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Computing the Teichmüller polynomial

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Abstract. The Teichmüller polynomial of a fibered 3-manifold, introduced in [McM00], plays a useful role in the construction of mapping classes having a small stretch factor. We provide a general algorithm for computing the Teichmüller polynomial given a pseudo-Anosov mapping class obtained as a loop in a train track automaton. As a byproduct, our algorithm allows us to derive all the relevant information on the topology of various fibers that belong to a fibered face.

Keywords. Teichmüller polynomial, pseudo-Anosov homeomorphism, Thurston norm

1. Introduction

A fibered hyperbolic 3-manifold M is a rich source of pseudo-Anosov mapping classes: Thurston's theory of fibered faces tells us that integer points in the *fibered cone* $\mathbb{R}^+ \cdot F \subset H^1(M,\mathbb{R})$ over the *fibered face* F of the Thurston norm unit ball correspond to fibrations of M over the circle. If M is hyperbolic, the monodromy of each such fibration is a pseudo-Anosov class $[\psi]$ with stretch factor $\lambda(\psi) > 1$. These stretch factors are packaged in the Teichmüller polynomial, defined in [McM00]. This is an element $\Theta_F = \sum_{g\in G} a_g g$ in the group ring $\mathbb{Z}[H_1(M,\mathbb{Z})/\mathrm{Torsion}]$, which is associated to the fibered face F and that is used to compute the stretch factor $\lambda(\psi)$ effectively. More precisely, if $[\alpha] \in H^1(M,\mathbb{Z})$ is the integer class corresponding to ψ in the fibered cone and $\xi_\alpha \in H_1(M,\mathbb{Z})$ is its dual, then the largest root of the Laurent polynomial $\Theta_F(\alpha) := \sum_{g\in G} a_g \cdot t^{\,\xi_\alpha(g)} \in \mathbb{Z}[t,t^{-1}]$ (in absolute value) is the stretch factor $\lambda(\psi)$. The Teichmüller polynomial has been used as a natural source of pseudo-Anosov homeomorphisms having small *normalized stretch factors*: infinite families of pseudo-Anosov homeomorphisms $[\psi] \in \mathrm{Mod}(\Sigma_g)$ satisfying $\lambda(\psi)^g = O(1)$ as $g \to \infty$. In particular, it has been intensively used in the papers by Hironaka [Hir10], Hironaka–Kin [HK06], Kin–Takasawa [KT11, KT13] and Kin–Kojima–Takasawa [KKT13]. Most of the known pseudo-Anosov homeomorphisms having a small normalized stretch factor come from fibrations of two very particular hyperbolic mani-

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folds: the mapping torus of the simplest hyperbolic braid [McM00, Hir10] and the "magic manifold" [KT11].

The Teichmüller polynomial was originally defined as the generator of the Fitting ideal of a module of transversals (defined by a lamination) over $\mathbb{Z}[H_1(M,\mathbb{Z})/\mathrm{Torsion}]$. However, it is a result of McMullen [McM00] that this polynomial can also be defined in terms of the transition matrix of an infinite train track associated to a fibration on the fibered face F.

The main goal of our paper, based on this second definition, is to present an algorithm to compute the Teichmüller polynomial explicitly and to give a unified presentation of the aforementioned papers. More precisely, we will denote the mapping torus of $[\psi] \in \mathrm{Mod}(S)$ by

$$M_\psi := S \times [0,1]/(x,1) \sim (\psi(x),0)$$

and we will suppose that the first Betti number of M_ψ is at least 2.

Using results of Penner and Papadopoulos [PP87] on train tracks and elementary operations (folding operations in the present paper), we provide an algorithm that

(1) computes the Teichmüller polynomial Θ_F of the fibered face F of M_ψ where $[\psi] \in \mathrm{Mod}(S)$ is a pseudo-Anosov class;
(2) computes the topology (genus, number of singularities and type) of the fibers of fibrations in the cone $\mathbb{R}^+ \cdot F$.

We will present our algorithm (and examples) in the case where S is the n-punctured disc $\mathbf{D}_n$. Then $\mathrm{Mod}(S)$ is naturally isomorphic to the braid group $B(n)$. Let $\beta \in B(n)$ and let $[f_\beta]$ be the corresponding mapping class in $\mathrm{Mod}(\mathbf{D}_n)$. We shall show:

Theorem 1.1. *For any pseudo-Anosov class $[f_\beta] \in \mathrm{Mod}(\mathbf{D}_n)$ represented by a path in some automaton*

$$\tau_1 \xrightarrow{T_1} \tau_2 \xrightarrow{T_2} \cdots \xrightarrow{T_{n-1}} \tau_n \xrightarrow{T_n} \tau_{n+1},$$

with transition matrices $M_i = M(T_i) \in \mathrm{GL}(\mathbb{Z}^r)$, the Teichmüller polynomial $\Theta_F(\mathbf{t},u)$ of the associated fibered face F determined by $[f_\beta]$ is

$$\Theta_F(\mathbf{t},u) = \det(u \cdot \mathrm{Id} - M_1 D_1 \cdot M_2 D_2 \cdots M_n D_n R)$$

where the diagonal matrices $D_i \in \mathrm{GL}(\mathbb{Z}[\mathbf{t}]^r)$ are uniquely determined in terms of the path in the automaton and $R \in \mathrm{GL}(\mathbb{Z}^r)$ is a relabeling matrix.

Remark 1.2. All steps of the algorithm that we present can be extended to a general surface. However, for simplicity, we will specify the discussion to the case of the punctured disc. This will avoid several technical difficulties.

For a more precise statement, in particular the nature of the variables u and $\mathbf{t}$, see Theorem 5.4 and below (see also Section 4 for the definition of the automaton). Observe that Bestvina and Handel [BH95] have introduced an effective algorithm that determines whether a given homeomorphism $f \in \mathrm{Mod}(\mathbf{D}_n)$ is pseudo-Anosov. See also [Bri00] and [Hal] for implementations of the algorithm; in the pseudo-Anosov case the algorithm generates the train tracks and a path in some corresponding automaton.

Reader's guide

In Section 2 we recall Thurston's theory of fibered faces and we review basic definitions and properties of the Teichmüller polynomial and its relation to the stretch factor associated to the monodromy of a fibration $\Sigma \to M \to S^1$. In Section 3 we describe a general strategy to compute the Teichmüller polynomial Θ_F from a train track and a train track map (after [McM00]). In Section 4 we introduce the notion of automaton and we give several relevant examples. In particular we use the convention of labeled train tracks (similarly to what Kerckhoff and Marmi–Moussa–Yoccoz did for interval exchange transformations). Section 5 is devoted to the statement and proof of our main theorem. Finally, as a byproduct, our algorithm allows us also to derive all the relevant information on the topology of various fibers that belong to the face. This is the content of Section 6 (Proposition 6.1, Corollary 6.3 and Propositions 6.4–6.6). In Sections 7, 8 and Appendix A we apply our results to produce several examples, recovering the ones of McMullen [McM00] and Hironaka [Hir10], but also giving infinitely many new examples of Teichmüller polynomials defined by pseudo-Anosov braids in B_n for $n \geq 4$. Finally, a step by step description of the algorithm can be found in Section 5.4. This paper will not cover the details of its implementation, and a list of examples will be the subject of a forthcoming paper.

We end the introduction with a general description of the algorithm.

Key steps of the algorithm

To each pseudo-Anosov map $\psi : S \to S$ one can associate (in a non-unique way!) an invariant train track (an embedded graph τ in S) and a train track map $T : \tau \to \tau$.

McMullen's construction relates the construction of $\Theta_\psi(\mathbf{t}, u)$ to the lifts of τ and T to a cover $\hat{S}$ of S as follows. Let $H_\psi = \mathrm{Hom}(H^1(S, \mathbb{Z})^\psi, \mathbb{Z})$ where $H^1(S, \mathbb{Z})^\psi$ is the ψ-invariant cohomology of S. By evaluating cohomology classes on loops, we obtain a natural map $\pi_1(S) \to H_\psi$. Then $\hat{S}$ is the H_ψ-covering space of S. Pick a lift $\hat{\tau} \to \hat{S}$ and a lift $\hat{T} : \hat{\tau} \to \hat{\tau}$. Then $\hat{T}$ acts by a matrix $M(\hat{T})$ with coefficients in the free $\mathbb{Z}[H_\psi]$-module generated by the lifts of the edges of τ. We choose a multiplicative basis $\mathbf{t} = (t_1, \ldots, t_r)$ of H_ψ. Then for a suitable choice of τ, we have

$$\Theta_\psi(\mathbf{t}, u) = \det(u \cdot \mathrm{Id} - M(\hat{T})).$$

One of the main difficulties in the theory lies in the computation of the matrices $M(T)$ and $M(\hat{T})$. There is a general strategy, developed by Papadopoulos and Penner, that allows one to simplify the calculation of $M(T)$. Roughly speaking, they define two operations (folding/splitting) that split the graph map T into a sequence of train track maps:

$$\tau_1 \xrightarrow{T_1} \tau_2 \xrightarrow{T_2} \cdots \xrightarrow{T_{n-1}} \tau_n \xrightarrow{T_n} \tau_1$$

where $T = T_n \circ \cdots \circ T_1$ and $M(T) = M(T_1) \cdots M(T_n)$.

The purpose of this paper is to explain how one can lift the elementary operations $T_i : \tau_i \to \tau_{i+1}$ to $\hat{S}$ so that $\hat{T} = \hat{T}_n \circ \cdots \circ \hat{T}_1$ and how to deduce the formula

$$\Theta_\psi(\mathbf{t}, u) = \det(u \cdot \mathrm{Id} - M(\hat{T}_1) \cdots M(\hat{T}_n)),$$

The main difficulty lies in the *normalization*, i.e. the choice of the lifts of τ_i and T_i. This is explained in Sections 4 and 5.

Remark 1.3. We stress that all the above computations depend on H_ψ. To avoid this problem, in order to have an algorithm independent of ψ, we will use the maximal abelian covering of S given by $H = \mathrm{Hom}(H^1(S,\mathbb{Z}),\mathbb{Z})$, instead of H_ψ. The polynomial $\Theta_\psi(\mathbf{t}, u)$ will be obtained from the above algorithm (with H) by specifying some relations between the t_i. See Section 5.2 for details.

2. Thurston's theory of fibered faces and the Teichmüller polynomial

In this section we recall Thurston's theory of fibered faces. We also review the construction of the Teichmüller polynomial and its relation to the stretch factor associated to the monodromy of a fibration $\Sigma \to M \to S^1$.

We begin by fixing some notation. Let S be a surface (for which one might have $\partial S \neq \emptyset$). Let $[\psi]$ be a class in $\mathrm{Mod}(S)$. A deep result by Thurston (see e.g. [FM12, §13, Thm. 13.4]) tells us that M_ψ admits a hyperbolic metric if and only if the mapping class $[\psi]$ is pseudo-Anosov. By Mostow's rigidity theorem the isometry class of M_ψ does not depend on the choice of the representative or the conjugacy class of $[\psi] \in \mathrm{Mod}(S)$.

2.1. Thurston norm and fibered faces

Thurston introduced a very effective tool for studying essential surfaces in 3-manifolds: a norm on $H_2(M,\mathbb{R})$. For practical reasons, we will define this norm on $H^1(M,\mathbb{R})$. For nice references see e.g. [FLP79, exposé 14], [Cal07, Thu86].

For a compact connected surface S, let $\chi_-(S) = |\min\{0,\chi(S)\}|$. In general, if a surface S has r connected components $S_1,\dots,S_r$ we define $\chi_-(S)$ by $\sum_{i=1}^r \chi_-(S_i)$. This determines a function $\|\cdot\|_T : H^1(M,\mathbb{R}) \to \mathbb{N}\cup\{0\}$ as follows:

$$\|[\alpha]\|_T := \inf\{\chi_-(S) \mid S \text{ is a properly embedded oriented surface where } [S] \text{ is dual to } [\alpha]\},$$

where $[S]$ is in $H_2(M,\mathbb{Z})$ (or $H_2(M,\partial M;\mathbb{Z})$ if $\partial M \neq \emptyset$). So far this function just measures the minimal topological complexity of a surface dual to $[\alpha]$. However, if M is irreducible (i.e. every embedded sphere bounds a ball) then $\|\cdot\|_T$ satisfies the pseudo-norm properties. Therefore it has a unique continuous extension to a pseudo-norm on $H^1(M,\mathbb{R})$. If in addition M is atoroidal and $\chi(\partial M) = 0$, this continuous extension is a norm, called the *Thurston norm*. The unit ball of this norm has a very special geometry.

Theorem 2.1 ([Thu86]). *Let M be an irreducible and atoroidal manifold. Then the unit ball of the Thurston norm is a convex finite polytope.*

An avid reader can consult the proof of the preceding theorem in Calegari's book (see [Cal07, Theorem 5.10]). The most striking aspect of the Thurston norm is that it provides a very nice picture for homology classes representing fibrations of M over the circle.

2.2. *From homology classes to fibrations*

Let $[M, S^1]$ denote the set of homotopy classes of maps from M to S^1. Given a class $[f] \in [M, S^1]$ one can choose a smooth representative $f : M \to S^1$ and $d\theta$ the angle form on S^1. The pullback defines a class $[f^*d\theta]$ in $H^1(M, \mathbb{R})$. This correspondence defines a bijection between $H^1(M, \mathbb{Z})$ and $[M, S^1]$. We will call $[\alpha] \in H^1(M, \mathbb{Z})$ a *fibration* if the corresponding class in $[M, S^1]$ is a fibration. Set

$$\Phi(M) := \{[\alpha] \in H^1(M, \mathbb{Z}) \mid [\alpha] \text{ is a fibration}\}$$

and for every face F of the Thurston norm ball let $\mathbb{R}^+ \cdot F$ denote the positive cone in $H^1(M, \mathbb{R})$ whose basis is F.

Theorem 2.2 ([Thu86]). *Suppose that $b_1(M) \geq 2$. If $\Phi(M) \cap \mathbb{R}^+ \cdot F \neq \emptyset$ for some top-dimensional face F of the Thurston norm unit ball, then $\Phi(M) \cap \mathbb{R}^+ \cdot F = H^1(M, \mathbb{Z}) \cap \mathbb{R}^+ \cdot F$.*

When $\Phi(M) \cap \mathbb{R}^+ \cdot F \neq \emptyset$ we call F a *fibered face* and $\mathbb{R}^+ \cdot F$ a *fibered cone*. A fiber of a fibration minimizes the Thurston norm in its homology class [Cal07, Corollary 5.13].

2.3. *Hyperbolic manifolds*

If the manifold M is hyperbolic, then the monodromy of each fibration $\Sigma \to M \to S^1$ defines a pseudo-Anosov class in $\mathrm{Mod}(\Sigma)$. Hence we can think of each integer point in a fibered cone $\mathbb{R}^+ \cdot F$ as a pseudo-Anosov class (on a surface that is not necessarily connected). We want to compute, for a fixed fibered face F, the stretch factors of all pseudo-Anosov maps arising as monodromies of fibrations in the fibered cone $\mathbb{R}^+ \cdot F$. This can be done by using an invariant of the fibered face called the *Teichmüller polynomial*. Roughly speaking, it is an element of the group ring $\mathbb{Z}[G]$, where $G = H_1(M, \mathbb{Z})/\mathrm{Tor}$ and Tor is the torsion subgroup of $H_1(M, \mathbb{Z})$. Following McMullen, we denote it by Θ_F.

We will now explain how Θ_F is used to calculate stretching factors of pseudo-Anosov monodromies and we will later deal with its definition. With any $[\alpha] \in H^1(M, \mathbb{Z})$ we can associate a morphism $(\xi_\alpha : H_1(M, \mathbb{Z}) \to \mathbb{Z}) \in \mathrm{Hom}(H_1(M, \mathbb{Z}), \mathbb{Z})$. Now Θ_F is an element of the group ring $\mathbb{Z}[G]$, thus it can be written as a formal sum:

$$\Theta_F = \sum_{g \in G} a_g g, \qquad a_g \in \mathbb{Z} \text{ for all } g \in G,$$

where at most a finite number of the coefficients a_g are different from zero. The evaluation of Θ_F on $[\alpha]$ is defined as follows:

$$\Theta_F(\alpha) := \sum_{g \in G} a_g \cdot t^{\xi_\alpha(g)} \in \mathbb{Z}[t, t^{-1}].$$

Note that $\Theta_F(\alpha)$ is a Laurent polynomial in $\mathbb{Z}[t, t^{-1}]$. Let $\lambda(\alpha)$ be the stretch factor of the pseudo-Anosov class in $\mathrm{Mod}(\Sigma)$ defined by the monodromy of the fibration corresponding to $[\alpha]$. The following theorem links the Laurent polynomial $\Theta_F(\alpha)$ to $\lambda(\alpha)$.

Theorem 2.3 ([McM00]). *For any fibration $[\alpha] \in \mathbb{R}^+ \cdot F$, the stretch factor $\lambda(\alpha)$ is given by the largest root (in absolute value) of the equation*

$$\Theta_F(\alpha) = \sum_{g \in G} a_g \cdot t^{\xi_\alpha(g)} = 0.$$

3. Teichmüller polynomial and train tracks

In this section we recall the construction of the Teichmüller polynomial Θ_F and basic facts on train tracks.

3.1. The Teichmüller polynomial of a fibered face

In what follows, G will denote $H_1(M, \mathbb{Z})/\mathrm{Tor}$. As before we assume that $b_1(M) \geq 2$. The pseudo-Anosov monodromy ψ of any fibration $[\alpha] \in \mathbb{R}^+ \cdot F$ with fiber Σ has an expanding invariant lamination $\lambda \subset \Sigma$ which is unique up to isotopy. Let $\mathcal{L}$ be the mapping torus of $\psi : \lambda \to \lambda$ and $\widetilde{\mathcal{L}}$ be the preimage of the lamination $\mathcal{L}$ on the covering space

$$\pi : \widetilde{M} \to M$$

corresponding to the kernel of the map $\pi_1(M) \to G$. As Fried explains (see [FLP79, ex. 14]), $\mathcal{L}$ is a compact lamination which, up to isotopy, depends only on the fibered face F. Using this fact, McMullen [McM00] defines the Teichmüller polynomial of the fibered face as

$$\Theta_F = \gcd(f : f \in I) \in \mathbb{Z}[G]$$

where I is the *Fitting ideal* of the finitely presented $\mathbb{Z}[G]$-module of transversals of the lamination $\widetilde{\mathcal{L}}$. Observe that Θ_F is well defined up to multiplication by a unit in $\mathbb{Z}[G]$. One of the main results of [McM00] that we exploit in this article is a formula that allows one to compute Θ_F explicitly. We recall how to derive this formula later.

3.2. The setting

The formula that allows us to compute Θ_F explicitly requires a particular splitting of the group G. Fix a fiber $\Sigma \hookrightarrow M$ and let $\psi : \Sigma \to \Sigma$ be the corresponding pseudo-Anosov monodromy. We will denote by $H_\psi = \mathrm{Hom}(H^1(\Sigma, \mathbb{Z})^\psi, \mathbb{Z})$ the dual of the ψ-invariant cohomology of Σ. The natural map $\pi_1(S) \to H_1(S, \mathbb{Z}) \to H_\psi$ determines an infinite abelian covering

$$\rho : \widetilde{\Sigma} \to \Sigma$$

with deck transformation group H_ψ. We can think of $\widetilde{\Sigma}$ as a component of the preimage of a fixed fiber Σ in the covering $\pi : \widetilde{M} \to M$, and of H_ψ as the subgroup of $\mathrm{Deck}(\pi) = G$ fixing $\widetilde{\Sigma}$. For every lift

$$\widetilde{\psi} : \widetilde{\Sigma} \to \widetilde{\Sigma} \tag{3.1}$$

of ψ, the 3-manifold $\widetilde{M}$ can be easily described in terms of $\widetilde{\Sigma}$ and $\widetilde{\psi}$ as follows. For every $k \in \mathbb{Z}$ let A_k denote a copy of $\widetilde{\Sigma} \times [0, 1]$. Then $\widetilde{M}$ is obtained from $\bigsqcup_{k \in \mathbb{Z}} A_k$ by identifying $(s, 1) \in A_k$ with $(\widetilde{\psi}(s), 0) \in A_{k+1}$, for every $k \in \mathbb{Z}$. In this setting, the deck transformation group of $\widetilde{M}$ splits as

$$G = H_\psi \oplus \mathbb{Z}\widetilde{\Psi}$$

where the map $\widetilde{\Psi}$ acts on $\widetilde{M}$ as $\widetilde{\Psi}(s,t) = (\widetilde{\psi}(s), t-1)$. Equipped with these coordinates, if $F \subset H^1(M,\mathbb{R})$ is the fibered face with $[\Sigma] \in \mathbb{R}^+ \cdot F$, then we can regard Θ_F as a Laurent polynomial $\Theta_F(t,u) \in \mathbb{Z}[G] = \mathbb{Z}[H_\psi] \oplus \mathbb{Z}[u]$ where $t = (t_1, \ldots, t_{b-1})$ is a basis of H_ψ and $u = \widetilde{\Psi}$.

Remark 3.1. If $\widetilde{\psi}_1$ and $\widetilde{\psi}_2$ are two different lifts of ψ to $\widetilde{\Sigma}$ then $\widetilde{\psi}_1 = t \cdot \widetilde{\psi}_2$ for some $t \in H_\psi = \mathrm{Deck}(\rho)$. Hence, taking a different lift in (3.1) translates into a change of variables of the form $u' = tu$. On the other hand, since the topology of $\widetilde{M}$ is independent of ψ, the topology of the non-compact surface $\widetilde{\Sigma}$ is also independent of ψ.

3.3. *Train tracks*

A train track is a connected graph with an additional "smooth" structure. More precisely, let τ be a graph and let $h : \tau \to \Sigma$ be an embedding so that the edges are tangent at the vertices. Since the vertices are smooth, at each vertex the edges can be partitioned into two sets, called ingoing and outgoing for convenience (the choice of the partition is arbitrary). We will also assume that at each vertex of τ we have a cyclic order (given by h). This gives the notion of *cusp* at a vertex: this is a region formed by a consecutive pair (in terms of a cyclic ordering) of either two ingoing or two outgoing edges.

The pair (τ, h) (often called simply τ if there is no confusion) is a *train track* if the following additional properties are fulfilled:

(1) τ has no vertex of valence 1 or 2;
(2) the connected components of $\Sigma \setminus h(\tau)$ are either polygons with at least one cusp or annuli with one boundary component contained in $\partial\Sigma$ and the other with one cusp.

A (transverse) *measure* μ on a train track is an assignment of a real number $\mu(e) \geq 0$ to each edge e of τ which satisfies the switch condition at each vertex: the sum of the measures of the edges in the ingoing set is the same as that for the outgoing set. The train track τ equipped with a measure μ will be called a *measured train track*, and will be denoted (τ, h, μ).

3.4. *Measured foliations and train tracks*

We can construct a (class of) measured foliation $\mathcal{F}$ from a measured train track (τ, h, μ) as follows. We replace each edge e of $h(\tau)$ by a rectangle, of arbitrary width and height $\mu(e)$, foliated by horizontal leaves. According to the switch condition, the rectangles glued together give a subsurface $\widetilde{\Sigma} \subset \Sigma$ (with boundaries) and a measured foliation $\mathcal{F}$ on $\widetilde{\Sigma}$. Now to define the foliation on the whole surface, one has to collapse the complementary regions. By assumption, no complementary components of $\widetilde{\Sigma}$ into Σ are smooth annuli, so that we can contract each boundary component to obtain a well-defined measured foliation on Σ (see [PP87,FLP79] for details). We will call the sides of the polygons of $\Sigma \backslash \tau$ around punctures or around the singularities of $\mathcal{F}$ *infinitesimal edges*.

Remark 3.2. There are many arbitrary parameters in the above construction, but the equivalence class $[\mathcal{F}]$ (up to isotopy and Whitehead moves) of $\mathcal{F}$ is well defined.

There is a converse to the above construction. Let $\mathcal{F}$ be a measured foliation representing $[\mathcal{F}]$ and let $p \in \mathcal{F}$ be a singularity. Consider a polygon Δ_p embedded into the surface Σ, where each side Δ_p is contained in a leaf of $\mathcal{F}$. We will say that the subsurface $\Sigma \setminus \bigcup_{p \in \text{sing.}} \Delta_p$ has a *partial foliation* (still denoted by $\mathcal{F}$) *induced by* $\mathcal{F}$. Since all complementary regions of this partial measured foliation have at least two cusps, we can collapse the leaves of this foliation to obtain a measured train track (τ, h, μ).

By considering small segments transversal to the horizontal leaves on the rectangles used in the above procedure, we obtain a fibered neighborhood $N(\tau) \subset \Sigma$ equipped with a retraction $N(\tau) \to \tau$. The neighborhood $N(\tau)$ has cusps on its boundary, and the fibers of the retraction are called *ties*. The train track τ can be recovered from $N(\tau)$ by collapsing every tie to a point. We will say that $\mathcal{F}$ is *carried by* τ (written $\mathcal{F} \prec \tau$) if $\mathcal{F}$ can be represented by a partial foliation contained in $N(\tau)$ whose leaves are transverse to the ties. If in addition no leaves of $\mathcal{F}$ connect cusps of $N(\tau)$, we say that τ is *suited to* $\mathcal{F}$.

The next sections are intended to make explicit some well-known relations between pseudo-Anosov homeomorphisms and train track morphisms.

3.5. Invariant train tracks

By definition, any representative of a pseudo-Anosov class $[\psi] \in \mathrm{Mod}(\Sigma)$ leaves invariant a pair of transverse measured foliations $(\mathcal{F}^s, \mathcal{F}^u)$. However, the action of ψ on these foliations is rather difficult to describe. A good tool to understand this action is given by train tracks (see e.g. [PP87, §4]). Let $h : \tau \hookrightarrow \Sigma$ be suited to $\mathcal{F}^u$. Since $\mathcal{F}^u$ is invariant under ψ, it follows that τ is invariant under ψ, namely:

(1) The foliation $\mathcal{F}^u$ can be represented by a partial measured foliation $\mathcal{F}$ whose support is a fibered neighborhood $N(\tau)$ of $h(\tau)$.
(2) The image $\psi(h(\tau))$ can be isotoped to a train track $h'(\tau')$ which is contained in a *fibered neighborhood* $N(h(\tau))$ of $h(\tau)$, *is transversal to the tie foliation of* $h(\tau)$ and has switches that are disjoint from the collection of central ties of $h(\tau)$.

If (2) holds, we will say that $\psi(\tau)$ is *carried by* τ and write $\psi(\tau) \prec \tau$.

Convention. In this paper, we will work with *labeled train tracks*, that is, triples (τ, h, ε), where $\varepsilon : E(\tau) \to \mathbb{A}$ is a labeling map from the set of edges of τ into a fixed finite alphabet $\mathbb{A}$. In the following we will abuse the notation and abbreviate (τ, h, ε) to τ whenever the embedding of the graph $h : \tau \hookrightarrow \Sigma$ and the labeling are clear from the context.

In order to make a distinction between infinitesimal edges and other edges, we make the following choice: we label infinitesimal edges by capital letters and other edges by lower case letters. We denote by $\mathbb{A}_{\text{prong}} \subset \mathbb{A}$ the n letters corresponding to the infinitesimal edges $E(\tau)_{\text{prong}}$ enclosing the punctures of $\mathbf{D}_n$. We also denote by $\mathbb{A}_{\text{real}} \subset \mathbb{A}$ the letters corresponding to non-infinitesimal edges $E(\tau)_{\text{real}}$.

3.6. *Incidence matrix*

In the above situation, if $\psi(\tau) = \sigma \prec \tau$, we naturally associate an *incidence matrix* $M(\psi) \in \mathrm{GL}(\mathbb{Z}^{\mathbb{A}})$ to ψ in the following way. For any edge e of τ we make a choice of a tie above an interior point e (called the *central tie* associated to the edge e). Let σ' isotopic to σ be such that $\sigma' \subset N(\tau)$. For any edge f of σ we have a corresponding edge f' of σ' given by the isotopy. We can furthermore isotope σ' slightly so that it is in general position with respect to the central ties of τ. Now for any pair (e, f) with labels (α, β) (i.e. $\varepsilon(e) = \alpha$ and $\varepsilon(f) = \beta$) we define $M_{\beta,\alpha}(\psi)$ as the geometric intersection of f' and the central tie associated to the edge e of τ.

A classical theorem [PP87, Theorem 4.1] asserts that in the pseudo-Anosov case, the leading eigenvalue of this matrix equals the stretch factor of the pseudo-Anosov class $[\psi]$ (if one restricts to a good set of edges, the corresponding matrix is a Perron–Frobenius matrix).

The determinant formula. Now consider a component $\widetilde{\tau} \subset \widetilde{\Sigma}$ of $\rho^{-1}(\tau)$ lying in the infinite surface $\widetilde{\Sigma}$, as defined in §3.2. This is an infinite train track whose sets of edges and of vertices can be identified with $E \times H_\psi$ and $V \times H_\psi$ respectively. Since τ is ψ-invariant, $\widetilde{\tau}$ is $\widetilde{\psi}$-invariant. This means that $\widetilde{\psi}(\widetilde{\tau})$ can be isotoped to a train track $\widetilde{\tau}'$ which lies in a tie neighborhood $N(\widetilde{\tau})$ of $\widetilde{\tau}$, is transverse to $\widetilde{\tau}$'s ties and whose switches are disjoint from the collection of central ties of $\widetilde{\tau}$. As with the train track τ and the map ψ, we can associate to $\widetilde{\tau}$ and $\widetilde{\psi}$ an incidence matrix $P_E(t) \in \mathrm{GL}(\mathbb{Z}[H_\psi]^{\mathbb{A}})$ with entries in $\mathbb{Z}[H_\psi]$. Analogously, there is a matrix $P_V(t)$ with entries in $\mathbb{Z}[H_\psi]$ associated to the set of vertices of $\widetilde{\tau}$. The next theorem states that the Teichmüller polynomial associated to the fibered face F can be recovered from the matrices $P_E(t)$ and $P_V(t)$.

Theorem 3.3 ([McM00]). *The Teichmüller polynomial of the fibered face F is given by*

$$\Theta_F(t, u) = \frac{\det(u \cdot \mathrm{Id} - P_E(t))}{\det(u \cdot \mathrm{Id} - P_V(t))}. \tag{3.2}$$

3.7. *Train track morphisms*

We begin with a classical definition (see e.g. [Los10]).

Definition 3.4. A map T between two train tracks (τ, h) and (τ', h') is a *train track morphism* if it is cellular and preserves the smooth structure. If in addition (τ, h) and (τ', h') are isomorphic as train tracks, we call T a *train track map*.

A train track morphism $T : \tau \to \tau'$ is a *representative* of $[f] \in \mathrm{Mod}(\Sigma)$ if in addition

(1) the diagram

$$\begin{array}{ccc} \tau & \xrightarrow{h} & \Sigma \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle f} \\ \tau' & \xrightarrow{h'} & \Sigma \end{array}$$

commutes, up to isotopy, and

(2) $f \circ h(\tau) \subset N(h'(\tau'))$ and $f \circ h(\tau)$ is transverse to the tie foliation of $h'(\tau')$.

To any train track morphism T one can associate an incidence matrix $M(T) \in \mathrm{GL}(\mathbb{Z}^{\mathbb{A}})$ in the following way: for any pair (e, e') with labels (α, β) (i.e. $\varepsilon(e) = \alpha$ and $\varepsilon(e') = \beta$) we define $M(T)_{\alpha,\beta}$ as the number of occurrences of e' in the edge path $T(e)$. It is clear from the definitions that if $T : \tau \to \tau$ is representative map of a homeomorphism $f : \Sigma \to \Sigma$ and if $f(\tau) \prec \tau$ then the incidence matrix $M(f)$ defined in the preceding section and the incidence matrix $M(T)$ are equal.

Theorem 3.5 ([PP87]). *Let ψ be pseudo-Anosov homeomorphism and let $\mathcal{F}^u$ be in the class of its unstable foliation. There exists a train track τ suited to $\mathcal{F}^u$ such that $\psi(\tau) \prec \tau$. Furthermore $\psi(\tau)$ is isotopic to a train track $\tau' \subset N(\tau)$ which is transverse to the ties so that the matrix describing the linear map from the space of weights on real edges of τ' to the space of weights on real edges of τ is primitive irreducible* (*i.e. some iterate has all entries strictly positive).*

3.8. Elementary operations

One of the main difficulties in using the aforementioned formulas (computing the matrix $M(\psi)$ and formula (3.2)) is that $\psi(\tau)$ (or $\widetilde{\psi}(\tilde{\tau})$) might look very complicated so that the isotopy needed to embed this train track in a tie neighborhood of τ transverse to the ties might be difficult to find. There is a general strategy that will simplify calculation, involving two natural (dual) operations on train tracks, called *folding* and *splitting*.

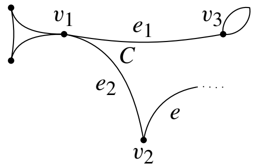
Roughly speaking, they are defined by folding or splitting a fibered neighborhood $N(\tau)$ along a cusp. For a more precise definition, see [Los10, PP87] (for splitting) and [SKL02] (for folding). In this paper we shall make use of the *folding operation* which will produce from a train track τ a new train track τ' with the property that $\tau \prec \tau'$. We now briefly describe the combinatorial folding operations. Observe that these operations first appear as (dual) right/left splits described in [PP87].

Let τ be a train track. Let e_1, e_2 be two edges of τ that are issued from the same vertex v_1 and that form a cusp C. We assume that one of the two edges (say e_1) is not infinitesimal. We describe the folding where edge e_1 *is folded onto edge* e_2 (the other case being similar). The edge e_2 (respectively, e_1) is incident to two vertices v_1 and v_2 (respectively, v_1 and v_3). The orientation around v_1 determines an edge e attached to v_2 (see Figure 1). If the cusp determined by e is on the same side as the cusp C then we cannot fold e_1 onto e_2. In the other case we form a new graph τ' in the following way: we identify the edges e_2 and e_1 so that the new graph we obtain has a new edge e_1' from v_3 to v_2. If e is an infinitesimal edge we then fold e_1' on e again. The new train track (τ', h') naturally inherits a labeling ε' induced from the one of τ: every edge of τ' shares the label with the corresponding edge of τ.

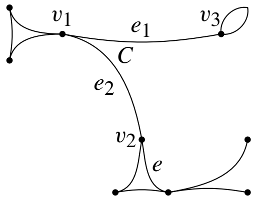
Definition 3.6. We will say that a train track τ *refines* to a train track σ if there exists a sequence

$$\sigma = \tau_1 \prec \tau_2 \prec \cdots \prec \tau_{k-1} \prec \tau_k = \tau \tag{3.3}$$

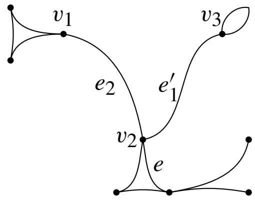
where τ_i is obtained from τ_{i-1} by a folding operation.



We cannot fold e_1 onto e_2



Folding e_1 onto e_2 allowed



Edge e_1 folded onto e_2

Fig. 1. The folding operation: edge e_1 folded onto e_2 produces a new train track.

Proposition 3.7 ([PP87]). *Suppose that $\mathcal{F} \prec \sigma \prec \tau$ where σ is contained in a fibered neighborhood $N(\tau)$ and τ is suited to $\mathcal{F}$. Then τ refines to σ.*

Sketch of proof of Proposition 3.7. We use splitting instead of folding, as it is easier to explain and the corresponding sequence of foldings is easy to derive. Up to isotopy, one can find a fibered neighborhood $N(\sigma)$ contained in the interior of $N(\tau)$ whose tie foliation is formed by subarcs of the tie foliation of $N(\tau)$. The number of cusps of $N(\sigma)$ and $N(\tau)$ is the same and one can define a pairing between these two sets of cusps with a family $\{\Gamma_i\}_{i=1}^{I}$ of disjointly embedded arcs contained in $N(\tau) \setminus \mathrm{Int}(N(\sigma))$ which are transverse to the ties [PP87, Lemma 2.1]. The sequence of splittings that defines the refinement is obtained by cutting $N(\sigma)$ along Γ_i, $i = 1, \ldots, n$. Each time the arc Γ_i crosses a singular tie of $N(\tau)$, the cutting along Γ_i defines a splitting operation on τ. The concatenation of these operations produces the sequence (3.3). □

The previous proposition has a simple but important consequence: the refinement of τ to $\psi(\tau)$ allows us to factorize the incidence matrix $M(\psi)$ as a product of matrices associated to folding operations. Below we explain how this can be done.

3.9. *Folding operations and train track morphisms*

Each folding operation from a train track (τ, h, ε) to a train track $(\tau', h', \varepsilon')$ produces a train track morphism $T : \tau \to \tau'$ that represents some $[f] \in \mathrm{Mod}(\Sigma)$ such that $f(h(\tau)) \prec h'(\tau')$. Hence our preceding discussion can be reformulated as follows.

Lemma 3.8 (Penner–Papadoupoulos [PP87]). *Every* (*labeled*) *train track map representing a class $[f] \in \mathrm{Mod}(\Sigma)$ is obtained by a finite sequence of train track maps induced by folding operations and then followed by a relabeling operation.*

Hence for any pseudo-Anosov class $[\psi]$ and any invariant train track τ, one can define a (non-unique) sequence of folding operations defined by the refinement sequence

$$\psi(\tau) = \tau_1 \prec \tau_2 \prec \cdots \prec \tau_{k-1} \prec \tau_k = \tau.$$

The sequence of folding operations defines a sequence of train track maps:

$$\psi(\tau) = (\tau_1, \varepsilon_1) \xrightarrow{T_1} (\tau_2, \varepsilon_2) \xrightarrow{T_2} \cdots \xrightarrow{T_{k-1}} (\tau_k, \varepsilon_k) \xrightarrow{T_k} (\tau_1, \varepsilon_{k+1}) \xrightarrow{R} (\tau_1, \varepsilon_1).$$

Here R is just a relabeling map. Therefore Lemma 3.8 in this context implies that $T = R \circ T_k \circ T_{k-1} \circ \cdots \circ T_1$. Hence the incidence matrix $M(\psi)$ or $M(T)$ is

$$M(T) = M(R \circ T_k \circ T_{k-1} \circ \cdots \circ T_1) = M(T_1) \cdots M(T_k)M(R). \tag{3.4}$$

Remark 3.9. Observe that since we work with *labeled train tracks*, all the transition matrices $M(T_i)$ have the form $\mathrm{Id} + E$ where E is a non-negative matrix.

To summarize, to each pseudo-Anosov homeomorphisms, one can associate a train track and a sequence of folding operations such that the corresponding product of matrices is irreducible, i.e. it has some power such that every entry has positive coefficients (Theorem 3.5). In general the converse is not true, but under a mild assumption one has:

Theorem 3.10. *Let $\tau = \tau_1 \prec \tau_2 \prec \cdots \prec \tau_{k-1} \prec \tau_k = \tau$ be a refinement sequence defined by folding operations such that the corresponding incidence matrix is irreducible and the Perron–Frobenius eigenvector satisfies the switch conditions of the train track τ. Then the train track map T associated to this sequence is the representative of a pseudo-Anosov homeomorphism.*

3.10. Elementary operations and standardness

As we have seen in the preceding subsections, every train track map $T : \tau \to \tau$ representing a class in $\mathrm{Mod}(\Sigma)$ is the product of train track maps defined by elementary operations. When Σ is the n-punctured disc $\mathbf{D}_n$, this product can be refined by requiring that every train track in (3.3) be *standardly embedded*. Since all the calculations that we present in the present paper are described in this context, we will discuss these notions in detail.

In §3.5 we made the convention that the n letters in $\mathbb{A}_{\text{prong}}$ label the infinitesimal edges enclosing punctures of $\mathbf{D}_n$, hence any labeling using these letters defines a labeling of the punctures of $\mathbf{D}_n$. We consider $\mathbf{D}_n$ to be modeled on the unit disc in $\mathbb{C}$ with n punctures along the real line $\mathbb{R}$. Let l_α be a vertical segment joining the puncture labeled by $\alpha \in \mathbb{A}_{\text{prong}}$ to the boundary of the disc. Now consider a train track $h : \tau \hookrightarrow \mathbf{D}_n$. If all the edges except these infinitesimal edges are embedded in the upper (or lower) half-disc, then we say that (τ, h) is *standard*. If only one open real edge of $h(\tau)$ intersects $\bigcup l_\alpha$ only once, and all the other real edges are embedded in the upper (or lower) half-disc, then we say that (τ, h) is *almost standard*. These notions were first introduced in [SKL02]; see Figure 2.

We consider the n-th braid group B_n with standard generators $\sigma_1, \ldots, \sigma_{n-1}$ and consider the natural map $B_n \to \mathrm{Mod}(\mathbf{D}_n)$ which associates to each braid β the mapping class f_β. If (τ, h) is standard and we perform a folding operation on $h(\tau)$, then the resulting train track (τ_1, h_1) is either standard or almost standard. In the latter situation we can easily turn (τ_1, h_1) into a standard marking.

Fig. 2. Standardly and almost standardly embedded train tracks.

Lemma 3.11 ([SKL02]). *Let (τ, h) be an almost standard train track in $\mathbf{D}_n$. Then there exists an n-braid of the form $\delta^{\pm}_{[l,m]} = (\sigma_{m-1}\sigma_{m-2}\cdots\sigma_l)^{\pm}$ (called a standardizing braid) such that $(\tau_1, f_{\delta^{\pm}_{[l,m]}} \circ h_1)$ is standard.*

In this context we say that $f_{\delta^{\pm}_{[l,m]}}$ is a *standardizing homeomorphism* for (τ_1, h_1).

Definition 3.12. Any infinitesimal edge around a puncture determines a cusp (enclosing the puncture). Any standardizing homeomorphism f_β acts on those edges by a permutation $\pi \in \mathrm{Sym}(\mathbb{A}_{\mathrm{prong}})$. Moreover if e, f are two infinitesimal edges (with labeling α, β respectively) and if $\pi(\alpha) = \beta$ then f_β acts as a rotation whose support is contained in a small neighborhood of the punctures. We encode this action by the rotation number $k \in \mathbb{Z}$ (under the convention that a counterclockwise rotation has positive sign) and we will write $\pi(\alpha) = \beta^k$.

Example 3.13. In Figure 3 we depict how the standardizing homeomorphism corresponding to $f_{\sigma_1^{-1}}$ acts on punctures in $\mathbf{D}_3$. If we identify punctures and infinitesimal edges enclosing the punctures (with labels A, B, C, from left to right), it is easy to deduce that $\pi(\sigma_1^{-1}) = \left(\begin{smallmatrix} A & B & C \\ B & A^{-1} & C \end{smallmatrix}\right)$. An analogous figure helps one to conclude that $\pi(\sigma_2) = \left(\begin{smallmatrix} A & B & C \\ A & C^{+1} & B \end{smallmatrix}\right)$. In particular, for any $k \in \mathbb{Z}$, $k > 0$, the permutation associated to σ_2^{2k} is $\pi(\sigma_2^{2k}) = \left(\begin{smallmatrix} A & B & C \\ A & B^k & C^k \end{smallmatrix}\right)$.

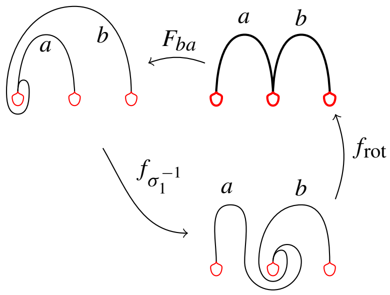


Fig. 3. Rotation around a puncture induced by a standardizing homeomorphism.

4. Construction of the automaton

In this section, for simplicity, we specify to the case of the punctured disc. However, all the discussion can be done for surfaces of higher genera. Let us fix $n > 2$, the number of punctures, and the singularity data of train tracks (i.e. the number and type of prongs).

We fix an alphabet $\mathbb{A}$. We will also fix the maximal abelian covering of $\mathbf{D}_n$ induced by $H = \mathrm{Hom}(H^1(\mathbf{D}_n, \mathbb{Z}), \mathbb{Z})$, and we fix a multiplicative basis $t_1, \dots, t_n$ of H.

4.1. Graphs of foldings

We start with the following observation: the number of labeled train tracks (τ, h, ε) of $\mathbf{D}_n$ where

- (τ, h) is standard,
- τ has prescribed singularity data and labeling $\varepsilon : E(\tau) \to \mathbb{A}$

(up to isotopy of $\mathbf{D}_n$ fixing the punctures) is finite.

Moreover this set is also (setwise) invariant under folding operations followed by standardness operations. Finally, given a tuple (τ, h, ε) in this finite set, since the number of cusps and edges is finite, there are only finitely many possible folding operations on $h(\tau)$. These three finiteness ingredients allow us to construct a graph in the following way.

(1) Vertices are tuples $(\tau, h, \varepsilon_{|E(\tau)_{\mathrm{real}}})$ where $h : \tau \to \mathbf{D}_n$ is standard (up to isotopy fixing the punctures).
(2) There is an edge between $(\tau_1, h_1, \varepsilon_1)$ and $(\tau_2, h_2, \varepsilon_2)$ if there is a folding operation from $(\tau_1, h_1, \varepsilon_1)$ to $(\tau_2, h'_1, \varepsilon_2)$ and either
  (a) $h'_1(\tau_2)$ is standard, then $h_2 = h'_1$, or
  (b) $h'_1(\tau_2)$ is almost standard, then $h_2 = f_\beta \circ h'_1$ where f_β is a standardizing braid.
(3) There is an edge between (τ, h, ε) and $(\tau, f_\beta \circ h, \varepsilon)$ where $\beta \in B_n$ and $f_\beta \circ h(\tau)$ is also standard.

The resulting directed graph is called the *folding automaton* associated to the number of marked points of $\mathbf{D}_n$ and the type of singularities. Observe that this graph is not necessarily strongly connected (or even connected). It would be nice to have a description of the topology of these graphs in general.

For any train track (τ, h, ε) we will denote by $\mathcal{N}^{\mathrm{lab}}(\tau, h, \varepsilon)$ the connected component of the folding automaton containing (τ, h, ε). One can also perform the same construction without labeling; the connected components containing (τ, h) are then denoted by $\mathcal{N}(\tau, h)$.

Example 4.1. See Figures 14–18 for examples of automata.

4.2. Closed loops in $\mathcal{N}(\tau, h)$ and pseudo-Anosov homeomorphisms

The labeling allows us to define for each edge of $\mathcal{N}^{\mathrm{lab}}(\tau, h, \varepsilon)$ a train track map and its associate transition matrix. Hence given a path η in $\mathcal{N}^{\mathrm{lab}}(\tau, h, \varepsilon)$ (not necessarily closed) represented by

$$(\tau_1, \varepsilon_1) \xrightarrow{T_1} (\tau_2, \varepsilon_2) \xrightarrow{T_2} \cdots \xrightarrow{T_{k-1}} (\tau_k, \varepsilon_k) \xrightarrow{T_k} (\tau_1, \varepsilon_{k+1})$$

one defines the matrix $M(\eta)$ by using formula (3.4):

$$M(\eta) = M(T_k \circ T_{k-1} \circ \cdots \circ T_1) = M(T_1) \cdots M(T_k).$$

Now if γ is a *loop* in $\mathcal{N}(\tau, h)$ starting at some point (τ_i, h_i), we can lift γ to some path $\hat{\gamma}$ in $\mathcal{N}^{\mathrm{lab}}(\tau, h, \varepsilon)$ starting at $(\tau_i, h_i, \varepsilon_i)$. Here ε is any labeling of (τ, h). The end point of $\hat{\gamma}$ (that is, $(\tau_i, h_i, \varepsilon_i')$) defines a train track map

$$R : (\tau_i, h_i, \varepsilon_i') \to (\tau_i, h_i, \varepsilon_i).$$

The associated matrix $M(R) \in \mathrm{GL}(\mathbb{Z}^{\mathbb{A}})$ is induced by a permutation, namely $R_{\alpha,\beta} = 1$ if $\pi(\alpha) = \beta$ and 0 otherwise, where $\pi = \varepsilon_i' \circ (\varepsilon_i)^{-1} \in \mathrm{Sym}(\mathbb{A})$. We then define

$$M(\gamma) := M(\hat{\gamma}) \cdot M(R).$$

Obviously the conjugacy class of the matrix $M(\gamma)$ does not depend on the choice of the labeling ε.

Remark 4.2. The above discussion allows us to reformulate Theorem 3.5 and Lemma 3.8 as follows: any pseudo-Anosov homeomorphism is obtained from a closed loop in some graph $\mathcal{N}(\tau, h)$ by using the above construction. The converse is almost true: this is Theorem 3.10.

We end this section with a useful description of the train track map representing the lift of homeomorphisms to $\widetilde{\mathbf{D}}_n$.

4.3. Lifting train tracks

Let $\rho : \widetilde{\mathbf{D}}_n \to \mathbf{D}_n$ be a normal covering of the punctured disc and $\mathfrak{H}$ the corresponding deck transformation group. The surface $\widetilde{\mathbf{D}}_n$ can be constructed by glueing $\mathfrak{H}$ copies of the simply connected domain obtained by cutting the base $\mathbf{D}_n$ along n disjoint segments from the punctures to the exterior boundary. The way of glueing is dictated by the monodromy of the covering. We call each of these simply connected domains a *leaf* of the covering $\rho : \widetilde{\mathbf{D}}_n \to \mathbf{D}_n$. In this paper we will usually choose a leaf in $\widetilde{\mathbf{D}}_n$, label it with $e_{\mathfrak{H}}$ (the identity element in $\mathfrak{H}$) and call it *the leaf at level zero*. As explained in Section 3.2, we will be considering infinite coverings $\rho : \widetilde{\mathbf{D}}_n \to \mathbf{D}_n$ where $\mathfrak{H}$ is given by the maximal abelian covering of $\mathbf{D}_n$ or by the dual of the invariant cohomology of $\mathbf{D}_n$ with respect to the action of a pseudo-Anosov class.

For each standard (τ, h, ε), $\widetilde{h}(\widetilde{\tau}) := \rho^{-1}(h(\tau))$ defines an infinite train track $\widetilde{h} : \widetilde{\tau} \to \widetilde{\mathbf{D}}_n$. The edges and vertices of $\widetilde{\tau}$ are in bijection with $E(\tau) \times \mathfrak{H}$ and $V(\tau) \times \mathfrak{H}$ respectively, and there are several ways to label the edges of $\widetilde{\tau}$.

Every permutation $\eta \in \mathrm{Sym}(\mathbb{A}_{\mathrm{prong}})$ defines a labeling of the edges of $\widetilde{\tau}$ as follows. For every edge $\mathbf{e}$ of $\widetilde{\tau}$ whose image under $\widetilde{h}$ is properly contained in the leaf at level zero we define $\widetilde{\varepsilon}(\mathbf{e}) = \varepsilon(e)$, where $\rho(\mathbf{e}) = e$. Now by the way we defined the leaves of the covering, and given that we are working with standardly embedded train tracks, there are exactly $2n$ edges of $\widetilde{\tau}$ whose images under $\widetilde{h}$ are not properly contained in the leaf of level zero. Moreover, these edges can be grouped into pairs $\{e^1, e^2\}$ where $\rho(e^1) = \rho(e^2) = e$, and e is an infinitesimal edge of τ around a puncture. For every such edge e we define $\widetilde{\varepsilon}(e^1) = \eta(\varepsilon(e))$ where $\varepsilon(e) \in \mathbb{A}_{\text{prong}}$. Finally, we extend $\widetilde{\varepsilon}$ to the remaining edges of $\widetilde{\tau}$ by using the $\mathfrak{H}$-monodromy action of the covering.

4.4. *Lifting train track maps*

Let (τ, h) and (τ', h') be two train tracks in $\mathbf{D}_n$ and let $T : \tau \to \tau'$ be a train track map representing a class $[f] \in \text{Mod}(\mathbf{D}_n)$. Now let $\widetilde{f} : \widetilde{\mathbf{D}}_n \to \widetilde{\mathbf{D}}_n$ be a lift of $[f]$ to the $H = \mathbb{Z}^{b_1(M)-1}$-covering of the punctured disc $\rho : \widetilde{\mathbf{D}}_n \to \mathbf{D}_n$, and let $(\widetilde{\tau}, \widetilde{h})$ and $(\widetilde{\tau}', \widetilde{h}')$ be lifts of (τ, h) and (τ', h') respectively to this covering. As with finite train tracks, a cellular map $\widetilde{T} : \widetilde{\tau} \to \widetilde{\tau}'$ that preserves the smooth structure will be called a *train track morphism*. If in addition the domain and image train tracks of the morphism are isomorphic as train tracks, we speak of a *train track map*. A train track morphism $\widetilde{T} : \widetilde{\tau} \to \widetilde{\tau}'$ is a representative of the lift $\widetilde{f}$ if:

(1) The diagram

$$\begin{array}{ccc} \widetilde{\tau} & \xrightarrow{\widetilde{h}} & \widetilde{\mathbf{D}}_n \\ {\scriptstyle \widetilde{T}}\downarrow & & \downarrow{\scriptstyle \widetilde{f}} \\ \widetilde{\tau}' & \xrightarrow{\widetilde{h}'} & \widetilde{\mathbf{D}}_n \end{array}$$

commutes, up to isotopy, and

(2) $\widetilde{f} \circ \widetilde{h}(\widetilde{\tau}) \subset N(\widetilde{h}'(\widetilde{\tau}'))$ and $\widetilde{f} \circ \widetilde{h}(\widetilde{\tau})$ is transverse to the tie foliation of $\widetilde{h}'(\widetilde{\tau}')$.

It is clear that for every lift $\widetilde{f}$ of f there is a train track map representing it.

Let $\eta \in \text{Sym}(\mathbb{A}_{\text{prong}})$ be any permutation and $\pi \in \text{Sym}(\mathbb{A}_{\text{prong}})$ be the permutation defined by f. These permutations define labelings $(\widetilde{\tau}, \widetilde{h}, \widetilde{\varepsilon})$ and $(\widetilde{\tau}', \widetilde{h}', \widetilde{\varepsilon}')$ by η and $\pi \circ \eta$ respectively. As in the case of finite train tracks we can associate to the train track map $\widetilde{T} : \widetilde{\tau} \to \widetilde{\tau}'$ representing $\widetilde{f}$ an incidence matrix. The matrix $M(\widetilde{T}) \in \text{GL}(\mathbb{Z}[H]^{\mathbb{A}})$ records how the edges of $\widetilde{f} \circ \widetilde{h}(\widetilde{\tau})$ intersect the central ties of $\widetilde{h}'(\widetilde{\tau}')$. Obviously by construction one has $M(\widetilde{T}) = M(T) \cdot \text{Diag}(v)$ for a suitable vector $v \in \mathbb{Z}[H]^{\mathbb{A}}$. In the next section we explain how to compute this vector in the particular situation where $T : \tau \to \tau'$ is an edge of the folding automaton.

5. Computing the Teichmüller polynomial

In this section we shall prove our main result. The statement uses what we call the *decorated folding automaton*. The idea is to enrich the folding automaton by adding additional information to each of its edges so that the computation of the Teichmüller polynomial can be carried out using just the decorated folding automaton. This represents a simplification of the problem of computing Θ_F, for with the method we propose there is no need to pass to an abelian infinite covering.

Recall that we have fixed the maximal abelian covering of $\mathbf{D}_n$ induced by $H = \mathrm{Hom}(H^1(\mathbf{D}_n, \mathbb{Z}), \mathbb{Z})$, and $t_1, \ldots, t_n$ is a multiplicative basis of H.

5.1. The decorated folding automaton

In the next subsections we define the extra piece of information needed to obtain the decorated folding automaton. Roughly speaking, this extra element is a vector v with entries in $\mathbb{Z}[H]$ that encodes the incidence matrix $M(\widetilde{T})$ of the lift of a train track map T coming from a folding operation (see §4.4).

Recall that when defining the folding automaton in §4.1, the labeling map ε in (τ, h, ε) is restricted to the set $E(\tau)_{\mathrm{real}}$ of real edges of τ. We will often choose the convention that, for *any* train track, the infinitesimal edges enclosing punctures are labeled by $\{A, B, C, \ldots\} = \mathbb{A}_{\mathrm{prong}}$ where the alphabetical order is set to match the order on the punctures of $\mathbf{D}_n$ induced by the natural order of $\mathbb{R}$.

Let $(\tau, h, \varepsilon) \xrightarrow{T} (\tau', h', \varepsilon')$ be a train track map associated to an edge in the folding automaton which corresponds to a folding operation F and that represents a standardizing homeomorphism f_β, where β is a braid in B_n. If $h'(\tau')$ is standardly embedded we say that the folding F is *standard*. For every edge in the folding automaton corresponding to a standard folding we define $v \in \mathbb{Z}[H]^{\mathbb{A}_{\mathrm{real}}}$ as the constant vector on which each entry is equal to 1.

Henceforth we assume that the folding F is *not standard*. There are two real edges $\{e, e'\} \subset E(\tau)$ and three vertices $\{v_0, v_1, v_2\} \subset V(\tau)$ involved when performing F. We observe that:

(1) there is a unique edge $f \in \{e, e'\}$ in $f_\beta(h(\tau))$ which is not properly embedded, i.e. it traverses to the lower half of the punctured disc $\mathbf{D}_n$,
(2) there exists a unique vertex $v_{\mathrm{fix}} \in \{v_0, v_1, v_2\}$ which is fixed by f_β, and
(3) after performing the folding operation on F, a new cusp in $(\tau', h', \varepsilon')$ appears. This cusp is incident to a vertex $v_{\mathrm{end}} \in \{v_0, v_1, v_2\}$. Let $X \in \mathbb{A}_{\mathrm{prong}}$ be the label of the unique infinitesimal edge of τ enclosing a puncture that is incident to v_{end}.

Definition 5.1. We denote by $N(T)$ the connected component of $\tau \setminus f$ which *does not* contain the vertex v_{fix} (possibly $N(T) = \emptyset$). We define $f' \in \{e, e'\}$ by $f' \neq f$. There are two cases to consider:

- **Case 1**: $f' \notin N(T)$. We define $v \in \mathbb{Z}[H]^{\mathbb{A}_{\mathrm{real}}}$ by

$$\begin{cases} v_\alpha = X^{\pm 1} & \text{if } \varepsilon(e) = \alpha \in \mathbb{A}_{\mathrm{real}} \text{ and } e \in N(T) \cup f, \\ v_\alpha = 1 & \text{otherwise.} \end{cases}$$

- **Case 2**: $f' \in N(T)$. We define $v \in \mathbb{Z}[H]^{\mathbb{A}_{\mathrm{real}}}$ by

$$\begin{cases} v_\alpha = X^{\pm 1} & \text{if } \varepsilon(e) = \alpha \in \mathbb{A}_{\mathrm{real}} \text{ and } e \in N(T), \\ v_\alpha = 1 & \text{otherwise.} \end{cases}$$

The sign of the exponent in $X^{\pm 1}$ is determined by the choice of the counterclockwise direction as positive direction for rotations on the disc. The *decorated folding automaton* $\mathcal{N}^{\mathrm{aug}}(\tau, h)$ is $\mathcal{N}^{\mathrm{lab}}(\tau, h, \varepsilon)$ where we add the information (π, v) at each edge ($\pi \in \mathrm{Sym}(\mathbb{A}_{\mathrm{prong}})$ is the permutation given by Definition 3.12).

Remark 5.2. A priori one would expect the vector v encoding the matrix $M(\widetilde{T})$ to be larger, that is, $v \in \mathbb{Z}[H]^{\mathbb{A}}$. However, as we will see in the next section, the contribution of infinitesimal edges to the determinant formula (3.2) cancels out with the contribution of the matrix $P_V(t)$, and hence one can restrict the computation of the Teichmüller polynomial to the subset $\mathbb{A}_{\mathrm{real}} \subset \mathbb{A}$ formed by the real edges.

If there is an edge between (τ, h, ε) and $(\tau, f_\beta \circ h, \varepsilon)$ where $\beta \in B_n$ and $f_\beta \circ h(\tau)$ is also standard then the vector $v \in \mathbb{Z}[H]^{\mathbb{A}}$ is defined by $v = (X^{\pm 1}, \dots, X^{\pm 1})$ depending the orientation of the braid β, and X is the label associated to the first, respectively last, prong of $\mathbf{D}_n$.

Example 5.3. We consider the folding automaton for the train track depicted in Figure 4.

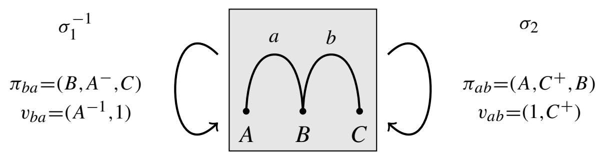


Fig. 4. The decorated folding automaton for B_3. The two edges represent σ_1^{-1} (left) and σ_2 (right).

We have depicted only real edges. Infinitesimal edges enclosing punctures are labeled by $\{A, B, C\} = \mathbb{A}_{\mathrm{prong}}$, where the alphabetical order is set to match the order on the punctures of $\mathbf{D}_3$ induced by the natural order of $\mathbb{R}$. This automaton has two edges. The one on the right corresponds to the non-standard folding F_{ab} that folds the real edge labeled with a over the real edge labeled with b. The standardizing homeomorphism in this case is given by f_{σ_2} and a direct computation shows that:

- v_{fix} is the vertex to which the infinitesimal edge A is incident and v_{end} is the vertex to which the infinitesimal edge C is incident. Therefore the label of the unique infinitesimal edge of τ enclosing a puncture that is incident to v_{end} is given by $X = C$.
- The edge f in Definition 5.1 is the edge labeled with a. Therefore $N(T_{ab})$ is the graph containing the infinitesimal edges B and C, the vertices to which they are incident and the real edge b. Thus, according to Definition 5.1, we are in Case 2.

Hence we deduce that the vector corresponding to this edge of the automaton is given by $v_{ab} = (1, C^{+1})$. We leave the rest of the computations to the reader. To deduce the signed permutations corresponding to the edges of the automaton it is useful to look at Figure 13.

5.2. *Maximal abelian covering and intermediate abelian coverings*

We stress that the Teichmüller polynomial depends on some covering induced by $H_\psi = \mathrm{Hom}(H^1(S,\mathbb{Z})^\psi, \mathbb{Z})$, whereas all the above computations depend only on $H = \mathrm{Hom}(H^1(S,\mathbb{Z}),\mathbb{Z})$.

In order to recall the action of $\psi = [f_\beta] \in \mathrm{Mod}(\mathbf{D}_n)$ on punctures, we appeal to the function

$$t : \mathbb{A}_{\mathrm{prong}} \to H_{f_\beta} = \mathrm{Hom}(H^1(\mathbf{D}_n, \mathbb{Z})^{f_\beta}, \mathbb{Z}) \tag{5.5}$$

constructed as follows. We have a collection of cycles $s_\alpha = [\partial U_\alpha]$, $\alpha \in \mathbb{A}_{\mathrm{prong}}$, that form a basis for $H_1(\mathbf{D}_n,\mathbb{Z})$. Moreover f_β acts on this basis: for every $\alpha \in \mathbb{A}_{\mathrm{prong}}$ we have $f_\beta(s_\alpha) = s_{\beta(\alpha)}$, where $\beta \in \mathrm{Sym}(\mathbb{A}_{\mathrm{prong}})$ is the permutation that β defines on the punctures. For each cycle σ of β let $t_\sigma = \sum_{\alpha\in\mathrm{Supp}(\sigma)} s_\alpha$. This defines a multiplicative basis for H, which we denote by $t_1,\dots,t_r$ for simplicity. The map $t : \mathbb{A}_{\mathrm{prong}} \to H_{f_\beta}$ is given by $t(\alpha) = t_\sigma$ whenever $\alpha \in \mathrm{Supp}(\sigma)$.

Convention. Let $\mathbb{A}^{\pm1}_{\mathrm{prong}} := \{\alpha^{+1}, \alpha^{-1}\}_{\alpha\in\mathbb{A}_{\mathrm{prong}}}$. We extend the function $t : \mathbb{A}_{\mathrm{prong}} \to H_{f_\beta}$ (respectively, $\pi \in \mathrm{Sym}(\mathbb{A}_{\mathrm{prong}})$) to a function $\{1\} \cup \mathbb{A}^{\pm1}_{\mathrm{prong}} \to H_{f_\beta}$ (respectively, a permutation of $\{1\} \cup \mathbb{A}^{\pm1}_{\mathrm{prong}}$) by $t(\alpha^{\pm1}) = t(\alpha)^{\pm1}$ if $\alpha \in \mathbb{A}_{\mathrm{prong}}$ and $t(1) = 1$ (respectively, $s(\alpha^{\pm1}) = s(\alpha)^{\pm1}$ and $s(1) = 1$).

Finally, if v is a vector with entries in $\{1, \alpha^{\pm1} \mid \alpha \in \mathbb{A}_{\mathrm{prong}}\}$, we define $t(v)$ (respectively, $\pi(v)$) as the vector that results from the evaluation of t (respectively, π) on each coordinate.

5.3. *Main result*

Theorem 5.4. *Let $[f_\beta]$ be a pseudo-Anosov class given by the loop*

$$(\tau_1,\varepsilon_1) \xrightarrow{T_1} (\tau_2,\varepsilon_2) \xrightarrow{T_2} \cdots \xrightarrow{T_{k-1}} (\tau_k,\varepsilon_k) \xrightarrow{T_k} (\tau_1,\varepsilon_{k+1}) \xrightarrow{R} (\tau_1,\varepsilon_1)$$

in the decorated folding automaton. Assume that the matrix describing the linear map on the space of weights on real edges is primitive irreducible. Then the Teichmüller polynomial $\Theta_F(t,u)$ of the associated fibered face F determined by $[f_\beta]$ is

$$\Theta_F(t_1,\dots,t_r,u) = \det(u\cdot \mathrm{Id} - M) \tag{5.6}$$

where

$$M = M(T_1)D_1 \cdots M(T_k)D_k \cdot M(R), \tag{5.7}$$

and, for $i = 1,\dots,k$,

$$\begin{cases} D_i = \mathrm{Diag}(t(w_i)) \in \mathrm{GL}(\mathbb{Z}[H]^{\mathbb{A}_{\mathrm{real}}}), \\ w_i = \eta_i(v_i), \\ \eta_1 = \mathrm{Id}_{\mathbb{A}_{\mathrm{prong}}} \text{ and } \eta_i = \pi_{i-1}\circ\eta_{i-1} \text{ for } i \geq 2. \end{cases} \tag{5.8}$$

Proof. We first observe that the assumption on the real edges implies that the contribution of the infinitesimal edges to the numerator in the determinant formula (3.2) cancels out with the denominator. This fact, together with the discussion in Sections 4.2–4.4, implies (5.6) and (5.7).

We now need to show that each diagonal matrix D_i is given by (5.8). Throughout this proof, $\widetilde{\mathbf{D}}_n$ denotes the infinite abelian covering with deck transformation group $H_{f_\beta} = \mathrm{Hom}(H^1(\mathbf{D}_n, \mathbb{Z})^{f_\beta}, \mathbb{Z})$ associated to the f_β-invariant cohomology of $\mathbf{D}_n$.

We will first prove that given an edge

$$(\tau_i, \varepsilon_i, h_i) \xrightarrow{T_i} (\tau_{i+1}, \varepsilon_{i+1}, h_{i+1}) \tag{5.9}$$

of the decorated automaton representing a standardizing homeomorphism f_β, the incidence matrix $M(\widetilde{T_i})$ associated to a lift of T_i to $\widetilde{\mathbf{D}}_n$ is of the form $M(T_i)\,\mathrm{Diag}(t(\eta_i(v_i)))$. This is the longest part of the proof and is done case-by-case depending on the train track $(\tau_i, \varepsilon_i, h_i)$. We deal with the recursive nature of formula (5.8) at the end. In order to present the list of cases, we observe that:

(1) *No train track of the decorated folding automaton defines a polygon whose sides are real edges*. More precisely, if $(\tau, h, \varepsilon) \in \mathcal{N}^{\mathrm{lab}}(\tau, h, \varepsilon)$, then no connected component of $\mathbf{D}_n \setminus h(\tau)$ homeomorphic to a disc with boundary formed by real edges of τ.
(2) *All standardizing homeomorphisms are simple in the sense of Ko–Los–Song* [SKL02]. More precisely, let $\delta_{[l,m]} := \sigma_{m-1}\sigma_{m-2}\cdots\sigma_l$, where the σ_i's are the standard Artin generators of the braid group B_n. Then we can suppose that all the homeomorphisms used to standardize train tracks in the augmented folding automaton are of the form $\delta^{\pm 1}_{[l,m]}$.

To understand the proof it will be useful to describe the covering $\widetilde{\mathbf{D}}_n$ discussed in Section 4.3 more precisely. Let D be the leaf of this covering obtained by cutting the base $\mathbf{D}_n$ along n disjoint vertical segments that go from the punctures to the lower part of the exterior boundary. Any labeling by elements of $\mathbb{A}_{\mathrm{prong}}$ of the infinitesimal edges surrounding punctures defines a natural labeling of these vertical segments. Denote them by ι_α where $\alpha \in \mathbb{A}_{\mathrm{prong}}$. The infinite covering $\widetilde{\mathbf{D}}_n$ is obtained by glueing the disjoint family of copies of D in the family $\{D_h\}_{h \in H_{f_\beta}}$ as follows: for every $h \in H_{f_\beta}$, crossing in D_h the segment ι_α in the counterclockwise direction takes one to $D_{t(\alpha)h}$, where t is the map defined in (5.5). From this detailed description we deduce that if the train track map corresponding to the edge (5.9) comes from a standard folding F_i then $M(\widetilde{T_i}) = M(T_i)$. Indeed, it is sufficient to remark that no real edge of $\widetilde{f_\beta}(\widetilde{\tau}_i)$ intersects a vertical segment ι_α.

Now suppose that the edge (5.9) comes from a non-standard folding F_i. We justify in detail the equality $M(\widetilde{T_i}) = M(T_i)\,\mathrm{Diag}(t(\eta_i(v_i)))$ in two illustrative cases. Then we explain how to proceed with all the cases that remain (see Appendix C).

Case A.1. This case is formed by an infinite family of train tracks arising from a graph Γ embedded in $\mathbf{D}_n$. This graph Γ will be called a *basic type* and consists of: two real edges $\{e, e'\}$, three vertices $\{v_0, v_1, v_2\}$ and at most three infinitesimal edges, each of which encloses a puncture of $\mathbf{D}_n$ and is incident to a vertex in $\{v_0, v_1, v_2\}$. For the situation of Case A.1, the graph Γ is depicted in Figure 5.

The idea now is to add edges and vertices to Γ to form vertices $(\tau_i, \varepsilon_i, h_i)$ of the folding automaton in $\mathbf{D}_n$. There are many ways to do this, some of which are depicted in Figure 5. In fact, given that the vertices of the automaton are properly embedded train

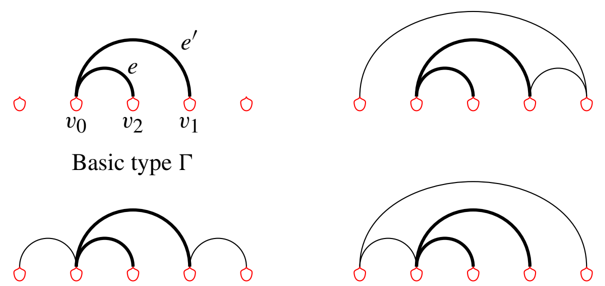


Fig. 5. Basic type A.1 and some train tracks that arise from it.

tracks, *there are only finitely many types* of train tracks that can be obtained this way (for a fixed number n of punctures in the disc). To illustrate what we mean by a *type of a train track* we display in Figure 6 all possible types of train tracks arising from the basic type Γ in Figure 5.[1] In Figure 6, each of the small boxes represents a subgraph of $(\tau_i, \varepsilon_i, h_i)$.

Now consider an edge of the automaton (5.9) where $(\tau_i, h_i, \varepsilon_i)$ is the type A.1.1 depicted in Figure 6 and the train track map T_i represents the homeomorphism f_β that standardizes the train track $F_{ee'}(\tau_i)$ that arises when folding edge e over edge e'. In Figure 7 we depict this edge of the automaton in detail. The numbers 0, 1, 2 in this figure represent the vertices $\{v_0, v_1, v_2\}$ respectively. Note that for this edge of the automaton we have $v_{\text{fix}} = v_0$ and $v_{\text{end}} = v_1$. Moreover, edges f, f' from Definition 5.1 are given by $f = e$,

Fig. 6. Types A.1.1 and A.1.2 arising from simple type A.1.

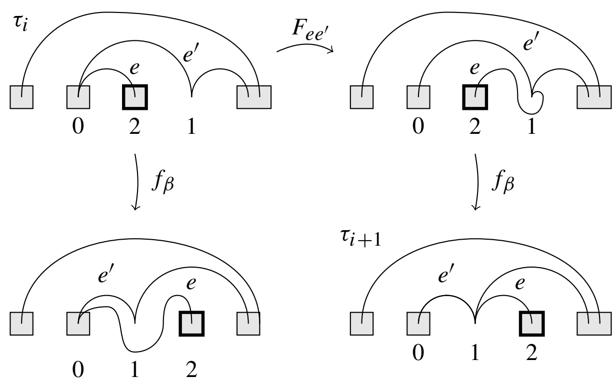


Fig. 7. A detailed edge of the automaton in Case A.1.1.

[1] For this particular case we depict Γ embedded in $\mathbf{D}_4$.

$f' = e'$ and the subgraph $N(T_i)$ is highlighted in bold. We observe that $f' \notin N(T_i)$, hence we are in Case 1 of that definition.

Now consider the lift $\widetilde{\mathbf{D}}_n \xrightarrow{\widetilde{f}_\beta} \widetilde{\mathbf{D}}_n$ which fixes the point in the fiber over v_{fix} contained in the leaf of level zero. Such a lift always exists, for f_β fixes v_{fix} downstairs by definition. We now consider the lift

$$(\widetilde{\tau}_i, \widetilde{\varepsilon}_i, \widetilde{h}_i) \xrightarrow{\widetilde{T}_i} (\widetilde{\tau}_{i+1}, \widetilde{\varepsilon}_{i+1}, \widetilde{h}_{i+1}) \tag{5.10}$$

representing $\widetilde{f}_\beta$. To compute $M(\widetilde{T}_i)$ we present Figure 8 where $\widetilde{f}_\beta(\widetilde{\tau}_i)$ and $\widetilde{h}_{i+1}(\widetilde{\tau}_{i+1})$ are both depicted at the leaf $D_{e_{H_{f_\beta}}}$ of level zero. Observe that *at the leaf of level zero*, except for the real edges in $\widetilde{f}_\beta(\widetilde{\tau}_i)$ depicted in bold, all real edges of $\widetilde{f}_\beta(\widetilde{\tau}_i)$ are labeled by $\widetilde{\varepsilon}_i$ with letters in $\mathbb{A}$. Labels given by $\widetilde{\varepsilon}_i$ to the edges in bold are of the form $t_i^{-1}x$, where x ranges over labels in $\mathbb{A}$ reserved for real edges in $N(T_i) \cup f$ and $t(X) = t_i \in H_{f_\beta}$. Here, $X \in \mathbb{A}_{\text{prong}}$ is the label (given by ε_i) of the unique infinitesimal edge of τ_i enclosing a puncture and incident to v_{end}. Now the projections to the base $\mathbf{D}_n$ of the real edges depicted in bold in Figure 8 are precisely the edges in $f_\beta(\tau_i)$ contained in $N(T_i) \cup f$.

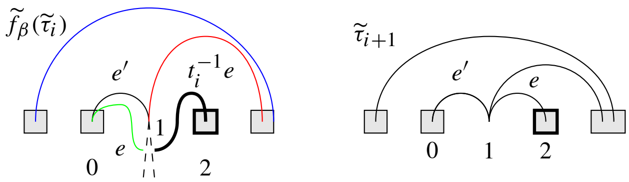


Fig. 8. Lifting an edge of the automaton in Case A.1.1.

From this data, a straightforward calculation yields $M(\widetilde{T}_i) = M(T_i)\,\mathrm{Diag}(t(\eta_i(v_i)))$, where all entries in the diagonal matrix $\mathrm{Diag}(t(\eta_i(v_i)))$ different from 1 are equal to t_i. The case when T_i represents the homeomorphism f_β that standardizes the train track $F_{ee'}(\tau_i)$ that arises when folding e' over e is treated in the same way.

Case B.1. This case is very similar to the preceding one. Consider the basic type Γ given by Figure 9. We consider all possible types of train tracks arising from Γ, which are displayed in the same figure. Among these, consider the edge of the automaton (5.9) where $(\tau_i, h_i, \varepsilon_i)$ is the type B.1.1 depicted in Figure 9 and the train track map T_i represents the homeomorphism f_β that standardizes the train track $F_{ee'}(\tau_i)$ that arises when folding edge e over edge e'.

In Figure 10 we depict this edge of the automaton in detail. Note that $v_{\text{fix}} = v_2$, $v_{\text{end}} = v_1$, $f = e$, $f' = e'$ and the subgraph $N(T_i)$ is highlighted in bold. We remark that $f \in N(T_i)$, hence we are in Case 2 of Definition 5.1. This is the main difference with Case A.1.

Now consider the lift $\widetilde{\mathbf{D}}_n \xrightarrow{\widetilde{f}_\beta} \widetilde{\mathbf{D}}_n$ which fixes the point in the fiber over v_{fix} contained in D_{e_H}. To compute $M(\widetilde{T}_i)$ we present Figure 11 where $\widetilde{f}_\beta(\widetilde{\tau}_i)$ and $\widetilde{h}_{i+1}(\widetilde{\tau}_{i+1})$ are both depicted at the leaf of level zero. Note that *at the leaf of level zero*, except for the real edges

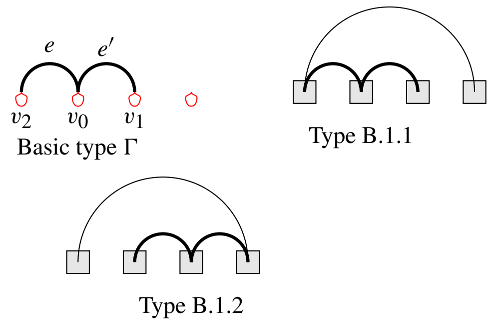


Fig. 9. Types B.1.1 and B.1.2 arising from simple type B.1.

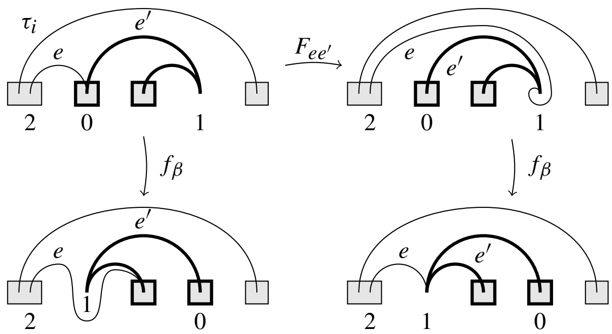


Fig. 10. A detailed edge of the automaton in Case B.1.1.

in $\widetilde{f}_\beta(\widetilde{\tau}_i)$ depicted in bold, all real edges in $\widetilde{f}_\beta(\widetilde{\tau}_i)$ are labeled by $\widetilde{\varepsilon}_i$ with letters in $\mathbb{A}$. Labels given by $\widetilde{\varepsilon}_i$ to the edges in bold are of the form $t_i^{-1}x$, where x ranges over labels in $\mathbb{A}$ reserved for real edges in $N(T_i) \cup f$ and $t(X) = t_i$. Here, $X \in \mathbb{A}_{\text{prong}}$ is the label (given by ε_i) of the unique infinitesimal edge of τ_i enclosing a puncture and incident to v_{end}. Now the projections of the real edges depicted in bold in Figure 11 to the base $\mathbf{D}_n$ are precisely the edges in $f_\beta(\tau_i)$ contained in $N(T_i)$. From this data, a straightforward calculation shows that $M(\widetilde{T}_i) = M(T_i)\,\mathrm{Diag}(t(\eta_i(v_i)))$, where all entries in $\mathrm{Diag}(t(\eta_i(v_i)))$ different from 1 are equal to t_i. The case when T_i represents the homeomorphism f_β that standardizes the train track $F_{ee'}(\tau_i)$ that arises when folding e' over e is treated in the same way.

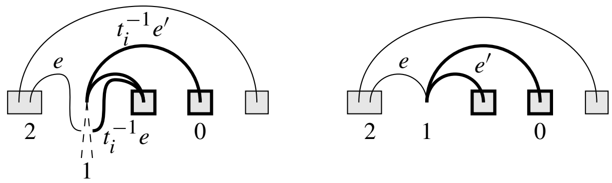


Fig. 11. Lifting an edge of the automaton in Case B.1.1.

The remaining cases are treated as follows:

(1) Pick a basic type Γ from the list presented in Appendix C and consider $(\tau_i, h_i, \varepsilon_i)$, one of the finitely many possible types of train tracks that can be constructed from Γ.
(2) Consider an edge (5.9) of the folding automaton starting from the train track chosen in the preceding step and where T_i comes from either the folding $F_{ee'}$ or $F_{e'e}$,[2] and represents a standardizing homeomorphism $\mathbf{D}_n \xrightarrow{f_\beta} \mathbf{D}_n$.
(3) Pick the lift $\widetilde{\mathbf{D}}_n \xrightarrow{\widetilde{f}_\beta} \widetilde{\mathbf{D}}_n$ which fixes the point in the fiber over v_{fix} contained in D_{e_H} and the corresponding train track map of the form (5.10) representing it. Then, depending on the case of Definition 5.1, perform analogous calculations to the ones presented in Cases A.1 and B.1.

To finish the proof consider any edge (5.9) in the decorated folding automaton with extra information (π_i, v_i). If e is an infinitesimal edge in τ_{i+1}, we define $\varepsilon_{i+1}(e) := \pi_i(\varepsilon_i(e))$. Hence $w_{i+1} = \pi_i \circ \cdots \circ \pi_1(v_{i+1})$. This implies the recursive nature of (5.8). $\square$

Remark 5.5. Let t be the variable of H_{f_β} and let u correspond to the lift $\widetilde{\psi}$. Then

$$\Theta_F(t, u) \in \mathbb{Z}[G] = \mathbb{Z}[t] \oplus \mathbb{Z}[u].$$

From Remark 3.1 we conclude that picking different lifts of the standardizing homeomorphism f_β results in multiplying $M(\widetilde{T})$ by some $t_0 \in H_{f_\beta}$. This does not affect the expression for the Teichmüller polynomial since $\Theta_F(t, u) = \det(u \cdot \mathrm{Id} - t_0 M(\widetilde{T})) = t_0^b \det(t_0^{-1} u \cdot \mathrm{Id} - M(\widetilde{T})) = t_0^{n-1} \Theta_F(u', t)$, where $u' = t_0^{-1} u$ is the coordinate corresponding to the other lift.

Remark 5.6. Observe that we can formally apply the above theorem to any isotopy class (not necessarily pseudo-Anosov). See the example below.

Example 5.7. Consider the class $[f_\beta]$ where $\beta = \sigma_2^2$ is a braid in B_3 (see Figure 4). The corresponding loop is

$$(\tau, \varepsilon) \xrightarrow{T_1} (\tau, \varepsilon) \xrightarrow{T_2} (\tau, \varepsilon).$$

In this example the map $t : \{A, B, C\} \to H$ is given by $t(\alpha) = t_\alpha$ for all $\alpha \in \mathbb{A}_{\text{prong}} = \{A, B, C\}$ since the permutation on the punctures defined by f_β is the identity. In other words, H is isomorphic to $\mathbb{Z}^3$. A direct calculation shows that $w_1 = \eta_1(1, C^+) = (1, C^+)$ since $\eta_1 = \mathrm{Id}_{|\mathbb{A}_{\text{prong}}}$. Hence $t(w_1) = (1, t_C)$. On the other hand, $w_2 = \eta_2(1, C^+) = (1, B^+)$ since $\eta_2(A, B, C) = (A, C, B)$. Hence $t(w_2) = (1, t_B)$. A direct calculation also shows that the matrix associated to T_i is $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ for $i = 1, 2$. Theorem 5.4 gives

$$M(\sigma_2^2) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & t_C \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & t_B \end{pmatrix} = \begin{pmatrix} 1 & t_B + t_B t_C \\ 0 & t_B t_C \end{pmatrix}.$$

Similarly for $\beta = \sigma_1^{-2}$ we have $t(\alpha) = t_\alpha$ and H isomorphic to $\mathbb{Z}^3$. A direct calculation shows that $w_1 = \eta_1(A^{-1}, 1) = (A^{-1}, 1)$. Hence $t(w_1) = (t_A^{-1}, 1)$. On the other hand,

[2] For some simple types only one of these two foldings is possible.

$w_2 = \eta_2(A^{-1}, 1) = (B^{-1}, 1)$ with $\eta_2(A, B, C) = (B, A, C)$. Hence $t(w_2) = (t_B^{-1}, 1)$. A direct calculation also shows that the matrix associated to T_i is $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ for $i = 1, 2$. Theorem 5.4 then gives

$$M(\sigma_1^{-2}) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} t_A^{-1} & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} t_B^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t_A^{-1} t_B^{-1} & 0 \\ t_A^{-1} t_B^{-1} + t_B^{-1} & 1 \end{pmatrix}.$$

Remark 5.8. Compare with the formula in [McM00, §11] (here the presentation of the group H is different):

$$M(\sigma_2^{-2}) = \begin{pmatrix} 1 & 0 \\ t_2^{-1} + t_2^{-1} t_3^{-1} & t_2^{-1} t_3^{-1} \end{pmatrix} \quad \text{and} \quad M(\sigma_1^2) = \begin{pmatrix} t_1 t_2 & t_1 t_2 + t_2 \\ 0 & 1 \end{pmatrix}.$$

For instance our algorithm applied to $\sigma_1^{-2}\sigma_2^6$ given by the loop

$$\left((\tau, \varepsilon) \xrightarrow{T_1} (\tau, \varepsilon)\right)^3 \xrightarrow{T_2} (\tau, \varepsilon)$$

(where T_1 represents σ_2^2 and T_2 represents σ_1^{-2}) shows that the Teichmüller polynomial is

$$\Theta_F(t_A, t_B, t_C, u) = \det\left(u \cdot \mathrm{Id} - \begin{pmatrix} 1 & t_B + t_B t_C \\ 0 & t_B t_C \end{pmatrix}^3 \cdot \begin{pmatrix} t_A^{-1} t_B^{-1} & 0 \\ t_A^{-1} t_B^{-1} + t_B^{-1} & 1 \end{pmatrix}\right).$$

5.4. Implementation of the algorithm

We end this section with a step-by-step description of the algorithm according to the proof of Theorem 5. The following is a pseudocode for a computer program to generate the decorated folding automata. This paper will not cover the details of its implementation, and a list of examples will be the subject of a forthcoming paper.

Input: Number of punctures $n > 2$, an alphabet $\mathbb{A}$ and a labeled train track $(\tau_0, h_0, \varepsilon_0)$ on $S = \mathbf{D}_n$ where (τ_0, h_0) is standard, and τ_0 has prescribed singularity data and labeling $\varepsilon_0 : E(\tau_0) \to \mathbb{A}$.

Output: Decorated folding automaton $\mathcal{N}^{\mathrm{aug}}(\tau_0, h_0)$.

Step 1. Initialization: Set $(\tau, h, \varepsilon) = (\tau_0, h_0, \varepsilon_0)$. The vertices of the folding automata $\mathcal{N}^{\mathrm{lab}}(\tau_0, h_0)$ are train tracks and the set of elementary folding maps forms the arrows. Recall that we consider labeling ε restricted to the set of *real* edges.

Step 2. Given (τ, h, ε) compute all the elementary folding operations followed by standardness operations. This gives edges from (τ, h, ε) to a finite collection of labeled train tracks $(\tau_i, h_i, \varepsilon_i)$ as explained in Section 4.1. Repeat Step 2 until $\mathcal{N}^{\mathrm{lab}}(\tau_0, h_0)$ is invariant under the folding operation.

Step 3. The resulting directed graph $\mathcal{N}^{\mathrm{lab}}(\tau_0, h_0)$ is connected by construction. To complete the construction of the folding automaton, for each arrow, compute the associated transition matrix, following Section 4.2.

Step 4. Choose a multiplicative basis $\mathbf{t} = (t_1, \dots, t_n)$ of $H = \mathrm{Hom}(H^1(S, \mathbb{Z}), \mathbb{Z})$. For each arrow of $\mathcal{N}^{\mathrm{lab}}(\tau_0, h_0)$, compute the associated vector $v \in \mathbb{Z}[H]^{\mathbb{A}_{\mathrm{real}}}$ following Definition 5.1 and the permutation $\pi \in \mathrm{Sym}(\mathbb{A}_{\mathrm{prong}})$ defined in Definition 3.12. This set of pairs (π, v) gives the decorated folding automaton $\mathcal{N}^{\mathrm{aug}}(\tau, h)$.

6. Topology of a fiber

In this section we provide a way to compute the topology (genus, number and type of singularities) of a fiber. We begin by introducing some notation that will be used throughout. As usual we will consider a mapping torus $M_\psi = \mathbf{D}_n \times [0, 1]/(x, 1) \sim (\psi(x), 0)$ induced by a pseudo-Anosov braid $\beta \in B_n$. It turns out that there is a natural model for M_ψ as a link complement $S^3 \setminus \mathcal{N}(L)$ where $\mathcal{N}(L)$ is a regular neighborhood of a link L in the 3-sphere. To construct the link $L = L_\beta$, simply close the braid β representing ψ while passing it through an unknot α (representing the boundary of the disc $\mathbf{D}_n$). Let $\Sigma \to M \to S^1$ be a fibration with monodromy $\varphi : \Sigma \to \Sigma$. Recall that if Σ has genus g and b boundary components, then $\chi_-(\Sigma) = 2g+b-2$. Hence the Thurston norm does not completely determine the topology of Σ. To achieve this we have to determine one of the numbers g or b (the surface Σ is orientable).

6.1. Computing the number of boundary components

Since M_ψ is homeomorphic to the link complement $S^3 \setminus \mathcal{N}(L)$, we can easily describe its homology group. First there is an embedding $i : \mathbf{D}_n \hookrightarrow M$ such that the image of the exterior boundary of $\mathbf{D}_n$ spans α and $i(\mathbf{D}_n)$ is punctured by the n strands of β. The boundary of M is a union of tori $T_1, \dots, T_r$, where $r = b_1(M)$ (viewed as the boundary of a regular neighborhood of the link components $\partial\mathcal{N}(L_i)$). Let $[S_1], \dots, [S_r]$ be a basis of $H_2(M, \partial M; \mathbb{R})$ (e.g. take Seifert surfaces whose boundary is T_i). By convention we normalize so that $S_r = i(\mathbf{D}_n)$. This normalization implies that T_r comes from the unknot α. The meridians of the components of L_β give a natural basis for $H_1(M, \mathbb{Z})$ [Hil12].

Now let $\{[m_i], [l_i]\}$ be a meridian and longitude basis for $H_1(T_i, \mathbb{R})$, where the orientation of $l_i \subset \partial S_i$ is induced by the orientation of $[S_i]$.We consider the long exact sequence of the homology groups of the pair $(M, \partial M)$. We write the boundary map

$$\partial_* : H_2(M, \partial M; \mathbb{R}) \to H_1(\partial M, \mathbb{R}).$$

On the chosen basis, for any $i = 1, \dots, r$, one has

$$\partial_*[S_i] = \sum_{j=1}^{r} (a_{ij}[m_j] + b_{ij}[l_j])$$

with $a_{ij}, b_{ij} \in \mathbb{Z}$. We set $A = (a_{ij})_{i,j=1}^r$ and $B = (b_{ij})_{i,j=1}^r$.

Proposition 6.1. *Let $\kappa = \sum_{i=1}^r c_i[S_i]$, where $c = (c_1, \dots, c_r) \in \mathbb{Z}^r$, be an integral homology class (not necessarily primitive). Then for any embedded surface $S \subset M_\psi$ (not necessarily minimal representative) such that $[S] = \kappa$, and for any $j = 1, \dots, r$, the number of connected components of $S \cap T_j$ is $\gcd(a_j, b_j)$ where $(a_1, \dots, a_r) = cA$ and $(b_1, \dots, b_r) = cB$.*

Proof. Write $[S] = \sum_{i=1}^r c_i[S_i] \in H_2(M, \partial M; \mathbb{R})$. Elementary linear algebra gives

$$\partial_*[S] = \sum_{i=1}^r \sum_{j=1}^r (c_i a_{ij}[m_j] + c_i b_{ij}[l_j]).$$

Now $S \cap T_j \subset \partial S$ is a union of oriented parallel simple closed curves. Hence its homology class is given by

$$\Big(\sum_{i=1}^r c_i a_{ij}\Big)[m_j] + \Big(\sum_{i=1}^r c_i b_{ij}\Big)[l_j] \in H_1(T_j, \mathbb{R}).$$

Thus the number of connected components of $S \cap T_j$ is

$$\gcd\Big(\sum_{i=1}^r c_i a_{ij}, \sum_{i=1}^r c_i b_{ij}\Big).$$

The proposition is proved. □

Remark 6.2. In our situation, since S_i is a Seifert surface whose boundary is the torus T_i, one has

$$\partial_*[S_i] = [l_i] - \sum_{j=1}^r \mathrm{Lk}(L_i, L_j)[m_j],$$

where $\mathrm{Lk}(L_i, L_j)$ is the linking number of the two closed curves L_i and L_j with orientations given by the orientations of $[l_i]$ and $[l_j]$. In other words, $B = \mathrm{Id}$ and $A = (\mathrm{Lk}(L_i, L_j)))_{i,j=1}^r$.

We end this section with the following corollary on the connected components of $\Sigma \cap T_i$.

Corollary 6.3. *For any $[\Sigma] = \sum_{i=1}^r c_i[S_i] \in H_2(M, \partial M; \mathbb{R})$ where $c = (c_1, \dots, c_r) \in \mathbb{Z}^r$, let $a = cA$ and $b = cB$ as above. Then each connected component of $\Sigma \cap T_j$ is identified to a curve (well defined up to isotopy)*

$$c_{p/q} = p[m_j] + q[l_j] \in H_1(T_j, \mathbb{R}) \quad \textit{with} \quad p = \frac{a_j}{\gcd(a_j, b_j)},\ q = \frac{b_j}{\gcd(a_j, b_j)}.$$

From the last corollary we make the following definition. If T is a torus, and if $H_1(T, \mathbb{Z})$ is equipped with its preferred basis given by meridian and longitude (denoted by $[m]$ and $[l]$), then the *slope* of an essential simple closed curve $[c] = p[m] + q[l]$ (with $\gcd(p, q) = 1$) is p/q. Conversely, for any $r \in \mathbb{Q} \cup \{\infty\}$ one defines the (isotopy class) c_r of the corresponding simple closed curve.

6.2. Computing the number and type of singularities of the fiber

Let $\Sigma \to M_\psi \to S^1$ be a fibration in $\mathbb{R}^+ \cdot F$ with pseudo-Anosov monodromy ϕ. In this section we explain how to compute the singularity data of the stable measured foliation of Σ that is invariant under ϕ using the singularity data of the stable measured foliation of ψ. The arguments are based on work of Fried.

In the following we denote by $\mathcal{F}$ the stable measured foliation invariant under ψ. Up to isotopy one can assume that $\psi(\mathcal{F}) = \mathcal{F}$. This determines a 2-dimensional lamination $\mathcal{L}_\psi = \mathcal{F} \times \mathbb{R}/\langle (s,t) \sim (\psi(s), t-1)\rangle$ obtained as the mapping torus of $\psi : \mathcal{F} \to \mathcal{F}$. The "vertical flow lines" $\{s\} \times \mathbb{R} \subset \Sigma \times \mathbb{R}$ descend to the leaves of a 1-dimensional foliation whose leaves will be called the *flow lines* of ψ. Hence $\mathcal{L}_\psi$ is swept out by the leaves of the flow lines passing through $\mathcal{F}$.

We distinguish two cases: $\mathcal{F}$ has no singularities in the interior of $\mathbf{D}_n$ (see Proposition 6.4) or $\mathcal{F}$ has some singularities in the interior of $\mathbf{D}_n$ (see Proposition 6.6). Any singularity s of $\mathcal{F}$ in the boundary of $\mathbf{D}_n$ determines a closed curve γ_s of slope p_s/q_s on the torus $T_s \subset \partial M$, given by the flow line passing through s. See for example [KT13, Figures 4 and 5].

Proposition 6.4 ($\mathcal{F}$ has no singularities in the interior of $\mathbf{D}_n$). *For any* $[\Sigma] \in \mathbb{R}^+ \cdot F \subset H_2(M, \partial M; \mathbb{R})$ *with monodromy* $\phi : \Sigma \to \Sigma$*, the singularity data of the stable foliation* $\mathcal{F}_\phi$ *of the pseudo-Anosov map* ϕ *is given by:*

(1) *$\mathcal{F}_\phi$ has no singularities in the interior of Σ.*
(2) *At each connected component of $\partial\Sigma \cap T_s$ (of slope p/q given by Corollary 6.3), there is a $k|p_s q - q_s p|$-prong singularity of $\mathcal{F}_\phi$ if s is a k-prong singularity.*

Remark 6.5. In all examples that we will be treating in this article, every singularity of $\mathcal{F}$ in the boundary of $\mathbf{D}_n$ that does not intersect T_r is a 1-prong. Except for the example in B_4 treated in Section 8, $\mathcal{F}_\phi$ has no singularities in the interior of Σ and for each $1 \le i < r$, if we write $\partial\Sigma \cap T_i = a[m_i] + b[l_i] \in H_1(T_i, \mathbb{R})$, $\mathcal{F}_\phi$ has $\gcd(a, b)$ singularities in T_i, each of which is a $|p_i q - q_i p|$-prong, where $p = a/\gcd(a, b)$, $q = b/\gcd(a, b)$ and p_i/q_i is the slope of the curve $\gamma_i \subset T_i$. The type of the remaining singularity can be determined using the Euler–Poincaré formula.

Proof of Proposition 6.4. Let $[\Sigma] \in \mathbb{R} \cdot F \subset H_2(M_\psi, \partial M_\psi; \mathbb{R})$ be a fiber of M with monodromy $\phi : \Sigma \to \Sigma$. We will use the following result of Fried [Fri82, McM00]. After an isotopy:

(1) The fiber Σ is transverse to the flow lines of ψ.
(2) The monodromy of the fibration determined by $[\Sigma]$ coincides with the first return map of the foliation $\mathcal{F}$.

Hence the monodromies of any two points in $\mathbb{R} \cdot F \subset H_2(M_\psi, \partial M_\psi; \mathbb{R})$ determine, up to isotopy, the same lamination $\mathcal{L}_\psi$. Let $\tau \hookrightarrow \mathbf{D}_n$ be a train track invariant under our given pseudo-Anosov homeomorphism ψ. Up to isotopy we assume that $\psi(\tau)$ is contained a fibered neighborhood of τ and transverse to the tie foliation. We assume that τ carries the measured foliation $\mathcal{F}$. Let $\mathcal{L}_\tau$ be the mapping torus of τ, that is, $\mathcal{L}_\tau = \tau \times [0, 1]/\langle (x, 1) \sim (\psi(x), 0)\rangle$.

The aforementioned result of Fried implies that the intersection $\mathcal{F}_\phi = \Sigma \cap \mathcal{L}_\psi$ defines an invariant measured foliation for ϕ and $\mathcal{F}_\phi$ is carried by the train track $\tau_\phi = \Sigma \cap \mathcal{L}_\tau$. By construction $\mathcal{L}_\psi$ is carried by the branched surface $\mathcal{L}_\tau$. For any singularity s of $\mathcal{F}$, one obtains a simple closed curve $\gamma_s \subset M$ which is the closed orbit of the flow line passing through s. Notice that the union of all γ_s is the branched locus of $\mathcal{L}_\tau$. Since $\mathcal{F}$ has no singularities in the interior of $\mathbf{D}_n$, all curves γ_s lie in T_i for some i. Hence all the singularities of $\mathcal{F}_\phi$ lie in $\partial\Sigma \cap T_i$, which proves the first point of the proposition.

Now we determine the number and type of prongs of τ_ϕ. For that we consider the number of prongs of $\mathcal{F}_\phi$ at each component of $\partial\Sigma \cap T_j$ for each j (clearly for a given j the type of the singularity at each component is the same). Let $c_{[p/q]} = p[m_j] + q[l_j] \in H_1(T_j, \mathbb{R})$ be the corresponding curve representing a connected component of $\partial\Sigma \cap T_j$ (see Corollary 6.3). By the aforementioned result of Fried, each intersection between $c_{[p/q]}$ and γ_s contributes to k-infinitesimal edges (if s is a k-prong singularity). Hence the total number of prongs of $\mathcal{F}_\phi$ at $\partial\Sigma \cap T_j$ is equal to

$$k \cdot i(c_{[p/q]}, \gamma_s).$$

Since the slope of γ_s is p_s/q_s, it follows that

$$i(c_{[p/q]}, \gamma_s) = |p_s q - q_s p|.$$

This ends the proof of Proposition 6.4. □

We now address the case when $\mathcal{F}$ has singularities in the interior of $\mathbf{D}_n$. Roughly speaking, the idea is to remove the interior singularities in order to be in the preceding case.

Note that in the definition of a pseudo-Anosov homeomorphism we can remove or add punctures while keeping the "same" map $\psi : S \to S$. More precisely when $\{\psi^i(x)\}$ is a periodic orbit of *unpunctured points*, puncturing at $\{\psi^i(x)\}$ means adding them to the puncture set $\{p_i\}$. Conversely, when $\{\psi^i(p)\}$ is a periodic orbit of k-prong punctured singularities for $k > 1$, capping them off means removing them from the puncture set. For pseudo-Anosov braids, puncturing or capping off corresponds to adding or removing some strands.

Proposition 6.6 ($\mathcal{F}$ has singularities in the interior of $\mathbf{D}_n$). *Puncturing at $\{\psi^i(s)\}$ for any singularity s of $\mathcal{F}$ in the interior of $\mathbf{D}_n$ gives rise to a pseudo-Anosov $\widetilde{\psi} : \mathbf{D}_m \to \mathbf{D}_m$ where $m > n$. By construction $\mathcal{F}_{\widetilde{\psi}}$ has no interior singularities. Moreover the injection $\mathbf{D}_n \to \mathbf{D}_m$ induces a map $M_\psi \to M_{\widetilde{\psi}} =: \widetilde{M}$ and each class $[\Sigma] \in \mathbb{R}^+ \cdot F \subset H_2(M, \partial M; \mathbb{R})$ (with monodromy ϕ) determines a class $[\widetilde{\Sigma}] \in \mathbb{R}^+ \cdot F \subset H_2(\widetilde{M}, \partial\widetilde{M}; \mathbb{R})$ with monodromy $\widetilde{\phi}$. The map ϕ is obtained by capping the singularities of $\widetilde{\phi}$ that lie in the interior of $\mathbf{D}_n$. In particular $\mathcal{F}_\phi$ and $\mathcal{F}_{\widetilde{\phi}}$ share the same number and type of singularities.*

Proof. This is clear from the definition of capping off and puncturing at singularities. □

6.3. Orientability of a singular foliation

In this section we determine whether or not the measured foliation $\mathcal{F}_\phi$ is orientable. For that we will use the following well known theorem of Thurston:

Theorem. *For any pseudo-Anosov homeomorphism ϕ on a surface Σ the following are equivalent:*

(1) *The stretch factor of ϕ is an eigenvalue of the linear map ϕ_* defined on $H_1(\Sigma, \mathbb{Z})$.*
(2) *The invariant measured foliation $\mathcal{F}_\phi$ of ϕ is orientable.*

To compute the homological dilatation we will make use of the Alexander polynomial. Just as the Teichmüller polynomial, the *Alexander polynomial* of M,

$$\Delta_M = \sum_{g \in G} b_g \cdot g,$$

is an element of the group ring $\mathbb{Z}[G]$, where $G = H_1(M, \mathbb{Z})/\mathrm{Tor}$. For a precise definition see [McM02, §2]. The Alexander polynomial can be evaluated on a homology class $[\Sigma] \in H_2(M, \partial M; \mathbb{R})$ using Poincaré–Lefschetz duality and then proceeding as with the Teichmüller polynomial in the corresponding dual cohomology class. We have the following classical result (see e.g. [Mil68]):

Theorem 6.7. *Let $[\alpha] \in \mathbb{R}^+ \cdot F \subset H^1(M, \mathbb{R})$ with monodromy $\phi : \Sigma \to \Sigma$. Then the characteristic polynomial of ϕ_* acting on $H_1(\Sigma, \mathbb{Z})$ is given by the Alexander polynomial Δ_M evaluated on $[\alpha]$.*

7. First examples in B_3

The goal of this section is to revisit classical examples, first studied by Hironaka [HK06], [Hir10], Kin–Takasawa [KT13] and McMullen [McM00]. We stress that the novelty in this section is the methods presented to perform calculations, and not the results of these. In the next sections we will address examples in B_n for $n \geq 4$.

Convention. The natural order on $\mathbb{R}$ induces an order on the punctures of $\mathbf{D}_n$ and thus a labeling. We label the infinitesimal edges enclosing punctures in $\mathbf{D}_3$, from left to right, by A, B, C so that $\mathbb{A}_{\text{prong}} = \{A, B, C\}$. Hence the standard generators σ_1, σ_2 (induced by left Dehn half-twists around loops enclosing the punctures) define the permutations $(A, B, C) \mapsto (B, A, C)$ and $(A, B, C) \mapsto (A, C, B)$ respectively.

7.1. The simplest pseudo-Anosov braid

We first consider the homeomorphism $\psi = f_{\sigma_1^{-1}\sigma_2}$ and treat this example in detail.

7.1.1. Invariant train track. It is well known that the isotopy class of ψ is pseudo-Anosov. Indeed, ψ leaves invariant the train track τ_0 presented in Figure 12. The map $f_{\sigma_1^{-1}\sigma_2}$ is then represented by the train track map $T : \tau_0 \to \tau_0$ defined by $a \mapsto aab$ and $b \mapsto ab$. The incidence matrix, $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, is irreducible.

We now quickly review how one can find a sequence of foldings discussed in the previous sections. For this purpose, consider the folding automaton and the two folding maps F_{ba}, F_{ab} corresponding to the two standardizing homeomorphisms $f_{\sigma_1^{-1}}$ and f_{σ_2} depicted in Figure 13.

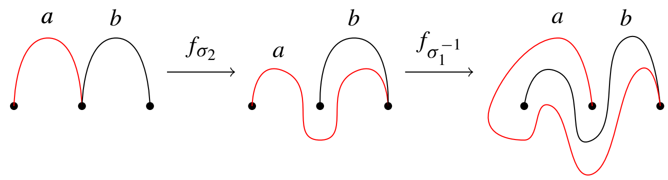


Fig. 12. An invariant train track τ_0 for $f_{\sigma_1^{-1}\sigma_2}$.

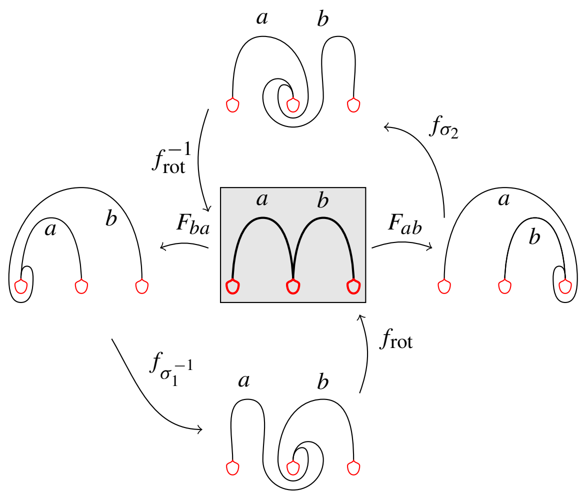


Fig. 13. The folding automaton for B_3. The map f_{rot} is an isotopy (rotation in the neighborhood of punctures). Observe that the folding F_{ba} induces a train track map T_{ab} that represents $f_{\sigma_1^{-1}}$. The same is true for F_{ba} with f_{σ_2}.

Remark 7.1. We will encode the folding automaton by representing the isotopy near the punctures by a permutation (see Definition 3.12 and Example 3.13). This defines a simpler automaton: see Figure 14.

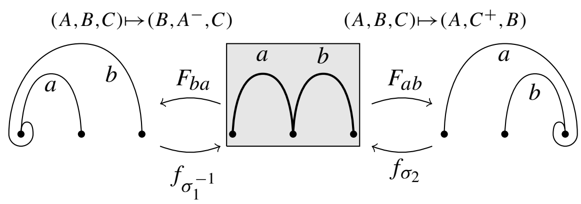


Fig. 14. The folding automaton for B_3. Note that the graphs $\mathcal{N}^{\mathrm{lab}}(\tau, h, \varepsilon)$ and $\mathcal{N}(\tau, h)$ coincide: they each have only one vertex and two edges represented by the two folding maps F_{ab} and F_{ba}. For each edge we have represented the action of the standardizing braids on the punctures.

The two foldings F_{ba} and F_{ab} define two train track maps T_{ba} and T_{ab} (representing the two homeomorphisms $f_{\sigma_1^{-1}}$ and f_{σ_2}, respectively):

$$T_{ba} : \tau_0 \to \tau_0, \quad a \mapsto a, \quad b \mapsto ba,$$
$$T_{ab} : \tau_0 \to \tau_0, \quad a \mapsto ab, \quad b \mapsto a,$$

whose incidence matrices are $M_{ba} := M(T_{ba}) = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ and $M_{ab} := M(T_{ab}) = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. To be more precise, one sees that in this very particular example τ_0 is invariant under both f_{σ_2} and $f_{\sigma_1^{-1}}$ and the associated train track maps are given by T_{ab} and T_{ba}. Hence the path in the automaton representing $f_{\sigma_1^{-1}\sigma_2} = f_{\sigma_1^{-1}} \circ f_{\sigma_2}$ has train track map given by $T_{ba} \circ T_{ab}$. Therefore the incidence matrix is

$$M(T_{ba} \circ T_{ab}) = M_{ab} \cdot M_{ba} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Observe that in this case the relabeling map involved is the identity map. In the above situation, the matrices belong to $\mathrm{GL}(\mathbb{Z}^{\mathbb{A}_{\mathrm{real}}}) = \mathrm{GL}(\mathbb{Z}^{\{a,b\}})$.

7.1.2. The Teichmüller polynomial. We now compute the Teichmüller polynomial of the fibered face containing the fibration defined by the suspension of $f_{\sigma_1^{-1}\sigma_2}$. Recall that $\mathbf{D}_3$ is the complement of three round discs D_A, D_B and D_C lying along a diameter of the closed unit disc. The rank of the group $H_{f_{\sigma_1^{-1}\sigma_2}}$ is given by the number of cycles of the permutation induced by the action of $f_{\sigma_1^{-1}\sigma_2}$ on the boundary $\{[\partial D_\alpha]\}_{\alpha \in \mathbb{A}_{\mathrm{prong}}}$. Since the braid $\beta = \sigma_1^{-1}\sigma_2$ permutes the three strands cyclically, H is isomorphic to $\mathbb{Z}$. Therefore $\pi : \widetilde{\mathbf{D}}_3 \to \mathbf{D}_3$ is a $\mathbb{Z}$-covering. The infinite surface $\widetilde{\mathbf{D}}_3$ can be constructed by glueing $\mathbb{Z}$ copies of the simply connected domain obtained by cutting $\mathbf{D}_3$ along three disjoint segments going from D_α to the exterior boundary of $\mathbf{D}_3$. These are called the leaves of the covering $\pi : \widetilde{\mathbf{D}}_3 \to \mathbf{D}_3$ (see §4.3). For our computations, we fix a labeling by $t \in \mathbb{Z}$ of the set of leaves forming $\widetilde{\mathbf{D}}_3$ that is coherent with the action of $\mathrm{Deck}(\pi)$. This labeling induces a labeling for the edges and vertices of the infinite train track $(\widetilde{\tau}_0, \widetilde{h})$.

As noted before, the path in the automaton $\mathcal{N}(h, \tau_0)$ representing $f_{\sigma_1^{-1}\sigma_2}$ is $T_{ba} \circ T_{ab}$. In Figure 15 we depict the lift to $\widetilde{\mathbf{D}}_3$ of each factor in this path.

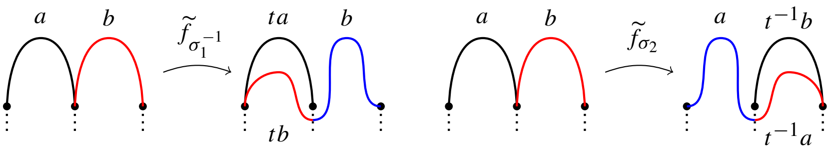


Fig. 15. The lift of homeomorphisms induced by folding operations.

The first train track map T_{ab} corresponds to the homeomorphism f_{σ_2}. We choose the lift $\widetilde{f}_{\sigma_2}$ of f_{σ_2} that fixes the vertex v_{fix}, which in Figure 15 is the vertex on the left. Equipped with this choice we get a train track map $\widetilde{T}_{ab} : \widetilde{\tau}_0 \to \widetilde{\tau}_0$ induced by $\widetilde{f}_{\sigma_2}(\widetilde{\tau}_0) \prec \widetilde{\tau}_0$. Similarly the train track map T_{ba} represents the homeomorphism $f_{\sigma_1^{-1}}$ and we choose the lift $\widetilde{f}_{\sigma_1^{-1}}$ of $f_{\sigma_1^{-1}}$ that fixes v_{fix}, which in Figure 15 is the vertex on the right. This lift is represented by the train track map $\widetilde{T}_{ba} : \widetilde{\tau}_0 \to \widetilde{\tau}_0$. As in Example 5.7, a direct calculation shows that

$$w_1 = \eta_1(1, C^+) = (1, C^+) \quad \text{since} \quad \eta_1(A, B, C) = (A, B, C),$$
$$w_2 = \eta_2(A^{-1}, 1) = (A^{-1}, 1) \quad \text{since} \quad \eta_2(A, B, C) = \pi_1 \circ \eta_1(A, B, C) = (A, C, B).$$

In this particular case, all punctures are permuted cyclically, hence $t(w_1) = (1, t)$ and $t(w_2) = (t^{-1}, 1)$. Theorem 5.4 then implies that the incidence matrix of the train track map $\widetilde{T}$ representing $\widetilde{f}_{\sigma_1^{-1}\sigma_2}$ is

$$M(\widetilde{T}) = M(\widetilde{T}_{ba} \circ \widetilde{T}_{ab}) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 + t^{-1} & t \\ 1 & t \end{pmatrix}.$$

Therefore the characteristic polynomial is

$$\Theta_F(t, u) = u^2 - (1 + t + t^{-1})u + 1.$$

Remark 7.2. From the preceding calculations it is easy to compute the Teichmüller polynomials associated to braids in B_3 that permute the strands cyclically (by considering products of $M(\widetilde{T}_{ba})$ and $M(\widetilde{T}_{ab})$. Compare with [McM00, §11].

7.1.3. Evaluating the Teichmüller polynomial of $\sigma_1^{-1}\sigma_2$. First, given a class $f_\beta \in \text{Mod}(\mathbf{D}_n)$ we explain how to assign coordinates on $H^1(M, \mathbb{Z})$ such that the cohomology class corresponding to the fibration defined f_β is $(0, \dots, 0, 1)$. Following Section 6.1, we choose an ordered basis $B = \{[m_1], \dots, [m_r]\}$ of $H_1(M, \mathbb{Z})$ formed by the meridians of the tori $T_1, \dots, T_r$ respectively. Since $H_1(M, \mathbb{Z})$ is torsion free, the base B defines a base $B^* = \{[s_1], \dots, [s_{r-1}], [y]\}$ for $H^1(M, \mathbb{Z}) \simeq \text{Hom}(H_1(M, \mathbb{Z}), \mathbb{Z})$ by duality. Here $s_i = m_i^*$ for $i = 1, \dots, r-1$ and $m_r = y$. Let $[S_r] = i(\mathbf{D}_n)$ denote the class of the fiber of the fibration defined by f_β. The intersection of $[S_r]$ with $[m_i]$ is given by δ_{rj}, and hence, using the Universal Coefficient Theorem and Poincaré duality, we deduce that the coordinates of $[S_r]^* \in H^1(M, \mathbb{Z}) \simeq H_2(M, \partial M; \mathbb{Z})$ for the basis B^* are precisely $(0, \dots, 0, 1)$. In the rest of the examples presented in this text we always choose the ordered basis B^*. Note that $\{[m_1], \dots, [m_{r-1}]\}$ generate the f_β-invariant homology of the $r-1$-punctured disc and $[m_r]$ corresponds to the natural lifting $\widetilde{f}_\beta$ of f_β, hence we can identify $\{[m_1], \dots, [m_{r-1}], [m_r]\}$ with the variables $\{t_1, \dots, t_{r-1}, u\}$ of the Teichmüller polynomial (see Section 3.1). In Figure 16 we depict the link complement defined by $\sigma_1^{-1}\sigma_2$.

We now determine the Thurston norm for $\beta = \sigma_1^{-1}\sigma_2$. We achieve this by computing first the Alexander norm of $M = M_{f_\beta}$. Direct computation shows that the Alexander polynomial of β is

$$\Delta_M(t, u) = u + u^{-1} - (-t^{-1} + 1 - t) \tag{7.11}$$

(well defined up to multiplication by a unit in $\mathbb{Z}[G]$; see [McM02]). The Newton polygon $N(\Delta_M)$ of this polynomial is the symmetric diamond forming the convex hull of the points $\{(0,\pm 1),(\pm 1,0),(0,0)\}$; its Newton polytope is the square of vertices $\{(\pm\frac{1}{2},\pm\frac{1}{2})\}$. By Remark B.3 the unit ball of the Alexander norm is the square of vertices $\{(\pm\frac{1}{2},\pm\frac{1}{2})\}$ (in $H^1(M,\mathbb{Z})$). Hence

$$\|(s,y)\|_A = \max(|2s|,|2y|) \quad \text{for all } (s,y)\in H^1(M,\mathbb{Z}).$$

By Theorems B.1–B.2 we conclude that the segment joining the points $(-\frac{1}{2},\frac{1}{2})$ and $(\frac{1}{2},\frac{1}{2})$ is the fibered face F of the Thurston norm ball whose cone $\mathbb{R}^+\cdot F$ contains the fibration defined by $\sigma_1^{-1}\sigma_2$. Hence

$$\|(s,y)\|_T = \max(|2s|,|2y|)$$

for all fibrations $(s,y)\in\mathbb{R}^+\cdot F\cap H^1(M,\mathbb{Z})$.

We finally explain how to evaluate Θ_F on a point $(s,y)\in H^1(M,\mathbb{Z})$. Let $f_s, f_y : H_1(M,\mathbb{Z})\to\mathbb{Z}$ be the duals of $[s]$ and $[y]$ respectively. Hence, the dual of a point $(s,y)\in H^1(M,\mathbb{Z})$ is given by $f_{(s,y)} := sf_s + yf_y$. Since by definition $f_{(s,y)}(u)=y$ and $f_{(s,y)}(t)=s$, one has

$$\begin{aligned}\Theta_F(s,y) &= X^{f_{(s,y)}(u^2)} - (1 + X^{f_{(s,y)}(t)} + X^{f_{(s,y)}(t^{-1})})X^{f_{(s,y)}(u)} + 1\\ &= X^{2y} - (1+X^s+X^{-s})X^y + 1.\end{aligned}$$

7.1.4. The topology of the fiber. Let Σ be the fiber of the fibration determined by the point $(s,y)\in\mathbb{R}^+\cdot F$, where F is the segment joining the points $(-\frac{1}{2},\frac{1}{2})$ and $(\frac{1}{2},\frac{1}{2})$. Since every fiber is (Thurston) norm minimizing in its homology class, we have

$$\|(s,y)\|_T = |y| = -\chi(\Sigma) = 2\,\mathrm{genus}(\Sigma) - 2 + \#\{\text{boundary components of } \Sigma\}. \tag{7.12}$$

We now calculate the number of boundary components of Σ as follows. We choose a basis $\{[S_1],[S_2]\}$ of $H_2(M,\partial M;\mathbb{Z})$ by taking two Seifert surfaces of the components of the 6^2_2 link shown in Figure 16. By Remark 6.2, we have

$$\partial_*[S_1] = l_1 - \mathrm{Lk}(L_1,L_2)m_2, \qquad \partial_*[S_2] = l_2 - \mathrm{Lk}(L_2,L_1)m_1.$$

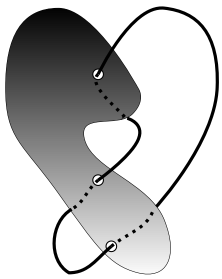


Fig. 16. The link 6^2_2 and the fiber of the fibration defined by $\sigma_1^{-1}\sigma_2$.

A straightforward computation shows that $|\mathrm{Lk}(L_1, L_2)| = |\mathrm{Lk}(L_2, L_1)| = 3$. Let $A = \left(\begin{smallmatrix} 0 & 3 \\ 3 & 0 \end{smallmatrix}\right)$. Proposition 6.1 implies that the number of connected components of $\partial\Sigma \cap T_j$ is $\gcd(a_j, b_j)$ where $a = (s, y)$, $A = (3y, 3s)$ and $b = (s, y)$. Therefore the total number of connected components of $\partial\Sigma$ is $\gcd(s, 3y) + \gcd(3s, y) = \gcd(3, s) + \gcd(3, y)$. Plugging this data into (7.12) we get

$$\mathrm{genus}(\Sigma) = |y| + 1 - \frac{\gcd(3, s) + \gcd(3, y)}{2}.$$

With the notation of Corollary 6.3 the slopes of the boundary components are $3/s$ and $3/y$.

7.1.5. The singularities of the fiber. We already observed that Σ has $\gcd(3, s)$ boundary components at T_1 and $\gcd(3, y)$ boundary components at T_2.

For any singularity s of $\mathcal{F}$ one needs to determine the slope of $\gamma_s \subset M$ where γ_s is the closed orbit of the flow line passing through s. Since $\mathcal{F}$ has no singularities in the interior of $\mathbf{D}_n$, all curves γ_s lie in T_i for some i. We label the prongs with the capital letters A, B, C. One sees that the braid β permutes the prongs (A, B, C) to (C, A, B). We denote the corresponding permutation $\pi(\beta)$. Since $\pi(\beta)$ has only one cycle, there is only one torus component (see Figure 17). Now when performing the pseudo-Anosov braid and isotopy, one needs to understand the rotation in the neighborhood of punctures (see Definition 3.12 and Example 3.12).

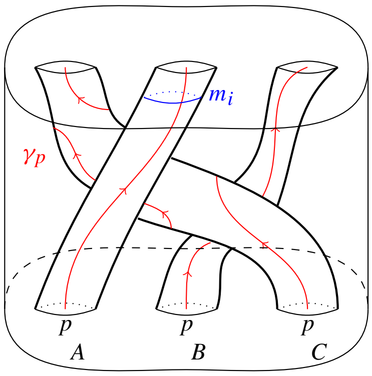


Fig. 17. Computing the slope of the curve γ_p.

As usual one can obtain the permutation $\pi(\beta)$ as follows. For each elementary step we have a permutation encoding how the isotopy (rotation) acts in the neighborhood of punctures. More precisely,

$$\pi(\sigma_2) : (A, B, C) \mapsto (A, C^+, B) \quad \text{and} \quad \pi(\sigma_1^{-1}) : (A, B, C) \mapsto (B, A^-, C).$$

Composition gives the desired slope:

$$\pi(\beta) = \pi(\sigma_1^{-1}\sigma_2) = \pi(\sigma_2) \circ \pi(\sigma_1^{-1}).$$

Hence $\pi(\beta) : (A, B, C) \mapsto (C^+, A^-, B)$ and so $\gamma = [l]$, i.e. its slope is $0/1$ (no Dehn twist around the meridian).

Concretely the slope of γ_p is $p/q = 0/1$. The slope for the other component T_2 is $1/0$. We apply Proposition 6.4 at each connected component of $\partial\Sigma \cap T_j$ as follows.

(1) For T_1 (coordinate t): One has $(a_1, b_1) = (3y, s)$. Thus at each of the $\gcd(3y, s)$ components there is a $3y/\gcd(3, s)$-prong singularity of $\mathcal{F}_\phi$.
(2) For T_2 (coordinate t): One has $(a_2, b_2) = (3s, y)$. Thus at each of the $\gcd(3s, y)$ components there is a $y/\gcd(3, y)$-prong singularity of $\mathcal{F}_\phi$.

7.1.6. Orientability of the singular foliation. To compute the homological dilatation we will use the Alexander polynomial $\Delta_M(t, u) = u^2 - u(-t^{-1} + 1 - t) + 1$ (up to a factor). By Theorem 6.7, the homological dilatation is the maximal root of Δ_M (in absolute value) evaluated on (s, y), that is, the maximal root (in absolute value) of the polynomial

$$Q(X) = X^{2y} - (1 - X^s - X^{-s})X^y + 1, \qquad y > s.$$

Recall that the stretch factor is the maximal root of

$$P(X) = X^{2y} - (1 + X^s + X^{-s})X^y + 1.$$

Since $Q(-X) = P(X)$ when s is odd and y is even, we infer that the invariant measured foliation is orientable if s is odd and y is even.

In the rest of this paper we will focus only on computing the Teichmüller polynomials, for the rest (Thurston norm, topology of fibers and type of singularities) can be found using the methods presented for the simplest pseudo-Anosov braid.

7.2. *The Teichmüller polynomial of* $\sigma_2\sigma_1^{-1}\sigma_2 \in B_3$

The link complement $M = \mathbf{S}^3 \setminus L(\beta)$ of the braid $\beta = \sigma_2\sigma_1^{-1}\sigma_2$ is homeomorphic to the *magic manifold* (see [KT13] for more details). This braid fixes one strand and permutes the other two, hence the $H_{\sigma_2\sigma_1^{-1}\sigma_2}$-covering $\widetilde{\mathbf{D}}_3$ is a $\mathbb{Z}^2$-covering. Denote by (t_A, t_B) the variables of the deck transformation group of $\pi : \widetilde{\mathbf{D}}_3 \to \mathbf{D}_3$ corresponding to the permuted and fixed strands, respectively. From the automaton of Figure 14 one sees that the path in the automaton $\mathcal{N}(h, \tau_0)$ representing $f_{\sigma_2\sigma_1^{-1}\sigma_2}$ is the composition of three folding maps. By Theorem 5.4,

$$\begin{aligned}
w_1 &= \eta_1(1, C^+) = (1, C^+) &&\text{since} \quad \eta_1(A, B, C) = (A, B, C),\\
w_2 &= \eta_2(A^{-1}, 1) = (A^{-1}, 1) &&\text{since} \quad \eta_2(A, B, C) = \pi_1 \circ \eta_1(A, B, C) = (A, C, B),\\
w_3 &= \eta_3(1, C^+) = (1, B^+) &&\text{since} \quad \eta_3(A, B, C) = \pi_2 \circ \pi_1(A, B, C) = (C, A, B).
\end{aligned}$$

Hence $t(w_1) = (1, t_A)$, $t(w_2) = (t_A^{-1}, 1)$ and $t(w_3) = (1, t_B)$, and the incidence matrix $M(\widetilde{T})$ of the train track map representing a lift of $f_{\sigma_2\sigma_1^{-1}\sigma_2}$ is

$$M(\widetilde{T}) = \begin{pmatrix} 1 & t_A \\ 0 & t_A \end{pmatrix} \begin{pmatrix} t_A^{-1} & 0 \\ t_A^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & t_B \\ 0 & t_B \end{pmatrix} = \begin{pmatrix} t_A^{-1} + 1 & t_A t_B + t_B + t_B t_A^{-1} \\ 1 & t_A t_B + t_B \end{pmatrix}.$$

Taking the characteristic polynomial we get

$$\Theta_F(t_A, t_B, u) = u^2 - (t_A t_B + t_B + 1 + t_A^{-1})u + t_B.$$

8. The Teichmüller polynomial of $\sigma_1^{-1}\sigma_2\sigma_3 \in B_4$

In this section we illustrate our algorithm, described in Section 5.4, on several examples defined on the punctured disc.

8.1. Invariant train track

The homeomorphism $f_{\sigma_1^{-1}\sigma_2\sigma_3}$ is a pseudo-Anosov homeomorphism: it leaves invariant the train track τ_1 (see Figure 18), and the train track map $T : \tau_1 \to \tau_1$ induced by $f_{\sigma_1^{-1}\sigma_2\sigma_3}(\tau_1) \prec \tau_1$ is given by

$$a \mapsto cbaa, \quad b \mapsto c, \quad c \mapsto d, \quad d \mapsto ba.$$

Its incidence matrix $M(T) = \begin{pmatrix} 2&1&1&0\\ 0&0&1&0\\ 0&0&0&1\\ 1&1&0&0 \end{pmatrix}$ is irreducible. In Figure 19 we depict part of the folding automaton $\mathcal{N}(\tau_1, h)$.

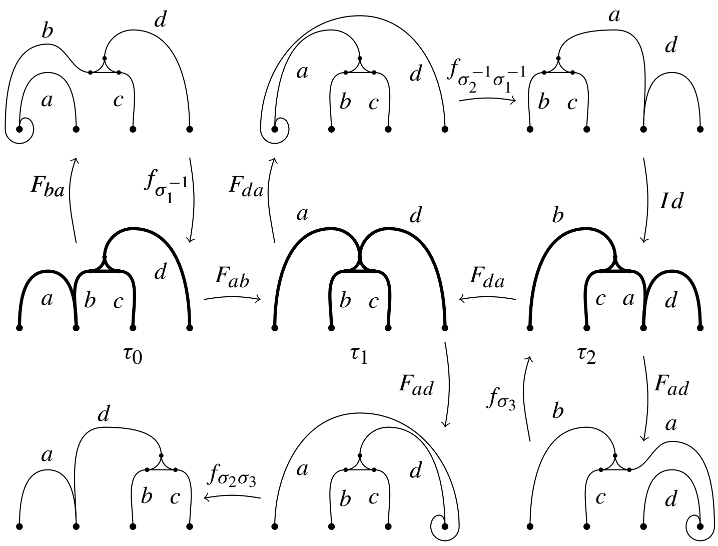


Fig. 18. Detail of foldings and standardizing braids in the automaton in B_4.

In this part we see three vertices (τ_i for $i = 1, 2, 3$, bolder train tracks). More precisely the (standard) folding F_{ab} induces a train track map $T_1 : \tau_1 \to \tau_2$ that represents $[\mathrm{Id}] \in \mathrm{Mod}(\mathbf{D}_4)$. On the other hand the folding F_{ad} induces a train track map $T_2 : \tau_2 \to \tau_1$ that represents $[f_{\sigma_2\sigma_3}] \in \mathrm{Mod}(\mathbf{D}_4)$. Finally, F_{ba} induces a train track map $T_3 : \tau_1 \to \tau_1$ that represents $[f_{\sigma_1^{-1}}] \in \mathrm{Mod}(\mathbf{D}_4)$. Hence the closed path representing $f_{\sigma_1^{-1}\sigma_2\sigma_3}$ is given by the sequence of train track maps in the labeled automaton

$$(\tau_1, \varepsilon_1) \xrightarrow{T_1} (\tau_2, \varepsilon_2) \xrightarrow{T_2} (\tau_1, \varepsilon_3) \xrightarrow{T_3} (\tau_1, \varepsilon_3) \xrightarrow{R} (\tau_1, \varepsilon_1)$$

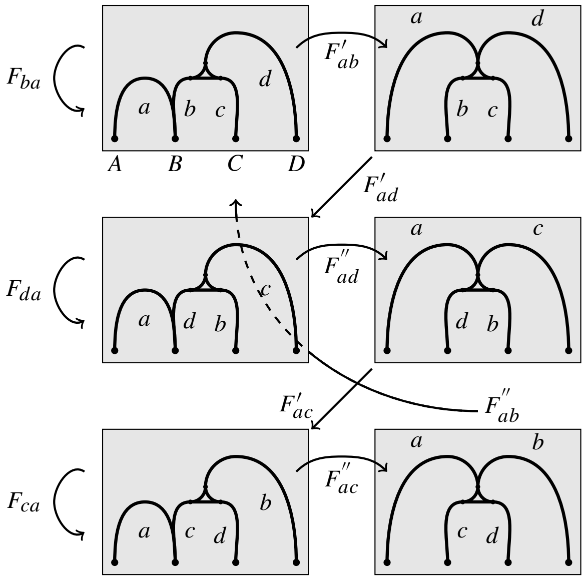


Fig. 19. Detail of the labeled folding automaton in B_4.

with the relabeling $R : (\tau_1, \varepsilon_3) \to (\tau_1, \varepsilon_1)$. Direct computation gives (in the ordered basis (a, b, c, d))

$$M(T_1) = \begin{pmatrix} 1&1&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}, \quad M(T_2) = \begin{pmatrix} 1&0&0&1\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}, \quad M(T_3) = \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 1&0&0&1 \end{pmatrix}, \quad M(R) = \begin{pmatrix} 1&0&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 0&1&0&0 \end{pmatrix}.$$

We recover our incidence matrix $M(T)$ as $M(R \circ T_3 \circ T_2 \circ T_1) = M(T_1)M(T_2)M(T_3)M(R)$.

8.2. *Teichmüller polynomial*

We now calculate the Teichmüller polynomial of the fibered face containing the fibration defined by the suspension of $f_{\sigma_1^{-1}\sigma_2\sigma_3}$. Since the braid permutes all the strands cyclically, $\pi : \widetilde{\mathbf{D}}_4 \to \mathbf{D}_4$ is a $\mathbb{Z}$-covering. We now apply Theorem 5.4 step by step.

(1) The folding F_{ab} is standard, hence $v_1 = (1, 1, 1, 1)$, which implies that D_1 is the identity matrix.

(2) The folding F_{ad} is not standard. By Definition 5.1, in (τ_2, ε_2) we have $f = a$, $f' = d \in N(T_2)$, hence we are in Case 2. We conclude that $v_2 = (1, D, D, D)$. Since $\sigma_1^{-1}\sigma_2\sigma_3$ permutes the strands cyclically, we have $w_2 = (1, t, t, t)$. Therefore

$$D_2 = \begin{pmatrix} 1&0&0&0\\ 0&t&0&0\\ 0&0&t&0\\ 0&0&0&t \end{pmatrix}.$$

(3) The folding F_{ba} is not standard. By Definition 5.1, in(τ_3, ε_3) we have $f = b$, $f' = a \notin N(T_2)$, hence we are in Case 1. We conclude that $v_3 = (A^{-1}, 1, 1, 1)$. Since

$\sigma_1^{-1}\sigma_2\sigma_3$ permutes the strands cyclically , we have $w_2 = (t^{-1}, 1, 1, 1)$. Therefore

$$D_3 = \begin{pmatrix} t^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Hence, the matrix whose characteristic polynomial is Θ_F is given by:

$$\begin{aligned} M(\widetilde{T}) &= M(T_1) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} M(T_2) \begin{pmatrix} t^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} M(T_3) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & t \end{pmatrix} M(R) \\ &= \begin{pmatrix} 1+t^{-1} & t & t & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & t \\ 1 & t & 0 & 0 \end{pmatrix}. \end{aligned}$$

We conclude that the Teichmüller polynomial of $f_{\sigma_1^{-1}\sigma_2\sigma_3}$ is

$$\Theta_F(t,u) = u^4 - (1+t^{-1})u^3 - (t^2+t^3)u + t^2.$$

This calculation can also be performed without the use of elementary operations: $\widetilde{\tau}_1$ is $\widetilde{f}_{\sigma_1^{-1}\sigma_2\sigma_3}$-invariant and the corresponding incidence matrix is precisely $M(\widetilde{T})$ (see Figure 20).

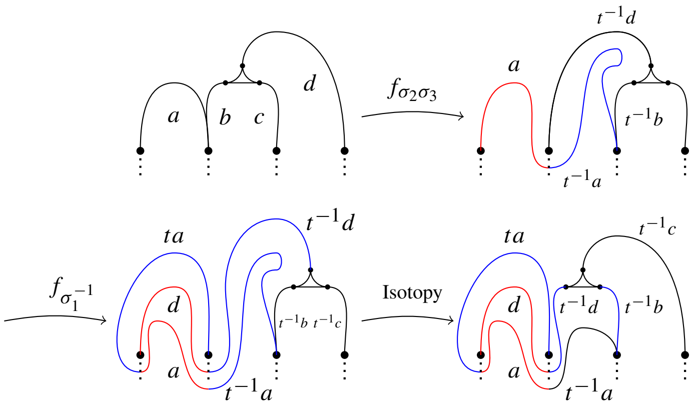


Fig. 20. The lift of $f_{\sigma_3^{-1}\sigma_2^{-1}\sigma_1}$ to $\widetilde{\mathbf{D}}_4$.

Appendix A. An infinite family of braids

We consider, for each $n \in \mathbb{N}$, $n > 0$, the braid $\beta_n \in B_{n+4}$ given by

$$\beta_n = \delta_n \sigma_4 \cdots \sigma_{n+1}\sigma_{n+2}\delta_n\sigma_1$$

where $\delta_n = (\sigma_1 \cdots \sigma_{n+3})^{-1}$. Consider the train track (τ_1, ε_1) given by Figure 21. We have

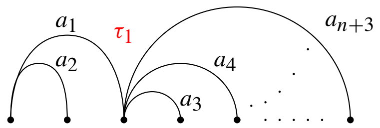


Fig. 21. The train track (τ_1, ε_1).

a train track map $(\tau_1, \varepsilon_1) \xrightarrow{T} (\tau_1, \varepsilon_1)$ representing f_{β_n} induced by the loop

$$(\tau_1, \varepsilon_1) \xrightarrow{T_1} (\tau_2, \varepsilon_2) \xrightarrow{T_2} \cdots \xrightarrow{T_n} (\tau_{n+1}, \varepsilon_{n+1}) \xrightarrow{T_{n+1}} (\tau_1, \varepsilon_{n+1}) \xrightarrow{T_{n+2}} (\tau_1, \varepsilon_{n+1}) \xrightarrow{R} (\tau_1, \varepsilon_1)$$

so that $T = R \circ T_{n+2} \circ T_{n+1} \circ \cdots \circ T_1$, where:

(1) The train track map T_1 is induced by folding the edge labeled a_1 onto the edge labeled a_2; it represents the braid σ_1.
(2) The train track morphism T_2 is induced by folding the edge labeled a_{n+3} onto the edge labeled a_1; it represents the braid δ_n.
(3) For every $i = 3, \ldots, n+1$ the train track morphism T_i is induced by folding the edge labeled a_{n+5-i} onto the edge labeled a_1 and then applying a standardizing braid σ_{n+5-i}.
(4) T_{n+2} is induced by the braid δ_n since $\delta_n^{-1} \circ h(\tau_1)$ is standard.
(5) R is the relabeling.

We easily obtain

$$M(T_1) = \mathrm{Id}_{\mathbb{A}_{\mathrm{real}}} + E_{a_1 a_2}, \quad M(T_i) = \mathrm{Id}_{\mathbb{A}_{\mathrm{real}}} + E_{a_{n+5-i} a_1} \quad \text{for } i = 2, \ldots, n+1,$$
$$M(T_{n+2}) = \mathrm{Id}_{\mathbb{A}_{\mathrm{real}}},$$

where $E_{\alpha\beta}$ is the matrix having all entries zero except at position (α, β) where the entry is 1. We also have (in the ordered basis $(a_1, \ldots, a_{n+3})$)

$$M(R) = \left(\begin{array}{ccc|cccc} 1 & 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & \cdots & 1 \\ 0 & 1 & 0 & 0 & \cdots & \cdots & 0 \\ \hline 0 & 0 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & 0 & 1 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{array}\right)_{(a_1, \ldots, a_{n+3})}.$$

Therefore the incidence matrix $M(T)$ is $M(T_1) \ldots M(T_{n+2}) M(R)$, with characteristic polynomial

$$P(X) = X^{n+3} - X^{n+2} - \cdots - X + 1.$$

We now calculate the Teichmüller polynomial of the fibered face F containing the fibration defined by the suspension of f_{β_n}. Since the braid permutes all the strands cyclically, $\pi : \widetilde{\mathbf{D}}_n \to \mathbf{D}_n$ is a $\mathbb{Z}$-covering and we fix a labeling by $t \in \mathbb{Z}$ of the set of leaves forming $\widetilde{\mathbf{D}}_n$ that is coherent with the action of $\mathrm{Deck}(\pi)$.

One needs to compute the vectors $w_i = \eta_i(v_i)$ for $i = 1, \dots, n+1$. The first case is similar to the situation discussed in other examples: $w_1 = v_1 = (1, B, 1, \dots, 1)$. Hence $t(w_1) = (1, t, 1, \dots, 1)$.

For the map T_2 one has $f = a_{n+3}$ and $f' = a_1$. On the other hand, $f' \in N(T_2)$, thus we are in Case 2, hence $v_2 = (X^{-1}, \dots, X^{-1}, 1)$. For the other vectors, for each $i = 3, \dots, n+1$ we have $f = a_{n+5-i}$ and $f' = a_1$. Since $N(T_i) = \emptyset$, $f' \notin N(T_i)$ and we are in Case 1. Hence $(v_i)_{a_{n+5-i}} = X$ and $(v_i)_\alpha = 1$ otherwise. Finally, for T_{n+2}, one has $(v_{n+2})_\alpha = X^{-1}$ for every α. In conclusion, a straightforward computation shows

$$
\begin{aligned}
t(w_1) &= (1, t, 1, \dots, 1),\\
t(w_2) &= (t^{-1}, \dots, t^{-1}, 1),\\
t(w_i) &= (1, \dots, 1, t, 1, \dots, 1) \quad \text{for } i = 3, \dots, n+1,\\
t(w_{n+2}) &= (t^{-1}, \dots, t^{-1}).
\end{aligned}
$$

where the entry t in $t(w_i)$ is at position $n+5-i$.

We can apply (5.6) to obtain $\Theta_F(t,u) = \det(u \cdot \mathrm{Id} - M)$ where $M = M(T_1)D_1 \cdots M(T_{n+2})D_{n+2}M(R)$ with $D_i = \mathrm{Diag}(t(w_i))$. Therefore

$$
M = \left(\begin{array}{ccc|cccc}
t^{-2} & 0 & 0 & 0 & \cdots & \cdots & t^{-1}\\
0 & 0 & 0 & 0 & \cdots & \cdots & t^{-1}\\
0 & t^{-2} & 0 & 0 & \cdots & \cdots & 0\\
\hline
t^{-2} & 0 & t^{-1} & 0 & \cdots & \cdots & 0\\
\vdots & \vdots & 0 & t^{-1} & \cdots & \cdots & 0\\
\vdots & \vdots & \vdots & 0 & \ddots & 0 & 0\\
t^{-2} & 0 & 0 & 0 & \cdots & t^{-1} & 0
\end{array}\right),
$$

and its associated characteristic polynomial, that is, the Teichmüller polynomial of f_β, is

$$
\Theta_F(t,u) = u^{n+3} - t^{-2}u^{n+2} - t^{-3}u^{n+1} - \cdots - t^{-(n+3)}u + t^{-(n+5)}.
$$

Appendix B. Computing the Thurston norm

The Thurston norm of a link complement can be computed directly in some simple examples (see for example [Thu86]). Our calculations will make use of the Alexander norm. Its definition makes use of the Alexander polynomial $\Delta_M = \sum_{g \in G} b_g \cdot g \in \mathbb{Z}[G]$, where $G = H_1(M, \mathbb{Z})/\mathrm{Tor}$. The *Alexander norm* is defined on $H^1(M, \mathbb{R})$ by

$$
\|\alpha\|_A := \sup_{b_g \neq 0 \neq b_h} \alpha(g - h). \tag{B.13}
$$

The next two theorems explain how the Alexander and Thurston norms are related.

Theorem B.1 ([McM02]). *Let M be a compact, orientable 3-manifold whose boundary, if any, is a union of tori. If $b_1(M) \geq 2$ then for all $[\alpha] \in H^1(M, \mathbb{Z})$,*

$$\|\alpha\|_A \leq \|\alpha\|_T.$$

Moreover, equality holds when $\alpha : \pi_1(M) \to \mathbb{Z}$ is represented by a fibration $\Sigma \to M_\psi \to S^1$, where Σ has non-positive Euler characteristic.

Theorem B.2 ([McM00]). *Let F be a fibered face in $H^1(M, \mathbb{R})$ with $b_1(M) \geq 2$. Then:*

(1) *$F \subset A$ for a unique face A of the Alexander unit norm ball.*
(2) *$F = A$ and Δ_M divides Θ_F if the lamination $\mathcal{L}$ associated to F is transversally orientable.*

In particular, *the Thurston and Alexander norms agree on integer classes in the cone over a fibered face of the Thurston norm ball*. The condition "the lamination $\mathcal{L}$ associated to F is transversally orientable" is equivalent to the following condition: there exists a fibration $\Sigma \to M_\psi \to S^1$ whose pseudo-Anosov monodromy fixes a projective measured lamination $[(l, \mu)] \in \mathbb{P}\mathcal{ML}(\Sigma)$ which is transversally orientable. Equivalently, this last condition is equivalent to the orientability of a train track τ carrying l. From these theorems we can deduce the following simple fact: if $b_1(M) = 2$ and all faces of B_T are fibered, then the Thurston and Alexander norms coincide. The effective calculation of the Alexander norm is possible thanks to the following obvious remark:

Remark B.3. Since the Alexander polynomial of a 3-manifold is symmetric, the Alexander norm ball is dual to the Newton polytope of the Alexander polynomial, scaled by a factor of 2.

For completeness we end this section by discussing the Teichmüller norm and how it can also be used to calculate the Thurston norm. Fix a fibered face $F \subset H^1(M, \mathbb{R})$ and let $\Theta_F = \sum_{g \in G} a_g \cdot g$ be the corresponding Teichmüller polynomial. The *Teichmüller norm* (relative to F) is defined by

$$\|\alpha\|_{\Theta_F} := \sup_{a_g \neq 0 \neq a_h} \alpha(g - h). \tag{B.14}$$

Compare with (B.13). The unit ball B_{Θ_F} of the Teichmüller norm is dual to the Newton polytope $N(\Theta_F)$ of the Teichmüller polynomial [McM00]. Moreover,

Theorem B.4 ([McM00]). *For any fibered face F of the Thurston norm ball, there exists a face D of the Teichmüller norm ball,*

$$D \subset \{[\alpha] \mid \|[\alpha]\|_{\Theta_F} = 1\},$$

such that $\mathbb{R}^+ \cdot F = \mathbb{R}^+ \cdot D$.

Appendix C. Basic types

In Figure 22 we present the basic types that remain to complete the proof of Theorem 5.4. To understand the picture it is important to consider:

(1) For each basic type depicted in the figure we omit the basic type obtained by performing a reflection with respect to a vertical line. We have to take these 'reflected' basic types into consideration for the proof.
(2) At most three infinitesimal edges are depicted, nevertheless the types presented can live in any punctured disc.
(3) The little black dot to which in some basic types the real edges are incident needs to be changed, when constructing a train track from the basic type, to either a vertex or a multigon formed by infinitesimal edges.

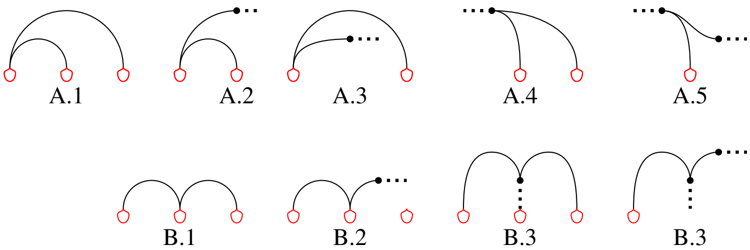


Fig. 22. Basic graphs.

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