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## Quaternionic covers and monodromy of the Kontsevich–Zorich cocycle in orthogonal groups

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**Abstract.** We give an example of a Teichmüller curve which contains, in a factor of its monodromy, a group which was not observed before. Namely, it has Zariski closure equal to the group  $SO^*(6)$  in its standard representation; up to finite index, this is the same as  $SU(3, 1)$  in its second exterior power representation.

The example is constructed using origamis (i.e. square-tiled surfaces). It can be generalized to give monodromy inside the group  $SO^*(2n)$  for all  $n$ , but in the general case the monodromy might split further inside the group.

Also, we take the opportunity to compute the multiplicities of representations in the  $(0, 1)$  part of the cohomology of regular origamis, answering a question of Matheus–Yoccoz–Zmiaikou.

**Keywords.** Translation surfaces, monodromy, orthogonal groups, square-tiled surfaces, moduli spaces of abelian differentials, Teichmüller flow, Hodge bundle, Kontsevich–Zorich cocycle

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## 1. Introduction

A *translation surface* is the data  $(M, \omega)$  of a compact Riemann surface  $M$  of genus  $g \geq 1$  and a non-zero Abelian differential (holomorphic 1-form)  $\omega$  on  $M$ .

The moduli spaces of translation surfaces possess a natural  $SL(2, \mathbb{R})$ -action whose dynamical features play a key role in some applications to dynamical systems and algebraic geometry such as the study of interval exchange transformations and translation flows, and the classification of commensurability classes of ball quotients introduced by Deligne and Mostow. See, for example, the works of Masur [15], Veech [20], Zorich [22], Forni [10], Delecroix–Hubert–Lelièvre [4], Athreya–Eskin–Zorich [1], and Kappes–Möller [14] for some illustrations.

A major actor in the investigation of the dynamics of the  $SL(2, \mathbb{R})$ -action on moduli spaces of translation surfaces is the so-called Kontsevich–Zorich cocycle (KZ cocycle for short): for instance, the properties of the KZ cocycle are a crucial ingredient in the celebrated recent work of Eskin–Mirzakhani [7] towards the classification of  $SL(2, \mathbb{R})$ -invariant measures on moduli spaces of translation surfaces.

A detailed study of the KZ cocycle was initiated by Forni [10, 11]. These works are concerned with formulas for exponents, spectral gap and non-triviality of exponents, as well as applications to dynamics on translation surfaces. Later, in [8] it was proved that the KZ cocycle is semisimple and its decomposition respects the Hodge structure. Using this property, in [9], an analysis of possible groups appearing in the Zariski closure of the monodromy (or the algebraic hull) was done. Up to finite index and compact factors, the list of groups and representations is:

- (i)  $Sp(2d, \mathbb{R})$  in the standard representation;
- (ii)  $SU_{\mathbb{C}}(p, q)$  in the standard representation;
- (iii)  $SU_{\mathbb{C}}(p, 1)$  in an exterior power representation;
- (iv)  $SO^*(2n)$  in the standard representation (see [13] and §4.2 for a description);
- (v)  $SO_{\mathbb{R}}(n, 2)$  in a spin representation.

In other words, this list of five representations *suffices* to account for all possibilities for the monodromy group of the KZ cocycle.

Nevertheless, our understanding of the KZ cocycle is not completely satisfactory yet: for instance, while the monodromy groups in items (i) and (ii) above appear in several families of examples in the literature (see, e.g., Eskin, Kontsevich, and Zorich [6] and McMullen [17]), Question 5.5 of [9] asks whether the monodromy groups in items (iii), (iv) and (v) *actually* occur in the context of the KZ cocycle.<sup>1</sup>

In this note, we give the following partial answer to this question.

**Theorem 1.1.** *There exists an origami  $\tilde{L}$  of genus 11 such that the restriction of the KZ cocycle over  $SL(2, \mathbb{R}) \cdot \tilde{L}$  to a certain  $SL(2, \mathbb{R})$ -irreducible piece of the corresponding semisimple decomposition acts (modulo finite-index subgroups) through a Zariski dense subgroup of  $SO^*(6)$  in its standard representation.*

**Remark 1.2.** This result says that a particular case of item (iv) above occurs for the KZ cocycle over the  $SL(2, \mathbb{R})$ -action on moduli spaces of translation surfaces. In fact, we have an exceptional isomorphism of real Lie algebras  $\mathfrak{so}^*(6) \cong \mathfrak{su}_{3,1}$ . This can be seen, for instance, by comparing the Satake diagrams at the back of [21].

Moreover, letting  $\mathfrak{so}^*(6)$  act in its standard representation identifies it with  $\mathfrak{su}_{3,1}$  acting in the second exterior power of its standard representation. The underlying vector space is  $\mathbb{C}^6$  viewed as  $\mathbb{R}^{12}$ .

In particular, up to finite center, the groups  $SO^*(6)$  and  $SU(3, 1)$  are isomorphic. This means that the example from [Theorem 1.1](#) is, in fact, also an example of  $SU(3, 1)$  acting in its second exterior power (thus, of item (iii) in the list above).

**Remark 1.3.** In principle, the computations from [Section 3](#) as well as the discussion in [Section 4](#) suggest how to find further examples of pieces of the monodromy in  $SO^*(2n)$  for any  $n$ . Namely, one can first look for quaternionic representations occurring in the cohomology of some regular origami (or more general family of translation surfaces with symmetries). The multiplicity  $n$  of the representation will constrain the monodromy to lie inside  $SO^*(2n)$ . One then has to check that the monodromy is irreducible, i.e. the cocycle does not split further. Note that the computations of Matheus, Yoccoz, and Zmiaikou [16] give rise to a large number of explicit examples.

On the other hand, the question of finding examples of  $SL(2, \mathbb{R})$ -orbits in moduli spaces of translation surfaces whose associated KZ cocycles have monodromy groups with non-compact factors realizing all cases in items (iii) or (v) seems more challenging in our opinion (cf. [footnote 1](#)).

Regarding the multiplicities of representations for the cohomology of regular origamis, we have the next result.

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<sup>1</sup> It is known that each item of this list can be realized *abstractly* as monodromy group of variations of Hodge structures over certain families of Riemann surfaces and Abelian varieties (see [9] for more explanations). However, it is not clear how to convert these abstract realizations into non-compact factors of the KZ cocycle over the closure of some  $SL(2, \mathbb{R})$ -orbit in the moduli space of translation surfaces.

**Theorem 1.4.** *Let  $S$  be a regular origami, determined by a group  $G$  and the two generators  $h, v \in G$  (see Section 2 for definitions). Let  $c := hvh^{-1}v^{-1}$  be their commutator, and let  $\pi$  be a complex irreducible representation of  $G$ . Then in the cohomology group  $H^1(S, \mathbb{C})$ , the representation  $\pi$  appears (see (3.3)) with multiplicity*

$$2\delta_{\pi=\text{triv}} + \dim \pi - m_0.$$

Here  $\delta_{\pi=\text{triv}}$  is 1 if  $\pi$  is trivial and zero otherwise, while  $m_0$  denotes the dimension of the vector subspace in  $\pi$  fixed by the element  $c$ .

In the cohomology group  $H^{0,1}(S)$ , the representation  $\pi$  appears (see (3.5)) with multiplicity

$$\delta_{\pi=\text{triv}} + \frac{1}{2}(\dim \pi - m_0) + \frac{1}{N} \sum_{i=1}^{N-1} \left(i - \frac{N}{2}\right) m_i.$$

Here  $m_i$  denotes the dimension of the space in the representation  $\pi$  on which the element  $c$  acts with eigenvalue  $\exp(2\pi\sqrt{-1}i/N)$ , while  $N$  is the order of  $c$  in  $G$ .

For regular origamis, the previous theorem answers a question posed in [16, Remark 5.13].

**Paper outline.** In Section 2 we recall some basic facts about origamis. Next, in Section 3 we explicitly compute the multiplicities of representations for regular origamis. This extends the computations in [16] to the multiplicities in the  $(1, 0)$  and  $(0, 1)$  parts of the Hodge decomposition. In Section 4, we discuss the general structure of (semisimple) local systems with symmetries. We also include a more detailed description of the group  $SO^*(2n)$ .

The construction of the example and the proof of Theorem 1.1 occupy the remainder of the note. We introduce in Section 5 the origami  $\tilde{L}$  (the main object of this note), and we determine the Lyapunov spectrum of the KZ cocycle over  $SL(2, \mathbb{R}) \cdot \tilde{L}$ . In particular, the structure of the Lyapunov spectrum allows us to show that the monodromy group of the KZ cocycle over  $SL(2, \mathbb{R}) \cdot \tilde{L}$  has a non-compact factor isomorphic to either  $SO^*(4)$  or  $SO^*(6)$  (modulo compact and finite-index subfactors; see Theorem 5.4 below). Finally, in Section 6 we complete the proof of Theorem 1.1 by ruling out the possibility in Theorem 5.4 of a  $SO^*(4)$  monodromy through the computation of certain matrices<sup>2</sup> of the KZ cocycle over  $SL(2, \mathbb{R}) \cdot \tilde{L}$ .

## 2. Preliminaries

In this section, we recall some useful facts about origamis (square-tiled surfaces) and the  $SL(2, \mathbb{R})$ -action on the moduli spaces of translation surfaces. For this sake, we will loosely follow the exposition in [16] for a large portion of this section.

<sup>2</sup> That is, the actions on the homology group  $H_1(\tilde{L}, \mathbb{R})$  of certain affine homeomorphisms of  $\tilde{L}$ .

2.1. *Origamis*

An *origami* (or *square-tiled surface*) is a translation surface  $X = (M, \omega)$  such that the Riemann surface  $M$  is obtained by a finite covering  $\pi : M \rightarrow \mathbb{T}^2$  of the torus  $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$  which is unramified off  $0 \in \mathbb{T}^2$ , and the (non-zero) Abelian differential  $\omega$  on  $M$  is the pullback  $\omega = \pi^*(dz)$  of the Abelian differential  $dz$  on  $\mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z}) \simeq \mathbb{T}^2$ .

Alternatively, an origami is a translation surface  $X = (M, \omega)$  determined by a pair of permutations  $h, v \in S_N$  of the set  $\{1, \dots, N\}$  through the following recipe. We take  $N$  copies  $sq_1, \dots, sq_N$  of the unit square  $[0, 1]^2 \subset \mathbb{R}^2$ , and, for each  $n \in \{1, \dots, N\}$ , we glue by translation the rightmost vertical side of  $sq_n$  to the leftmost vertical side of  $sq_{h(n)}$ , and the topmost horizontal side of  $sq_n$  to the bottommost horizontal side of  $sq_{v(n)}$ . In this way, after performing these identifications, we obtain a Riemann surface  $M$  that is naturally equipped with an Abelian differential  $\omega$  given by the pullback of  $dz$  on each  $sq_n$  (observe that this makes sense because the identifications are given by translations on  $\mathbb{R}^2$ ). Note that the translation surface  $X = (M, \omega)$  associated to a pair of permutations  $h, v \in S_N$  is connected if and only if the group generated by  $h$  and  $v$  acts transitively on  $\{1, \dots, N\}$ .

These two definitions of origamis are related as follows. On the one hand, given a translation surface  $X = (M, \omega)$  associated to a pair of permutations  $h, v \in S_N$ , the natural projection from each  $S_n, n = 1, \dots, N$ , to  $\mathbb{T}^2$  induces a finite covering  $\pi : M \rightarrow \mathbb{T}^2$  which is unramified off  $0 \in \mathbb{T}^2$  and such that  $\omega = \pi^*(dz)$ . On the other hand, given a finite covering  $\pi : M \rightarrow \mathbb{T}^2$  of degree  $N$  unramified off  $0 \in \mathbb{T}^2$ , the closures in  $M$  of each of the  $N$  connected components of  $\pi^{-1}((0, 1)^2)$  determine  $N$  copies of the unit square  $[0, 1]^2$ . After labeling  $sq_1, \dots, sq_N$  the connected components of  $\pi^{-1}((0, 1)^2)$  in some arbitrary way, we obtain a pair of permutations  $h, v \in S_N$  such that, for each  $n = 1, \dots, N$ , the neighbor to the right, resp. on the top, of  $sq_n$  is  $sq_{h(n)}$ , resp.  $sq_{v(n)}$ .

**Remark 2.1.** Observe that the particular choice of numbering of the connected components of  $\pi^{-1}((0, 1)^2)$  is not relevant from the point of view of translation surfaces: in other terms, by replacing a pair of permutations  $h, v \in S_N$  by another pair  $\phi h \phi^{-1}, \phi v \phi^{-1} \in S_N$  of permutations that are simultaneously conjugate to  $h, v$ , we obtain the same origami  $X = (M, \omega)$ .

2.2. *SL(2, ℝ)-action on strata of moduli spaces of translation surfaces*

Let  $X = (M, \omega)$  be a translation surface of genus  $g \geq 1$ . Since  $\omega$  is a non-zero Abelian differential (by definition), the set  $\Sigma$  of zeroes of  $\omega$  is finite, and by the Riemann–Roch theorem the list  $\kappa = (k_1, \dots, k_\sigma), \sigma = \#\Sigma$ , of the orders of the zeroes of  $\omega$  satisfies

$$\sum_{l=1}^{\sigma} k_l = 2g - 2.$$

Given a finite subset  $\Sigma \subset M$  and a list  $\kappa = (k_1, \dots, k_\sigma), \sigma = \#\Sigma$ , of natural numbers satisfying  $\sum_{l=1}^{\sigma} k_l = 2g - 2$ , the corresponding *stratum*  $\mathcal{H}(\kappa)$ , resp.  $\mathcal{T}(\kappa)$ , of the *moduli space*, resp. *Teichmüller space*, of translation surfaces of genus  $g$  is the space of orbits

of translation surfaces  $X = (M, \omega)$  such that  $\omega|_{\Sigma} = 0$  and the list of orders of zeroes of  $\omega$  is  $\kappa$  under the natural action of the group  $\text{Homeo}^+(M, \Sigma, \kappa)$ , resp.  $\text{Homeo}_0(M, \Sigma, \kappa)$ , of orientation-preserving homeomorphisms of  $M$  that preserve  $\Sigma$  and  $\kappa$ , resp. the connected component in  $\text{Homeo}^+(M, \Sigma, \kappa)$  of the identity element. Note that  $\mathcal{H}(\kappa) = \mathcal{T}(\kappa)/\Gamma(M, \Sigma, \kappa)$  where  $\Gamma(M, \Sigma, \kappa) := \text{Homeo}^+(M, \Sigma, \kappa)/\text{Homeo}_0(M, \Sigma, \kappa)$  is the so-called *mapping class group* of isotopy classes of orientation-preserving homeomorphisms of  $M$  respecting  $\Sigma$  and  $\kappa$ .

The group  $SL(2, \mathbb{R})$  acts on  $\mathcal{H}(\kappa)$  as follows. A translation surface  $X = (M, \omega) \in \mathcal{H}(\kappa)$  is determined by the so-called *translation charts* provided by the local primitives of  $\omega$  on  $M - \Sigma$ . Given  $g \in SL(2, \mathbb{R})$  and  $X = (M, \omega) \in \mathcal{H}(\kappa)$ , we define  $g(X)$  as the translation surface obtained by post-composition of the translation charts of  $X = (M, \omega)$  with  $g$ . In this setting, the action of the diagonal subgroup  $g_t = \text{diag}(e^t, e^{-t})$  of  $SL(2, \mathbb{R})$  on strata of moduli spaces of translation surfaces is called *Teichmüller geodesic flow*.

### 2.3. Veech groups and affine homeomorphisms

The  $SL(2, \mathbb{R})$ -orbits of origamis are particular examples of closed  $SL(2, \mathbb{R})$ -orbits in the strata of the moduli spaces of translation surfaces.

The geometry of the  $SL(2, \mathbb{R})$ -orbit of an origami  $(M, \omega)$  admits the following simple description. Let  $SL(M, \omega)$  be the *Veech group* of  $(M, \omega)$ , i.e., the stabilizer of  $(M, \omega)$  with respect to the  $SL(2, \mathbb{R})$ -action. The Veech group  $SL(M, \omega)$  of an origami  $(M, \omega)$  is commensurable to  $SL(2, \mathbb{Z})$ . In particular, if  $(M, \omega)$  is an origami, the hyperbolic surface  $\mathbb{H}/SL(M, \omega)$  has finite area. As it turns out, the  $SL(2, \mathbb{R})$ -orbit  $SL(2, \mathbb{R}) \cdot (M, \omega)$  of an origami  $(M, \omega)$  is naturally isomorphic to the unit cotangent bundle  $SL(2, \mathbb{R})/SL(M, \omega)$  of a finite-area hyperbolic surface.

The Veech group  $SL(M, \omega)$  relates to the flat geometry of translation surfaces via the notion of *affine homeomorphisms*. The group  $\text{Aff}(M, \omega)$  of affine homeomorphisms of  $X = (M, \omega)$  consists of all orientation-preserving homeomorphisms of  $M$  (respecting  $\Sigma$ ) whose expressions in translation charts (of local primitives of  $\omega$  on  $M - \Sigma$ ) are affine transformations of  $\mathbb{R}^2$ . By extracting the linear part of these affine transformations, we obtain a homomorphism from  $\text{Aff}(X)$  to  $SL(2, \mathbb{R})$  whose kernel  $\text{Aut}(X)$  is the so-called *automorphism group* of  $X = (M, \omega)$ . Moreover, this homomorphism fits into an exact sequence

$$\{\text{Id}\} \rightarrow \text{Aut}(X) \rightarrow \text{Aff}(X) \rightarrow SL(X) \rightarrow \{\text{Id}\}.$$

In particular, the Veech group  $SL(X)$  is the subgroup of  $SL(2, \mathbb{R})$  capturing all linear parts of affine homeomorphisms of  $X$ .

The group  $\text{Aff}(X)$  of affine homeomorphisms is the stabilizer in the mapping class group  $\Gamma(M, \Sigma, \kappa)$  of the  $SL(2, \mathbb{R})$ -orbit of  $X = (M, \omega)$ .

### 2.4. KZ cocycle over the $SL(2, \mathbb{R})$ -orbit of an origami

The *Kontsevich–Zorich cocycle* over the  $SL(2, \mathbb{R})$ -orbit of a translation surface  $X = (M, \omega)$  is the quotient of the trivial cocycle

$$SL(2, \mathbb{R}) \cdot X \times H_1(M, \mathbb{R}) \rightarrow SL(2, \mathbb{R}) \cdot X \times H_1(M, \mathbb{R})$$

(over the tautological dynamics of  $SL(2, \mathbb{R})$  on  $SL(2, \mathbb{R}) \cdot X$ ) by the natural action (on both factors of the trivial cocycle) of the stabilizer  $\text{Aff}(X)$  in the mapping class group of the  $SL(2, \mathbb{R})$ -orbit of  $X$  in a stratum of the moduli space of translation surfaces. Equivalently, the KZ cocycle over  $SL(2, \mathbb{R}) \cdot X \simeq SL(2, \mathbb{R})/SL(X)$  acts on  $H_1(M, \mathbb{R})$  via appropriate elements of  $\text{Aff}(X)$ .

Suppose from now on that  $X = (M, \omega)$  is an origami and let  $\pi : M \rightarrow \mathbb{T}^2$  be the associated finite ramified covering. In this case, the KZ cocycle respects the following decomposition into ( $\text{Aff}(X)$ -invariant) subspaces defined over  $\mathbb{Q}$ :

$$H_1(M, \mathbb{R}) = H_1^{\text{st}}(M, \mathbb{R}) \oplus H_1^{(0)}(M, \mathbb{R}),$$

where  $H_1^{(0)}(M, \mathbb{R})$  is the kernel of  $\pi_* : H_1(M, \mathbb{R}) \rightarrow H_1(\mathbb{T}^2, \mathbb{R})$ , and  $H_1^{\text{st}}(M, \mathbb{R})$  is the symplectic orthogonal of  $H_1^{(0)}(M, \mathbb{R})$  with respect to the usual (symplectic) intersection form.

The subspace  $H_1^{\text{st}}(M, \mathbb{R})$  is naturally identified with  $\mathbb{R}^2 \simeq H_1(\mathbb{T}^2, \mathbb{R})$  because it is generated by the absolute homology classes  $\sigma = \sum_s \sigma_s$  and  $\zeta = \sum_s \zeta_s$  where  $s$  runs through the set  $Sq(X)$  of squares of  $X$  (i.e., the closures of connected components of  $\pi^{-1}((0, 1)^2)$ ), and  $\sigma_s$ , resp.  $\zeta_s$ , is the bottommost, resp., leftmost, horizontal, resp. vertical, side of  $s$ . Furthermore, under this identification, the affine group  $\text{Aff}(X)$  acts on  $H_1^{\text{st}}(M, \mathbb{R}) \simeq \mathbb{R}^2$  via the composition of the homomorphism  $\text{Aff}(X) \rightarrow SL(X)$  and the standard action of  $SL(X) \subset SL(2, \mathbb{R})$  on  $\mathbb{R}^2$ . In other terms, the monodromy group of the restriction of the KZ cocycle to  $H_1^{\text{st}}(M, \mathbb{R})$  is  $SL(2, \mathbb{R})$  (up to finite index).

Denote by  $G = \text{Aut}(X)$  the (finite) group of automorphisms of the origami  $X = (M, \omega)$ . Then the subspaces  $H_1^{\text{st}}(M, \mathbb{R})$  and  $H_1^{(0)}(M, \mathbb{R})$  are  $G$ -modules. In particular, we can write

$$H_1^{(0)}(M, \mathbb{R}) = \bigoplus_{a \in \text{Irr}_{\mathbb{R}}(G)} W_a$$

where  $\text{Irr}_{\mathbb{R}}(G)$  is the set of (isomorphism classes of) irreducible representations of  $G$ , and  $W_a$  is the isotypical component of  $a \in \text{Irr}_{\mathbb{R}}(G)$  in the  $G$ -module  $H_1^{(0)}(M, \mathbb{R})$ .

The isotypical components  $W_a$ ,  $a \in \text{Irr}_{\mathbb{R}}(G)$ , of  $H_1^{(0)}(M, \mathbb{R})$  are permuted by the group  $\text{Aff}(X)$  of affine homeomorphisms of the origami  $X = (M, \omega)$ . More precisely,  $\text{Aff}(X)$  acts on  $G = \text{Aut}(X)$  by conjugation, that is, we have a natural homomorphism  $\text{Aff}(X) \rightarrow \text{Aut}(G)$ . Next, we recall that one has a canonical homomorphism  $\text{Aut}(G) \rightarrow \text{Out}(G) := \text{Aut}(G)/\text{Inn}(G)$  where  $\text{Inn}(G)$  is the (normal) subgroup of inner automorphisms of  $G$ . Furthermore,  $\text{Out}(G)$  acts on  $\text{Irr}_{\mathbb{R}}(G)$ . In this context, the composition  $\text{Aff}(X) \rightarrow \text{Aut}(G) \rightarrow \text{Out}(G)$  of the two homomorphisms described above induces an action of  $\text{Aff}(X)$  on  $\text{Irr}_{\mathbb{R}}(G)$  such that the elements of  $\text{Aff}(X)$  permute the isotypical components  $W_a$  according to this action (i.e.,  $A(W_a) = W_{A \cdot a}$  for every  $A \in \text{Aff}(X)$  and  $a \in \text{Irr}_{\mathbb{R}}(G)$ ).

Denote by  $\text{Aff}_{**}(X)$  the kernel of the homomorphism  $\text{Aff}(X) \rightarrow \text{Aut}(G)$ . By definition,  $\text{Aff}_{**}(X)$  is a finite-index subgroup of  $\text{Aff}(X)$  whose elements fix the isotypical components  $W_a$  of  $H_1^{(0)}(M, \mathbb{R})$ , and moreover the restrictions of these elements to each  $W_a$  are  $G$ -module automorphisms (because the elements of  $\text{Aff}_{**}(X)$  commute with

$G = \text{Aut}(X)$ ). Since the isotypical components  $W_a$  are mutually orthogonal with respect to the symplectic intersection form on  $H_1(M, \mathbb{R})$  and the restriction of that form to each  $W_a$  is also a symplectic form, we deduce the following restriction on the monodromy group of the KZ cocycle: the elements of  $\text{Aff}_{**}(X)$  act via a subgroup of the product of the groups  $Sp(W_a)$  of automorphisms of the  $G$ -modules  $W_a$  preserving the symplectic intersection form.

**Remark 2.2.** In principle, the action of  $\text{Aff}_{**}(X)$  in a given isotypical component  $W_a$  might not be irreducible. In this case, Möller [18] (see also [8] for a more general version) showed the following “Deligne’s semisimplicity theorem”:  $W_a$  can be further decomposed into  $\text{Aff}_{***}(X)$ -irreducible symplectic subspaces (respecting the so-called Hodge structure of  $H^1(M, \mathbb{R})$ ), where  $\text{Aff}_{***}(X)$  is an appropriate finite-index subgroup of  $\text{Aff}_{**}(X)$ .

**Definition 2.3.** The *monodromy* of the Kontsevich–Zorich cocycle is the image of the group  $\text{Aff}(X)$  inside the product of the symplectic groups  $Sp(W_a)$ . Since the groups  $\text{Aff}_{**}(X)$  and  $\text{Aff}_{***}(X)$  are finite-index in  $\text{Aff}(X)$ , they map with finite index to the monodromy.

Therefore, the connected component of the Zariski closure of the monodromy is independent of which group in the family  $\text{Aff}_\bullet(X)$  we consider for the action.

The long-term behavior of the KZ cocycle (action of  $\text{Aff}(X)$  on homology) is described by its *Lyapunov spectrum*. More precisely, the  $SL(2, \mathbb{R})$ -orbit of an origami  $X = (M, \omega)$  is isomorphic to  $SL(2, \mathbb{R})/SL(X)$  where  $SL(X)$  is commensurable to  $SL(2, \mathbb{Z})$ . Thus,  $SL(2, \mathbb{R}) \cdot X$  carries a unique  $SL(2, \mathbb{R})$ -invariant ergodic probability measure  $\mu$ . In this setting, the Oseledets theorem says that, for  $\mu$ -almost every  $x \in SL(2, \mathbb{R}) \cdot X$ , the sequence  $(A_n)_{n \in \mathbb{Z}} \subset \text{Aff}_{**}(X)$  of matrices of the KZ cocycle (depending on  $x$ ) along the orbit  $g_t \cdot x$  of  $x$  under the Teichmüller geodesic flow  $g_t = \text{diag}(e^t, e^{-t})$  has the following asymptotic behavior: for all  $v \in H_1(M, \mathbb{R}) - \{0\}$ , we have well-defined quantities

$$\lim_{n \rightarrow \pm\infty} \frac{\log \|A_n(v)\|}{\log \|A_n\|} = \theta$$

independently of the choice of the norm  $\|\cdot\|$  on  $H_1(M, \mathbb{R})$ ; furthermore, the collection of numbers  $\theta$  obtained in this way is a finite list of numbers (with multiplicities)  $\{\theta_1 \geq \dots \geq \theta_{2g}\}$  of cardinality  $2g = \dim_{\mathbb{R}} H_1(M, \mathbb{R})$  which is independent of  $x$ . In the literature, the list  $\{\theta_1 \geq \dots \geq \theta_{2g}\}$  is called the Lyapunov spectrum, the quantities  $\theta_\alpha$  are called the *Lyapunov exponents*, and

$$E(\theta_\alpha, x) := \left\{ v \in H_1(M, \mathbb{R}) - \{0\} \mid \lim_{n \rightarrow \pm\infty} \frac{\log \|A_n(v)\|}{\log \|A_n\|} = \theta_\alpha \right\}$$

are called the *Oseledets subspaces*.<sup>3</sup>

The Lyapunov spectrum of the KZ cocycle is symmetric with respect to the origin, i.e.,  $\theta_{2g-\alpha+1} = \theta_\alpha$  for all  $\alpha = 1, \dots, g$ . Indeed, this is a consequence of the fact that the

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<sup>3</sup> These subspaces depend measurably on  $x$ .



KZ cocycle is symplectic (the action of  $\text{Aff}(X)$  on  $H_1(M, \mathbb{R})$  preserves the symplectic intersection form). In particular, the Lyapunov spectrum of the KZ cocycle is always determined by its non-negative Lyapunov exponents  $\theta_1 \geq \dots \geq \theta_g (\geq 0)$ .

It is known that the top Lyapunov exponent  $\theta_1$  equals 1, it is simple (i.e.,  $\theta_1 > \theta_2$ ), and it comes from the action of the Teichmüller geodesic flow  $g_t$  on  $H_1^{\text{st}}(M, \mathbb{R})$ .

On the other hand, the remaining Lyapunov exponents might exhibit multiplicities depending on the “symmetries” of  $X = (M, \omega)$ . For example, for any  $A \in \text{Aff}(X)$ , the Lyapunov spectra of the (restrictions of the KZ cocycle to the) isotypical components  $W_a$  and  $W_{A \cdot a}$ ,  $a \in \text{Irr}_{\mathbb{R}}(G)$ , are the same. Moreover, for each isotypical component  $W_a$ , the associated Oseledets subspaces  $W_a(\theta, x) = W_a \cap E(\theta, x)$  are  $G$ -modules. In particular, if the representation  $a \in \text{Irr}_{\mathbb{R}}(G)$  is complex, resp. quaternionic, then each Lyapunov exponent of  $W_a$  has multiplicity at least 2, resp. 4.

Closing this section, we recall that Eskin–Kontsevich–Zorich [5] proved the following explicit formula for the sum of the non-negative Lyapunov exponents of the KZ cocycle over the  $SL(2, \mathbb{R})$ -orbit of an origami. Let  $X = (M, \omega) \in \mathcal{H}(\kappa)$ ,  $\kappa = (k_1, \dots, k_\sigma)$ , be an origami of genus  $g$ . Consider the  $SL(2, \mathbb{Z})$ -orbit of  $X$ ; this is a finite set<sup>4</sup> of origamis that one can compute by successively applying the generators  $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  and  $S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  of  $SL(2, \mathbb{Z})$  to the origamis. In terms of pairs of permutations  $h, v \in S_N$  associated to origamis,  $T$  and  $S$  act as  $T(h, v) = (h, vh^{-1})$  and  $S(h, v) = (hv^{-1}, v)$ , and this allows<sup>5</sup> one to calculate the  $SL(2, \mathbb{Z})$ -orbits of origamis. In this context, the sum of the non-negative Lyapunov exponents  $1 = \theta_1 \geq \dots \geq \theta_g$  of the KZ cocycle over  $SL(2, \mathbb{R}) \cdot X$  is

$$\theta_1 + \dots + \theta_g = \frac{1}{12} \sum_{l=1}^{\sigma} \frac{k_l(k_l + 2)}{k_l + 1} + \frac{1}{\#SL(2, \mathbb{Z}) \cdot X} \sum_{\substack{Y \in SL(2, \mathbb{Z}) \cdot X \\ c \text{ is a cycle of } h_Y}} \frac{1}{\text{length of } c}$$

where  $(h_Y, v_Y)$  is a pair of permutations associated to the origami  $Y$ .

### 3. Multiplicities of representations for regular origamis

This section calculates the multiplicities of representations that occur in the cohomology of regular origamis. For the topological case, these calculations were done by Matheus, Yoccoz, and Zmiaikou [16]. We also compute the multiplicities for the holomorphic 1-forms (cf. [16, Remark 5.13]).

Two tools appear in the computation. One is the Lefschetz fixed point formula and its holomorphic version. The second is the Frobenius reciprocity formula—it reduces the calculation to a cyclic subgroup of the origami symmetries.

<sup>4</sup> The cardinality of  $SL(2, \mathbb{Z}) \cdot X$  is the index of the Veech group  $SL(X)$  in  $SL(2, \mathbb{Z})$  when the origami  $X = (M, \omega)$  is reduced (i.e., the covering  $\pi : M \rightarrow \mathbb{T}^2$  such that  $\pi^*(dz) = \omega$  does not factor through a cover  $p : \mathbb{T}^2 \rightarrow \mathbb{T}^2$  of degree  $> 1$ ).

<sup>5</sup> Here, one has to keep in mind that two pairs of permutations give the same origami when they are simultaneously conjugate, that is,  $(h, v)$  and  $(\phi h \phi^{-1}, \phi v \phi^{-1})$  correspond to the same origami.

**Setup.** Calculations will take the algebraic point of view on origamis (cf. §2.1). Let  $G$  be a finite group generated by two elements  $h, v \in G$  and let their commutator be  $c := [h, v] = hvh^{-1}v^{-1}$ . Let  $S$  be the corresponding regular origami. The squares of  $S$  correspond to the elements of  $G$ , and there are two edges in the glued surface per element of  $G$ .

On the origami, the ramification points (i.e. vertices of the square) are in natural bijection with the coset space  $G/\langle c \rangle$ . Here  $\langle c \rangle$  denotes the cyclic subgroup of  $G$  generated by  $c$ . If we denote its cardinality by  $N := |\langle c \rangle|$ , the genus of the surface satisfies

$$\text{genus}(S) = \frac{1}{2} \left( |G| - \frac{|G|}{N} \right) + 1.$$

### 3.1. Preliminaries from representation theory

We now recall some necessary facts from the representation theory of finite groups. This material is available in Serre's textbook [19, Ch. 7].

Consider two finite groups  $H \subset G$ . For two complex-valued functions  $\phi, \psi$  on  $G$ , define

$$\langle \phi, \psi \rangle_G = \frac{1}{|G|} \sum_{g \in G} \phi(g) \overline{\psi(g)}.$$

A similar definition applies to  $\langle \cdot, \cdot \rangle_H$ .

For  $G$ -representations  $V$  and  $\pi$  (with  $\pi$  irreducible), denoting their characters by  $\chi_V$  and  $\chi_\pi$  we have

$$\langle \chi_\pi, \chi_V \rangle_G = \dim \text{Hom}_G(\pi, V).$$

In other words, the product of characters gives the multiplicity of  $\pi$  in  $V$ .

**Restriction.** Given a representation  $\pi$  of  $G$ , let  $\text{Res}_H^G(\pi)$  denote the representation of  $H$  obtained by restricting  $\pi$  to  $H$ . If the character of  $\pi$  is  $\chi_\pi$ , let  $\text{Res}_H^G \chi_\pi$  denote the character of the restricted representation. We will often omit the notation of  $G$  and  $H$  from  $\text{Res}$ , since we will work with only one such pair. Note that  $\text{Res} \chi_\pi$  is simply the restriction of the function  $\chi_\pi$  on  $G$  to the subset (and subgroup)  $H$ .

**Induction.** Given a representation  $\pi$  of  $H$ , let  $\text{Ind}_H^G(\pi)$  denote the representation of  $G$  induced from  $\pi$ . If the character of  $\pi$  is  $\chi_\pi$ , then the character of the induced representation is given by the formula

$$\text{Ind}_H^G \chi_\pi(g) = \frac{1}{|H|} \sum_{\substack{s \in G \\ sgs^{-1} \in H}} \chi_\pi(sgs^{-1}). \quad (3.1)$$

The factor of  $1/|H|$  accounts for the action of  $H$  on the set of  $s \in G$  such that  $sgs^{-1} \in H$ .

**Frobenius reciprocity.** The relation between restriction and induction is given by the Frobenius reciprocity formula:

$$\langle \text{Ind}_H^G \chi_{\pi_1}, \chi_{\pi_2} \rangle = \langle \chi_{\pi_1}, \text{Res}_H^G \chi_{\pi_2} \rangle_H.$$

Here,  $\pi_1$  is a representation of  $H$  and  $\pi_2$  a representation of  $G$ .

We shall apply this formula to the group  $G$  coming from the regular origami  $S$  and to its cyclic subgroup  $H = \langle c \rangle$ . These will be omitted from the notation in Res and Ind. Finally, observe that inducing the trivial representation of  $H$  to  $G$  gives the representation of  $G$  on functions on  $G/H$ .

**Real, complex, quaternionic representations.** Let  $\pi$  be an irreducible representation of  $G$  on a complex vector space. We have the following possibilities for  $\pi$ :

- Real representation. The character is real-valued and the representation is induced from the complexification of a real representation.
- Complex representation. The character is complex-valued. The representation  $\pi \oplus \bar{\pi}$  is induced from a representation on a real vector space.
- Quaternionic representation. The character is real-valued, but the representation is *not* induced from the complexification of any real representation. The representation  $\pi \oplus \pi$ , however, is induced from a representation on a real vector space.

Let now  $\pi$  be an irreducible representation of  $G$  on a real vector space. Unlike the case of complex vector spaces above,  $\pi$  can have non-trivial endomorphisms. We have the following possibilities:

- Real representation. The endomorphisms equal  $\mathbb{R}$ , and after complexification,  $\pi$  is still irreducible.
- Complex representation. The endomorphisms equal  $\mathbb{C}$ , and after complexification,  $\pi$  becomes isomorphic to  $\pi_1 \oplus \bar{\pi}_1$ , where  $\pi_1$  is a complex representation in the sense above.
- Quaternionic representation. The endomorphisms equal  $\mathbb{H}$ , and after complexification,  $\pi$  becomes isomorphic to  $\pi_1 \oplus \pi_1$ , where  $\pi_1$  is a quaternionic representation in the sense above.

### 3.2. Isotypical components in topological cohomology

In this section, we shall compute the multiplicity of an irreducible representation of  $G$  in the first cohomology of the origami  $S$ .

**Lefschetz fixed point formula.** Recall the following statement, described for instance in the monograph of Griffiths and Harris [12, p. 421]. Let  $g$  be a smooth diffeomorphism with isolated fixed points of a compact manifold  $S$ . Then we have the following formula for the alternating sum of traces for the action on cohomology:

$$\sum_{i=0}^{\dim S} (-1)^i \text{tr}(g^*|_{H^i(S)}) = \sum_{g(p)=p} \text{index}(p).$$

**The case of origamis.** In our case,  $S$  is a smooth surface and all fixed points have index 1 (since the diffeomorphisms respect a complex structure). We have a full action of the group  $G$  and let  $\chi_{H^1}$  be the character of the  $G$ -representation of the first cohomology group  $H^1(S)$  (coefficients in  $\mathbb{C}$ ).

Applying the Lefschetz fixed point formula to all  $g$  not the identity, we find

$$\chi_{H^1}(g) = \begin{cases} 2 \cdot \text{genus}(S) & \text{if } g = \text{id}, \\ 2 - \#\text{fixed pts.}(g) & \text{otherwise.} \end{cases}$$

**Counting fixed points.** Assume  $g \in G$  is not the identity. Then the fixed points of the action of  $g$  on  $S$  can only be among the vertices of the squares tiling  $S$ . Moreover, the action of  $g$  on them is the same as the action on the coset  $G/\langle c \rangle$ , which is naturally identified with the set of vertices. If we let  $\chi_0$  denote the character of the trivial representation of  $\langle c \rangle$ , the number of fixed points of  $g$  acting on  $S$  is then given by  $\text{Ind}_{\langle c \rangle}^G \chi_0(g)$ .

Therefore, the character of the  $G$ -representation on  $H^1(S)$  can be written as

$$\chi_{H^1} = 2 - \text{Ind } \chi_0 + (\text{Ind } \chi_0(\text{id}) - 2 + 2\text{genus}(S))\delta_{\text{id}}. \quad (3.2)$$

Here  $\delta_{\text{id}}$  is the delta-function at the identity on  $G$ . Note also that  $\text{Ind } \chi_0(\text{id}) = \dim(\text{Ind } \chi_0) = |G|/N$ .

**Computing multiplicities.** The multiplicity of an irreducible representation  $\pi$  of  $G$  in  $\chi_{H^1}$  is given by  $\langle \chi_{H^1}, \chi_\pi \rangle_G$ . Frobenius reciprocity will reduce the computation to a computation on the cyclic group generated by  $c$ .

We have the following identities for inner products of functions on  $G$  (where ‘‘const’’ denotes a constant function):

$$\langle \delta_{\text{id}}, \chi_\pi \rangle_G = \frac{\dim \pi}{|G|}, \quad \langle \text{const}, \chi_\pi \rangle_G = \begin{cases} \text{const} & \text{if } \pi \text{ is trivial,} \\ 0 & \text{otherwise.} \end{cases}$$

Taking the product of  $\chi_\pi$  with  $\chi_{H^1}$ , using (3.2) and the formulas above, we find

$$\begin{aligned} \langle \chi_{H^1}, \chi_\pi \rangle &= 2\delta_{\pi=\text{triv}} - \langle \text{Ind } \chi_0, \chi_\pi \rangle_G + \frac{\dim \pi}{|G|} \left( \frac{|G|}{N} - 2 + \left( |G| - \frac{|G|}{N} + 2 \right) \right) \\ &= 2\delta_{\pi=\text{triv}} + \dim \pi - \langle \text{Ind } \chi_0, \chi_\pi \rangle_G. \end{aligned}$$

Here,  $\delta_{\pi=\text{triv}}$  is 1 if  $\pi$  is the trivial representation, and 0 otherwise. Using Frobenius reciprocity from  $G$  to the subgroup  $\langle c \rangle$  to evaluate the last term, we find

$$\langle \chi_{H^1}, \chi_\pi \rangle = 2\delta_{\pi=\text{triv}} + \dim \pi - \langle \chi_0, \text{Res } \chi_\pi \rangle_{\langle c \rangle}. \quad (3.3)$$

The term  $\langle \chi_0, \text{Res } \chi_\pi \rangle_{\langle c \rangle}$  counts the multiplicity of the trivial representation of the restriction of  $\pi$  to the cyclic group  $\langle c \rangle$ . This is exactly the dimension of the space of vectors fixed by  $c \in G$  in the representation  $\pi$ . Note that (3.3) recovers a result of Matheus–Yoccoz–Zmiaikou [16, Cor. 3.5].

3.3. *Isotypical components in the holomorphic forms*

The pattern for the calculation is similar to the one above. The Lefschetz fixed point formula has a holomorphic version, which is a bit more involved. To have a more convenient algebraic framework to analyze it, we introduce an extra parameter and expand our functions in power series.

**Holomorphic Lefschetz fixed point formula.** Recall the following statement, described for instance in [12, p. 426]. Let  $g$  be a holomorphic self-map of a complex manifold  $S$ , with isolated fixed points. Then the alternating sum of traces on the Dolbeault cohomology groups is given by

$$\sum_{i=0}^{\dim_{\mathbb{C}} S} (-1)^i \operatorname{tr}(g^*|_{H^{0,i}(S)}) = \sum_{\substack{g(p)=p \\ (Dg)_p: T_p S \rightarrow T_p S}} \frac{1}{\det(1 - (Dg)_p)}.$$

In our situation,  $S$  is a compact Riemann surface, so the formula becomes

$$1 - \operatorname{tr}(g^*|_{H^{0,1}(S)}) = \sum_{g \cdot p=p} \frac{1}{1 - \mu_p(g)}$$

where  $\mu_p(g)$  is the derivative of  $g$  at its fixed point  $p$  (thus, a complex scalar). Note that  $\mu_p(g)$  is never 1 when  $g$  is not the identity.

**Derivative at fixed points for origamis.** In our situation, the fixed points are among the elements of the coset  $G/\langle c \rangle$ . Corresponding to one such  $p \in S$  we have a coset  $h_p \langle c \rangle$ . This means that  $g \cdot h_p \cdot \langle c \rangle = h_p \cdot \langle c \rangle$ . Therefore, there exists  $k_p \in \mathbb{N}$  such that  $gh_p = h_p c^{k_p}$ . So we have  $g = h_p c^{k_p} h_p^{-1}$ .

Moreover, the scaling factor  $\mu_p$  of the action of  $g$  on the tangent space at  $p$  is

$$\mu_p(g) = \exp(2\pi\sqrt{-1} k_p/N). \tag{3.4}$$

As before,  $N$  is the cardinality of the cyclic group  $\langle c \rangle$ . The holomorphic Lefschetz fixed point formula gives

$$\chi_{H^{0,1}}(g) = \begin{cases} \operatorname{genus}(S) & \text{if } g = \operatorname{id}, \\ 1 - \sum_{g \cdot p=p} \frac{1}{1 - \mu_p(g)} & \text{otherwise.} \end{cases}$$

For the group  $\langle c \rangle$  we have the characters  $\chi_j$  defined on the generator via

$$\chi_j(c) := \exp(2\pi\sqrt{-1} j/N).$$

Then we have, for any  $j \in \mathbb{N}$ , the following relation between local multipliers and characters:

$$\sum_{g \cdot p=p} \mu_p(g)^j = \operatorname{Ind}_{\langle c \rangle}^G \chi_j(g).$$

This follows from (3.4) and the formula (3.1) for the induced character.

**An auxiliary function.** To compute the multiplicity of  $\pi$  in  $H^{0,1}$ , we introduce an auxiliary function. Its power series expansion will allow us to use the above formula for the powers of multipliers. For  $r \in \mathbb{C}$ , define

$$\chi_{H^{0,1}}(g, r) = \begin{cases} \text{genus}(S) & \text{if } g = \text{id}, \\ 1 - \sum_{g \cdot p=p} \frac{1}{1-r\mu_p(g)} & \text{otherwise.} \end{cases}$$

For fixed  $g \in G$ , this is a meromorphic function of  $r$ . It has finitely many poles on the unit circle, but there is no pole at 1. The value of  $\chi_{H^{0,1}}(g, r)$  at  $r = 1$  is exactly the character we want to understand.

We shall next perform a power series expansion for  $|r| < 1$  and manipulate the function as a uniformly convergent power series. The result will have a limit as  $r \rightarrow 1$  and this will give the desired multiplicity.

We have, for  $g \neq \text{id}$  and  $|r| < 1$ ,

$$\chi_{H^{0,1}}(g, r) = 1 - \sum_{g \cdot p=p} \sum_{i \geq 0} r^i \mu_p(g)^i = 1 - \sum_{i \geq 0} r^i \text{Ind}_{(c)}^G \chi_i(g).$$

This means that in general, for all  $g \in G$  and  $|r| < 1$ ,

$$\begin{aligned} \chi_{H^{0,1}}(g, r) &= 1 - \sum_{i \geq 0} r^i \text{Ind}_{(c)}^G \chi_i(g) + \delta_{\text{id}} \left( \frac{|G|}{N} \left( \sum_{i \geq 0} r^i \right) - 1 + \text{genus}(S) \right) \\ &= 1 - \sum_{i \geq 0} r^i \text{Ind}_{(c)}^G \chi_i(g) + \delta_{\text{id}} |G| \left( \frac{1}{1-r} \cdot \frac{1}{N} + \frac{N-1}{2N} \right). \end{aligned}$$

**Multiplicities of representations.** To compute the multiplicity of a representation  $\pi$  in  $H^{0,1}$ , we take the product of the corresponding characters. We shall use  $\chi_{H^{0,1}}(r)$  with its power series expansion, and will evaluate the result as  $r \rightarrow 1$ . We have the following expression (using Frobenius reciprocity at the last step):

$$\begin{aligned} \langle \chi_{H^{0,1}}(r), \chi_\pi \rangle_G &= \left\langle 1 - \sum_{i \geq 0} r^i \text{Ind}_{(c)}^G \chi_i(g) + \delta_{\text{id}} |G| \left( \frac{1}{1-r} \cdot \frac{1}{N} + \frac{N-1}{2N} \right), \chi_\pi \right\rangle_G \\ &= \delta_{\pi=\text{triv}} - \sum_{i \geq 0} r^i \langle \text{Ind } \chi_i, \chi_\pi \rangle_G + \dim \pi \left( \frac{1}{1-r} \cdot \frac{1}{N} + \frac{N-1}{2N} \right) \\ &= \delta_{\pi=\text{triv}} + \frac{1}{2} \dim \pi - \sum_{i \geq 0} r^i \langle \chi_i, \text{Res } \chi_\pi \rangle_{(c)} + \frac{\dim \pi}{N} \left( \frac{1}{1-r} - \frac{1}{2} \right). \end{aligned}$$

Define now  $m_i := \langle \chi_i, \text{Res } \chi_\pi \rangle_{(c)}$ , which is also the number of eigenvalues of  $c$  in the representation  $\pi$  that are equal to  $\exp(2\pi\sqrt{-1}i/N)$ . We then have

$$m_0 + m_1 + \cdots + m_{N-1} = \dim \pi.$$

We also extend the sequence periodically by setting  $m_{j+N} = m_j$ . The multiplicity of the representation then becomes

$$\begin{aligned} & \delta_{\pi=\text{triv}} + \frac{1}{2} \dim \pi - \sum_{i \geq 0} r^i m_i + \frac{\dim \pi}{N} \left( \frac{1}{r-1} - \frac{1}{2} \right) \\ & = \delta_{\pi=\text{triv}} + \frac{1}{2} \dim \pi - \frac{m_0 + r m_1 + \cdots + r^{N-1} m_{N-1}}{1-r^N} + \frac{\dim \pi}{N} \left( \frac{1}{1-r} - \frac{1}{2} \right). \end{aligned}$$

We would like to evaluate the sum of the last two terms at  $r = 1$ . For this, we first rearrange:

$$\begin{aligned} & \frac{m_0 + r m_1 + \cdots + r^{N-1} m_{N-1}}{1-r^N} - \frac{\dim \pi}{N} \cdot \frac{1}{1-r} \\ & = \frac{1}{1-r} \left( \frac{m_0 + r m_1 + \cdots + r^{N-1} m_{N-1}}{1+r+\cdots+r^{N-1}} - \frac{\dim \pi}{N} \right). \end{aligned}$$

Recalling that  $m_0 + \cdots + m_{N-1} = \dim \pi$ , we can evaluate the limit as  $r \rightarrow 1$  via L'Hopital's rule:

$$\begin{aligned} & \left( \frac{m_0 + \cdots + r^{N-1} m_{N-1}}{1+r+\cdots+r^{N-1}} \right)' \Big|_{r=1} \\ & = \frac{1}{N^2} \left( (m_1 + 2m_2 + \cdots + (N-1)m_{N-1})N - \frac{N(N-1)}{2} (m_0 + m_1 + \cdots + m_{N-1}) \right) \\ & = \frac{1}{N} \sum_{i=0}^{N-1} \left( i - \frac{N-1}{2} \right) m_i. \end{aligned}$$

Plugging this value into the above calculation of the multiplicity, we find that  $\pi$  appears in  $H^{0,1}$  with multiplicity

$$\delta_{\pi=\text{triv}} + \frac{1}{2}(\dim \pi - m_0) + \frac{1}{N} \sum_{i=1}^{N-1} \left( i - \frac{N}{2} \right) m_i. \tag{3.5}$$

**Remark 3.1.** (1) The term  $\delta_{\pi=\text{triv}} + \frac{1}{2}(\dim \pi - m_0)$  in (3.5) is exactly half the multiplicity of  $\pi$  in  $H^1$ , which appears in (3.3).

(2) Denote  $\Delta_\pi := \sum_{i=0}^{N-1} (i - N/2) m_i$ . Then for the complex-conjugate representation  $\bar{\pi}$  we have  $\Delta_{\bar{\pi}} = -\Delta_\pi$ . So the multiplicity in  $H^{0,1}$  combined with that for  $H^{1,0}$  gives the correct multiplicity for  $H^1$ .

(3) If  $\chi_\pi$  is real-valued, then necessarily  $\Delta_\pi = 0$ . Indeed, in this case the representations  $\pi$  and  $\bar{\pi}$  are isomorphic since they have the same character. The character  $\chi_\pi$  takes real values if and only if  $\pi$  is a real or quaternionic representation. Therefore,  $\Delta_\pi$  can be non-zero only for purely complex representations of the finite group  $G$ .

#### 4. General considerations about monodromy

This section contains a general discussion of two aspects relevant to our constructions in Section 5. First, in §4.1 we discuss the general structure of semisimple local systems with a finite symmetry group. The main consequences are for the structure of the Lyapunov spectrum and monodromy representation. Next, in §4.2 we discuss the group  $SO^*(2n)$  in more detail. In particular, we describe it both as a group of quaternionic and complex matrices.

##### 4.1. Local systems with symmetries

**Setup.** Consider a local system  $\mathbb{W} \rightarrow X$  over some base. Assume that  $\mathbb{W}$  has the usual semisimplicity properties: any sublocal system has a complement. Consider the situation when a finite group  $G$  acts on the fibers of the local system  $\mathbb{W}$ . We would like to understand the decomposition of  $\mathbb{W}$  for these symmetries, as well as the consequences for the Lyapunov spectrum (when the base  $X$  carries a flow).

**Isotypical components of representations.** First, we consider the canonical decomposition of any  $G$ -representation  $W$ . To fix notation, for an isomorphism class of irreducible representation  $\pi$ , let  $R_\pi$  be a vector space with  $G$ -action realizing this isomorphism class. Thus, we have a chosen map  $G \rightarrow GL(R_\pi)$ .

Further, let  $A_\pi$  denote the algebra of endomorphisms of  $R_\pi$  viewed as a  $G$ -representation. When the field of scalars is  $\mathbb{R}$ , the possibilities for  $A_\pi$  are  $\mathbb{R}$ ,  $\mathbb{C}$  or  $\mathbb{H}$  (see §3.1).

For a given representation  $W$ , we can form *the space of isotypical components* corresponding to  $\pi$ , defined by

$$V_\pi := \text{Hom}_G(R_\pi, W).$$

In other words,  $V_\pi$  is the linear space of maps from  $R_\pi$  to  $W$  which commute with the  $G$ -action. Note that  $V_\pi$  *does not* carry an action of  $G$ , but it does carry an action of  $A_\pi$ . Indeed, given  $a \in A_\pi$ ,  $\phi \in V_\pi$  and  $r \in R_\pi$ , define

$$a \cdot \phi(r) := \phi(ar).$$

This makes  $V_\pi$  into a *right*  $A_\pi$ -module, since  $a_1 \cdot (a_2 \cdot \phi) = (a_2 \cdot a_1) \cdot \phi$ . We shall therefore write the action of  $A_\pi$  on  $V_\pi$  on the right.

Next, we have a natural evaluation map:

$$\text{ev} : V_\pi \otimes R_\pi \rightarrow W, \quad \phi \otimes r \mapsto \phi(r).$$

This map surjects onto the space of isotypical components of  $W$  isomorphic to  $R_\pi$ . However, it also has a kernel. Namely, given  $a \in A_\pi$  (recall that it acts on  $V_\pi$  on the right), we have

$$\text{ev}(\phi \cdot a \otimes r) = \phi(ar) = \text{ev}(\phi \otimes a \cdot r).$$

We therefore have a factorization of the evaluation map through

$$\text{ev} : V_\pi \otimes_{A_\pi} R_\pi \rightarrow W$$



where  $V_\pi \otimes_{A_\pi} R_\pi := V_\pi \otimes R_\pi / \{\phi a \otimes r = \phi \otimes ar\}$ . Moreover, this map is a natural isomorphism onto the collection of isotypical components of  $W$  isomorphic to  $R_\pi$ .

To summarize the discussion, recall we started with an arbitrary representation  $W$  of  $G$ . For each irreducible representation  $\pi$  of  $G$ , we constructed the space  $V_\pi$  which carried a right action of  $A_\pi$ —the endomorphisms of  $R_\pi$ . This gave us a natural isomorphism

$$W = \bigoplus_{\pi \in \text{Irr}_{\mathbb{R}}(G)} V_\pi \otimes_{A_\pi} R_\pi.$$

**Isotypical components of local systems.** We now extend the above discussion to local systems.

Given the local system  $\mathbb{W} \rightarrow X$  carrying an action of the finite group  $G$ , we can form the local systems of isotypical components (for each representation  $\pi$  of  $G$ )

$$\mathbb{V}_\pi := \text{Hom}_G(R_\pi, \mathbb{W}).$$

As before,  $\mathbb{V}_\pi$  carries an action of the endomorphisms  $A_\pi$  of  $R_\pi$ , but no canonical action of the group  $G$ . We have the canonical decomposition of  $\mathbb{W}$  as

$$\mathbb{W} = \bigoplus_{\pi \in \text{Irr}_{\mathbb{R}}(G)} \mathbb{V}_\pi \otimes_{A_\pi} R_\pi. \tag{4.1}$$

If  $\mathbb{W}$  carried a variation of Hodge structures, then so will  $\mathbb{V}_\pi$  (and the weight will be the same). Moreover, now the variation on  $\mathbb{V}_\pi$  will carry an action of the endomorphism algebra  $A_\pi$ , rather than the group  $G$  which was on  $\mathbb{W}$ .

**Lyapunov exponents. Monodromy.** If the base of the local system  $\mathbb{W} \rightarrow X$  carries a flow, we can consider the associated Lyapunov exponents. To understand them, recall the canonical decomposition (4.1). Let  $r$  be the rank of  $R_\pi$  viewed as a module over  $A_\pi$ . This will control the multiplicities of the answers for  $\mathbb{W}$  in terms of those for  $\mathbb{V}_\pi$ .

Each local system  $\mathbb{V}_\pi$  will have its own Lyapunov exponents  $\{\lambda_{i,\pi}\}$ . Then for  $\mathbb{W}$ , the Lyapunov exponent  $\lambda_{i,\pi}$  will appear with multiplicity  $r$ . Moreover, the Oseledets subspaces for  $\mathbb{W}$  can also be read off from (4.1) and the Oseledets subspaces of  $\mathbb{V}_\pi$ . This is compatible with the fact that the Oseledets subspaces of  $\mathbb{W}$  must themselves carry an action of  $G$ .

Finally, the monodromy for  $\mathbb{W}$  can be understood in terms of the monodromy for  $\mathbb{V}_\pi$  as follows. The local system  $\mathbb{V}_\pi$  will have some monodromy group  $\mathbb{G}$ , which will commute with the action of  $A_\pi$ . Then the monodromy of  $\mathbb{W}$  will equal  $r$  copies of the same group.

**The example from Theorem 1.1.** In this case, we will have a local system, called  $W_{\chi_2}$  later, but let us call it  $\mathbb{W}$  for this discussion. It will correspond to a quaternionic representation  $\chi_2$  occurring with multiplicity 3. This means that the space of isotypical components of  $\mathbb{W}$  (denoted  $\mathbb{V}_{\chi_2}$  according to the above discussion) will be of rank 3 over the endomorphisms of  $\chi_2$ , which are the quaternions  $\mathbb{H}$ .

Since the real dimension of the representation will be 4, the total real dimension of  $\mathbb{W}$  will be 12. We also see that the real dimension of  $\mathbb{V}_{\chi_2}$  will be 12, but its monodromy can be viewed as quaternionic  $3 \times 3$  matrices.

#### 4.2. A description of $SO^*(2n)$

This section describes the group  $SO^*(2n)$  in more detail. It is a real Lie group, but can be viewed as a group of either quaternionic or complex matrices, preserving certain (skew-) hermitian forms.

**Conventions.** Quaternions will be denoted

$$\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}.$$

If  $x = a + bi + cj + dk$  is a quaternion, its conjugate is  $\bar{x} := a - bi - cj - dk$ . Elements of  $\mathbb{H}^n$  are quaternionic column vectors, and if  $x \in \mathbb{H}^n$  is one such, then  $x^t$  denotes the corresponding row vector. Denote adjoints of vectors by  $x^\dagger := \bar{x}^t$ .

**Quaternionic description of  $SO^*(2n)$ .** View  $\mathbb{H}^n$  as a *right*  $\mathbb{H}$ -module and define the quaternionic-linear group by

$$GL_n(\mathbb{H}) := \{f : \mathbb{H}^n \rightarrow \mathbb{H}^n \text{ invertible} \mid f(x\lambda) = f(x)\lambda, \forall x \in \mathbb{H}^n, \lambda \in \mathbb{H}\}$$

We can define the quaternionic skew-hermitian form  $C(-, -)$  for  $x, y \in \mathbb{H}^n$  via

$$C(x, y) := x^\dagger \cdot j \cdot y.$$

Here  $j \in \mathbb{H}$  is just the element of the quaternions,  $x^\dagger$  is a row vector and  $y$  is a column vector. Note that we have the following property of the bilinear form, for  $\lambda, \mu \in \mathbb{H}$ :

$$C(x \cdot \lambda, y \cdot \mu) = -\bar{\lambda} \cdot \overline{C(y, x)} \cdot \mu.$$

The group is now defined by the requirement to preserve this form:

$$SO^*(2n) := \{f \in GL_n(\mathbb{H}) \mid C(x, y) = C(f(x), f(y)), \forall x, y \in \mathbb{H}^n\}.$$

Another common notation for this group is  $U_n^*(\mathbb{H})$ .

**Complex description of  $SO^*(2n)$ .** For a complex description, identify  $x \in \mathbb{H}$  with the vector  $\begin{bmatrix} a_x \\ b_x \end{bmatrix} \in \mathbb{C}^2$  by the requirement that  $x = a_x + jb_x$  in  $\mathbb{H}$ . Further make the identification of  $x \in \mathbb{H}^n$  with  $\begin{bmatrix} a_x \\ b_x \end{bmatrix} \in \mathbb{C}^{2n}$  via  $x = a_x + jb_x$  (thus  $a_x, b_x \in \mathbb{C}^n$ ).

Note that  $\mathbb{C}$  acts on both spaces on the right in a natural way, giving both the structure of a  $\mathbb{C}$ -vector space. On  $\mathbb{C}^{2n}$  we have a  $\mathbb{C}$ -antilinear operator  $R_j$  induced from multiplication by  $j$  on the right:

$$R_j \begin{bmatrix} a_x \\ b_x \end{bmatrix} = \begin{bmatrix} -\bar{b}_x \\ \bar{a}_x \end{bmatrix}.$$

For two quaternion vectors  $x, y \in \mathbb{H}^n$  written as  $x = a_x + jb_x$  and  $y = a_y + jb_y$ , we have

$$C(x, y) = x^\dagger \cdot j \cdot y = (b_x^\dagger a_y - a_x^\dagger b_y) + j(a_x^t a_y + b_x^t b_y).$$

On  $\mathbb{C}^{2n}$  we can thus define two forms

$$\omega\left(\begin{bmatrix} a_x \\ b_x \end{bmatrix}, \begin{bmatrix} a_y \\ b_y \end{bmatrix}\right) = b_x^\dagger a_y - a_x^\dagger b_y, \quad g\left(\begin{bmatrix} a_x \\ b_x \end{bmatrix}, \begin{bmatrix} a_y \\ b_y \end{bmatrix}\right) = a_x^t a_y + b_x^t b_y.$$

Note that  $\omega$  is skew-hermitian, i.e.  $\omega(v, w) = -\overline{\omega(w, v)}$ , and  $g$  is symmetric. Letting  $h(v, w) := i\omega(v, w)$ , we see that  $h$  is in fact hermitian, i.e.  $h(v, w) = \overline{h(w, v)}$ , and a linear transformation of  $\mathbb{C}^{2n}$  preserves  $h$  if and only if it preserves  $\omega$ .

It is clear that any quaternionic matrix preserving the quaternionic skew-hermitian form  $C$  above, viewed as a matrix in  $GL_{2n}(\mathbb{C})$ , will necessarily preserve the symmetric form  $g$  and the hermitian form  $h$ .

However, the converse is also true. If a matrix  $A \in GL_{2n}(\mathbb{C})$  preserves both  $g$  and  $h$ , then it must come from a quaternionic matrix preserving  $C$ . For this, it suffices to check that such an  $A$  will have to commute with the operator  $R_j$  defined above. More generally, any two of the following conditions imply the third:

- $A \in GL_{2n}(\mathbb{C})$  preserves the symmetric form  $g$ .
- $A \in GL_{2n}(\mathbb{C})$  preserves the hermitian form  $h$ .
- $A \in GL_{2n}(\mathbb{C})$  commutes with the  $\mathbb{C}$ -antilinear operator  $R_j$ .

In particular, we see that the group  $SO^*(2n)$  can be alternatively described as the intersection of a complex orthogonal group and a unitary group. Writing out the matrices for  $g$  and  $h$  explicitly, we find that

$$SO^*(2n) = \left\{ A \in GL_{2n}(\mathbb{C}) \mid AA^t = 1 \text{ and } A^\dagger \cdot \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} A = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \right\}.$$

### 5. An interesting quaternionic cover $\tilde{L}$

In this section, we shall describe the origami  $\tilde{L}$ . Then, using known results about Lyapunov exponents, we shall compute its Lyapunov spectrum. In this particular case, all exponents can be computed explicitly. This will constrain the monodromy to some extent, and in Section 6 we shall prove that in fact the non-trivial piece has  $SO^*(2n)$  as the Zariski closure of the monodromy.

#### 5.1. Definition of $\tilde{L}$

Let  $L^0 \in \mathcal{H}(2)$  be the  $L$ -shaped genus 2 origami associated to the pair of permutations  $h_0 = (1, 2)(3)$  and  $v_0 = (1, 3)(2)$ .

Consider the following covering  $\tilde{L}$  of  $L^0$ . For each element  $g$  of the quaternion group  $Q = \{1, -1, i, -i, j, -j, k, -k\}$ , take a copy  $L_g$  of  $L^0$ . The origami  $\tilde{L}$  is obtained by gluing (by translation) the two topmost horizontal sides of  $L_g$  with the corresponding two bottommost horizontal sides of  $L_{gi}$ , and the two rightmost vertical sides of  $L_g$  with the corresponding two leftmost vertical sides of  $L_{gj}$  (for each  $g \in Q$ ). Alternatively, we

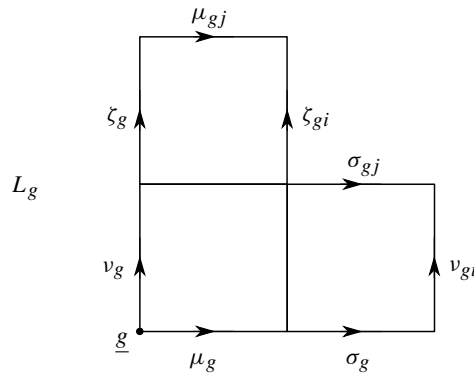


Fig. 1. Labels of the sides of the  $L_g, g \in Q$ .

label the sides of  $L_g$  according to Figure 1, and we glue by translation the sides with the same labels.

Denoting by  $\underline{g} \in \tilde{L}$  the bottommost and leftmost corner of  $L_g$  (as indicated in Figure 1), we have  $\underline{g} = -g$  for each  $g \in Q$ , and moreover the set  $\Sigma = \{\underline{1}, \underline{i}, \underline{j}, \underline{k}\}$  of four distinct points consists of all conical singularities of  $\tilde{L}$ . Furthermore, it is not hard to check that the conical angle around each of these singularities is  $12\pi$ . Thus,  $\tilde{L} \in \mathcal{H}(5, 5, 5, 5)$ , so that  $\tilde{L}$  has genus 11.

### 5.2. The group of automorphisms of $\tilde{L}$

Given  $h \in Q$ , we obtain an automorphism of  $\tilde{L}$  by sending (by translation)  $L_g$  to  $L_{hg}$ . As it turns out, this accounts for all automorphisms of  $\tilde{L}$ , i.e.,  $\text{Aut}(\tilde{L}) \simeq Q$ .

Recall that the non-trivial subgroups of  $\text{Aut}(\tilde{L}) \simeq Q$  are its center  $Z = \{1, -1\}$ , and  $\langle i \rangle = \{1, -1, i, -i\}$ ,  $\langle j \rangle = \{1, -1, j, -j\}$  and  $\langle k \rangle = \{1, -1, k, -k\}$ .

The quotient  $L_{\pm}$  of  $\tilde{L}$  by  $Z$  is an origami (of genus 5) in  $\mathcal{H}(2, 2, 2, 2)$  with  $\text{Aut}(L_{\pm})$  isomorphic to Klein’s group  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \simeq Q/Z = \{1_{\pm}, i_{\pm}, j_{\pm}, k_{\pm}\}$ . Moreover, the quotients  $L_{\langle i_{\pm} \rangle}, L_{\langle j_{\pm} \rangle}$  and  $L_{\langle k_{\pm} \rangle}$  (resp.) of  $L_{\pm}$  by the subgroups  $\langle i_{\pm} \rangle = \{1_{\pm}, i_{\pm}\}$ ,  $\langle j_{\pm} \rangle = \{1_{\pm}, j_{\pm}\}$  and  $\langle k_{\pm} \rangle = \{1_{\pm}, k_{\pm}\}$  (resp.) of  $\text{Aut}(L_{\pm}) \simeq Q/Z$  give rise to three origamis (of genus 3) in  $\mathcal{H}(2, 2)$  that are precisely the quotients of  $\tilde{L}$  by the subgroups  $\langle i \rangle, \langle j \rangle$  and  $\langle k \rangle$  of  $\text{Aut}(\tilde{L}) \simeq Q$ . Furthermore, the origamis  $L_{\langle i_{\pm} \rangle}, L_{\langle j_{\pm} \rangle}$  and  $L_{\langle k_{\pm} \rangle}$  are unramified double covers of the origami  $L^0 \in \mathcal{H}(2)$ . In some sense, these characteristics of the origami  $L_{\pm}$  are similar to the so-called *wind-tree model* [4] except that *all* origamis  $L_{\langle i_{\pm} \rangle}, L_{\langle j_{\pm} \rangle}$  and  $L_{\langle k_{\pm} \rangle}$  belong to the odd connected component  $\mathcal{H}(2, 2)^{\text{odd}}$  of the stratum  $\mathcal{H}(2, 2)$ .

### 5.3. The first absolute homology group of $\tilde{L}$

The intermediate cover  $\tilde{L} \rightarrow L_{\pm} = \tilde{L}/Z$  of  $\tilde{L} \rightarrow \mathbb{T}^2$  induces a decomposition

$$H_1(\tilde{L}, \mathbb{R}) = H_1^+(\tilde{L}, \mathbb{R}) \oplus H_1^-(\tilde{L}, \mathbb{R})$$

where  $-1 \in Z \subset Q \simeq \text{Aut}(\tilde{L})$  acts by id, resp.  $-\text{id}$ , on  $H_1^+(\tilde{L}, \mathbb{R})$ , resp.  $H_1^-(\tilde{L}, \mathbb{R})$ , and  $H_1^+(\tilde{L})$  is naturally isomorphic to  $H_1(L_\pm, \mathbb{R})$ . Note that  $H_1^+(\tilde{L}, \mathbb{R})$  is a 10-dimensional subspace of the 22-dimensional space  $H_1(\tilde{L}, \mathbb{R})$  (since  $L_\pm$  has genus 5 and  $\tilde{L}$  has genus 11).

Similarly, the intermediate covers  $L_\pm \rightarrow L_{(i_\pm)} \rightarrow L^0$ ,  $L_\pm \rightarrow L_{(j_\pm)} \rightarrow L^0$  and  $L_\pm \rightarrow L_{(k_\pm)} \rightarrow L^0$  induce a decomposition

$$H_1(L_\pm, \mathbb{R}) = H_1^{+,+}(L_\pm, \mathbb{R}) \oplus H_1^{+,-}(L_\pm, \mathbb{R}) \oplus H_1^{-,+}(L_\pm, \mathbb{R}) \oplus H_1^{-,-}(L_\pm, \mathbb{R})$$

where  $i_\pm \in \text{Aut}(L_\pm)$ , resp.  $j_\pm \in \text{Aut}(L_\pm)$ , acts on  $H_1^{\varepsilon i_\pm, \varepsilon j_\pm}(L_\pm, \mathbb{R})$ ,  $\varepsilon_{i_\pm}, \varepsilon_{j_\pm} \in \{+, -\}$ , by  $(\varepsilon_{i_\pm})\text{id}$ , resp.  $(\varepsilon_{j_\pm})\text{id}$ , and

- $H_1^{+,+}(L_\pm, \mathbb{R}) \simeq H_1(L^0, \mathbb{R}) = H_1^{\text{st}}(L^0, \mathbb{R}) \oplus H_1^{(0)}(L^0, \mathbb{R})$ ,
- $H_1^{+,-}(L_\pm, \mathbb{R}) \oplus H_1^{-,+}(L_\pm, \mathbb{R}) \simeq H_1(L_{(i_\pm)}, \mathbb{R})$ ,
- $H_1^{-,+}(L_\pm, \mathbb{R}) \oplus H_1^{+,-}(L_\pm, \mathbb{R}) \simeq H_1(L_{(j_\pm)}, \mathbb{R})$ ,
- $H_1^{+,-}(L_\pm, \mathbb{R}) \oplus H_1^{-,+}(L_\pm, \mathbb{R}) \simeq H_1(L_{(k_\pm)}, \mathbb{R})$ .

By means of the isomorphism  $H_1(L_\pm, \mathbb{R}) \simeq H_1^+(\tilde{L}, \mathbb{R})$ , we obtain a decomposition

$$H_1(\tilde{L}, \mathbb{R}) \simeq H_1^{\text{st}}(L^0, \mathbb{R}) \oplus H_1^{(0)}(L^0, \mathbb{R}) \oplus \bigoplus_{\substack{\alpha, \beta \in \{+, -\}, \\ (\alpha, \beta) \neq (+, +)}} H_1^{\alpha, \beta}(L_\pm, \mathbb{R}) \oplus H_1^-(\tilde{L}, \mathbb{R})$$

where all summands are symplectic subspaces that are mutually symplectically orthogonal, and all summands are 2-dimensional subspaces of  $H_1(\tilde{L}, \mathbb{R})$  except for the 12-dimensional subspace  $H_1^-(\tilde{L}, \mathbb{R})$ .

Observe that the action of  $\text{Aff}(\tilde{L})$  on  $H_1(\tilde{L}, \mathbb{R})$  respects each summand of the decomposition above: these summands were defined in terms of deck transformations of certain intermediate coverings of  $\tilde{L} \rightarrow \mathbb{T}^2$  which act by pre-composition with translation charts of  $\tilde{L}$ , and  $\text{Aff}(\tilde{L})$  acts by post-composition with translation charts of  $\tilde{L}$ . In particular, this is a decomposition of the  $\text{Aut}(\tilde{L})$ -module  $H_1(\tilde{L}, \mathbb{R})$  into  $\text{Aut}(\tilde{L})$ -submodules.

Note that this decomposition refines  $H_1(\tilde{L}, \mathbb{R}) = H_1^{\text{st}}(\tilde{L}, \mathbb{R}) \oplus H_1^{(0)}(\tilde{L}, \mathbb{R})$  in the sense that

$$H_1^{(0)}(\tilde{L}, \mathbb{R}) \simeq H_1^{(0)}(L^0, \mathbb{R}) \oplus \bigoplus_{\substack{\alpha, \beta \in \{+, -\}, \\ (\alpha, \beta) \neq (+, +)}} H_1^{\alpha, \beta}(L_\pm, \mathbb{R}) \oplus H_1^-(\tilde{L}, \mathbb{R}). \quad (5.1)$$

As it turns out, this is precisely the decomposition of  $H_1^{(0)}(\tilde{L}, \mathbb{R})$  into isotypical components. More precisely, the quaternion group  $\text{Aut}(\tilde{L}) \simeq Q$  has five irreducible representations  $\chi_1, \chi_i, \chi_j, \chi_k, \chi'_2$  (over  $\mathbb{C}$ ) whose characters are given by the following table:

	1	-1	$\pm i$	$\pm j$	$\pm k$
$\chi_1$	1	1	1	1	1
$\chi_i$	1	1	1	-1	-1
$\chi_j$	1	1	-1	1	-1
$\chi_k$	1	1	-1	-1	1
$\text{tr } \chi'_2$	2	-2	0	0	0

Furthermore, the representation  $\chi'_2$  is quaternionic, while  $\chi_1, \chi_i, \chi_j, \chi_k$  are real. A quick comparison between this table and the actions of the elements of  $Q$  on the summands of this decomposition of  $H_1^{(0)}(\tilde{L}, \mathbb{R})$  reveals that

$$\begin{aligned} H_1^{(0)}(L^0, \mathbb{R}) &= W_{\chi_1} \simeq 2\chi_1, & H_1^{+,-}(L_{\pm}, \mathbb{R}) &= W_{\chi_i} \simeq 2\chi_i, \\ H_1^{-,+}(L_{\pm}, \mathbb{R}) &= W_{\chi_j} \simeq 2\chi_j, & H_1^{-,-}(L_{\pm}, \mathbb{R}) &= W_{\chi_k} \simeq 2\chi_k, \\ H_1^-(\tilde{L}, \mathbb{R}) &= W_{\chi_2} \simeq 3\chi_2, \end{aligned}$$

where  $W_a$  stands for the isotypical component of  $a$  in  $H_1^{(0)}(\tilde{L}, \mathbb{R})$ ,  $W_a \simeq \ell_a a$  with  $\ell_a \in \mathbb{N}$  means that  $a$  appears with multiplicity  $\ell_a$  in  $H_1^{(0)}(\tilde{L}, \mathbb{R})$ , and  $\chi_2 := 2\chi'_2$ .

5.4. *Lyapunov spectrum of the KZ cocycle over  $SL(2, \mathbb{R}) \cdot \tilde{L}$*

The Lyapunov spectrum of the restriction to  $H_1^+(\tilde{L}, \mathbb{R})$  of the KZ cocycle over  $SL(2, \mathbb{R}) \cdot \tilde{L}$  is not difficult to understand thanks to the intermediate covers  $L_{\pm} \rightarrow L_{(*)} \rightarrow L^0, * \in \{i_{\pm}, j_{\pm}, k_{\pm}\}$ , and some results of Bainbridge and Chen–Möller.

More concretely, since  $L^0 \in \mathcal{H}(2)$ , the work of Bainbridge [2] ensures that the subbundles  $H_1^{\text{st}}(\tilde{L}, \mathbb{R})$ , resp.  $W_{\chi_1}$ , of

$$H_1^{\text{st}}(\tilde{L}, \mathbb{R}) \oplus W_{\chi_1} \simeq H_1^{+,+}(L_{\pm}, \mathbb{R}) \simeq H_1(L^0, \mathbb{R}) = H_1^{\text{st}}(L^0, \mathbb{R}) \oplus H_1^{(0)}(L^0, \mathbb{R})$$

contribute with the Lyapunov exponents 1 and  $-1$ , resp.  $1/3$  and  $-1/3$ .

Next, since  $L_{(*)} \in \mathcal{H}(2, 2)^{\text{odd}}$ ,  $* \in \{i_{\pm}, j_{\pm}, k_{\pm}\}$ , the work of Chen–Möller [3] guarantees that the sum of the non-negative Lyapunov exponents associated to each of the subbundles

$$H_1(L_{(*)}, \mathbb{R}) \simeq H_1^{+,+}(L_{\pm}, \mathbb{R}) \oplus H_1^{\alpha,\beta}(L_{\pm}, \mathbb{R}) \simeq (H_1^{\text{st}}(\tilde{L}, \mathbb{R}) \oplus W_{\chi_1}) \oplus W_{\rho}$$

for the appropriate choices of  $\alpha, \beta \in \{+, -\}$ ,  $(\alpha, \beta) \neq (+, +)$ , and  $\rho \in \{\chi_i, \chi_j, \chi_k\}$  depending on  $* \in \{i_{\pm}, j_{\pm}, k_{\pm}\}$  is  $5/3$ . By combining this information with our knowledge of the Lyapunov exponents associated to  $H_1^{+,+}(L_{\pm}, \mathbb{R})$ , we deduce that each of the subbundles  $W_{\rho}$ ,  $\rho \in \{\chi_i, \chi_j, \chi_k\}$ , contributes with the Lyapunov exponents  $1/3$  and  $-1/3$ .

In summary, the Lyapunov spectrum of the restriction of the KZ cocycle over  $SL(2, \mathbb{R}) \cdot \tilde{L}$  to the 10-dimensional symplectic subspace  $H_1^+(\tilde{L}, \mathbb{R})$  is

$$1 > \frac{1}{3} = \frac{1}{3} = \frac{1}{3} = \frac{1}{3} > -\frac{1}{3} = -\frac{1}{3} = -\frac{1}{3} = -\frac{1}{3} > -1.$$

**Remark 5.1.** It is possible to show that  $\text{Aff}(\tilde{L})$  acts on each  $W_{\rho}$ ,  $\rho \in \{\chi_1, \chi_i, \chi_j, \chi_k\}$ , through a Zariski dense subgroup of  $SL(2, \mathbb{R})$ . In particular, the monodromy of the restriction of the KZ cocycle over  $SL(2, \mathbb{R}) \cdot \tilde{L}$  to each  $W_{\rho}$ ,  $\rho \in \{\chi_1, \chi_i, \chi_j, \chi_k\}$ , is given by item (i) (with  $d = 1$ ) in the first author’s classification of KZ cocycle monodromies mentioned in Section 1.

The Lyapunov spectrum of the restriction of the KZ cocycle over  $SL(2, \mathbb{R}) \cdot \tilde{L}$  to  $H_1^-(\tilde{L}, \mathbb{R}) = W_{\chi_2} \simeq 3\chi_2$  is determined by the quaternionic nature of the symplectic  $\text{Aut}(\tilde{L})$ -module

$W_{\chi_2} \simeq 3\chi_2$  and the sum of the non-negative Lyapunov exponents of the KZ cocycle over  $SL(2, \mathbb{R}) \cdot \tilde{L}$ .

More concretely, since  $\chi_2$  is a quaternionic representation, each Lyapunov exponent associated to  $W_{\chi_2}$  has multiplicity four (at least). Because  $\text{Aff}(\tilde{L})$  respects the symplectic intersection form on the symplectic module  $W_{\chi_2} \simeq 3\chi_2$ , it follows that the Lyapunov spectrum of the restriction of the KZ cocycle over  $SL(2, \mathbb{R}) \cdot \tilde{L}$  to  $W_{\chi_2}$  has the form

$$\lambda = \lambda = \lambda = \lambda \geq 0 = 0 = 0 = 0 \geq -\lambda = -\lambda = -\lambda = -\lambda.$$

Hence, the sum  $\theta_1 + \dots + \theta_{11}$  of the 11 non-negative Lyapunov exponents of the KZ cocycle acting on  $H_1(\tilde{L}, \mathbb{R}) = H_1^+(\tilde{L}, \mathbb{R}) \oplus H_1^-(\tilde{L}, \mathbb{R})$  is

$$\theta_1 + \dots + \theta_{11} = 1 + 4 \times \frac{1}{3} + 4 \times \lambda + 2 \times 0 = \frac{7}{3} + 4\lambda.$$

We will use the Eskin–Kontsevich–Zorich formula [5] in order to determine the numerical value of  $\lambda$ :

**Theorem 5.2.** *The numerical value of the sum of the non-negative Lyapunov exponents of the KZ cocycle over  $SL(2, \mathbb{R}) \cdot \tilde{L}$  is 3. In particular,  $\lambda = 1/6$ .*

*Proof.* Recall that Eskin–Kontsevich–Zorich [5] showed that the sum of the non-negative Lyapunov exponents of the KZ cocycle over the  $SL(2, \mathbb{R})$ -orbit of an origami  $X \in \mathcal{H}(k_1, \dots, k_\sigma)$  is given by

$$\frac{1}{12} \sum_{l=1}^{\sigma} \frac{k_l(k_l + 2)}{k_l + 1} + \frac{1}{\#SL(2, \mathbb{Z}) \cdot X} \sum_{\substack{Y \in SL(2, \mathbb{Z}) \cdot X, \\ c \text{ is a cycle of } h_Y}} \frac{1}{\text{length of } c}$$

where  $(h_Y, v_Y)$  is a pair of permutations associated to the origami  $Y$ .

Since  $\tilde{L} \in \mathcal{H}(5, 5, 5)$ , we deduce that  $\theta_1 + \dots + \theta_{11}$  is equal to

$$\begin{aligned} & \frac{1}{12} \left( 4 \times \frac{5 \times 7}{6} \right) + \frac{1}{\#SL(2, \mathbb{Z}) \cdot \tilde{L}} \sum_{\substack{Y \in SL(2, \mathbb{Z}) \cdot \tilde{L}, \\ c \text{ is a cycle of } h_Y}} \frac{1}{\text{length of } c} \\ &= \frac{35}{18} + \frac{1}{\#SL(2, \mathbb{Z}) \cdot \tilde{L}} \sum_{\substack{Y \in SL(2, \mathbb{Z}) \cdot \tilde{L}, \\ c \text{ is a cycle of } h_Y}} \frac{1}{\text{length of } c}. \end{aligned} \quad (5.2)$$

Therefore, our task is reduced to the computation of the  $SL(2, \mathbb{Z})$ -orbit of  $\tilde{L}$ .

For this sake, we note that the squares of  $\tilde{L}$  can be labeled in such a way that the square of  $L_1$  in Figure 1 with sides  $\mu_1$  and  $\nu_1$  has number 1 and the remaining squares of  $\tilde{L}$  are numbered in a compatible manner with the pair of permutations

$$h_{\tilde{L}} = (1, 2, 13, 14, 7, 8, 19, 20)(3, 15, 9, 21)(4, 5, 22, 23, 10, 11, 16, 17)(6, 24, 12, 18),$$

$$v_{\tilde{L}} = (1, 3, 4, 6, 7, 9, 10, 12)(2, 5, 8, 11)(13, 15, 16, 18, 19, 21, 22, 24)(14, 17, 20, 23).$$

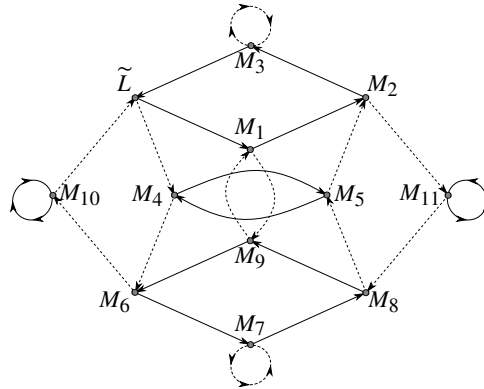
In this setting, recall that the  $SL(2, \mathbb{Z})$ -orbit of  $\tilde{L}$  is obtained by successively applying the generators  $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  and  $S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  of  $SL(2, \mathbb{Z})$  while keeping in mind that  $T$  and  $S$

act on pairs of permutations as  $T(r, u) = (r, ur^{-1})$  and  $S(r, u) = (ru^{-1}, u)$ , and the pairs of permutations  $(r, u)$  and  $(\phi r \phi^{-1}, \phi u \phi^{-1})$  define the same origami.

By performing this straightforward computation, one finds that  $SL(2, \mathbb{Z}) \cdot \tilde{L}$  has the structure described by Figure 2, namely:

- $SL(2, \mathbb{Z}) \cdot \tilde{L} = \{\tilde{L}, M_1, \dots, M_{11}\}$  has cardinality 12;
- $M_1 = T(\tilde{L}), M_2 = T^2(\tilde{L}), M_3 = T^3(\tilde{L}), M_4 = S(\tilde{L}), M_5 = T(M_4), M_6 = S(M_4), M_7 = T(M_6), M_8 = T^2(M_6), M_9 = T^3(M_6), M_{10} = S(M_2)$  and  $M_{11} = S(M_6)$ ;
- the  $T$ -orbits are  $\{\tilde{L}, M_1, M_2, M_3\}, \{M_4, M_5\}, \{M_6, M_7, M_8, M_9\}, \{M_{10}\}$  and  $\{M_{11}\}$ ;
- the  $S$ -orbits are  $\{\tilde{L}, M_4, M_6, M_{11}\}, \{M_1, M_9\}, \{M_2, M_{10}, M_8, M_5\}, \{M_3\}$  and  $\{M_7\}$ ;
- the origamis  $M_l, l \in \{4, 6, 10, 11\}$ , are associated to pairs of permutations  $(h_{M_l}, v_{M_l}), l \in \{4, 6, 10, 11\}$ , with

$$\begin{aligned}
 h_{M_4} &= (1, 18, 17, 7, 24, 23)(2, 16, 9, 8, 22, 3)(4, 15, 14, 10, 21, 20)(5, 13, 12, 11, 19, 6), \\
 h_{M_6} &= (1, 11, 22, 20, 7, 5, 16, 14)(2, 19, 17, 10, 8, 13, 23, 4)(3, 18, 9, 24)(6, 15, 12, 21), \\
 h_{M_{10}} &= (1, 24, 11)(2, 4, 21)(3, 14, 16)(5, 7, 18)(6, 23, 13)(8, 10, 15)(12, 17, 19)(20, 22, 9) \\
 h_{M_{11}} &= (1, 21, 17)(2, 22, 6)(3, 11, 13)(4, 18, 14)(5, 19, 9)(7, 15, 23)(8, 16, 12)(10, 24, 20).
 \end{aligned}$$



**Fig. 2.**  $SL(2, \mathbb{Z})$ -orbit of  $\tilde{L}$ . The full arrows indicate the action of  $T$  and the dashed arrows indicate the action of  $S$ .

Observe that the first permutation  $h_Y$  in a pair  $(h_Y, v_Y)$  associated to an origami  $Y$  is constant along the  $T$ -orbit of  $Y$ , i.e.,  $h_Z = h_Y$  whenever the origamis  $Z$  and  $Y$  belong to the same  $T$ -orbit (since  $T(r, u) = (r, ur^{-1})$ ). It follows that

$$\begin{aligned}
 & \frac{1}{\#SL(2, \mathbb{Z}) \cdot \tilde{L}} \sum_{\substack{Y \in SL(2, \mathbb{Z}) \cdot \tilde{L}, \\ c \text{ is a cycle of } h_Y}} \frac{1}{\text{length of } c} \\
 &= \frac{1}{12} \left\{ 4 \times \left( \frac{1}{8} + \frac{1}{4} + \frac{1}{8} + \frac{1}{4} \right) + 2 \times \left( \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} \right) \right. \\
 & \quad + 4 \times \left( \frac{1}{8} + \frac{1}{8} + \frac{1}{4} + \frac{1}{4} \right) + \left( \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} \right) \\
 & \quad \left. + \left( \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} \right) \right\} = \frac{19}{18}. \tag{5.3}
 \end{aligned}$$



By combining (5.2) and (5.3), we conclude that

$$\theta_1 + \cdots + \theta_{11} = \frac{35}{18} + \frac{19}{18} = 3.$$

This completes the proof of the theorem.  $\square$

**Remark 5.3.** The homological dimension of the unique  $SL(2, \mathbb{R})$ -probability measure supported on  $SL(2, \mathbb{R}) \cdot \tilde{L}$  in the sense of Forni’s paper [11] is 8: this is not hard to deduce from the description of  $SL(2, \mathbb{Z}) \cdot \tilde{L}$  given above. In particular, Forni’s criterion in [11] says that the KZ cocycle over  $SL(2, \mathbb{R}) \cdot \tilde{L}$  has eight positive Lyapunov exponents at least. In other terms, Forni’s criterion falls short of predicting the correct number (i.e., 9) of positive Lyapunov exponents in the particular case of the KZ cocycle over  $SL(2, \mathbb{R}) \cdot \tilde{L}$ .

5.5. A dichotomy for the monodromy of the KZ cocycle on  $W_{\chi_2}$

Recall from Remark 2.2 that we can split  $W_{\chi_2}$  into a direct sum of  $\text{Aff}_{***}(\tilde{L})$ -irreducible symplectic  $\text{Aut}(\tilde{L})$ -submodules, where  $\text{Aff}_{***}(\tilde{L})$  is a finite-index subgroup of  $\text{Aff}(\tilde{L})$ . Since  $W_{\chi_2} \simeq 3\chi_2$ , this gives us three possibilities:

- $\text{Aff}_{***}(\tilde{L})$  acts irreducibly on  $W_{\chi_2}$ ;
- $W_{\chi_2} = U \oplus V$  where  $U$  and  $V$  are  $\text{Aff}_{***}(\tilde{L})$ -irreducible  $\text{Aut}(\tilde{L})$ -submodules with  $U \simeq 2\chi_2$  and  $V \simeq \chi_2$ ;
- $W_{\chi_2} = W' \oplus W'' \oplus W'''$  where  $W', W''$  and  $W'''$  are  $\text{Aff}_{***}(\tilde{L})$ -irreducible  $\text{Aut}(\tilde{L})$ -submodules with  $W' \simeq W'' \simeq W''' \simeq \chi_2$ .

Furthermore, the discussion in §4.1 and the classification of the monodromy groups of the KZ cocycle in [9] imply that the Zariski closure of the monodromy group (modulo compact and finite-index factors) of the restriction of the KZ cocycle to  $W_{\chi_2}$  is:

- $SO^*(6)$  if  $\text{Aff}_{***}(\tilde{L})$  acts irreducibly on  $W_{\chi_2}$ ;
- a subgroup of  $SO^*(4) \times SO^*(2)$  if  $W_{\chi_2} = U \oplus V$  for some  $\text{Aff}_{***}(\tilde{L})$ -irreducible  $\text{Aut}(\tilde{L})$ -submodules  $U$  and  $V$ ;
- a subgroup of  $SO^*(2) \times SO^*(2) \times SO^*(2)$  if  $W_{\chi_2} = W' \oplus W'' \oplus W'''$  for some  $\text{Aff}_{***}(\tilde{L})$ -irreducible  $\text{Aut}(\tilde{L})$ -submodules  $W', W''$  and  $W'''$ .

Actually, the possibility in the last item above cannot occur. Indeed, all Lyapunov exponents in  $W_{\chi_2}$  would then vanish, contradicting Theorem 5.2 saying that the Lyapunov spectrum of the KZ cocycle on  $W_{\chi_2}$  contains  $\lambda = 1/6$ .

In summary, we have shown the following result:

**Theorem 5.4.** *The Zariski closure of the monodromy group of the restriction of the KZ cocycle to  $W_{\chi_2}$  is either*

- $SO^*(6)$  if  $\text{Aff}_{***}(\tilde{L})$  acts irreducibly on  $W_{\chi_2}$ , or
- a subgroup of  $SO^*(4) \times SO^*(2)$  if  $W_{\chi_2} = U \oplus V$  for some  $\text{Aff}_{***}(\tilde{L})$ -irreducible  $\text{Aut}(\tilde{L})$ -submodules  $U \simeq 2\chi_2$  and  $V \simeq \chi_2$ .

**Remark 5.5.** This result suffices to show that an “exotic” monodromy group of the type described in item (iv) of Section 1 occurs for the restriction to  $W_{\chi_2}$  of the KZ cocycle over  $SL(2, \mathbb{R}) \cdot \tilde{L}$ . Nevertheless, we will complete the statement of Theorem 5.4 by showing (in the next section) that  $\text{Aff}_{***}(\tilde{L})$  acts irreducibly on  $W_{\chi_2}$ .

## 6. $\text{Aff}_{***}(\tilde{L})$ -irreducibility of $W_{\chi_2}$

In this section, we improve [Theorem 5.4](#) by showing the following result:

**Theorem 6.1.** *Any finite-index subgroup  $\text{Aff}_{***}(\tilde{L})$  of  $\text{Aff}(\tilde{L})$  acts irreducibly on  $W_{\chi_2}$ . In particular, the monodromy group of the KZ cocycle on  $W_{\chi_2}$  is  $SO^*(6)$ .*

Note that [Theorem 1.1](#) is a direct consequence of this theorem, so that our task is reduced to proving [Theorem 6.1](#). In this direction, we will compute the action on  $W_{\chi_2}$  of certain Dehn multitwists in a suitable basis of  $W_{\chi_2}$ .

### 6.1. A choice of basis of $W_{\chi_2}$

The cycles  $\mu_g, \sigma_g, \nu_g, \zeta_g$  displayed in [Figure 1](#) form a generating set of the relative homology group  $H_1(\tilde{L}, \Sigma, \mathbb{R})$ . Note that we have a relation

$$\square_g := \mu_g + \sigma_g + \nu_{gi} - \sigma_{gj} + \zeta_{gi} - \mu_{gj} - \zeta_g - \nu_g = 0$$

for each  $L_g, g \in Q$ . Observe that this gives us seven independent<sup>6</sup> relations because  $\sum_{g \in Q} \square_g = 0$ .

It is not difficult to check that, for each  $g \in Q$ , the cycles  $\sigma_g$  and  $\zeta_g$  are absolute cycles, and the cycles  $\mu_g$  and  $\nu_g$  are relative cycles with boundaries

$$\partial\mu_g = \overline{gi} - \overline{g} \quad \text{and} \quad \partial\nu_g = \overline{gj} - \overline{g}$$

For each  $g \in Q$ , consider the following absolute cycles:

$$\widehat{\mu}_g := \mu_g - \mu_{-g}, \quad \widehat{\sigma}_g := \sigma_g - \sigma_{-g}, \quad \widehat{\nu}_g := \nu_g - \nu_{-g}, \quad \widehat{\zeta}_g := \zeta_g - \zeta_{-g},$$

By definition,  $\{\widehat{\mu}_g, \widehat{\sigma}_g, \widehat{\nu}_g, \widehat{\zeta}_g\}_{g \in Q}$  is a generating set for  $W_{\chi_2}$ , and it is not hard to verify that the list

$$\mathcal{B} = \{\widehat{\sigma}_1, \widehat{\sigma}_i, \widehat{\sigma}_j, \widehat{\sigma}_k, \widehat{\zeta}_1, \widehat{\zeta}_i, \widehat{\zeta}_j, \widehat{\zeta}_k, \widehat{\mu}_1, \widehat{\mu}_i, \widehat{\nu}_1, \widehat{\nu}_j\}$$

of 12 absolute cycles is a basis of  $W_{\chi_2} \simeq 3\chi_2$ .

For later use, let us express  $\widehat{\mu}_j, \widehat{\mu}_k, \widehat{\nu}_i, \widehat{\nu}_k$  in terms of the basis  $\mathcal{B}$ . To do so, we expand the relations  $\square_g - \square_{-g} = 0$  for  $g = 1, i, j, k$ :

$$\widehat{\mu}_1 + \widehat{\sigma}_1 + \widehat{\nu}_i - \widehat{\sigma}_j + \widehat{\zeta}_i - \widehat{\mu}_j - \widehat{\zeta}_1 - \widehat{\nu}_1 = 0, \quad (6.1)$$

$$\widehat{\mu}_i + \widehat{\sigma}_i - \widehat{\nu}_1 - \widehat{\sigma}_k - \widehat{\zeta}_1 - \widehat{\mu}_k - \widehat{\zeta}_i - \widehat{\nu}_i = 0, \quad (6.2)$$

$$\widehat{\mu}_j + \widehat{\sigma}_j - \widehat{\nu}_k + \widehat{\sigma}_1 - \widehat{\zeta}_k + \widehat{\mu}_1 - \widehat{\zeta}_j - \widehat{\nu}_j = 0, \quad (6.3)$$

$$\widehat{\mu}_k + \widehat{\sigma}_k + \widehat{\nu}_j + \widehat{\sigma}_i + \widehat{\zeta}_j + \widehat{\mu}_i - \widehat{\zeta}_k - \widehat{\nu}_k = 0. \quad (6.4)$$

<sup>6</sup> In particular, we can extract from the generating set  $\{\mu_g, \sigma_g, \nu_g, \zeta_g\}_{g \in Q}$  of 32 cycles a basis of  $H_1(\tilde{L}, \Sigma, \mathbb{R})$  with  $32 - 7 = 25$  cycles. Of course, this agrees with the facts that  $\tilde{L}$  has genus 11 and  $\#\Sigma = 4$ , so that  $H_1(\tilde{L}, \Sigma, \mathbb{R})$  has dimension  $2 \times 11 + 4 - 1 = 25$ .

By adding together these four equations, we deduce that

$$2\widehat{\sigma}_1 + 2\widehat{\sigma}_i - 2\widehat{\zeta}_1 - 2\widehat{\zeta}_k + 2\widehat{\mu}_1 + 2\widehat{\mu}_i - 2\widehat{v}_1 - 2\widehat{v}_k = 0,$$

that is,

$$\widehat{v}_k = \widehat{\sigma}_1 + \widehat{\sigma}_i - \widehat{\zeta}_1 - \widehat{\zeta}_k + \widehat{\mu}_1 + \widehat{\mu}_i - \widehat{v}_1. \quad (6.5)$$

By substituting (6.5) into (6.3) and (6.4), we obtain

$$\widehat{\mu}_j = \widehat{\sigma}_i - \widehat{\sigma}_j - \widehat{\zeta}_1 + \widehat{\zeta}_j + \widehat{\mu}_i - \widehat{v}_1 + \widehat{v}_j, \quad (6.6)$$

$$\widehat{\mu}_k = \widehat{\sigma}_1 - \widehat{\sigma}_k - \widehat{\zeta}_1 - \widehat{\zeta}_j + \widehat{\mu}_1 - \widehat{v}_1 - \widehat{v}_j. \quad (6.7)$$

Finally, by substituting (6.6) into (6.1) (or equivalently (6.7) into (6.2)), we get

$$\widehat{v}_i = -\widehat{\sigma}_1 + \widehat{\sigma}_i - \widehat{\zeta}_i + \widehat{\zeta}_j - \widehat{\mu}_1 + \widehat{\mu}_i + \widehat{v}_j. \quad (6.8)$$

## 6.2. Dehn multitwist in the direction (1, 1)

The straight lines in the direction (1, 1) decompose  $\widetilde{L}$  into eight cylinders  $a+$ ,  $a-$ ,  $b+$ ,  $b-$ ,  $c+$ ,  $c-$ ,  $d+$ ,  $d-$  such that:

- for each  $* \in \{a, b, c, d\}$ , the cylinder  $*-$  is the image of  $*+$  under the automorphism  $-1 \in Q$ ;
- the cylinders  $a+$ ,  $b+$  and  $c+$  cross the cycles  $\mu_1$ ,  $v_1$  and  $\zeta_1$  (resp.) in  $L_1$ , and the cylinder  $d+$  crosses the cycle  $v_j$  in  $L_j$ .

Note that the ratio between the height and the width of each of these cylinders is  $1/3$ . Thus, the matrix  $\underline{A} \in SL(2, \mathbb{Z})$  such that  $\underline{A}(1, 1) = (1, 1)$  and  $\underline{A}(1, 0) = (1, 0) + 3(1, 1) = (4, 3)$ , i.e.,  $\underline{A} = \begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix}$ , belongs to the Veech group  $SL(\widetilde{L})$ . By a slight abuse of notation, we also write  $\underline{A}$  for the element of the affine group  $\text{Aff}(\widetilde{L})$  with derivative  $\underline{A} \in SL(2, \mathbb{Z})$  fixing pointwise the elements of  $\Sigma$ .

For each  $* \in \{a, b, c, d\}$ , let  $\rho_{*+}$ , resp.  $\rho_{*-}$ , be the homology class of the waist curve of the cylinder  $*+$ , resp.  $*-$ . Observe that the absolute cycle  $\rho_{*-}$  is the image under the automorphism  $-1 \in Q$  of the absolute cycle  $\rho_{*+}$ . In particular, the cycles  $\widehat{\rho}_* := \rho_{*+} - \rho_{*-}$  belong to  $W_{\chi_2}$ .

Denote by  $A$  the action on the relative homology group  $H_1(\widetilde{L}, \Sigma, \mathbb{R})$  induced by  $\underline{A}$ . We have

$$\begin{aligned} A(\sigma_1) &= \sigma_1 + \rho_{c-}, & A(\sigma_i) &= \sigma_i + \rho_{b-}, & A(\sigma_j) &= \sigma_j + \rho_{a+}, & A(\sigma_k) &= \sigma_k + \rho_{d+}, \\ A(\zeta_1) &= \zeta_1 - \rho_{c+}, & A(\zeta_i) &= \zeta_i - \rho_{b+}, & A(\zeta_j) &= \zeta_j - \rho_{a-}, & A(\zeta_k) &= \zeta_k - \rho_{d-}, \\ A(\mu_1) &= \mu_1 + \rho_{a+}, & A(\mu_i) &= \mu_i + \rho_{d+}, & A(v_1) &= v_1 - \rho_{b+}, & A(v_j) &= v_j - \rho_{d+}. \end{aligned}$$

Therefore, the action of  $A$  on  $W_{\chi_2}$  is described by the formulas

$$\begin{aligned} A(\widehat{\sigma}_1) &= \widehat{\sigma}_1 - \widehat{\rho}_c, & A(\widehat{\sigma}_i) &= \widehat{\sigma}_i - \widehat{\rho}_b, & A(\widehat{\sigma}_j) &= \widehat{\sigma}_j + \widehat{\rho}_a, & A(\widehat{\sigma}_k) &= \widehat{\sigma}_k + \widehat{\rho}_d, \\ A(\widehat{\zeta}_1) &= \widehat{\zeta}_1 - \widehat{\rho}_c, & A(\widehat{\zeta}_i) &= \widehat{\zeta}_i - \widehat{\rho}_b, & A(\widehat{\zeta}_j) &= \widehat{\zeta}_j + \widehat{\rho}_a, & A(\widehat{\zeta}_k) &= \widehat{\zeta}_k + \widehat{\rho}_d, \\ A(\widehat{\mu}_1) &= \widehat{\mu}_1 + \widehat{\rho}_a, & A(\widehat{\mu}_i) &= \widehat{\mu}_i + \widehat{\rho}_d, & A(\widehat{v}_1) &= \widehat{v}_1 - \widehat{\rho}_b, & A(\widehat{v}_j) &= \widehat{v}_j - \widehat{\rho}_d. \end{aligned} \quad (6.9)$$

This information allows us to write the matrix of  $A$  in our preferred basis  $\mathcal{B}$  once we express each  $\widehat{\rho}_*$ ,  $*$   $\in \{a, b, c, d\}$  in terms of the elements of  $\mathcal{B}$ . To do so, we note that the symmetry provided by the automorphism  $-1 \in Q$  reduces this task to calculating  $\rho_{*+}$ ,  $*$   $\in \{a, b, c, d\}$ .

A direct inspection of [Figure 1](#) reveals that

$$\begin{aligned} \rho_{a+} &= \mu_1 + \sigma_1 + \nu_i + \nu_{-k} + \zeta_{-k} + \mu_i, & \rho_{c+} &= \mu_j + \sigma_j + \nu_{-k} + \nu_{-i} + \zeta_{-i} + \mu_{-k}, \\ \rho_{b+} &= \zeta_1 + \mu_j + \mu_k + \sigma_k + \nu_j + \nu_1, & \rho_{d+} &= \zeta_j + \mu_{-1} + \mu_i + \sigma_i + \nu_{-1} + \nu_j. \end{aligned}$$

Hence,

$$\begin{aligned} \widehat{\rho}_a &= \widehat{\sigma}_1 - \widehat{\zeta}_k + \widehat{\mu}_1 + \widehat{\mu}_i + \widehat{\nu}_i - \widehat{\nu}_k, & \widehat{\rho}_c &= \widehat{\sigma}_j - \widehat{\zeta}_i + \widehat{\mu}_j - \widehat{\mu}_k - \widehat{\nu}_i - \widehat{\nu}_k, \\ \widehat{\rho}_b &= \widehat{\sigma}_k + \widehat{\zeta}_1 + \widehat{\mu}_j + \widehat{\mu}_k + \widehat{\nu}_1 + \widehat{\nu}_j, & \widehat{\rho}_d &= \widehat{\sigma}_i + \widehat{\zeta}_j - \widehat{\mu}_1 + \widehat{\mu}_i - \widehat{\nu}_1 + \widehat{\nu}_j. \end{aligned}$$

By substituting (6.5)–(6.8) into the previous equations, we deduce that the formulas for  $\widehat{\rho}_*$ ,  $*$   $\in \{a, b, c, d\}$ , in terms of  $\mathcal{B}$  are

$$\begin{aligned} \widehat{\rho}_a &= -\widehat{\sigma}_1 + \widehat{\zeta}_1 - \widehat{\zeta}_i + \widehat{\zeta}_j - \widehat{\mu}_1 + \widehat{\mu}_i + \widehat{\nu}_1 + \widehat{\nu}_j, \\ \widehat{\rho}_b &= \widehat{\sigma}_1 + \widehat{\sigma}_i - \widehat{\sigma}_j - \widehat{\zeta}_1 + \widehat{\mu}_1 + \widehat{\mu}_i - \widehat{\nu}_1 + \widehat{\nu}_j, \\ \widehat{\rho}_c &= -\widehat{\sigma}_1 - \widehat{\sigma}_i + \widehat{\sigma}_k + \widehat{\zeta}_1 + \widehat{\zeta}_j + \widehat{\zeta}_k - \widehat{\mu}_1 - \widehat{\mu}_i + \widehat{\nu}_1 + \widehat{\nu}_j, \\ \widehat{\rho}_d &= \widehat{\sigma}_i + \widehat{\zeta}_j - \widehat{\mu}_1 + \widehat{\mu}_i - \widehat{\nu}_1 + \widehat{\nu}_j. \end{aligned} \tag{6.10}$$

By combining (6.9) and (6.10), we find that the matrix of  $A$  in the basis  $\mathcal{B}$  is

$$A = \begin{pmatrix} 2 & -1 & -1 & 0 & 1 & -1 & -1 & 0 & -1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 & 1 & -1 & 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & -1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & -1 & 0 & 2 & 1 & 1 & 1 & 0 & -1 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 0 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 2 & -1 & -1 \\ -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & 2 & 1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & 0 \end{pmatrix}. \tag{6.11}$$

### 6.3. Dehn multitwist in the direction $(3, -1)$

The straight lines in the direction  $(3, -1)$  decompose  $\widetilde{L}$  into eight cylinders  $\alpha+$ ,  $\alpha-$ ,  $\beta+$ ,  $\beta-$ ,  $\gamma+$ ,  $\gamma-$ ,  $\delta+$ ,  $\delta-$  such that:

- for each  $*$   $\in \{\alpha, \beta, \gamma, \delta\}$ , the cylinder  $*-$  is the image of  $*+$  under the automorphism  $-1 \in Q$ ;
- the cylinders  $\alpha+$  and  $\beta+$  cross the cycles  $\mu_1$  and  $\sigma_1$  (resp.) in  $L_1$ , and the cylinder  $\gamma+$ , resp.  $\delta+$ , crosses the cycles  $\nu_1$  and  $\nu_i$ , resp.  $\zeta_1$  and  $\nu_i$ , in  $L_1$ .

Note that the matrix  $\underline{B} \in SL(2, \mathbb{Z})$  such that  $\underline{B}(3, -1) = (3, -1)$  and  $\underline{B}(1, 0) = (1, 0) + 3(3, -1) = (10, -3)$ , i.e.,  $\underline{B} = \begin{pmatrix} 10 & 27 \\ -3 & -8 \end{pmatrix}$ , belongs to the Veech group  $SL(\tilde{L})$ . By a slight abuse of notation, we also write  $\underline{B}$  for the element of the affine group  $\text{Aff}(\tilde{L})$  with derivative  $\underline{B} \in SL(2, \mathbb{Z})$  fixing pointwise the elements of  $\Sigma$ .

For each  $* \in \{\alpha, \beta, \gamma, \delta\}$ , let  $\rho_{*+}$ , resp.  $\rho_{*-}$ , be the homology class of the waist curve of the cylinder  $*+$ , resp.  $*-$ . Observe that the absolute cycle  $\rho_{*-}$  is the image under the automorphism  $-1 \in Q$  of the absolute cycle  $\rho_{*+}$ . In particular, the cycles  $\widehat{\rho}_* := \rho_{*+} - \rho_{*-}$  belong to  $W_{\chi_2}$ .

Denote by  $B$  the action on the relative homology group  $H_1(\tilde{L}, \Sigma, \mathbb{R})$  induced by  $\underline{B}$ . We have

$$\begin{aligned} B(\sigma_1) &= \sigma_1 + \rho_{\beta+}, & B(\sigma_i) &= \sigma_i + \rho_{\delta+}, \\ B(\sigma_j) &= \sigma_j + \rho_{\alpha-}, & B(\sigma_k) &= \sigma_k + \rho_{\gamma-}, \\ B(\zeta_1) &= \zeta_1 + \rho_{\delta+} + \rho_{\beta-} + \rho_{\delta-}, & B(\zeta_i) &= \zeta_i + \rho_{\beta-} + \rho_{\delta-} + \rho_{\beta+}, \\ B(\zeta_j) &= \zeta_j + \rho_{\gamma+} + \rho_{\alpha+} + \rho_{\gamma-}, & B(\zeta_k) &= \zeta_k + \rho_{\alpha-} + \rho_{\gamma+} + \rho_{\alpha+}, \\ B(\mu_1) &= \mu_1 + \rho_{\alpha+}, & B(\mu_i) &= \mu_i + \rho_{\gamma+}, \\ B(\nu_1) &= \nu_1 + \rho_{\alpha+} + \rho_{\beta+} + \rho_{\gamma+}, & B(\nu_j) &= \nu_j + \rho_{\beta+} + \rho_{\alpha-} + \rho_{\delta-}. \end{aligned}$$

Therefore, the action of  $B$  on  $W_{\chi_2}$  is described by

$$\begin{aligned} B(\widehat{\sigma}_1) &= \widehat{\sigma}_1 + \widehat{\rho}_\beta, & B(\widehat{\sigma}_i) &= \widehat{\sigma}_i + \widehat{\rho}_\delta, \\ B(\widehat{\sigma}_j) &= \widehat{\sigma}_j - \widehat{\rho}_\alpha, & B(\widehat{\sigma}_k) &= \widehat{\sigma}_k - \widehat{\rho}_\gamma, \\ B(\widehat{\zeta}_1) &= \widehat{\zeta}_1 - \widehat{\rho}_\beta, & B(\widehat{\zeta}_i) &= \widehat{\zeta}_i - \widehat{\rho}_\delta, \\ B(\widehat{\zeta}_j) &= \widehat{\zeta}_j + \widehat{\rho}_\alpha, & B(\widehat{\zeta}_k) &= \widehat{\zeta}_k + \widehat{\rho}_\gamma, \\ B(\widehat{\mu}_1) &= \widehat{\mu}_1 + \widehat{\rho}_\alpha, & B(\widehat{\mu}_i) &= \widehat{\mu}_i + \widehat{\rho}_\gamma, \\ B(\widehat{\nu}_1) &= \widehat{\nu}_1 + \widehat{\rho}_\alpha + \widehat{\rho}_\beta + \widehat{\rho}_\gamma, & B(\widehat{\nu}_j) &= \widehat{\nu}_j - \widehat{\rho}_\alpha + \widehat{\rho}_\beta - \widehat{\rho}_\delta. \end{aligned} \tag{6.12}$$

This permits us to write the matrix of  $B$  in our preferred basis  $\mathcal{B}$  once we express each  $\widehat{\rho}_*$ ,  $* \in \{\alpha, \beta, \gamma, \delta\}$  in terms of the elements of  $\mathcal{B}$ . To do so, we note that the symmetry provided by the automorphism  $-1 \in Q$  reduces this task to calculating  $\rho_{*+}$ ,  $* \in \{\alpha, \beta, \gamma, \delta\}$ .

A direct inspection of [Figure 1](#) shows that

$$\begin{aligned} \rho_{\alpha+} &= \sigma_i + \sigma_{-i} + \sigma_{-j} - \zeta_{-j} + \mu_1 + \mu_{-1} + \mu_i + 2\mu_{-i} + \mu_{-j} - \nu_{-i} - \nu_{-j}, \\ \rho_{\beta+} &= \sigma_1 + \sigma_k + \sigma_{-k} - \zeta_1 + \mu_1 + \mu_j + \mu_{-j} + 2\mu_k + \mu_{-k} - \nu_1 - \nu_k, \\ \rho_{\gamma+} &= \sigma_1 + \sigma_{-1} + \sigma_{-k} - \zeta_{-k} + 2\mu_1 + \mu_{-1} + \mu_i + \mu_{-i} + \mu_{-k} - \nu_1 - \nu_{-k}, \\ \rho_{\delta+} &= \sigma_i + \sigma_j + \sigma_{-j} - \zeta_i + \mu_i + \mu_j + 2\mu_{-j} + \mu_k + \mu_{-k} - \nu_i - \nu_{-j}. \end{aligned}$$

Hence,

$$\begin{aligned} \widehat{\rho}_\alpha &= -\widehat{\sigma}_j + \widehat{\zeta}_j - \widehat{\mu}_i - \widehat{\mu}_j + \widehat{\nu}_i + \widehat{\nu}_j, \\ \widehat{\rho}_\beta &= \widehat{\sigma}_1 - \widehat{\zeta}_1 + \widehat{\mu}_1 + \widehat{\mu}_k - \widehat{\nu}_1 - \widehat{\nu}_k, \\ \widehat{\rho}_\gamma &= -\widehat{\sigma}_k + \widehat{\zeta}_k + \widehat{\mu}_1 - \widehat{\mu}_k - \widehat{\nu}_1 + \widehat{\nu}_k, \\ \widehat{\rho}_\delta &= \widehat{\sigma}_i - \widehat{\zeta}_i + \widehat{\mu}_i - \widehat{\mu}_j - \widehat{\nu}_i + \widehat{\nu}_j. \end{aligned}$$

By substituting (6.5)–(6.8) into the previous equations, we deduce that the formulas for  $\widehat{\rho}_*$ ,  $*$   $\in \{\alpha, \beta, \gamma, \delta\}$  in terms of  $\mathcal{B}$  are

$$\begin{aligned}\widehat{\rho}_\alpha &= -\widehat{\sigma}_1 + \widehat{\zeta}_1 - \widehat{\zeta}_i + \widehat{\zeta}_j - \widehat{\mu}_1 - \widehat{\mu}_i + \widehat{\nu}_1 + \widehat{\nu}_j, \\ \widehat{\rho}_\beta &= \widehat{\sigma}_1 - \widehat{\sigma}_i - \widehat{\sigma}_k - \widehat{\zeta}_1 - \widehat{\zeta}_j + \widehat{\zeta}_k + \widehat{\mu}_1 - \widehat{\mu}_i - \widehat{\nu}_1 - \widehat{\nu}_j, \\ \widehat{\rho}_\gamma &= \widehat{\sigma}_i + \widehat{\zeta}_j + \widehat{\mu}_1 + \widehat{\mu}_i - \widehat{\nu}_1 + \widehat{\nu}_j, \\ \widehat{\rho}_\delta &= \widehat{\sigma}_1 - \widehat{\sigma}_i + \widehat{\sigma}_j + \widehat{\zeta}_1 - 2\widehat{\zeta}_j + \widehat{\mu}_1 - \widehat{\mu}_i + \widehat{\nu}_1 - \widehat{\nu}_j.\end{aligned}\tag{6.13}$$

By combining (6.12) and (6.13), we see that the matrix of  $B$  in the basis  $\mathcal{B}$  is

$$B = \begin{pmatrix} 2 & 1 & 1 & 0 & -1 & -1 & -1 & 0 & -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ -1 & 1 & -1 & 0 & 2 & -1 & 1 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & -1 & 0 & -1 & 1 \\ -1 & -2 & -1 & -1 & 1 & 2 & 2 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 0 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 2 & -1 & 1 \\ -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 0 & -3 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}.\tag{6.14}$$

#### 6.4. Dehn multitwist in the direction $(-1, 3)$

The origami  $\widetilde{L}$  has a “symmetry” given by the (orientation-reversing) matrix  $\underline{R} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in GL(2, \mathbb{Z})$ .

More precisely, by applying this matrix to each  $L_g$ ,  $g \in Q$ , in Figure 1, we exchange the roles of  $\mu_g, \mu_{gj}$ , resp.  $\sigma_g, \sigma_{gj}$ , and  $\nu_g, \nu_{gi}$ , resp.  $\zeta_g, \zeta_{gi}$ . By relabeling the sides  $\nu_g, \zeta_g, \mu_g, \sigma_g$  of  $\underline{R}(L_g)$  as  $\mu_{\phi(g)}, \sigma_{\phi(g)}, \nu_{\phi(g)}, \zeta_{\phi(g)}$  (resp.) where  $\phi : Q \rightarrow Q$  is the (outer) automorphism of  $Q$  with  $\phi(i) = j$ ,  $\phi(j) = i$ ,  $\phi(k) = -k$ , we recover the origami  $\widetilde{L}$ .

Note that this “symmetry” exchanges the roles of the directions  $(3, -1)$  and  $(-1, 3)$  in the origami  $\widetilde{L}$ . In particular, the eight cylinders  $*+, *-$  with  $*$   $\in \{\alpha, \beta, \gamma, \delta\}$  of  $\widetilde{L}$  in the direction  $(3, -1)$  are associated under this “symmetry” to the eight cylinders  $*+, *-$  with  $*$   $\in \{\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta}\}$ .

The matrix  $\underline{C} = \begin{pmatrix} -8 & -3 \\ 27 & 10 \end{pmatrix}$  deduced from  $\underline{B}$  by conjugation with  $\underline{R}$  belongs to the Veech group  $SL(\widetilde{L})$ . Denote by  $C$  the action on homology of the affine homeomorphism with linear part  $\underline{C}$  and fixing  $\Sigma$  pointwise.

For each  $*$   $\in \{\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta}\}$ , let  $\rho_{*+}$ , resp.  $\rho_{*-}$ , be the absolute homology classes of the waist curves of the cylinder  $*+$ , resp.  $*-$ , so that the cycles  $\widehat{\rho}_* := \rho_{*+} - \rho_{*-}$  belong to  $W_{\chi_2}$ .

By “symmetry”, we deduce from the analogous formulas for  $B$  that the action of  $C$  on  $W_{\chi_2}$  is

$$\begin{aligned} C(\widehat{\xi}_1) &= \widehat{\xi}_1 + \widehat{\rho}_{\overline{\beta}}, & C(\widehat{\xi}_j) &= \widehat{\xi}_j + \widehat{\rho}_{\overline{\delta}}, \\ C(\widehat{\xi}_i) &= \widehat{\xi}_i - \widehat{\rho}_{\overline{\alpha}}, & C(\widehat{\xi}_k) &= \widehat{\xi}_k + \widehat{\rho}_{\overline{\gamma}}, \\ C(\widehat{\sigma}_1) &= \widehat{\sigma}_1 - \widehat{\rho}_{\overline{\beta}}, & C(\widehat{\sigma}_j) &= \widehat{\sigma}_j - \widehat{\rho}_{\overline{\delta}}, \\ C(\widehat{\sigma}_i) &= \widehat{\sigma}_i + \widehat{\rho}_{\overline{\alpha}}, & C(\widehat{\sigma}_k) &= \widehat{\sigma}_k - \widehat{\rho}_{\overline{\gamma}}, \\ C(\widehat{v}_1) &= \widehat{v}_1 + \widehat{\rho}_{\overline{\alpha}}, & C(\widehat{v}_j) &= \widehat{v}_j + \widehat{\rho}_{\overline{\gamma}}, \\ C(\widehat{\mu}_1) &= \widehat{\mu}_1 + \widehat{\rho}_{\overline{\alpha}} + \widehat{\rho}_{\overline{\beta}} + \widehat{\rho}_{\overline{\gamma}}, & C(\widehat{\mu}_i) &= \widehat{\mu}_i - \widehat{\rho}_{\overline{\alpha}} + \widehat{\rho}_{\overline{\beta}} - \widehat{\rho}_{\overline{\delta}}, \end{aligned}$$

where

$$\begin{aligned} \widehat{\rho}_{\overline{\alpha}} &= -\widehat{\xi}_1 + \widehat{\sigma}_1 - \widehat{\sigma}_j + \widehat{\sigma}_i - \widehat{v}_1 - \widehat{v}_j + \widehat{\mu}_1 + \widehat{\mu}_i, \\ \widehat{\rho}_{\overline{\beta}} &= \widehat{\xi}_1 - \widehat{\xi}_j + \widehat{\xi}_k - \widehat{\sigma}_1 - \widehat{\sigma}_i - \widehat{\sigma}_k + \widehat{v}_1 - \widehat{v}_j - \widehat{\mu}_1 - \widehat{\mu}_i, \\ \widehat{\rho}_{\overline{\gamma}} &= \widehat{\xi}_j + \widehat{\sigma}_i + \widehat{v}_1 + \widehat{v}_j - \widehat{\mu}_1 + \widehat{\mu}_i, \\ \widehat{\rho}_{\overline{\delta}} &= \widehat{\xi}_1 - \widehat{\xi}_j + \widehat{\xi}_i + \widehat{\sigma}_1 - 2\widehat{\sigma}_i + \widehat{v}_1 - \widehat{v}_j + \widehat{\mu}_1 - \widehat{\mu}_i. \end{aligned}$$

Therefore, the matrix of  $C$  in the basis  $\mathcal{B}$  is

$$C = \begin{pmatrix} 2 & 1 & -1 & 0 & -1 & -1 & 1 & 0 & 0 & -3 & 1 & 0 \\ 1 & 2 & 2 & -1 & -1 & -1 & -2 & 1 & 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ -1 & -1 & -1 & 0 & 2 & 1 & 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 0 & -3 & 1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 0 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 2 \end{pmatrix}. \quad (6.15)$$

### 6.5. End of proof of [Theorem 6.1](#)

By [Theorem 5.4](#), the proof of [Theorem 6.1](#) is complete once we show that  $W_{\chi_2}$  cannot be decomposed as

$$W_{\chi_2} = U \oplus V$$

for some symplectic  $\text{Aff}_{***}(\widetilde{L})$ -invariant  $\text{Aut}(\widetilde{L})$ -submodules  $U \simeq 2\chi_2$  and  $V \simeq \chi_2$ , where  $\text{Aff}_{***}(L)$  is a finite-index subgroup of  $\text{Aff}(\widetilde{L})$ .

Suppose that  $W_{\chi_2}$  admits such a decomposition. Recall from [Subsection 5.5](#) that the action of  $\text{Aff}_{***}(\widetilde{L})$  on  $V \simeq \chi_2$  occurs through a subgroup of the compact group  $SO^*(2)$ . Thus,  $V$  is the central eigenspace of the action of any element of  $\text{Aff}_{***}(\widetilde{L})$  on  $W_{\chi_2}$  whose spectrum is “simple” (i.e., it has three eigenvalues of multiplicity four whose moduli are distinct).

Hence, we reach a contradiction if there are two elements of  $\text{Aff}_{***}(\tilde{L})$  acting on  $W_{\chi_2}$  with simple spectrum such that their central eigenspaces are distinct.

We claim that, for some  $k, l \in \mathbb{N}$ , the matrices  $(A.B)^k$  and  $(C.B)^l$  associated to the actions on  $W_{\chi_2}$  of the powers  $(\underline{A} \circ \underline{B})^k$  and  $(\underline{C} \circ \underline{B})^l$  of the affine homeomorphisms  $\underline{A} \circ \underline{B}$  and  $\underline{C} \circ \underline{B}$  have the desired properties.

Indeed, a direct computation with (6.11) and (6.14) shows that the matrix

$$A.B = \begin{pmatrix} 5 & 2 & 0 & 2 & -2 & -4 & -2 & -2 & -2 & -2 & -1 & 1 \\ 2 & 1 & 2 & 0 & 0 & -2 & -2 & 2 & -2 & 2 & -1 & 1 \\ -2 & 2 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & -1 & -3 \\ -2 & -2 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & -1 & 1 \\ -4 & 0 & 0 & -2 & 3 & 2 & 2 & 2 & 2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 2 & 0 & 1 & -2 & -2 & -2 & -2 & -3 & 1 \\ -2 & -4 & 2 & -2 & 0 & 4 & 1 & 4 & 0 & 4 & 1 & 3 \\ 0 & -2 & 0 & 0 & -2 & 2 & 0 & 1 & 0 & 0 & 1 & 3 \\ 4 & 2 & -2 & 0 & -2 & -4 & 0 & -2 & 1 & -2 & 2 & 0 \\ 2 & 0 & 4 & -2 & 0 & -2 & -2 & 4 & -2 & 5 & 0 & 2 \\ -4 & 0 & -2 & -2 & 2 & 2 & 4 & 0 & 4 & 0 & 3 & -4 \\ 0 & -4 & 2 & -2 & -2 & 2 & 0 & 4 & 0 & 4 & 2 & 5 \end{pmatrix}$$

has characteristic polynomial

$$\begin{aligned} P_{AB}(x) &= x^{12} - 28x^{11} + 322x^{10} - 1964x^9 + 6895x^8 - 14392x^7 \\ &\quad + 18332x^6 - 14392x^5 + 6895x^4 - 1964x^3 + 322x^2 - 28x + 1 \\ &= (x-1)^4(x^2-6x+1)^4. \end{aligned}$$

In particular,  $A.B$  has three eigenvalues (each with multiplicity four), namely,  $3 + 2\sqrt{2}$ , 1 and  $3 - 2\sqrt{2}$ , so that  $A.B$  has “simple” spectrum. Furthermore, it is not hard to see that the central eigenspace  $V_{AB}$  of  $A.B$  (associated to the eigenvalue 1) is spanned by the following four vectors:

$$\begin{aligned} v_{AB}^{(1)} &= (-1, 1, -1, 1, 1, -1, 1, 1, 0, 0, 0, 2), \\ v_{AB}^{(2)} &= (-1, -1, 1, 1, 1, -1, -1, -1, 0, 0, 2, 0), \\ v_{AB}^{(3)} &= (0, 0, 0, 0, 0, 0, 0, -1, 0, 1, 0, 0), \\ v_{AB}^{(4)} &= (0, 0, 0, 0, 0, 0, -1, 0, 1, 0, 0, 0). \end{aligned}$$

Similarly, an immediate calculation with (6.14) and (6.15) reveals that the matrix

$$C.B = \begin{pmatrix} 5 & 2 & -4 & 2 & -4 & -2 & 4 & -2 & 3 & -5 & 5 & -1 \\ 4 & 7 & 4 & -2 & -4 & -6 & -4 & 2 & -1 & 1 & 3 & 1 \\ 0 & -2 & 3 & 0 & 0 & 2 & -2 & 0 & -3 & 1 & -3 & 3 \\ 2 & 0 & 0 & 3 & -2 & 0 & 0 & -2 & -1 & -3 & -1 & 1 \\ -4 & -4 & -2 & -2 & 5 & 4 & 2 & 2 & 1 & 3 & -1 & -1 \\ 0 & -2 & -2 & 0 & 0 & 3 & 2 & 0 & 1 & -1 & 1 & 1 \\ 4 & 0 & 2 & -2 & -4 & 0 & -1 & 2 & -1 & 1 & 3 & 5 \\ -2 & 0 & 0 & -2 & 2 & 0 & 0 & 3 & 1 & 3 & 1 & -1 \\ 2 & 2 & -4 & 4 & -2 & -2 & 4 & -4 & 3 & -6 & 2 & -4 \\ 4 & 4 & 2 & -2 & -4 & -4 & -2 & 2 & 0 & 1 & 4 & 2 \\ -2 & -4 & -2 & -4 & 2 & 4 & 2 & 4 & 2 & 4 & 3 & 2 \\ 4 & -2 & 4 & -2 & -4 & 2 & -4 & 2 & -4 & 2 & 0 & 9 \end{pmatrix}$$



has characteristic polynomial

$$\begin{aligned} P_{CB}(x) &= x^{12} - 44x^{11} + 770x^{10} - 6780x^9 + 31471x^8 - 76120x^7 \\ &\quad + 101404x^6 - 76120x^5 + 31471x^4 - 6780x^3 + 770x^2 - 44x + 1 \\ &= (x-1)^4(x^2-10x+1)^4. \end{aligned}$$

In particular,  $C.B$  has three eigenvalues (each with multiplicity four), namely,  $5 + 2\sqrt{6}$ , 1 and  $5 - 2\sqrt{6}$ , so that  $C.B$  also has “simple” spectrum. Furthermore, it is not hard to see that the central eigenspace  $V_{CB}$  of  $C.B$  (associated to the eigenvalue 1) is spanned by the following four vectors:

$$\begin{aligned} v_{CB}^{(1)} &= (0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0), \\ v_{CB}^{(2)} &= (0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0), \\ v_{CB}^{(3)} &= (0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0), \\ v_{CB}^{(4)} &= (1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0). \end{aligned}$$

It follows that  $V_{AB}$  and  $V_{CB}$  are distinct (and actually  $\{v_{AB}^{(n)}, v_{CB}^{(m)}\}_{1 \leq n, m \leq 4}$  span an 8-dimensional subspace). Moreover, the same properties are true for any powers  $(A.B)^k$  and  $(C.B)^l$ : the matrices  $(A.B)^k$  and  $(C.B)^l$  have “simple spectrum” and  $V_{(A.B)^k} = V_{A.B}$  and  $V_{(C.B)^l} = V_{C.B}$  are distinct for all  $k, l \in \mathbb{N}$ . By taking  $k, l \in \mathbb{N}$  so that  $(\underline{A} \circ \underline{B})^k$  and  $(\underline{C} \circ \underline{B})^l$  belong to  $\text{Aff}_{***}(\tilde{L})$  (this is always possible because  $\text{Aff}_{***}(\tilde{L})$  is a finite-index subgroup of  $\text{Aff}(\tilde{L})$ ), we conclude that there is no decomposition  $W_{\chi_2} = U \oplus V$  with the features described at the beginning of this subsection. This completes the proof of [Theorem 6.1](#).  $\square$

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