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The vanishing conjecture for maps of Tor and derived splinters

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Abstract.
We say an excellent local domain $(S, \mathfrak{n})$ *satisfies the vanishing conditions for maps of Tor* if for every $A \to R \to S$ with A regular and $A \to R$ a module-finite torsion-free extension, and every A-module M, the map $\operatorname{Tor}_i^A(M, R) \to \operatorname{Tor}_i^A(M, S)$ vanishes for every $i \geq 1$. Hochster–Huneke's conjecture (theorem in equal characteristic) states that regular rings satisfy such vanishing conditions [HH95]. The main theorem of this paper shows that, in equal characteristic, rings that satisfy the vanishing conditions for maps of Tor are exactly *derived splinters* in the sense of Bhatt [Bha12]. In particular, rational singularities in characteristic 0 satisfy the vanishing conditions. This greatly generalizes Hochster–Huneke's result [HH95] and Boutot's theorem [Bou87]. Moreover, our result leads to a new (and surprising) characterization of rational singularities in terms of splittings in module-finite extensions.

Keywords. The vanishing conjecture for maps of Tor, derived splinters, rational singularities

1. Introduction

Hochster and Huneke proved the following extremely strong vanishing result in equal characteristic:

Theorem 1.1 (cf. [HH95, Theorem 4.1]). *Let A be an equal characteristic regular domain, let R be a module-finite and torsion-free extension of A, and let $R \to S$ be any homomorphism from R to a regular ring S. Then for every A-module M and every $i \geq 1$, the map $\operatorname{Tor}_i^A(M, R) \to \operatorname{Tor}_i^A(M, S)$ vanishes.*

They also conjectured that Theorem 1.1 holds in mixed characteristic. This is one of the well-known homological conjectures: *the vanishing conjecture for maps of Tor*. The importance of Theorem 1.1, as well as the corresponding conjecture in mixed characteristic, lies in the fact that, in any characteristic, it implies both the direct summand conjecture and the conjecture that direct summands of regular rings are Cohen–Macaulay [HH95]. Indeed, it was shown in [Ran00] that the vanishing conjecture for maps of Tor is equivalent to a strong form of the direct summand conjecture (we refer to [Ran00] for details).

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In fact, results very similar to Theorem 1.1 were first proved in [HH93], in characteristic $p > 0$ only, using tight closure and phantom homology theory.[1] The proof given in [HH95] makes crucial use of the existence of weakly functorial balanced big Cohen–Macaulay algebras in equal characteristic. In characteristic $p > 0$, the existence of such algebras follows directly from [HH92], where it was shown that the absolute integral closure R^+ is such an algebra. In characteristic 0, the construction of weakly functorial balanced big Cohen–Macaulay algebras depends on a very delicate and difficult reduction to a characteristic $p > 0$ argument (we refer to [HH95, Section 3] for details). In mixed characteristic, the analogy of Theorem 1.1 is known when A, R, S all have dimension less than or equal to three [Hoc02], based on Heitmann's results [Hei02]. However, in general the vanishing conjecture for maps of Tor is wide open in mixed characteristic.

In this paper, we investigate Theorem 1.1 in some new and different ways. We study the "converse" of Theorem 1.1 in the following sense: in a given characteristic, for which local domain S, does the map $\mathrm{Tor}_i^A(M,R) \to \mathrm{Tor}_i^A(M,S)$ vanish for every $A \to R \to S$ and every A-module M (where A is regular and $A \to R$ is a module-finite torsion-free extension)? We will say such an S *satisfies the vanishing conditions for maps of Tor* (see Section 2 for precise definitions). We will show that, in all characteristics, such vanishing conditions imply S has only pseudo-rational singularities, which is a characteristic-free analogue of rational singularities. Our main result in equal characteristic is the following:

Theorem 1.2 (= Theorem 5.5). *Let S be a local domain that is essentially of finite type over a field. The following are equivalent:*

(1) *S satisfies the vanishing conditions for maps of Tor.*
(2) *S is a derived splinter.*
(3) *For every regular local ring A with $S = A/P$ and every module-finite torsion-free extension $A \to B$ with $Q \in \operatorname{Spec} B$ lying over P, the map $P \to Q$ splits as a map of A-modules.*

We note that derived splinters are formally introduced by Bhatt [Bha12], and are well understood in equal characteristic: they are equivalent to rational singularities in characteristic 0 [Kov00], [Bha12], and in characteristic $p > 0$, they turn out to be the same as splinters [Bha12] (see Section 2 for precise definitions of splinters and derived splinters). In fact, at least in characteristic 0, the idea of derived splinters plays a crucial role in our proofs.

As regular local rings in equal characteristic are derived splinters, Theorem 1.2[(1)⇔(2)] greatly extends Theorem 1.1. We will see in Remark 5.7 that Theorem 1.2 also generalizes Boutot's theorem that direct summands of rational singularities are rational singularities [Bou87] (Boutot's theorem follows from the vanishing of Tor applied

[1] It is pointed out in the introduction of [HH93] that by reduction to characteristic $p > 0$, one can develop the corresponding theory in characteristic 0. The full results in [HH93] are, in some sense, even stronger than Theorem 1.1, but are slightly technical to state here. However, we point out that all these (stronger) results can be established by the argument used in [HH95]. Our method can also provide generalizations of these results, both in characteristic $p > 0$ and characteristic 0 (see Remark 5.6).

to $M = E_A$, the injective hull of A). Moreover, as an immediate consequence of Theorem 1.2[(2)⇔(3)], we obtain the following new characterization of rational singularities. We find this characterization surprising as it only addresses splittings in module-finite extensions.

Corollary 1.3 (= Corollary 5.8). *Let $(S, \mathfrak{n})$ be a local domain essentially of finite type over a field of characteristic 0. Then S has rational singularities if and only if for every regular local ring A with $S = A/P$, every module-finite torsion-free extension $A \to T$, and every $Q \in \operatorname{Spec} T$ lying over P, the map $P \to Q$ splits as a map of A-modules.*

This paper is organized as follows. In Section 2, we recall and review the basic theories, and we introduce two important concepts: the vanishing conditions for maps of Tor and the vanishing conditions for maps of local cohomology. The rest of the paper is devoted to the proof of Theorem 1.2. In Section 3 we show that the vanishing conditions for maps of Tor implies pseudo-rationality. In Section 4 we prove (1)⇔(3) of Theorem 1.2. Finally, in Section 5 we prove (1)⇔(2) of Theorem 1.2 and we also prove some partial results in mixed characteristic: for example, we show that the vanishing conjecture for maps of Tor implies the derived direct summand conjecture. Throughout this paper, unless otherwise stated, we will make the following assumptions on commutative rings and schemes (we will sometimes repeat and emphasize these conditions):

(1) All rings are Noetherian, excellent and are homomorphic images of regular rings.
(2) All schemes are Noetherian, separated, excellent and admit dualizing complexes.
(3) In characteristic 0, all rings and schemes are essentially of finite type over a field.

We point out that (1) and (2) are very mild conditions (e.g., all rings essentially of finite type over a complete local ring satisfy (1)). We make the assumption (3) mainly because we need to apply the Grauert–Riemenschneider type vanishing theorems [GR70], [Kol86] in characteristic 0.

2. Definitions and preliminaries

We begin with some basic definitions of plus closure. Let S be an integral domain and $I \subseteq S$ be an ideal. The *plus closure* of I, I^+, is the set of elements $x \in S$ such that $x \in IT$ for some module-finite extension T of S; and I is called *plus closed* if $I^+ = I$. The *absolute integral closure* of S, denoted by S^+, is the integral closure of S in the algebraic closure of the fraction field of S, which is also the direct limit of all the module-finite domain extensions of S [HH92]. It follows that $I^+ = IS^+ \cap S$. The plus closure of 0 in $H^d_{\mathfrak{n}}(S)$, the top local cohomology module, is defined as $0^+_{H^d_{\mathfrak{n}}(S)} = \ker(H^d_{\mathfrak{n}}(S) \to H^d_{\mathfrak{n}}(S^+))$.

A domain S (resp., an integral scheme X) is called a *splinter* if for every module-finite extension T of S (resp., every finite surjective map $Y \to X$), the natural map $S \to T$ (resp., $\mathcal{O}_X \to \mathcal{O}_Y$) is split in the category of S-modules (resp., $\mathcal{O}_X$-modules). It is easy to see that S is a splinter if and only if every ideal in S is plus closed.

Let $(S, \mathfrak{n})$ be an excellent local domain of characteristic $p > 0$. The top local cohomology module $H^d_{\mathfrak{n}}(S)$ has a natural Frobenius action. In this situation, there is a unique largest proper submodule of $H^d_{\mathfrak{n}}(S)$ that is stable under the Frobenius action, namely $0^*_{H^d_{\mathfrak{n}}(S)}$, the tight closure of 0 in $H^d_{\mathfrak{n}}(S)$ [Smi97]. $(S,\mathfrak{n})$ is called *F-rational* if it is Cohen–Macaulay and $0^*_{H^d_{\mathfrak{n}}(S)}=0$ [HH94], [Smi97]. This is not the original definition of F-rationality, but it turns out to be extremely useful in many applications. It is worth mentioning that a deep result of Smith [Smi94] shows that $0^*_{H^d_{\mathfrak{n}}(S)}=0^+_{H^d_{\mathfrak{n}}(S)}$, which we will need in Section 5.

We make some more comments on splinters. In equal characteristic 0, using the trace map, it is straightforward to check that splinters are exactly normal schemes. However, even in equal characteristic $p>0$ in the affine case, splinters are quite mysterious. It is known that affine splinters in characteristic $p>0$ are always F-rational [Smi94], [Bha12], and it is conjectured that they are *F-regular*, which is a natural strengthening of F-rationality and an important concept in tight closure theory.[2] We refer to [Sin99] and [CEMS18] for the best partial results on this conjecture. In mixed characteristic, our knowledge about splinters is minimal: Hochster's famous *direct summand conjecture* asserts that regular local rings are splinters. This conjecture is known to be true in dimension ≤ 3 [Hei02], and is open (in mixed characteristic) in dimension ≥ 4.

Following [Bha12], we say an integral scheme X is a *derived splinter* if for any proper surjective map $f\colon Y\to X$, the pull-back map $\mathcal{O}_X\to \mathbf{R}f_*\mathcal{O}_Y$ is split in the derived category $D(\mathrm{Coh}(X))$ of coherent sheaves on X. This is the same as requiring $\mathcal{O}_X\to \mathbf{R}f_*\mathcal{O}_Y$ to split in $D(\mathrm{QCoh}(X))$, the derived category of quasi-coherent sheaves on X. It is easy to see that derived splinters are splinters. It was first observed in [Kov00] that derived splinters in characteristic 0 coincide with rational singularities,[3] while it was shown in [Bha12] that, quite surprisingly, derived splinters are equivalent to splinters in characteristic $p>0$.

Next we recall pseudo-rational singularities [LT81]: A d-dimensional local ring $(R,\mathfrak{m})$ is called *pseudo-rational* if it is normal, Cohen–Macaulay, analytically unramified (i.e., the completion $\widehat{R}$ is reduced), and if for every proper, birational map $\pi: W\to \operatorname{Spec} R$ with W normal, the canonical map $H^d_{\mathfrak{m}}(R)\to H^d_E(W,\mathcal{O}_W)$ is injective where $E=\pi^{-1}(\mathfrak{m})$ denotes the closed fiber. Pseudo-rationality is a property of local rings which is an analog of rational singularities for more general schemes, e.g., rings which may not have a desingularization. When the ring is essentially of finite type over a field of characteristic 0, pseudo-rational singularities are the same as rational singularities. In characteristic p, pseudo-rationality is slightly weaker than F-rationality [Smi97], [Har98].

We summarize the relations between these concepts. In characteristic 0, we have

$$\text{derived splinter} = \text{rational singularities} = \text{pseudo-rational} \Rightarrow \text{splinter}.$$

In characteristic $p>0$, we have

$$\text{derived splinter} = \text{splinter} \Rightarrow F\text{-rational} \Rightarrow \text{pseudo-rational}.$$

[2] As we will not use deep results in tight closure theory, we omit the precise definition of F-regularity (and the original definition of F-rationality). We refer to [HH90] for details on tight closure theory.

[3] This was proved in [Kov00] when $Y\to X$ has connected fibers (which was sufficient for the applications in [Kov00]). A complete proof was given in [Bha12, Theorem 2.12].

Now we introduce the central concepts that we will study in this paper:

Definition 2.1. We say a local domain $(S, \mathfrak{n})$ satisfies *the vanishing conditions for maps of Tor* if for every $A \to R \to S$ such that A is a regular domain, $A \to R$ is a module-finite torsion-free extension, and A, R, S have the same characteristic,[4] the natural map $\mathrm{Tor}_i^A(M, R) \to \mathrm{Tor}_i^A(M, S)$ vanishes for every A-module M and every $i \geq 1$.

It is also quite natural to ask: if $(R, \mathfrak{m}) \twoheadrightarrow (S, \mathfrak{n})$ is a surjection of local domains, when does $H^j_{\mathfrak{m}}(R) \to H^j_{\mathfrak{n}}(S)$ vanish for every $j < \dim R$? (this is inspired by [HH93, Corollary 4.24], which is itself a consequence of Theorem 1.1). Hence similar to the vanishing conditions for maps of Tor, we want to introduce certain vanishing conditions for maps of local cohomology. Since there are several equivalent ways to define this, we summarize them into a proposition.

Proposition/Definition 2.2. *Let $(S, \mathfrak{n})$ be a local domain of dimension d. Then the following are equivalent (we always assume R and S have the same characteristic):*

(1) *For every surjection $(R, \mathfrak{m}) \twoheadrightarrow (S, \mathfrak{n})$ with $(R, \mathfrak{m})$ equidimensional, the induced map $H^j_{\mathfrak{m}}(R) \to H^j_{\mathfrak{n}}(S)$ vanishes for every $j < \dim R$.*
(2) *For every surjection $(R, \mathfrak{m}) \twoheadrightarrow (S, \mathfrak{n})$ with $(R, \mathfrak{m})$ a local domain, the induced map $H^j_{\mathfrak{m}}(R) \to H^j_{\mathfrak{n}}(S)$ vanishes for every $j < \dim R$.*
(3) *S is Cohen–Macaulay and for every surjection $(R, \mathfrak{m}) \twoheadrightarrow (S, \mathfrak{n})$ such that $(R, \mathfrak{m})$ is a local domain with $\dim R > d$, the induced map $H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{n}}(S)$ vanishes.*
(4) *S is Cohen–Macaulay and for every surjection $(R, \mathfrak{m}) \twoheadrightarrow (S, \mathfrak{n})$ such that $\dim R/P > d$ for every minimal prime of P of R, the induced map $H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{n}}(S)$ vanishes.*

We say $(S, \mathfrak{n})$ satisfies the vanishing conditions for maps of local cohomology *if it satisfies the above equivalent conditions.*

Proof. (1)⇒(2): This is obvious.

(2)⇒(3): Applying (2) to $R = S$, we see that the identity map $H^j_{\mathfrak{n}}(S) \to H^j_{\mathfrak{n}}(S)$ vanishes for every $j < \dim S = d$. Thus S is Cohen–Macaulay. The remaining part is obvious (note that one cannot apply (3) to $R = S$, because the hypothesis on R in (3) forces $\dim R > d$).

(3)⇒(4): Since S is a domain, every surjection $R \twoheadrightarrow S$ factors as $R \twoheadrightarrow R' \twoheadrightarrow S$, where $R' = R/P$ for some minimal prime P of R. Now (3) implies $H^d_{\mathfrak{m}}(R') \to H^d_{\mathfrak{n}}(S)$ vanishes because $\dim R' = \dim R/P > d$. Thus $H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{n}}(S)$ also vanishes.

(4)⇒(1): If $\dim R = \dim S = d$ (i.e., $R = S$) in (1), then $H^j_{\mathfrak{m}}(R) \to H^j_{\mathfrak{n}}(S)$ vanishes for every $j < \dim R = d$ because $H^j_{\mathfrak{n}}(S) = 0$ (S is Cohen–Macaulay). Otherwise we have $\dim R > d$. Since R is equidimensional, $\dim R/P > d$ for every minimal prime P of R. Thus applying (4), we know that $H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{n}}(S)$ vanishes. □

Remark 2.3. One cannot expect that $H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{n}}(S)$ vanishes for all $R \twoheadrightarrow S$ with $\dim R > d$, even when S is regular. For example, let $R = \frac{k[[x,y,z]]}{(x,y)\cap(z)}$ and $S = k[[z]]$. We

[4] This means A, R, S all have *equal characteristic*, i.e., they all contain a field, or they all have *mixed characteristic* (i.e., the characteristic of the ring is different from that of its residue field).

know that $\dim R = 2$ and $\dim S = 1$. But it is easy to check that $H^1_{\mathfrak{m}}(R) \to H^1_{\mathfrak{n}}(S)$ is surjective and hence does not vanish. The trouble here is that there is a component of R that has the same dimension as S. Thus the hypotheses in Definition 2.2(1)–(4) are necessary.

We will see in later sections that the vanishing conditions for Tor and for local cohomology are deeply related (Proposition 3.4, Theorem 5.10).

3. Vanishing of Tor, vanishing of local cohomology and pseudo-rationality

In this section we will show that the vanishing conditions for maps of Tor implies pseudo-rationality, which will be a crucial ingredient in proving (1)⇒(2) in Theorem 1.2. We also obtain many characteristic-free results of independent interest.

Lemma 3.1. *Let $(S, \mathfrak{n})$ be a local domain that is a homomorphic image of a regular ring. Then*

$$\sum_R \operatorname{im}(H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{n}}(S)) \supseteq 0^+_{H^d_{\mathfrak{n}}(S)} \tag{3.1.1}$$

where the sum is taken over all $R \twoheadrightarrow S$ such that $\dim R/P > \dim S = d$ for every minimal prime P of R. In particular, if $(S, \mathfrak{n})$ satisfies the vanishing conditions for maps of local cohomology, then $0^+_{H^d_{\mathfrak{n}}(S)} = 0$.

Proof. Since $0^+_{H^d_{\mathfrak{n}}(S)} = \ker(H^d_{\mathfrak{n}}(S) \to H^d_{\mathfrak{n}}(S^+)) = \bigcup_T \ker(H^d_{\mathfrak{n}}(S) \to H^d_{\mathfrak{n}}(T))$ where T runs over all module-finite domain extensions of S, it suffices to show that the inclusion $\sum_R \operatorname{im}(H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{n}}(S)) \supseteq \ker(H^d_{\mathfrak{n}}(S) \to H^d_{\mathfrak{n}}(T))$ holds for every such T.

We write $S = A/P$ for some regular local ring A such that $\dim A \geq d + 1$. Let $t_1, \dots, t_n$ be a set of generators of T over S. Since T is integral over S, each t_i satisfies a monic polynomial f_i over S. We can lift each f_i to A and form the ring $B = A[x_1, \dots, x_n]/(f_1, \dots, f_n)$. We have a natural surjective map $B \twoheadrightarrow T$ with kernel $Q \in \operatorname{Spec} B$. It is clear that Q lies over P in A. Let $R = A + Q \subseteq B$. We know that $R/Q = A/P = S$. In sum, we have

$$\begin{tikzcd}
0 \arrow[r] & Q \arrow[r] & B \arrow[r] & T \arrow[r] & 0 \\
0 \arrow[r] & Q \arrow[r] \arrow[u, "\cong"] & R \arrow[r] \arrow[u] & S \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

Let $\mathfrak{m}$ be the pre-image of $\mathfrak{n}$ in R. Because B is free over A and R is a subring of B, R is torsion-free over A. Now localizing at $\mathfrak{m}$ if necessary, we know that $(R, \mathfrak{m})$ is equidimensional and $\dim R = \dim B = \dim A \geq d + 1$. This guarantees that $\dim R/P > d$

for every minimal prime P of R. The induced long exact sequences on local cohomology gives

$$\begin{tikzcd} H^d_{\mathfrak{m}}(B) \arrow[r] & H^d_{\mathfrak{m}}(T) = H^d_{\mathfrak{n}}(T) \arrow[r] & H^{d+1}_{\mathfrak{m}}(Q) \arrow[r] & H^{d+1}_{\mathfrak{m}}(B) \\ H^d_{\mathfrak{m}}(R) \arrow[r] & H^d_{\mathfrak{m}}(S) = H^d_{\mathfrak{n}}(S) \arrow[r] \arrow[u] & H^{d+1}_{\mathfrak{m}}(Q) \arrow[u, "\cong"'] & \end{tikzcd}$$

By chasing this diagram, it is easy to see that

$$\ker(H^d_{\mathfrak{n}}(S) \to H^d_{\mathfrak{n}}(T)) \subseteq \operatorname{im}(H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{n}}(S)).$$

This proves (3.1.1). Finally, if S satisfies the vanishing conditions for maps of local cohomology, then the left hand side of (3.1.1) is 0 by Definition 2.2(4), thus $0^+_{H^d_{\mathfrak{n}}(S)} = 0$. □

Corollary 3.2. *If* $(S, \mathfrak{n})$ *satisfies the vanishing conditions for maps of local cohomology, then every ideal generated by a full system of parameters in S is plus closed. In particular this implies S is normal.*

Proof. Let $I = (x_1, \ldots, x_d)$ be any ideal generated by a full system of parameters of S. Consider the commutative diagram

$$\begin{tikzcd} H^d_{\mathfrak{n}}(S) \arrow[r, hook] & H^d_{\mathfrak{n}}(S^+) \\ S/I \arrow[r] \arrow[u, hook] & S^+/IS^+ \arrow[u] \end{tikzcd}$$

The left vertical map is injective because S is Cohen–Macaulay by Definition 2.2, and the map in the top row is injective by Lemma 3.1. Chasing this diagram we find that $S/I \hookrightarrow S^+/IS^+$ is injective. This proves that I is plus closed.

Finally, if every ideal generated by a system of parameters is plus closed then every ideal generated by part of a system of parameters is plus closed: Suppose that $(x_1, \ldots, x_t)$ is part of a system of parameters, contained in $(x_1, \ldots, x_t, x_{t+1}, \ldots, x_d)$. If $y \in (x_1, \ldots, x_t)^+$, then $y \in (x_1, \ldots, x_t, x^s_{t+1}, \ldots, x^s_d)^+ = (x_1, \ldots, x_t, x^s_{t+1}, \ldots, x^s_d)$ for every $s > 0$. So

$$y \in \bigcap_s (x_1, \ldots, x_t, x^s_{t+1}, \ldots, x^s_d) = (x_1, \ldots, x_t).$$

In particular, every principal ideal is plus closed. Let $y \in \overline{(x)}$, the integral closure of the ideal generated by x. Then $y \in \overline{(x)R^+} = (x)R^+$ because R^+ is integrally closed. So $y \in (x)^+ = (x)$. This proves every principal ideal is integrally closed, and hence S is normal. □

Lemma 3.3. *If S satisfies the vanishing conditions for maps of local cohomology, then S is pseudo-rational.*

Proof. By our general assumption on commutative rings, S is an excellent local domain, hence is analytically unramified. By Definition 2.2 and Corollary 3.2, S is Cohen–Macaulay and normal. To check the last condition of pseudo-rationality, we let $W \to \operatorname{Spec} S$ be a proper birational map with W normal, and we can assume this map is projective and birational by Chow's Lemma. Therefore $W \to \operatorname{Spec} S$ is just the blow-up of some ideal J in S, i.e.,

$$W = \operatorname{Proj} S \oplus Jt \oplus J^2t^2 \oplus \cdots =: \operatorname{Proj} R.$$

Now we apply the Sancho de Salas exact sequence (see [SdS87, p. 202], or take cohomology of (5.11.1) below) to $W = \operatorname{Proj} R \to \operatorname{Spec} S$ to get ($d = \dim S$)

$$\begin{tikzcd} H_E^{d-1}(W, \mathcal{O}_W) \arrow[r] & {[H^d_{\mathfrak{n}+R_{>0}}(R)]_0} \arrow[r] \arrow[d, hook] & H^d_{\mathfrak{n}}(S) \arrow[r] \arrow[d, "\cong"] & H^d_E(W, \mathcal{O}_W) \arrow[r] & {[H^{d+1}_{\mathfrak{n}+R_{>0}}(R)]_0} \\ & H^d_{\mathfrak{n}+R_{>0}}(R) \arrow[r, "0"] & H^d_{\mathfrak{n}}(S) & & \end{tikzcd}$$

Since R has dimension $d+1$ and S satisfies the vanishing conditions for maps of local cohomology, the bottom map is the zero map. By the commutativity of the diagram, the map $[H^d_{\mathfrak{n}+R_{>0}}(R)]_0 \to H^d_{\mathfrak{n}}(S)$ vanishes. Therefore $H^d_{\mathfrak{n}}(S) \to H^d_E(W, \mathcal{O}_W)$ is injective. This finishes the proof. $\square$

Proposition 3.4. *If S is a complete local domain, then S satisfying the vanishing conditions for maps of Tor implies S satisfies the vanishing conditions for maps of local cohomology. Hence both imply S has only pseudo-rational singularities.*

Proof. Let $(R, \mathfrak{m}) \twoheadrightarrow (S, \mathfrak{n})$ be a surjection with R a domain. We may complete R to get $\widehat{R} \twoheadrightarrow \widehat{S} = S$. Since R is excellent by our general assumptions on commutative rings, $\widehat{R}$ is equidimensional. Since S is a domain, the map $\widehat{R} \twoheadrightarrow S$ factors as $\widehat{R} \twoheadrightarrow R' \twoheadrightarrow S$ where $R' = \widehat{R}/P$ for P a minimal prime of $\widehat{R}$. Thus in order to show $H^j_{\mathfrak{m}}(R) \to H^j_{\mathfrak{n}}(S)$ vanishes for $j < \dim R$, it suffices to show $H^j_{\mathfrak{m}}(R') \to H^j_{\mathfrak{n}}(S)$ vanishes for $j < \dim R'$. Hence without loss of generality, we may replace R by R' and assume R is a complete local domain.

Now by Cohen's structure theorem, we have a module-finite extension $(A, \mathfrak{m}_0) \hookrightarrow (R, \mathfrak{m})$ with $(A, \mathfrak{m}_0)$ regular local. Let $E_A = E_A(A/\mathfrak{m}_0) \cong H^n_{\mathfrak{m}_0}(A)$ be the injective hull of the residue field of A. Since the Čech complex gives a flat resolution of E_A, we know that $\operatorname{Tor}_i^A(E_A, R) \cong H^{n-i}_{\mathfrak{m}}(R)$ and $\operatorname{Tor}_i^A(E_A, S) \cong H^{n-i}_{\mathfrak{m}}(S)$. Since S satisfies the vanishing conditions for maps of Tor, by considering the map $A \to R \to S$, we find that $\operatorname{Tor}_i^A(E_A, R) \to \operatorname{Tor}_i^A(E_A, S)$ vanishes for every $i \geq 1$. Hence $H^j_{\mathfrak{m}}(R) \to H^j_{\mathfrak{n}}(S)$ vanishes for $j < n = \dim R$. The last assertion then follows from Lemma 3.3. $\square$

Remark 3.5. We assume $(S, \mathfrak{n})$ is complete in the proof of Proposition 3.4 because we use Cohen's structure theorem to find $A \to R$ module-finite with A regular. Hence the conclusion of Proposition 3.4 still holds when we work with rings that are essentially of finite type over a field (we can use Noether normalization instead).

4. Vanishing conditions for maps of Tor and the splitting property

The goal of this section is to prove (1)⇔(3) in Theorem 1.2. As a corollary we will see that if S satisfies the vanishing conditions for maps of Tor then S is a splinter. We start with a lemma restating [HH95, (4.5)]. This was only stated in the complete case, but the same argument works for rings essentially of finite type over a field (one needs to replace in [HH95] Cohen's structure theorem by Noether normalization).

Lemma 4.1 (cf. [HH95, (4.5)]). *Let $(S, \mathfrak{n})$ be either complete or essentially of finite type over a field. To show $(S, \mathfrak{n})$ satisfies the vanishing conditions for maps of Tor in a given characteristic, we may assume A is local, R is a domain, $A \to S$ is surjective and M is finitely generated. Furthermore, it suffices to prove the vanishing of Tor for $i = 1$.*

Remark 4.2. Suppose $A \to R \to S$ has the property that $A \to R$ is module-finite and torsion-free, and the composite map $A \to S$ is *surjective*. In this situation, if we set $S = A/P = R/\widetilde{P}$, then modulo $\widetilde{P}$, elements of R come from elements of A. Thus in this case $R = A + \widetilde{P}$.

Theorem 4.3. *Let $(S, \mathfrak{n})$ be either complete or essentially of finite type over a field.*

(1) *S satisfies the vanishing conditions for maps of Tor.*
(2) *For every regular local ring A with $S = A/P$, and every module-finite torsion-free extension $A \to B$ with $Q \in \operatorname{Spec} B$ lying over P, the map $P \to Q$ splits as a map of A-modules.*
(3) *For every regular local ring A with $S = A/P$, every module-finite torsion-free extension $A \to B$ that splits as a map of A-modules, and every $Q \in \operatorname{Spec} B$ lying over P, the map $P \to Q$ splits as a map of A-modules.*

Then (2)⇒(1)⇒(3). *In particular,* (1)⇔(2)⇔(3) *in equal characteristic.*

Proof. Let $S = A/P$ with A regular. Let $A \to B$ be a module-finite torsion-free extension with $Q \in \operatorname{Spec} B$ lying over P. We form the ring $R_0 = A + Q \subseteq B$. Then R_0 is also a module-finite torsion-free extension of A and we have $R_0/Q = A/P = S$. Now we look at the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & Q \arrow[r] & R_0 \arrow[r] & S \arrow[r] & 0 \\ 0 \arrow[r] & P \arrow[r] \arrow[u, "\alpha"] & A \arrow[r] \arrow[u, "\beta"] & S \arrow[r] \arrow[u, "\cong"] & 0 \end{tikzcd}$$

Tensoring it with an arbitrary A-module M, we get

$$\begin{tikzcd} \operatorname{Tor}_1^A(M, R_0) \arrow[r, "\varphi_M"] & \operatorname{Tor}_1^A(M, S) \arrow[r] & Q \otimes_A M \arrow[r] & R_0 \otimes_A M \arrow[r] & S \otimes_A M \arrow[r] & 0 \\ 0 \arrow[r] & \operatorname{Tor}_1^A(M, S) \arrow[r] \arrow[u, "\cong"] & P \otimes_A M \arrow[r] \arrow[u, "\alpha\otimes \mathrm{id}_M"] & A \otimes_A M \arrow[r] \arrow[u, "\beta\otimes \mathrm{id}_M"] & S \otimes_A M \arrow[r] \arrow[u, "\cong"] & 0 \end{tikzcd}$$

By diagram chasing, one can see that

$$\alpha \otimes \mathrm{id}_M \text{ is injective } \Leftrightarrow \varphi_M = 0 \text{ and } \beta \otimes \mathrm{id}_M \text{ is injective.} \tag{4.3.1}$$

(1)⇒(3): Suppose $A \to B$ splits (i.e., we are in condition (3)); then $A \to R_0$ also splits, in particular $\beta \otimes \mathrm{id}_M$ is injective. Applying the vanishing conditions for maps of Tor to $A \to R_0 \to S$, we know $\varphi_M = 0$. Now (4.3.1) implies $P \otimes_A M \xrightarrow{\alpha\otimes \mathrm{id}_M} Q \otimes_A M$ is injective for every M. But this implies $P \to Q$ splits by [HR76, Corollary 5.2] since Q/P is a finitely generated A-module.

(2)⇒(1): By Lemma 4.1, we may assume $A \to R \to S$ has $A \twoheadrightarrow S$ surjective. Now we set $B = R$ and $Q = \ker(R \to S)$. By Remark 4.2, we have $Q = \widetilde{P}$ and hence $R = A + \widetilde{P} = A + Q = R_0$ in this situation. Now (2) implies $P \to Q$ splits, in particular $\alpha \otimes \mathrm{id}_M$ is injective. Thus (4.3.1) implies

$$\varphi_M \colon \operatorname{Tor}_1^A(M, R) = \operatorname{Tor}_1^A(M, R_0) \to \operatorname{Tor}_1^A(M, S)$$

vanishes for every M. This proves S satisfies the vanishing conditions for maps of Tor, since it is enough to check the vanishing of Tor for $i = 1$ by Lemma 4.1.

Finally, in equal characteristic, every module-finite extension $A \to B$ splits when A is regular. So (2)⇔(3) and hence (1)⇔(2)⇔(3). □

Lemma 4.4. *Let $A \to B$ be a module-finite extension. Suppose $Q \in \operatorname{Spec} B$ lies over $P \in \operatorname{Spec} A$. If $P \to Q$ splits as a map of A-modules and $\operatorname{depth}_P A \geq 2$, then $A \to B$ splits compatibly with $P \to Q$, i.e., there exists a splitting $\theta \colon B \to A$ such that $\theta(Q) = P$. In particular, $A/P \to B/Q$ splits as a map of A-modules.*

Proof. Let $\phi \colon Q \to P$ be a splitting. The exact sequences $0 \to Q \to B \to B/Q \to 0$ and $0 \to P \to A$ induce a commutative diagram

$$\begin{tikzcd} \operatorname{Hom}_A(B, A) \arrow[r] & \operatorname{Hom}_A(Q, A) \arrow[r] & \operatorname{Ext}^1_A(B/Q, A) \\ & \operatorname{Hom}_A(Q, P) \arrow[u, hook] & \end{tikzcd}$$

Since B/Q is a finitely generated A-module annihilated by P, and $\operatorname{depth}_P A \geq 2$, we know that $\operatorname{Ext}^1_A(B/Q, A) = 0$ by [Eis95, Proposition 18.4]. Hence $\operatorname{Hom}_A(B, A)$ maps onto $\operatorname{Hom}_A(Q, A)$, in particular it maps onto the image of $\operatorname{Hom}_A(Q, P)$. Thus there is a map $\theta \colon B \to A$ such that $\theta|_Q = \phi$. We show that θ has to be a splitting from B to A. Suppose $\theta(1) = a \in A$. Then for every nonzero element $r \in P$ we have

$$ra = r\theta(1) = \theta(r) = \phi(r) = r.$$

So $a = 1$ and hence θ is a splitting from B to A such that $\theta(Q) = P$. Finally, $\overline{\theta}$ gives a splitting $B/Q \to A/P$. □

Corollary 4.5. *If S satisfies the vanishing conditions for maps of Tor, then S is a splinter.*

Proof. We use a construction similar to the one used in the proof of Lemma 3.1. We write $S = A/P$ with A a regular local ring with $\operatorname{depth}_P A \geq 2$ (this can be achieved, for example, by adding indeterminates). Let $S \to T$ be a module-finite domain extension. Let $t_1, \ldots, t_n$ be a set of generators of T over $S = A/P$. Each t_i is a zero of a monic polynomial f_i over S. We lift each f_i to A and form the ring $B = A[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$. We have a natural surjection $B \twoheadrightarrow T$ with kernel $Q \in \operatorname{Spec} B$. It is straightforward to check that Q lies over P.

Since B is finite free over A, we know that $A \to B$ splits as a map of A-modules. Now by (1)⇒(3) of Theorem 4.3, $P \to Q$ is split. Since $\operatorname{depth}_P A \geq 2$, by Lemma 4.4, $S = A/P \to B/Q = T$ splits as a map of A-modules (hence also as a map of S-modules). As this is true for any module-finite domain extension T of S, S is a splinter. $\square$

Remark 4.6. Applying Corollary 4.5 with S a regular local ring, we see that the vanishing conjecture for maps of Tor implies the direct summand conjecture in all characteristics. Although this is well known, we want to point out that the original proof given in [HH95] and [Ran00] depends on applying the vanishing conjecture to the map $A \to R \to S = R/\mathfrak{m}$, i.e., studying the map from a mixed characteristic ring $(R, \mathfrak{m})$ to its residue field $S = R/\mathfrak{m}$. Such a map, though being very natural, does *not* preserve the characteristic of the rings! Our theorem above gives a totally different proof, and it shows that the vanishing conjecture for maps of Tor, even if we restrict to $A \to R \to S$ all of the same characteristic, still implies the direct summand conjecture.

5. Main results

In this section we prove our main theorem in equal characteristic. We begin by recalling some facts about dualizing complexes. For an integral scheme X, a *normalized dualizing complex* $\omega_X^\bullet$ is an object in $D^b_{\mathrm{Coh}}(X)$ which has finite injective dimension, the canonical map $O_X \to \mathbf{R}\mathscr{H}om_X(\omega_X^\bullet, \omega_X^\bullet)$ is an isomorphism in $D^b_{\mathrm{Coh}}(X)$, and the first nonzero cohomology of $\omega_X^\bullet$ lies in degree $-\dim X$. Note that under this definition, if $\omega_X^\bullet$ is a normalized dualizing complex, then so is $\omega_X^\bullet \otimes \mathscr{L}$ for any line bundle $\mathscr{L}$ (in fact, this is all the ambiguity for a connected scheme [Har66]).

To clear this ambiguity, notice that all our rings and schemes in this section are essentially of finite type over a field k (or over a fixed scheme $\operatorname{Spec} S$ as in Theorem 5.11 and Remark 5.13). Therefore we simply *define* $\omega_X^\bullet = \pi^! k$ (resp., $\pi^! \omega_S^\bullet$ for some chosen $\omega_S^\bullet$), where $\pi\colon X \to \operatorname{Spec} k$ (resp., $X \to \operatorname{Spec} S$) is the structural map and $\pi^!$ is the functor from Grothendieck duality theory [Har66]. By standard duality theory, $\omega_X^\bullet$ is a normalized dualizing complex discussed in the previous paragraph. Moreover, after this choice, we have

$$\mathbf{R}\mathscr{H}om_X(\mathbf{R} f_* O_Y, \omega_X^\bullet) \cong \mathbf{R} f_* \omega_Y^\bullet$$

for any proper and dominant morphism $f\colon Y \to X$ of integral schemes, where X, Y are both essentially of finite type over a field k or over a fixed scheme $\operatorname{Spec} S$. We refer to [Har66] for details on Grothendieck duality theory and to [BST15, Section 2.3] for a very nice summary.

Next we recall that for an excellent local domain A, a complex

$$F_\bullet : 0 \to A^{b_n} \xrightarrow{\alpha_n} A^{b_{n-1}} \xrightarrow{\alpha_{n-1}} \cdots \xrightarrow{\alpha_2} A^{b_1} \xrightarrow{\alpha_1} A^{b_0} \to 0$$

of finitely generated free A-modules is said to *satisfy the standard conditions on rank and height* (resp. *rank and depth*) if, for every $1 \le i \le n$, $\operatorname{rank}\alpha_i + \operatorname{rank}\alpha_{i+1} = b_i$ and the height (resp. the depth) of the ideal $I_{\operatorname{rank}\alpha_i}(\alpha_i)$ is at least i where $I_t(\alpha_i)$ denotes the ideal generated by the size t minors of a matrix for α_i; it is independent of the choice of bases for F_i and F_{i-1} ($\operatorname{rank}\alpha_i$ is the largest integer r such that $I_r(\alpha_i) \neq 0$). We use the convention that $I_0(\alpha_i) = R$ and the unit ideal has height infinity.

Remark 5.1. Let $F_\bullet$ be a complex of finite free A-modules. It is well known that $F_\bullet$ is acyclic (which means $F_\bullet$ is exact except possibly in degree 0) if and only if $F_\bullet$ satisfies the standard conditions on rank and depth. Moreover, in characteristic $p > 0$, $F_\bullet$ satisfies the standard conditions on rank and height if and only if $F_\bullet$ is *phantom acyclic*. We refer to [HH93] and [Abe94] for details on phantom homology.

Now we are ready to state and prove our key theorem, which implies (and is in fact much stronger than) (2)⇒(1) of Theorem 1.2.

Theorem 5.2 (Key Theorem). *Let $(A, \mathfrak{m})$ be a local domain that is essentially of finite type over a field and $F_\bullet : 0 \to F_n \to \cdots \to F_1 \to F_0 \to 0$ be a complex of finitely generated free A-modules that satisfies the standard conditions on rank and height. Let $X \xrightarrow{f_0} Y \xrightarrow{f} \operatorname{Spec} A$ be maps of integral schemes such that $Y \to \operatorname{Spec} A$ is proper surjective and X is a derived splinter. Then the natural map*

$$h^{-i}(F_\bullet \otimes \mathbf{R}f_*O_Y) \to h^{-i}(F_\bullet \otimes \mathbf{R}f_*\mathbf{R}f_{0*}O_X)$$

induced by the pull-back f_0^* is the zero map for every $i > 0$ (note that F_i has cohomological degree $-i$).

Proof. As the methods we use in characteristic 0 and $p > 0$ are very different, we separate the proof in two parts.

Proof in characteristic* 0*: We assume $(A, \mathfrak{m})$ is of equal characteristic 0. Let $p\colon Z \to Y$ be a resolution of singularities and let $W = X \times_Y Z$. We have the following diagram:

$$\begin{tikzcd}
 & Z \arrow[dl, "g"'] \arrow[d, "p"] & W \arrow[l, "f_1"'] \arrow[d, "q"] \\
\operatorname{Spec} A & Y \arrow[l, "f"'] & X \arrow[l, "f_0"']
\end{tikzcd}$$

Since p is a resolution of singularities, the map q obtained by base change is proper and surjective. Because X is a derived splinter, the natural map $q^*\colon O_X \to \mathbf{R}q_*O_W$ has a splitting η in the derived category $D(\mathrm{Coh}(X))$, i.e., $\eta \circ q^* = \mathrm{id}$. Therefore we have the following commutative diagram in $D(\mathrm{QCoh}(\mathrm{Spec}\, A))$:

$$\begin{tikzcd}
\mathbf{R}g_* O_Z \arrow[r, "f_1^*"] & \mathbf{R}g_*\mathbf{R}f_{1*}O_W = \mathbf{R}f_*\mathbf{R}f_{0*}\mathbf{R}q_*O_W \arrow[d, dotted, bend left, "\eta"] \\
\mathbf{R}f_*O_Y \arrow[u, "p^*"] \arrow[r, "f_0^*"] & \mathbf{R}f_*\mathbf{R}f_{0*}O_X \arrow[u, "q^*"]
\end{tikzcd}$$

Now we tensor the above diagram with $F_\bullet$ in $D(\mathrm{QCoh}(\mathrm{Spec}\, A))$ and take cohomology in negative degree; we get a commutative diagram (since $F_\bullet$ is a complex of free A-modules, $\otimes^{\mathbf{L}}$ is the same as $\otimes$)

$$\begin{tikzcd}
h^{-i}(F_\bullet \otimes \mathbf{R}g_*O_Z) \arrow[r, "f_1^*"] & h^{-i}(F_\bullet \otimes \mathbf{R}f_*\mathbf{R}f_{0*}\mathbf{R}q_*O_W) \arrow[d, dotted, bend left, "\eta"] \\
h^{-i}(F_\bullet \otimes \mathbf{R}f_*O_Y) \arrow[u, "p^*"] \arrow[r, "f_0^*"] & h^{-i}(F_\bullet \otimes \mathbf{R}f_*\mathbf{R}f_{0*}O_X) \arrow[u, "q^*"]
\end{tikzcd}$$

Since $\eta \circ q^* = \mathrm{id}$, in order to show f_0^* induces the zero map, it is enough to show $\eta \circ f_1^* \circ p^*$ induces the zero map by the above commutative diagram. We will show this by proving that $h^{-i}(F_\bullet \otimes \mathbf{R}g_*O_Z) = 0$ for every $i > 0$, when $F_\bullet$ satisfies the standard conditions on rank and height and $Z \to \mathrm{Spec}\, A$ is proper surjective with Z smooth.

We use induction on the dimension of A. When $\dim A = 0$, $(A, \mathfrak{m})$ is Artinian and it is easy to see that every complex $F_\bullet$ that satisfies the standard conditions on rank and height is split exact except at the zeroth spot. Hence the complex $F_\bullet \otimes \mathbf{R}g_*O_Z \cong A^n \otimes \mathbf{R}g_*O_Z = \bigoplus^n \mathbf{R}g_*O_Z$ has no negative degree part, so $h^{-i}(F_\bullet \otimes \mathbf{R}g_*O_Z) = 0$ for every $i > 0$.

Now let $\dim A = d$. We set $G^\bullet = F_\bullet \otimes \mathbf{R}g_*O_Z$. Let $\underline{x} = x_1, \ldots, x_d$ denote a full system of parameters of A and let $C^\bullet(\underline{x}, A)$ be the Čech complex associated to $\underline{x}$. We also let

$$\widetilde{G}^\bullet = G^\bullet \otimes^{\mathbf{L}} C^\bullet(\underline{x}, A) = G^\bullet \otimes C^\bullet(\underline{x}, A).$$

We compute $h^{-i}(\widetilde{G}^\bullet)$ using spectral sequences of the double complex

$$\begin{tikzcd}
 & & & \\
\cdots \arrow[r] & G^{p+1} \otimes C^q(\underline{x}, A) \arrow[r] \arrow[u] & G^{p+1} \otimes C^{q+1}(\underline{x}, A) \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & G^{p} \otimes C^q(\underline{x}, A) \arrow[r] \arrow[u] & G^{p} \otimes C^{q+1}(\underline{x}, A) \arrow[r] \arrow[u] & \cdots \\
 & \arrow[u] & \arrow[u] &
\end{tikzcd}$$

Each $C^q(\underline{x}, A)$ is a direct sum of localizations of A, in particular it is flat over A. So when we take the cohomology of the columns, we get

$$E_1^{pq} = h^p(G^\bullet) \otimes C^q(\underline{x}, A).$$

Note that when $q = 0$, this is just $h^p(G^\bullet)$, while when $q > 0$, this is a direct sum of proper localizations of $h^p(G^\bullet)$. However, when $p < 0$, $h^p(G^\bullet) = h^p(F_\bullet \otimes \mathbf{R}g_* O_Z)$ is supported only at the maximal ideal $\mathfrak{m}$ by the induction hypothesis. Because if it is supported at another prime ideal, say P, then $h^p((F_\bullet)_P \otimes \mathbf{R}g_* O_{Z \times_{\mathrm{Spec}\, R} \mathrm{Spec}\, R_P}) \neq 0$. But $(F_\bullet)_P$ satisfies the standard conditions on rank and height as a free complex over R_P (the ranks of all the F_i are preserved, and the height of an ideal does not decrease when we localize), and $Z \times_{\mathrm{Spec}\, R} \mathrm{Spec}\, R_P$ is smooth; we thus get a contradiction since $\dim R_P < d$. Hence we know that $E_1^{pq} = 0$ when $p < 0$ and $q > 0$. In sum, the E_1-stage of the spectral sequence looks like

$$\begin{array}{c}
\cdots \\
E_1^{00} = h^0(G^\bullet) \longrightarrow E_1^{01} \longrightarrow E_1^{02} \longrightarrow \cdots \longrightarrow E_1^{0d} \\
E_1^{-1,0} = h^{-1}(G^\bullet) \longrightarrow E_1^{-1,1} = 0 \longrightarrow E_1^{-1,2} = 0 \longrightarrow \cdots \longrightarrow E_1^{-1,d} = 0 \\
E_1^{-2,0} = h^{-2}(G^\bullet) \longrightarrow E_1^{-2,1} = 0 \longrightarrow E_2^{-2,2} = 0 \longrightarrow \cdots \longrightarrow E_1^{-2,d} = 0 \\
\cdots
\end{array}$$

From this we know that

$$h^{-i}(\widetilde{G}^\bullet) = E_1^{-i,0} = h^{-i}(G^\bullet) \tag{5.2.1}$$

for every $i > 0$. This is because $E_1^{-i,0} = h^{-i}(G^\bullet)$ is the only nontrivial term that contributes to $h^{-i}(\widetilde{G}^\bullet)$ when $i > 0$, and all the further differentials at this spot,

$$E_r^{-i+r-1,-r} \to E_r^{-i,0} \to E_r^{-i-r+1,r},$$

vanish since $E_r^{-i+r-1,-r} = E_r^{-i-r+1,r} = 0$ when $i > 0$ and $r \geq 1$.

Rewriting (5.2.1), we have

$$h^{-i}(F_\bullet \otimes \mathbf{R}g_* O_Z) = h^{-i}(F_\bullet \otimes \mathbf{R}g_* O_Z \otimes C^\bullet(\underline{x}, A)). \tag{5.2.2}$$

Since we have functorial isomorphism $\mathbf{R}\Gamma_{\mathfrak{m}}(-) \xrightarrow{\cong} C^\bullet(\underline{x}, A) \otimes -$ in $D(\mathrm{QCoh}(\mathrm{Spec}\, A))$ by [Lip02, Proposition 3.1.2], the above yields

$$h^{-i}(F_\bullet \otimes \mathbf{R}g_* O_Z) \cong h^{-i}(F_\bullet \otimes \mathbf{R}\Gamma_{\mathfrak{m}} \mathbf{R}g_* O_Z). \tag{5.2.3}$$

By local duality,

$$h^j(\mathbf{R}\Gamma_{\mathfrak{m}} \mathbf{R}g_* O_Z) = h^{-j}(\mathbf{R}\,\mathrm{Hom}(\mathbf{R}g_* O_Z, \omega_A^\bullet))^\vee = h^{-j}(\mathbf{R}g_* \omega_Z^\bullet)^\vee \tag{5.2.4}$$

where $\omega_A^\bullet$, $\omega_Z^\bullet$ are the normalized dualizing complexes of $\mathrm{Spec}\, A$ and Z. Since Z is smooth, $\omega_Z^\bullet \cong \omega_Z[n]$ where $n = \dim Z$. Hence $h^{-j}(\mathbf{R}g_* \omega_Z^\bullet) = h^{n-j}(\mathbf{R}g_* \omega_Z) = 0$ when $n - j > n - d$ (equivalently, $j < d$) by [Kol86, Theorem 2.1].[5] Now from (5.2.4), we know that

$$h^j(\mathbf{R}\Gamma_{\mathfrak{m}} \mathbf{R}g_* O_Z) = 0, \quad \forall j < d. \tag{5.2.5}$$

[5] In [Kol86], the main theorem requires that the schemes are projective, but this is not essential. One can refer to [EV92, Corollary 6.11].

On the other hand, we know that $F_\bullet$ satisfies the standard conditions on rank and height. This implies $I_{\operatorname{rank}\alpha_n}(\alpha_n)$ must be the unit ideal when $n > d$, because there are no proper ideals in R with height strictly larger than the dimension of R. Hence $F_\bullet$ is split exact at cohomological degree $-n$ when $n > d$ by [BE73, Lemma 1]. Therefore, in $D(\mathrm{Coh}(\operatorname{Spec} A))$ or $D(\mathrm{QCoh}(\operatorname{Spec} A))$, $F_\bullet$ is quasi-isomorphic to a free complex

$$H_\bullet : 0 \to H_k \to H_{k-1} \to \cdots \to H_1 \to H_0 \to 0$$

with $k \le d$ and H_j has cohomological degree $-j$. Now from (5.2.5), it is straightforward to check that

$$h^{-i}(F_\bullet \otimes \mathbf{R}\Gamma_{\mathfrak{m}}\mathbf{R}g_* \mathcal{O}_Z) = h^{-i}(H_\bullet \otimes \mathbf{R}\Gamma_{\mathfrak{m}}\mathbf{R}g_* \mathcal{O}_Z) = 0$$

for every $i > 0$. Hence by (5.2.3) we know that $h^{-i}(F_\bullet \otimes \mathbf{R}g_* \mathcal{O}_Z) = 0$ for every $i > 0$. This finishes our proof in equal characteristic 0.

Proof in characteristic $p > 0$***:*** Now we assume $(A, \mathfrak{m})$ is of equal characteristic $p > 0$. By [Bha12, Theorem 1.5], there exists a finite surjective morphism $\pi : Z \to Y$ such that the pull-back $\pi^*_{\ge 1} : \tau_{\ge 1}\mathbf{R}f_* \mathcal{O}_Y \to \tau_{\ge 1}\mathbf{R}f_*\pi_* \mathcal{O}_Z$ is the zero map. From this it follows (see [Bha12, proof of Theorem 1.4] or [BST15, Lemma 5.1]) that the natural map $\mathbf{R}f_* \mathcal{O}_Y \to \mathbf{R}f_*\pi_* \mathcal{O}_Z$ factors as

$$\mathbf{R}f_* \mathcal{O}_Y \xrightarrow{\theta} (f \circ \pi)_* \mathcal{O}_Z \xrightarrow{\iota} \mathbf{R}f_*\pi_* \mathcal{O}_Z. \tag{5.2.6}$$

Let $g = f \circ \pi$. We know that $g_* \mathcal{O}_Z$ is a module-finite extension of A, as $Z \to \operatorname{Spec} A$ is proper. Let $W = Z \times_Y X$. We have the commutative diagram

$$\begin{tikzcd}
\operatorname{Spec}(g_* \mathcal{O}_Z) \arrow[d] & Z \arrow[l] \arrow[d, "\pi"] \arrow[dl, "g"'] & W \arrow[l, "f_1"'] \arrow[d] \\
\operatorname{Spec} A & Y \arrow[l, "f"'] & X \arrow[l, "f_0"']
\end{tikzcd}$$

This together with (5.2.6) tell us that there is a commutative diagram in $D(\mathrm{QCoh}(\operatorname{Spec} A))$

$$\begin{tikzcd}
g_* \mathcal{O}_Z \arrow[r, "\iota"] & \mathbf{R}g_* \mathcal{O}_Z \arrow[r, "f_1^*"] & \mathbf{R}f_*\mathbf{R}f_{0*}\pi_* \mathcal{O}_W \\
& \mathbf{R}f_* \mathcal{O}_Y \arrow[ul, "\theta"] \arrow[u, "\pi^*"'] \arrow[r, "f_0^*"] & \mathbf{R}f_*\mathbf{R}f_{0*} \mathcal{O}_X \arrow[u]
\end{tikzcd} \tag{5.2.7}$$

Now we pick an arbitrary element $x \in h^{-i}(F_\bullet \otimes \mathbf{R}f_* \mathcal{O}_Y)$ for an arbitrary $i > 0$. We want to show that x maps to 0 in $h^{-i}(F_\bullet \otimes \mathbf{R}f_*\mathbf{R}f_{0*} \mathcal{O}_X)$. To prove this, we first look at the image of x under the map

$$h^{-i}(F_\bullet \otimes \mathbf{R}f_* \mathcal{O}_Y) \xrightarrow{\theta} h^{-i}(F_\bullet \otimes g_* \mathcal{O}_Z).$$

Let $y = \theta(x)$. Since we are in equal characteristic $p > 0$ and $A \to g_* \mathcal{O}_Z$ is a module-finite extension, $A^+ = (g_* \mathcal{O}_Z)^+$ is a balanced big Cohen–Macaulay algebra over A

[HH92]. Since $F_\bullet$ satisfies the standard conditions on rank and height and $(g_*\mathcal{O}_Z)^+$ is big Cohen–Macaulay, it follows from the generalized Buchsbaum–Eisenbud criterion [Abe94, Theorem 1.2.3] that $h^{-i}(F_\bullet \otimes (g_*\mathcal{O}_Z)^+) = 0$ for $i > 0$. In particular, there exists a module-finite extension B of $g_*\mathcal{O}_Z$ such that the image of y in $h^{-i}(F_\bullet \otimes B)$ is 0. Let $W' = W \times_{\mathrm{Spec}(g_*\mathcal{O}_Z)} \mathrm{Spec}\, B$. We know $\pi'\colon W' \to W \to X$ is a finite surjective map of schemes. Since X is a derived splinter, in particular a splinter, we know that $\mathcal{O}_X \to \pi'_*\mathcal{O}_{W'}$ has a splitting η. In sum, after tensoring (5.2.7) with $F_\bullet$ in $D(\mathrm{QCoh}(\mathrm{Spec}\, A))$ and taking cohomology, we get a commutative diagram

$$\begin{tikzcd}
h^{-i}(F_\bullet \otimes B) \arrow[r] & h^{-i}(F_\bullet \otimes \mathbf{R}f_*\mathbf{R}f_{0*}\pi'_*\mathcal{O}_{W'}) \arrow[d, dotted, bend left=60, "\eta"] \\
h^{-i}(F_\bullet \otimes g_*\mathcal{O}_Z) \arrow[u] \arrow[r, "f_1^*\circ\iota"] & h^{-i}(F_\bullet \otimes \mathbf{R}f_*\mathbf{R}f_{0*}\pi_*\mathcal{O}_W) \arrow[u] \\
h^{-i}(F_\bullet \otimes \mathbf{R}f_*\mathcal{O}_Y) \arrow[u, "\theta"] \arrow[r, "f_0^*"] & h^{-i}(F_\bullet \otimes \mathbf{R}f_*\mathbf{R}f_{0*}\mathcal{O}_X) \arrow[u]
\end{tikzcd}$$

From this diagram, it is easy to see that the image of $x \in h^{-i}(F_\bullet \otimes \mathbf{R}f_*\mathcal{O}_Y)$ maps to 0 under f_0^*, because by our construction, the image of x in $h^{-i}(F_\bullet \otimes B)$ is 0. Since our choices of x and $i > 0$ are arbitrary, this proves that $f_0^*\colon h^{-i}(F_\bullet \otimes \mathbf{R}f_*\mathcal{O}_Y) \to h^{-i}(F_\bullet \otimes \mathbf{R}f_*\mathbf{R}f_{0*}\mathcal{O}_X)$ is the zero map for every $i > 0$. This finishes our proof in equal characteristic $p > 0$. □

Remark 5.3. Note that in the above proof, in equal characteristic $p > 0$, we only need to assume X is a splinter. But splinters and derived splinters are the same in characteristic $p > 0$ [Bha12]. In fact, in the course of our proof we use Theorem 1.5 of [Bha12], which is a key ingredient in proving splinters and derived splinters are equivalent in characteristic $p > 0$. We refer to [Bha12] for details.

In the case of maps between rings instead of schemes, our Key Theorem 5.2 specializes to the following corollary:

Corollary 5.4. *Let $(A, \mathfrak{m})$ be a local domain that is essentially of finite type over a field and let M be a finitely generated A-module of finite projective dimension. Let $A \to R \to S$ be ring homomorphisms such that $A \to R$ is a module-finite domain extension and S is a derived splinter. Then the natural map*

$$\mathrm{Tor}_i^A(M, R) \to \mathrm{Tor}_i^A(M, S)$$

is the zero map for every $i > 0$.

Proof. Since M has a finite projective dimension, it has a finite free resolution $F_\bullet$. As $F_\bullet$ is acyclic, it satisfies the standard conditions on rank and depth and hence also the standard conditions on rank and height. Applying Theorem 5.2 to $F_\bullet$, $Y = \mathrm{Spec}\, R$, $X = \mathrm{Spec}\, S$ and noticing that there are no higher direct images because all the maps are affine, we find that $h^{-i}(F_\bullet \otimes R) \to h^{-i}(F_\bullet \otimes S)$ vanishes for every $i > 0$. But $\mathrm{Tor}_i^A(M, R) = h^{-i}(F_\bullet \otimes R)$ and $\mathrm{Tor}_i^A(M, S) = h^{-i}(F_\bullet \otimes S)$, so the conclusion follows. □

Now we state and prove our main theorem.

Theorem 5.5. *Let S be a local domain that is essentially of finite type over a field. The following are equivalent:*

(1) *S satisfies the vanishing conditions for maps of Tor.*
(2) *S is a derived splinter.*
(3) *For every regular local ring A with $S = A/P$ and every module-finite torsion-free extension $A \to B$ with $Q \in \operatorname{Spec} B$ lying over P, the map $P \to Q$ splits as a map of A-modules.*

Proof. We already know (1)⇔(3) by Theorem 4.3. Moreover, recall that derived splinters are the same as rational singularities (equivalently, pseudo-rational singularities) in characteristic 0, and are equivalent to splinters in characteristic $p > 0$. Hence (1)⇒(2) follows from Remark 3.5 and Corollary 4.5 in characteristic 0 and characteristic $p > 0$ respectively.

Finally, we prove (2)⇒(1). Suppose we have $A \to R \to S$ with A regular and R module-finite and torsion-free over A. To check $\mathrm{Tor}_i^A(M, R) \to \mathrm{Tor}_i^A(M, S)$ vanishes, we can assume A is local, R is a domain and M is a finitely generated A-module by Lemma 4.1. Since A is regular, M has finite projective dimension. Hence the vanishing of $\mathrm{Tor}_i^A(M, R) \to \mathrm{Tor}_i^A(M, S)$ follows immediately from Corollary 5.4. □

Remark 5.6. (1) We point out that in Theorem 5.5, (2)⇒(1) in characteristic $p > 0$ also follows directly from the fact that R^+ is a balanced big Cohen–Macaulay algebra: one can use the same argument as in [HH95, Theorem 4.1] and simply notice that the map $S \to S^+$ is always pure when S is a splinter in characteristic $p > 0$.

(2) However, our method in characteristic 0 is of great interest: it gives the first proof of Theorem 1.1 (even in the regular case) in characteristic 0 without using reduction to characteristic $p > 0$. In fact, our result in characteristic 0 does not even seem to follow from reduction to characteristic $p > 0$. It is well known from [Smi97] and [Har98] that a local ring essentially of finite type over a field of characteristic 0 has rational singularities if and only if its mod p reductions, for all sufficiently large p, are F-rational. But F-rationality is known to be weaker than being a derived splinter in characteristic $p > 0$, and hence F-rational rings do not satisfy the vanishing conditions for maps of Tor in general by Theorem 5.5.

(3) Moreover, equal characteristic regular local rings satisfying the vanishing conditions for maps of Tor is a very special case of our Key Theorem 5.2, the case where A is regular with $F_\bullet$ a free resolution of a finitely generated A-module M, $Y \to \operatorname{Spec} A$ is finite surjective and X is regular affine. So our Theorem 5.2 greatly extends Hochster–Huneke's Theorem 1.1, and actually it also generalizes the main theorems of [HH93].

Remark 5.7. We point out that Boutot's theorem that direct summands of rational singularities are rational singularities [Bou87] follows from our vanishing conditions for maps of Tor applied to $M = E_A$, the injective hull of A. Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a split map of local rings essentially of finite type over a field of characteristic 0, and let S have rational singularities. For every surjection $(B, \mathfrak{m}_1) \twoheadrightarrow (R, \mathfrak{m})$ with B equidimensional, we can find $(A, \mathfrak{m}_0) \to (B, \mathfrak{m}_1)$ module-finite with $(A, \mathfrak{m}_0)$ regular by Noether normalization. Now we consider the map $A \to B \to R \to S$. Since S has rational singularities, it satisfies the vanishing conditions for maps of Tor by Theorem 5.5. Hence $\mathrm{Tor}_i^A(E_A, B) \to \mathrm{Tor}_i^A(E_A, R) \to \mathrm{Tor}_i^A(E_A, S)$ vanishes for $i \geq 1$. This implies $\mathrm{Tor}_i^A(E_A, B) \to \mathrm{Tor}_i^A(E_A, R)$ vanishes for $i \geq 1$ because $R \to S$ splits. Hence R satisfies the vanishing conditions for maps of local cohomology (recall that $\mathrm{Tor}_i^A(E_A, B) = H_{\mathfrak{m}}^{d-i}(B)$). Therefore by Lemma 3.3, R has rational singularities.

As a consequence of Theorem 5.5, we obtain a new characterization of rational singularities:

Corollary 5.8. *Let $(S, \mathfrak{n})$ be a local domain essentially of finite type over a field of characteristic 0. Then S has rational singularities if and only if for every regular local ring A with $S = A/P$, every module-finite torsion-free extension $A \to T$, and every $Q \in \operatorname{Spec} T$ lying over P, the map $P \to Q$ splits as a map of A-modules.*

Proof. This follows immediately from (2)$\Leftrightarrow$(3) in Theorem 5.5, and the result that derived splinters are exactly rational singularities in equal characteristic 0. □

We next want to prove a theorem that characterizes the vanishing conditions for maps of local cohomology in equal characteristic. We first prove a lemma that is of independent interest. Recall that in characteristic $p > 0$, $0^*_{H_{\mathfrak{n}}^d(S)}$ (the tight closure of 0) is the largest proper submodule of $H_{\mathfrak{n}}^d(S)$ that is stable under the natural Frobenius action [Smi97].

Lemma 5.9. *Let $(S, \mathfrak{n})$ be a local domain of equal characteristic $p > 0$. Then*

$$\sum_R \operatorname{im}(H_{\mathfrak{m}}^d(R) \to H_{\mathfrak{n}}^d(S)) = 0^*_{H_{\mathfrak{n}}^d(S)} \tag{5.9.1}$$

where the sum is taken over all $(R, \mathfrak{m}) \twoheadrightarrow (S, \mathfrak{n})$ such that $\dim R/P > \dim S = d$ for every minimal prime P of R.

Proof. Take a surjection $(R, \mathfrak{m}) \to (S, \mathfrak{n})$. We first prove that the image of $H_{\mathfrak{m}}^d(R) \to H_{\mathfrak{n}}^d(S)$ is contained in $0^*_{H_{\mathfrak{n}}^d(S)}$. Since $\dim R/P > d$ for every minimal prime P of R, $R \to S$ obviously factors as $R \to R' \to S$ for some domain R' with $\dim R' = d + 1$. Hence the image of $H_{\mathfrak{m}}^d(R) \to H_{\mathfrak{n}}^d(S)$ is contained in the image of $H_{\mathfrak{m}}^d(R') \to H_{\mathfrak{n}}^d(S)$, which is clearly a submodule of $H_{\mathfrak{n}}^d(S)$ stable under the Frobenius action. Thus it is contained in $0^*_{H_{\mathfrak{n}}^d(S)}$ as long as it is not equal to $H_{\mathfrak{n}}^d(S)$. Therefore, it suffices to show that $H_{\mathfrak{m}}^d(R') \to H_{\mathfrak{n}}^d(S)$ is not surjective. Writing $S = R'/Q$ for some height one prime ideal Q of R' we have the long exact sequence of local cohomology

$$\cdots \to H_{\mathfrak{m}}^d(R') \to H_{\mathfrak{n}}^d(S) \to H_{\mathfrak{m}}^{d+1}(Q) \to H_{\mathfrak{m}}^{d+1}(R'). \tag{5.9.2}$$

Since R' has dimension $d+1$ and Q is a height one prime, $H^{d+1}_{\mathfrak{m}}(Q) \to H^{d+1}_{\mathfrak{m}}(R')$ is not injective by [Ma14, Lemma 3.3]. Therefore $H^d_{\mathfrak{m}}(R') \to H^d_{\mathfrak{n}}(S)$ is not surjective by (5.9.2). Hence

$$\sum_R \operatorname{im}(H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{n}}(S)) \subseteq 0^*_{H^d_{\mathfrak{n}}(S)}.$$

On the other hand, Lemma 3.1 shows that

$$\sum_R \operatorname{im}(H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{n}}(S)) \supseteq 0^+_{H^d_{\mathfrak{n}}(S)} = 0^*_{H^d_{\mathfrak{n}}(S)},$$

where the last equality follows from the main theorem of [Smi94]. □

Theorem 5.10. *Let $(S, \mathfrak{n})$ be a local domain that is essentially of finite type over a field. In characteristic 0, S satisfying the vanishing conditions for maps of local cohomology if and only if S has rational singularities, while in characteristic $p > 0$, S satisfies the vanishing conditions for maps of local cohomology if and only if S is F-rational.*

Proof. The characteristic 0 assertion follows (implicitly) from the proof of Theorem 5.5, as S satisfying the vanishing conditions for maps of local cohomology implies S has rational singularities by Lemma 3.3. It remains to prove the characteristic $p > 0$ statement. But this follows immediately from Lemma 5.9 and Definition 2.2, since when S is Cohen–Macaulay, S is F-rational if and only if $0^*_{H^d_{\mathfrak{n}}(S)} = 0$ [HH94], [Smi97]. □

Finally, it is quite natural to ask whether our main theorem, Theorem 5.5, holds in mixed characteristic. By Theorem 4.3, (3)⇒(1) always holds, and the main obstruction to (1)⇒(3) is the direct summand conjecture in mixed characteristic. Below we give a partial answer for (1)⇒(2). We believe this result and its proof are of independent interest (for example, see Remarks 5.12 and 5.13).

Theorem 5.11. *If $(S, \mathfrak{n})$ is a quasi-Gorenstein complete local domain (of mixed characteristic) that satisfies the vanishing condition for maps of Tor, then S is a derived splinter.*

Proof. We first note that the conditions imply S is Cohen–Macaulay (and thus Gorenstein) by Proposition 3.4 because S is complete and satisfies the vanishing conditions for maps of Tor.

Let $\pi\colon X \to \operatorname{Spec} S$ be a proper surjective map, we want to show $S \to \mathbf{R}\pi_* O_X$ splits in the derived category of S-modules. By Chow's Lemma we may assume that X is projective. We claim we may reduce to the case where $X \to \operatorname{Spec} S$ is a projective and generically finite map between integral schemes.[6] We first find an irreducible component W of X and an affine open $U = \operatorname{Spec} B$ of W that dominates $\operatorname{Spec} S$. It follows that B is a domain containing S, finitely generated as an S-algebra. Let L be the fraction field of S. We have $\dim(L \otimes B) = \dim B - \dim S$ by [Eis95, Theorem 13.8]. Hence if $\dim B - \dim S \geq 1$, then $\dim(L \otimes B) \geq 1$. This means there exist nonzero primes in B that contract to 0 in S. Pick such a prime Q; then $S \hookrightarrow B/Q$ is injective. Thus $V = \operatorname{Spec} B/Q$

[6] This should be well known. We provide the argument because we cannot find a good reference for this in mixed characteristic.

still dominates Spec S. Let X be the closure of V in W. Then $X' \to \operatorname{Spec} S$ is projective and surjective with $\dim X' < \dim X$. We repeat this process until we get $\dim X = \dim S$, i.e., $X \to \operatorname{Spec} S$ is projective and generically finite. Next we consider the Stein factorization

$$X \to \operatorname{Spec}(\pi_* O_X) \to \operatorname{Spec} S.$$

Let $T = \pi_* O_X$. We know that T is a module-finite domain extension of S. In particular, since S is complete, this implies T is a local ring and $\mathfrak{n}T$ is primary to the maximal ideal of T. The map $X \to \operatorname{Spec} T$ is projective and birational, thus it is just the blow-up of some ideal $J \subseteq T$. Let $R = T[Jt] = T \oplus Jt \oplus J^2t^2 \oplus \cdots$; then we have $X = \operatorname{Proj} R$.

Pick $f_1, \ldots, f_n \in Jt = [R]_1$ such that $U = \{U_i = \operatorname{Spec}[R_{f_i}]_0\}$ is an affine open cover of X. We have an exact sequence of chain complexes [Lip94, p. 150]

$$0 \to \check{C}^\bullet(U, O_X)[-1] \to [C^\bullet(f_1, \ldots, f_n, R)]_0 \to T \to 0.$$

Since $\check{C}^\bullet(U, O_X) \cong \mathbf{R}\pi_* O_X$, the above sequence gives (after rotating) an exact triangle

$$[\mathbf{R}\Gamma_{R_{>0}} R]_0 = [C^\bullet(f_1, \ldots, f_n, R)]_0 \to T \to \mathbf{R}\pi_* O_X \xrightarrow{+1} .$$

Applying $\mathbf{R}\Gamma_{\mathfrak{n}}$, we get

$$[\mathbf{R}\Gamma_{\mathfrak{n}+R_{>0}} R]_0 \to \mathbf{R}\Gamma_{\mathfrak{n}} T \to \mathbf{R}\Gamma_{\mathfrak{n}} \mathbf{R}\pi_* O_X \xrightarrow{+1} . \tag{5.11.1}$$

Let $d = \dim S = \dim T$ and $d + 1 = \dim R$. Next we prove two claims:

Claim 5.11.2. $[H^{d+1}_{\mathfrak{n}+R_{>0}}(R)]_0 = 0$*, thus* $[\mathbf{R}\Gamma_{\mathfrak{n}+R_{>0}} R]_0$ *lives in degree* $[0, 1, \ldots, d]$.

Proof. This is well known, because the a-invariant of the Rees ring is always -1 (for example, see [HS03, 2.4.2 and 2.5.2]). For the sake of completeness we point out that this also follows from (5.11.1). By local duality, the dual of $h^d(\mathbf{R}\Gamma_{\mathfrak{n}} T) \to h^d(\mathbf{R}\Gamma_{\mathfrak{n}} \mathbf{R}\pi_* O_X)$ is the natural inclusion $\pi_*\omega_X \hookrightarrow \omega_T$ (since $X \to \operatorname{Spec} T$ is birational). Hence

$$[H^{d+1}_{\mathfrak{n}+R_{>0}}(R)]_0 = h^{d+1}([\mathbf{R}\Gamma_{\mathfrak{n}+R_{>0}} R]_0) = 0. \qquad \square$$

Claim 5.11.3. *There exists an S-linear surjection $\phi\colon T \twoheadrightarrow S$ such that the composite*

$$[H^d_{\mathfrak{n}+R_{>0}}(R)]_0 \to H^d_{\mathfrak{n}}(T) \xrightarrow{\phi} H^d_{\mathfrak{n}}(S)$$

is the zero map (*the first map is induced by the natural surjection* $R \twoheadrightarrow T$).

Proof. Let $R' = S \oplus Jt \oplus J^2t^2 \oplus \cdots$ be the subring of R (they only differ at the degree 0 spot). We note that since J is a finitely generated S-module, R' is a Noetherian graded domain over S. We have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & R_{>0} \arrow[r] & R \arrow[r] & T \arrow[r] & 0 \\
0 \arrow[r] & R'_{>0} \arrow[r] \arrow[u, "\cong"] & R' \arrow[r] \arrow[u] & S \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

If we view everything as modules or algebras over R', the above diagram induces a commutative diagram in local cohomology:

$$\begin{tikzcd}
{[H^d_{\mathfrak{n}+R_{>0}}(R)]_0} \arrow[r, "f"] & H^d_{\mathfrak{n}}(T) \arrow[r] & {[H^{d+1}_{\mathfrak{n}+R_{>0}}(R_{>0})]_0} \arrow[r] & {[H^{d+1}_{\mathfrak{n}+R_{>0}}(R)]_0 = 0} \\
{[H^d_{\mathfrak{n}+R_{>0}}(R')]_0} \arrow[r, "0"] \arrow[u] & H^d_{\mathfrak{n}}(S) \arrow[r, hook] \arrow[u] & {[H^{d+1}_{\mathfrak{n}+R_{>0}}(R'_{>0})]_0} \arrow[u, "\cong"'] &
\end{tikzcd}$$

The rightmost 0 on the first line comes from Claim 5.11.2, and the first map on the second line is 0 because S is complete and satisfies the vanishing conditions for maps of Tor, hence in particular it satisfies the vanishing conditions for local cohomology by Proposition 3.4. By chasing the diagram it follows immediately that $H^d_{\mathfrak{n}}(S) \hookrightarrow H^d_{\mathfrak{n}}(T)/\mathrm{im}(f)$. Since S is quasi-Gorenstein, $H^d_{\mathfrak{n}}(S) \cong E_S$ is an injective S-module. So there is a map $g\colon H^d_{\mathfrak{n}}(T)/\mathrm{im}(f) \to H^d_{\mathfrak{n}}(S)$ such that the composite

$$H^d_{\mathfrak{n}}(S) \to H^d_{\mathfrak{n}}(T) \to H^d_{\mathfrak{n}}(T)/\mathrm{im}(f) \xrightarrow{g} H^d_{\mathfrak{n}}(S)$$

is the identity. In particular, there is a splitting $H^d_{\mathfrak{n}}(T) \xrightarrow{\phi} H^d_{\mathfrak{n}}(S)$ of $H^d_{\mathfrak{n}}(S) \hookrightarrow H^d_{\mathfrak{n}}(T)$ such that the composite $[H^d_{\mathfrak{n}+R_{>0}}(R)]_0 \xrightarrow{f} H^d_{\mathfrak{n}}(T) \xrightarrow{\phi} H^d_{\mathfrak{n}}(S)$ is the zero map. However, it follows from the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_S(H^d_{\mathfrak{n}}(T), H^d_{\mathfrak{n}}(S)) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_S(H^d_{\mathfrak{n}}(S), H^d_{\mathfrak{n}}(S)) \arrow[d, "\cong"] \\
\mathrm{Hom}_S(T, S) \arrow[r] & \mathrm{Hom}_S(S, S)
\end{tikzcd}$$

that every splitting $H^d_{\mathfrak{n}}(T) \xrightarrow{\phi} H^d_{\mathfrak{n}}(S)$ comes from some surjection $T \xrightarrow{\phi} S$. □

Now we return to the proof of Theorem 5.11. We claim that the composite

$$[\mathbf{R}\Gamma_{\mathfrak{n}+R_{>0}}R]_0 \to \mathbf{R}\Gamma_{\mathfrak{n}}T \to H^d_{\mathfrak{n}}(T)[-d] \xrightarrow{\phi} H^d_{\mathfrak{n}}(S)[-d] \cong \mathbf{R}\Gamma_{\mathfrak{n}}S$$

is the zero map *in the derived category*: it induces zero on the d-th cohomology by Claim 5.11.3, but by Claim 5.11.2, $[\mathbf{R}\Gamma_{\mathfrak{n}+R_{>0}}R]_0$ lives in degree $[0, 1, \ldots, d]$ while $H^d_{\mathfrak{n}}(S)[-d]$ lives only in degree d, so the map is zero in the derived category. The last isomorphism follows because S is Cohen–Macaulay.

Let ϕ be the surjection in Claim 5.11.3. There exists $t \in T$ such that $\phi(t) = 1 \in S$, in particular the composite $S \xrightarrow{\cdot t} T \xrightarrow{\phi} S$ is the identity. From the above discussion and (5.11.1), we have a natural diagram in the derived category of S-modules

$$\begin{tikzcd}
{[\mathbf{R}\Gamma_{\mathfrak{n}+R_{>0}}R]_0} \arrow[r] \arrow[dr, dotted, "0"] & \mathbf{R}\Gamma_{\mathfrak{n}}T \arrow[r] \arrow[d, bend right, "\phi"'] & \mathbf{R}\Gamma_{\mathfrak{n}}\mathbf{R}\pi_*\mathcal{O}_X \arrow[r, "+1"] & {} \\
& \mathbf{R}\Gamma_{\mathfrak{n}}S \arrow[u, bend right, "\cdot t"'] & &
\end{tikzcd}$$

Taking Matlis dual and applying local duality, we get

$$\begin{tikzcd} \mathbf{R}\pi_*\omega_X^\bullet \arrow[r] & \omega_T^\bullet \arrow[r] \arrow[d, bend right, "\cdot t^\vee"'] & {[\mathbf{R}\Gamma_{\mathfrak{n}+R_{>0}}R]_0^\vee} \arrow[r, "+1"] & {} \\ & \omega_S^\bullet \arrow[u, bend right, "\phi^\vee"'] \arrow[ur, dotted, "0"] & & \end{tikzcd} \tag{5.11.4}$$

From (5.11.4) it follows that $\phi^\vee$ factors through a map $\omega_S^\bullet \to \mathbf{R}\pi_*\omega_X^\bullet$ such that the composite

$$\omega_S^\bullet \to \mathbf{R}\pi_*\omega_X^\bullet \to \omega_T^\bullet \xrightarrow{\cdot t^\vee} \omega_S^\bullet$$

is the identity. Applying $\mathbf{R}\operatorname{Hom}_S(-, \omega_S^\bullet)$, we obtain

$$S \xrightarrow{\cdot t} T = \pi_* O_X \to \mathbf{R}\pi_* O_X \to S$$

such that the composite is the identity. But this implies $S \to \mathbf{R}\pi_* O_X \xrightarrow{\cdot t} \mathbf{R}\pi_* O_X \to S$ is the identity (the second map is induced by $O_X \xrightarrow{\cdot t} O_X$ by viewing t as a section of $X \to \operatorname{Spec} S$). Hence $S \to \mathbf{R}\pi_* O_X$ splits in the derived category of S-modules. Therefore S is a derived splinter, as desired. □

At the moment we do not see how to drop the quasi-Gorenstein hypothesis on S in Theorem 5.11; the subtle point here seems to be Claim 5.11.3. However, the above result and its proof already have some interesting consequences.

Remark 5.12. Since regular local rings are certainly quasi-Gorenstein, Theorem 5.11 immediately implies that Hochster–Huneke's vanishing conjecture for maps of Tor (or equivalently, the strong direct summand conjecture [Ran00]) implies the derived direct summand conjecture of Bhatt [Bha12].

Remark 5.13. The argument used in Theorem 5.11 gives a new proof that splinters and derived splinters are the same in characteristic $p > 0$ for all local rings that are homomorphic images of Gorenstein local rings. First of all it is well known that splinters are Cohen–Macaulay in characteristic $p > 0$ (we do not need completeness [HL07], [HH92]). Now the only place in the argument where we use the vanishing conditions for maps of Tor and the quasi-Gorenstein hypothesis seriously is in the proof of Claim 5.11.3. But this claim is clear in characteristic $p > 0$ and we give a short argument as follows: by [HL07, Theorem 2.1] we know that there exists a module-finite extension B of R such that the induced map $H^d_{\mathfrak{n}+R_{>0}}(R) \to H^d_{\mathfrak{n}+R_{>0}}(B)$ is zero. Since $B \otimes_R T$ is a module-finite extension of T and hence a module-finite extension of S, the map $S \to B \otimes_R T$ splits as a map of S-modules. Let

$$\phi : T \to B \otimes_R T \xrightarrow{g} S$$

be the composite map for some splitting $g\colon B \otimes_R T \to S$. We have the commutative diagram

$$\begin{tikzcd} H^d_{\mathfrak{n}+R_{>0}}(B) \arrow[r] & H^d_{\mathfrak{n}}(B \otimes_R T) \arrow[dr, "g"] & \\ {[H^d_{\mathfrak{n}+R_{>0}}(R)]_0} \arrow[u] \arrow[r] & H^d_{\mathfrak{n}}(T) \arrow[u] \arrow[r, dotted, "\phi"] & H^d_{\mathfrak{n}}(S) \end{tikzcd}$$

Since the left vertical map is the zero map by our choice of B, chasing through the diagram shows that the composite $[H^d_{\mathfrak{n}+R_{>0}}(R)]_0 \to H^d_{\mathfrak{n}}(T) \xrightarrow{\phi} H^d_{\mathfrak{n}}(S)$ is the zero map. Hence Claim 5.11.3 holds in characteristic $p > 0$ as long as S is a splinter (without any quasi-Gorenstein or complete hypothesis).

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