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A sharp quantitative version of Alexandrov's theorem via the method of moving planes

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Abstract. We prove the following quantitative version of the celebrated *Soap Bubble Theorem* of Alexandrov. Let S be a C^2 closed embedded hypersurface of $\mathbb{R}^{n+1}$, $n \geq 1$, and denote by $\operatorname{osc}(H)$ the oscillation of its mean curvature. We prove that there exists a positive ε, depending on n and upper bounds on the area and the C^2-regularity of S, such that if $\operatorname{osc}(H) \leq \varepsilon$ then there exist two concentric balls B_{r_i} and B_{r_e} such that $S \subset \overline{B}_{r_e} \setminus B_{r_i}$ and $r_e - r_i \leq C \operatorname{osc}(H)$, with C depending only on n and upper bounds on the surface area of S and the C^2-regularity of S. Our approach is based on a quantitative study of the method of moving planes, and the quantitative estimate on $r_e - r_i$ we obtain is optimal.

As a consequence, we also prove that if $\operatorname{osc}(H)$ is small then S is diffeomorphic to a sphere, and give a quantitative bound which implies that S is C^1-close to a sphere.

Keywords. Alexandrov Soap Bubble Theorem, method of moving planes, stability, mean curvature, pinching

Contents

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1. Introduction

The *Soap Bubble Theorem* proved by Alexandrov [A2] has been the object of many investigations. In its simplest form it states that

> *The n-dimensional sphere is the only compact connected embedded hypersurface of $\mathbb{R}^{n+1}$ with constant mean curvature.*

As is well-known, the embeddedness condition is necessary, as implied by the celebrated counterexamples by Hsiang–Teng–Yu [HYY] and Wente [W]. There have been several extensions of the rigidity result of Alexandrov to more general settings. Alexandrov proved his theorem in a more general setting; in particular, the Euclidean space can be replaced by any space of constant curvature (see also [A3] where he discussed several possible generalizations). Montiel and Ros [MR] and Korevaar [K] studied the case of hypersurfaces with constant higher order mean curvatures embedded in space forms. Alexandrov's Theorem has also been studied for warped product manifolds by Montiel [Mo], Brendle [B] and Brendle and Eichmair [BE]. There are many other related results; the interested reader can refer to [CFSW, CFMN, CY, DCL, HY, Re, Ros1, Ros2, Y] and references therein.

To prove the Soap Bubble Theorem, Alexandrov introduced the *method of moving planes*, a very powerful technique which has been the source of many insights in analysis and differential geometry. Serrin understood that the method can be applied to partial differential equations. Indeed, in his seminal paper [Se] he obtained a symmetry result for the torsion problem which gave rise to a huge amount of results for overdetermined problems (the interest reader can consult the references in [CMS1]). Gidas, Ni and Nirenberg [GNN] refined Serrin's argument to obtain several symmetry results for positive solutions of second order elliptic equations in bounded and unbounded domains (see also [Li1] and [Li2]). The method was further employed by Caffarelli, Gidas and Spruck [CGS] to prove asymptotic radial symmetry of positive solutions for the conformal scalar curvature equation and other semilinear elliptic equations (see also [KMPS]). The moving planes were also used to obtain several celebrated results in differential geometry: Schoen [Sch] characterized the catenoid, and Meeks [Me] and Korevaar, Kusner and Solomon [KKS] showed that a complete connected properly embedded constant mean curvature surface in the Euclidean space with two annuli ends is rotationally symmetric. There are a large number of other interesting papers on these topics which are not mentioned here.

Alexandrov's proof in the Euclidean space works as follows: (i) one shows that for any direction ω there exists a critical hyperplane orthogonal to ω which is a hyperplane of symmetry for the surface S; (ii) since the center $\mathcal{O}$ of mass of S lies on each hyperplane of symmetry, every hyperplane passing through $\mathcal{O}$ is a hyperplane of reflection symmetry for S; (iii) since any rotation about $\mathcal{O}$ can be written as a composition of $n+1$ reflections, S is rotationally invariant, which implies that S is the n-dimensional sphere. The crucial step in this proof is (i), which is obtained by applying the method of moving planes and the maximum principle (see Theorem A in Subsection 2.2).

In this paper we study a quantitative version of the Soap Bubble Theorem, that is, we assume that the oscillation of the mean curvature $\operatorname{osc}(H)$ is *small* and we prove that S

is *close to a sphere*. More precisely, let S be an n-dimensional, C^2-regular, connected, closed hypersurface embedded in $\mathbb{R}^{n+1}$, and denote by $|S|$ the area of S. Since S is C^2-regular, it satisfies a uniform touching sphere condition of (optimal) radius ρ. We orient S according to the inner normal. Given $p \in S$, we denote by $H(p)$ the mean curvature of S at p, and we let

$$\operatorname{osc}(H) = \max_{p \in S} H(p) - \min_{p \in S} H(p).$$

Our main result is the following theorem.

Theorem 1.1. *Let S be an n-dimensional, C^2-regular, connected, closed hypersurface embedded in $\mathbb{R}^{n+1}$. There exist constants $\varepsilon, C > 0$ such that if*

$$\operatorname{osc}(H) \le \varepsilon, \tag{1.1}$$

then there are two concentric balls B_{r_i} and B_{r_e} such that

$$S \subset \overline{B}_{r_e} \setminus B_{r_i}, \tag{1.2}$$

$$r_e - r_i \le C \operatorname{osc}(H). \tag{1.3}$$

The constants ε and C depend only on n and upper bounds on ρ^{-1} and $|S|$.

Under the assumption that S bounds a convex domain, there exist some results in the spirit of Theorem 1.1 in the literature. In particular, when the domain is an ovaloid, the problem was studied by Koutroufiotis [Kou], Lang [L] and Moore [Moo]. Other stability results can be found in Schneider [Sch] and Arnold [Ar]. These results were improved by Kohlmann [Ko] who proved an explicit Hölder type stability in (1.3). In Theorem 1.1, we do not consider any convexity assumption and we obtain the optimal rate of stability in (1.3), as can be proven by a simple calculation for ellipsoids.

Theorem 1.1 has a quite interesting consequence which we now explain. It is well-known (see for instance [G]) that if every principal curvature κ_i of S is pinched between two positive numbers, i.e.

$$1/r \le \kappa_i \le (1+\delta)/r, \qquad i = 1, \dots, n,$$

then S is close to a sphere of radius r. Following Gromov [G, Remark (c), pp. 67–68], one can ask what happens when only the mean curvature is pinched. We have the following result.

Corollary 1.2. *Let $\rho_0, A_0 > 0$ and $n \in \mathbb{N}$ be fixed. There exists a positive constant ε, depending on n, ρ_0 and A_0, such that if S is a connected closed C^2 hypersurface embedded in $\mathbb{R}^{n+1}$ with $|S| \le A_0$ and $\rho \ge \rho_0$, whose mean curvature H satisfies*

$$\operatorname{osc}(H) < \varepsilon,$$

then S is diffeomorphic to a sphere. Moreover S is C^1-close to a sphere, i.e. there exists a C^1-map $F = \mathrm{Id} + \Psi\nu : \partial B_{r_i} \to S$ such that

$$\|\Psi\|_{C^1(\partial B_{r_i})} \le C(\operatorname{osc}(H))^{1/2} \tag{1.4}$$

where C depends only on n and upper bounds on ρ^{-1} and $|S|$.

Before explaining the proof of Theorem 1.1, we give a couple of remarks on the bounds on ρ and $|S|$ in Theorem 1.1 and Corollary 1.2. The upper bound on ρ^{-1} controls the C^2-regularity of the hypersurface, which is crucial for obtaining an estimate like (1.3). Indeed, if we assume that ρ is not bounded from below, it is possible to construct a family of closed surfaces embedded in $\mathbb{R}^3$, not diffeomorphic to a sphere, with $\mathrm{osc}(H)$ arbitrarily small and such that (1.3) fails to hold (see Remark 5.2 and [CM]). The upper bound on $|S|$ is a control on the constants ε and C, which clearly change under dilatations.

We remark that Corollary 1.2 can be obtained by a compactness argument by using the theory of varifolds by Allard [All] and Almgren [Alm]. Indeed, by Allard's compactness theorem every sequence of closed hypersurfaces satisfying (uniformly) the assumptions of Corollary 1.2 admits a subsequence which, up to translations, converges to a hypersurface which satisfies a touching ball condition and hence is $C^{1,1}$ regular. By standard regularity theory, the hypersurface is smooth and is a sphere by the classical Alexandrov theorem. We think that also the stability estimates in Theorem 1.1 can be obtained by using Allard's regularity theorem.

There are other possible strategies to obtain quantitative estimates for almost constant mean curvature hypersurfaces and give results in the spirit of Theorem 1.1. Indeed, as already mentioned, there are several proofs of the rigidity result of Alexandrov (i.e. when H is constant). Besides the method of moving planes (which will be our approach), one could try to quantitatively study the proofs in [MR], [Re] and [Ros2], which are based on integral identities. For instance, the approach in [CM] starts from [Ros2] and finds quantitative estimates on the closeness of the hypersurface to a compound of tangent balls. As explained in [CM, Appendix A], another possible approach would be to start from the proof in [MR] and then study almost umbilical hypersurfaces, as in [DLM1] and [DLM2]. However, these approaches based on integral identities do not seem to lead to optimal estimates as in our Theorem 1.1 (see [CM] for a detailed discussion).

Our approach, instead, is based on a quantitative analysis of the method of moving planes and uses arguments from elliptic PDE theory. Since the proof of symmetry is based on the maximum principle, our proof of the stability result will make use of Harnack's and Carleson's (or boundary Harnack's) inequalities and the Hopf Lemma, which can be considered as the quantitative counterpart of the strong and boundary maximum principles. We emphasize that the stability estimate (1.3) is optimal and that our proof permits computing the constants explicitly.

A quantitative study of the method of moving planes was first performed in [ABR], where the authors obtained a stability result for Serrin's overdetermined problem [Se], and it has been used in a series of paper by the first author [CMS2, CMV1, CMV2] to study the stability of radial symmetry for Serrin's and other overdetermined problems (see also [BNST] for an approach based on integral identities).

In this paper, we follow the approach of [ABR], but the setting here is complicated by the fact that we have to deal with manifolds. As we will show, the main goal is to prove an approximate symmetry result for one (arbitrary) direction. With that at hand, the approximate radial symmetry is well-established and follows by an argument in [ABR]. To prove the approximate symmetry in one direction, we apply the method of moving planes and show that the union of the maximal cap and of its reflection provides a set that *fits* S *well*. This is the main point of our paper and is achieved by developing the following argument. Assume that the surface and the reflected cap are tangent at some point p_0 which is an interior point of the reflected cap, and write the two surfaces as graphs of functions in a neighborhood of p_0. The difference w of these two functions satisfies an elliptic equation $Lw = f$, where $\|f\|_\infty$ is bounded by $\operatorname{osc}(H)$. By applying Harnack's inequality and interior regularity estimates, we have a bound on the C^1 norm of w, which says that the two graphs are no more than some constant times $\operatorname{osc}(H)$ distant in C^1 norm. It is important to observe that this estimate implies that the two surfaces are close to each other and also that the two corresponding Gauss maps are close (in some sense) in that neighborhood of p_0. Then we connect any point p of the reflected cap to p_0 and we show that such closeness propagates at p. Since we are dealing with a manifold, we have to change local parametrization when moving from p_0 to p, and we have to prove that the closeness information is preserved. By using careful estimates and making use of interior and boundary Harnack inequalities, we show that this is possible if we assume that $\operatorname{osc}(H)$ is smaller than some fixed constant.

The paper is organized as follows. In Section 2 we prove some preliminary results about hypersurfaces in $\mathbb{R}^{n+1}$, we recall some results on classical solutions to mean curvature type equations, and we give a sketch of the proof of the symmetry result of Alexandrov. In Section 3 we prove some technical lemmas which will be used in proving the stability result. In Sections 4 and 5 we prove Theorem 1.1 and Corollary 1.2, respectively.

2. Notation and preliminary results

In this section we collect some preliminary results. Although some of them are known, we sketch their proofs for the sake of completeness and in order to explain the notation which will be adopted.

Let S be a C^2-regular, connected, closed hypersurface embedded in $\mathbb{R}^{n+1}$, $n \geq 1$, and let Ω be the relatively compact domain of $\mathbb{R}^{n+1}$ bounded by S. We denote by T_pS the tangent hyperplane to S at p and by ν_p the inward normal vector. Given a point $\xi \in \mathbb{R}^{n+1}$ and an $r > 0$, we denote by $B_r(\xi)$ the ball in $\mathbb{R}^{n+1}$ of radius r centered at ξ. When a ball is centered at the origin O, we simply write B_r instead of $B_r(O)$.

Let $\operatorname{dist}_S \colon \mathbb{R}^{n+1} \to \mathbb{R}$ be the distance function from S, i.e.

$$\operatorname{dist}_S(\xi) = \begin{cases} \operatorname{dist}(\xi, S) & \text{if } \xi \in \Omega, \\ -\operatorname{dist}(\xi, S) & \text{if } \xi \in \mathbb{R}^{n+1} \setminus \Omega; \end{cases}$$

it is clear that $S = \{\xi \in \mathbb{R}^{n+1} : \operatorname{dist}_S(\xi) = 0\}$. Moreover, it is well-known (see e.g. [GT]) that dist_S is Lipschitz continuous with Lipschitz constant 1, and that it is of class C^2 in an open neighborhood of S. Therefore the *implicit function theorem* implies that, given a point $p \in S$, S can be locally represented as a graph over the tangent hyperplane T_pS: there exist an open neighborhood $\mathcal{U}_r(p)$ of p in S and a C^2 map $u \colon B_r \cap T_pS \to \mathbb{R}$ such that

$$\mathcal{U}_r(p) = \{p + x + u(x)\nu_p : x \in B_r \cap T_pS\}. \tag{2.1}$$

Moreover, if $q = p + x + u(x)\nu_p$ with $x \in B_r(p) \cap T_pS$, we have

$$\nu_q = \frac{\nu_p - \nabla u(x)}{\sqrt{1 + |\nabla u(x)|^2}}, \tag{2.2}$$

where

$$\nabla u(x) = \sum_{i=1}^{N} \partial_{e_i} u(x) e_i$$

and $\{e_1, \dots, e_n\}$ is an arbitrary *orthonormal* basis of T_pS. We notice that, according to the definition above, $\nabla u(x)$ is a vector in $\mathbb{R}^{n+1}$ for every x in the domain of u. Moreover $\nu_q \cdot \nu_p > 0$ for every $q \in B_r \cap T_pS$, and if $|\nabla u|$ is uniformly bounded in $B_r \cap T_pS$, then u can be extended to $B_{r'} \cap T_pS$ with $r' > r$.

Since S is C^2-regular, the domain Ω satisfies a uniform touching ball condition, and we denote by ρ the optimal radius, that is, for any $p \in S$ there exist two balls of radius ρ centered at $c^- \in \Omega$ and $c^+ \in \mathbb{R}^{n+1} \setminus \overline{\Omega}$ such that $B_\rho(c^-) \subset \Omega$, $B_\rho(c^+) \subset \mathbb{R}^{n+1} \setminus \overline{\Omega}$, and $p \in \partial B_\rho(c^\pm)$. The balls are called, respectively, the *interior* and *exterior* touching balls at p.

In the following lemma we show that we may assume $r = \rho$ in (2.1), and we give some bounds in terms of ρ which will be useful.

Lemma 2.1. *Let $p \in S$. There exists a C^2 map $u : B_\rho \cap T_pS \to \mathbb{R}$ such that*

$$\mathcal{U}_\rho(p) = \{p + x + u(x)\nu_p : x \in B_\rho \cap T_pS\}$$

is a relatively open subset of S and

$$|u(x)| \le \rho - \sqrt{\rho^2 - |x|^2}, \tag{2.3}$$

$$|\nabla u(x)| \le \frac{|x|}{\sqrt{\rho^2 - |x|^2}}, \tag{2.4}$$

for every $x \in B_\rho \cap T_pS$. Moreover

$$\nu_p \cdot \nu_q \ge \frac{1}{\rho}\sqrt{\rho^2 - |x|^2} \quad \text{and} \quad |\nu_p - \nu_q| \le \sqrt{2}\,\frac{|x|}{\rho} \tag{2.5}$$

for every $q = p + x + u(x)\nu_p$ in $\mathcal{U}_\rho(p)$.

Proof. By the implicit function theorem, there exist $r > 0$, $u : B_r \cap T_pS \to \mathbb{R}$ and $\mathcal{U}_r(p)$ as in (2.1). We may assume that $r \le \rho$. The bound (2.3) in $B_r \cap T_pS$ easily follows from the definition of the interior and exterior touching balls at p. We now prove the estimate (2.4) in $B_r \cap T_pS$, which allows us to enlarge the domain of u to $B_\rho \cap T_pS$. Let

$$q = p + x + u(x)\nu_p$$

with $|x| < r$ be an arbitrary point of $\mathcal{U}_r(p)$ (notice that $\nu_p \cdot \nu_q > 0$). Since

$$B_\rho(p + \rho\nu_p) \cap B_\rho(q - \rho\nu_q) = \emptyset,$$

we have

$$|p + \rho \nu_p - q + \rho \nu_q| \geq 2\rho.$$

Analogously, $B_\rho(p - \rho\nu_p) \cap B_\rho(q + \rho\nu_q) = \emptyset$ gives

$$|q + \rho \nu_q - p + \rho \nu_p| \geq 2\rho.$$

By adding the squares of the last two inequalities we obtain

$$|p - q|^2 + 2\rho^2(\nu_p \cdot \nu_q) \geq 2\rho^2,$$

and from (2.3) we get (2.5). From (2.2) and (2.5) we obtain (2.4) in $B_r \cap T_pS$. Since $|\nabla u|$ is bounded in $\overline{B}_r \cap T_pS$, we can extend u to a larger ball where (2.4) is still satisfied. It is clear that we can choose $r = \rho$ and (2.3)–(2.5) hold. □

Given $p, q \in S$ we denote by $d_S(p, q)$ their intrinsic distance inside S, and if A is an arbitrary subset of S, we define

$$d_S(p, A) = \inf_{q \in A} d_S(p, q).$$

Lemma 2.2. *Let $p \in S$, $q \in \mathcal{U}_\rho(p)$ and let x be the orthogonal projection of q onto the hyperplane T_pS. Then*

$$|x| \leq d_S(p, q) \leq \rho \arcsin(|x|/\rho). \tag{2.6}$$

Proof. The first inequality is trivial. In order to prove the second inequality we consider the curve $\gamma : [0, 1] \to S$ joining p to q defined by $\gamma(t) = p + tx + u(tx)\nu_p$, $t \in [0, 1]$. Then

$$\dot\gamma(t) = x + (\nabla u(tx) \cdot x)\nu_p;$$

since $x \in T_pS$, by the Cauchy–Schwarz inequality we obtain

$$|\dot\gamma(t)| \leq |x|\sqrt{1 + |\nabla u(tx)|^2}.$$

Therefore inequality (2.4) implies

$$|\dot\gamma(t)| \leq \frac{\rho|x|}{\sqrt{\rho^2 - t^2|x|^2}}.$$

Since $d_S(p, q) \leq \int_0^1 |\dot\gamma(t)|\, dt$, we obtain

$$d_S(p, q) \leq |x|\rho \int_0^1 \frac{1}{\sqrt{\rho^2 - t^2|x|^2}}\, dt,$$

which gives (2.6). □

Let $p \in S$ and let $u : B_\rho \cap T_pS \to S$ be as in Lemma 2.1. It is well-known (see [GT]) that u is a classical solution to

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = nH \quad \text{in } B_\rho \cap T_pS, \tag{2.7}$$

where H is the mean curvature of S regarded as a map on $B_\rho \cap T_pS$. We notice that $\nabla u \in T_pS$ and the divergence is meant in local coordinates on T_pS: if $\{e_1, \dots, e_n\}$ is an orthonormal basis of T_pS and $F = \sum_{i=1}^n F_i e_i$, then

$$\operatorname{div} F = \sum_{i=1}^n \frac{\partial F_i}{\partial e_i}.$$

Moreover, (2.7) is *uniformly elliptic* once u is regarded as a regular map in an open set of $\mathbb{R}^n$ and has bounded gradient, since

$$|\xi|^2 \le \frac{\partial}{\partial \zeta_j}\left(\frac{\zeta_i}{\sqrt{1+|\zeta|^2}}\right)\xi_i\xi_j \le (1+|\zeta|^2)|\xi|^2 \tag{2.8}$$

for every $\xi = (\xi_1, \dots, \xi_n)$, $\zeta = (\zeta_1, \dots, \zeta_n)$ in $\mathbb{R}^n$.

2.1. *Classical solutions to the mean curvature equation*

In this subsection we collect some results about classical solutions to (2.7) which will be used in the next sections.

Let B_r be the ball of $\mathbb{R}^k$ centered at the origin and having radius r. Given a differentiable map $u : B_r \to \mathbb{R}$, we denote by Du the gradient of u in $\mathbb{R}^k$:

$$Du = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_k}\right).$$

We remark that this notation differs from the one in the rest of the paper, where we use the ∇ symbol to denote a vector in $\mathbb{R}^{n+1}$.

Let $H_0, H_1 \in C^0(B_r)$ and u_0 and u_1 be two classical solutions of

$$\operatorname{div}\left(\frac{Du_j}{\sqrt{1+|Du_j|^2}}\right) = kH_j \quad \text{in } B_r,\ j = 0, 1. \tag{2.9}$$

It is well-known (see [GT]) that $w = u_1 - u_0$ satisfies the linear elliptic equation

$$Lw = k(H_1 - H_0), \tag{2.10}$$

where

$$Lw = \sum_{i,j=1}^k \frac{\partial}{\partial x_j}\left(a^{ij}(x)\frac{\partial w}{\partial x_i}\right) \tag{2.11}$$

with

$$a^{ij}(x) = \int_0^1 \frac{\partial}{\partial \zeta_j}\left(\frac{\zeta_i}{\sqrt{1+|\zeta|^2}}\right)\Big|_{\zeta = Du_t(x)} dt,$$
$$u_t(x) = tu_1(x) + (1-t)u_0(x), \qquad x \in B_r.$$

From (2.8), we find that

$$|\xi|^2 \le a^{ij}(x)\xi_i\xi_j \le |\xi|^2 \int_0^1 (1+|Du_t(x)|^2)\,dt, \tag{2.12}$$

where we have used the Einstein summation convention. The following Harnack type inequality will be one of the crucial tools for proving the stability result.

Lemma 2.3. *Let u_j, $j=0,1$, be classical solutions of (2.9) with $u_1-u_0\ge 0$ in B_r, and assume that*

$$\|Du_j\|_{C^1(B_r)} \le M, \qquad j=0,1, \tag{2.13}$$

for some positive constant M. Then there exists a constant K_1, depending only on the dimension k and M, such that

$$\|u_1-u_0\|_{C^1(B_{r/4})} \le K_1\Big(\inf_{B_{r/2}}(u_1-u_0) + \|H_1-H_0\|_{C^0(B_r)}\Big). \tag{2.14}$$

Proof. We have already observed that $w=u_1-u_0$ satisfies (2.10) in B_r. From (2.12) and (2.13), we find that Lw is uniformly elliptic with continuous bounded coefficients:

$$|\xi|^2 \le a^{ij}(x)\xi_i\xi_j \le |\xi|^2(1+M^2),$$

and

$$\left|\frac{\partial}{\partial x_j}a^{ij}(x)\right| \le M'$$

for some positive M' depending only on M.

From [GT, Theorems 8.17 and 8.18], we obtain the following Harnack inequality:

$$\sup_{B_{r/2}} w \le C_1\Big(\inf_{B_{r/2}} w + \|H_1-H_0\|_{C^0(B_r)}\Big).$$

Then we use [GT, Theorem 8.32] to obtain

$$|w|_{C^{1,\alpha}(B_{r/4})} \le C_2(\|w\|_{C^0(B_{r/2})} + \|H_1-H_0\|_{C^0(B_{r/2})}),$$

where $|\cdot|_{C^{1,\alpha}(B_{r/4})}$ is the $C^{1,\alpha}$ seminorm in $B_{r/4}$ with $\alpha\in(0,1)$. By combining the last two inequalities, we obtain (2.14) at once. □

Another crucial tool for our result is the following boundary Harnack type inequality (or *Carleson estimate* [CS]).

Lemma 2.4. *Let E be a domain in $\mathbb{R}^k$ and let T be an open subset of ∂E which is of class C^2. Let $u_j \in C^2(\overline{E})$, $j=0,1$, be solutions of*

$$\operatorname{div}\left(\frac{Du_j}{\sqrt{1+|Du_j|^2}}\right) = kH_j \quad \text{in } E,\ \ j=0,1, \tag{2.15}$$

satisfying $\|Du_j\|_{C^1(E)} \le M$ for some positive M. Let $x_0 \in T$ and $r > 0$ be such that $B_r(x_0) \cap \partial E \subset T$, and assume that

$$u_1 - u_0 \ge 0 \quad \text{in } B_r(x_0) \cap E, \qquad u_1 - u_0 \equiv 0 \quad \text{on } B_r \cap \partial E.$$

Assume further that e_1 is the interior normal to E at x_0. Then there exists a constant $K_2 > 0$ such that

$$\sup_{B_{r/4}(x_0)\cap E} (u_1 - u_0) \le K_2\Big((u_1 - u_0)\Big(x_0 + \frac{r}{2}e_1\Big) + \|H_1 - H_0\|_{C^0(B_r)}\Big), \tag{2.16}$$

where the constant K_2 depends only on the dimension k, M and the C^2-regularity of T.

Proof. The proof is analogous to the one of Lemma 2.3, where we use [BCN, Theorem 1.3] and [GT, Corollary 8.36] in place of [GT, Theorems 8.17, 8.18 and 8.32]. □

We conclude this subsection with a quantitative version of the Hopf Lemma. We start with a statement which is valid for a general second order elliptic operator of the form

$$\mathcal{L}w = \sum_{i,j=1}^{k} a^{ij} w_{x_i x_j} + \sum_{i=1}^{k} b^i w_{x_i} \tag{2.17}$$

satisfying the ellipticity conditions

$$a^{ij}\zeta_i\zeta_j \ge \lambda|\zeta|^2 \quad \text{and} \quad |a^{ij}|, |b^i| \le \Lambda, \qquad i, j = 1, \dots, k, \tag{2.18}$$

for some $\lambda, \Lambda > 0$.

Lemma 2.5. *Let $r > 0$ and $\gamma \ge 0$. Assume that $w \in C^2(B_r) \cap C^0(\overline{B}_r)$ fulfills the conditions*

$$\mathcal{L}w \le \gamma \quad \text{and} \quad w \ge 0 \quad \text{in } B_r,$$

with $\mathcal{L}$ given by (2.17). Then there exists a positive constant C depending on k, λ, Λ, and an upper bound on γ such that for any $x_0 \in \partial B_r$ we have

$$\sup_{B_{r/2}} w \le C\Big(\frac{w((1 - t/r)x_0)}{t} + \gamma\Big) \quad \text{for any } 0 < t \le r/2. \tag{2.19}$$

Moreover, if $w(x_0) = 0$ then

$$\sup_{B_{r/2}} w \le C\Big(\frac{\partial w(x_0)}{\partial\nu} + \gamma\Big), \tag{2.20}$$

where ν denotes the inward normal to ∂B_r.

Proof. In the annulus $A = B_r \setminus \overline{B}_{r/2}$, we consider the auxiliary function

$$v(x) = \Big(\min_{\overline{B}_{r/2}} w\Big) \frac{e^{-\alpha|x|^2} - e^{-\alpha r^2}}{e^{-\alpha(r/2)^2} - e^{-\alpha r^2}} + e^{\beta|x|^2} - e^{\beta r^2},$$

where

$$\alpha = \frac{(k + r\sqrt{k})\Lambda}{2\lambda^2}, \qquad \beta = \gamma\big[k\lambda - \sqrt{k}\,\Lambda r + \sqrt{(k\lambda - \sqrt{k}\,\Lambda r)^2 + \gamma\lambda r^2}\big]^{-1}.$$

Here, the constants α and β are chosen in such a way that $\mathcal{L}v \geq \gamma$. We notice that

$$\frac{v((1 - t/r)x_0)}{t} \geq \frac{\alpha r e^{-\alpha r^2}}{e^{-\alpha(r/2)^2} - e^{-\alpha r^2}} \Big(\min_{\overline{B}_{r/2}} w\Big) - 2\beta r e^{\beta r^2}. \tag{2.21}$$

Since $v = 0$ on ∂B_r and $v \leq \min_{\partial B_{r/2}} w$ on $\partial B_{r/2}$, the function $w - v$ satisfies

$$\begin{cases} \mathcal{L}(w - v) \leq 0 & \text{in } A, \\ w - v \geq 0 & \text{on } \partial A. \end{cases}$$

Hence, by the maximum principle, $w - v \geq 0$ in $\overline{A}$, and from (2.21) we obtain

$$\min_{\overline{B}_{r/2}} w \leq \frac{e^{3\alpha r^2/4} - 1}{\alpha r} \Big(\frac{w((1 - t/r)x_0)}{t} + 2\beta r e^{\beta r^2}\Big) \tag{2.22}$$

for $0 < t < r/2$. As in the proof of Lemma 2.3, we use [GT, Theorems 8.17 and 8.18] to get

$$\max_{\overline{B}_{r/2}} w \leq C_1\Big(\min_{\overline{B}_{r/2}} w + \gamma\Big),$$

and from (2.22) we obtain (2.19) and (2.20). □

We will use Lemma 2.5 in the following form.

Lemma 2.6. *Let E, T, u_0, u_1, M, and x_0 be as in Lemma 2.4, with*

$$u_1 - u_0 \geq 0 \quad \text{in } E.$$

Assume that there exists $B_r(c) \subset E$ with $x_0 \in \partial B_r(c) \cap T$. Let $\ell = (c - x_0)/r$. Then there exists a constant K_3 such that

$$\|u_1 - u_0\|_{C^1(B_{r/4}(c))} \leq K_3\Big(\frac{(u_1 - u_0)(x_0 + t\ell)}{t} + \|H_1 - H_0\|_{C^0(B_r(c))}\Big) \tag{2.23}$$

for every $t \in (0, r/2)$, and

$$\|u_1 - u_0\|_{C^1(B_{r/4}(c))} \leq K_3\Big(\frac{\partial(u_1 - u_0)}{\partial\ell}(x_0) + \|H_1 - H_0\|_{C^0(B_r(c))}\Big) \tag{2.24}$$

for $t = 0$. The constant K_3 depends only on the dimension k, M, ρ, and an upper bound on $\|H_1 - H_0\|_{C^0(B_r(c))}$.

Proof. As shown in the proof of Lemma 2.3, $w = u_1 - u_0$ satisfies (2.10), which is uniformly elliptic. Moreover, by letting

$$\gamma = \|H_1 - H_0\|_{C^0(B_r(c))},$$

we have $Lw \leq \gamma$. Hence, we can apply Lemma 2.5, and conclude the proof by using Lemma 2.3. □

2.2. *The symmetry result of Alexandrov*

In order to make the paper self-contained, we give a sketch of the proof of the Soap Bubble Theorem by Alexandrov. This will be the occasion to set up some necessary notation.

Let S be a C^2-regular, connected, closed hypersurface embedded in $\mathbb{R}^{n+1}$, $n \geq 1$, and let Ω be the relatively compact domain of $\mathbb{R}^{n+1}$ bounded by S. Let $\omega \in \mathbb{R}^{n+1}$ be a unit vector and $\lambda \in \mathbb{R}$ a parameter. For an arbitrary set A, we define the following objects:

$$\begin{aligned}
&\pi_\lambda = \{\xi \in \mathbb{R}^{n+1} : \xi \cdot \omega = \lambda\}, && \text{a hyperplane orthogonal to } \omega,\\
&A^\lambda = \{p \in A : p \cdot \omega > \lambda\}, && \text{the right-hand cap of } A,\\
&\xi^\lambda = \xi - 2(\xi \cdot \omega - \lambda)\,\omega, && \text{the reflection of } \xi \text{ about } \pi_\lambda,\\
&A_\lambda = \{p \in \mathbb{R}^{n+1} : p^\lambda \in A^\lambda\}, && \text{the reflected cap about } \pi_\lambda,\\
&\hat{A}_\lambda = \{p \in A : p \cdot \omega < \lambda\}, && \text{the portion of } A \text{ in the left half-plane.}
\end{aligned} \tag{2.25}$$

Set $\mathcal{M} = \max\{p \cdot \omega : p \in S\}$, the extent of S in the direction ω; if $\lambda < \mathcal{M}$ is close to $\mathcal{M}$, the reflected cap Ω_λ is contained in Ω. Set

$$m = \inf\{\mu : \Omega_\lambda \subset \Omega \text{ for all } \lambda \in (\mu, \mathcal{M})\}. \tag{2.26}$$

Then for $\lambda = m$ at least one of the following two cases occurs:

(i) S_m becomes internally tangent to S at some point $p \in S \setminus \pi_m$;
(ii) π_m is orthogonal to S at some point $p \in S \cap \pi_m$.

Theorem A (Alexandrov Soap Bubble Theorem). *Let S be a C^2-regular, closed, connected hypersurface embedded in $\mathbb{R}^{n+1}$. If the mean curvature H of S is constant, then S is a sphere.*

Proof. Let ω be a fixed direction. We apply the method of moving planes in the direction ω and we find a critical position for $\lambda = m$.

If case (i) occurs, then we locally write S_m and S as graphs of functions u_1 and u_0, respectively, over $B_r \cap T_pS$ (which coincides with T_pS_m), where p is the tangency point. It is clear that $w = u_1 - u_0$ is non-negative, and since H is constant, w satisfies

$$Lw = 0 \quad \text{in } B_r \cap T_pS$$

for some $r > 0$, where L is given by (2.11). Since $w(0) = 0$, by the strong maximum principle we obtain $w \equiv 0$ in $B_r \cap T_pS$, that is, S and S_m coincide in an open neighborhood of p.

If case (ii) occurs, then we locally write S_m and S as the graphs of functions u_1 and u_0, respectively, over $T_pS \cap \{x \cdot \omega \le m\}$. As for case (i), we find that there exists $r > 0$ such that

$$\begin{cases} Lw = 0 & \text{in } B_r \cap T_pS \cap \{x \cdot \omega < m\}, \\ w = 0 & \text{on } B_r \cap T_pS \cap \{x \cdot \omega = m\}. \end{cases}$$

Since $\nabla w(0) = 0$, from the Hopf Lemma (see for instance [GT]) we deduce that $w \equiv 0$ in $B_r \cap T_pS \cap \{x \cdot \omega \le m\}$.

Hence, in both cases (i) and (ii) the set of tangency points (that is, those points for which case (i) or (ii) occurs) is open. Since it is also closed and non-empty, we must have $S_m = \hat{S}_m$, that is, S is symmetric about the hyperplane π_m. Since ω is arbitrary, we find that S is symmetric in every direction.

Up to a translation, we can assume that the origin O is the center of mass of S. Since O belongs to every axis of symmetry and every rotation can be written as a composition of reflections, we see that S is invariant under rotations, which implies that it is a sphere. □

2.3. *Curvatures of projected surfaces*

Before giving the results of this subsection, we need to recall some basic facts about hypersurfaces in $\mathbb{R}^{n+1}$, in particular about the interplay between the normal and the principal curvatures. Let U be an orientable hypersurface of class C^2 embedded in $\mathbb{R}^{n+1}$ (which in the proof of Theorem 1.1 will be an open subset of the surface S). The choice of an orientation on U is equivalent to the choice of a Gauss map $\nu : U \to \mathbb{S}^n$ (in this general context there is no canonical orientation). Fixing a point $q \in U$, we denote by $W_q : T_qU \to T_qU$ the shape operator $W_q = -d\nu_q$. It is symmetric and its eigenvalues $\kappa_i(q)$ are the principal curvatures of U at q. We assume that $\kappa_1(q) \le \cdots \le \kappa_n(q)$. The first and the last principal curvature can be obtained as the minimum and maximum of the normal curvature. Here we recall that, given a non-zero vector $v \in T_qU$, its normal curvature $\kappa(q, v)$ is defined as

$$\kappa(q, v) = \frac{1}{|v|^2} W_q(v) \cdot v.$$

$\kappa(q, v)$ can be alternatively written in terms of curves as

$$\kappa(q, v) = \frac{1}{|\dot{\alpha}(0)|^2}\, \nu_{\alpha(0)} \cdot \ddot{\alpha}(0)$$

where $\alpha : I \to U$ is an arbitrary curve satisfying $\alpha(0) = 0$ and $\dot{\alpha}(0) = v$.

In order to perform a quantitative study of moving planes, we need to handle the following situation: given a hypersurface U of class C^2 in $\mathbb{R}^{n+1}$, we consider its intersection U' with an affine hyperplane π_1 (in the proof of Theorem 4.1, π_1 will be the critical hyperplane in the direction ω). If π_1 intersects U transversally, $U' = U \cap \pi_1$ is a hypersurface of class C^2 of π_1 and we consider its projection U'' onto another hyperplane π_2 of $\mathbb{R}^{n+1}$ (which will be tangent to the reflected cap at some point close to the critical hyperplane). An example in $\mathbb{R}^3$ is shown in Figure 1. The next two propositions allow us

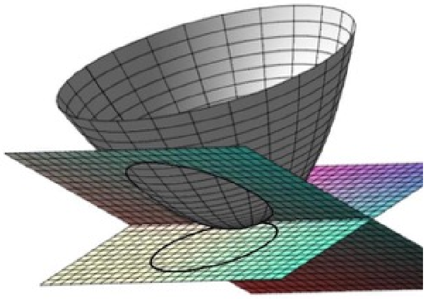


Fig. 1. In the figure U is the paraboloid $z = x^2 + y^2$, π_1 is the affine plane $z = 2 + 8y$ and π_2 is the plane $z = 0$. In this case U' is the ellipse in π_1, while U'' is the circle projected in π_2.

to control the principal curvatures of U'' in terms of the principal curvatures of U and the normal vectors to U and U'.

Proposition 2.7. *Let U be an orientable hypersurface of class C^2 embedded in $\mathbb{R}^{n+1}$ with principal curvatures κ_j, $j = 1, \dots, n$, and Gauss map ν. Let π be a hyperplane of $\mathbb{R}^{n+1}$ intersecting U transversally and let $U' = U \cap \pi$. Then U' is an orientable hypersurface of class C^2 embedded in π, and once a Gauss map $\nu' \colon U' \to \mathbb{S}^{n-1}$ is fixed, its principal curvatures κ'_i satisfy*

$$\frac{1}{\nu_q \cdot \nu'_q}\kappa_1(q) \le \kappa'_i(q) \le \frac{1}{\nu_q \cdot \nu'_q}\kappa_n(q) \tag{2.27}$$

for every $q \in U'$ and $i = 1, \dots, n-1$.

Proof. First of all we observe that U' is of class C^2 by the implicit function theorem, and it is orientable since the map $\nu' \colon U' \to \mathbb{S}^{n-1}$ defined by

$$\nu'_q = (-1)^{n+1}\,\mathrm{vers}(*(*(\nu_q \wedge \omega) \wedge \omega)) \tag{2.28}$$

is a Gauss map on U', where $*$ denotes the Hodge "star" operator in $\mathbb{R}^{n+1}$ computed with respect to the standard metric and the standard orientation.

In order to prove (2.27), fix $q \in U'$ and consider an arbitrary unit vector $v \in T_qU'$. Let $\kappa(q, v)$ be the normal curvature of U at (q, v). Then

$$\kappa(q, v) = \nu_q \cdot \ddot{\alpha}(0)$$

where α is an arbitrary smooth curve in U' parametrized by arc length and such that $\alpha(0) = q$ and $\dot{\alpha}(0) = v$. Since ν_q is orthogonal to T_qU', it belongs to the plane generated by ω and ν'_q and we can write

$$\nu_q = (\nu'_q \cdot \omega)\omega + (\nu_q \cdot \nu'_q)\nu'_q.$$

Therefore

$$\kappa(q, v) = \nu_q \cdot \ddot{\alpha}(0) = (\nu_q \cdot \nu'_q)(\nu'_q \cdot \ddot{\alpha}(0)) = (\nu_q \cdot \nu'_q)\kappa'(q, v),$$

where $\kappa'(q, v)$ is the normal curvature of U' at (q, v), and the claim follows. □

We observe that in Proposition 2.7 we can choose ν to be the Gauss map defined by (2.28) and have

$$\nu_q \cdot \nu'_q = \sqrt{1 - (\nu_q \cdot \omega)^2}. \tag{2.29}$$

Indeed, we fix a positive oriented orthonormal basis $\{e_1, \ldots, e_n, \nu_q\}$ of $\mathbb{R}^{n+1}$ such that the first n vectors are an orthonormal basis of $T_q U$ and $\langle e_1, \ldots, e_{n-1}\rangle = \omega^\perp$. Then

$$|{*}({*}(\nu_q \wedge \omega) \wedge \omega)| = |{*}(\nu_q \wedge \omega) \wedge \omega| = \omega \cdot e_n$$

and

$$(\omega \cdot e_n)(\nu_q \cdot \nu'_q) = (-1)^{n+1} {*}({*}(\nu_q \wedge \omega) \wedge \omega) \cdot \nu_q = (-1)^{n+1} {*}(\nu_q \wedge \omega) \wedge \omega \cdot {*}\nu_q.$$

Moreover,

$${*}(\nu_q \wedge \omega) = -(\omega \cdot e_n) e_1 \wedge \cdots \wedge e_{n-1}, \qquad {*}\nu_q = (-1)^n \, e_1 \wedge \cdots \wedge e_n,$$

and

$$\nu_q \cdot \nu'_q = (e_1 \wedge \cdots \wedge e_{n-1} \wedge \omega) \cdot (e_1 \wedge \cdots \wedge e_n) = \omega \cdot e_n = \sqrt{1 - (\nu_q \cdot \omega)^2},$$

as required.

Therefore, if ν'_q is given by (2.28), then (2.27) reads

$$\frac{1}{\sqrt{1 - (\nu_q \cdot \omega)^2}} \kappa_1(q) \le \kappa'_i(q) \le \frac{1}{\sqrt{1 - (\nu_q \cdot \omega)^2}} \kappa_n(q) \tag{2.30}$$

for $i = 1, \ldots, n-1$.

Proposition 2.8. *Let ω_1 and ω_2 be unit vectors in $\mathbb{R}^{n+1}$, denote by π_1 a hyperplane orthogonal to ω_1, and let π_2 be the hyperplane orthogonal to ω_2 passing through the origin of $\mathbb{R}^{n+1}$. Let U' be a C^2-regular oriented hypersurface of π_1 such that ω_2 is not tangent to U' at any point. Denote by κ'_i, for $i = 1, \ldots, n-1$, the principal curvatures of U' and denote by ν' the normal vector to U'. Then the orthogonal projection U'' of U' onto π_2 is a C^2-regular hypersurface of π_2 with a canonical orientation. Moreover, for any $q \in U'$ we have*

$$|\kappa''_i(\mathrm{pr}(q))| \le \frac{|\omega_1 \cdot \omega_2|}{[(\omega_1 \cdot \omega_2)^2 + (\omega_2 \cdot \nu'_q)^2]^{3/2}} \max\{|\kappa'_1(q)|, |\kappa'_{n-1}(q)|\} \tag{2.31}$$

for every $i = 1, \ldots, n-1$, where $\mathrm{pr}(q)$ is the projection of q onto π_2, and $\{\kappa''_i\}$ are the principal curvatures of U''.

Proof. If X is a local positive oriented parametrization of U', then $Y = X - (X \cdot \omega_2)\omega_2$ is a local parametrization of U'', and

$$\nu'' \circ Y := \mathrm{vers}(*(Y_1 \wedge \cdots \wedge Y_{n-1} \wedge \omega_2))$$

defines a Gauss map for U'', where Y_k is the k^{th} derivative of Y with respect to the coordinates of its domain. Therefore U'' is a C^2-regular hypersurface of π_2 oriented by the map ν''.

Now we prove inequalities (2.31). Fix a point $q \in U'$ and let $\mathrm{pr}(q) = q - (q \cdot \omega_2)\omega_2$ be its projection onto U''. Let X be a local positive oriented parametrization of U' around q and $Y = X - (X \cdot \omega_2)\omega_2$ be the induced parametrization of U'' around $\mathrm{pr}(q)$.

Let $\beta\colon (-\delta, \delta) \to U''$ be an arbitrary regular curve contained in U'' such that $\beta(0) = \mathrm{pr}(q)$ and let

$$v = \frac{\dot\beta(0)}{|\dot\beta(0)|}, \qquad g = \frac{1}{|\dot\beta|^2}\nu''_\beta \cdot \ddot\beta.$$

Then

$$g(0) = \kappa''(\mathrm{pr}(q), v),$$

where $\kappa''(\mathrm{pr}(q), v)$ is the normal curvature of U'' at (q, v). The curve β can be seen as the projection of a regular curve α in U' passing through p. Since ν''_β is orthogonal to ω_2, we have

$$g = \frac{1}{|\dot\beta|^2}\nu''_\beta \cdot \ddot\alpha.$$

Note that since

$$Y_k = X_k - (X_k \cdot \omega_2)\omega_2,$$

we have

$$\nu'' \circ Y = \mathrm{vers}(*(X_1 \wedge \cdots \wedge X_{n-1} \wedge \omega_2))$$

and

$$g = \frac{(*(X_1(\alpha) \wedge \cdots \wedge X_{n-1}(\alpha) \wedge \omega_2)) \cdot \ddot\alpha}{|\dot\beta|^2 |X_1(\alpha) \wedge \cdots \wedge X_{n-1}(\alpha) \wedge \omega_2|}.$$

Now, it is easy to prove that

$$(*(X_1(\alpha) \wedge \cdots \wedge X_{n-1}(\alpha) \wedge \omega_2)) \cdot \ddot\alpha = (\omega_1 \cdot \omega_2)\ *(X_1(\alpha) \wedge \cdots \wedge X_{n-1}(\alpha) \wedge \omega_1) \cdot \ddot\alpha,$$

and therefore

$$g = \frac{\omega_1 \cdot \omega_2}{|\dot\beta|^2}\,\frac{(*(X_1(\alpha) \wedge \cdots \wedge X_{n-1}(\alpha) \wedge \omega_1)) \cdot \ddot\alpha}{|X_1(\alpha) \wedge \cdots \wedge X_{n-1}(\alpha) \wedge \omega_2|},$$

which implies

$$g = (\nu'_\alpha \cdot \ddot\alpha)\frac{\omega_1 \cdot \omega_2}{|\dot\beta|^2}\,\frac{|X_1(\alpha) \wedge \cdots \wedge X_{n-1}(\alpha) \wedge \omega_1|}{|X_1(\alpha) \wedge \cdots \wedge X_{n-1}(\alpha) \wedge \omega_2|}.$$

We may assume that α is parametrized by arc length and so

$$|\dot\beta|^2 = 1 - (\dot\alpha \cdot \omega_2)^2,$$

which implies

$$g = (\nu'_\alpha \cdot \ddot{\alpha}) \frac{\omega_1 \cdot \omega_2}{1 - (\dot{\alpha} \cdot \omega_2)^2} \frac{|X_1(\alpha) \wedge \cdots \wedge X_{n-1}(\alpha) \wedge \omega_1|}{|X_1(\alpha) \wedge \cdots \wedge X_{n-1}(\alpha) \wedge \omega_2|}.$$

Moreover a standard computation yields

$$\frac{|X_1(\alpha) \wedge \cdots \wedge X_{n-1}(\alpha) \wedge \omega_1|}{|X_1(\alpha) \wedge \cdots \wedge X_{n-1}(\alpha) \wedge \omega_2|} = \frac{1}{((\omega_1 \cdot \omega_2)^2 + (\omega_2 \cdot \nu_\alpha)^2)^{1/2}},$$

and hence

$$g(0) = \kappa'(q, \dot{\alpha}(0)) \frac{\omega_1 \cdot \omega_2}{((\omega_1 \cdot \omega_2)^2 + (\omega_2 \cdot \nu'_q)^2)^{1/2}} \frac{1}{1 - (\dot{\alpha}(0) \cdot \omega_2)^2},$$

where $\kappa'(q, \dot{\alpha}(0))$ is the normal curvature of U' at $(q, \dot{\alpha}(0))$. Therefore

$$\kappa''_1(\mathrm{pr}(q)) = \frac{\omega_1 \cdot \omega_2}{((\omega_1 \cdot \omega_2)^2 + (\omega_2 \cdot \nu'_q)^2)^{1/2}} \inf_{v \in \mathbb{S}^{n-1}_q} \frac{\kappa'(q, v)}{1 - (v \cdot \omega_2)^2}, \tag{2.32}$$

$$\kappa''_{n-1}(\mathrm{pr}(q)) = \frac{\omega_1 \cdot \omega_2}{((\omega_1 \cdot \omega_2)^2 + (\omega_2 \cdot \nu'_q)^2)^{1/2}} \sup_{v \in \mathbb{S}^{n-1}_q} \frac{\kappa'(q, v)}{1 - (v \cdot \omega_2)^2}, \tag{2.33}$$

where $\mathbb{S}^{n-1}_q = \{v \in T_q U' : |v| = 1\}$. Since

$$|\kappa''_i(\mathrm{pr}(q))| \le \max\{|\kappa''_1(\mathrm{pr}(q))|\, |\kappa''_{n-1}(\mathrm{pr}(q))|\}, \qquad i = 1, \ldots, n-1,$$

from (2.32) and (2.33) we obtain

$$|\kappa''_i(\mathrm{pr}(q))| \le \frac{|\omega_1 \cdot \omega_2|}{((\omega_1 \cdot \omega_2)^2 + (\omega_2 \cdot \nu'_q)^2)^{1/2}} \sup_{v \in \mathbb{S}^{n-1}_q} \frac{|\kappa'(q, v)|}{1 - (v \cdot \omega_2)^2},$$

and since $\mathbb{R}^{n+1} = T_q U' \oplus \langle \nu'_q \rangle \oplus \langle \omega_2 \rangle$ with

$$1 - (v \cdot \omega_2)^2 \ge (\omega_1 \cdot \omega_2)^2 + (\omega_2 \cdot \nu_q)^2,$$

we have

$$|\kappa''_i(\mathrm{pr}(q))| \le \frac{|\omega_1 \cdot \omega_2|}{((\omega_1 \cdot \omega_2)^2 + (\omega_2 \cdot \nu'_q)^2)^{3/2}} \sup_{v \in \mathbb{S}^{n-1}_q} |\kappa'(q, v)|$$

for every $i = 1, \ldots, n-1$, which implies (2.31). □

3. Technical lemmas

Let S be a connected closed C^2-regular hypersurface embedded in $\mathbb{R}^{n+1}$ and let ρ be the radius of the uniform touching sphere.

Let S_m and π_m be as in (2.25) and let $\partial S_m = S \cap \pi_m$. It will be useful to define

$$S_m^\delta = \{p \in S_m : d_S(p, \partial S_m) > \delta\} \quad \text{for } \delta > 0. \tag{3.1}$$

Lemma 3.1. *Let* $0 < \delta < \rho$ *and set* $\sigma = \rho \sin(\delta/\rho)$*. Then:*

(i) *For any* $p \in S_m^\delta$ *we have* $\mathcal{U}_\sigma(p) \subset S_m$*.*
(ii) *For any* $q \in S_m \setminus S_m^\delta$ *there exist* $p \in \partial S_m$ *and* $x \in B_\delta \cap T_pS$ *such that*

$$q = p + x + u(x)\nu_p.$$

Here u *and* $\mathcal{U}$ *are as in* (2.1)*.*

Proof. (i) Let $x \in B_\sigma \cap T_pS$ and let $q = p + x + u(x)\nu_p$. Since

$$d_S(q, \partial S_m) \geq d_S(p, \partial S_m) - d_S(p, q),$$

(2.6) implies

$$d_S(q, \partial S_m) \geq \delta - \rho \arcsin(|x|/\rho).$$

The assumption $|x| < \sigma$ implies the conclusion.

(ii) Let $p \in \partial S_m$ be such that $d_S(q, \partial S_m) = d_S(p, q)$, and let x be the orthogonal projection of q onto T_pS. Since $|x| \leq d_S(p, q) < \delta$ and $\delta < \rho$, we have $|x| < \rho$ and Lemma 2.1 implies the statement. □

In the next lemma we show that any two points in S_m^δ can be joined by a piecewise geodesic curve, and we give a bound on its length. An analogous lemma was proved in [ABR] in the special case when S_m^δ is contained in a hyperplane.

Lemma 3.2. *Let* $0 < \delta < \rho$*, and set*

$$L = \frac{|S|2^n}{\omega_n \delta^{n-1}} \tag{3.2}$$

where ω_n *is the volume of the unit ball in* $\mathbb{R}^n$*. Let* p, q *be in a connected component of* S_m^δ*. Then there exists a piecewise geodesic path* $\gamma : [0, 1] \to S_m^{\delta/2}$ *satisfying* $\gamma(0) = p$ *and* $\gamma(1) = q$ *and with length bounded by* L*. Moreover,* γ *can be built by joining* $\mathcal{N}$ *minimal geodesics of length* δ*, with*

$$\mathcal{N}\delta \leq L, \tag{3.3}$$

and one minimal geodesic of length $\leq \delta$*.*

Proof. We can join p and q by a path $\tilde{\gamma} : [0, 1] \to S_m^\delta$ such that $\tilde{\gamma}(0) = p$ and $\tilde{\gamma}(1) = q$. Given a point $z_0 \in S$, we denote by $D_r(z_0)$ the set of points on S with intrinsic distance from z_0 less than r, i.e.

$$D_r(z_0) = \{z \in S : d_S(z, z_0) < r\}.$$

When $r < \rho$, (2.6) implies

$$|D_r(p)| \geq \omega_n r^n. \tag{3.4}$$

Then we consider the increasing sequence $\{t_0, t_1, \dots, t_I\}$ in [0, 1] recursively defined as follows: $t_0 = 0$, and

$$t_{i+1} = \inf\Big\{t \in [0,1] : D_{\delta/2}(\tilde\gamma(s)) \cap \bigcup_{j=0}^{i} D_{\delta/2}(\tilde\gamma(t_j)) = \emptyset,\ \forall s \in [t,1]\Big\} \tag{3.5}$$

if the set in braces is non-empty, and $t_{i+1} = t_I$ otherwise. Therefore $\{t_0, t_1, \dots, t_I\}$ is an increasing sequence in [0, 1] satisfying

$$D_{\delta/2}(\tilde\gamma(t_i)) \cap D_{\delta/2}(\tilde\gamma(t_j)) = \emptyset \quad \text{for } i \neq j,\ i, j = 0, \dots, I, \tag{3.6}$$

and

$$D_{\delta/2}(\tilde\gamma(t_i)) \subset S_m^{\delta/2}, \quad i = 0, \dots, I.$$

We complete the sequence by adding $t_{I+1} = 1$ as the last term. Since

$$\Big|\bigcup_{i=0}^{I} D_{\delta/2}(\tilde\gamma(t_i))\Big| \leq |S|,$$

from (3.4) and (3.6) we obtain

$$I + 1 \leq \frac{2^n}{\omega_n \delta^n}|S|. \tag{3.7}$$

From (3.5), it is clear that

$$\overline{D}_{\delta/2}(\tilde\gamma(t_i)) \cap \bigcup_{j=0}^{i-1} \overline{D}_{\delta/2}(\tilde\gamma(t_j)) \neq \emptyset$$

for every $i = 1, \dots, I$. Let

$$\sigma(i) = \max\{j > i : \overline{D}_{\delta/2}(\tilde\gamma(t_i)) \cap \overline{D}_{\delta/2}(\tilde\gamma(t_j)) \neq \emptyset\}.$$

Then we set $\sigma^2(i) = \sigma(\sigma(i))$, $\sigma^3(i) = \sigma(\sigma(\sigma(i)))$ and so on, and fix $\tau \in \mathbb{N}$ such that $\sigma^\tau(0) = I$. We define γ_1 as a minimal geodesic joining p and $\tilde\gamma(t_{\sigma(0)})$ and such that

$$\gamma_1 \subset \overline{D}_{\delta/2}(p) \cup \overline{D}_{\delta/2}(\tilde\gamma(t_{\sigma(0)}));$$

for $i = 2, \dots, \tau$, we let γ_i be a minimal geodesic joining $\tilde\gamma(t_{\sigma^i(0)})$ and $\tilde\gamma(t_{\sigma^{i+1}(0)})$ and such that

$$\gamma_i \subset \overline{D}_{\delta/2}(\tilde\gamma(t_{\sigma^i(0)})) \cup \overline{D}_{\delta/2}(\tilde\gamma(t_{\sigma^{i+1}(0)})).$$

Moreover, we let $\gamma_{\tau+1}$ be a minimal geodesic joining $\tilde\gamma(t_I)$ and q and such that

$$\gamma_{\tau+1} \subset \overline{D}_{\delta/2}(\tilde\gamma(t_{\sigma^{\tau+1}(0)})) \cup \overline{D}_{\delta/2}(q).$$

Let γ be the piecewise geodesic obtained as the union of $\gamma_1, \ldots, \gamma_{\tau+1}$. It is clear that each γ_i has length δ for $i = 1, \ldots, \tau$, and $\leq \delta$ for $i = \tau + 1$. Since $\tau \leq I$, from (3.7) we obtain

$$\mathrm{length}(\gamma) \leq (\tau + 1)\delta \leq \frac{2^n}{\omega_n \delta^{n-1}}|S|,$$

which implies (3.2) and (3.3), and the proof is complete. □

It will be useful to define the following two numbers:

$$\varepsilon_0 = \min\left(\frac{1}{2}, \frac{\rho}{16L} \sin \frac{\delta}{2\rho}\right), \tag{3.8}$$

$$N_0 = 1 + \left[\log_{(1-\varepsilon_0)} \frac{1}{2}\right], \tag{3.9}$$

where L is given by (3.2) and $[\cdot]$ is the integer part function. We have the following lemma.

Lemma 3.3. *Let $\delta \in (0, \rho)$, $\varepsilon \in (0, \varepsilon_0)$ with ε_0 given by (3.8), and set*

$$r_i = (1 - \varepsilon)^i \rho \sin \frac{\delta}{2\rho} \tag{3.10}$$

for $i \in \mathbb{N}$. Let p and q be any two points in a connected component of S_m^δ. Then there exist an integer $N \leq N_0$ with N_0 given by (3.9) and a sequence $\{p_1, \ldots, p_N\}$ of points in $S_m^{\delta/2}$ such that

$$p, q \in \textstyle\bigcup_{i=0}^{n} \overline{\mathcal{U}}_{r_i/4}(p_i), \tag{3.11}$$

$$\mathcal{U}_{r_0}(p_i) \subset S_m, \qquad i = 0, \ldots, N, \tag{3.12}$$

$$p_{i+1} \in \overline{\mathcal{U}_{r_i/4}(p_i)}, \qquad i = 0, \ldots, N - 1, \tag{3.13}$$

where $\mathcal{U}_{r_i}(p_i)$ are defined as in (2.1).

Proof. Let γ be a path as in Lemma 3.2 and denote by s its arc length. Set $p_0 = p$ and define $p_i = \gamma(r_i/4)$ for each $i = 1, \ldots, N - 1$, and $p_N = q$. Here, N is the largest integer such that

$$\sum_{i=0}^{N-1} \frac{r_i}{4} \leq L.$$

Since $\varepsilon < \varepsilon_0$, we have

$$\sum_{i=0}^{N_0-1} \frac{r_i}{4} > 2L,$$

and hence such an N exists and we can assume that $N \leq N_0$, where N_0 is defined by (3.9). Since $\gamma \subset S_m^{\delta/2}$, the assertion of the theorem easily follows from (2.6). □

For a fixed direction $\ell \in \mathbb{S}^n$, we denote by $\ell^\perp$ the orthogonal subspace to ℓ, i.e.

$$\ell^\perp = \{z \in \mathbb{R}^{n+1} : z \cdot \ell = 0\}.$$

Lemma 3.4. *Let $p \in S$ and $u\colon B_r \cap T_pS \to \mathbb{R}$ be a C^2 map as in (2.1) with $r < \rho$. Let $\ell \in \mathbb{S}^n$ be such that*

$$\nu_p \cdot \ell > 0 \quad \text{and} \quad |\ell - \nu_p| < \varepsilon \tag{3.14}$$

for some $0 \le \varepsilon < 1$. There exists a C^2 function $v : B_{r\sqrt{1-\varepsilon^2}} \cap \ell^\perp \to \mathbb{R}$ such that the set

$$V = \{p + y + v(y)\ell : y \in B_{r\sqrt{1-\varepsilon^2}} \cap \ell^\perp\} \tag{3.15}$$

is contained in $\mathcal{U}_r(p)$. Moreover,

$$\|v\|_\infty \le \|u\|_\infty + \sqrt{2}\,\varepsilon r. \tag{3.16}$$

Proof. Let $q = p + x + u(x)\nu_p$ be a point in $\mathcal{U}_r(p)$ with

$$|x| < r\sqrt{1-\varepsilon^2}. \tag{3.17}$$

By the implicit function theorem, if $\nu_q \cdot \ell > 0$, then S can be locally represented as the graph of a function near q over the hyperplane $\ell^\perp$. Let $A \in \mathrm{SO}(n+1)$ be a special orthogonal matrix such that $A\nu_p = \ell$, and let $y \in \ell^\perp$ be such that $y = Ax$. Since $A \in \mathrm{SO}(n+1)$, we have $|x| = |y|$ and so

$$|y| < r\sqrt{1-\varepsilon^2}.$$

From the triangle and Cauchy–Schwarz inequalities we have

$$\nu_q \cdot \ell \ge \nu_q \cdot \nu_p - |\ell - \nu_p|;$$

(2.5) and (3.14) yield

$$\nu_q \cdot \ell \ge \sqrt{1 - |x|^2/\rho^2} - \varepsilon,$$

which implies that $\nu_q \cdot \ell > 0$ on account of (3.17). Therefore any point $q \in V$ can be written both as $q = p + x + u(x)\nu_p$ and as $q = p + y + v(y)\ell$ for some $x \in T_pS$ and $y \in \ell^\perp$. In particular

$$y + v(y)\ell = x + u(x)\nu_p,$$

and since $y = Ax$, we have

$$(I - A)x + u(x)\nu_p = v(y)\ell.$$

By taking the scalar product with ℓ, we readily obtain

$$|v(\xi)| \le |I - A|\,|x| + |u(x)|. \tag{3.18}$$

The matrix A can be chosen such that $|I - A| \le 2\sqrt{1 - \ell \cdot \nu_p} \le \sqrt{2}\,\varepsilon$, and (3.18) implies the last part of the statement. □

It will be important to compare the normal vectors to two surfaces which are graphs of functions over the same domain. We have the following lemma.

Lemma 3.5. *Let $u_1, u_2 \in C^1(B_r \cap e_{n+1}^{\perp})$ and assume that*

$$|\nabla u_2(x_0) - \nabla u_1(x_0)| < \varepsilon$$

for some $x_0 \in B_r \cap e_{n+1}^{\perp}$. Let $p_i = x_0 + u_i(x_0)e_{n+1}$, $i = 1, 2$. Then

$$|\nu_{p_1} - \nu_{p_2}| \leq \tfrac{1}{2}\sqrt{5}\,\varepsilon, \tag{3.19}$$

where

$$\nu_{p_i} = \frac{-\nabla u_i(x_0) + e_{n+1}}{\sqrt{1 + |\nabla u_i(x_0)|^2}}$$

is the inward normal to the graph of u_i at p_i, $i = 1, 2$.

Proof. Since the eigenvalues of the Hessian of the function $x \mapsto \sqrt{1 + |x|^2}$ are uniformly bounded by 1, its gradient is Lipschitz continuous with constant 1 and we have

$$\left| \frac{\nabla u_1(x)}{\sqrt{1 + |\nabla u_1(x)|^2}} - \frac{\nabla u_2(x)}{\sqrt{1 + |\nabla u_2(x)|^2}} \right| \leq |\nabla u_1(x) - \nabla u_2(x)|. \tag{3.20}$$

Moreover,

$$\left| \frac{1}{\sqrt{1 + |\nabla u_1(x)|^2}} - \frac{1}{\sqrt{1 + |\nabla u_2(x)|^2}} \right| \leq \frac{1}{2}\big||\nabla u_1(x)| - |\nabla u_2(x)|\big|. \tag{3.21}$$

From the triangle inequality and from (3.20) and (3.21) we readily obtain (3.19). □

4. Proof of Theorem 1.1

The proof of Theorem 1.1 relies upon a quantitative study of the method of moving planes and it consists of several steps, which we now sketch.

Step 1. We fix a direction ω, apply the method of moving planes, and find a critical position which defines a critical hyperplane π_m, as described in Subsection 2.2. By using the smallness of $\operatorname{osc}(H)$, we can prove that (up to a connected component) the surface S and the reflected cap S_m are close. Hence, the union of the cap and the reflected cap provides a symmetric set in the direction ω which gives information about the approximate symmetry of S in that direction. It is important to notice that the estimates do not depend on the chosen direction.

Step 2. We apply Step 1 in $n + 1$ orthogonal directions and we obtain a point $\mathcal{O}$ as the intersection of the corresponding $n + 1$ critical hyperplanes. Since the estimates in Step 1 do not depend on the direction, the point $\mathcal{O}$ can be chosen as an approximate center of symmetry. Moreover, any critical hyperplane in any other direction is less than some constant times $\operatorname{osc}(H)$ away from $\mathcal{O}$.

Step 3. Again by using the estimates in Step 1, we can define two balls centered at $\mathcal{O}$ such that estimate (1.3) holds.

We notice that once we have the approximate symmetry in one direction, i.e. Step 1, then the argument for proving Steps 2 and 3 is well-established [ABR, Section 4]. In the following we will prove Step 1, which is our main result of this section, and for the sake of completeness, we give a sketch of the proof for Steps 2 and 3.

4.1. Step 1. Approximate symmetry in one direction

We apply the moving plane procedure as described in Subsection 2.2. Let $\omega \in \mathbb{S}^n$ be a direction in $\mathbb{R}^{n+1}$ and let S_m, $\hat{S}_m$ be defined as in (2.25). Let p_0 be a tangency point between S_m and $\hat{S}_m$, and denote by Σ and $\hat{\Sigma}$ the connected components of S_m and $\hat{S}_m$, respectively, containing p_0 or having p_0 on their boundary. Let S^* be the reflection of S about π_m. For a point p in S (or S^*), we denote by ν_p the normal vector to S (or to S^*) at p. We will use this notation when it does not lead to ambiguity: the choice of the vector normal and of the surface is implied by the point itself. If $p \in S \cap S^*$ is a point of tangency between S and S^*, then the normal vector at p is the same for both the surfaces, and the notation is coherent. When ambiguity occurs, i.e. for non-tangency points in $S \cap S^*$, we will specify the dependence on the surface. For points on $\partial\Sigma$ (or $\partial\hat{\Sigma}$) we will denote by ν the Gauss map on $\partial\Sigma$ (or $\partial\hat{\Sigma}$) which is induced by the one on S^* (or S).

The main goal of Step 1 is to prove the following result of approximate symmetry in one direction.

Theorem 4.1. *There exists a positive constant ε such that if* $\operatorname{osc}(H) \leq \varepsilon$, *then for any $p \in \Sigma$ there exists $\hat{p} \in \hat{\Sigma}$ such that*

$$|p - \hat{p}| + |\nu_p - \nu_{\hat{p}}| \leq C \operatorname{osc}(H). \tag{4.1}$$

Here, the constants ε and C depend only on n, ρ, $|S|$ and do not depend on the direction ω.

Before giving the proof of Theorem 4.1, we provide two preliminary results about the geometry of Σ. For $t > 0$ we set

$$\Sigma^t = \{p \in \Sigma : d_\Sigma(p, \partial\Sigma) > t\}.$$

The following two lemmas show some conditions implying that Σ^t is connected for t small enough.

Lemma 4.2. *Assume that there exists $\mu \leq 1/2$ such that*

$$\nu_p \cdot \omega \leq \mu \tag{4.2}$$

for every p on the boundary of Σ. Then Σ^t is connected for any $0 < t \leq t_0$, where

$$t_0 = \frac{\rho}{2\sqrt{n}}\sqrt{1 - 2\mu^2}.$$

Proof. Let S^* be the reflection of S about π_m. We notice that, by construction of the moving planes, Σ and π_m enclose a bounded simply connected domain of $\mathbb{R}^{n+1}$. Moreover, $\nu_p \cdot \omega \geq 0$ on $\partial\Sigma$ and (4.2) implies that π_m intersects S^* transversally. Hence, the boundary of Σ is a manifold of class C^2. We prove that the boundary of Σ^t lies in a tubular neighborhood of the boundary of Σ in S^*. Then, since Σ is connected, any two points in Σ^t can be joined by a curve in Σ which can be pushed into Σ^t by using the normal vector field to the boundary Σ.

Following Section 2.3, we denote the boundary of Σ by Σ' and we orient Σ' by the Gauss map satisfying

$$\nu_p \cdot \nu'_p = 1 - (\nu_p \cdot \omega)^2$$

(see (2.28)). Hence, from (4.2), we have

$$\nu_p \cdot \nu'_p \geq 1 - \mu^2.$$

Since the principal curvatures of S are bounded by ρ^{-1}, from Proposition 2.7 the principal curvatures κ'_i of Σ' satisfy

$$|\kappa'_i| \leq \frac{1}{\rho(1-\mu^2)}, \qquad i = 1, \ldots, n-1. \tag{4.3}$$

From Lemma 3.4, we can write S^* as the graph of a function $u : B_r \cap (\nu'_p)^\perp \to \mathbb{R}$ with $r = \rho\sqrt{1-2\mu^2}$. Moreover, (4.3) and Lemma 2.1 imply that Σ' is locally the graph of u restricted to $B_r \cap T_p\Sigma'$. Taking into account that $(\nu'_p)^\perp = T_p\Sigma' \oplus \langle\omega\rangle$, we consider the subset of S^* given by

$$Q(p) = \{q = p + \xi + s\omega + u(\xi + s\omega)\nu'_p : \xi \in B_r \cap T_p\Sigma',\ |s| \leq t_0\},$$

which contains a tubular neighborhood of $\Sigma' \cap B_{t_0}(p)$ of radius at least t_0. Hence, the set $\mathcal{Q} = \bigcup_{p\in\Sigma'} Q(p)$ contains a tubular neighborhood of Σ' in S^* of radius at least t_0, which concludes the proof. □

Lemma 4.3. *Let* $0 < \delta \leq \rho(8\sqrt{n})^{-1}$. *If there exists a connected component* Γ^δ *of* Σ^δ *satisfying*

$$0 \leq \nu_p \cdot \omega \leq 1/8 \quad \text{for any } p \in \partial\Gamma^\delta,$$

then Σ^δ *is connected.*

Proof. To simplify the notation we let $\mu_0 = 1/8$. Notice that the interior and exterior touching balls at every boundary point of Γ^δ intersect π_m. By using this argument and after elementary but tedious calculations, we can prove that for any $q \in \Sigma \setminus \Gamma^\delta$,

$$d_\Sigma(q, \Gamma^\delta) \leq \rho \arcsin((1 + 2\mu_0)\delta/\rho).$$

In particular, for any $q \in \partial\Sigma$ there exists $p \in \partial\Sigma^\delta$ such that

$$d_\Sigma(q, p) \leq \rho \arcsin((1 + 2\mu_0)\delta/\rho),$$

and from Lemma 2.1 we obtain

$$|\nu_p - \nu_q| \leq \sqrt{2} \arcsin((1 + 2\mu_0)\delta/\rho).$$

By writing $\nu_q \cdot \omega = \nu_p \cdot \omega - (\nu_q - \nu_p) \cdot \omega$ and by the triangle inequality we get

$$|\nu_q \cdot \omega| \leq \mu_0 + \sqrt{2} \arcsin((1 + 2\mu_0)\delta/\rho);$$

our assumptions on δ imply the following (rougher but simpler) bound:

$$|\nu_q \cdot \omega| \le 2\mu_0 + 1/2.$$

Now we use Lemma 4.2 by setting $\mu = 2\mu_0 + 1/2$ and taking $\delta \le t_0$. □

Now, we focus on the proof of Theorem 4.1. It will be divided into four cases, which we study in the consecutive subsections. In each case, δ will be fixed to be

$$\delta = \min\left(\frac{\rho}{2^6}, \frac{\rho}{8\sqrt{n}}\right).$$

Moreover, the constants ε and C can be chosen as

$$\varepsilon = \min\{\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3\} \quad \text{and} \quad C = \tfrac{5}{4} C_1 K_1 K_2 K_3.$$

Here, ε_0 is given by (3.8), and $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and C_1 will be defined below. Moreover, K_1, K_2, K_3 are given by Lemmas 2.3, 2.4, 2.6, respectively, where M is chosen according to Lemma 2.1 by assuming that $|x| \le \rho/2$. Hence, the constants ε and C depend only on n and upper bounds on ρ^{-1} and $|S|$.

4.1.1. Case 1: $d_\Sigma(p_0, \partial\Sigma) > \delta$ and $d_\Sigma(p, \partial\Sigma) \ge \delta$. In this case we assume that p_0 and p are interior points of Σ, which are more than δ away from $\partial\Sigma$. We remark that in this case, p_0 is an interior touching point between Σ and $\hat{\Sigma}$, so that case (i) in the method of moving planes occurs. We first assume that p_0 and p are in the same connected component of Σ^δ; then Lemma 4.3 will be used to show that Σ^δ is in fact connected.

Let

$$r_0 = \rho \sin \frac{\delta}{2\rho}.$$

Since p and p_0 are in a connected component of Σ^δ, there exist: $\{p_1, \dots, p_N\}$ in the connected component of $\Sigma^{\delta/2}$ containing p_0, a chain $\{\mathcal{U}_{r_0}(p_i)\}_{i=0}^N$ of open subsets of Σ and a sequence of maps $u_i : B_{r_0} \cap T_{p_i}\Sigma \to \mathbb{R}$, $i = 0, \dots, N$, as in Lemma 3.3, where $r_i = (1-\varepsilon)^i r_0$. We notice that Σ and $\hat{\Sigma}$ are tangent at p_0, and in particular the normal vectors to Σ and $\hat{\Sigma}$ at p_0 coincide. We stress that $\hat{\Sigma} \subset S$, and since $r_0 < \rho$, from Lemma 2.1 we know that S is locally represented near p_0 as the graph of a map $\hat{u}_0 \colon B_{r_0} \cap T_{p_0}S \to \mathbb{R}$.

Lemma 2.1 implies that $|\nabla u_0|, |\nabla \hat{u}_0| \le M$ in $B_{r_0} \cap T_{p_0}\Sigma$, where M is some constant which depends only on r_0, i.e. only on ρ. Now, we use Lemma 2.3: since $u_0(0) = \hat{u}_0(0)$ and $u_0 \ge \hat{u}_0$, (2.14) gives

$$\|u_0 - \hat{u}_0\|_{C^1(B_{r_0/4} \cap T_{p_0}\Sigma)} \le K_1 \operatorname{osc}(H), \tag{4.4}$$

where K_1 depends only on n and M. We notice that from (3.13) we have $p_1 \in \overline{\mathcal{U}}_{r_0/4}(p_0)$. Let x_1 be the projection of p_1 onto $T_{p_0}\Sigma$ and let

$$\hat{p}_1^* := p_0 + x_1 + \hat{u}_0(x_1)\nu_{p_0} \in \hat{\Sigma}.$$

From (4.4) we obtain

$$|\nabla u_0(x_1) - \nabla \hat{u}_0(x_1)| \leq K_1 \mathrm{osc}(H),$$

and therefore Lemma 3.5 yields

$$|\nu_{p_1} - \nu_{\hat{p}_1^*}| \leq \tfrac{1}{2}\sqrt{5}\, K_1 \,\mathrm{osc}(H). \tag{4.5}$$

Let $\hat{p}_1$ be the nearest point to p_1 in $\hat{\Sigma}$ which can be written as $\hat{p}_1 = p_1 - \tau \nu_{p_1}$ for some $\tau \geq 0$. Since $|x_1| \leq r_0/4$, from (2.5) we have $\nu_p \cdot \nu_{p_1} \geq \sqrt{1-(r_0/\rho)^2}$. From (4.4), (4.5) and by a simple geometrical argument, we obtain

$$\tau \leq \frac{2K_1\,\mathrm{osc}(H)}{\nu_p \cdot \nu_{p_1}} \quad \text{and} \quad |x_1 - \hat{x}_1| \leq 2K_1 \frac{\delta}{4\rho}\,\mathrm{osc}(H),$$

where $\hat{x}_1$ is the projection of $\hat{p}_1$ onto $T_{p_0}\Sigma$. This implies that

$$|p_1 - \hat{p}_1| + |\nu_{p_1} - \nu_{\hat{p}_1}| \leq cK_1\,\mathrm{osc}(H), \tag{4.6}$$

where c depends only on n and ρ.

As already mentioned, we have a local parametrization of Σ in a neighborhood of p_1 as the graph of the C^2 function $u_1\colon B_{r_0} \cap T_{p_1}\Sigma \to \mathbb{R}$. Lemma 3.4 and (4.5) imply that S can be locally parametrized by the graph of a function $\hat{u}_1\colon B_{r_1} \cap T_{p_1}\Sigma \to \mathbb{R}$, where $r_1 < r_0\sqrt{1 - c^2K_1^2\varepsilon^2}$ since $\varepsilon \leq \varepsilon_1$ with

$$\varepsilon_1 = (1 + c^2K_1^2)^{-1}. \tag{4.7}$$

From the definition of $\hat{p}_1$, (4.6) and since $u_1 - \hat{u}_1 \geq 0$ by construction, we find that

$$0 \leq u_1(0) - \hat{u}_1(0) \leq cK_1\,\mathrm{osc}(H).$$

We use Lemma 2.3 to deduce that

$$\|u_1 - \hat{u}_1\|_{C^1(B_{r_1/4}\cap T_{p_1}\Sigma)} \leq K_1[cK_1 + 1]\,\mathrm{osc}(H). \tag{4.8}$$

Now, (4.8) is the analogue of (4.4) with p_1 instead of p_0, and we can iterate until we obtain two functions $u_N, \hat{u}_N : B_{r_N} \cap T_p\Sigma \to \mathbb{R}$ such that

$$\|u_N - \hat{u}_N\|_{C^1(B_{r_N/4}\cap T_p\Sigma)} \leq C_1 \mathrm{osc}(H). \tag{4.9}$$

A choice of $\hat{p}$ as in the statement of Theorem 4.1 is then given by $\hat{p} = p + \hat{u}_N(0)\nu_p$, since (4.1) is implied by (4.9) and Lemma 3.5.

We notice that a choice of the constant C_1 in (4.9) is given by

$$C_1 = (cK_1 + 1)^{N_0+1}, \tag{4.10}$$

where N_0 is given by (3.9). Hence the constant C_1 depends only on n, δ/ρ, and an upper bound on $|S|$.

Once we have (4.9) for any p in a connected component of Σ^δ, we have in fact

$$\nu_q \cdot \omega \leq 1/8$$

for any point q at the boundary of such a connected component, as follows from

Lemma 4.4. *Let $q \in \Sigma$ be such that $d_\Sigma(q, \partial\Sigma) \leq \delta$. Assume that the point $\hat{q} = q - \alpha\nu_q$ is on $\hat{\Sigma}$ and*

$$|\nu_q - \nu_{\hat{q}}| \leq \alpha \tag{4.11}$$

with $\alpha + 2\delta < \rho$. Then

$$0 \leq \nu_q \cdot \omega \leq \sqrt{8\delta^2/\rho^2 + \alpha/2}. \tag{4.12}$$

Proof. Let q^m be the reflection of q about π_m and let

$$t = \nu_q \cdot \omega.$$

By construction of the moving planes, it is clear that $t \geq 0$ and the first inequality in (4.12) follows. We denote by ν_{q^m} the inner normal vector to S at q^m. Since $\nu_q \cdot \omega = -\nu_{q^m} \cdot \omega$ and $\nu_q - \nu_{q^m} = 2t\omega$, we have

$$\nu_q \cdot \nu_{q^m} = 1 - 2t^2. \tag{4.13}$$

We notice that q^m and $\hat{q}$ both lie in S and $|q^m - \hat{q}| \leq \alpha + 2\delta$, which implies that $\hat{q} \in \mathcal{U}_\rho(q^m)$ provided that $\alpha + 2\delta < \rho$. Hence, (2.5) yields

$$\nu_{\hat{q}} \cdot \nu_{q^m} \geq \sqrt{1 - \left(\frac{\alpha + 2\delta}{\rho}\right)^2}.$$

From (4.11) and (4.13) we find that

$$1 - 2t^2 \geq \sqrt{1 - \left(\frac{\alpha + 2\delta}{\rho}\right)^2} - \alpha,$$

which gives

$$t^2 \leq \frac{1}{2}\left(\frac{\alpha + 2\delta}{\rho}\right)^2 + \frac{\alpha}{2},$$

and we obtain the second inequality in (4.12). □

The conclusion of Case 1 follows from the following argument. From (4.9) we know that for any q on the boundary of the connected component of Σ^δ containing p_0 there exists $\hat{q} \in \hat{\Sigma}$ such that

$$|q - \hat{q}| + |\nu_q - \nu_{\hat{q}}| \leq C_1 \operatorname{osc}(H).$$

We apply Lemma 4.4 by letting $\alpha = C_1 \operatorname{osc}(H)$; since $\varepsilon \leq \varepsilon_2$ with

$$\varepsilon_2 \leq 1/(2^6 C_1),$$

we obtain $0 \leq \nu_q \cdot \omega \leq 1/8$. Hence, from Lemma 4.3 we find that Σ^δ is connected.

4.1.2. Case 2: $d_\Sigma(p_0, \partial\Sigma) \geq \delta$ and $d_\Sigma(p, \partial\Sigma) < \delta$. Here the idea consists in extending the estimate of Subsection 4.1.1 to the whole Σ. This will be done by using Carleson type estimates given by Lemma 2.4. We remark that its application is not trivial, since we need more information on how S intersects π_m.

Following (2.25), for a given point $p \in \Sigma$ such that $d_\Sigma(p, \partial\Sigma) \leq \delta$, we denote by p^m the point of S obtained by reflecting p about π_m. The surface S can be locally written as the graph of a function $u\colon B_\rho \cap T_pS \to \mathbb{R}$. For $0 < r < \rho$, we define $U_r^*(p)$ as the reflection of $\mathcal{U}_r(p^m)$ about π_m and we denote by $U_r(p)$ the subset of Σ obtained by

$$U_r(p) = U_r^*(p) \cap \{q \in \mathbb{R}^{n+1} : q \cdot \omega < m\}.$$

Moreover, we denote by E_r the open subset of $B_r \cap T_p\Sigma$ such that

$$U_r(p) = \{p + x + u(x)\nu_p : x \in E_r\}. \tag{4.14}$$

The next result is a consequence of Propositions 2.7 and 2.8.

Lemma 4.5. *Let $q \in \Sigma$ be such that $d_\Sigma(q, \partial\Sigma) = \delta$ and $0 \leq \nu_q \cdot \omega \leq 1/4$. Let $U' = U^*_{\sqrt{2}\rho/8}(q) \cap \pi_m$ and U'' be the orthogonal projection of U' onto $T_q\Sigma$. Then U'' is a hypersurface of class C^2 of $T_q\Sigma$ whose principal curvatures are bounded by*

$$\mathcal{K} = 4\delta/\rho^2.$$

Proof. We notice that since $d_\Sigma(q, \partial\Sigma) = \delta$, we have $U' \neq \emptyset$. Let $\zeta \in U'$. Since the projection $\mathrm{pr}(\zeta)$ of ζ on $T_q\Sigma$ is in $\overline{B}_{\sqrt{2}\rho/8}$, from (2.5) we know that

$$|\nu_q - \nu_\zeta| \leq 1/4. \tag{4.15}$$

Since $\nu_\zeta \cdot \omega = \nu_q \cdot \omega + (\nu_\zeta - \nu_q) \cdot \omega$, we have

$$|\nu_\zeta \cdot \omega| \leq 1/2, \tag{4.16}$$

which implies that π_m intersects $U^*_{\sqrt{2}\rho/8}(q)$ transversally, and so U'' is a hypersurface of $T_q\Sigma$. Since the principal curvatures of S are bounded by $1/\rho$, (2.31) implies that the principal curvatures of U'' satisfy

$$|\kappa_i''(\mathrm{pr}(\zeta))| \leq \frac{1}{\rho|\nu_\zeta \cdot \nu_\zeta'|} \cdot \frac{\omega \cdot \nu_q}{[(\omega \cdot \nu_q)^2 + (\nu_q \cdot \nu_\zeta')^2]^{3/2}}, \qquad i = 1, \ldots, n-1,$$

where ν' is the Gauss map of U' viewed as a hypersurface of π_m satisfying

$$\nu_\zeta \cdot \nu_\zeta' = \sqrt{1 - (\nu_\zeta \cdot \omega)^2}. \tag{4.17}$$

Hence,

$$|\kappa_i''(\mathrm{pr}(\zeta))| \leq \frac{\omega \cdot \nu_q}{\rho|\nu_\zeta \cdot \nu_\zeta'|\,|\nu_q \cdot \nu_\zeta'|^3}, \qquad i = 1, \ldots, n-1. \tag{4.18}$$

From (4.16) and (4.17), we obtain

$$\nu_\zeta \cdot \nu_\zeta' \geq \sqrt{3}/2. \tag{4.19}$$

By writing $\nu_q \cdot \nu'_\zeta = (\nu_q - \nu_\zeta) \cdot \nu'_\zeta + \nu_\zeta \cdot \nu'_\zeta$ and using (4.15) and (4.19) we get

$$\nu_q \cdot \nu'_\zeta \geq 1/2,$$

and from (4.18) and (4.19) we obtain the assertion. □

In the next lemma we give a bound which will be useful later.

Lemma 4.6. *Let q and α be as in Lemma 4.4. Then*

$$0 \leq \nu_\zeta \cdot \omega \leq \sqrt{8\delta^2/\rho^2 + \alpha/2} + (\sqrt{2}/\rho) d_\Sigma(q, \zeta) \tag{4.20}$$

for any $\zeta \in \overline{U}_\rho(q)$, where $U_\rho(q)$ is defined as in (4.14).

Proof. Let $\zeta \in \overline{U}_\rho(q)$. By construction we have $\nu_\zeta \cdot \omega \geq 0$. Since

$$\nu_\zeta \cdot \omega \leq \nu_q \cdot \omega + |\nu_\zeta - \nu_q|,$$

from (2.5) and (4.12) we get the assertion. □

Now we are ready to prove Theorem 4.1 for Case 2. Let

$$\varepsilon_3 = \delta/(\rho C_1)$$

where C_1 is given by (4.10). We assume that $d_\Sigma(p_0, \partial\Sigma) \geq \delta$ and $d_\Sigma(p, \partial\Sigma) < \delta$. By arguing as in Case 1, we see that Σ^δ is connected. Let $q \in \Sigma$ and $\bar{p} \in \partial\Sigma$ be such that

$$d_\Sigma(p, q) + d_\Sigma(p, \partial\Sigma) = \delta \quad \text{and} \quad d_\Sigma(p, \bar{p}) = d_\Sigma(p, \partial\Sigma)$$

(we notice that our choice of δ implies that q and $\bar{p}$ exist).

Since $d_\Sigma(q, \partial\Sigma) = \delta$, from Case 1 we find that there exists $\hat{q} \in \hat{\Sigma}$ such that

$$|q - \hat{q}| + |\nu_q - \nu_{\hat{q}}| \leq C_1 \operatorname{osc}(H) \tag{4.21}$$

(see (4.9)). From the proof of Case 1, it is clear that $\hat{q}$ can be chosen as

$$\hat{q} = q - \alpha\nu_q$$

for some $0 \leq \alpha \leq C_1 \operatorname{osc}(H)$. Let

$$r = \rho/8. \tag{4.22}$$

We define the sets $U_r(q) \subset \Sigma$, $E_r \subseteq B_r \cap T_q\Sigma$, and the map $u : E_r \to \mathbb{R}$ as in (4.14) with q in place of p. Since $\hat{q} \in \hat{\Sigma} \subset S$ and $|\nu_q - \nu_{\hat{q}}| \leq C_1 \operatorname{osc}(H)$, from Lemma 3.4 we infer that S can be locally written (around $\hat{q}$) as the graph of a function $\hat{u}$ over $T_q\Sigma \cap B_{\rho\sqrt{1-C_1^2\varepsilon_3^2}}$ and in particular over $T_q\Sigma \cap B_r$ (which is justified by our choice of ε_3).

We notice that Lemma 2.2 implies that $p, \bar{p} \in \overline{U}_r(q)$. Let ∂E_r be the boundary of E_r in $T_q\Sigma$ and let $\bar{x} \in \partial E_r$ be the projection of $\bar{p}$. Since $d_\Sigma(q, \bar{p}) = \delta$, from Lemma 2.2 we have

$$\rho \sin(\delta/\rho) \leq |\bar{x}| \leq \delta. \tag{4.23}$$

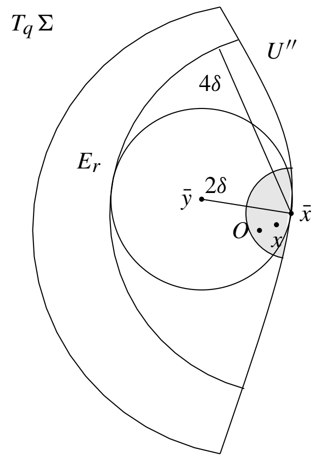


Fig. 2. Case 2 in the proof of Theorem 4.1. The shadow region is $B_\delta(\bar{x}) \cap E_r$.

Let $U' = U_r^*(q) \cap \pi_m$ and let U'' be the projection of U' onto $T_q\Sigma$ (as in Lemma 4.5). Notice that by definition, $U'' \subset \partial E_r$, and in particular $u = \hat{u}$ on U''. From Lemmas 4.4 and 4.5, the principal curvatures of U'' are uniformly bounded by $\mathcal{K}$. We notice that our choice of δ implies that $\mathcal{K} \leq 1/(16\rho)$.

Let x be the projection of p over $T_q\Sigma$. From (4.23) we have $B_{4\delta}(\bar{x}) \cap \partial E_r \subset U''$ and we can apply Lemma 2.4 to deduce that

$$\sup_{B_\delta(\bar{x})\cap E_r} (u - \hat{u}) \leq K_2\big((u - \hat{u})(\bar{y}) + \operatorname{osc}(H)\big) \tag{4.24}$$

with $\bar{y} = \bar{x} + 2\delta\nu''_{\bar{x}}$, where $\nu''_{\bar{x}}$ is the interior normal to U'' at $\bar{x}$ (see Figure 2). We notice that $x \in B_\delta(\bar{x}) \cap E_r$, and so from (4.24) we find that

$$(u - \hat{u})(x) \leq K_2\big((u - \hat{u})(\bar{y}) + \operatorname{osc}(H)\big). \tag{4.25}$$

Since $2\delta < \mathcal{K}^{-1}$, the point $\bar{y}$ has distance 2δ from the boundary of E_r, and by Lemma 2.2 the point

$$\bar{q} = q + \bar{y} + u(\bar{y})\nu_q$$

satisfies

$$d_\Sigma(\bar{q}, \partial\Sigma) \geq 2\delta.$$

Hence, from Case 1 (applied to p_0 and $\bar{q}$) we obtain the estimate

$$(u - \hat{u})(\bar{y}) \leq C_1 \operatorname{osc}(H),$$

and from (4.25) we get

$$(u - \hat{u})(x) \leq C_1 K_2 \operatorname{osc}(H).$$

By letting $\hat{p} = q + x + \hat{u}(x)\nu_q$, and since $d_\Sigma(p, \partial\Sigma) > 0$, a standard application of Lemmas 2.3 and 3.5 yields the estimate

$$|p - \hat{p}| + |\nu_p - \nu_{\hat{p}}| \leq \tfrac{1}{2}\sqrt{5}\, C_1 K_1 K_2 \operatorname{osc}(H),$$

and the proof of Case 2 is complete.

4.1.3. Case 3: $0 < d_\Sigma(p_0, \partial\Sigma) < \delta$. Since p_0 is the tangency point, it is easy to show that the center of the interior touching sphere of radius ρ to S at p_0 lies in the half-space $\{q \in \mathbb{R}^{n+1} : q \cdot \omega \leq m\}$ (see for instance [CMV1, Lemma 2.1]). From this, and since

$$|p_0 \cdot \omega - m| \leq d_\Sigma(p_0, \partial\Sigma) \leq \delta,$$

by Lemma 4.4 (with $\alpha = 0$) we obtain

$$\nu_{p_0} \cdot \omega \leq 3\delta/\rho.$$

As in Case 2 (with q replaced by p_0), we locally write Σ and $\hat{\Sigma}$ as the graphs of functions $u, \hat{u} \colon E_r \to \mathbb{R}$, respectively, where $E_r \subseteq T_{p_0}\Sigma$ is defined as in the introduction to this subsection, and r is given by (4.22). Moreover, we denote by U'' the portion of ∂E_r which is obtained by projecting $U_r^*(p_0) \cap \pi_m$ onto $T_{p_0}\Sigma$. We remark that $u = \hat{u}$ on U'' and that the principal curvatures of U'' are bounded by $\mathcal{K}$.

Let $\bar{x} \in U''$ be a point such that

$$|\bar{x}| = \min_{x \in U''} |x|.$$

Notice that $|\bar{x}| \leq d_\Sigma(p_0, \partial\Sigma) < \delta$. Let $\nu''_{\bar{x}}$ be the interior normal to U'' at $\bar{x}$, and set

$$y = \bar{x} + 2\delta\nu''_{\bar{x}}$$

(see Figure 3). We notice that the principal curvatures of U'' are bounded by $\mathcal{K}$, and

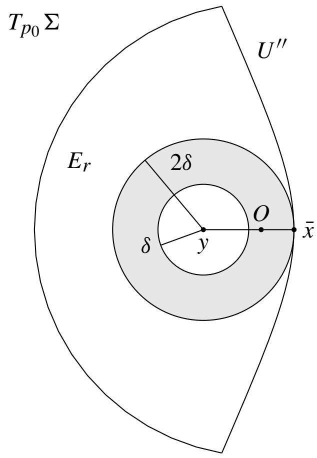


Fig. 3. Case 3 in the proof of Theorem 4.1.

$2\delta \le \mathcal{K}^{-1}$ and the ball $B_{2\delta}(y) \cap T_{p_0}\Sigma$ is contained in E_r and tangent to U'' at $\bar{x}$, with $\nu''_{\bar{x}} = -\bar{x}/|\bar{x}|$. Hence, the origin O of $T_{p_0}\Sigma$ (i.e. the projection of p_0 over $T_{p_0}\Sigma$) lies in the annulus $(B_{2\delta}(y) \setminus B_\delta(y)) \cap T_{p_0}\Sigma$. Therefore, we can apply (2.23) (where we set $x_0 = \bar{x}$, $c = y$ and $r = 2\delta$), and since $u(0) = \hat{u}(0)$, we find that

$$\|u - \hat{u}\|_{C^1(B_{\delta/2}(y)\cap T_{p_0}\Sigma)} \le K_3 \operatorname{osc}(H). \tag{4.26}$$

Let

$$q = p_0 + y + u(y)\nu_{p_0} \quad \text{and} \quad \hat{q} = p_0 + y + \hat{u}(y)\nu_{p_0}.$$

We notice that (4.26) and Lemma 3.5 imply that

$$|q - \hat{q}| + |\nu_q - \nu_{\hat{q}}| \le \tfrac{1}{2}\sqrt{5}\, K_3 \operatorname{osc}(H).$$

Since y has distance 2δ from ∂E_r, we have $d_\Sigma(q, \partial\Sigma) \ge 2\delta$, and we can apply Cases 1 and 2 to conclude the proof.

4.1.4. Case 4: $p_0 \in \partial\Sigma$. This case is the limiting case of Case 3 for $d_\Sigma(p_0, \partial\Sigma) \to 0$. Indeed, in this case we can write Σ and $\hat{\Sigma}$ as graphs of functions over a half-ball on $T_{p_0}\Sigma$. Hence the argument used in Case 3 can be easily adapted by using (2.24) instead of (2.23).

4.2. *Steps 2–3. Approximate radial symmetry and conclusion*

We consider $n+1$ orthogonal directions $e_1, \dots, e_{n+1}$, and we denote by $\pi_1, \dots, \pi_{n+1}$ the corresponding critical hyperplanes. Let

$$\mathcal{O} = \bigcap_{i=1}^{n+1} \pi_i,$$

and denote by $\mathcal{R}(p)$ the reflection of p in $\mathcal{O}$. The following lemma extends Theorem 4.1.

Lemma 4.7. *For any $p \in S$ there exists $q \in S$ such that*

$$|\mathcal{R}(p) - q| \le (n+1)C \operatorname{osc}(H).$$

Proof. We write $\mathcal{R} = \mathcal{R}_{n+1} \circ \cdots \circ \mathcal{R}_1$, where $\mathcal{R}_i$ is the reflection about π_i, $i = 1, \dots, N+1$. By iterating Theorem 4.1 $n+1$ times, we conclude the proof. ◻

As in [ABR, Proposition 6], for every direction ω,

$$\operatorname{dist}(\mathcal{O}, \pi_m) \le C \operatorname{osc}(H), \tag{4.27}$$

where π_m is the critical hyperplane in the direction ω and C is a constant that depends only on ρ and $\operatorname{diam} S = \max_{p,q\in S} |p-q|$. We notice that $\operatorname{diam} S$ can be bounded in terms of $|S|$ and ρ^{-1}. Indeed, let $p, q \in S$ be such that $|p - q| = \operatorname{diam} S$. By arguing as in the proof of Lemma 3.2, we can find a piecewise geodesic path on S joining p and q, and with length bounded by (3.2) (with $\delta = \rho/2$ there); therefore,

$$\operatorname{diam} S \le \frac{|S| 2^{2n}}{\omega_n \rho^{n-1}}.$$

Hence, the constant C in (4.27) can be bounded in terms of the dimension n and upper bounds on ρ^{-1} and $|S|$.

Finally, the bound on the difference of the radii (1.3) of the approximating balls is obtained by arguing as in [ABR, Proposition 7]. Indeed, if we define

$$r_i = \min_{p \in S} |p - \mathcal{O}| \quad \text{and} \quad r_e = \max_{p \in S} |p - \mathcal{O}|,$$

and assume that the minimum and maximum are attained at p_i and p_e, respectively, we obtain

$$r_e - r_i \le 2 \operatorname{dist}(\mathcal{O}, \pi),$$

where π is the critical hyperplane in the direction $(p_e - p_i)/|p_e - p_i|$. By (4.27) we conclude the proof.

5. Proof of Corollary 1.2

Lemma 5.1. *Let S be a closed C^2 hypersurface embedded in $\mathbb{R}^{n+1}$ and assume*

$$S \subset \overline{B}_{r_e} \setminus B_{r_i} \quad \text{with } r_e - r_i \le 2\rho.$$

Then

$$\frac{p}{|p|} \cdot \nu_p \le -1 + \frac{1}{\rho}(r_e - r_i) \quad \text{for every } p \in S.$$

Proof. Without loss of generality we may assume that B_{r_e} and B_{r_i} are centered at the origin. Let $p \in S$ and let c^- and c^+ be the centers of the interior and the exterior touching balls of radius ρ tangent at p, respectively. Then

$$\left| c^- + \frac{c^-}{|c^-|}\rho \right| = \sup_{q \in B_\rho(c^-)} |q| \le r_e, \qquad \left| c^+ - \frac{c^+}{|c^+|}\rho \right| = \inf_{q \in B_\rho(c^+)} |q| \ge r_i,$$

and so

$$\left| c^- + \frac{c^-}{|c^-|}\rho \right|^2 - \left| c^+ - \frac{c^+}{|c^+|}\rho \right|^2 \le r_e^2 - r_i^2.$$

Therefore

$$|c^-|^2 + 2\rho|c^-| - |c^+|^2 + 2\rho|c^+| \le r_e^2 - r_i^2.$$

Taking into account that $c^+ = p - \rho\nu_p$ and $c^- = p + \rho\nu_p$, we get

$$4\rho\, p \cdot \nu_p + 2\rho(|c^-| + |c^+|) \le r_e^2 - r_i^2,$$

and so

$$\frac{p}{|p|}\cdot\nu(p)\le-\frac{|c^-|+|c^+|}{2|p|}+\frac{r_e+r_i}{4\rho|p|}(r_e-r_i).$$

Since $|c^-|+|c^+|\ge|c^-+c^+|=2|\rho|$, and $r_e=r_i+(r_e-r_i)\le|p|+(r_e-r_i)$, we have

$$\frac{p}{|p|}\cdot\nu_p\le-1+\frac{r_e-r_i}{2\rho}+\frac{(r_e-r_i)^2}{4\rho^2}\le-1+\frac{r_e-r_i}{\rho},$$

as required. □

Proof of Corollary 1.2. Step 1: S is diffeomorphic to a sphere. In view of Theorem 1.1, there exist $\tilde\varepsilon$ and C such that if $\operatorname{osc}(H)<\tilde\varepsilon$, then (1.2) and (1.3) hold. We may assume the concentric balls B_{r_e} and B_{r_i} are centered in the origin. Let

$$\varepsilon=\min\{\tilde\varepsilon,\rho/(2C)\}. \tag{5.1}$$

Hence the assumptions in Lemma 5.1 are satisfied. We consider the map $\varphi\colon S\to\partial B_{r_i}$ defined by

$$\varphi(p)=r_i\,p/|p|.$$

We show that φ a diffeomorphism. It is clear that φ is smooth. Since B_{r_i} is contained in the bounded domain enclosed by S, φ is surjective. Indeed, if $\zeta\in\partial B_{r_i}$, then

$$\operatorname{dist}_S(\zeta)\le 0,\qquad \operatorname{dist}_S((r_e-r_i)\zeta)\ge 0,$$

and, by continuity, there exists a $t\ge 0$ such that $\operatorname{dist}_S((1+t)\zeta)=0$, i.e. $\zeta\in\varphi(S)$. Hence, assumption (1.1) plays a role only for proving the injectivity of φ. Let $p,q\in S$ and assume for contradiction that $\varphi(p)=\varphi(q)$. Then we may assume that $|p|<|q|$. Let $c^+=p-\rho\nu_p$ be the center of the exterior touching ball to S at p. Since $p/|p|=q/|q|$, we have

$$|q-c^+|^2=\left|(|q|-|p|)\frac{p}{|p|}+\rho\nu(p)\right|^2=(|q|-|p|)^2+\rho^2+2\rho(|q|-|p|)\frac{p}{|p|}\cdot\nu_p.$$

From Lemma 5.1 and since $|q|-|p|\le r_e-r_i$, we obtain

$$|q-c^+|^2\le(r_e-r_i)^2+\rho^2+2\rho(r_e-r_i)\left(-1+\frac{r_e-r_i}{\rho}\right)=\rho^2-(r_e-r_i)(2\rho-3(r_e-r_i)).$$

The choice of ε as in (5.1) implies that $|q-c^+|<\rho$, which gives a contradiction.

Step 2: proof of (1.4). We denote by $F\colon\partial B_{r_i}\to S$ the inverse of the map $\varphi\colon S\to\partial B_{r_i}$ considered in the first step. We can write $F(\zeta)=\zeta+\Psi(\zeta)\zeta/r_i$ for every ζ in ∂B_{r_i}, and from Step 1 and Theorem 1.1 it follows that $\|\Psi\|_{C^0(\partial B_{r_i})}\le C\operatorname{osc}(H)$. In order to prove a quantitative bound on the C^0-norm of the derivatives of Ψ, we work in the same fashion as in the proof of Lemma 3.4.

Let ζ be a fixed point on ∂B_{r_i} and set $p = F(\zeta)$ (i.e. $\zeta = r_i p/|p|$). Let T_ζ and T_p be the tangent spaces to ∂B_{r_i} at ζ and to S at p, respectively. We can locally write S around p as

$$q = p + x + u(x)\nu_p,$$

where x belongs to a small neighborhood of the origin O and u is a C^2 map satisfying $u(O) = 0$ and $\nabla u(O) = 0$. Without loss of generality, we can assume that $\zeta = r_i e_{n+1}$ so that

$$T_\zeta = \{x \in \mathbb{R}^{n+1} : x_{n+1} = 0\},$$

and we locally write ∂B_{r_i} as $\zeta' = \zeta + x + \eta(x)\nu_\zeta$, where $\eta(x) = r_i - \sqrt{r_i^2 - |x|^2}$.

As in the proof of Lemma 3.4, we can choose $A \in \mathrm{SO}(n+1)$ satisfying $A(\zeta) = -r_i \nu_p$ (we recall that $\nu_\zeta = -\zeta/r_i$), and we can locally write

$$p + Ax + u(Ax)\nu_p = p + x + v(x)\nu_\zeta; \tag{5.2}$$

furthermore, A is such that

$$|A - I| \leq 2\sqrt{1 - \nu_\zeta \cdot \nu_p}. \tag{5.3}$$

We first prove that

$$\partial_{x_k}\psi(O) = -\frac{1}{r_i}\partial_{x_k} v(O), \qquad k = 1, \ldots, n. \tag{5.4}$$

Indeed, by setting $\psi = \Psi \circ \eta$, we have

$$p + x + v(x)\nu_\zeta = \eta(x) - \psi(x)\nu_{\eta(x)},$$

which implies

$$p \cdot \nu_{\eta(x)} + x \cdot \nu_{\eta(x)} + v(x)\nu_\zeta \cdot \nu_{\eta(x)} - \eta(x) \cdot \nu_{\eta(x)} = -\psi(x),$$

i.e.

$$\frac{1}{r_i} p \cdot \eta(x) + \frac{1}{r_i} x \cdot \eta(x) + \frac{1}{r_i} v(x)\nu_\zeta \cdot \eta(x) - r_i = \psi(x),$$

where we have used $\nu_{\eta(x)} = -\eta(x)/r_i$. From $\eta(O) = \zeta$ and $v(O) = 0$ we obtain (5.4).

Now, we give a bound on the derivatives of v at O in terms of the difference $r_e - r_i$. We notice that (5.2) implies

$$v(x) = (A - I)x \cdot \nu_\zeta + u(Ax)\nu_p \cdot \nu_\zeta,$$

and since $|\nabla u(O)| = 0$, we obtain

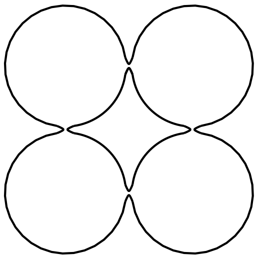
$$|\partial_{x_k} v(O)| \leq |A - I|, \qquad k = 1, \ldots, n.$$

From (5.3) and Lemma 5.1 we deduce that

$$|\partial_{x_k} v(O)| \leq 2\sqrt{\frac{r_e - r_i}{\rho}}, \qquad k = 1, \ldots, n,$$

and from (1.3) and (5.4) we find (1.4). □

Remark 5.2. As emphasized in the Introduction, if we assume that ρ is not bounded from below, it is possible to construct a family of closed surfaces embedded in $\mathbb{R}^3$, not diffeomorphic to a sphere, with $\operatorname{osc}(H)$ arbitrarily small and such that (1.3) fails. For instance one can consider the following example, suggested by A. Ros, obtained by gluing pieces of suitable small perturbations of unduloids.



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