) [MR 0109367](#)
- [28] Kesten, H.: Aspects of first passage percolation. In: *École d’été de probabilités de Saint-Flour, XIV—1984*, Lecture Notes in Math. 1180, Springer, Berlin, 125–264 (1986) [Zbl 0602.60098](#) [MR 0876084](#)
- [29] Kim, I.-J., Shader, B. L.: Smith normal form and acyclic matrices. *J. Algebraic Combin.* **29**, 63–80 (2009) [Zbl 1226.05158](#) [MR 2470115](#)
- [30] Kirkpatrick, S., Eggarter, T. P.: Localized states of a binary alloy. *Phys. Rev. B* **6**, 3598–3609 (1972)
- [31] Lehner, F., Neuhauser, M., Woess, W.: On the spectrum of lamplighter groups and percolation clusters. *Math. Ann.* **342**, 69–89 (2008) [Zbl 1185.60115](#) [MR 2415315](#)
- [32] Lück, W.: L^2 -invariants: Theory and Applications to Geometry and K -theory. *Ergeb. Math. Grenzgeb.* 44, Springer, Berlin (2002) [Zbl 1009.55001](#) [MR 1926649](#)
- [33] Lyons, R.: Identities and inequalities for tree entropy. *Combin. Probab. Comput.* **19**, 303–313 (2010) [Zbl 1215.05031](#) [MR 2593624](#)
- [34] Mohar, B., Woess, W.: A survey on spectra of infinite graphs. *Bull. London Math. Soc.* **21**, 209–234 (1989) [Zbl 0645.05048](#) [MR 0986363](#)
- [35] Nelson, E.: Notes on non-commutative integration. *J. Funct. Anal.* **15**, 103–116 (1974) [Zbl 0292.46030](#) [MR 0355628](#)
- [36] Reed, M., Simon, B.: *Methods of Modern Mathematical Physics. I. Functional Analysis*. 2nd ed., Academic Press, New York (1980) [Zbl 0459.46001](#) [MR 0751959](#)
- [37] Salez, J.: *Quelques conséquences de la convergence locale faible pour les graphes aléatoires*. PhD thesis, <http://www.proba.jussieu.fr/~salez/> (2011)

-
- [38] Schultz, H.: Brown measures of sets of commuting operators in a type II_1 factor. *J. Funct. Anal.* **236**, 457–489 (2006) [Zbl 1101.47004](#) [MR 2240171](#)
- [39] Thom, A.: Sofic groups and Diophantine approximation. *Comm. Pure Appl. Math.* **61**, 1155–1171 (2008) [Zbl 1149.43003](#) [MR 2417890](#)
- [40] Veselić, I.: Spectral analysis of percolation Hamiltonians. *Math. Ann.* **331**, 841–865 (2005) [Zbl 1117.81064](#) [MR 2148799](#)
- [41] Watts, D., Strogatz, S.: Collective dynamics of ‘small-world’ networks. *Nature* **393**, no. 6684, 440–442 (2011) [Zbl 1368.05139](#)