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# The wild McKay correspondence and $p$-adic measures

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**Abstract.** We prove a version of the wild McKay correspondence by using $p$-adic measures. This result provides new proofs of mass formulas for extensions of a local field by Serre, Bhargava and Kedlaya.

**Keywords.** McKay correspondence, $p$-adic measures, wild quotient singularities, stringy invariants, mass formulas

## Contents

## 1. Introduction

The aim of this paper is to prove a version of the wild McKay correspondence, the McKay correspondence in positive or mixed characteristic where a given finite group may have order dividing the characteristic of the base field or the residue field. Our main tool is the $p$-adic measure.

By the McKay correspondence, we mean an equality between a certain invariant of a $G$-variety $V$ with $G$ a finite group and a similar invariant of the quotient variety $V/G$ or a desingularization of it. There are different versions for different invariants. Our concern is the one using motivic invariants or their realizations. In characteristic zero, such a version was studied by Batyrev [Bat99] and Denef–Loeser [DL02]. Recently, after examining a

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special case in [Yas14], the author started to try to generalize it to positive or mixed characteristic, and formulated a conjecture in [Yas] for linear actions on affine spaces over a complete discrete valuation ring with algebraically closed residue field. Later, variants and generalizations were formulated in [WY15, Yas16]. In [WY15], the situation was considered where the residue field is only perfect. Moreover, when the residue field is finite, the point-counting realization was discussed. In [Yas16], non-linear actions on affine normal varieties were treated.

In the present paper, we consider non-linear actions on normal quasi-projective varieties over a complete discrete valuation ring with finite residue field and prove a version of the wild McKay correspondence at the level of point-counting realization, with a little dissatisfaction at the formulation in the non-affine case.

Let $\mathcal{O}_K$ be a complete discrete valuation ring, $K$ its fraction field and $k$ its residue field, which is supposed to be finite. For the pair $(X, D)$ of an $\mathcal{O}_K$-variety $X$ and a $\mathbb{Q}$-divisor $D$ on $X$ such that $K_X + D$ is $\mathbb{Q}$-Cartier with $K_X$ the canonical divisor of $X$ over $\mathcal{O}_K$, we define the *stringy point count* of $(X, D)$,

$$\sharp_{\mathrm{st}}(X, D) \in \mathbb{R}_{\geq 0} \cup \{\infty\},$$

as the volume of $X(\mathcal{O}_K)$ with respect to a certain $p$-adic measure. When $D = 0$, identifying the pair $(X, 0)$ with the variety $X$ itself, we write $\sharp_{\mathrm{st}}(X, 0) = \sharp_{\mathrm{st}} X$. Roughly, the stringy point count is the point-count realization of the motivic counterpart of the stringy $E$-function introduced by Batyrev [Bat98, Bat99]. Its principal properties are as follows.

- When $X$ is $\mathcal{O}_K$-smooth, we have $\sharp_{\mathrm{st}} X = \sharp X(k)$.
- There exists a decomposition into contributions of $k$-points,

$$\sharp_{\mathrm{st}}(X, D) = \sum_{x \in X(k)} \sharp_{\mathrm{st}}(X, D)_x.$$

- If $f : Y \to X$ is a proper birational morphism of normal $\mathcal{O}_K$-varieties which induces a crepant map $(Y, E) \to (X, D)$ of pairs, then

$$\sharp_{\mathrm{st}}(X, D) = \sharp_{\mathrm{st}}(Y, E).$$

We generalize the invariant to pairs having a finite group action. Let $(V, E)$ be a pair as above and suppose that a finite group $G$ acts faithfully on $V$ and the divisor $E$ is stable under the action. Let $M$ be a $G$-étale $K$-algebra, that is, $\operatorname{Spec} M \to \operatorname{Spec} K$ is an étale $G$-torsor, and let $\mathcal{O}_M$ be its integer ring. We define the *$M$-stringy point count* of $(V, E)$, denoted by $\sharp_{\mathrm{st}}^M(V, E)$, as a certain $p$-adic volume of the set of $G$-equivariant $\mathcal{O}_K$-morphisms $\operatorname{Spec} \mathcal{O}_M \to V$, and define the *$G$-stringy point count* by

$$\sharp_{\mathrm{st}}^G(V, E) = \sum_M \sharp_{\mathrm{st}}^M(V, E),$$

where $M$ runs over the isomorphism classes of $G$-étale $K$-algebras. Thus $\sharp_{\mathrm{st}}^G(V, E)$ is the weighted count of $G$-étale $K$-algebras $M$ with weights $\sharp_{\mathrm{st}}^M(V, E)$. Basic properties of the $G$-stringy point count are as follows.

- When $G = 1$, we have $\sharp_{\mathrm{st}}^{G}(V, E) = \sharp_{\mathrm{st}}(V, E)$.
- When $V = \mathbb{A}^n_{\mathcal{O}_K}$, $E = 0$ and the $G$-action is linear, then

$$\sharp_{\mathrm{st}}^{M} V = q^{n-\mathbf{v}_V(M)}/\sharp \mathrm{Aut}^G(M/K),$$

  where $\mathbf{v}_V$ is a function associated to the $G$-action on $V$, and $\mathrm{Aut}^G(M/K)$ is the group of $G$-equivariant $K$-automorphisms of $M$.

The invariant $\sharp_{\mathrm{st}}^{G}(V, E)$ is roughly the point-counting realization of a motivic invariant studied in [Yas16], which is a simultaneous refinement and generalization of the orbifold $E$-function and the stringy $E$-function considered by Batyrev [Bat99].

If we set $X := V/G$, there exists a unique $\mathbb{Q}$-divisor $D$ on $X$ such that the natural morphism $(V, E) \to (X, D)$ of pairs is crepant. Our main result is as follows.

**Theorem 1.1.** *We have*

$$\sharp_{\mathrm{st}}(X, D) = \sharp_{\mathrm{st}}^{G}(V, E).$$

This is the point-counting version of a conjecture in [Yas16]. The proof basically follows the strategy presented in [Yas, Yas16], which generalizes arguments in characteristic zero by Denef–Loeser [DL02], except that we use $p$-adic measures instead of motivic integration. This switch, from motives to numbers of points, and from motivic integration to $p$-adic measures, enables us to avoid the use of conjectural moduli spaces which the author relied on in [Yas, Yas16]. It also makes a large part of the arguments much simpler. Although the author believes that we will have the desired moduli spaces and prove more general and stronger results by means of motivic integration in near future, it is nice to have an elementary and short proof of a result a little weaker but still strong enough for many applications. In the text, we prove a slightly more general result than the theorem above: $\sharp_{\mathrm{st}}(X, D)_{\overline{C}} = \sharp_{\mathrm{st}}^{G}(V, E)_C$ for a $G$-stable constructible subset $C$ of $V \otimes_{\mathcal{O}_K} k$ and its image $\overline{C}$ in $X \otimes_{\mathcal{O}_K} k$.

It is suggestive to write the equality of the theorem as

$$\sum_{x \in X(k)} \sharp_{\mathrm{st}}(X, D)_x = \sum_{M} \sharp_{\mathrm{st}}^{M}(V, E),$$

an equality between a weighted count of $k$-points of $X$ and one of $G$-étale $K$-algebras. A particularly interesting case of the theorem is as follows. We suppose that $V = \mathbb{A}^n_{\mathcal{O}_K}$, the $G$-action is linear and has no pseudo-reflection, and there exists a crepant proper birational morphism $Y \to X$ with $Y$ regular. If we denote the $\mathcal{O}_K$-smooth locus of $Y$ by $Y_{\mathrm{sm}}$, then the theorem reduces to the form

$$\sharp Y_{\mathrm{sm}}(k) = \sum_{M} q^{n-\mathbf{v}_V(M)}/\sharp \mathrm{Aut}^G(M/K). \tag{1.1}$$

When it is possible to count $k$-points of $Y_{\mathrm{sm}}$ explicitly, we readily obtain a *mass formula* for $G$-étale $K$-algebras with respect to weights $q^{n-\mathbf{v}_V(M)}/\sharp \mathrm{Aut}^G(M/K)$.

Serre [Ser78] proved a beautiful mass formula for totally ramified field extensions $L/K$ of fixed degree with respect to weights determined by discriminants. He gave two different proofs. Krasner [Kra79] gave an alternative proof by using a formula which had been obtained by himself. As far as the author knows, these have been all known proofs of Serre's mass formula. Bhargava [Bha07] proved a similar mass formula for all étale $K$-algebras of fixed degree, using Serre's formula. Kedlaya [Ked07] interpreted Bhargava's formula as a mass formula for local Galois representations (that is, continuous homomorphisms $\mathrm{Gal}(K^{\mathrm{sep}}/K) \to S_n \subset \mathrm{GL}_n(\mathbb{C})$) with respect to the Artin conductor. He then studied the case where $S_n$ is replaced with other groups.

Wood and Yasuda [WY15] showed a close relation between the function $\mathbf{v}_V$ and the Artin conductor. Using this and a desingularization by the Hilbert scheme of points, we can deduce Bhargava's formula as a special case of formula (1.1). A similar relation between Bhargava's formula and the Hilbert scheme of points was discussed in [WY15]. Then, using for instance the exponential formula relating Serre's and Bhargava's formulas [Ked07, p. 8], we can give a new proof of Serre's formula. In a similar way, we can also prove Kedlaya's mass formula for the group of signed permutation matrices in $\mathrm{GL}_n(\mathbb{C})$ [Ked07], unless $K$ has residual characteristic two. Detailed computation of the last example will be given in [WY17]. These new proofs of mass formulas are not as easy as the original ones. However, they are interesting because they fit into the general framework of the wild McKay correspondence and reduce the problem to explicit computation of desingularization, which seems unrelated at first glance.

The paper is organized as follows. In Section 2 we set our conventions and notation. In Section 3 we recall $K$-analytic manifolds with $K$ a local field and $p$-adic measures on them associated to differential forms. In Section 4 we define stringy point counts of log pairs. In Section 5 we show a certain one-to-one correspondence of points associated to a Galois cover of varieties. In Section 6 we discuss the untwisting technique, which is the technical core of the proof of our main result. In Section 7 we prove the main result. In Section 8 we discuss the case where a finite group acts linearly on an affine space and its application to mass formulas.

## 2. Conventions and notation

**2.1.** Throughout the paper, we fix a non-archimedean local field $K$, that is, a finite extension of either $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$. We denote its integer ring by $\mathcal{O}_K$, its residue field by $k$, the cardinality of $k$ by $q$, and the maximal ideal of $\mathcal{O}_K$ by $\mathfrak{m}_K$.

**2.2.** We usually denote by $M$ an $G$-étale $K$-algebra (Section 5.1) and by $\operatorname{Spec} L$ a connected component of $\operatorname{Spec} M$ such that $L$ is a finite separable field extension of $K$. We denote by $\mathcal{O}_M$ and $\mathcal{O}_L$ the rings of integers of $M$ and $L$ respectively. We denote by $H$ the stabilizer subgroup of $G$ of this component.

**2.3.** If $R$ is either $K$, $\mathcal{O}_K$ or $k$, an *$R$-variety* means a reduced quasi-projective $R$-scheme $X$ such that

- $X$ is flat and of finite type over $R$,

- $X$ is equi-dimensional over $R$: all irreducible components have the same relative dimension over $R$, and
- the structure morphism $X \to \operatorname{Spec} R$ is smooth on an open dense subscheme of $X$.

The *dimension* of an $R$-variety always means its relative dimension over $R$. We usually denote the dimension of a variety by $d$ (or $n$ when the variety is an affine space). For an $\mathcal{O}_K$-variety $X$, we define $X_K := X \otimes_{\mathcal{O}_K} K$ and $X_k := X \otimes_{\mathcal{O}_K} k$. Note that $X_k$ is not generally reduced or a $k$-variety.

**2.4.** For an $\mathcal{O}_K$-variety $X$, we denote by $X(\mathcal{O}_K)^\circ$ the set of $\mathcal{O}_K$-points $\operatorname{Spec} \mathcal{O}_K \to X$ that send the generic point of $\operatorname{Spec} \mathcal{O}_K$ into the locus where $X$ is $\mathcal{O}_K$-smooth.

**2.5.** Groups act on schemes on the left, and on rings, fields and modules on the right, unless otherwise noted. Thus, for an affine scheme $\operatorname{Spec} R$, if a group $G$ acts on $\operatorname{Spec} R$ and if $g : \operatorname{Spec} R \to \operatorname{Spec} R$ is the automorphism induced by $g \in G$, then we have the corresponding ring automorphism $g^* : R \to R$ and the same group $G$ naturally acts on $R$ by $r \cdot g := g^*(r)$, $r \in R$. Conversely, a $G$-action on $R$ gives a natural $G$-action on $\operatorname{Spec} R$ in a similar way.

## 3. $p$-adic measures on $K$-analytic manifolds

In this section, we review basic material on the Haar measure on $K^d$ for a local field $K$ and a measure on a $K$-analytic manifold induced by a differential form.

**3.1.** Let $K$ be a non-archimedean local field. Recall that we always denote the cardinality of the residue field $k$ by $q$. Let $|\cdot|$ be the normalized absolute value on $K$ so that $|\varpi| = q^{-1}$ for a uniformizer $\varpi \in \mathcal{O}_K$. For an integer $d \geq 0$, we define $\mu_{K^d}$ to be the Haar measure of $K^d$ normalized so that $\mu_{K^d}(\mathcal{O}_K^d) = 1$. A function $f : U \to K$ on an open subset $U \subset K^d$ is called *$K$-analytic* if in a neighborhood of every point of $U$, $f$ is expressed as a convergent Taylor series.

**Lemma 3.1.** *Let $f : U \to K$ be a $K$-analytic function defined on an open compact subset $U \subset K^d$. Suppose that $f$ is nowhere locally constant. Then $\mu_{K^d}(f^{-1}(0)) = 0$.*

*Proof.* This result should be well-known to specialists, though the author could not find a reference. For the sake of completeness, we give a proof here, which follows a suggestion of a referee. (A similar result is found in Igusa's book [Igu00, Lemma 8.3.1]. However, he proves it only for characteristic zero.)

The proof is by induction on $d$. If $d = 0$, there is nothing to prove. If $d = 1$, then looking at the Taylor expansion, we see that $f^{-1}(0)$ is a discrete subset of $U$ and has measure zero. For $d > 1$, it suffices to show that if $f^{-1}(0)$ contains the origin $o \in K^d$, then there exists an open neighborhood $V$ of $o$ such that $f^{-1}(0) \cap V$ has measure zero. There exists a line $L \cong K$ passing through $o$ such that $f|_{L \cap U}$ is not locally constant around $o$. Indeed, if there is no such line and if $f$ is expressed as a power series on a neighborhood $V$ of $o$, then $f$ is constant on $V$, which contradicts the assumption. By a suitable linear transform, we may assume that $L$ is given by $x_1 = \cdots = x_{d-1} = 0$. From the Weierstrass preparation theorem (for instance, see [Igu00, Theorem 2.3.1]), $f$ is of the form

$$g(x_1, \ldots, x_d)\big(x_d^m + f_1(x_1, \ldots, x_{d-1})x_d^{m-1} + \cdots + f_m(x_1, \ldots, x_{d-1})\big)$$

on a neighborhood $V$ of $o$, where $g, f_1, \ldots, f_m$ are convergent power series such that $g$ is nowhere vanishing on $V$. Define $h : V \to K$ by

$$h := f/g = x_d^m + f_1(x_1, \ldots, x_{d-1})x_d^{m-1} + \cdots + f_m(x_1, \ldots, x_{d-1}),$$

which has the same zero locus as $f|_V$. Let $\pi : V \to K^{d-1}$ be the projection to the first $d-1$ coordinates. For every $y \in \pi(V)$, $h|_{\pi^{-1}(y)}$ is a function given by a nonzero polynomial, in particular, it is nowhere locally constant. From the case $d = 1$, its zero locus has measure zero with respect to $\mu_K$. Now the Fubini–Tonelli theorem (for instance, see [Hal50, Theorems A and B, p. 147]) shows that $f^{-1}(0) \cap V = h^{-1}(0)$ has measure zero with respect to $\mu_{K^d}$, which completes the proof. □

**3.2.** *$K$-analytic manifolds* are defined in a similar way to ordinary manifolds. For details, we refer the reader to [Igu00, Section 2.4]. We can similarly define *$K$-analytic differential forms* as well. Let $X$ be a $K$-analytic manifold of dimension $d$. For a $K$-analytic $d$-form $\omega$ on $X$ and an open compact subset $U$ of $X$, one can define the integral

$$\int_U |\omega| \in \mathbb{R}_{\geq 0}.$$

When $\omega$ is written as $f(x)dx_1 \wedge \cdots \wedge dx_d$ for a $K$-analytic function $f(x)$ and local coordinates $x_1, \ldots, x_d$ on $U$, then

$$\int_U |\omega| = \int_{U'} |f(x)|\, d\mu_{K^d},$$

where $U'$ is the open subset of $K^n$ corresponding to $U$. Thus a $K$-analytic $d$-form defines a measure $\mu_\omega$ on $X$: for a compact open subset $U \subset X$,

$$\mu_\omega(U) := \int_U |\omega| \in \mathbb{R}_{\geq 0},$$

and for an arbitrary open subset $U \subset X$,

$$\mu_\omega(U) := \sup\{\mu_\omega(U') \mid U' \subset U\text{: open and compact}\} \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

We need to generalize this slightly as in [Ito04, Wan98]. Let $\omega$ be an $r$-fold $d$-form, that is, a section of $(\Omega_X^d)^{\otimes r}$. Here $\Omega_X^d$ is the sheaf of $K$-analytic $d$-forms and the tensor product is taken over the sheaf of $K$-analytic functions. Locally $\omega$ is written as $f(x)(dx_1 \wedge \cdots \wedge dx_d)^{\otimes r}$ with $f(x)$ a $K$-analytic function, say on an open compact subset $U$. We then define

$$\int_U |\omega|^{1/r} := \int_{U'} |f(x)|^{1/r}\, d\mu_{K^d}$$

and extend the definition to an arbitrary $r$-fold $d$-form on $X$ in the obvious way. We define the measure $\mu_\omega$ by

$$\mu_\omega(U) := \int_U |\omega|^{1/r}$$

for an open compact $U$, and similarly for an arbitrary open subset.

## 4. Log pairs

Using measures on $K$-analytic manifolds considered in the preceding section, we introduce, in this section, the notion of *stringy point count* for log pairs and study its basic properties.

**4.1.** Let $X$ be a $d$-dimensional $\mathcal{O}_K$-variety. We write $X_k := X \otimes_{\mathcal{O}_K} k$ and $X_K := X \otimes_{\mathcal{O}_K} K$. Let $X_{K,\mathrm{sm}}$ be the $K$-smooth locus of $X_K$ and let

$$X(\mathcal{O}_K)^\circ := X(\mathcal{O}_K) \cap X_{K,\mathrm{sm}}(K),$$

thinking of $X(\mathcal{O}_K)$ as a subset of $X(K) = X_K(K)$. This set $X(\mathcal{O}_K)^\circ$ has a natural structure of a $K$-analytic manifold.

Let $\mathcal{I}$ be an invertible $\mathcal{O}_X$-submodule of $(\Omega^d_{X/\mathcal{O}_K})^{\otimes r} \otimes K(X)$, where $K(X)$ is the sheaf of total quotient rings of $X$. We define a measure $\mu_\mathcal{I}$ on $X(\mathcal{O}_K)^\circ$ as follows. Let $X = \bigcup U_i$ be a Zariski open cover so that $X(\mathcal{O}_K)^\circ = \bigcup U_i(\mathcal{O}_K)^\circ$ and $\mathcal{I}|_{U_i}$ is a free $\mathcal{O}_{U_i}$-module. Let $\omega_i \in \mathcal{I}|_{U_i}$ be a generator. It defines an $r$-fold $d$-form $\omega_i^{\mathrm{an}}$ on the $K$-analytic manifold $U_i(\mathcal{O}_K)^\circ$ in the obvious way, and the measure $\mu_{\omega_i^{\mathrm{an}}}$. If $\omega_i'$ is another generator of $\mathcal{I}|_{U_i}$, then there exists a nowhere vanishing regular function $f$ on $U_i$ such that $\omega_i = f\omega_i'$. If $f^{\mathrm{an}}$ is the corresponding $K$-analytic function on $U_i(\mathcal{O}_K)^\circ$, then $|f^{\mathrm{an}}| \equiv 1$. Therefore the measure $\mu_{\omega_i^{\mathrm{an}}}$ does not depend on the choice of generator. Now it is clear that the measures $\mu_{\omega_i^{\mathrm{an}}}$ for different $i$ glue together and define a measure on the entire space $X(\mathcal{O}_K)^\circ$; we denote it by $\mu_\mathcal{I}$. We further extend $\mu_\mathcal{I}$ to $X(\mathcal{O}_K)$ by declaring all subsets of $X(\mathcal{O}_K) \setminus X(\mathcal{O}_K)^\circ$ to have measure zero.

The following lemma is a slight generalization of [Wei82, Theorem 2.2.5].

**Lemma 4.1.** *If $X$ is $\mathcal{O}_K$-smooth and $\mathcal{I} = (\Omega^d_{X/\mathcal{O}_K})^{\otimes r}$, then*

$$\mu_\mathcal{I}(X(\mathcal{O}_K)) = \sharp X(k)/q^d.$$

*Proof.* For $x \in X(k)$, let $X(\mathcal{O}_K)_x$ be the set of $\mathcal{O}_K$-points which induce $x$ by composition with $\operatorname{Spec} k \to \operatorname{Spec} \mathcal{O}_K$. If $x_1, \dots, x_d$ are local coordinates around $x$, then they give a bijection from $X(\mathcal{O}_K)_x$ onto $\mathfrak{m}_K^d \subset K^d$. On the other hand, $\mathcal{I}$ has a local generator $(dx_1 \wedge \cdots \wedge dx_d)^{\otimes r}$. Therefore

$$\mu_\mathcal{I}(X(\mathcal{O}_K)_x) = \int_{\mathfrak{m}_K^d} 1\, d\mu_{K^d} = \mu_{K^d}(\mathfrak{m}_K^d) = q^{-d},$$

and the lemma follows. □

**Lemma 4.2.** *For an integer $s > 0$, regarding $\mathcal{I}^{\otimes s}$ as an $\mathcal{O}_K$-submodule of $(\Omega^d_{X/\mathcal{O}_K})^{\otimes sr} \otimes K(X)$, we have $\mu_{\mathcal{I}} = \mu_{\mathcal{I}^{\otimes s}}$.*

*Proof.* If $\mathcal{I}$ has a local generator $f(x)(dx_1 \wedge \cdots \wedge dx_d)^{\otimes r}$, then $f(x)^s(dx_1 \wedge \cdots \wedge dx_d)^{\otimes sr}$ is a local generator of $\mathcal{I}^{\otimes s}$, and we have

$$\int |f(x)|^{1/r}\, d\mu_{K^d} = \int |f(x)^s|^{1/(rs)}\, d\mu_{K^d},$$

where the integrals are taken over a suitable open compact subset of $K^d$. The lemma easily follows. □

**Lemma 4.3.** *For a subscheme $Y \subset X$ of positive codimension,*

$$\mu_{\mathcal{I}}(Y(\mathcal{O}_K)) = 0.$$

*Proof.* This is a direct consequence of Lemma 3.1. □

**4.2.** For a normal $\mathcal{O}_K$-variety $X$, the canonical sheaf $\omega_X = \omega_{X/\mathcal{O}_K}$ is defined as in [Kol13, pp. 7–8]. It is a reflexive sheaf, in particular, locally free in codimension one, and coincides with $\Omega^d_{X/\mathcal{O}_K} := \bigwedge^d \Omega_{X/\mathcal{O}_K}$ on the $\mathcal{O}_K$-smooth locus. We denote the corresponding divisor by $K_X$, which is determined up to linear equivalence.

**Definition 4.4.** A *log pair* is a pair $(X, D)$ of a normal $\mathcal{O}_K$-variety $X$ and a $\mathbb{Q}$-divisor $D$ (a Weil divisor with rational coefficients) on $X$ such that $K_X + D$ is $\mathbb{Q}$-Cartier. We sometimes call a log pair $(X, D)$ a *log structure* on $X$.

We identify a normal $\mathbb{Q}$-Gorenstein ($K_X$ is $\mathbb{Q}$-Cartier) $\mathcal{O}_K$-variety $X$ with the log pair $(X, 0)$.

Let $(X, D)$ be a log pair and let $r \in \mathbb{N}$ be such that $r(K_X + D)$ is Cartier. Then the invertible sheaf $\mathcal{O}_X(r(K_X + D))$ is naturally a subsheaf of $(\Omega^d_{X/\mathcal{O}_K})^{\otimes r} \otimes K(X)$.

**Definition 4.5.** A *morphism* of log pairs, $f : (Y, E) \to (X, D)$, is a morphism of the underlying varieties, $f : Y \to X$. We say that a morphism $f : (Y, E) \to (X, D)$ is *crepant* if for $r \in \mathbb{N}$ such that $r(K_X + D)$ and $r(K_Y + E)$ are both Cartier, the canonical morphism $f^*(\Omega^d_{X/\mathcal{O}_K})^{\otimes r} \to (\Omega^d_{Y/\mathcal{O}_K})^{\otimes r}$ induces an isomorphism $f^*\mathcal{O}_X(r(K_X+D)) \to \mathcal{O}_Y(r(K_Y + E))$.

**Lemma 4.6.** *Let $(X, D)$ be a log pair and let $f : Y \to X$ be a generically étale morphism of normal varieties. There exists a unique $\mathbb{Q}$-divisor $E$ on $Y$ such that the morphism $f : (Y, E) \to (X, D)$ is crepant.*

*Proof.* The pull-back $f^*\mathcal{O}_X(r(K_X + D))$ is an invertible subsheaf of $(\Omega^d_{Y/\mathcal{O}_K})^{\otimes r} \otimes K(Y)$. Hence there exists a unique Weil divisor $E'$ such that $\omega^{\otimes r}_{Y/\mathcal{O}_K}(E')$ coincides with $f^*\mathcal{O}_X(r(K_X + D))$ in codimension one as subsheaves of $(\Omega^d_{Y/\mathcal{O}_K})^{\otimes r} \otimes K(Y)$. Now $(1/r)E'$ is the desired $\mathbb{Q}$-divisor. □

**4.3.** In this subsection we consider measures related to log pairs.

**Definition 4.7.** Let $(X, D)$ be a log pair and $r \in \mathbb{N}$ such that $r(K_X + D)$ is Cartier. We define the measure $\mu_{X,D}$ on $X(\mathcal{O}_K)$ to be $\mu_{\mathcal{O}_X(r(K_X+D))}$. For a normal $\mathbb{Q}$-Gorenstein $\mathcal{O}_K$-variety $X$, we write $\mu_{X,0}$ simply as $\mu_X$.

From Lemma 4.2, the definition does not depend on the choice of $r$. If $X$ is $\mathcal{O}_K$-smooth, from Lemma 4.1, we have

$$\mu_X(X(\mathcal{O}_K)) = \sharp X(k)/q^d.$$

We use the following theorem many times in later sections:

**Theorem 4.8.** *Let $f : (Y, E) \to (X, D)$ be a crepant morphism of log pairs. Suppose that a finite group $G$ acts faithfully on $Y$ and trivially on $X$ so that the morphism $f : Y \to X$ of the underlying varieties is $G$-equivariant. Suppose that the induced morphism $Y/G \to X$ is birational. Then, for a $G$-stable open subset $A \subset Y(\mathcal{O}_K)$, we have*

$$\frac{1}{\sharp G}\mu_{Y,E}(A) = \mu_{X,D}(f(A)) \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

*Proof.* From Lemma 4.3, removing measure zero subsets from $A$ and $f(A)$, we may suppose that the map $A \to f(A)$ is a $G$-equivariant étale morphism of $K$-analytic manifolds with $A/G \cong f(A)$. Each point $x \in f(A)$ has a $K$-analytic open neighborhood $U$ such that $f^{-1}(U) \subset A$ is isomorphic to the disjoint union of $\sharp G$ copies of $U$. Let $r \in \mathbb{N}$ be such that $r(K_X + D)$ is Cartier and $\mathcal{I} := \mathcal{O}_X(r(K_X + D))$. For a Zariski open $V \subset X$ such that $U \subset V(\mathcal{O}_K)$, if $\omega \in \Gamma(V, \mathcal{I})$ is a generator of $\mathcal{I}|_V$, we clearly have

$$\frac{1}{\sharp G}\int_{f^{-1}(U)} |f^{-1}\omega|^{1/r} = \int_U |\omega|^{1/r}.$$

It is now easy to deduce the theorem. □

**4.4.** Finally, we define the stringy point count:

**Definition 4.9.** Let $(X, D)$ be a log pair, $d$ the dimension of the $\mathcal{O}_K$-variety $X$ and $C \subset X_k$ a constructible subset. Let $X(\mathcal{O}_K)_C$ be the set of $\mathcal{O}_K$-points sending the closed point of Spec $\mathcal{O}_K$ into $C$. We define the *stringy point count* of a log pair $(X, D)$ along $C$ by

$$\sharp_{\mathrm{st}}(X, D)_C := q^d \cdot \mu_{X,D}(X(\mathcal{O}_K)_C).$$

When $C = X_k$, we omit the subscript $C$ and write $\sharp_{\mathrm{st}}(X, D)$. When $C$ consists of a single $k$-point $x$, we write the subscript simply as $x$.

We obviously have

$$\sharp_{\mathrm{st}}(X, D) = \sum_{x \in X(k)} \sharp_{\mathrm{st}}(X, D)_x,$$

showing that $\sharp_{\mathrm{st}}(X, D)$ is the count of $k$-points with weights $\sharp_{\mathrm{st}}(X, D)_x$. If $X$ is a smooth variety, identified with the log pair $(X, 0)$, then $\sharp_{\mathrm{st}} X = \sharp X(k)$.

**Corollary 4.10.** *Let $f : (Y, E) \to (X, D)$ be a crepant proper birational morphism of log pairs and $C \subset X_k$ a constructible subset. We have*

$$\sharp_{\mathrm{st}}(Y, E)_{f^{-1}(C)} = \sharp_{\mathrm{st}}(X, D)_C.$$

*Proof.* We apply Theorem 4.8 to the case where $G = 1$ and $A = Y(\mathcal{O}_K)_{f^{-1}(C)}$. From the valuative criterion for properness, $f(A)$ coincides with $X(\mathcal{O}_K)_C$ modulo measure zero subsets, and the corollary follows. □

**4.5.** We now give an explicit formula for stringy point counts under a certain assumption, an analogue of Denef's formula [Den87, Theorem 3.1] and the definition of the stringy $E$-function [Bat98]. Although we do not use it in the rest of the paper, the formula is useful for applications. Firstly we suppose that $X$ is regular. We also suppose that $D$ is simple normal crossing in the following sense: if we write $D = \sum a_i D_i$ with $D_i$ prime divisors and $a_i \neq 0$, then

- for every $i$ and every $k$-point $x \in D_i$ where $X$ is $\mathcal{O}_K$-smooth, the completion of $D_i$ is irreducible, and
- for every $k$-point $x \in X$ where $X$ is $\mathcal{O}_K$-smooth, there exists a regular system of parameters $x_0 = \varpi, x_1, \dots, x_d$ on a neighborhood $U$ of $x$ with $\varpi$ a uniformizer of $K$ such that the support of $D$ is defined by a product $\prod_{j=1}^m x_{i_j}$ $(0 \le i_1 < \cdots < i_m \le d)$.

We then rewrite $D$ as

$$D = \sum_{h=1}^{l} a_h A_h + \sum_{i=1}^{m} b_i B_i + \sum_{j=1}^{n} c_j C_j$$

such that

- for every $h$, $a_h \neq 0$, $A_h \subset X_k$ and $X$ is $\mathcal{O}_K$-smooth at the generic point of $A_h$,
- for every $i$, $b_i \neq 0$, $B_i \subset X_k$ and $X$ is *not* $\mathcal{O}_K$-smooth at the generic point of $B_i$, and
- for every $j$, $c_j \neq 0$ and $C_j$ dominates $\operatorname{Spec} \mathcal{O}_K$.

Such a decomposition of $D$ is unique. For each $h$, we let $A_h^\circ$ be the locus in $A_h$ where $X$ is $\mathcal{O}_K$-smooth. For each subset $J \subset \{1, \dots, n\}$, we set

$$C_J^\circ := \Big(\bigcap_{j \in J} C_j\Big) \setminus \Big(\bigcup_{j \in \{1,\dots,n\} \setminus J} C_j\Big).$$

Let $X_{\mathrm{sm}}$ be the $\mathcal{O}_K$-smooth locus of $X$.

**Proposition 4.11.** *The stringy point count $\sharp_{\mathrm{st}}(X, D)_C$ is finite if and only if $c_j < 1$ for every $j$ such that $(C_j \cap C \cap X_{\mathrm{sm}})(k) \neq \emptyset$. If these equivalent conditions hold, we have*

$$\sharp_{\mathrm{st}}(X, D)_C = \sum_{h=1}^{l} q^{a_h} \sum_{J \subset \{1,\dots,n\}} \sharp(C \cap A_h^\circ \cap C_J^\circ)(k) \prod_{j \in J} \frac{q-1}{q^{1-c_j} - 1}.$$

*Proof.* The proof follows the one of [Ito04, Proposition 3.4]. There is no $\mathcal{O}_K$-point passing through a $k$-point where $X$ is not $\mathcal{O}_K$-smooth. Therefore

$$\sharp_{\mathrm{st}}(X, D)_C = \sum_{x \in (C \cap X_{\mathrm{sm}})(k)} \sharp_{\mathrm{st}}(X, D)_x.$$

We fix $x \in (C \cap X_{\mathrm{sm}})(k)$ and suppose that $C_1, \dots, C_\lambda$ are those prime divisors among $C_1, \dots, C_n$ containing $x$. Let $a$ be $a_h$ if $A_h$ contains $x$, and zero if none of $A_h$ contains $x$. To show the proposition, it suffices to show:

**Claim.** *If $c_j \geq 1$ for some $j$ with $1 \leq j \leq \lambda$, then $\sharp_{\mathrm{st}}(X, D)_x = \infty$, and otherwise*

$$\sharp_{\mathrm{st}}(X, D)_x = q^a \cdot \prod_{j=1}^{\lambda} \frac{q-1}{q^{1-c_j}-1}.$$

To see this, using local coordinates $x_0 = \varpi, x_1, \dots, x_d$, we suppose that $A_h$ containing $x$ (if any) is defined by $x_0$, and for $1 \leq i \leq \lambda$, $C_i$ is defined by $x_i$. Then, for an integer $r > 0$,

$$\begin{aligned}\sharp_{\mathrm{st}}(X, D)_x &= q^d \int_{X(\mathcal{O}_K)_x} |\varpi^{-ra} x_1^{-rc_1} \cdots x_\lambda^{-rc_\lambda}|^{1/r} dx_1 \wedge \cdots \wedge dx_d \\ &= q^a \cdot \prod_{j=1}^{d} q \int_{\mathfrak{m}_K} |x|^{-c_j}\, dx,\end{aligned}$$

with $c_j := 0$ for $j > \lambda$. For any $c \in \mathbb{R}$, we have

$$\begin{aligned}\int_{\mathfrak{m}_K} |x|^{-c}\, dx &= \sum_{i=1}^{\infty} q^{ic} \cdot \mu_K(\mathfrak{m}_K^i \setminus \mathfrak{m}_K^{i+1}) = \sum_{i=1}^{\infty} q^{ic} \cdot (q^{-i} - q^{-i-1}) \\ &= \begin{cases} q^{-1} \cdot \frac{q-1}{q^{1-c}-1} & (c < 1) \\ \infty & (c \geq 1). \end{cases}\end{aligned}$$

This shows the claim and the proposition. □

## 5. Group actions

In this section, we consider a $G$-cover of varieties $V \to X = V/G$ with $G$ a finite group and show a correspondence of $\mathcal{O}_K$-points of $X$ and equivariant $\mathcal{O}_M$-points with $M$ $G$-étale $K$-algebras. From now on, $G$ denotes a finite group.

**5.1.** We first define some basic notions.

**Definition 5.1.** A *$G$-étale $K$-algebra* means a finite $K$-algebra $M$ of degree $\sharp G$ endowed with a (right) $G$-action such that the subset of $G$-invariant elements, $M^G$, is identical to $K$. An *isomorphism* $M \to N$ of $G$-étale $K$-algebras is a $K$-algebra isomorphism compatible with the given $G$-actions on $M$ and $N$. We denote the set of representatives of isomorphism classes of $G$-étale $K$-algebras by $G$-Ét$(K)$. We denote the automorphism group of a $G$-étale $K$-algebra $M$ by $\mathrm{Aut}^G(M/K)$.

Let $M \in G$-Ét$(K)$. There exists a field extension $L$ of $K$ such that $M$ is isomorphic to the product $L^c$ of $c$ copies of $L$ as an $K$-algebra for some positive integer $c$. Geometrically we can write

$$\operatorname{Spec} M = \overbrace{\operatorname{Spec} L \sqcup \cdots \sqcup \operatorname{Spec} L}^{c}.$$

Let $H \subset G$ be the stabilizer of one connected component of $\operatorname{Spec} M$. The subgroup $H$ depends on the choice of the component, but is unique up to conjugation in $G$. The automorphism group $\operatorname{Aut}^G(M/K)$ is isomorphic to $C_G(H)^{\mathrm{op}}$, the opposite group of the centralizer $H$ (for instance, see [Yas16]).

**Remark 5.2.** $G$-étale $K$-algebras $M$ correspond to $G$-conjugacy classes of continuous homomorphisms $\rho : \operatorname{Gal}(K^{\mathrm{sep}}/K) \to G$ with $K^{\mathrm{sep}}$ a separable closure of $K$. The stabilizer $H$ of a connected component of $\operatorname{Spec} M$ coincides with the image of the corresponding map $\rho$ up to conjugation. Since the Galois group of a finite Galois extension of a local field is always solvable (see [Ser79, p. 68]), if $G$ is not solvable, then no $G$-étale $K$-algebra $M$ is a field.

**5.2.** Let $V$ be an $\mathcal{O}_K$-variety endowed with a faithful $G$-action and let $X := V/G$ be the quotient variety. For $M \in G\text{-}\acute{\mathrm{E}}\mathrm{t}(K)$, we let $V(\mathcal{O}_M)^G$ be the set of $G$-equivariant $\mathcal{O}_M$-points of $V$, that is, $G$-equivariant $\mathcal{O}_K$-morphisms $\operatorname{Spec} \mathcal{O}_M \to V$. A $G$-equivariant $\mathcal{O}_M$-point $\operatorname{Spec} \mathcal{O}_M \to V$ induces a natural morphism between the quotients of the source and target, $\operatorname{Spec} \mathcal{O}_K \to X$. This defines a map

$$V(\mathcal{O}_M)^G \to X(\mathcal{O}_K).$$

Let $H \subset G$ be the stabilizer of a connected component of $\operatorname{Spec} M$ as above. The restriction of the $G$-action on $V$ to $C_G(H)$ induces a (left) $C_G(H)$-action on $V(\mathcal{O}_M)^G$, namely $g \in C_G(H)$ sends a $G$-equivariant point $\alpha : \operatorname{Spec} \mathcal{O}_M \to V$ to $g \circ \alpha$. After identifying $\operatorname{Aut}^G(M/K)$ with $C_G(H)^{\mathrm{op}}$, this action is identical to the left $\operatorname{Aut}^G(M/K)^{\mathrm{op}}$-action corresponding to the right $\operatorname{Aut}^G(M/K)$-action induced from the $\operatorname{Aut}^G(M/K)$-action on $\operatorname{Spec} \mathcal{O}_M$. The map $V(\mathcal{O}_M)^G \to X(\mathcal{O}_K)$ factors through $V(\mathcal{O}_M)^G/C_G(H)$.

**Definition 5.3.** Let $X(\mathcal{O}_K)^\natural$ be the set of $\mathcal{O}_K$-points $\operatorname{Spec} \mathcal{O}_K \to X$ sending the generic point into the unramified locus of $V \to X$ in $X$, and let $V(\mathcal{O}_M)^{G,\natural}$ be the set of $G$-equivariant $\mathcal{O}_M$-points sending the generic points into the unramified locus of the same morphism in $V$.

The map $V(\mathcal{O}_M)^G/C_G(H) \to X(\mathcal{O}_K)$ restricts to $V(\mathcal{O}_M)^{G,\natural}/C_G(H) \to X(\mathcal{O}_K)^\natural$.

**Proposition 5.4.** *The map*

$$\bigsqcup_{M \in G\text{-}\acute{\mathrm{E}}\mathrm{t}(K)} V(\mathcal{O}_M)^{G,\natural}/C_G(H) \to X(\mathcal{O}_K)^\natural$$

*is bijective.*

*Proof.* We first show the injectivity. Let $\alpha : \operatorname{Spec} \mathcal{O}_M \to V$ be a $G$-equivariant point in $V(\mathcal{O}_M)^\natural$ and $\beta : \operatorname{Spec} \mathcal{O}_K \to X$ its image in $X(\mathcal{O}_K)^\natural$. Then the class of $\alpha$ modulo the $C_G(H)$-action is reconstructed from $\beta$ as the normalization of $\operatorname{Spec} \mathcal{O}_K \times_{\beta,X,\psi} V$ with $\psi$ the quotient morphism $V \to X$. Indeed the induced morphism $\operatorname{Spec} M \to \operatorname{Spec} K \times_{\beta,X,\psi} V$ is a morphism of étale $G$-torsors over $\operatorname{Spec} K$, hence is an isomorphism. This shows the injectivity.

Given a point $\beta : \operatorname{Spec} \mathcal{O}_K \to X$ in $X(\mathcal{O}_K)^\natural$, the normalization of $\operatorname{Spec} \mathcal{O}_K \times_{\beta,X,\psi} V$ is the spectrum of $\mathcal{O}_M$ for a $G$-étale $K$-algebra $M$. This shows the surjectivity. □

## 6. Untwisting

The key for the formulation and the proof of our main results is *untwisting*, which makes a correspondence between a set of equivariant $\mathcal{O}_M$-points of a $G$-variety $V$ and a set of $\mathcal{O}_K$-points of another variety $V^{|M|}$ constructed by twisting $V$ somehow. At the cost of the twist of $V$, the study of equivariant points (twisted arcs) reduces to the one of ordinary points (ordinary arcs).

**6.1.** We fix an $\mathcal{O}_K$-linear faithful $G$-action on an affine space

$$\mathbb{A}^n_{\mathcal{O}_K,\boldsymbol{x}} := \operatorname{Spec} \mathcal{O}_K[\boldsymbol{x}] = \operatorname{Spec} \mathcal{O}_K[x_1, \dots, x_n].$$

Let $\mathcal{O}_K[\boldsymbol{x}]_1$ be the linear part of $\mathcal{O}_K[\boldsymbol{x}]$. We introduce a notion playing the central role in the untwisting technique.

**Definition 6.1.** We define the *tuning module* $\Xi_M$ by

$$\Xi_M := \{\phi \in \operatorname{Hom}_{\mathcal{O}_K}(\mathcal{O}_K[\boldsymbol{x}]_1, \mathcal{O}_M) \mid \forall g \in G,\ \phi \circ g = g \circ \phi\}.$$

The module $\Xi_M$ is actually identified with $\mathbb{A}^n_{\mathcal{O}_K,\boldsymbol{x}}(\mathcal{O}_M)^G$ by the map

$$\mathbb{A}^n_{\mathcal{O}_K,\boldsymbol{x}}(\mathcal{O}_M)^G \to \Xi_M, \quad \gamma \mapsto \gamma^*|_{\mathcal{O}_K[\boldsymbol{x}]_1}.$$

It turns out that the tuning module $\Xi_M$ is a free $\mathcal{O}_K$-submodule of $\operatorname{Hom}_{\mathcal{O}_K}(\mathcal{O}_K[\boldsymbol{x}]_1, \mathcal{O}_M) = \mathbb{A}^n_{\mathcal{O}_K,\boldsymbol{x}}(\mathcal{O}_M)$ of rank $n$ [Yas, WY15].

**Remark 6.2.** Our definition of the tuning module follows the one in [Yas16]. Noting that throughout the related literature [WY15, Yas, Yas16], a finite group always acts on an affine space $\mathbb{A}^n_{\mathcal{O}_K,\boldsymbol{x}}$ on the left, if we think that the tuning module is associated to $M$ and the $G$-variety $\mathbb{A}^n_{\mathcal{O}_K,\boldsymbol{x}}$ (rather than its coordinate ring) and if $\mathbb{A}^n_{\mathcal{O}_K,\boldsymbol{x}}(\mathcal{O}_M)$ is identified with $\mathcal{O}_M^{\oplus n}$, then our definition of $\Xi_M$ coincides with the tuning submodule considered in [Yas, WY15]. On the other hand, the coordinate ring $\mathcal{O}_K[\boldsymbol{x}]$ has a right or left action, depending on the paper, and one has to be careful about the caused notational difference.

We fix a basis $\phi_1, \dots, \phi_n \in \Xi_M$ and let $y_1, \dots, y_n$ be its dual basis so that

$$\operatorname{Hom}_{\mathcal{O}_K}(\Xi_M, \mathcal{O}_K) = \bigoplus_j \mathcal{O}_K \cdot y_j \quad \text{and} \quad \operatorname{Hom}_{\mathcal{O}_K}(\Xi_M, \mathcal{O}_M) = \bigoplus_j \mathcal{O}_M \cdot y_j.$$

We think of these modules as the linear parts of the polynomial rings $\mathcal{O}_K[\boldsymbol{y}] = \mathcal{O}_K[y_1, \dots, y_n]$ and $\mathcal{O}_M[\boldsymbol{y}] = \mathcal{O}_M[y_1, \dots, y_n]$. We set

$$\mathbb{A}^n_{\mathcal{O}_K,\boldsymbol{y}} := \operatorname{Spec} \mathcal{O}_K[\boldsymbol{y}] \quad \text{and} \quad \mathbb{A}^n_{\mathcal{O}_M,\boldsymbol{y}} := \operatorname{Spec} \mathcal{O}_M[\boldsymbol{y}].$$

**Definition 6.3.** We define an $\mathcal{O}_K$-algebra morphism $u^* : \mathcal{O}_K[\boldsymbol{x}] \to \mathcal{O}_M[\boldsymbol{y}]$ by

$$u^*(x_i) = \sum_{j=1}^n \phi_j(x_i) y_j$$

and the corresponding morphism of schemes

$$u : \mathbb{A}^n_{\mathcal{O}_M,\boldsymbol{y}} \to \mathbb{A}^n_{\mathcal{O}_K,\boldsymbol{x}}.$$

The linear part of $u^*$ is identical to the canonical map

$$\mathcal{O}_K[\boldsymbol{x}]_1 \to \mathrm{Hom}_{\mathcal{O}_K}(\Xi_M, \mathcal{O}_M) \quad f \mapsto (\phi \mapsto \phi(f)),$$

which gives an intrinsic description of $u$. This is useful in order to show that some derived maps are equivariant.

**Lemma 6.4.** *The restriction of* $u$*,* $\mathbb{A}^n_{M,\boldsymbol{y}} \to \mathbb{A}^n_{K,\boldsymbol{x}}$*, is étale.*

*Proof.* The chosen basis $\phi_1, \ldots, \phi_n$ of $\Xi_M$ is an $M$-basis of the free $M$-module

$$\mathrm{Hom}_{\mathcal{O}_K}(\mathcal{O}_K[\boldsymbol{x}]_1, M) = \mathrm{Hom}_M(M[\boldsymbol{x}]_1, M)$$

and its dual basis $y_1, \ldots, y_n$ is naturally regarded as a basis of

$$\mathrm{Hom}_M(\mathrm{Hom}_M(M[\boldsymbol{x}]_1, M), M).$$

Therefore the map $K[\boldsymbol{x}]_1 \to M[\boldsymbol{y}]_1$ corresponding to the morphism of the lemma is identified with

$$K[\boldsymbol{x}]_1 \to M[\boldsymbol{x}]_1 \to \mathrm{Hom}_M(\mathrm{Hom}_M(M[\boldsymbol{x}]_1, M), M),$$

the composition of the scalar extension and the canonical morphism to the double dual space. This proves the lemma. □

**6.2.** We introduce another notion:

**Definition 6.5.** The given $G$-action on $\mathrm{Spec}\,\mathcal{O}_M$ and the trivial $G$-action on $\mathbb{A}^n_{\mathcal{O}_K,\boldsymbol{y}}$ defines a $G$-action on the fiber product $\mathbb{A}^n_{\mathcal{O}_M,\boldsymbol{y}} = \mathrm{Spec}\,\mathcal{O}_M \times_{\mathrm{Spec}\,\mathcal{O}_K} \mathbb{A}^n_{\mathcal{O}_K,\boldsymbol{y}}$ over $\mathcal{O}_K$; we call it the *Galois G-action.*

**Lemma 6.6.** *The map* $u$ *is equivariant with respect to the given* $G$*-action on* $\mathbb{A}^n_{\mathcal{O}_K,\boldsymbol{x}}$ *and the Galois* $G$*-action on* $\mathbb{A}^n_{\mathcal{O}_M,\boldsymbol{y}}$.

*Proof.* For $1 \le i \le n$ and $g \in G$, we have

$$u^*(x_i g) = \sum_{j=1}^{n} \phi_j(x_i g) y_j.$$

From the definition of $\Xi_M$, we have

$$u^*(x_i g) = \sum_{j=1}^{n} \phi_j(x_i g) y_j = \sum_{j=1}^{n} (\phi_j(x_i) g) y_j = u^*(x_i) g,$$

and the lemma follows. □

**Proposition 6.7.** *We have*

$$u^*(\mathcal{O}_K[\boldsymbol{x}]^G) \subset \mathcal{O}_K[\boldsymbol{y}].$$

*Proof.* Note that $\mathcal{O}_K[\boldsymbol{y}]$ is identical to the invariant subring $\mathcal{O}_M[\boldsymbol{y}]^G$ with respect to the Galois $G$-action as above. Since $u^*$ is $G$-equivariant, all $G$-invariant elements of $\mathcal{O}_K[\boldsymbol{x}]$ map into $\mathcal{O}_K[\boldsymbol{y}]$. □

Let $\Psi : \mathbb{A}^n_{\mathcal{O}_K,\boldsymbol{x}} \to \mathbb{A}^n_{\mathcal{O}_K,\boldsymbol{x}}/G$ be the quotient morphism and $\Psi^{|M|} : \mathbb{A}^n_{\mathcal{O}_K,\boldsymbol{y}} \to \mathbb{A}^n_{\mathcal{O}_K,\boldsymbol{x}}/G$ the morphism corresponding to $u^* : \mathcal{O}_K[\boldsymbol{x}]^G \to \mathcal{O}_K[\boldsymbol{y}]$. We obtain the following commutative diagram:

$$\begin{tikzcd}
& \mathbb{A}^n_{\mathcal{O}_M,\boldsymbol{y}} \arrow[ld, "u"'] \arrow[rd, "\otimes_{\mathcal{O}_K}\mathcal{O}_M"] & \\
\mathbb{A}^n_{\mathcal{O}_K,\boldsymbol{x}} \arrow[rd, "\Psi"'] & & \mathbb{A}^n_{\mathcal{O}_K,\boldsymbol{y}} \arrow[ld, "\Psi^{|M|}"] \\
& \mathbb{A}^n_{\mathcal{O}_K,\boldsymbol{x}}/G &
\end{tikzcd}$$

**6.3.** By contrast, we will call some actions "non-Galois":

**Definition 6.8.** The action of $C_G(H) = \mathrm{Aut}(M)^{\mathrm{op}}$ on $\Xi_M$ induces $C_G(H)$-actions on $\mathcal{O}_K[\boldsymbol{y}]_1 = \mathrm{Hom}_{\mathcal{O}_K}(\Xi_M, \mathcal{O}_K)$ and $\mathcal{O}_M[\boldsymbol{y}]_1 = \mathrm{Hom}_{\mathcal{O}_K}(\Xi_M, \mathcal{O}_M)$, which are $\mathcal{O}_K$-linear and $\mathcal{O}_M$-linear respectively. In turn, they induce $C_G(H)$-actions on $\mathbb{A}^n_{\mathcal{O}_K,\boldsymbol{y}}$ and $\mathbb{A}^n_{\mathcal{O}_M,\boldsymbol{y}}$; we call all these actions *non-Galois actions*.

**Lemma 6.9.** (1) *The map $u$ is equivariant with respect to the restriction of the given $G$-action on $\mathbb{A}^n_{\mathcal{O}_K,\boldsymbol{x}}$ to $C_G(H)$ and the non-Galois $C_G(H)$-action on $\mathbb{A}^n_{\mathcal{O}_M,\boldsymbol{y}}$.*

(2) *The map $\mathbb{A}^n_{\mathcal{O}_M,\boldsymbol{y}} \to \mathbb{A}^n_{\mathcal{O}_K,\boldsymbol{y}}$ given by scalar extension is equivariant with respect to the non-Galois $C_G(H)$-actions.*

*Proof.* The natural map

$$\mathcal{O}_K[\boldsymbol{x}]_1 \to \mathcal{O}_M[\boldsymbol{y}]_1 = \mathrm{Hom}_{\mathcal{O}_K}(\Xi_M, \mathcal{O}_M)$$

is clearly $C_G(H)$-equivariant, and the first assertion of the lemma follows. The second assertion is trivial. □

**Definition 6.10.** Let $\mathbb{A}^n_{\mathcal{O}_M,\boldsymbol{y}}(\mathcal{O}_M)^G$ be the set of $G$-equivariant $\mathcal{O}_M$-morphisms $\mathrm{Spec}\,\mathcal{O}_M \to \mathbb{A}^n_{\mathcal{O}_M,\boldsymbol{y}}$. The sets $\mathbb{A}^n_{\mathcal{O}_M,\boldsymbol{y}}(\mathcal{O}_M)^G$ and $\mathbb{A}^n_{\mathcal{O}_K,\boldsymbol{y}}(\mathcal{O}_K)$ both have $C_G(H)$-actions by $g \cdot \gamma = g \circ \gamma$; we call these again *non-Galois*.

**Lemma 6.11.** *The map obtained by scalar extension,*

$$\mathbb{A}^n_{\mathcal{O}_K,\boldsymbol{y}}(\mathcal{O}_K) \to \mathbb{A}^n_{\mathcal{O}_M,\boldsymbol{y}}(\mathcal{O}_M)^G,$$

*is bijective and $C_G(H)$-equivariant with respect to the non-Galois actions.*

*Proof.* A $G$-equivariant (with respect to the Galois action on $\mathbb{A}^n_{\mathcal{O}_M,\boldsymbol{y}}$) $\mathcal{O}_M$-point $\gamma$ is determined by $\gamma(y_j)$, $1 \leq j \leq n$, which must lie in $\mathcal{O}_K$. This shows the bijectivity. The equivariance follows from the equivariance of $\mathbb{A}^n_{\mathcal{O}_M,\boldsymbol{y}} \to \mathbb{A}^n_{\mathcal{O}_K,\boldsymbol{y}}$. □

If $\gamma \in \mathbb{A}^n_{\mathcal{O}_M,\boldsymbol{y}}(\mathcal{O}_M)^G$, then $u \circ \gamma \in \mathbb{A}^n_{\mathcal{O}_K,\boldsymbol{x}}(\mathcal{O}_M)^G$. This defines a map

$$\alpha : \mathbb{A}^n_{\mathcal{O}_M,\boldsymbol{y}}(\mathcal{O}_M)^G \to \mathbb{A}^n_{\mathcal{O}_K,\boldsymbol{x}}(\mathcal{O}_M)^G, \qquad \gamma \mapsto u \circ \gamma.$$

**Lemma 6.12.** *The map $\alpha$ is a $C_G(H)$-equivariant bijection with respect to the non-Galois action on $\mathbb{A}^n_{\mathcal{O}_M,\boldsymbol{y}}(\mathcal{O}_M)^G$.*

*Proof.* The map is clearly $C_G(H)$-equivariant. As for the bijectivity, the point is that both sets $\mathbb{A}^n_{\mathcal{O}_M,\boldsymbol{y}}(\mathcal{O}_M)^G$ and $\mathbb{A}^n_{\mathcal{O}_K,\boldsymbol{x}}(\mathcal{O}_M)^G$ are naturally identified with $\Xi_M$. A point $\gamma \in \mathbb{A}^n_{\mathcal{O}_M,\boldsymbol{y}}(\mathcal{O}_M)$ is $G$-equivariant if and only if $\gamma^*(y_i) \in \mathcal{O}_K = (\mathcal{O}_M)^G$. Recalling that $y_1, \dots, y_n$ is the dual basis of $\phi_1, \dots, \phi_n$, we identify $\mathbb{A}^n_{\mathcal{O}_M,\boldsymbol{y}}(\mathcal{O}_M)^G$ with $\bigoplus_{i=1}^n \mathcal{O}_K \cdot \phi_i = \Xi_M$. By Definition 6.3, $(u(\phi_i))^*$ sends $x_{i'}$ to

$$\sum_{j=1}^n \phi_j(x_{i'})\phi_i(y_j) = \phi_i(x_{i'}).$$

Namely the restriction of $(u(\phi_i))^*$ to the linear part $\mathcal{O}_K[\boldsymbol{x}]_1$ is $\phi_i$. Thus, with the obvious identification $\mathbb{A}^n_{\mathcal{O}_K,\boldsymbol{x}}(\mathcal{O}_M)^G = \Xi_M$, the map $\alpha$ corresponds to the identity map of $\Xi_M$. In particular, $\alpha$ is bijective. $\square$

We have obtained two one-to-one correspondences which are $C_G(H)$-equivariant:

$$\begin{tikzcd} & \mathbb{A}^n_{\mathcal{O}_M,\boldsymbol{y}}(\mathcal{O}_M)^G \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\ \mathbb{A}^n_{\mathcal{O}_K,\boldsymbol{x}}(\mathcal{O}_M)^G & & \mathbb{A}^n_{\mathcal{O}_K,\boldsymbol{y}}(\mathcal{O}_K) \end{tikzcd}$$

which induce one-to-one correspondences

$$\begin{tikzcd} & \mathbb{A}^n_{\mathcal{O}_M,\boldsymbol{y}}(\mathcal{O}_M)^G/C_G(H) \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\ \mathbb{A}^n_{\mathcal{O}_K,\boldsymbol{x}}(\mathcal{O}_M)^G/C_G(H) & & \mathbb{A}^n_{\mathcal{O}_K,\boldsymbol{y}}(\mathcal{O}_K)/C_G(H) \end{tikzcd}$$

**Definition 6.13.** Let $\mathbb{A}^n_{\mathcal{O}_K,\boldsymbol{y}}(\mathcal{O}_K)^\natural$ (resp. $\mathbb{A}^n_{\mathcal{O}_M,\boldsymbol{y}}(\mathcal{O}_M)^{G,\natural}$) be the set of $\mathcal{O}_K$-points (resp. $G$-equivariant $\mathcal{O}_M$-points) sending the generic point(s) into the locus where $\mathbb{A}^n_{\mathcal{O}_K,\boldsymbol{y}} \to \mathbb{A}^n_{\mathcal{O}_K,\boldsymbol{x}}/G$ (resp. $\mathbb{A}^n_{\mathcal{O}_M,\boldsymbol{y}} \to \mathbb{A}^n_{\mathcal{O}_K,\boldsymbol{x}}/G$) is étale.

From Lemma 6.4, the above correspondences restrict to the correspondences

$$\begin{tikzcd} & \mathbb{A}^n_{\mathcal{O}_M,\boldsymbol{y}}(\mathcal{O}_M)^{G,\natural}/C_G(H) \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\ \mathbb{A}^n_{\mathcal{O}_K,\boldsymbol{x}}(\mathcal{O}_M)^{G,\natural}/C_G(H) & & \mathbb{A}^n_{\mathcal{O}_K,\boldsymbol{y}}(\mathcal{O}_K)^\natural/C_G(H). \end{tikzcd}$$

From Proposition 5.4, we obtain:

**Proposition 6.14.** *The natural map*

$$\bigsqcup_{M\in G\text{-}\mathrm{\acute{E}t}(K)} \mathbb{A}^n_{\mathcal{O}_K,\boldsymbol{y}}(\mathcal{O}_K)^\natural/C_G(H) \to (\mathbb{A}^n_{\mathcal{O}_K,\boldsymbol{x}}/G)(\mathcal{O}_K)^\natural,$$

*induced by the morphisms* $\Psi^{|M|}:\mathbb{A}^n_{\mathcal{O}_K,\boldsymbol{y}} \to \mathbb{A}^n_{\mathcal{O}_K,\boldsymbol{x}}/G$*, is bijective. Here* $H$*,* $\mathbb{A}^n_{\mathcal{O}_K,\boldsymbol{y}}$ *and* $\Psi^{|M|}$ *vary, depending on* $M$*.*

**6.4.** Let $V \subset \mathbb{A}^n_{\mathcal{O}_K,\boldsymbol{x}}$ be a normal closed $\mathcal{O}_K$-subvariety stable under the $G$-action such that the induced $G$-action on $V$ is faithful. Let $X$ be the quotient variety $V/G$ and $\overline{X}$ be the image of $V$ in $\mathbb{A}^n_{\mathcal{O}_K,\boldsymbol{x}}/G$. The canonical morphism $X \to \overline{X}$ is finite and birational. If $U \subset X$ and $\overline{U} \subset \overline{X}$ denote the loci where $V \to X$ and $V \to \overline{X}$ are unramified, then the morphism $X \to \overline{X}$ induces an isomorphism $U \to \overline{U}$.

**Definition 6.15.** For $M \in G\text{-}\mathrm{\acute{E}t}(K)$, we define $V^{|M|}$ to be the preimage of $\overline{X}$ in $\mathbb{A}^n_{\mathcal{O}_K,\boldsymbol{y}}$ and $V^{\langle M\rangle}$ to be the preimage of $\overline{X}$ in $\mathbb{A}^n_{\mathcal{O}_M,\boldsymbol{y}}$; both are given the reduced scheme structures.

Let $V^{\langle M\rangle,\nu}$ and $V^{|M|,\nu}$ be the normalizations of $V^{\langle M\rangle}$ and $V^{|M|}$ respectively. The normalization morphisms $V^{\langle M\rangle,\nu} \to V^{\langle M\rangle}$ and $V^{|M|,\nu} \to V^{|M|}$ are isomorphisms over $V^{\langle M\rangle} \otimes M$ and $V^{|M|} \otimes K$ respectively. We have the following diagram:

$$\begin{tikzcd}
 & V^{\langle M\rangle,\nu} \arrow[d] \arrow[dr] & \\
 & V^{\langle M\rangle} \arrow[dl] \arrow[dr] & V^{|M|,\nu} \arrow[d] \arrow[ddl] \\
V \arrow[dr] \arrow[ddr, "\psi"'] & & V^{|M|} \arrow[ddl, "\psi^{|M|}"] \\
 & X \arrow[d] & \\
 & \overline{X} &
\end{tikzcd}$$

We name morphisms as in the diagram. All morphisms here are generically étale. Let $\overline{X}(\mathcal{O}_K)^\natural$ be the set of $\mathcal{O}_K$-points of $\overline{X}$ sending the generic point into the unramified locus of $V \to \overline{X}$ (equivalently of $V^{|M|} \to \overline{X}$), and let $V^{|M|}(\mathcal{O}_K)^\natural$ (resp. $V^{|M|,\nu}(\mathcal{O}_K)^\natural$) be the set of $\mathcal{O}_K$-points of $V^{|M|}$ (resp. $V^{|M|,\nu}$) sending the generic point into the unramified locus of $V^{|M|} \to \overline{X}$ (resp. $V^{|M|,\nu} \to \overline{X}$).

**Proposition 6.16.** *The natural maps*

$$\bigsqcup_{M\in G\text{-}\mathrm{\acute{E}t}(K)} \frac{V^{|M|}(\mathcal{O}_K)^\natural}{C_G(H)} \to \overline{X}(\mathcal{O}_K)^\natural \quad \textit{and} \quad \bigsqcup_{M\in G\text{-}\mathrm{\acute{E}t}(K)} \frac{V^{|M|,\nu}(\mathcal{O}_K)^\natural}{C_G(H)} \to X(\mathcal{O}_K)^\natural$$

*are bijective. Here the subgroup* $H \subset G$ *varies, depending on* $M$*.*

*Proof.* The first map is a restriction of the map in Proposition 6.14 and is easily seen to be bijective. Since $V^{|M|,\nu} \otimes K \to V^{|M|} \otimes K$ and $U \to \overline{U}$ are isomorphisms, we have natural bijections $V^{|M|,\nu}(\mathcal{O}_K)^\natural \to V^{|M|}(\mathcal{O}_K)^\natural$ and $X(\mathcal{O}_K)^\natural \to \overline{X}(\mathcal{O}_K)^\natural$. It follows that the second map of the proposition is also bijective. □

## 7. Main results

Using the untwisting, we introduce the notion of *$G$-stringy point counts* for $G$-log pairs, prove our main results.

**7.1.** We first define log pairs with a $G$-action.

**Definition 7.1.** A *$G$-log pair* is a log pair $(V, E)$ with a faithful $G$-action on $V$ such that $E$ is $G$-stable, that is, for every $g \in G$, $g_* E = E$.

Let $(V, E)$ be a $G$-log pair, let $X := V/G$ be the quotient scheme, and let $\pi : V \to X$ be the quotient morphism.

**Lemma 7.2.** *There exists a unique $\mathbb{Q}$-divisor $D$ such that $(X, D)$ is a log pair and the induced morphism $(V, E) \to (X, D)$ is crepant.*

*Proof.* Let $K_{V/X}$ be the ramification divisor of $\pi$, defined so that the equality of subsheaves of $\omega_V$,

$$\omega_V(-K_{V/X}) = \pi^* \omega_X,$$

holds in codimension one. We set $D := (1/\sharp G)\pi_*(E - K_{V/X})$. The pull-back $\pi^*(K_X + D)$ is defined at least in codimension one and coincides with $K_V + E$. Since the pull-back map $\pi^*$ gives a one-to-one correspondence of $\mathbb{Q}$-Cartier divisors on $X$ and $G$-stable $\mathbb{Q}$-Cartier divisors on $V$, we conclude that $K_X + D$ is $\mathbb{Q}$-Cartier. Thus $(X, D)$ is a log variety such that $(V, E) \to (X, D)$ is crepant. The uniqueness of $D$ is obvious. □

**7.2.** In this subsection, we suppose that $(V, E)$ is a $G$-log pair with $V$ affine. Then there exist an affine variety $\mathbb{A}^n_{\mathcal{O}_K}$ with a faithful $\mathcal{O}_K$-linear $G$-action and a $G$-equivariant closed embedding $V \hookrightarrow \mathbb{A}^n_{\mathcal{O}_K}$. Indeed, if the coordinate ring $\mathcal{O}_V$ of $V$ is generated by $f_1, \dots, f_l$ as an $\mathcal{O}_K$-algebra, then we let $S$ be the union of the $G$-orbits of the generators $f_i$ and consider the polynomial ring $\mathcal{O}_K[x_s \mid s \in S]$ with indeterminates corresponding to elements of $S$. Now $G$ acts on this polynomial ring faithfully and $\mathcal{O}_K$-linearly, and the natural map

$$\mathcal{O}_K[x_s \mid s \in S] \to \mathcal{O}_V, \qquad x_s \mapsto s,$$

defines a desired embedding $V \hookrightarrow \mathbb{A}^{\sharp S}_{\mathcal{O}_K}$.

We fix such an embedding $V \subset \mathbb{A}^n_{\mathcal{O}_K, \boldsymbol{x}}$ and follow the notation of Section 6. In particular, for each $M \in G\text{-}\acute{\mathrm{E}}\mathrm{t}(K)$, we obtain the diagram of Section 6.4. We define the

$G$-log structure $(V^{\langle M\rangle,\nu}, E^{\langle M\rangle,\nu})$ and log structures $(V^{|M|,\nu}, E^{|M|,\nu})$ and $(X, D)$ so that all the solid arrows in the diagram

$$\begin{tikzcd}
& (V^{\langle M\rangle,\nu}, E^{\langle M\rangle,\nu}) \arrow[dl] \arrow[dr] & \\
(V, E) \arrow[dr] & & (V^{|M|,\nu}, E^{|M|,\nu}) \arrow[dl, dashed] \\
& (X, D) &
\end{tikzcd}$$

are crepant. Then the dashed arrow is also crepant, which shows $(V^{|M|,\nu}, E^{|M|,\nu})$ is a $C_G(H)$-log pair. For a $G$-stable constructible subset $C \subset V_k$ and $M \in G\text{-Ét}(K)$, we define $C^{\langle M\rangle,\nu} \subset V_k^{\langle M\rangle,\nu}$ to be the preimage of $C$, and $C^{|M|,\nu} \subset V_k^{|M|,\nu}$ to be its image.

**Definition 7.3.** For $M \in G\text{-Ét}(K)$, we define the *$M$-stringy point count* of $(V, E)$ along $C$ by

$$\sharp_{\mathrm{st}}^{M}(V, E)_C := \sharp_{\mathrm{st}}(V^{|M|,\nu}, E^{|M|,\nu})_{C^{|M|,\nu}}/\sharp C_G(H),$$

and the *$G$-stringy point count* along $C$ by

$$\sharp_{\mathrm{st}}^{G}(V, E)_C := \sum_{M\in G\text{-Ét}(K)} \sharp_{\mathrm{st}}^{M}(V, E)_C.$$

Again we omit the subscript $C$ when $C = V_k$.

**Theorem 7.4.** *For a $G$-log pair $(V, E)$ with $V$ affine, let $(X, D)$ be as above. Let $C \subset V_k$ be a $G$-stable subset and $\overline{C} \subset X_k$ its image. Then*

$$\sharp_{\mathrm{st}}^{G}(V, E)_C = \sharp_{\mathrm{st}}(X, D)_{\overline{C}}.$$

*In particular,*

$$\sharp_{\mathrm{st}}^{G}(V, E) = \sharp_{\mathrm{st}}(X, D).$$

*Proof.* Let $d$ be the dimension of the $\mathcal{O}_K$-variety $V$. From Lemma 4.3, Theorem 4.8 and Proposition 6.16, we have

$$\begin{aligned}
\sharp_{\mathrm{st}}(X, D)_{\overline{C}} = q^d \mu_{X,D}(X(\mathcal{O}_K)_{\overline{C}}) &= \sum_{M\in G\text{-Ét}(K)} \frac{q^d \mu_{V^{|M|,\nu},E^{|M|,\nu}}(V^{|M|,\nu}(\mathcal{O}_K)_{C^{|M|,\nu}})}{\sharp C_G(H)} \\
= \sum_{M\in G\text{-Ét}(K)} \frac{\sharp_{\mathrm{st}}(V^{|M|,\nu}, E^{|M|,\nu})_{C^{|M|,\nu}}}{\sharp C_G(H)} &= \sum_{M\in G\text{-Ét}(K)} \sharp_{\mathrm{st}}^{M}(V, E)_C = \sharp_{\mathrm{st}}^{G}(V, E)_C. \qquad \square
\end{aligned}$$

**Corollary 7.5.** *Let $V$ be a normal $\mathbb{Q}$-Gorenstein affine $\mathcal{O}_K$-variety endowed with a faithful $G$-action, and $X := V/G$ its quotient scheme. Suppose that the quotient morphism $V \to X$ is étale in codimension one. Then, for a $G$-stable constructible subset $C \subset V_k$, we have*

$$\sharp^G_{\mathrm{st}}(V)_C = \sharp_{\mathrm{st}}(X)_{\overline{C}}.$$

*In particular,*

$$\sharp^G_{\mathrm{st}} V = \sharp_{\mathrm{st}} X.$$

*Proof.* From the assumption, the morphism $V = (V, 0) \to X = (X, 0)$ is crepant. Therefore the corollary is a special case of the preceding theorem. □

**7.3.** We now consider an arbitrary $G$-log pair $(V, E)$ ($V$ is not necessarily affine but quasi-projective from our definition of varieties in Section 2). Let us take an affine open cover $V = \bigcup_i V_i$ such that each $V_i$ is $G$-stable. For each $M \in G$-Ét$(K)$, let $\mu^M_{V,E,i}$ be the measure on $V_i(\mathcal{O}_M)^{G,\circ}$ corresponding to

$$\mu_{V_i^{|M|,\nu},(E|_{V_i})^{|M|,\nu}}$$

through the correspondence $V_i(\mathcal{O}_M)^{G,\circ} \leftrightarrow V_i^{|M|}(\mathcal{O}_K)^\circ$. The argument of the proof of Theorem 7.4 shows that the measures $\mu^M_{V,E,i}$ and $\mu^M_{V,E,j}$ coincide on $(V_i \cap V_j)(\mathcal{O}_M)^{G,\circ}$, and we obtain a measure on $V(\mathcal{O}_M)^{G,\circ}$, and one on $V(\mathcal{O}_M)^G$ by extending it so that all subsets of $V(\mathcal{O}_M)^G \setminus V(\mathcal{O}_M)^{G,\circ}$ have measure zero; we denote it by $\mu^M_{V,E}$.

**Definition 7.6.** For a $G$-stable constructible subset $C \subset V_k$, we define

$$\sharp^G_{\mathrm{st}}(V, E)_C := q^d \sum_{M \in G\text{-}\mathrm{\acute{E}t}(K)} \mu^M_{V,E}(V(\mathcal{O}_M)^G_C)/\sharp C_G(H),$$

with $d$ the dimension of $V$.

With this definition, we obviously have:

**Theorem 7.7.** *Theorem 7.4 and Corollary 7.5 hold without the assumption that $V$ is affine.*

**Remark 7.8.** Our construction of the measure $\mu^M_{V,E}$ and the definition of $\sharp^G_{\mathrm{st}}(V, E)_C$ are not completely satisfactory, because they depend on Theorem 7.4 for the affine case, and then Theorem 7.7 is somewhat tautological. Therefore it is an interesting problem to construct the measure intrinsically, in particular, without gluing affine pieces.

## 8. Linear actions and mass formulas

In this section, we consider the case $V = \mathbb{A}^n_{\mathcal{O}_K,x}$ and compute the $M$-stringy point counts $\sharp^M_{\mathrm{st}} V$ for $M \in G$-Ét$(K)$. Then we briefly illustrate how our main results in this case prove the mass formulas by Serre, Bhargava and Kedlaya. However, we should note that it is impossible to locally linearize a given group action, unlike the case of an algebraically closed base field of characteristic zero.

**8.1.** We now suppose that $V = \mathbb{A}^n_{\mathcal{O}_K,\boldsymbol{x}} = \operatorname{Spec} \mathcal{O}_K[x_1, \ldots, x_n]$ and that $G$ acts $\mathcal{O}_K$-linearly on it, and consider the trivial log structure $\mathcal{V} = (V, E = 0)$. For each $M \in G\text{-Ét}(K)$, let $\Xi_M$ be the tuning module (see Section 6.1), which is a submodule of $\operatorname{Hom}_{\mathcal{O}_K}(\mathcal{O}_K[\boldsymbol{x}]_1, \mathcal{O}_M)$. We denote the origin of $V_k$ by $o$.

**Definition 8.1.** We define

$$\mathbf{v}_{\mathcal{V}}(M) := \frac{1}{\sharp G} \text{length} \frac{\operatorname{Hom}_{\mathcal{O}_K}(\mathcal{O}_K[\boldsymbol{x}]_1, \mathcal{O}_M)}{\mathcal{O}_M \cdot \Xi_M}.$$

Let $\eta_k : V^{\langle M \rangle} \otimes_{\mathcal{O}_K} k \to V_k$ be the base change of $\eta : V^{\langle M \rangle} \to V$ from $\mathcal{O}_K$ to $k$. We define

$$\mathbf{w}_{\mathcal{V}}(M) := \dim \eta_k^{-1}(o) - \mathbf{v}_{\mathcal{V}}(M).$$

**Remark 8.2.** Our definition of $\mathbf{v}_{\mathcal{V}}$ as a function associated to the $G$-variety $V$ is identical to the one given in [WY15] (see also Remark 6.2). Our definition of $\mathbf{w}_{\mathcal{V}}$ is slightly different from the one given in [WY15]. However, the following lemma shows that they coincide in three important cases.

**Lemma 8.3.** *Suppose that one of the following conditions holds:*

(1) $p \nmid \sharp G$,
(2) $K = k((t))$ *and the* $G$*-action on* $V$ *is the base change of one on* $V_k$,
(3) *the* $G$*-action on* $V$ *is permutation of coordinates.*

*For* $M \in G\text{-Ét}(K)$, *let* $H_0 \subset G$ *be the stabilizer of a geometric connected component of* Spec $M$, *that is, a component of* Spec $M \otimes_{\mathcal{O}_K} \mathcal{O}_L$ *with* $L$ *the maximal unramified extension of* $K$. *Then*

$$\dim \eta_k^{-1}(o) = \operatorname{codim}((V_k)^{H_0} \subset V_k).$$

*Proof.* Our construction of $V^{|M|}$ and $C^{|M|}$ is compatible with the base change by $\mathcal{O}_{K'}/\mathcal{O}_K$ for a finite unramified extension $K'/K$. Hence we may suppose that a connected component Spec $L$ of Spec $M$ is a geometric connected component as well. It follows that $H_0 = H$ and $L$ has residue field $k$. Let $\eta_k : V_k^{\langle M \rangle} \to V_k$ be the base change of the morphism $\eta : V^{\langle M \rangle} \to V$ by Spec $k \to$ Spec $\mathcal{O}_K$. The reduced scheme $(V_k^{\langle M \rangle})_{\mathrm{red}}$ associated to $V_k^{\langle M \rangle}$ is the disjoint union of $[G : H]$ copies of $\mathbb{A}_k^n$, and the restriction of $\eta_k$ to each connected component $(V_k^{\langle M \rangle})_{\mathrm{red}}$ is a $k$-linear map. Moreover we can identify $V_k^{|M|}$ with the connected component of $(V_k^{\langle M \rangle})_{\mathrm{red}}$ corresponding to Spec $L$, which we denote by $V_{k,0}^{\langle M \rangle}$. We denote the corresponding component of $V^{\langle M \rangle}$ by $V_0^{\langle M \rangle}$. To prove the lemma, it suffices to show that the map

$$\eta_k|_{V_{k,0}^{\langle M \rangle}} : V_{k,0}^{\langle M \rangle} \to V_k$$

has image $(V_k)^H$.

Let $v$ be an arbitrary $k$-point of $V_{k,0}^{\langle M\rangle} = V_k^{|M|}$. This point lifts to an $\mathcal{O}_K$-point $\tilde{v}$ of $V^{|M|}$ and hence to an $H$-equivariant $\mathcal{O}_L$-point $\hat{v}$ of $V_0^{\langle M\rangle}$. The image of $\hat{v}$ on $V$ is also $H$-equivariant, and hence the $k$-point

$$\operatorname{Spec} k \hookrightarrow \operatorname{Spec} \mathcal{O}_K \xrightarrow{\hat{v}} V$$

lies in $(V_k)^H$. From the construction, this $k$-point is the image of $v$, which shows that the image of $\eta_k|_{V_{k,0}^{\langle M\rangle}}$ is contained in $(V_k)^H$.

Next let $w$ be an arbitrary $H$-fixed $k$-point of $V_k$. From either of the three conditions in the proposition, there exists an $H$-fixed $\mathcal{O}_K$-point $\tilde{w}$ of $V$ which is a lift of $w$. Then the composition

$$\hat{w} : \operatorname{Spec} \mathcal{O}_L \to \operatorname{Spec} \mathcal{O}_K \xrightarrow{\tilde{w}} V$$

is $H$-equivariant. It then lifts to an $H$-equivariant $\mathcal{O}_L$-point $\check{w}$ of $V_0^{\langle M\rangle}$. The induced $k$-point

$$\operatorname{Spec} k \hookrightarrow \operatorname{Spec} \mathcal{O}_L \xrightarrow{\check{w}} V_0^{\langle M\rangle}$$

maps to $w$ by $V^{\langle M\rangle} \to V$. This proves that the image of $\eta_k|_{V_{k,0}^{\langle M\rangle}}$ contains $(V_k)^H$, and completes the proof of the lemma. □

Let $(V^{|M|}, E^{|M|})$ be the log structure on $V^{|M|}$ defined as in Section 7.2, where we do not need the normalization as $V^{|M|} = \mathbb{A}^n_{\mathcal{O}_{K,y}}$ is clearly normal.

**Lemma 8.4.** *Regarding $V_k^{|M|} = \mathbb{A}^n_{k,y}$ as a prime divisor on $V^{|M|}$, we have*

$$E^{|M|} = -\mathbf{v}_V(M) \cdot V_k^{|M|}.$$

*Proof.* This is a special case of [Yas16, Lemma 6.5] (except that $k$ is finite in the present paper, while it is algebraically closed in the cited paper). The outline of the proof is as follows. Let $m$ be the quotient of $\mathcal{O}_M$ by the Jacobson radical. Then $\operatorname{Spec} m$ is the union of the closed points of $\operatorname{Spec} \mathcal{O}_M$ with reduced structure. We set $V_m^{\langle M\rangle} := V^{\langle M\rangle} \otimes_{\mathcal{O}_M} m$. Let $\operatorname{Spec} L$ be a connected component of $\operatorname{Spec} M$ and $H \subset G$ its stabilizer. Let $\delta_{L/K}$ be the different exponent of $L/K$ (the different of $L/K$ is $\mathfrak{m}_L^{\delta_{L/K}}$). If $(V^{\langle M\rangle}, E^{\langle M\rangle})$ is the induced log structure on $V^{\langle M\rangle}$, then

$$E^{\langle M\rangle} = -(\sharp H \cdot \mathbf{v}_V(M) + \delta_{L/K}) V_m^{\langle M\rangle},$$

where $\sharp H \cdot \mathbf{v}_V(M)$ is the contribution of the morphism $V^{\langle M\rangle} \to V \otimes \mathcal{O}_M$, and $\delta_{L/K}$ is the contribution of the morphism $V \otimes \mathcal{O}_M \to V$. Now we have

$$E^{|M|} = \frac{1}{\sharp G}(\theta_* E^{\langle M\rangle}) = -\mathbf{v}_V(M) \cdot V_k^{|M|}$$

with $\theta$ the natural morphism $V^{\langle M\rangle} \to V^{|M|}$. □

**Proposition 8.5.** *We have*

$$\sharp_{\mathrm{st}}^{M} V = \frac{q^{n-\mathbf{v}_V(M)}}{\sharp C_G(H)}, \qquad \sharp_{\mathrm{st}}^{G} V = \sum_{M \in G\text{-Ét}(K)} \frac{q^{n-\mathbf{v}_V(M)}}{\sharp C_G(H)},$$
$$\sharp_{\mathrm{st}}^{M} (V)_o = \frac{q^{\mathbf{w}_V(M)}}{\sharp C_G(H)}, \qquad \sharp_{\mathrm{st}}^{G} (V)_o = \sum_{M \in G\text{-Ét}(K)} \frac{q^{\mathbf{w}_V(M)}}{\sharp C_G(H)}.$$

*Proof.* If we write $\mathbf{v}_V(M) = s/r$ with some integers $s \geq 0$ and $r > 0$, then $\mathcal{O}_{V^{|M|}}(r(K_{V^{|M|}} + E^{|M|}))$ has a generator $\varpi^s (dy_1 \wedge \cdots \wedge dy_n)^{\otimes r}$ with $\varpi$ a uniformizer of $K$. We have

$$\begin{aligned}\sharp_{\mathrm{st}}^{M} V &= \frac{\sharp_{\mathrm{st}}(V^{|M|}, E^{|M|})}{\sharp C_G(H)} = \frac{q^n}{\sharp C_G(H)} \int_{\mathcal{O}_K^n} |\varpi^s|^{1/r} \\ &= \frac{q^{n-s/r}}{\sharp C_G(H)} \mu_{K^n}(\mathcal{O}_K^n) = \frac{q^{n-s/r}}{\sharp C_G(H)},\end{aligned}$$

showing the first two equalities of the proposition. If we set $a := \dim \eta_k^{-1}(o)$, the subset $C^{|M|} \subset V_k^{|M|}$ is a linear subspace of dimension $a$. Therefore

$$\begin{aligned}\sharp_{\mathrm{st}}^{M} (V)_o &= \frac{\sharp_{\mathrm{st}}(V^{|M|}, E^{|M|})_{C^{|M|}}}{\sharp C_G(H)} = \frac{q^n}{\sharp C_G(H)} \int_{\mathcal{O}_K^a \times \mathfrak{m}_K^{n-a}} |\varpi^s|^{1/r} \\ &= \frac{q^{n-s/r}}{\sharp C_G(H)} \mu_{K^n}(\mathcal{O}_K^a \times \mathfrak{m}_K^{n-a}) = \frac{q^{a-s/r}}{\sharp C_G(H)}.\end{aligned}$$

This shows the last two equalities of the proposition. □

**8.2.** Serre [Ser78] proved a beautiful mass formula: for each integer $n \geq 2$,

$$\sum_{\substack{L/K\colon \text{tot. ram.} \\ [L:K]=n}} \frac{q^{-d_{L/K}}}{\sharp \mathrm{Aut}(L/K)} = q^{1-n},$$

where $L$ runs over the isomorphism classes of totally ramified field extensions of a fixed local field $K$ with $[L:K] = n$, $d_{L/K}$ is the discriminant exponent of $L/K$ (the discriminant of $L/K$ is $\mathfrak{m}_K^{d_{L/K}}$) and $\mathrm{Aut}(L/K)$ the group of $K$-automorphisms of $L$. Bhargava [Bha07] proved a similar formula: for each $n \geq 2$, denoting by $n$-Ét$(K)$ the set of isomorphism classes of étale $K$-algebras of degree $n$, we have

$$\sum_{L \in n\text{-Ét}(K)} \frac{q^{-d_{L/K}}}{\sharp \mathrm{Aut}(L/K)} = \sum_{i=0}^{n-1} P(n, n-i) q^{-i},$$

where $P(n, n-i)$ is the number of partitions of the integer $n$ into exactly $n-i$ parts.

**8.3.** Let $S_n$ be the $n$-th symmetric group acting on $\{1,\dots,n\}$. We embed $S_{n-1}$ into $S_n$ as the stabilizer subgroup of 1. Let $n$-Ét$(K)$ be the set of isomorphism classes of étale $K$-algebras of degree $n$. We have a bijection

$$S_n\text{-Ét}(K)\to n\text{-Ét}(K),\qquad M\mapsto M^{S_{n-1}}.$$

Moreover the automorphism group of $M$ as an $S_n$-étale $K$-algebra is isomorphic to the automorphism group of the étale $K$-algebra $M^{S_{n-1}}$.

Suppose that $S_n$ acts on $V=\mathbb{A}^{2n}_{\mathcal{O}_K}$ by the direct sum of two copies of the standard permutation representation. Wood and Yasuda [WY15] showed

$$d_{M^{S_{n-1}}/K}=\mathbf{v}_V(M).$$

Therefore the left hand side of Bhargava's formula can be written as

$$\sum_{M\in S_n\text{-Ét}(K)}\frac{q^{-\mathbf{v}_V(M)}}{\sharp C_{S_n}(H)},$$

where $H\subset S_n$ is the stabilizer of a component of Spec $M$.

The quotient variety $V/S_n$ is identical to the $n$-th symmetric product of $\mathbb{A}^2_{\mathcal{O}_K}$ over $\mathcal{O}_K$. Let $\mathrm{Hilb}^n(\mathbb{A}^2_{\mathcal{O}_K})$ be the Hilbert scheme of $n$ points of $\mathbb{A}^2_{\mathcal{O}_K}$ defined relatively over $\mathcal{O}_K$, which is a smooth $\mathcal{O}_K$-variety. From [BK05, 7.4.6] the Hilbert–Chow morphism $\mathrm{Hilb}^n(\mathbb{A}^2_{\mathcal{O}_K})\to V/S_n$ is proper, birational and crepant. Therefore

$$\sharp_{\mathrm{st}}(V/S_n)=\sharp_{\mathrm{st}}\mathrm{Hilb}^n(\mathbb{A}^2_{\mathcal{O}_K})=\sharp\mathrm{Hilb}^n(\mathbb{A}^2_{\mathcal{O}_K})(k).$$

Using a stratification of $\mathrm{Hilb}^n(\mathbb{A}^2_k)$ into affine spaces (or Gröbner basis theory), we can count the $k$-points of $\mathrm{Hilb}^n(\mathbb{A}^2_k)$ and get

$$\sharp\mathrm{Hilb}^n(\mathbb{A}^2_{\mathcal{O}_K})(k)=\sum_{i=0}^{n-1}P(n,n-i)q^{2n-i}.$$

Proposition 8.5 gives

$$\sum_{M\in S_n\text{-Ét}(K)}\frac{q^{2n-\mathbf{v}_V(M)}}{\sharp C_{S_n}(H)}=\sharp_{\mathrm{st}}^{S_n}V=\sharp_{\mathrm{st}}(V/S_n)=\sharp_{\mathrm{st}}\mathrm{Hilb}^n(\mathbb{A}^2_{\mathcal{O}_K})=\sum_{i=0}^{n-1}P(n,n-i)q^{2n-i}.$$

Dividing these by $q^{2n}$, we reprove Bhargava's mass formula as a consequence of the wild McKay correspondence. This computation will be revisited in [WY17] in relation to dualities discussed there. Kedlaya [Ked07] obtained a similar formula for the group of signed permutation matrices. In [WY17], Kedlaya's formula is also deduced from the wild McKay correspondence in a very similar way except in the case of residual characteristic two.

We can deduce Serre's mass formula from Bhargava's, for instance, by using the exponential formula in the direction opposite to [Ked07]. Let us write

$$N(K,n) := \sum_{\substack{L/K:\,\text{tot. ram.}\\ [L:K]=n}} \frac{q^{-d_{L/K}}}{\sharp\mathrm{Aut}(L/K)} \quad \text{and} \quad M(K,n) := \sum_{L\in n\text{-Ét}(K)} \frac{q^{-d_{L/K}}}{\sharp\mathrm{Aut}(L/K)}.$$

Kedlaya [Ked07, pp. 7–8] showed

$$\sum_{n=0}^{\infty} M(K,n)x^n = \exp\left(\sum_{n=1}^{\infty} \frac{x^n}{n} \sum_{f\mid n} \frac{N(K_f, n/f)}{f}\right),$$

where $K_f$ is the unramified extension of $K$ of degree $f$. He used Serre's formula for $N(K_f, n/f)$ to get Bhargava's formula. It is easy to show that given the values of $M(K_f, n/f)$ for all $f$ and $n$, the equality above determines the values of $N(K_f, n)$ for all $f$ and $n$. In particular, $M(K,n)$ must be equal to $q^{1-n}$, and Serre's formula holds.

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