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Strong minimality and the j-function

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Abstract. We show that the order three algebraic differential equation over $\mathbb{Q}$ satisfied by the analytic j-function defines a non-$\aleph_0$-categorical strongly minimal set with trivial forking geometry relative to the theory of differentially closed fields of characteristic zero, answering a long-standing open problem about the existence of such sets. The theorem follows from Pila's modular Ax–Lindemann–Weierstrass theorem with derivatives using Seidenberg's embedding theorem. As a by-product of this analysis, we obtain a more general version of the modular Ax–Lindemann–Weierstrass theorem, which, in particular, applies to automorphic functions for arbitrary arithmetic subgroups of $\mathrm{SL}_2(\mathbb{Z})$. We then apply the results to prove effective finiteness results for intersections of subvarieties of products of modular curves with isogeny classes. For example, we show that if $\psi : \mathbb{P}^1 \to \mathbb{P}^1$ is any non-identity automorphism of the projective line and $t \in \mathbb{A}^1(\mathbb{C}) \setminus \mathbb{A}^1(\mathbb{Q}^{\mathrm{alg}})$, then the set of $s \in \mathbb{A}^1(\mathbb{C})$ for which the elliptic curve with j-invariant s is isogenous to the elliptic curve with j-invariant t, and the elliptic curve with j-invariant $\psi(s)$ is isogenous to the elliptic curve with j-invariant $\psi(t)$, has size at most $2^{38} \cdot 3^{14}$. In general, we prove that if V is a Kolchin closed subset of $\mathbb{A}^n$, then the Zariski closure of the intersection of V with the isogeny class of a tuple of transcendental elements is a finite union of weakly special subvarieties. We bound the sum of the degrees of the irreducible components of this union by a function of the degree and order of V.

Keywords. j-function, strong minimality, forking triviality, Schwarzian derivative

1. Introduction

According to Sacks, "[t]he least misleading example of a totally transcendental theory is the theory of differentially closed fields of characteristic 0 (DCF_0)" [28]. This observation has been borne out through the discoveries that a prime differential field need not be minimal [27], the theory DCF_0 has the ENI-DOP property [15], and Morley rank and Lascar rank differ in differentially closed fields [11], amongst others. However, the theory of differentially closed fields of characteristic zero does enjoy some properties not shared by all totally transcendental theories, most notably the Zilber trichotomy holds for its minimal types [12] and there are infinite definable families of strongly minimal sets for which the induced structure on each such definable set is $\aleph_0$-categorical and orthogonality

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between the fibers is definable [7, 9]. Early in the study of the model theory of differential fields, Lascar asked whether the induced structure on every strongly minimal set orthogonal to the constants must be $\aleph_0$-categorical [14] (Lascar's formulation of the question was slightly different, though implies the condition we stated; also, Lascar attributed the question to Poizat, but the question does not appear in the paper to which Lascar refers). From the existence of Manin kernels, one knows that there are strongly minimal sets relative to DCF_0 which are not $\aleph_0$-categorical [8], but the question of whether there are non-$\aleph_0$-categorical strongly minimal sets with trivial forking geometry has remained open (see [26] for instance). We exhibit an explicit equation defining a set with these properties.

The analytic j-function, $j : \mathfrak{h} \to \mathbb{C}$, which has been known to mathematicians for quite some time, appearing implicitly in the work of Gauss already in the late eighteenth century [6], satisfies a differential equation over $\mathbb{Q}$ which when evaluated in a differentially closed field defines a non-$\aleph_0$-categorical strongly minimal set with trivial forking geometry. The specific differential equation satisfied by the j-function is given by the vanishing of a differential rational function. In addition to the fiber of the function above zero (the equation of the j-function), we prove that all fibers are strongly minimal, trivial, and pairwise orthogonal.

Besides the applications to differential-algebraic geometry, we give some number-theoretic applications. Specifically, we use our differential-algebraic approach to prove effective bounds on the size of the intersection of Hecke orbits of transcendental points on products of modular curves with non-weakly special varieties.

Mazur posed some effective finiteness questions in connection with a recent theorem of Orr [22]. Of course, knowing the isogeny class of an elliptic curve determines that curve only to within a countably infinite set. Mazur surmised that the data of the isogeny class of an elliptic curve E and of the isogeny class of some other naturally associated (but not so naturally associated as to respect the Hecke correspondences) elliptic curve might pin down E or at least constrain it to a finite set. In fact, it is a consequence of the main theorem of [22] that if $C \subseteq \mathbb{A}^2_{\mathbb{C}}$ is an irreducible affine plane curve which is not modular or horizontal or vertical, then for any point $(a, b) \in C(\mathbb{C})$ there are only finitely many other points $(c, d) \in C(\mathbb{C})$ for which the elliptic curve coded by a is isogenous to the elliptic curve given by c and the curve corresponding to b is isogenous to that coded by d. In this sense, if we regard C as a correspondence which associates to an elliptic curve E with j-invariant a one of the elliptic curves having j-invariant b with $(a, b) \in C(\mathbb{C})$, then the data of the isogeny class of E and of the C-associated elliptic curve determine E up to a finite set.

Orr's theorem applies to arbitrary points without any hypothesis on the degree of the point over $\mathbb{Q}$. However, this generality incurs a cost: his argument follows the Pila–Zannier strategy for proving diophantine geometric finiteness theorems which depends in an essential way on ineffective results in the Pila–Wilkie o-minimal counting theorem and in class field theory. On the other hand, by restricting attention to transcendental points, we may compute explicit bounds on the sizes of these finite sets. While our proof that the sets in question are finite also passes through the Pila–Wilkie o-minimal counting theorem in the guise of Pila's modular Ax–Lindemann–Weierstrass theorem with derivatives, this appeal does not leave a trace of ineffectivity.

Let us describe the basic tactics involved in our approach to the general problem. The key point is to replace the Hecke orbits by solution sets to particular differential equations. This approach already appears in Buium's article [2]. The obvious downside to this move is that (as referenced above) the Kolchin (differential Zariski) topology has wild behavior compared to the Zariski topology. This is mitigated by our model-theoretic work understanding the differential equation satisfied by the j-function. Here we use the strong minimality and triviality of the differential equation which the j-function satisfies in order to establish the finiteness of its intersection with non-weakly special algebraic varieties. The advantage of this approach is uniformity—we replace an arithmetic object by a (differential) variety. The finiteness of certain intersections then follows by our proof of strong minimality, and the actual bounds come from an effective version of uniform bounding for definable sets in differential fields due to Hrushovki and Pillay [10] (with improvements due to Binyamini [1]); essentially these bounds come from doing intersection theory (of algebraic varieties) in jet spaces of algebraic varieties. The actual bounds are rather tractable, being doubly exponential in the various inputs—the degrees and dimensions of certain associated algebraic varieties.

This paper is organized as follows. In Section 2, we recall some of the basic theory of the j-function, including the theory of the Schwarzian derivative and the differential equation satisfied by j. With Section 3 we complete the proof of our main theorem and draw some corollaries. The main ingredients of the proof are Seidenberg's embedding theorem, Pila's modular Ax–Lindemann–Weierstrass theorem with derivatives and a construction of a nonlinear order three differential rational operator χ for which $\chi(j) = 0$. In Section 4, we show via a change of variables trick and some basic forking calculus that for any parameter a the set defined by $\chi(x) = a$ is strongly minimal. In Section 4.2 we show that these fibers are orthogonal. The final section is devoted to arithmetic applications, where we use our main theorem to get bounds on the intersections of non-weakly special varieties with Hecke orbits in products of modular curves.

2. Basic theory and the j-function

In this section we summarize some of the basic theory of the j-function and of geometric stability theory.

We denote the upper half-plane by

$$\mathfrak{h} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}.$$

We write t for the variable ranging over $\mathfrak{h}$ (or some open subdomain).

The *j-function* is an analytic function on $\mathfrak{h}$ whose Fourier expansion begins with

$$j(t) = \exp(-2\pi i t) + 744 + 196\,884 \exp(2\pi i t) + 21\,493\,760 \exp(4\pi i t) + \cdots .$$

The algebraic group $\mathrm{SL}_2(\mathbb{C})$ acts on the projective line via linear fractional transformations, and the restriction of this action to $\mathrm{SL}_2(\mathbb{R})$ induces an action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathfrak{h}$. The j-function is a modular function for $\mathrm{SL}_2(\mathbb{Z})$ in the sense that $j(\gamma \cdot t) = j(t)$ for each

$\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Indeed, more is true: for $a, b \in \mathfrak{h}$ one has $j(a) = j(b)$ if and only if there is some $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ with $\gamma \cdot a = b$.

The differential equation satisfied by j is best expressed using the Schwarzian derivative. We shall write x' for $\partial x/\partial t$. More generally, in any differential ring (R, ∂) we shall write x' for $\partial(x)$. We define the *Schwarzian* by

$$S(x) = \left(\frac{x''}{x'}\right)' - \frac{1}{2}\left(\frac{x''}{x'}\right)^2.$$

When dealing with the Schwarzian derivative associated with a particular derivation ∂, we will use the notation S_∂, but when ∂ is fixed or clear from the context, we will drop the subscript.

The Schwarzian satisfies a chain rule:

$$S(f \circ g) = (g')^2 S(f) \circ g + S(g).$$

A characteristic feature of the Schwarzian is that if (K, ∂) is a differential field of characteristic zero with field of constants $C = \{x \in K : x' = 0\}$ and f and g are two elements of K, then one has $S(f) = S(g)$ if and only if $f = \frac{ag+b}{cg+d}$ for some constants a, b, c and d. In particular, one computes immediately from the formula for the Schwarzian that if $z' = 1$, then $S(z) = 0$ so that the solutions to the equation $S(x) = 0$ are precisely the degree one rational functions in z with coefficients from C.

The following is an order three algebraic differential equation satisfied by j (see [17, p. 20]):

$$S(y) + R(y)(y')^2 = 0, \qquad (\star)$$

where

$$R(y) = \frac{y^2 - 1968y + 2\,654\,208}{2y^2(y - 1728)^2}.$$

For the remainder of this paper, when we speak of the differential equation satisfied by j, we mean equation ($\star$). We will also make use of the differential rational function which gives the equation; throughout the paper, we will denote

$$\chi(y) := S(y) + R(y)(y')^2.$$

Similarly, when there is some ambiguity or choice about the particular derivation ∂ with which we are working, we will write χ_∂ for the resulting differential rational function.

Our model-theoretic notation is standard and generally follows that of [25].

Non-forking (for which we use the symbol $\mathop{\smile\!\!\!\!\!\mid}$) is a notion of independence in model theory, which we will not define in general, but we will describe its manifestation in differential fields of characteristic zero [16, Section 2].

Let $(\mathbb{U}, \partial) \supset (K, \partial)$ be an extension of differential fields and a be a tuple of elements from $\mathbb{U}$. Let L/K be an extension of differential fields inside $\mathbb{U}$. Then $a \not\mathop{\smile\!\!\!\!\!\mid}_K L$ if and only if for some $k \in \mathbb{N}$,

$$\mathrm{tr.deg}_L\, L(a, \partial(a), \ldots, \partial^k(a)) < \mathrm{tr.deg}_K\, K(a, \partial(a), \ldots, \partial^k(a)). \qquad (\dagger)$$

Read more algebraically, if we let $I(a/L)$ be the differential ideal of all differential polynomials over L which vanish on a and K is algebraically closed, then $a \mathop{\smile\!\!\!\!\!\!|}_K L$ if and only if $I(a/L)$ is generated by differential polynomials with coefficients from K.

If $a \not\mathop{\smile\!\!\!\!\!\!|}_K L$, then in the language of types, we say that $\mathrm{tp}(a/L)$ is a *forking extension* of $\mathrm{tp}(a/K)$.

When p is a type over A, the *Lascar rank* of p, denoted $\mathrm{U}(p)$, is defined as follows. In general, $\mathrm{U}(p) \geq 0$ for every consistent type p, $\mathrm{U}(p) \geq \alpha + 1$ if and only if there is a forking extension q of p with $\mathrm{U}(q) \geq \alpha$, and $U(p) \geq \lambda$ for λ a limit ordinal just in case $U(p) \geq \alpha$ for all $\alpha < \lambda$. Generally, Lascar rank is ordinal-valued, when defined, but in this paper, we only deal with finite Lascar rank types. By our characterization of forking in DCF_0, transcendence calculations (as in (†)) control forking and thus Lascar rank.

Let us recall a basic principle in stability theory, called in [4] the "Shelah reflection principle". In a stable theory, let p be a stationary type; then the canonical base of p is definable from a Morley sequence in p. A proof of this principle in the more general context of simple theories may be found in [3, Proposition 17.24]. A proof in the stable case may be found in [25, Lemma 2.28]. We will use a consequence of this principle. Namely, if $A \subset B$ are subsets of some model of a stable theory and a is a tuple from the model, then if $a \not\mathop{\smile\!\!\!\!\!\!|}_A B$ there is a Morley sequence $(d_i)_{i=0}^{\infty}$ in $\mathrm{tp}(a/B)$ such that (d_i) is not independent over A. In particular, there is a finite initial segment of the Morley sequence which is not independent over A.

3. Minimality and the j-function

In this section, we deduce our main theorem on the strong minimality of the set defined by equation ($\star$). We regard the differential field $\mathbb{C}\langle j \rangle = \mathbb{C}(j, j', j'')$ as a subdifferential field of some differentially closed field with field of constants $\mathbb{C}$.

Let us recall Seidenberg's embedding theorem [30].

Theorem 3.1 (Seidenberg). *Let $K = \mathbb{Q}\langle u_1, \dots, u_n \rangle$ be a differential field generated by n elements over $\mathbb{Q}$ and let $K_1 = K\langle v \rangle$ be a differential field extension of K generated by a single element v. Suppose $U \subset \mathbb{C}$ is an open ball and $\iota : K \to \mathcal{M}(U)$ is a differential field embedding of K into the differential field of meromorphic functions on U. Then there is an open ball $V \subseteq U$ and an extension of ι to a differential field embedding of K_1 into $\mathcal{M}(V)$.*

We will use Seidenberg's theorem in conjunction with analyses of Morley sequences (as discussed in Section 2).

Lemma 3.2. *We can realize any finite sequence $\{d_1, \dots, d_n\}$ of solutions to ($\star$) as $\{j(g_1 t), \dots j(g_n t)\}$ where $g_i \in \mathrm{GL}_2(\mathbb{C})$ for $i = 1, \dots, n$.*

Remark 3.3. The first author discussed portions of this argument with Ronnie Nagloo, who made several essential suggestions.

Proof of Lemma 3.2. By using Theorem 3.1, we may assume that $\{d_1, \ldots, d_n\}$ are realized as meromorphic functions on some domain U contained in $\mathfrak{h}$. Since the j-function is a surjective analytic function from $\mathfrak{h}$ to $\mathbb{C}$, it follows that there are holomorphic functions $\psi_i : U \to \mathfrak{h}$ such that $j(\psi_i(t)) = d_i(t)$. By hypothesis, $j(\psi_i(t))$ satisfies ($\star$). Hence,

$$\begin{aligned}0 &= \chi(j \circ \psi_i) = S(j \circ \psi_i) + R(j \circ \psi_i)((j \circ \psi_i)')^2 \\ &= (S(j) \circ \psi_i) \cdot (\psi_i')^2 + S(\psi_i) + R(j \circ \psi_i)(j' \circ \psi_i)^2 \cdot (\psi_i')^2 \\ &= (\chi(j) \circ \psi_i) \cdot (\psi_i')^2 + S(\psi_i) = S(\psi_i).\end{aligned}$$

Thus, if $j \circ \psi_i$ is a solution to $\chi(x) = 0$, then $S(\psi_i) = 0$. As we noted in Section 2, all such solutions are rational functions of degree one. That is, there is some $g_i \in \mathrm{GL}_s(\mathbb{C})$ for which $\psi_i(t) = g_i \cdot t$, showing that d_i is realized as $j(g_i t)$ as required. □

The second main ingredient in the proof of our main theorem is a result of Pila [24]. Before stating the theorem, we define some terminology.

Let $W \subseteq \mathbb{A}^m_{\mathbb{C}}$ be an irreducible affine algebraic variety. Let $a_1, \ldots, a_n \in \mathbb{C}(W)$ be rational functions on W and suppose that $p \in W(\mathbb{C})$ is a point for which $a_i(p) \in \mathfrak{h}$ for all $i \le n$. Then the maps $j(a_1), \ldots, j(a_n)$ can be considered as functions in a neighborhood of the point p. The functions $a_1, \ldots, a_n$ are *geodesically independent* if the a_i are all non-constant and they satisfy no relations of the form $a_i = g a_j$ for some $g \in \mathrm{GL}_2^+(\mathbb{Q})$. Under these circumstances, Pila proved:

Theorem 3.4 (Pila). *The $3n$ functions $j(a_i)$, $j'(a_i)$ and $j''(a_i)$ for $i = 1, \ldots, n$ are algebraically independent over $\mathbb{C}(W)$.*

Remark 3.5. If W is a curve and $\partial : \mathbb{C}(W) \to \mathbb{C}(W)$ is a non-zero $\mathbb{C}$-derivation on $\mathbb{C}(W)$, then using the chain rule and the algebraic independence of $j(a_i)$, $j'(a_i)$, and $j''(a_i)$ over $\mathbb{C}(W)$, we deduce the algebraic independence of $j(a_i)$, $\partial(j(a_i))$, and $\partial^2(j(a_i))$ over $\mathbb{C}(W)$.

Remark 3.6. In fact, [24] proves a stronger result giving the independence of the above $3n$ functions over a larger field which includes exponential functions and Weierstrass $\wp$ functions. One can deduce this more general form of the Ax–Lindemann–Weierstrass theorem from the results of this paper, as we will explain in Remark 3.9 below.

With the next theorem we deduce from Theorem 3.4 that $\mathrm{tp}(j/\mathbb{C})$ is minimal.

Theorem 3.7. $\mathrm{U}(\mathrm{tp}(j/\mathbb{C})) = 1$.

Proof. By the superstability of DCF_0, we may find a finitely generated algebraically closed subfield $A \subseteq \mathbb{C}$ for which $\mathrm{U}(j/A) = \mathrm{U}(j/\mathbb{C})$.

We need to check that any forking extension of $\mathrm{tp}(j/A)$ is algebraic. By the finite character of forking, it suffices to consider extensions of the type to finitely generated extensions of A. If $B \supseteq A$ is any such finitely generated differential field extension in our differentially closed field for which $\mathrm{tp}(j/B)$ forks over A, then by the Shelah reflection principle described in Section 2, we may find a finite initial segment of a Morley sequence $\{d_1, \ldots, d_n\}$ in $\mathrm{tp}(j/B)$ which is not independent over A. By Lemma 3.2, we may realize

$d_1, \ldots, d_n$ as $j(g_1 t), \ldots, j(g_n t)$ for some $g_i \in \mathrm{GL}_2(\mathbb{C})$. By Theorem 3.4 with $W = \mathbb{A}^1$ and $a_i(t) := g_i \cdot t$, if $g_1, \ldots, g_n$ are in distinct cosets of $\mathrm{GL}_2(\mathbb{Q})$, then $j(g_1 t), \ldots, j(g_n t)$ are forking independent over $\mathbb{C}$. On the other hand, if g_i and g_j are in the same coset of $\mathrm{GL}_2(\mathbb{Q})$, then $j(g_i t)$ and $j(g_j t)$ are interalgebraic over $\mathbb{Q}$ as witnessed by an appropriate modular polynomial $F_N(x, y) \in \mathbb{Z}[x, y]$ [18, pp. 183–186]; we will refer to this relation (between solutions of the differential equation ($\star$)) as a *Hecke correspondence*; Pila [24, 23] calls these modular relations. The only way that the elements of a Morley sequence may be interalgebraic is if the type itself is algebraic. Hence, from the dependence of the Morley sequence we deduce that $\mathrm{tp}(j/B)$ is algebraic, as required. □

Via another use of Pila's theorem, we strengthen Theorem 3.7 to the conclusion that E defines a *strongly minimal set*, that is, every definable subset is either finite or cofinite.

Theorem 3.8. *The set defined by the differential equation* ($\star$) *is strongly minimal.*

Proof. As ($\star$) is of degree one in the order three variable, it suffices to show that any differential specialization of j over $\mathbb{C}$ satisfies no lower order differential equation. By Lemma 3.2 we may realize f as $f = j(gt)$ for some $g \in \mathrm{GL}_2(\mathbb{C})$. Applying Theorem 3.4 with $W = \mathbb{A}^1$, $n = 1$, and $a_1(t) = g \cdot t$, we see that $\mathrm{tr.deg}_{\mathbb{C}(t)}(f, f', f'') = 3$. (Alternatively, this result follows from the main theorem of [21].) □

Remark 3.9. Recall that in Remark 3.6 we stated that Pila proved a more general statement than the independence of j, j', and j'' over the function field of some variety W. In fact, he proves that the conclusion holds over $\mathbb{C}(W)$ for the derivatives of the j-functions along with a collection of exponential and Weierstrass $\wp$ functions. In fact, the conclusion for the entire collection follows from minimality of the type $\mathrm{tp}(j/\mathbb{C})$. Indeed, this depends very little on the nature of the exponential and Weierstrass $\wp$-functions. Similar conclusions hold for any function (or collection of functions) $f(t)$ so that $f(t)$ satisfies an order two (or lower) differential equation over $\mathbb{C}$.

To see this, we show by induction on n that if $f_1, \ldots, f_n$ is a finite sequence of functions all of which satisfy differential equations of order at most two, then j is independent from $f_1, \ldots, f_n$ over $\mathbb{C}$. The case of $n = 0$ is trivial. For the inductive case of $n + 1$, let $K := \mathbb{C}\langle f_1, \ldots, f_n\rangle$ be the differential field generated by $f_1, \ldots, f_n$ over $\mathbb{C}$. By induction, j is free from K over $\mathbb{C}$ so that $\mathrm{tp}(j/K)$ is minimal as well, implying that if j depends on $f_1, \ldots, f_{n+1}$ over $\mathbb{C}$, then $j \in K\langle f_{n+1}\rangle^{\mathrm{alg}}$, but $\mathrm{tr.deg}_K(K\langle j\rangle) = 3 > 2 \geq \mathrm{tr.deg}_K(K\langle f_{n+1}\rangle^{\mathrm{alg}})$.

The main theorem of Hrushovski's manuscript [7] is that if the definable set X is defined by an order one differential equation over the constants and is orthogonal to the constants, then the induced structure on X over any finite set of parameters over which it is defined is $\aleph_0$-categorical. Under additional technical assumptions, Rosen [26] proved the theorem without the hypothesis that X is defined over the constants (however, the technical assumptions are of a nature such that it is not obvious if they ever hold). It has been known since the identification of Manin kernels that not every strongly minimal which is orthogonal to the constants must have $\aleph_0$-categorical induced structure, but the question of whether a strongly minimal set with trivial forking geometry must have $\aleph_0$-categorical induced structure has remained open until now.

Theorem 3.10. *Let X be the set defined by equation ($\star$). For each natural number n, every definable subset of X^n is defined by a Boolean combination of formulae of the form $x_\ell = \zeta$ for some $\zeta \in X$ and $\ell \leq n$ and $F_N(x_i, x_k) = 0$ for some $N \in \mathbb{Z}_+$ and $j, k \leq n$. Moreover, the definable sets $\{(x, y) \in X^2 : F_N(x, y) = 0\}$ as N ranges through $\mathbb{Z}_+$ give infinitely many distinct 0-definable subsets of X^2. Consequently, X is strongly minimal, has trivial forking geometry, but is not $\aleph_0$-categorical.*

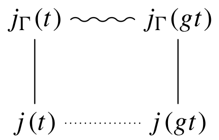
Proof. Our main theorem, Theorem 3.8, asserts that X is strongly minimal. Triviality of the forking geometry of the generic type of X (and hence of X itself) is an immediate consequence of Pila's Theorem 3.4.

We claim that if $(a_1, \ldots, a_n) \in X^n$ and $(b_1, \ldots, b_n) \in X^n$ and they agree with respect to all formulae of the form $F_N(x_i, x_k) = 0$ for some $N \in \mathbb{Z}_+$ and $j, k \leq n$, then they have the same type over $\mathbb{Q}^{\mathrm{alg}}$. Indeed, reordering the coordinates if need be, we may assume that no equation of the form $F_N(a_i, a_k) = 0$ holds with $i \leq m$ but that m is maximal with this property. It is an immediate consequence of Theorem 3.4 that $(a_1, \ldots, a_m)$ realizes the generic type of X^m as does $(b_1, \ldots, b_m)$. On the other hand, the remaining coordinates are algebraic over the first m, and their algebraic types are described by which modular polynomials they satisfy.

Thus, the 0-definable sets are given by finite Boolean combinations of modular relations. Since X is stably embedded (because DCF_0 is stable), the induced structure on X over any other set of parameters is given by an expansion by constants of the 0-definable structure.

Since the set X is closed under isogeny in the sense that if $x \in X$ and y is the j-invariant of an elliptic isogenous to an elliptic curve with j-invariant x, then $y \in X$ (see [2]), it is clear that the binary relations on X^2 given by the modular polynomials are infinite and distinct. □

Remark 3.11. Suppose that $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ is an arithmetic subgroup. One might inquire about the differential-algebraic properties of j_Γ, where j_Γ is the analytic function expressing the quotient space $\Gamma\backslash\mathfrak{h}$ as an algebraic curve. Since j_Γ is interalgebraic with j over $\mathbb{C}$, the type of j_Γ is strongly minimal. Further, for $g \in \mathrm{SL}_2(\mathbb{C})$, we have the following diagram:



The solid vertical lines indicate interalgebraicity. The relationship of $j(t)$ and $j(gt)$ is completely controlled by Theorem 3.4. It follows from the interalgebraicity of $j(gt)$ and $j_\Gamma(gt)$ that if $g_1, \ldots, g_n$ lie in distinct cosets of $\mathrm{SL}_2(\mathbb{Q})$, then the functions $j_\Gamma(g_1 t), \ldots, j_\Gamma(g_n t)$ are differentially algebraically independent. That is, the relationship (in terms of algebraic closure in the sense of differential fields) indicated by the top (curly) line is completely controlled by the modular Ax–Lindemann–Weierstrass theorem and the results of this paper; naturally, one obtains as a by-product the Ax–Lindemann–Weierstrass theorem with derivatives for $j_\Gamma(t)$.

4. The other fibers

In the previous sections, we investigated the properties of the algebraic differential equation satisfied by the j-function, or in the language of [2], we investigated the fiber of χ above 0. In this section, we will investigate the other fibers as well as the possible algebraic relations across fibers in order to prove finiteness results. The general problem will be reduced to an analytic one via Seidenberg's theorem combined with the special nature of the differential equations in question.

4.1. Minimality, strong minimality and other trivialities

Fix a_s, an element in some differential field extension of $\mathbb{Q}$. (Here the subscript "s" is meant to suggest the Schwarzian. The reason for this choice of notation should become clear shortly.) By Seidenberg's Theorem 3.1 we may realize the abstract differential field $\mathbb{Q}\langle a_s \rangle$ as a differential subfield of $\mathcal{M}(U)$, the field of meromorphic functions on some connected open subset U of $\mathfrak{h}$. We shall write the variable ranging over U as t and will write $a_s(t)$ when we wish to regard a_s as a meromorphic function. Perhaps at the cost of shrinking the open domain U, we may find some $\tilde{a}(t)$, an analytic function on some U, such that $\chi(j(\tilde{a}(t))) = a_s(t)$ as functions of t. Alternatively, from the analytic description of χ we see that a_s is the Schwarzian of $\tilde{a}$. Define $a := j(\tilde{a}) \in \mathcal{M}(U)$.

For a given derivation ∂, we remind the reader of our notation from the introduction:

$$\chi_\partial(x) := S_\partial(x) + \frac{x^2 - 1968x + 2\,654\,208}{2x^2(x-1728)^2}(\partial x)^2.$$

The following obvious observation (which is an immediate consequence of the chain rule) will be used throughout the remainder of the section.

Lemma 4.1. *If $V \subseteq U$ is a small enough connected open domain on which $\tilde{a}$ is holomorphic and one-to-one and K is a d/dt-differential subfield of $\mathcal{M}(V)$ containing a and $\tilde{a}$, then K is also a d/du-differential field where $u = \tilde{a}(t)$. Furthermore, $\chi_{d/du}(a) = 0$.*

Proposition 4.2. *The set defined by the formula $\chi(x) = a_s$ is strongly minimal. Moreover, if $a_1, \dots, a_n$ satisfy $\chi_\delta(a_i) = a_s$ and B is any algebraically closed differential field containing a_s, then $\{a_1, \dots, a_n\}$ is independent over B, unless there is some k with $a_k \in B$ or there is a pair $i < \ell$ for which $F_N(a_i, a_\ell) = 0$ for some modular polynomial F_N.*

Proof. Let us first address strong minimality. It suffices to show that in some differentially closed field $\mathbb{U}$ extending $\mathbb{Q}\langle a_s \rangle$ the set $F_{a_s} := \{x \in \mathbb{U} : \chi_{d/dt}(x) = a_s\}$ is infinite but every differentially constructible subset is finite or cofinite. Taking $\mathbb{U}$ to be a differential closure of $\mathbb{Q}\langle a_s, \tilde{a} \rangle$ and using Seidenberg's Theorem 3.1 repeatedly, we may realize $\mathbb{U}$ as a differential field of germs of meromorphic functions. By Lemma 4.1, the differential field $\mathbb{U}$ is also a differential field with respect to d/du and the set F_{a_s} is equal to $\{x \in \mathbb{U} : \chi_{d/du}(x) = 0\}$. By the strong minimality of the equation for the j-function, this latter set is infinite and every d/du-differentially constructible subset is finite or cofinite. In particular, since every d/dt-differentially constructible set is d/du-differentially constructible, F_{a_s} is strongly minimal.

For the "moreover" clause describing dependence amongst the solutions to $\chi(x)=a_s$, replacing $a_1, \dots, a_n$ with realizations of the non-forking extension of $\text{tp}(a_1, \dots, a_n/B)$ to $B\langle\tilde{a}\rangle$, we may assume that $a_1, \dots, a_n$ is independent from $\tilde{a}$ over B. Then as in the proof of strong minimality, we see that $a_1, \dots, a_n$ satisfy $\chi_{d/du}(x) = 0$. Regarded now in the differential field $(\mathbb{U}, d/du)$, any dependence amongst $\{a_1, \dots, a_n\}$ must come from either a_k being algebraic over $B\langle\tilde{a}\rangle$ for some k, or $F_N(a_i, a_\ell) = 0$ for some $i < \ell$ and natural number N. By transitivity of forking independence, algebraicity over $B\langle\tilde{a}\rangle$ would imply algebraicity over B. □

Remark 4.3. Proposition 4.2 characterizes algebraic closure in $\chi_\delta^{-1}(a_s)$ and shows that the sets have trivial forking geometry which is not $\aleph_0$-categorical.

4.2. *Orthogonality*

We begin this section with some standard notation from differential-algebraic geometry, which we will require in the proof of our result. The constructions of prolongation spaces and of the corresponding differential sections are valid in a much more general context than what we present here where we specialize to embedded affine varieties and work with coordinates. Further details may be found in [20].

Let (K, ∂) be a differential field of characteristic zero and n and ℓ a pair of natural numbers. We define the ℓ^{th} *prolongation space* of affine n space, $\tau_\ell \mathbb{A}^n$, to be the $\mathbb{A}^{n(\ell+1)}$ where if the coordinates on $\mathbb{A}^n$ are $x_1, \dots, x_n$, then the coordinates on $\tau_\ell \mathbb{A}^n$ are $x_{i,j}$ for $1 \leq i \leq n$ and $0 \leq j \leq \ell$. We define $\nabla_\ell : \mathbb{A}^n(K) \to \tau_\ell \mathbb{A}^n(K)$ by the rule

$$(a_1, \dots, a_n) \mapsto (a_1, \dots, a_n; a_1', \dots, a_n'; \dots; a_1^{(\ell)}, \dots, a_n^{(\ell)})$$

where as above we write x' for $\partial(x)$ and $x^{(\ell)}$ for $\partial^\ell(x)$. If (L, ∂) is a differential field extension of (K, ∂), we continue to write ∇_ℓ for the corresponding map on $\mathbb{A}^n(L)$.

If $X \subseteq \mathbb{A}^n$ is an embedded affine variety and $T \subset S \subseteq \tau_\ell \mathbb{A}^n$ are two subvarieties of the prolongation space, then we define the differential constructible set $(X, S \smallsetminus T)^\sharp$ by

$$(X, S \smallsetminus T)^\sharp(K) := \{a \in X(K) : \nabla(a) \in (S \smallsetminus T)(K)\}.$$

The particular differential-algebraic varieties which interest us are given by the fibers of χ:

$$\chi_\partial(x) := S_\partial(x) + \frac{x^2 - 1968x + 2\,654\,208}{2x^2(x - 1728)^2}(\partial x)^2.$$

Thus, $\chi_\delta^{-1}(a_s)$ is given by the set of x such that

$$\begin{aligned}\left(x'''x' - \tfrac{3}{2}(x'')^2\right)(2x^2(x - 1728)^2) + (x^2 - 1968x + 2\,654\,208)(x')^4 \\ = a_s \cdot (x')^2(2x^2(x - 1728)^2)\end{aligned}$$

and $x' \neq 0$ (note that this implies that $2x^2(x - 1728)^2 \neq 0$). In this case, the variety S is given by the above algebraic equation on $\tau_3(\mathbb{A}^1) = \mathbb{A}^4$ and T is given by the equation $x' = 0$ in the same space. We note that S is an irreducible hypersurface of degree 6.

When analyzing possible algebraic relations between collections of solutions (and their derivatives) to various fibers of χ, the previous section gives a complete account of the algebraic relations within a given fiber. In this section, we prove that there are no algebraic relations across fibers.

Theorem 4.4. *If the parameters b_s and c_s are distinct, then in the sense of stability theory the definable sets $\chi^{-1}(b_s)$ and $\chi^{-1}(c_s)$ are orthogonal.*

Proof. By Proposition 4.2, each of the definable sets $\chi^{-1}(b_s)$ and $\chi^{-1}(c_s)$ is strongly minimal. Hence, non-orthogonality of $\chi^{-1}(b_s)$ and $\chi^{-1}(c_s)$ would be equivalent to the existence of a finite-to-finite definable correspondence between these fibers, possibly defined over new parameters. We shall show that any finite-to-finite correspondence between two fibers of χ must be given by a modular polynomial. Since the fibers of χ are preserved by Hecke correspondences, it will then follow that there can be no finite-to-finite correspondences between different fibers.

For the sake of this argument, we work with models of our equations over finitely generated rings (rather than fields) so that we may specialize parameters.

Using Seidenberg's Theorem 3.1 and then shrinking the domain if need be to avoid poles, we may fix some domain $U \subseteq \mathfrak{h}$ so that we may realize b_s and c_s as elements of $\mathcal{O}(U)$, the differential ring of holomorphic functions on U. Moreover, we may assume that there are $\tilde{b}, \tilde{c} \in \mathcal{O}(U)$ with $b_s = S_{d/dt}(\tilde{b})$ and $c_s = S_{d/dt}(\tilde{c})$. Note that in particular we have arranged that $\frac{d}{dt}(\tilde{b})$ and $\frac{d}{dt}(\tilde{c})$ have no zeros on U. We set $b := j(\tilde{b})$ and $c := j(\tilde{c})$.

By quantifier elimination, the description of algebraic closure in differential fields, and the fact that the third derivative of a solution to a fiber of χ is rational over the previous derivatives, we may assume that the finite-to-finite correspondence Γ_0 is given by $\nabla_2^{-1}\Gamma$ where $\Gamma \subseteq \tau_2(\mathbb{A}^1 \times \mathbb{A}^1) = \mathbb{A}^3 \times \mathbb{A}^3$. By induction (it is only necessary to get a contradiction for irreducible correspondences) and possibly further shrinking U, we may assume that Γ is an absolutely integral $\mathcal{O}(U)$-scheme and that Γ gives a finite-to-finite correspondence on $\mathbb{A}^3_{\mathcal{O}(U)} \times \mathbb{A}^3_{\mathcal{O}(U)}$.

Take $t_0 \in U$ and consider the fiber $\Gamma_{t_0}(\mathbb{C}) \subseteq \tau_2(\mathbb{A}^1 \times \mathbb{A}^1)(\mathbb{C})$.

For a review of the prolongation spaces and their relation to differential-geometric jet spaces, see [29, Sections 2.1 and 2.2]. Taking differential-geometric jets we obtain a map $J_2(j) : J_2(\mathfrak{h}) \to \tau_2(\mathbb{A}^1)(\mathbb{C})$ which fits into the following commutative diagram:

$$\begin{array}{ccccc}
\mathfrak{h} \times \mathfrak{h} & \xleftarrow{\ \pi\ } & J_2(\mathfrak{h} \times \mathfrak{h}) & = & J_2(\mathfrak{h}) \times J_2(\mathfrak{h}) \\
\downarrow{\scriptstyle j \times j} & & \downarrow{\scriptstyle J_2(j \times j)} & & \downarrow{\scriptstyle J_2(j) \times J_2(j)} \\
(\mathbb{A}^1 \times \mathbb{A}^1)(\mathbb{C}) & \xleftarrow{\ \pi\ } & \tau_2(\mathbb{A}^1 \times \mathbb{A}^1)(\mathbb{C}) & = & (\mathbb{A}^3 \times \mathbb{A}^3)(\mathbb{C})
\end{array}$$

Claim 4.5. *The set*

$$A := \{(x(t_0), x'(t_0), x''(t_0), y(t_0), y'(t_0), y''(t_0)) \in \Gamma(\mathbb{C}) : \chi(x) = b_s,\ \chi(y) = c_s\} \subseteq \tau_3\mathbb{A}^2$$

is Zariski dense in Γ_{t_0}.

Proof of Claim. The first projection of A to $\tau_3\mathbb{A}^1$ contains the set

$$B := \left\{ \left(j(g \cdot \tilde{b}(t_0)), \frac{d}{dt} j(g \cdot \tilde{b}(t))\Big|_{t=t_0}, \frac{d^2}{dt^2} j(g \cdot \tilde{b}(t))\Big|_{t=t_0} \right) \;\Big|\; g \in \mathrm{GL}_2^+(\mathbb{R}) \right\},$$

which is Zariski dense in $\tau_3\mathbb{A}^1$. Indeed, because $\mathrm{GL}_2^+(\mathbb{R})$ acts transitively on $\mathfrak{h}$, we see that the projection of B to the first coordinate is all of $\mathbb{C}$. The fibers in the tangent space over any such point are obtained by restricting g to the stabilizer of some point in $\mathfrak{h}$. From the formula for the derivative, it is clear that the image of B is dense in these fibers as well. Likewise, the stabilizer of any point in $J_1(\mathfrak{h})$ is one-dimensional, and again the formula for the second derivative shows that the image of A in the fibers of $J_2(\mathfrak{h})$ over $J_1(\mathfrak{h})$ is dense.

Since Γ_{t_0} is an irreducible finite-to-finite correspondence, A is Zariski dense in Γ_{t_0}. ✠

Note that the action of $\mathrm{GL}_2^+(\mathbb{R}) \times \mathrm{GL}_2^+(\mathbb{R})$ on $\mathfrak{h} \times \mathfrak{h}$ extends canonically to an action on $J_2(\mathfrak{h} \times \mathfrak{h})$ via the jets. Let $\widetilde{\Gamma}_{t_0}$ be a component of $(J_2(j) \times J_2(j))^{-1}\Gamma_{t_0}$ and let H_{t_0} be the setwise stabilizer of $\widetilde{\Gamma}_{t_0}$ in $\mathrm{GL}_2^+(\mathbb{R}) \times \mathrm{GL}_2^+(\mathbb{R})$.

Claim 4.6. *For each $\gamma \in \mathrm{GL}_2^+(\mathbb{Q})$ there is some $\delta \in \mathrm{GL}_2^+(\mathbb{Q})$ with $(\gamma, \delta) \in H_{t_0}$ and likewise for each $\delta \in \mathrm{GL}_2^+(\mathbb{Q})$ there is some $\gamma \in \mathrm{GL}_2^+(\mathbb{Q})$ with $(\gamma, \delta) \in H_{t_0}$.*

Proof of Claim. Let $\gamma \in \mathrm{GL}_2^+(\mathbb{Q})$. We know that the image under $j \times j$ of the graph of the action of γ on $\mathfrak{h}$ is an algebraic correspondence on $\mathbb{A}^1 \times \mathbb{A}^1$ which restricts to a finite-to-finite correspondence from $\chi^{-1}(b_s)$ to itself. The image of this correspondence under $\nabla_2^{-1}(\Gamma_{t_0})$ is thus a finite-to-finite correspondence from $\chi^{-1}(c_s)$ to itself. By Proposition 4.2, this new correspondence must be given by a finite union of Hecke relations which are themselves images under $j \times j$ of graphs of the action of some $\delta_1, \ldots, \delta_n \in \mathrm{GL}_2^+(\mathbb{Q})$.

By Claim 4.5, there is a Zariski dense set of points (x, y) in $\Gamma_{t_0}(\mathbb{C})$ such that for any u with (x, u) in the Hecke correspondence coming from γ, there is some v with (y, v) in the Hecke correspondence coming from δ_i for some $i \le n$ and $(u, v) \in \Gamma_{t_0}(\mathbb{C})$. As this is an algebraic condition, it holds everywhere on Γ_{t_0}. Thus, for any $(x, y) \in \widetilde{\Gamma}_{t_0}$, there is some $i \le n$ and some $\epsilon \in \mathrm{SL}_2(\mathbb{Z})$ such that $(\gamma \cdot x, \epsilon\delta_i \cdot y) \in \widetilde{\Gamma}_{t_0}$. For any given $\delta \in \mathrm{GL}_2^+(\mathbb{Q})$ the set $\widetilde{\Gamma}_{t_0} \cap (\gamma^{-1}, \delta^{-1}) \cdot \widetilde{\Gamma}_{t_0}$ is a closed analytic subset of $\widetilde{\Gamma}_{t_0}$. As $\widetilde{\Gamma}_{t_0}$ is irreducible and may be expressed as the countable union of such intersections we have $\widetilde{\Gamma}_{t_0} = \widetilde{\Gamma}_{t_0} \cap (\gamma^{-1}, \delta^{-1}) \cdot \widetilde{\Gamma}_{t_0}$ for some $\delta \in \mathrm{GL}_2^+(\mathbb{Q})$. That is, $(\gamma, \delta) \in H_{t_0}$. Arguing with the first and second coordinates reversed, we obtain the "likewise" clause. ✠

Let us write $\overline{H}_{t_0}$ for the image of H_{t_0} in $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$. Note that $\overline{H}_{t_0}$ is the setwise stabilizer of $\widetilde{\Gamma}_{t_0}$ in $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$. From Claim 4.6 and the fact that the image of $\mathrm{GL}_2^+(\mathbb{Q})$ is dense in $\mathrm{PSL}_2(\mathbb{R})$ we see that the projection of $\overline{H}_{t_0}$ to each $\mathrm{PSL}_2(\mathbb{R})$ is surjective. Since $\widetilde{\Gamma}_{t_0}$ is a finite-to-finite correspondence between $J_2(\mathfrak{h})$ and $J_2(\mathfrak{h})$, necessarily $\overline{H}_{t_0}$ is a proper subgroup of $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$. Arguing as in [13] we see that $\overline{H}_{t_0}$ is the graph of an automorphism of $\mathrm{PSL}_2(\mathbb{R})$. Since every automorphism of $\mathrm{PSL}_2(\mathbb{R})$ is inner, we can find some $g \in \mathrm{PSL}_2(\mathbb{R})$ for which $\overline{H}_{t_0} = \{(\gamma, \gamma^g) : \gamma \in \mathrm{PSL}_2(\mathbb{R})\}$.

Let us consider some point $(x, y) \in \widetilde{\Gamma}_{t_0}$. Write $\pi(x, y) =: (x_0, y_0) \in \pi(\widetilde{\Gamma}_{t_0}) \subseteq \mathfrak{h} \times \mathfrak{h}$. Let $k \in \mathrm{PSL}_2(\mathbb{R})$ with $k \cdot x_0 = y_0$. We will show that we may take $k = g$.

Let us write the stabilizer of x in $\mathrm{PSL}_2(\mathbb{R})$ as S_x. Note that if $h \in S_x$, then because $(h, h^g) \in H$, we have $(x, h^g \cdot y) = (h, h^g) \cdot (x, y) \in \widetilde{\Gamma}_{t_0}$. Since $\widetilde{\Gamma}_{t_0}$ is a finite-to-finite correspondence, the fiber of $\widetilde{\Gamma}_{t_0}$ above x is finite. Hence, the orbit $S_x^g \cdot y$ is finite. That is, the group $S_x^g \cap S_y$ has finite index in S_x^g, but as this last group is connected, it follows that $S_x^g \leq S_y$. Projecting $\pi : J_2(\mathfrak{h}) \to \mathfrak{h}$ we conclude that $S_{x_0}^g = \pi(S_x^g) \leq \pi(S_y) = S_{y_0} = S_{x_0}^k$. Since the group S_x is self-normalizing, we conclude that $gS_{x_0} = kS_{x_0}$. That is, it would have been possible to take $k = g$. Thus, $\pi(\widetilde{\Gamma}_{t_0})$ is the graph of the action of g on $\mathfrak{h}$.

Since $J_2(j \times j)(\widetilde{\Gamma}_{t_0}) = \Gamma_{t_0}$ is an algebraic variety, necessarily $g \in \mathrm{GL}_2^+(\mathbb{Q})$. As there are only countably many Hecke relations, it follows that one must hold for the generic fiber of Γ. This finishes the proof of the theorem, because it contradicts $b_s \neq c_s$. □

5. Effective finiteness results

In this section, we compute explicit upper bounds on certain intersections of isogeny classes of products of elliptic curves with algebraic varieties. As we explained in the introduction to this paper, the questions we address were posed to us by Mazur in connection with theorems of Orr in line with the Zilber–Pink conjectures. In [22], Orr proves the following theorem.

Theorem 5.1 (Orr, [22, Theorem 1.3]). *Let Λ be the isogeny class of a point $s \in \mathcal{A}_g(\mathbb{C})$, the moduli space of principally polarized abelian varieties of dimension g. Let Z be an irreducible closed subvariety of $\mathcal{A}_g$ such that $Z \cap \Lambda$ is Zariski dense in Z and* $\dim(Z) > 0$. *Then there is a special subvariety $S \subseteq \mathcal{A}_g$ which is isomorphic to a product of Shimura varieties $S_1 \times S_2$ with* $\dim S_1 > 0$ *and such that*

$$Z = S_1 \times Z' \subseteq S$$

for some irreducible closed $Z' \subseteq S_2$.

If Z is a curve, Theorem 5.1 implies that Z must be a weakly special variety. We refer the reader to the original paper for a discussion of special and weakly special varieties, but note that if $S \subseteq \mathcal{A}_g$ is the subvariety corresponding to the abelian varieties expressible as a product of g elliptic curves, then on identifying S with $\mathbb{A}^n$, the special subvarieties of S are the components of varieties defined by equations of the form $F_N(x_i, x_k) = 0$ where F_N is a modular polynomial and $1 \leq i \leq k \leq n$. The weakly special varieties are obtained by allowing in addition equations of the form $x_\ell = \zeta$ for some $\zeta \in \mathbb{A}^1(\mathbb{C})$.

Taking the contrapositive of Theorem 5.1, again for curves, one sees that if $Z \subseteq \mathcal{A}_g$ is an algebraic curve which is not weakly special, then $Z \cap \Lambda$ is finite. One might wonder how large this finite set is. Since Orr's argument depends on ineffective constants coming from the Pila–Wilkie o-minimal counting theorem, it does not yield a method to compute a bound on $Z \cap \Lambda$. Using differential-algebraic methods, we can find explicit upper bounds depending only on geometric data, but we must restrict our attention to transcendental points.

Let us begin with a specific example before giving a general theorem. The general idea we are following is a familiar one in the model theory of fields (e.g. [8, 2]). Take a set A which has some arithmetic meaning (in our case, isogeny classes viewed in a moduli space); we wish to study the intersection of A with varieties. Instead of considering the intersections directly, take the closure $\overline{A}^{\mathrm{Kol}}$ of A in the Kolchin topology, and study intersections of $\overline{A}^{\mathrm{Kol}}$ with varieties. The sacrifice which one makes in moving to the Kolchin closure is offset by a reasonable understanding of the properties of the closure. The advantage is that the object in question is now a variety in the sense of differential-algebraic geometry, so we can apply tools and uniformities from the general theory.

The sort of problem which we are attacking has, on the face of it, nothing to do with differential algebra. This allows us a good deal of freedom in equipping the fields over which we are working with a derivation. Equip $\mathbb{C}$ with a derivation ∂ so that $(\mathbb{C}, \partial)$ is differentially closed and the field of constants of $(\mathbb{C}, \partial)$ is $\mathbb{Q}^{\mathrm{alg}}$. Given a particular isogeny class viewed in the moduli space of elliptic curves, in order to apply the results of the previous sections, we must know that the elements in the class satisfy the differential equation $\chi(x) = a$ for some a in the differential field. This is possible precisely when the element is transcendental.

For background on the theory of moduli spaces of elliptic curves, we refer to [19].

One key tool is an effective finiteness theorem of Hrushovski and Pillay [10] (for which some gaps are filled by León-Sanchez and Freitag [5]).

Theorem 5.2. *Let X be a closed subvariety of $\mathbb{A}^n$, with $\dim(X) = m$, and let $T \subset S \subseteq \tau_\ell \mathbb{A}^n$ be closed subvarieties (not necessarily irreducible) of $\tau_\ell \mathbb{A}^n$. Then the degree of the Zariski closure of $(X, S \smallsetminus T)^\sharp(\mathbb{C}, \partial)$ is at most $\deg(X)^{\ell 2^{m\ell}} \deg(S)^{2^{m\ell}-1}$. In particular, if $(X, S \smallsetminus T)^\sharp(\mathbb{C}, \partial)$ is a finite set, this expression bounds the number of points in that set.*

Remark 5.3. In [16, Dave Marker's differential fields article], there is a non-effective proof of the non-finite cover property in differential fields, that is, the assertion that for a differentially constructible family $\{X_b\}_{b \in B}$ of differentially constructible sets over a differentially closed field $\mathbb{U}$ of characteristic zero there is a bound N such that for any b, if $X_b(\mathbb{U})$ is finite, then $|X_b(\mathbb{U}))| \leq N$.

Remark 5.4. In a recent preprint [1], using the theory of Newton polyhedra, Binyamini establishes much better bounds than those in Theorem 5.2. From this improvement and the remaining finiteness results of the present paper, he deduces sharper estimates than we give here.

Convention 5.5. When we compute degrees of closed subsets of affine space in what follows, we take the *definition* of degree to be the sum of the geometric degrees of the irreducible components.

Our second ingredient is the fact that the fibers of χ are invariant under isogeny. In fact, by Theorem 4.2, we know that for any non-constant a, the equation $\chi(x) = \chi(a)$ holds on the isogeny class of a. (In fact, by a theorem of Buium [2], it defines the Kolchin closure of the isogeny class of a.) The third ingredient is our characterization of algebraic relations across the fibers of χ. We begin with a specific example.

5.1. Automorphisms of the Riemann sphere

Fix some non-identity element of $\mathrm{GL}_2(\mathbb{C})$, which we will write as $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Throughout this section, we let E_x denote an elliptic curve with j-invariant x and we write $x \sim y$ to mean that the elliptic curves E_x and E_y are isogenous.

Fix τ transcendental. The goal here is to establish an upper bound on the number of elliptic curves E_η such that $E_\tau \sim E_\eta$ and $E_{\alpha\cdot\tau} \sim E_{\alpha\cdot\eta}$.

Unless $\alpha \cdot \tau$ is algebraic (a case we will consider separately), since the fibers of χ are closed under isogeny, all such η belong to the set $\{z \in \mathbb{A}^1(\mathbb{C}) : \chi(z) = \chi(\tau)$ and $\chi(\alpha\cdot z)) = \chi(\alpha\cdot\tau)\}$ which is the projection to the first coordinate of the intersection of the graph of α (regarded as a linear fractional transformation) with $\chi^{-1}(\chi(\tau)) \times \chi^{-1}(\chi(\alpha\cdot\tau))$. From Theorem 4.4, because the graph of α is not a modular relation, we know that this intersection is finite. If $\alpha\cdot\tau$ is algebraic, the isogeny class of $\alpha\cdot\tau$ is contained in the set of constants, which is itself a strongly minimal set orthogonal to $\chi^{-1}(\chi(\tau))$, because the former has non-trivial forking geometry while the latter does have trivial forking geometry. Hence, for the same reason this set is finite.

Next, we apply Theorem 5.2. We explain the details in the case where $\alpha \cdot \tau$ is transcendental. The other case is even easier.

Let $X = \mathbb{A}^1$ and let $\ell = 3$, and write $\tau_3\mathbb{A}^1$ in coordinates $(z, \dot{z}, \ddot{z}, \dddot{z})$. Let S be given by the equations $\chi(z) = \chi(\tau)$ and $\chi(\alpha \cdot z) = \chi(\alpha \cdot \tau)$ re-expressed as algebraic equations in $z, \dot{z}, \ddot{z}, \dddot{z}$. By Bézout's theorem, S is of degree at most 36. By Theorem 5.2, $|(X, S)^\sharp| \leq 2^{24} \cdot 36^7 = 2^{38} \cdot 3^{14}$. Hence, given an elliptic curve E_τ with transcendental j-invariant, there are at most $2^{38} \cdot 3^{14}$ elliptic curves E_η in the isogeny class of E_τ for which $E_{\alpha\cdot\eta}$ is in the isogeny class of $E_{\alpha\cdot\tau}$.

Remark 5.6. We will state our results in full generality, but we should point out that in certain special cases, better bounds are available via comparably elementary reasoning.

Proposition 5.7. *Let $C \subseteq \mathbb{A}^2$ be some non-weakly-special irreducible curve defined over $\mathbb{Q}^{\mathrm{alg}}$ and let $P = (a, b)$ be some transcendental point. Then there can be at most one point in the isogeny class of P on C.*

Proof. Without loss of generality, we may assume that $P \in C$. Suppose that $(a', b') \in C$ is distinct from (a, b) but isogenous to (a, b) via isogenies of degrees n and m, respectively. Then P would belong to the intersection of C with the transform of C by the correspondence $\{(x, y), (u, v) : F_n(x, u) = 0 \;\&\; F_m(y, v) = 0\}$. As C is not weakly special, this intersection is zero-dimensional and defined over $\mathbb{Q}^{\mathrm{alg}}$, contradicting the presence of P on the intersection. □

5.2. A general finiteness result

For the remainder of the paper, we will be considering Kolchin closed subvarieties V of $\prod_{i=1}^n \chi^{-1}(a_i)$, so we may assume, without loss of generality, that V is written as $\nabla_3^{-1} S \cap \prod_{i=1}^n \chi^{-1}(a_i)$ for an algebraic subvariety S of $\tau_3(\mathbb{A}^n)$. For the purposes of stating the theorem, we will *define* $\deg(V) := \deg(S)$.

Theorem 5.8. *Let $V \subseteq \mathbb{A}^n$ be a Kolchin closed subset. Let $\mathbf{a} = (a_1, \ldots, a_n)$ be an n-tuple of transcendental points. Let*

$$\mathrm{Iso}(\mathbf{a}) := \{(b_1, \ldots, b_n) \in \mathbb{C}^n : a_i \sim b_i \text{ for } i \leq n\}$$

be the isogeny class of $\mathbf{a}$. *Let W be the Zariski closure of $V \cap \mathrm{Iso}(\mathbf{a})$. Then:*

(1) *W is a finite union of weakly special subvarieties of $\mathbb{A}^n$.*
(2) *The degree of W is bounded by $(6^n \cdot \deg(V))^{2^{3n}-1}$.*
(3) $V \cap \mathrm{Iso}(\mathbf{a}) = W \cap \mathrm{Iso}(\mathbf{a})$.

Proof. Since the fibers of χ are closed under isogeny, $V \cap \mathrm{Iso}(\mathbf{a})$ is contained in $V \cap \prod_{i=1}^n \chi^{-1}(a_i)$. By our orthogonality Theorem 4.4 and our description of dependence within fibers from Theorem 4.2, $V \cap \prod_{i=1}^n \chi^{-1}(a_i)$ is equal to $\bigcup_{j=1}^m X_j \cap \prod_{i=1}^n \chi^{-1}(a_i)$ where each X_i is an irreducible weakly special variety. It is easy to see that if an irreducible weakly special variety X meets $\mathrm{Iso}(\bar{a})$ non-trivially, then $X \cap \mathrm{Iso}(\mathbf{a})$ is Zariski dense in X. Hence W, the Zariski closure of $V \cap \mathrm{Iso}(\mathbf{a})$, is equal to $\bigcup_{j \in J} X_j$ for some $J \subseteq \{1, \ldots, m\}$.

Write V as $(\mathbb{A}^n, \Xi_{\mathbf{a}} \cap S)^\sharp$ where $\Xi_{\mathbf{a}}$ is given by the equations $\chi(x_j) = \chi(a_j)$ for $j \leq n$ in which $\chi(x_j)$ is re-expressed as a rational function in $x_j, \dot{x}_j, \ddot{x}_j$ and $\dddot{x}_j$. Examining the explicit equations for χ, one sees that $\deg(\Xi_{\mathbf{a}}) = 6^n$. By computing degrees and applying Theorem 5.2, the degree of the Zariski closure of V is bounded by $(6^n \cdot \deg(V))^{2^{3n}-1}$. As W is a union of some of the components of this Zariski closure, this number also bounds $\deg(W)$. ◻

If V is actually an algebraic variety, then we have $S = \tau_3 V$, so that $\deg(V)$ as defined with $\deg(S)$ is the same as $\deg(V)$ as usually defined. Thus, we obtain:

Corollary 5.9. *Let $V \subseteq \mathbb{A}^n$ be a Zariski closed subset. Let $\mathbf{a} = (a_1, \ldots, a_n)$ be an n-tuple of transcendental points. Let W denote the Zariski closure of $V \cap \mathrm{Iso}(\mathbf{a})$. Then W is a finite union of weakly special subvarieties. The degree of W is at most $(6^n \cdot \deg(V))^{2^{3n}-1}$.*

Remark 5.10. As we noted in Remark 5.4, using his improvement on Theorem 5.2 and our finiteness theorem, Binyamini has established much better estimates. In particular, he shows that in Corollary 5.9 the bound may be taken to be *singly* rather than *doubly* exponential in n.

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