J. Eur. Math. Soc. 20, 391–457

DOI 10.4171/JEMS/770

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Rational invariant tori and band edge spectra for non-selfadjoint operators

Received February 24, 2015

Abstract. We study semiclassical asymptotics for spectra of non-selfadjoint perturbations of selfadjoint analytic h-pseudodifferential operators in dimension 2, assuming that the classical flow of the unperturbed part is completely integrable. Complete asymptotic expansions are established for all individual eigenvalues in suitable regions of the complex spectral plane, near the edges of the spectral band, coming from rational flow-invariant Lagrangian tori.

Keywords. Non-selfadjoint, eigenvalue, spectral asymptotics, resolvent, semiclassical limit, completely integrable, Lagrangian torus, rational torus, secular perturbation theory, pseudospectrum, exponential weight, FBI transform

Contents

1. Introduction

Spectra for semiclassical non-selfadjoint operators often display fascinating features, from lattices of low-lying eigenvalues for operators of Kramers–Fokker–Planck

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Mathematics Subject Classification (2010): 35P15, 35P20, 37J35, 37J40, 53D22, 58J37, 58J40

type [8], [10] to eigenvalues for operators with analytic coefficients in dimension one, concentrated on unions of curves [26], [28], [6], [13]. The work [25] has established that for wide and stable classes of non-selfadjoint analytic pseudodifferential operators in dimension two, the individual eigenvalues can be determined accurately in the semiclassical limit, by means of a complex Bohr–Sommerfeld quantization condition, and form a distorted two-dimensional lattice. Now in many natural situations [20], [32], [31], [34], one encounters non-selfadjoint operators of the form

$$P_\varepsilon = p(x, hD_x) + i\varepsilon q(x, hD_x), \qquad 0 \le \varepsilon \ll 1, \tag{1.1}$$

considered on $\mathbb{R}^n$ or a compact real analytic manifold, with $P_{\varepsilon=0}$ being selfadjoint. Here $0 < h \ll 1$ is the semiclassical parameter and the second small parameter ε represents the strength of the non-selfadjoint perturbation. The principal symbol of P_ε in (1.1) is of the form $p_\varepsilon(x,\xi) = p(x,\xi) + i\varepsilon q(x,\xi)$, where p is real, and let us also assume, to fix ideas, that q is real. Both p and q are assumed to be analytic, with p elliptic near infinity. The spectrum of P_ε near the origin is confined to a band of width $\mathcal{O}(\varepsilon)$, and the general problem is to understand the distribution of the eigenvalues of P_ε near 0 in the semiclassical limit $h \to 0^+$. To this end, let us assume that 0 is a regular value of p, so that the energy surface $p^{-1}(0)$ is a smooth compact submanifold of the phase space. We then know [22], [23] that the real parts of the eigenvalues of P_ε near 0 are distributed according to the same Weyl law as those for the unperturbed operator $P_{\varepsilon=0}$. In order to study the distribution of the imaginary parts of the eigenvalues, following the method of averaging [38], [2], we let H_p be the Hamilton vector field of p and introduce the time averages

$$\langle q\rangle_T = \frac{1}{T}\int_{-T/2}^{T/2} q\circ \exp(tH_p)\,dt, \qquad T>0, \tag{1.2}$$

of q along the H_p-trajectories. It follows from [20], [32], [16] that if $z \in \mathrm{Spec}(P_\varepsilon)$ is such that $|\mathrm{Re}\, z| \le \delta$, then

$$\lim_{T\to\infty}\inf_{p^{-1}(0)}\langle q\rangle_T - o(1) \le \frac{\mathrm{Im}\, z}{\varepsilon} \le \lim_{T\to\infty}\sup_{p^{-1}(0)}\langle q\rangle_T + o(1) \quad \text{ as } (\varepsilon,\delta,h)\to 0^+. \tag{1.3}$$

The spectral analysis for non-selfadjoint operators of the form (1.1) has been pursued by the authors in the series of papers [11]–[16], the latter work jointly with S. Vũ Ngọc, when $n = 2$ and the H_p-flow is either periodic or completely integrable. Let us focus, from now on, on the completely integrable case. In this case, the energy surface $p^{-1}(0)$ is foliated by invariant Lagrangian tori, along with possibly some other more complicated flow-invariant sets. If $\Lambda \subset p^{-1}(0)$ is an invariant torus such that the rotation number of H_p along Λ is Diophantine, i.e. poorly approximated by rational numbers, or more generally, irrational, then the time averages $\langle q\rangle_T$ along Λ converge to the space average $\langle q\rangle(\Lambda)$ of q over Λ as $T \to \infty$. If Λ is a torus with a rational rotation number, or a singular set in the foliation of $p^{-1}(0)$, then in analogy with (1.3), we introduce the compact interval

$$Q_\infty(\Lambda) = \left[\lim_{T\to\infty}\inf_{\Lambda}\langle q\rangle_T, \lim_{T\to\infty}\sup_{\Lambda}\langle q\rangle_T\right] \tag{1.4}$$

of limits of the time averages above. The definition (1.4) will also be used when Λ is a torus with an irrational rotation number, in which case we have $Q_\infty(\Lambda) = \{\langle q\rangle(\Lambda)\}$.

Let $F_0 \in \mathbb{R}$ be such that $F_0 = \langle q\rangle(\Lambda_d)$ for a single Diophantine Lagrangian torus $\Lambda_d \subset p^{-1}(0)$, and assume that

$$F_0 \notin Q_\infty(\Lambda) \tag{1.5}$$

for any other invariant set $\Lambda \neq \Lambda_d$ in $p^{-1}(0)$. It was then shown in [16] that the spectrum of P_ε can be determined completely, modulo $\mathcal{O}(h^\infty)$, in a rectangle of the form $[-h^\delta/C, h^\delta/C] + i\varepsilon[F_0 - h^\delta/C, F_0 + h^\delta/C]$, where $\delta > 0$ and ε satisfies $h^K < \varepsilon \ll 1$, for $K \gg 1$. Similarly to [25], the spectrum has a structure of a distorted two-dimensional lattice, with the horizontal spacing $\sim h$ and the vertical one $\sim \varepsilon h$. A closely related result was obtained in [15], giving a Weyl type asymptotic formula for the number of eigenvalues of P_ε in an intermediate spectral band, bounded from above and from below by Diophantine levels, such as F_0 above. It turned out that the distribution of the imaginary parts of the eigenvalues of P_ε is governed by a Weyl law, expressed in terms of phase space volumes associated to p and the long time averages of q.

Having elucidated the role played by flow-invariant Diophantine tori in the spectral analysis of P_ε, let us now turn to spectral contributions of tori that are rational, which constitutes the subject of the present work. Let $F_0 \in \mathbb{R}$ be such that $F_0 = \langle q\rangle(\Lambda_d)$ for a Diophantine torus Λ_d as above, and rather than demanding (1.5), let us assume that there exists a rational torus $\Lambda_r \subset p^{-1}(0)$ such that $F_0 \in Q_\infty(\Lambda_r)$, $F_0 \neq \langle q\rangle(\Lambda_r)$, while $F_0 \notin Q_\infty(\Lambda)$, for $\Lambda \neq \Lambda_d, \Lambda_r$. An attempt to determine the individual eigenvalues of P_ε near $i\varepsilon F_0$ was made by the authors in [14], by means of the normal form techniques. As a result, the normal forms near Λ_r that we obtained were given by a family of one-dimensional "resonant" non-selfadjoint operators, and the possibility of quite serious pseudospectral phenomena for this family [4] prevented us from computing the eigenvalues individually. Correspondingly, the main result of [14] was weaker, establishing that the spectrum of P_ε near $i\varepsilon F_0$ was of the form $E_d \cup E_r$, where the "Diophantine" contribution E_d is a distorted lattice that can be described explicitly, as in [16], and the cardinality of the "rational" contribution E_r is $\ll$ than that of E_d.

Subsequently, in the course of some numerical experiments, the authors have encountered peculiar pictures of the spectra of P_ε, where the eigenvalues had the form of a "centipede", with the body agreeing with the range of torus averages of q — see Section 8 for the illustrations and the details of the numerical computations. The legs of the centipede were more mysterious at first, but things became clearer when we realized that they represented the influence of suitable rational tori. It then became natural to hope that the eigenvalues near the extremities of the legs could be determined asymptotically in a rigorous way, since the pseudospectral effects should become more moderate near the edges of the spectral band [4]. The main result of the present work, giving a complete asymptotic description of the individual extremal eigenvalues of P_ε, can be considered as a justification of this hope.

Let us conclude the introduction by formulating, in a rough way, the main result of the paper — see Theorem 2.1 below for the precise statement. Let $\Lambda_0 \subset p^{-1}(0)$ be a rational Lagrangian torus such that

$$\inf Q_\infty(\Lambda_0) < \inf_{\Lambda\subset p^{-1}(0),\, \Lambda\neq\Lambda_0} (\inf Q_\infty(\Lambda))\,. \tag{1.6}$$

The restriction of the H_p-flow to Λ_0 is periodic with primitive period $T_0 > 0$, and the time average $\langle q\rangle_{T_0}$ in (1.2) can naturally be viewed as a function on the space $\Lambda_0/\exp(\mathbb{R}H_p)$ of closed orbits. Let us assume that $\langle q\rangle_{T_0}$, viewed as a function on $\Lambda_0/\exp(\mathbb{R}H_p)$, has a unique minimum which is non-degenerate, and restrict ε to a suitable interval of the form $h^{1+\eta} \le \varepsilon \le h^{1-\eta}$, $\eta > 0$. Then for any fixed $C_0 > 0$ the eigenvalues of P_ε in the region

$$\left\{z \in \mathbb{C};\ |\mathrm{Re}\, z| < \frac{h}{C_0\sqrt{\varepsilon}},\ \frac{\mathrm{Im}\, z}{\varepsilon} \le \inf Q_\infty(\Lambda_0) + C_0\frac{h}{\sqrt{\varepsilon}}\right\}$$

can be determined completely, modulo $\mathcal{O}(h^\infty)$, and are given by

$$\lambda_{j,k} = a(\xi_2) + i\varepsilon b(\xi_2) + \varepsilon^{1/2} h\Lambda_{j,k}, \quad \xi_2 = h\left(j - \frac{k_0}{4}\right) - \frac{S}{2\pi}, \quad j \in \mathbb{Z},\ k \in \mathbb{N}. \tag{1.7}$$

Here the functions a and b do not depend on h, and we have $a(0) = 0$, $a'(0) > 0$, $b(0) = \inf Q_\infty(\Lambda_0)$. The quantities S and k_0 in (1.7) are the classical action and the Maslov index of a primitive closed H_p-trajectory in Λ_0, respectively, and we have a complete asymptotic expansion for $\Lambda_{j,k}$ in integer powers of $\widetilde{h} = h/\sqrt{\varepsilon}$,

$$\Lambda_{j,k} \sim \sum_{\nu=0}^{\infty} \widetilde{h}^\nu \lambda_k^\nu(\xi_2, \sqrt{\varepsilon}), \tag{1.8}$$

where

$$\lambda_k^0(0,0) = de^{i\pi/4}(2k+1), \quad d > 0. \tag{1.9}$$

Remark. Neglecting the corrections given by the classical quantities S and k_0, let us remark that the description of the eigenvalues of P_ε given in (1.7)–(1.9), as well as in Theorem 2.1 below, agrees, in a special case and to the leading order, with the spectrum of the model operator

$$\widetilde{P}_\varepsilon(x_1, hD_{x_1}, hD_{x_2}) = hD_{x_2} + (hD_{x_1})^2 + i\varepsilon x_1^2/2, \tag{1.10}$$

acting on $L^2(\mathbb{R}_{x_1} \times \mathbb{T}_{x_2})$, $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, which is given by

$$\lambda_{j,k} = hj + \varepsilon^{1/2} he^{i\pi/4} 2^{-1/2}(2k+1), \quad j \in \mathbb{Z},\ k \in \mathbb{N}$$

(see [3], [33]). Speaking heuristically, it is the occurrence of the complex harmonic oscillator $(hD_{x_1})^2 + i\varepsilon x_1^2/2$, as a part of some crucial normal form in our analysis, that ultimately gives rise to the "centipede" structure of the eigenvalues close to the edges of the spectral band.

2. Statement of the main results

2.1. General assumptions

We shall start by describing the general assumptions on our operators, which will be the same as in [14], [16], as well as in the earlier papers mentioned above. Let M denote either the space $\mathbb{R}^2$ or a real analytic compact manifold of dimension 2. When $M = \mathbb{R}^2$, let

$$P_\varepsilon = P^w(x, hD_x, \varepsilon; h), \qquad 0 < h \le 1, \tag{2.1}$$

be the h-Weyl quantization on $\mathbb{R}^2$ of a symbol $P(x, \xi, \varepsilon; h)$ (i.e. the Weyl quantization of $P(x, h\xi, \varepsilon; h)$), depending smoothly on $\varepsilon \in \mathrm{neigh}(0, \mathbb{R})$ and taking values in the space of holomorphic functions of (x, ξ) in a tubular neighborhood of $\mathbb{R}^4$ in $\mathbb{C}^4$, with

$$|P(x, \xi, \varepsilon; h)| \le \mathcal{O}(1) m(\mathrm{Re}(x, \xi)) \tag{2.2}$$

there. Here $1 \le m \in C^\infty(\mathbb{R}^4)$ is an order function, in the sense that

$$m(X) \le C_0 \langle X - Y\rangle^{N_0} m(Y), \qquad X, Y \in \mathbb{R}^4, \tag{2.3}$$

for some $C_0, N_0 > 0$, where we write $\langle X - Y\rangle = (1 + |X - Y|^2)^{1/2}$. We shall assume, as we may, that m belongs to its own symbol class, so that $\partial^\alpha m = \mathcal{O}_\alpha(m)$ for each $\alpha \in \mathbb{N}^4$. Then for $h > 0$ small enough and when equipped with the domain $H(m) := (m^w(x, hD))^{-1}(L^2(\mathbb{R}^2))$, P_ε becomes a closed densely defined operator on $L^2(\mathbb{R}^2)$.

Remark. The analyticity assumptions will allow us to treat the case when $\varepsilon \asymp h^\delta$ for $0 < \delta < 1$. When $\varepsilon = \mathcal{O}(h)$, standard C^∞-microlocal analysis would have been sufficient.

Assume furthermore that

$$P(x, \xi, \varepsilon; h) \sim \sum_{j=0}^{\infty} h^j p_{j,\varepsilon}(x, \xi) \tag{2.4}$$

in the space of holomorphic functions depending smoothly on $\varepsilon \in \mathrm{neigh}(0, \mathbb{R})$ and satisfying (2.2) in a fixed tubular neighborhood of $\mathbb{R}^4$. Explicitly, the assumption (2.4) states that for each $(N, k) \in \mathbb{N} \times \mathbb{N}$ there exists $C_{N,k}$ such that for all (x, ξ) in the tubular neighborhood, all $\varepsilon \in \mathrm{neigh}(0, \mathbb{R})$, and all $h \in (0, 1]$, we have

$$\left|\partial_\varepsilon^k \Big(P(x, \xi, \varepsilon; h) - \sum_{j=0}^{N-1} h^j p_{j,\varepsilon}(x, \xi)\Big)\right| \le C_{N,k} h^N m(\mathrm{Re}(x, \xi)). \tag{2.5}$$

We assume that $p_{0,\varepsilon}$ is elliptic near infinity,

$$|p_{0,\varepsilon}(x, \xi)| \ge \frac{1}{C} m(\mathrm{Re}(x, \xi)), \qquad |(x, \xi)| \ge C, \tag{2.6}$$

for some $C > 0$.

When M is a real analytic compact manifold, for simplicity we shall take P_ε to be a differential operator on M which for every choice of local coordinates centered at some point of M takes the form

$$P_\varepsilon = \sum_{|\alpha|\le m} a_{\alpha,\varepsilon}(x;h)(hD_x)^\alpha, \tag{2.7}$$

where $a_{\alpha,\varepsilon}(x;h)$ is a smooth function of $\varepsilon \in \mathrm{neigh}(0,\mathbb{R})$ with values in the space of bounded holomorphic functions in a complex neighborhood of $x = 0$, independent of h when $|\alpha| = m$. We further assume that

$$a_{\alpha,\varepsilon}(x;h) \sim \sum_{j=0}^{\infty} a_{\alpha,\varepsilon,j}(x)h^j, \quad h \to 0, \tag{2.8}$$

in the space of such functions, uniformly in ε, similarly to (2.5). The semiclassical principal symbol $p_{0,\varepsilon}$, defined on T^*M, takes the form

$$p_{0,\varepsilon}(x,\xi) = \sum_{|\alpha|\le m} a_{\alpha,\varepsilon,0}(x)\xi^\alpha, \tag{2.9}$$

if (x,ξ) are the canonical coordinates on T^*M. We make the ellipticity assumption,

$$|p_{0,\varepsilon}(x,\xi)| \ge \frac{1}{C}\langle\xi\rangle^m, \quad (x,\xi)\in T^*M, \quad |\xi| \ge C, \tag{2.10}$$

for some large $C > 0$. Here we assume that M has been equipped with some real analytic Riemannian metric so that $|\xi|$ and $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$ are well-defined.

Remark. The restriction to differential operators with analytic coefficients in the compact manifold case above is principally made in order to avoid developing a global discussion of semiclassical analytic pseudodifferential operators on a compact real analytic manifold. See also [31].

Sometimes, we write p_ε for $p_{0,\varepsilon}$ and simply p for $p_{0,0}$. We make the assumption that

$$P_{\varepsilon=0} \text{ is formally selfadjoint.}$$

When M is compact, we let the underlying Hilbert space be $L^2(M,\mu(dx))$ where $\mu(dx)$ is the Riemannian volume element.

The assumptions above imply that the spectrum of P_ε in a fixed neighborhood of $0 \in \mathbb{C}$ is discrete when $0 < h \le h_0$, $0 \le \varepsilon \le \varepsilon_0$, with $h_0 > 0$, $\varepsilon_0 > 0$ sufficiently small. Moreover, if $z \in \mathrm{neigh}(0,\mathbb{C})$ is an eigenvalue of P_ε then $\mathrm{Im}\, z = \mathcal{O}(\epsilon)$.

We furthermore assume that the real energy surface $p^{-1}(0)\cap T^*M$ is connected and that

$$dp \ne 0 \quad \text{along } p^{-1}(0)\cap T^*M.$$

In what follows we shall write

$$p_\varepsilon = p + i\varepsilon q + \mathcal{O}(\varepsilon^2) \tag{2.11}$$

in a neighborhood of $p^{-1}(0)\cap T^*M$, and for simplicity we shall assume throughout the paper that q is real-valued on the real domain. (In the general case, we should simply replace q below by $\mathrm{Re}\, q$.) We let $H_p = p'_\xi\cdot\partial_x - p'_x\cdot\partial_\xi$ be the Hamilton vector field of p.

2.2. Assumptions related to complete integrability

As in [16], [14], let us assume that there exists an analytic real-valued function f near $p^{-1}(0) \cap T^*M$ such that $H_p f = 0$, with the differentials df and dp being linearly independent on an open and dense set $\subset$ neigh$(p^{-1}(0) \cap T^*M, T^*M)$. For each $E \in$ neigh$(0, \mathbb{R})$, the level sets $\Lambda_{a,E} = f^{-1}(a) \cap p^{-1}(E) \cap T^*M$ are invariant under the H_p-flow and form a singular foliation of the 3-dimensional hypersurface $p^{-1}(E) \cap T^*M$. At each regular point (i.e. non-critical point for the restriction of f to $p^{-1}(E)$), the leaves of this foliation are two-dimensional analytic Lagrangian submanifolds, and each regular leaf is a finite union of tori. In what follows we shall use the word "leaf" and notation Λ for a connected component of some $\Lambda_{a,E}$. Let J be the set of all leaves in $p^{-1}(0) \cap T^*M$. Then we have a disjoint union decomposition

$$p^{-1}(0) \cap T^*M = \bigsqcup_{\Lambda \in J} \Lambda, \tag{2.12}$$

where Λ are compact connected H_p-invariant sets. The set J has a natural structure of a graph whose edges correspond to families of regular leaves and the set S of vertices is composed of singular leaves. The union of edges $J \setminus S$ possesses a natural real analytic structure and the corresponding tori depend analytically on $\Lambda \in J \setminus S$ with respect to that structure. See [14, Section 7] for an explicit description of the Lagrangian foliation when M is an analytic surface of revolution in $\mathbb{R}^3$.

In what follows, we shall assume that the graph J is finite. We shall identify each edge of J analytically with a real bounded interval, and this determines a distance on J in the natural way. Assume that the following continuity property holds:

$$\begin{aligned}&\text{For every } \Lambda_0 \in J \text{ and } \varepsilon > 0, \text{ there exists } \delta > 0 \text{ such that if } \Lambda \in J \\ &\text{and } \mathrm{dist}_J(\Lambda, \Lambda_0) < \delta, \text{ then } \Lambda \subset \{\rho \in p^{-1}(0) \cap T^*M;\ \mathrm{dist}(\rho, \Lambda_0) < \varepsilon\}.\end{aligned} \tag{2.13}$$

Remark. Let us assume that f is a Morse–Bott function when restricted to $p^{-1}(0) \cap T^*M$, in the sense that the set of critical points of the restriction of f to $p^{-1}(0) \cap T^*M$ is a disjoint union of connected submanifolds, with the transversal Hessian of f being non-degenerate along each of the submanifolds. In this case, the structure of the singular leaves is known [37]. The set J is then a finite connected graph and the property (2.13) holds.

Each torus $\Lambda \in J \setminus S$ carries real analytic coordinates (x_1, x_2), identifying Λ with $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$, so that along Λ, we have

$$H_p = a_1 \partial_{x_1} + a_2 \partial_{x_2}, \tag{2.14}$$

where $a_1, a_2 \in \mathbb{R}$. The rotation number is defined as the ratio

$$\omega(\Lambda) = [a_1 : a_2] \in \mathbb{RP}^1,$$

and it depends analytically on $\Lambda \in J \setminus S$. We say that the torus Λ is rational/irrational if a_1/a_2 has the corresponding property. While the rotation number of Λ depends on the choice of the coordinates (x_1, x_2) on the torus, the fundamental property of being rational/irrational is independent of this choice [17]. Recall also that the leading perturbation q has been introduced in (2.11). For each torus $\Lambda \in J \setminus S$, we define the torus average $\langle q\rangle(\Lambda)$ obtained by integrating $q|_\Lambda$ with respect to the natural smooth measure on Λ.

We introduce the time averages

$$\langle q\rangle_T = \frac{1}{T}\int_{-T/2}^{T/2} q \circ \exp(tH_p)\, dt, \qquad T > 0, \tag{2.15}$$

and consider the compact intervals $Q_\infty(\Lambda) \subset \mathbb{R}$, $\Lambda \in J$, defined as in [16],

$$Q_\infty(\Lambda) = \Big[\lim_{T\to\infty} \inf_{\Lambda} \langle q\rangle_T, \lim_{T\to\infty} \sup_{\Lambda} \langle q\rangle_T\Big]. \tag{2.16}$$

Notice that when $\Lambda \in J \setminus S$ and $\omega(\Lambda) \notin \mathbb{Q}$ then $Q_\infty(\Lambda) = \{\langle q\rangle(\Lambda)\}$. In the rational case, we write $\omega(\Lambda) = m/n$, where $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ are relatively prime. When $k(\omega(\Lambda)) := |m| + |n|$ is the height of $\omega(\Lambda)$, we recall from [16, Proposition 7.1] that

$$Q_\infty(\Lambda) \subset \langle q\rangle(\Lambda) + \mathcal{O}\left(\frac{1}{k(\omega(\Lambda))^\infty}\right)[-1, 1]. \tag{2.17}$$

Remark. As $J \setminus S \ni \Lambda \to \Lambda_0 \in S$, the set of all accumulation points of $\langle q\rangle(\Lambda)$ is contained in the interval $Q_\infty(\Lambda_0)$. See the related remark in [14, Section 2].

From [16, Theorem 7.6] we recall that

$$\frac{1}{\varepsilon}\,\mathrm{Im}\big(\mathrm{Spec}(P_\varepsilon) \cap \{z;\ |\mathrm{Re}\, z| \le \delta\}\big) \subset \Big[\inf \bigcup_{\Lambda\in J} Q_\infty(\Lambda) - o(1), \sup \bigcup_{\Lambda\in J} Q_\infty(\Lambda) + o(1)\Big] \tag{2.18}$$

as $(\varepsilon, h, \delta) \to 0$.

2.3. *The main result*

Let $\Lambda_0 \in J \setminus S$ be a rational invariant Lagrangian torus, so that as above, $\omega_0 := \omega(\Lambda_0) = m/n \in \mathbb{Q}$, and assume, as we may, that $n \neq 0$. Assume that the isoenergetic condition holds:

$$(d_\Lambda \omega)(\Lambda_0) \neq 0. \tag{2.19}$$

We recall from [14, Section 2] the behavior of the interval $Q_\infty(\Lambda)$ when $\Lambda \neq \Lambda_0$ is a rational torus in a neighborhood of Λ_0. Writing $\omega(\Lambda) = p/q$ where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ are relatively prime, $p = \mathcal{O}(q)$, we get, using the fact that $\omega(\Lambda) \neq \omega_0$,

$$|\omega(\Lambda) - \omega_0| \ge \frac{1}{nq} \ge \frac{1}{nk(\omega(\Lambda))}, \tag{2.20}$$

and therefore, in view of (2.17),

$$Q_\infty(\Lambda) \subset \langle q\rangle(\Lambda) + \mathcal{O}\big(\mathrm{dist}(\omega(\Lambda), \omega_0)^\infty\big)[-1, 1]. \tag{2.21}$$

This estimate is uniform in ω_0 provided that we have a uniform upper bound on the height of the rotation number $\omega_0 \in \mathbb{Q}$.

Let us assume that the chosen rational torus Λ_0 is such that

$$\inf Q_\infty(\Lambda_0) < \inf_{\Lambda \in J \setminus \{\Lambda_0\}} \inf Q_\infty(\Lambda). \tag{2.22}$$

The result below remains valid, with the obvious modifications, if we replace (2.22) by the assumption

$$\sup Q_\infty(\Lambda_0) > \sup_{\Lambda \in J \setminus \{\Lambda_0\}} \sup Q_\infty(\Lambda). \tag{2.23}$$

Applying a suitable linear transformation in $\mathrm{GL}(2, \mathbb{Z})$, we may choose action-angle coordinates (x, ξ) near Λ_0, so that Λ_0 is given by $\{\xi = 0\}$ in $\mathbb{T}_x^2 \times \mathbb{R}_\xi^2$, $p = p(\xi)$, and

$$p(0) = 0, \quad \partial_{\xi_1} p(0) = 0, \quad \partial_{\xi_2} p(0) > 0, \quad \partial_{\xi_1}^2 p(0) \neq 0. \tag{2.24}$$

Here the last property follows from (2.19), and in order to fix ideas, we shall assume that $\partial_{\xi_1}^2 p(0) > 0$. By the implicit function theorem we have

$$\partial_{\xi_1} p(\xi) = 0 \iff \xi_1 = f(\xi_2), \tag{2.25}$$

where f is an analytic function with $f(0) = 0$, and we obtain an analytic family of rational Lagrangian tori $\Lambda_E \subset p^{-1}(E)$, $E \in \mathrm{neigh}(0, \mathbb{R})$, given by

$$\xi_2 = \xi_2(E), \quad \xi_1 = f(\xi_2(E)). \tag{2.26}$$

Here $\xi_2 = \xi_2(E)$ is the unique smooth solution of the equation $p(f(\xi_2), \xi_2) = E$, close to 0, such that $\xi_2(0) = 0$.

Writing $q = q(x, \xi)$ in terms of the action-angle coordinates (x, ξ), let

$$\langle q \rangle_2(x_1, \xi) = \frac{1}{2\pi} \int_0^{2\pi} q(x, \xi)\, dx_2, \quad \xi \in \mathrm{neigh}(0, \mathbb{R}^2), \tag{2.27}$$

be the average of q with respect to x_2. We assume that

$$\mathbb{T} \ni x_1 \mapsto \langle q \rangle_2(x_1, 0) \text{ has a unique minimum which is non-degenerate.} \tag{2.28}$$

In order to give an invariant description of the assumption (2.28), notice that when restricted to Λ_0, the Hamilton flow of p is periodic of primitive period $T_0 > 0$ and the average $\langle q \rangle_2(x_1, 0)$ can naturally be viewed as the flow average $\langle q \rangle_{T_0}$, defined as in (2.15), considered as a function on the space of closed H_p-orbits in Λ_0,

$$\Lambda_0 / \exp(\mathbb{R} H_p) \simeq \mathbb{T}.$$

In its invariant formulation, the assumption (2.28) therefore states that the flow average $\langle q \rangle_{T_0}$, viewed as a function on $\Lambda_0/\exp(\mathbb{R} H_p)$, has a unique minimum which is non-degenerate.

It follows from (2.28) that the function $\mathbb{T} \ni x_1 \mapsto \langle q \rangle_2(x_1, \xi)$ has a unique minimum $x_1 = x_1(\xi)$ which is non-degenerate for $\xi \in \mathrm{neigh}(0, \mathbb{R}^2)$. The range of $\langle q \rangle_2(\cdot, 0)$ is equal to $Q_\infty(\Lambda_0)$, so the minimal value $\langle q\rangle_2(x_1(0),0) = \inf Q_\infty(\Lambda_0)$ is situated strictly below $\inf_{\Lambda\in J\setminus\{\Lambda_0\}} \inf Q_\infty(\Lambda)$.

In this paper, we shall work under the assumption that the subprincipal symbol of the unperturbed operator $P_{\varepsilon=0}$ vanishes,

$$p_{1,0}(x,\xi) = 0. \tag{2.29}$$

Here in the compact manifold case, in order to have an invariant definition of the subprincipal symbol of $P_{\varepsilon=0}$, we choose the local coordinates in (2.7) so that the Riemannian volume element agrees with the Lebesgue measure [35].

The following is the main result of this work.

Theorem 2.1. *We adopt the assumptions above, in particular,* (2.19)*,* (2.22)*,* (2.28)*, and* (2.29)*. Set $x_1(\xi_2) = x_1(f(\xi_2),\xi_2)$. Fix $\delta \in (1/18, 1/9)$ and assume that*

$$h^{1/(1-\delta)} \ll \varepsilon \ll h^{6/(5+12\delta)}. \tag{2.30}$$

Set

$$\widetilde{h} = h/\sqrt{\varepsilon}.$$

Then for every $C_0 > 0$, we have the following description of the eigenvalues of P_ε in the region

$$\left\{z \in \mathbb{C};\ |\mathrm{Re}\, z| < \frac{h}{C_0\sqrt{\varepsilon}},\ \frac{\mathrm{Im}\, z}{\varepsilon} \le \inf Q_\infty(\Lambda_0) + C_0\frac{h}{\sqrt{\varepsilon}}\right\}, \tag{2.31}$$

valid for all $h > 0$ small enough: the eigenvalues are simple and given modulo $\mathcal{O}(h^\infty)$ by

$$\begin{aligned}\lambda_{j,k} = {} & p(f(h(j-\theta_2)),h(j-\theta_2)) + i\varepsilon\langle q\rangle_2\big(x_1(h(j-\theta_2)),\, f(h(j-\theta_2)),h(j-\theta_2)\big)\\ & + \sqrt{\varepsilon}\, h(\lambda^0_{j,k} + \lambda^1_{j,k}\widetilde{h} + \lambda^2_{j,k}\widetilde{h}^2 + \cdots)\end{aligned} \tag{2.32}$$

with $j\in\mathbb{Z}$, $h(j-\theta_2) = \mathcal{O}(h/\sqrt{\varepsilon})$, $\mathbb{N}\ni k \le \mathcal{O}(1)$, where $\lambda^\nu_{j,k} = \lambda^\nu_k(h(j-\theta_2),\sqrt{\varepsilon})$ is a smooth function of $\xi_2 = h(j-\theta_2) \in \mathrm{neigh}(0,\mathbb{R})$ and $\sqrt{\varepsilon}\in \mathrm{neigh}(0,\overline{\mathbb{R}_+})$, and

$$\lambda^0_k(\xi_2,0) = e^{i\pi/4}(\partial^2_{\xi_1} p(f(\xi_2),\xi_2))^{1/2}\big(\partial^2_{x_1}\langle q\rangle_2(x_1(\xi_2),f(\xi_2),\xi_2)\big)^{1/2}(k+1/2). \tag{2.33}$$

Here we have written $\theta_2 = k_0(\alpha_2)/4 + S_2/(2\pi)h$, where $k_0(\alpha_2)$ and S_2 are the Maslov index and the classical action, respectively, of the fundamental cycle in Λ_0 given by a closed H_p-trajectory of minimal period.

Remark. In this paper, in the spirit of the previous works [11]–[16] in this series, we work under the analyticity assumptions, which seem essential when $\varepsilon \gg h\log(1/h)$ [4]. It may therefore be of some interest to verify that Theorem 2.1 allows us to reach some such values. When doing so, let us choose $\delta\in(1/18,1/9)$ in Theorem 2.1 to be close to $1/9$. We then see from (2.30) that the description of the eigenvalues in Theorem 2.1 in the region (2.31) is valid in the range

$$h^{9/8-\eta} \ll \varepsilon \ll h^{18/19+\eta},$$

when $\eta > 0$ is small, thus including some cases when $\varepsilon \gg h$. See also the discussion at the end of Section 3 below, where the conjectural optimal range for the perturbation parameter ε is given in (3.19).

Remark. The result of Theorem 2.1 admits a natural extension to the case when $\operatorname{Re} z \in \operatorname{neigh}(0,\mathbb{R})$ varies in a sufficiently small but fixed neighborhood of $0 \in \mathbb{R}$. Indeed, let us recall the family of rational Lagrangian tori $\Lambda_E \subset p^{-1}(E)$, $E \in \operatorname{neigh}(0,\mathbb{R})$, introduced in (2.26). A natural analog of the assumption (2.22) is then valid for $\inf Q_\infty(\Lambda_E)$, relative to the Lagrangian foliation in $p^{-1}(E)$, provided that $|E|$ is small enough. It follows therefore from Theorem 2.1 that the description (2.32) of the spectrum of P_ε remains valid when

$$|\operatorname{Re} z - E| \leq \frac{h}{C_0\sqrt{\varepsilon}}, \qquad \frac{\operatorname{Im} z}{\varepsilon} \leq \inf Q_\infty(\Lambda_E) + C_0\frac{h}{\sqrt{\varepsilon}},$$

uniformly in $E \in \operatorname{neigh}(0,\mathbb{R})$. We conclude therefore that the result of Theorem 2.1 extends to the spectral region

$$\left\{ z \in \mathbb{C};\ |\operatorname{Re} z| < \frac{1}{C}, \frac{\operatorname{Im} z}{\varepsilon} \leq \inf Q_\infty(\Lambda_{\operatorname{Re} z}) + \mathcal{O}\left(\frac{h}{\sqrt{\varepsilon}}\right)\right\},$$

for $C > 1$ large enough.

The plan of the paper is as follows. Section 3 is devoted to a general outline of the proof. In Section 4 we construct a global compactly supported weight function G such that the leading symbol of P_ε, acting on the weighted space associated to G, becomes $\approx p + i\varepsilon(q - H_p G)$, with the imaginary part avoiding the value $\varepsilon \inf Q_\infty(\Lambda_0)$ on $p^{-1}(0)$, away from the rational torus Λ_0. This effectively microlocalizes the spectral problem for P_ε to a small neighborhood of Λ_0. The quantum normal form construction for P_ε in the rational region is carried out in Section 5, using the techniques of secular perturbation theory, thereby reducing the analysis to the study of a one-parameter family of non-selfadjoint operators in dimension one, having double characteristics with elliptic quadratic approximations. In Section 6 we recall the computation of low-lying eigenvalues for such operators, following [8] and [10], and extend the results there to the parameter-dependent case. The final step in the proof of Theorem 2.1 is taken in Section 7, where we carry out a pseudospectral analysis for the family of the one-dimensional operators in question, controlling the resolvent norms and obtaining the spectral localization. It then becomes possible to complete the proof by solving a suitable globally well-posed Grushin problem for P_ε in a weighted space, using the ideas and techniques of [11], [16]. In Section 8 we present the results of numerical computations illustrating Theorem 2.1. The Appendix establishes some subelliptic resolvent bounds for non-selfadjoint operators of Schrödinger type, playing a principal role in the pseudospectral analysis of Section 7 in the main text. These bounds seem to be of some independent interest, and their proofs are very much based on the techniques developed in [8], [10].

3. Outline of the proof

In this section we shall give a general outline of the proof of Theorem 2.1. Some of the techniques come from the previous works [16], [14], and the presentation below will naturally focus on the new difficulties of pseudospectral nature, encountered in the analysis in the rational region. We shall then also describe heuristically some of the essential ideas employed in overcoming those difficulties, referring to Section 7 and to the Appendix for a detailed rigorous discussion.

The principal symbol of the operator P_ε in (2.1), (2.7) is of the form

$$p_\varepsilon = p + i\varepsilon q + \mathcal{O}(\varepsilon^2) \tag{3.1}$$

in a neighborhood of $p^{-1}(0) \cap T^*M$, and thanks to the ellipticity assumptions (2.6), (2.10) it suffices to make a microlocal study in the region where p is small. Recalling the assumption (2.22) and replacing q by $q - \inf Q_\infty(\Lambda_0)$, in the following discussion we shall assume, for notational simplicity only, that $\inf Q_\infty(\Lambda_0) = 0$. The first step in the argument is a construction of a global weight function $G \in C_0^\infty(T^*M)$ such that away from a small neighborhood of the rational Lagrangian torus Λ_0 in the region $p^{-1}([-E_0, E_0])$, $0 < E_0 \ll 1$, we have

$$q - H_p G \geq c_0 > 0. \tag{3.2}$$

Away from Λ_0, the weight G satisfies

$$H_p G = q - \langle q \rangle_T,$$

where $\langle q \rangle_T$ has been introduced in (2.15), and when constructing G in a neighborhood of Λ_0, we introduce action-angle coordinates $(x, \xi) \in T^*\mathbb{T}^2$ so that $\Lambda_0 = \{\xi = 0\} \subset T^*\mathbb{T}^2$ and

$$p_\varepsilon(x, \xi) = p(\xi) + i\varepsilon q(x, \xi) + \mathcal{O}(\varepsilon^2), \tag{3.3}$$

where the frequencies $\partial_{\xi_1} p(0)$ and $\partial_{\xi_2} p(0)$ are commensurable. After a linear change of variables, we get $\partial_{\xi_1} p(0) = 0$, and the isoenergetic condition (2.19) shows that $\partial_{\xi_1}^2 p(0) \neq 0$. In the following discussion, to fix ideas, we shall consider the model case $p(\xi) = \xi_2 + \xi_1^2$, which suffices to illustrate the difficulties. The weight function G near $\xi = 0$ satisfies the cohomological equation

$$H_p G = q - \widetilde{q} \tag{3.4}$$

modulo $\mathcal{O}(\xi^\infty)$, where $\widetilde{q} = \widetilde{q}(x_1, \xi)$ is independent of x_2 and is such that

$$\widetilde{q}(x_1, 0) = \frac{1}{2\pi} \int_0^{2\pi} q(x, 0)\, dx_2 \tag{3.5}$$

is the average of $q(x, 0)$ in the x_2-direction. From (2.28) we then know that $\widetilde{q}(x_1, 0) \geq 0$ and that $x_1 \mapsto \widetilde{q}(x_1, 0)$ has a unique minimum which is non-degenerate. The partial Birkhoff normal form construction utilized in solving (3.4) may be continued, first at the principal symbol level, and then at the level of operators, leading to the conclusion that microlocally in the rational region, when acting on an exponentially weighted space, the operator P_ε is unitarily equivalent to an operator of the form

$$P(x_1, hD_x, \varepsilon; h) + R(x, hD_x, \varepsilon; h) : L^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2). \tag{3.6}$$

We refer to Proposition 7.1 in Section 7 for the precise statement. Here the full symbol of $P(x_1, hD_{x_1}, \varepsilon; h)$ is independent of x_2 and is given by

$$P(x_1, \xi, \varepsilon; h) = p(\xi) + i\varepsilon\widetilde{q}(x_1, \xi) + \mathcal{O}(\varepsilon^2 + h^2). \tag{3.7}$$

The contribution $R(x, \xi, \varepsilon; h) = \mathcal{O}((\varepsilon, \xi, h)^\infty)$ in (3.6) is a remainder, which becomes $\mathcal{O}(h^\infty)$ when restricting attention to the region $\xi = \mathcal{O}(\varepsilon^\delta)$ for a suitable small fixed $\delta > 0$ — as we shall see, understanding this region suffices for the description of the eigenvalues in Theorem 2.1. In particular, since ξ becomes small, in the following heuristic discussion we shall make the simplifying assumption that $\widetilde{q}$ in (3.7) is independent of ξ altogether, depending on x_1 only. Let us also suppress the error term $\mathcal{O}(\varepsilon^2 + h^2)$ in (3.7), for simplicity. In Section 7, it will be treated entirely as a perturbation.

Taking a Fourier series decomposition in x_2, we may view the operator P in (3.6) as a one-parameter family of operators $P(x_1, hD_{x_1}, \xi_2, \varepsilon; h) = P(\xi_2)$, acting on $L^2(\mathbb{T})$, such that

$$P(\xi_2) = \xi_2 + L_\varepsilon, \quad \xi_2 = hj, \quad j \in \mathbb{Z}, \tag{3.8}$$

where

$$L_\varepsilon = (hD_{x_1})^2 + i\varepsilon\widetilde{q}(x_1), \quad \widetilde{q} \geq 0, \tag{3.9}$$

is a one-dimensional non-selfadjoint Schrödinger operator with $\varepsilon\widetilde{q}$ as a potential. We are interested in the spectrum of the family (3.8) in the region where $\operatorname{Re} z$ is small and $|\operatorname{Im} z| \leq \mathcal{O}(h\sqrt{\varepsilon})$, and the first observation is that the eigenvalues of the operator

$$\frac{1}{\varepsilon}L_\varepsilon = (\widetilde{h}D_{x_1})^2 + i\widetilde{q}(x_1), \quad \widetilde{h} = \frac{h}{\sqrt{\varepsilon}},$$

can be determined asymptotically in any disc $|w| < C\widetilde{h}$ by means of the harmonic approximation, provided that $\widetilde{h} \ll 1$. See [8], [10], and the discussion in Section 6 below. The eigenvalues of $\varepsilon^{-1}L_\varepsilon$ in this region are of the form

$$\mu_k(\widetilde{h}) \sim \sum_{j=0}^{\infty} \mu_{k,j}\widetilde{h}^{j+1}, \quad k \in \mathbb{N}, \tag{3.10}$$

where

$$\mu_{k,0} = (2\partial_{x_1}^2\widetilde{q}(x_1^{\min}))^{1/2}e^{i\pi/4}(k + 1/2) \tag{3.11}$$

are the eigenvalues of the globally elliptic quadratic operator

$$D_y^2 + \frac{i}{2}(\partial_{x_1}^2\widetilde{q}(x_1^{\min}))y^2$$

acting on $L^2(\mathbb{R})$ [3], [33]. Here $x_1^{\min} \in \mathbb{T}$ is the unique point such that $\widetilde{q}(x_1^{\min}) = 0$. The corresponding eigenvalues of $P(\xi_2)$ in (3.8) are given by $\xi_2 + \varepsilon\mu_k(\widetilde{h})$, and from [8], [10] we also know that

$$\|(P(\xi_2) - z)^{-1}\|_{\mathcal{L}(L^2, L^2)} \leq \mathcal{O}\left(\frac{1}{\sqrt{\varepsilon}\, h}\right), \tag{3.12}$$

provided that $|z-\xi_2| \le Ch\sqrt{\varepsilon}$ and $(z-\xi_2)/(h\sqrt{\varepsilon})$ avoids the quadratic eigenvalues $\mu_{k,0}$ in (3.11).

Now the direct sum decomposition (3.8) is only a microlocal approximation of P_ε, and in order to be able to absorb the error terms, when constructing the resolvent of P_ε globally it is of crucial importance to control the resolvent norms of L_ε also in the region $\operatorname{Re} z \in [Ch\sqrt{\varepsilon}, 1/\mathcal{O}(1))$, $|\operatorname{Im} z| \le \mathcal{O}(h\sqrt{\varepsilon})$. To get such a control, since the spectral parameter remains close to the boundary of the range of the symbol of L_ε, we apply the method of "bounded exponential weights", which in effect consists in replacing L_ε by a new operator for which the infimum of the imaginary part is increased in the non-elliptic region for $\operatorname{Re}(L_\varepsilon - \omega)$. This method has been carried out in closely related situations in [4], [8], [10], and we apply some of those works in the actual proof in the Appendix. Here we shall merely recall the essential ideas. See also [19], [27].

Let $G(x_1,\xi_1) \in C^\infty$ be real-valued and odd in ξ_1. Let us consider the formally conjugate operator

$$\widetilde{L}_\varepsilon = e^{-\varepsilon G(x_1,hD_{x_1})/h} \circ L_\varepsilon \circ e^{\varepsilon G(x_1,hD_{x_1})/h},$$

acting on L^2, or equivalently the operator L_ε acting on the weighted Hilbert space $e^{\varepsilon G(x_1,hD_{x_1})/h}L^2$. We want this space to be equal to L^2, with its norm

$$\|e^{-\varepsilon G(x_1,hD_{x_1})/h}u\|_{L^2}$$

uniformly equivalent to the standard L^2-norm. This is the case if the weight function G satisfies suitable symbol estimates and has the fundamental property

$$\varepsilon G(x_1,\xi_1)/h = \mathcal{O}(1), \tag{3.13}$$

uniformly with respect to the various parameters involved.

We view $e^{\varepsilon G(x_1,hD_{x_1})/h}$ as a Fourier integral operator with the associated canonical transformation $\exp(i\varepsilon H_G)$, approximately equal to $(x_1,\xi_1) \mapsto (x_1,\xi_1) + i\varepsilon H_G(x_1,\xi_1)$, since ε will be small. Here $H_G = G'_{\xi_1}\cdot\partial_{x_1} - G'_{x_1}\cdot\partial_{\xi_1}$ is the Hamilton vector field of G. By Egorov's theorem we expect $\widetilde{L}_\varepsilon$ to be an h-pseudodifferential operator with symbol

$$\begin{aligned}\widetilde{L}_\varepsilon(x_1,\xi_1) &\approx L_\varepsilon(\exp(i\varepsilon H_G(x_1,\xi_1))) \approx L_\varepsilon\big((x_1,\xi_1) + i\varepsilon H_G(L_\varepsilon)(x_1,\xi_1)\big)\\ &\approx L_\varepsilon(x_1,\xi_1) - i\varepsilon H_{L_\varepsilon}(G).\end{aligned}$$

Here $L_\varepsilon(x_1,\xi_1) = \xi_1^2 + i\varepsilon\widetilde{q}(x_1)$ is the symbol of L_ε in (3.9). With $\ell(x_1,\xi_1) = \xi_1^2$, we get

$$\widetilde{L}_\varepsilon(x_1,\xi_1) \approx \xi_1^2 + i\varepsilon(\widetilde{q} - H_\ell(G))(x_1,\xi_1) =: \xi_1^2 + i\varepsilon\widehat{q}(x_1,\xi_1).$$

When considering $\widetilde{L}_\varepsilon - \omega$ for $\operatorname{Re}\omega \ge h\sqrt{\varepsilon}$, the most critical region is the one where $\xi_1^2 \approx \operatorname{Re}\omega$, and it is here that we want to increase $\inf_{x_1}\widetilde{q}$ as much as possible. Naturally, that will not be enough for the complete analysis, but in the following heuristic discussion we shall restrict attention to the region where $\xi_1^2 = \operatorname{Re}\omega$. Here, we get

$$\widehat{q}(x_1,\xi_1) = \widetilde{q}(x_1) - 2\xi_1\partial_{x_1}G(x_1,\xi_1) = \widetilde{q}(x_1) - 2\sqrt{\operatorname{Re}\omega}\,\partial_{x_1}G(x_1,(\operatorname{Re}\omega)^{1/2}),$$

where we recall that G is odd in ξ_1, so that $\widehat{q}$ is even in the same variable. Then

$$\partial_{x_1} G(x_1) = \frac{\widetilde{q}(x_1) - \widehat{q}(x_1)}{2\sqrt{\operatorname{Re}\omega}},$$

omitting $\xi_1 = \sqrt{\operatorname{Re}\omega}$ in the argument of G. We want

$$\inf_{x_1} \widehat{q} - \inf_{x_1} \widetilde{q} \asymp \gamma^2 \tag{3.14}$$

for a suitable small parameter γ, which we wish to be as large as possible, and to achieve this, we clearly have to modify $\widetilde{q}$ in a γ-neighborhood of $x_1^{\min}$. Since we also wish $|G|$ to be as small as possible, we require

$$\operatorname{supp} G \subset [x_1^{\min} - \gamma, x_1^{\min} + \gamma],$$

and it is not hard to see that we can find such a G with

$$\partial_{x_1} G = \mathcal{O}\Big(\frac{\gamma^2}{\sqrt{\operatorname{Re}\omega}}\Big), \qquad G = \mathcal{O}\Big(\frac{\gamma^3}{\sqrt{\operatorname{Re}\omega}}\Big).$$

The condition (3.13) is fulfilled, provided that

$$\frac{\varepsilon\gamma^3}{h\sqrt{\operatorname{Re}\omega}} = \mathcal{O}(1) \ \Leftrightarrow\ \gamma = \mathcal{O}(1)\frac{h^{1/3}(\operatorname{Re}\omega)^{1/6}}{\varepsilon^{1/3}}. \tag{3.15}$$

Let $C \gg 1$ and choose

$$\gamma = \frac{1}{C}\min\Big(1, \frac{h^{1/3}(\operatorname{Re}\omega)^{1/6}}{\varepsilon^{1/3}}\Big). \tag{3.16}$$

It follows from the heuristic discussion above that in the region $h\sqrt{\varepsilon} \le \operatorname{Re}\omega \le 1/\mathcal{O}(1)$ we obtain the spectral gain

$$\varepsilon\gamma^2/\mathcal{O}(1) \asymp \min(\varepsilon, h^{2/3}(\operatorname{Re}\omega)^{1/3}\varepsilon^{1/3}) \ge h\sqrt{\varepsilon}, \tag{3.17}$$

in the sense that the resolvent $(L_\varepsilon - \omega)^{-1}$ is well defined in the region

$$h\sqrt{\varepsilon} \le \operatorname{Re}\omega \le 1/\mathcal{O}(1), \qquad \operatorname{Im}\omega \le \frac{1}{C}\min(h^{2/3}(\operatorname{Re}\omega)^{1/3}\varepsilon^{1/3}, \varepsilon),$$

and in that region,

$$\|(L_\varepsilon - \omega)^{-1}\|_{\mathcal{L}(L^2,L^2)} \le \frac{\mathcal{O}(1)}{\min(h^{2/3}\varepsilon^{1/3}(\operatorname{Re}\omega)^{1/3}, \varepsilon)}. \tag{3.18}$$

The resolvent estimates such as (3.18) are established in the Appendix, using the machinery of bounded exponential weights and relying on the techniques of [8], [10] — see Propositions A.2 and A.4 there, in particular. With the bounds (3.18) available, we get the corresponding pseudospectral control over the family $P(x_1, hD_{x_1}, \xi_2, \varepsilon; h)$ of (3.8) in the region where $|\mathrm{Re}\, z-\xi_2| \geq Ch\sqrt{\varepsilon}$, $\mathrm{Im}\, z \leq \mathcal{O}(h\sqrt{\varepsilon})$, and this allows us, eventually, to construct the resolvent of P_ε globally in this region. We therefore obtain some crucial spectral localization, making it possible to carry out the spectral analysis of P_ε working with one quantum number $\xi_2 = hj$ at a time, roughly speaking. A globally well-posed Grushin problem for P_ε is finally built from the corresponding one-dimensional Grushin problems for the operator L_ε in (3.9), and solving it along the same lines as in [11], [16], [8], we complete the proof of Theorem 2.1.

Remark. Our heuristic arguments seem to indicate that the optimal range for the perturbation parameter ε could be

$$h^2 \ll \varepsilon \ll h^{2/3}, \tag{3.19}$$

as we need $\widetilde{h} = h/\sqrt{\varepsilon} \ll 1$ and $\varepsilon \ll \widetilde{h}$. Due to many technicalities, we get a smaller range of values around $\varepsilon \approx h$, and leave the extension to the range (3.19) as an open problem for future work.

4. Secular reduction and the global weight

The purpose of this section is to construct a globally defined compactly supported weight function which will allow us to microlocalize the spectral problem for P_ε to a small neighborhood of the rational torus Λ_0. In doing so, we shall proceed similarly to [16], with the essential difference that when working near the torus, the basic cohomological equation will have quite different properties, compared to the Diophantine analysis of [16], and will be treated using the secular perturbation theory [21], [14]. The main result of this section is Proposition 4.2 below.

Let us keep all the assumptions of Section 2 and consider the operator P_ε with leading symbol p_ε in (2.11) in a neighborhood of $p^{-1}(0) \cap T^*M$. Let

$$\kappa_0 : \mathrm{neigh}(\Lambda_0, T^*M) \to \mathrm{neigh}(\xi = 0, T^*\mathbb{T}^2) \tag{4.1}$$

be a real analytic canonical transformation, given by the action-angle variables, such that the properties (2.24) hold for $p \circ \kappa_0^{-1}$, which we identify with p. By Taylor expansion and (2.24), we have

$$p(\xi) = p(f(\xi_2), \xi_2) + g(\xi)(\xi_1 - f(\xi_2))^2, \qquad g(0) > 0, \tag{4.2}$$

where f is the analytic function introduced in (2.25).

Implementing κ_0 in (4.1) by means of a microlocally unitary multi-valued h-Fourier integral operator with a real phase, as explained in [11, Theorem 2.4], and conjugating P_ε by this operator, we obtain a new h-pseudodifferential operator, still denoted by P_ε, defined microlocally near $\xi = 0$ in $T^*\mathbb{T}^2$. The full symbol of P_ε is holomorphic in a fixed complex neighborhood of $\xi = 0$, and the leading symbol is given by

$$p_\varepsilon(x, \xi) = p(\xi) + i\varepsilon q(x, \xi) + \mathcal{O}(\varepsilon^2) \tag{4.3}$$

with $p(\xi)$ of the form (4.2). The function q in (4.3) is real on the real domain. At the operator level, P_ε acts on the space of microlocally defined Floquet periodic functions on $\mathbb{T}^2$, $L^2_\theta(\mathbb{T}^2) \subset L^2_{\mathrm{loc}}(\mathbb{R}^2)$, elements u of which satisfy

$$u(x-\nu) = e^{i\theta\cdot\nu}u(x), \qquad \theta = \frac{S}{2\pi h} + \frac{k_0}{4}, \qquad \nu \in 2\pi\mathbb{Z}^2. \tag{4.4}$$

Here $S = (S_1, S_2)$ is given by the classical actions,

$$S_j = \int_{\alpha_j} \eta\, dy, \qquad j = 1, 2,$$

with α_j forming a system of fundamental cycles in Λ_0 such that

$$\kappa_0(\alpha_j) = \beta_j, \qquad j = 1, 2, \qquad \beta_j = \{x \in \mathbb{T}^2;\ x_{3-j} = 0\}.$$

The couple $k_0 = (k_0(\alpha_1), k_0(\alpha_2)) \in \mathbb{Z}^2$ stands for the Maslov indices of the cycles α_j, $j = 1, 2$.

Remark. Using (4.2), we see, using the implicit function theorem, that the energy surface $p(\xi) = E$, for $E \in \mathrm{neigh}(0, \mathbb{R})$, is given by

$$\xi_2 + \ell(\xi_1, E) = 0, \tag{4.5}$$

where ℓ is analytic with $\ell(\xi_1, 0) \sim \xi_1^2$ and $\ell'_E(0,0) < 0$.

Working near the zero section $\xi = 0$ in $T^*\mathbb{T}^2$ and following the method of normal forms [16], [14], we shall now discuss the cohomological equation

$$H_p G = q - \widetilde{q}, \tag{4.6}$$

where we want the remainder $\widetilde{q}$ to be simpler than q. Here we have

$$H_p = p'_\xi \cdot \partial_x,$$

and thus (4.6) can be written more explicitly as follows:

$$\partial_{\xi_2} p(\xi)\partial_{x_2} G + \partial_{\xi_1} p(\xi)\partial_{x_1} G = q - \widetilde{q}.$$

To simplify, we divide this equation by $\partial_{\xi_2} p$. Writing $u = G$, $v = (\partial_{\xi_2} p)^{-1} q$, $\widetilde{v} = (\partial_{\xi_2} p)^{-1}\widetilde{q}$, we get

$$(\partial_{x_2} + a(\xi)\partial_{x_1})u = v - \widetilde{v}, \tag{4.7}$$

where $a(\xi) = \partial_{\xi_1} p(\xi)/\partial_{\xi_2} p(\xi)$. To simplify further, we replace the variables ξ by

$$\eta = (\eta_1, \eta_2) = (\xi_1 - f(\xi_2), \xi_2), \tag{4.8}$$

and write, abusing the notation slightly, $u = u(x, \eta)$, $v = v(x, \eta)$, $\widetilde{v} = \widetilde{v}(x, \eta)$. It follows from (4.2) that the Taylor expansion of a has the form

$$a(\eta) = a_1(\eta_2)\eta_1 + a_2(\eta_2)\eta_1^2 + \cdots, \qquad a_1(0) \neq 0, \tag{4.9}$$

and let us Taylor expand u, v and $\widetilde{v}$ similarly,

$$\begin{aligned} u(x,\eta) &= \sum_{k=0}^{\infty} u_k(x,\eta_2)\eta_1^k, \\ v(x,\eta) &= \sum_{k=0}^{\infty} v_k(x,\eta_2)\eta_1^k, \\ \widetilde{v}(x,\eta) &= \sum_{k=0}^{\infty} \widetilde{v}_k(x,\eta_2)\eta_1^k. \end{aligned} \tag{4.10}$$

Inserting these equations into (4.7) and identifying the powers of η_1, we get

$$\partial_{x_2} u_0 = v_0 - \widetilde{v}_0, \tag{4.11}$$

$$\partial_{x_2} u_1 + a_1 \partial_{x_1} u_0 = v_1 - \widetilde{v}_1, \tag{4.12}$$

$$\partial_{x_2} u_2 + a_1 \partial_{x_1} u_1 + a_2 \partial_{x_1} u_0 = v_2 - \widetilde{v}_2, \tag{4.13}$$

and so on. The general equation is of the form

$$\partial_{x_2} u_k + a_1 \partial_{x_1} u_{k-1} + a_2 \partial_{x_1} u_{k-2} + \dots + a_k \partial_{x_1} u_0 = v_k - \widetilde{v}_k. \tag{4.14}$$

The parameter η_2 plays no essential role here and we sometimes suppress it from the notation. For a function u on the torus $\mathbb{T}^2$, we introduce its averages in x_k, $k = 1, 2$, and its total average by

$$\langle u \rangle_k (x_{3-k}) = \frac{1}{2\pi} \int_0^{2\pi} u(x_1, x_2)\, dx_k, \quad \langle\langle u \rangle\rangle = \langle\langle u \rangle_1\rangle_2 = \frac{1}{(2\pi)^2} \iint_{\mathbb{T}^2} u(x_1, x_2)\, dx_1\, dx_2.$$

Proposition 4.1. *Let $v_0, v_1, \ldots \in C^\infty(\mathbb{T}^2)$ be smooth functions on $\mathbb{T}^2$. A necessary and sufficient condition on $\widetilde{v}_0, \widetilde{v}_1, \ldots \in C^\infty(\mathbb{T}^2)$ for the existence of $u_0, u_1, \ldots \in C^\infty(\mathbb{T}^2)$ solving (4.11) and (4.14) for $k \geq 1$ is*

$$\langle \widetilde{v}_0 \rangle_2 = \langle v_0 \rangle_2, \quad \langle\langle \widetilde{v}_k \rangle\rangle = \langle\langle v_k \rangle\rangle, \quad k \geq 1. \tag{4.15}$$

Proof. The necessity of (4.15) follows from taking the x_2-mean of (4.11) and the total mean of (4.14).

Assume that the first equation in (4.15) holds, so that $\langle v_0 - \widetilde{v}_0 \rangle_2 = 0$. Then (4.11) has a solution $u_0 = u_0^0 \in C^\infty(\mathbb{T}^2)$ given by

$$u_0^0(x) = \int_0^{x_2} (v_0 - \widetilde{v}_0)(x_1, t)\, dt. \tag{4.16}$$

The general solution of (4.11) is of the form $u_0^0(x) + f_0(x_1)$, where $f_0(x_1)$ is any smooth periodic function.

We next consider (4.12) (i.e. (4.14) with $k = 1$), which we write as

$$\partial_{x_2} u_1 = v_1 - \widetilde{v}_1 - a_1 \partial_{x_1} u_0^0 - a_1 \partial_{x_1} f_0(x_1). \tag{4.17}$$

Here the total average of $v_1 - \widetilde{v}_1 - a_1 \partial_{x_1} u_0^0$ vanishes,

$$\langle\langle v_1 - \widetilde{v}_1 - a_1 \partial_{x_1} u_0^0 \rangle_2 \rangle_1 = 0,$$

and hence we can find a periodic smooth function $f_0(x_1)$, unique up to a constant, such that

$$\langle v_1 - \widetilde{v}_1 - a_1 \partial_{x_1} u_0^0 \rangle_2 - a_1 \partial_{x_1} f_0(x_1) = 0.$$

Equivalently,

$$\langle v_1 - \widetilde{v}_1 - a_1 \partial_{x_1} u_0^0 - a_1 \partial_{x_1} f_0(x_1) \rangle_2 = 0,$$

and we can therefore find a solution $u_1^0 \in C^\infty(\mathbb{T}^2)$ to (4.17), and hence to (4.12).

Assume by induction that we have found $u_0, u_1, \ldots, u_{k-1}$ solving (4.11) and (4.14) with k there replaced by $j = 1, \ldots, k-1$. We notice that the general solution of (4.14) with k replaced by $k-1$ is of the form $u_{k-1} = u_{k-1}^0 + f_{k-1}(x_1)$ for any smooth periodic function f_{k-1}. We rewrite (4.14) as

$$\partial_{x_2} u_k = w_k - a_1 \partial_{x_1} f_{k-1}(x_1), \tag{4.18}$$

where

$$w_k = v_k - \widetilde{v}_k - a_1 \partial_{x_1} u_{k-1}^0 - a_2 \partial_{x_1} u_{k-2} - \cdots - a_k \partial_{x_1} u_0,$$

and we notice that $\langle\langle w_k \rangle_2 \rangle_1 = \langle\langle w_k \rangle\rangle = 0$. Choose f_{k-1} such that $\langle w_k \rangle_2 = a_1 \partial_{x_1} f_{k-1}(x_1)$, or equivalently $\langle w_k - a_1 \partial_{x_1} f_{k-1} \rangle_2 = 0$. Then there is a smooth periodic solution $u_k = u_k^0$ to (4.18) and hence to (4.14). □

Remark. Observe that Proposition 4.1 has a natural extension to the real analytic category.

An application of Proposition 4.1 together with the remark above allows us to conclude that for any fixed $N \in \mathbb{N}$, there exists an analytic function G_0, defined in a fixed neighborhood of $\xi = 0$, such that

$$H_p G_0 = q - \widetilde{q} + \mathcal{O}((\xi_1 - f(\xi_2))^N) \tag{4.19}$$

for any analytic periodic function $\widetilde{q}$ which satisfies

$$\langle \widetilde{q}(\cdot, \xi) \rangle_2 = \langle q(\cdot, \xi) \rangle_2 \quad \text{when } \xi_1 = f(\xi_2) \tag{4.20}$$

and

$$\langle\langle \widetilde{q}(\cdot, \xi) \rangle\rangle = \langle\langle q(\cdot, \xi) \rangle\rangle, \quad \xi \in \text{neigh}(0, \mathbb{R}^2). \tag{4.21}$$

The following choice is convenient and will be made in what follows: Let $\chi : \mathbb{R} \to \mathbb{R}$ be real analytic such that $\chi(0) = 0$. We can then take

$$\widetilde{q}(x_1, \xi) = \big(1 - \chi(\xi_1 - f(\xi_2))\big) \langle q \rangle_2 (x_1, \xi) + \chi(\xi_1 - f(\xi_2)) \langle\langle q(\cdot, \xi) \rangle\rangle, \tag{4.22}$$

which is independent of x_2. Here $\langle q \rangle_2 (x_1, \xi)$ has been introduced in (2.27).

Let us now restrict attention to the energy surface $p^{-1}(0)$. According to (4.5), we have $p(\xi)=0 \Leftrightarrow \xi_2+\ell(\xi_1,0)=0$, $\ell(\xi_1,0)\sim \xi_1^2$. It follows from (4.22) that when $p(\xi)=0$, we may write

$$\inf_{x_1} \widetilde{q}(x_1,\xi)=(1-\psi(\xi_1))k(\xi_1)+\psi(\xi_1)g(\xi_1),$$

where

$$k(\xi_1)=\inf_{x_1}\langle q\rangle_2(x_1,\xi_1,-\ell(\xi_1,0))=\langle q\rangle_2\big(x_1(\xi_1,-\ell(\xi_1,0)),\xi_1,-\ell(\xi_1,0)\big),$$

$g(\xi_1)=\langle\langle q\rangle\rangle(\xi_1,-\ell(\xi_1,0))$, and ψ is an analytic function such that $\psi(0)=0$, $\psi'(0)=\chi'(0)$. We next compute

$$\begin{aligned}\partial_{\xi_1}\inf_{x_1}\widetilde{q}(x_1,\xi)&=k'(\xi_1)+\psi'(\xi_1)(g(\xi_1)-k(\xi_1))+\psi(\xi_1)(g'(\xi_1)-k'(\xi_1)),\\ \partial_{\xi_1}^2\inf_{x_1}\widetilde{q}(x_1,\xi)&=k''(\xi_1)+\psi''(\xi_1)(g(\xi_1)-k(\xi_1))+2\psi'(\xi_1)(g'(\xi_1)-k'(\xi_1))\\ &\quad+\psi(\xi_1)(g''(\xi_1)-k''(\xi_1)).\end{aligned}$$

Using the fact that

$$k(0)=\inf_{x_1\in\mathbb{T}}\langle q\rangle_2(x_1,0)=\inf Q_\infty(\Lambda_0)<\langle q\rangle(\Lambda_0)=\langle\langle q\rangle\rangle(0)=g(0),$$

we see that the derivatives $\chi'(0)$ and $\chi''(0)$ of the analytic function χ in (4.22) can be chosen so that when $p(\xi)=0$, we have

$$\inf_{x_1\in\mathbb{T}}\widetilde{q}(x_1,\xi)\geq\inf_{x_1\in\mathbb{T}}\langle q\rangle_2(x_1,0)+C\xi^2, \tag{4.23}$$

where the constant $C>0$ is large. In other words, for $\Lambda\in\operatorname{neigh}(\Lambda_0,J)$, we get

$$\inf_{\Lambda}\widetilde{q}\geq\inf Q_\infty(\Lambda_0)+\frac{1}{\mathcal{O}(1)}\operatorname{dist}(\Lambda,\Lambda_0)^2. \tag{4.24}$$

Remark. In the preceding discussion, we do not have to restrict ourselves to the energy surface $p^{-1}(0)$. Indeed, introducing the variables

$$\eta=(\eta_1,\eta_2)=(\xi_1-f(\xi_2),\xi_2)$$

as in (4.8), and repeating the computations above, we get

$$\inf_{x_1\in\mathbb{T}}\widetilde{q}(x_1,\xi)\geq\inf_{x_1\in\mathbb{T}}\langle q\rangle_2(x_1,f(\xi_2),\xi_2)+C(\xi_1-f(\xi_2))^2$$

when $p(\xi)=E$, for $E\in\operatorname{neigh}(0,\mathbb{R})$. Introducing the rational Lagrangian tori $\Lambda_E\subset p^{-1}(E)$ defined in (2.26), we therefore obtain, on $p^{-1}(E)$,

$$\inf_{x_1\in\mathbb{T}}\widetilde{q}(x_1,\xi)\geq\inf Q_\infty(\Lambda_E)+C_1\operatorname{dist}(\Lambda,\Lambda_E)^2. \tag{4.25}$$

We shall now construct a suitable global weight function. In doing so, let G_T be an analytic function defined in a neighborhood of $p^{-1}(0) \cap T^*M$ such that

$$H_p G_T = q - \langle q \rangle_T, \tag{4.26}$$

where

$$\langle q \rangle_T = \frac{1}{T} \int_{-T/2}^{T/2} q \circ \exp(t H_p)\, dt, \qquad T > 0,$$

has been introduced in (2.15). We refer to [16, Section 1] for the construction of an analytic solution of (4.26). An application of [16, Lemma 2.4] together with the assumption (2.22) allows us to conclude that outside an arbitrarily small neighborhood of Λ_0 in $p^{-1}(0) \cap T^*M$, we have

$$\inf(q - H_p G_T) \geq \inf Q_\infty(\Lambda_0) + 1/C_0, \tag{4.27}$$

provided that T is taken large enough. Here $C_0 > 0$ is independent of the neighborhood taken. In these considerations, we are allowed to vary the real energy a little, and we conclude that for any fixed neighborhood W of

$$\bigcup_{|E| \leq E_0} \Lambda_E, \qquad 0 < E_0 \ll 1, \tag{4.28}$$

in $p^{-1}([-E_0, E_0])$ there exists T large enough such that

$$\inf_{p^{-1}([-E_0,E_0])\backslash W} (q - H_p G_T) \geq \inf_{|E| \leq E_0} \inf Q_\infty(\Lambda_E) + 1/C_0. \tag{4.29}$$

Here $C_0 > 0$ is independent of the neighborhood chosen.

The global weight function will be obtained by gluing together the functions $G_T := G_T \circ \kappa_0^{-1}$ and G_0, both viewed as analytic functions defined in a neighborhood of the zero section $\xi = 0$ in $T^*\mathbb{T}^2$. Let $\psi = \psi(\xi) \in C^\infty(\mathrm{neigh}(0, \mathbb{R}^2); [0, 1])$ depend on ξ only, and assume that $\psi = 1$ near the rational region (4.28), and with support in a small neighborhood of that set. Set

$$G = (1 - \psi) G_T + \psi G_0. \tag{4.30}$$

It follows that

$$q - H_p G = \psi(q - H_p G_0) + (1 - \psi)\langle q \rangle_T. \tag{4.31}$$

In a neighborhood of the rational region (4.28), we have

$$q - H_p G = \widetilde{q} + \mathcal{O}((\xi_1 - f(\xi_2))^N)$$

with $\widetilde{q}$ given in (4.22), while further away from this set we have $q - H_p G = \langle q \rangle_T$. In order to understand the behavior of $\langle q \rangle_T$ near $\xi = 0$ for T large, we write

$$\langle q \rangle_T(x, \xi) = \frac{1}{T} \int_{-T/2}^{T/2} q(x + t p'(\xi), \xi)\, dt,$$

and expanding $q(\cdot,\xi)$ in a Fourier series, we obtain

$$\langle q\rangle_T(x,\xi)=\sum_{k=(k_1,k_2)\in\mathbb{Z}^2} e^{ik\cdot x}\widehat{q}(k,\xi)\widehat{K}(Tk\cdot p'(\xi)). \tag{4.32}$$

Here $\widehat{K}$ is the Fourier transform of the characteristic function K of $[-1/2,1/2]$. Let us decompose

$$\langle q\rangle_T(x,\xi)=\sum_{k_2\neq 0}\widehat{K}(Tp'(\xi)\cdot k)\widehat{q}(k,\xi)e^{ix\cdot k}+\sum_{k_2=0}\widehat{K}(Tp'(\xi)\cdot k)\widehat{q}(k,\xi)e^{ix\cdot k}=\mathrm{I}+\mathrm{II}, \tag{4.33}$$

with the natural definitions of I and II. When estimating I, we use (4.2) and notice that when $k_2\neq 0$, we have

$$|p'(\xi)\cdot k|\geq |p'_{\xi_2}k_2|-\mathcal{O}(1)|\xi_1-f(\xi_2)|\,|k_1|\geq 1-C|\xi_1-f(\xi_2)|\,|k|,\qquad C>0.$$

Here for notational simplicity we assume that the derivative of $\xi_2\mapsto p(f(\xi_2),\xi_2)$ is ≥ 1 near 0. It follows that

$$|p'(\xi)\cdot k|\geq 1/2,$$

provided that $2C|\xi_1-f(\xi_2)|\,|k|\leq 1$. Let now $0\leq\chi\in C_0^\infty((-1,1))$ be such that $\chi=1$ on $[-1/2,1/2]$ and write

$$\begin{aligned}
\mathrm{I}&=\sum_{k_2\neq 0}\chi(2C|\xi_1-f(\xi_2)|\,|k|)\widehat{K}(Tp'(\xi)\cdot k)\widehat{q}(k,\xi)e^{ix\cdot k}\\
&\quad+\sum_{k_2\neq 0}\big(1-\chi(2C|\xi_1-f(\xi_2)|\,|k|)\big)\widehat{K}(Tp'(\xi)\cdot k)\widehat{q}(k,\xi)e^{ix\cdot k}\\
&=\sum_{k_2\neq 0}\chi(2C|\xi_1-f(\xi_2)|\,|k|)\mathcal{O}\left(\frac{1}{T|p'(\xi)\cdot k|}\right)\widehat{q}(k,\xi)e^{ix\cdot k}\\
&\quad+\sum_{k_2\neq 0}\big(1-\chi(2C|\xi_1-f(\xi_2)|\,|k|)\big)\widehat{K}(Tp'(\xi)\cdot k)\widehat{q}(k,\xi)e^{ix\cdot k}.
\end{aligned} \tag{4.34}$$

It is now easy to see, using the smoothness of q, that

$$\mathrm{I}=\mathcal{O}(1/T+|\xi_1-f(\xi_2)|^\infty),\qquad T\geq 1. \tag{4.35}$$

When considering the contribution coming from II, we notice that

$$\mathrm{II}=\langle q\rangle_2(x_1,\xi)+\sum_{k_2=0,\,k_1\neq 0}(\widehat{K}(Tp'_{\xi_1}k_1)-1)e^{ix_1k_1}\widehat{q}(k,\xi). \tag{4.36}$$

Here $|p'_{\xi_1}|\sim|\xi_1-f(\xi_2)|$, in view of (4.2), and we conclude that in the rational region where $\xi_1=f(\xi_2)$, $\xi_2\in\mathrm{neigh}(0,\mathbb{R})$, we get

$$\langle q\rangle_T(x,\xi)=\langle q\rangle_2(x_1,\xi)+\mathcal{O}(1/T).$$

Away from the rational region $\xi_1=f(\xi_2)$, we see directly from (4.33) that II converges to the torus average $\langle\langle q\rangle\rangle(\xi)$ as $T\to\infty$.

Combining the equations and estimates (4.19), (4.24), (4.26), (4.29), and (4.30), we may summarize the discussion above in the following proposition.

Proposition 4.2. *Let us make the assumption* (2.22). *Let G_0 be an analytic solution near $\xi = 0$ of the equation* (4.6) *with $\widetilde{q}$ being of the form* (4.22) *modulo $\mathcal{O}((\xi_1 - f(\xi_2))^N)$ for some fixed N large enough. There exists a real-valued function $G \in C_0^\infty(T^*M)$ such that $G = G_0 \circ \kappa_0$ in a neighborhood of Λ_0, and away from a small neighborhood of Λ_0 in the region $p^{-1}([-E_0, E_0])$, $0 < E_0 \ll 1$, we have*

$$q - H_p G \geq \inf Q_\infty(\Lambda_0) + 1/C_0, \quad C_0 > 0. \tag{4.37}$$

When $\Lambda \subset p^{-1}(0)$, $\Lambda \in \mathrm{neigh}(\Lambda_0, J)$, we have furthermore

$$\inf_\Lambda (q - H_p G) \geq \inf Q_\infty(\Lambda_0) + \frac{1}{C}\mathrm{dist}(\Lambda, \Lambda_0)^2.$$

Associated to the weight function G defined in Proposition 4.2, we shall now introduce a suitable small but globally defined deformation of the real phase space T^*M into the complex domain. When doing so, let $\widetilde{M}$ be a complexification of M, and let $\widetilde{G} \in C_0^\infty(T^*\widetilde{M})$ be an almost holomorphic extension of G. Set

$$\Lambda_{\varepsilon G} = \exp(\varepsilon H^{\mathrm{Im}\,\sigma}_{\mathrm{Re}\,\widetilde{G}})(T^*M) \subset T^*\widetilde{M}. \tag{4.38}$$

Here σ is the complex symplectic (2, 0)-form on $T^*\widetilde{M}$, and $H^{\mathrm{Im}\,\sigma}_{\mathrm{Re}\,\widetilde{G}}$ is the Hamilton vector field of $\mathrm{Re}\,\widetilde{G}$ computed with respect to the real symplectic form $\mathrm{Im}\,\sigma$ on $T^*\widetilde{M}$. It follows that the manifold $\Lambda_{\varepsilon G}$ is I-Lagrangian, and being a small deformation of T^*M, it is also R-symplectic, i.e. an IR-manifold. From [24] and [29], we recall the general relation

$$\widehat{i\varepsilon H_{\widetilde{G}}} = \varepsilon H^{\mathrm{Im}\,\sigma}_{\mathrm{Re}\,\widetilde{G}},$$

valid to infinite order along the real domain T^*M. Here $\widehat{i\varepsilon H_{\widetilde{G}}}$ stands for the real vector field in $T^*\widetilde{M}$, naturally associated to the complex (1, 0) vector field

$$i\varepsilon H_{\widetilde{G}} = i\varepsilon \sum_{j=1}^{2} \left(\frac{\partial \widetilde{G}}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial \widetilde{G}}{\partial x_j} \frac{\partial}{\partial \xi_j} \right).$$

It follows that in the region where G is analytic, including a sufficiently small but fixed neighborhood of Λ_0, we have

$$\Lambda_{\varepsilon G} = \exp(i\varepsilon H_G)(T^*M),$$

where we write G also for the holomorphic extension and recall that $\exp(i\varepsilon H_G)$ is a holomorphic canonical transformation.

Associated to the IR-manifold $\Lambda_{\varepsilon G}$ is the microlocally exponentially weighted Hilbert space $H(\Lambda_{\varepsilon G})$, defined using the FBI-Bargmann approach, by modifying the exponential weight on the FBI transform side. We refer to [25], [15] for the detailed definition of $H(\Lambda_{\varepsilon G})$ for $M = \mathbb{R}^2$, and to [31] and the Appendix of [11] for the case when M is compact. Following [24], [25], [31], let us introduce a microlocally unitary h-Fourier integral operator

$$U_G : L^2(M) \to H(\Lambda_{\varepsilon G}), \tag{4.39}$$

defined microlocally near $p^{-1}(0) \cap T^*M$ and associated to a suitable canonical transformation

$$\kappa_G : \mathrm{neigh}(p^{-1}(0), T^*M) \to \mathrm{neigh}(p^{-1}(0), \Lambda_{\varepsilon G})$$

such that $\kappa_G = \exp(i\varepsilon H_G)$ near Λ_0. It follows that the operator

$$P_\varepsilon : H(\Lambda_{\varepsilon G}) \to H(\Lambda_{\varepsilon G}) \tag{4.40}$$

is microlocally near $p^{-1}(0)$ unitarily equivalent to the conjugate operator

$$U_G^{-1} P_\varepsilon U_G : L^2 \to L^2$$

with leading symbol

$$p_\varepsilon|_{\Lambda_{\varepsilon G}} \simeq p + i\varepsilon(q - H_p G) + \mathcal{O}(\varepsilon^2).$$

Letting

$$U_0 : L^2(M) \to L^2_\theta(\mathbb{T}^2)$$

be the semiclassical microlocally unitary Fourier integral operator with a real phase associated to the canonical transformation κ_0 in (4.1), and using the operator $U_0 U_G^{-1}$ associated to the canonical transformation

$$\kappa_0 \circ \kappa_G^{-1} : \mathrm{neigh}(\exp(i\varepsilon H_G)(\Lambda_0), \Lambda_{\varepsilon G}) \to \mathrm{neigh}(\xi = 0, T^*\mathbb{T}^2),$$

we find that microlocally near the Lagrangian torus $\exp(i\varepsilon H_G)(\Lambda_0) \subset \Lambda_{\varepsilon G}$, the operator in (4.40) is unitarily equivalent to an operator $\widetilde{P}_\varepsilon$, acting on $L^2_\theta(\mathbb{T}^2)$, defined microlocally near $\xi = 0$ in $T^*\mathbb{T}^2$ by

$$\widetilde{P}_\varepsilon \sim \sum_{\nu=0}^{\infty} h^\nu \widetilde{p}_\nu(x, \xi, \varepsilon). \tag{4.41}$$

Here $\widetilde{p}_\nu$ are holomorphic functions in a fixed complex neighborhood of $\xi = 0$, smooth in $\varepsilon \in \mathrm{neigh}(0, \mathbb{R})$, and

$$\widetilde{p}_0 = p(\xi) + i\varepsilon\widetilde{q}(x_1, \xi) + \mathcal{O}(\varepsilon^2) + \varepsilon\mathcal{O}((\xi_1 - f(\xi_2))^N) \tag{4.42}$$

with $\widetilde{q}(x_1, \xi)$ independent of x_2 and of the form (4.22). Furthermore, the assumption (2.29) implies that

$$\widetilde{p}_1(x, \xi, \varepsilon) = \mathcal{O}(\varepsilon).$$

We may illustrate the microlocal unitary equivalence above by the commutative diagram

$$\begin{array}{ccc} H(\Lambda_{\varepsilon G}) & \xrightarrow{P_\varepsilon} & H(\Lambda_{\varepsilon G}) \\ {\scriptstyle U_0 U_G^{-1}}\big\downarrow & & \big\downarrow{\scriptstyle U_0 U_G^{-1}} \\ L^2_\theta(\mathbb{T}^2) & \xrightarrow{\widetilde{P}_\varepsilon} & L^2_\theta(\mathbb{T}^2) \end{array} \tag{4.43}$$

In what follows, we shall drop the tildes from the notation in (4.41) and write simply P_ε and p_ν, $\nu \geq 0$.

5. Quantum normal forms near rational tori

In this section, we shall be concerned with a classical h-pseudodifferential operator $P_\varepsilon(x, hD_x; h)$, defined microlocally near $\xi = 0$ in $T^*\mathbb{T}^2$, given by the expansion (4.41), with leading symbol of the form (4.42). Our purpose here is to obtain a normal secular reduction of P_ε, also at the level of lower order symbols, and this will be accomplished in a way very similar to [16], [14]. The main result of this section is stated in Proposition 5.2 below.

Let us first discuss the normal form construction at the level of principal symbols. In doing so, we let $\widetilde{q}_0 := \widetilde{q}$ in (4.42), and write

$$p_0(x, \xi, \varepsilon) = p(\xi) + i\varepsilon\widetilde{q}_0(x_1, \xi) + i\varepsilon^2 q_1(x, \xi) + \mathcal{O}\big(\varepsilon^3 + \varepsilon(\xi_1 - f(\xi_2))^N\big). \tag{5.1}$$

Arguing as in Section 3, we can construct an analytic function G_1, defined near $\xi = 0$, such that modulo $\mathcal{O}((\xi_1 - f(\xi_2))^N)$, we have

$$H_p G_1 = q_1 - \widetilde{q}_1,$$

where $\widetilde{q}_1$ is any analytic function satisfying (4.20), (4.21) with q there replaced by q_1. It follows that

$$p_0(\exp(i\varepsilon^2 H_{G_1})(x, \xi)) = p(\xi) + i\varepsilon\widetilde{q}_0(x_1, \xi) + i\varepsilon^2\widetilde{q}_1(x_1, \xi) + \mathcal{O}\big(\varepsilon^3 + \varepsilon(\xi_1 - f(\xi_2))^N\big).$$

Continuing this procedure, we get the following result.

Proposition 5.1. *Let $p_0(x, \xi, \varepsilon) = p(\xi) + i\varepsilon\widetilde{q}_0(x_1, \xi) + \mathcal{O}(\varepsilon^2) + \varepsilon\mathcal{O}((\xi_1 - f(\xi_2))^N)$ be analytic defined near $\xi = 0$, depending smoothly on $\varepsilon \in$ neigh$(0, \mathbb{R})$. Here $N \geq 2$ is arbitrarily large but fixed. Assume that*

$$p(\xi) = p(f(\xi_2), \xi_2) + g(\xi)(\xi_1 - f(\xi_2))^2, \qquad g(0) > 0, \qquad f(0) = 0,$$

where $p(f(\xi_2), \xi_2) = \alpha\xi_2 + \mathcal{O}(\xi_2^2)$, $\alpha > 0$. There exists a holomorphic canonical transformation $\kappa_\varepsilon^{(N)}$ of the form

$$\kappa_\varepsilon^{(N)} = \exp(i\varepsilon^2 H_{G_1}) \circ \cdots \circ \exp(i\varepsilon^N H_{G_{N-1}}) \tag{5.2}$$

with G_j analytic near $\xi = 0$, $1 \leq j \leq N-1$, such that modulo an error term of the form $\mathcal{O}(\varepsilon^{N+1} + \varepsilon(\xi_1 - f(\xi_2))^N)$, the function

$$p_0(\kappa_\varepsilon^{(N)}(x, \xi)) \equiv p(\xi) + i\varepsilon\widetilde{q}_0(x_1, \xi) + i\varepsilon^2(\widetilde{q}_1(x_1, \xi) + \cdots + \varepsilon^{N-2}\widetilde{q}_{N-1})$$

is independent of x_2. Here, as discussed before, $\widetilde{q}_0$ is any analytic function satisfying (4.20), (4.21), and inductively $\widetilde{q}_k$ is any analytic function satisfying (4.20), (4.21) with q there replaced by a certain function q_k that depends on the previously chosen $\widetilde{q}_0, \ldots, \widetilde{q}_{k-1}$.

As will be discussed in Section 7, the complex canonical transformation $\kappa_\varepsilon^{(N)}$ in (5.2) can be quantized by means of an elliptic classical h-Fourier integral operator U_ε in the complex domain, depending smoothly on $\varepsilon \in \mathrm{neigh}(0, \mathbb{R})$, introduced rigorously on the FBI transform side. In this section, we shall proceed formally, and an application of Egorov's theorem allows us to conclude that the operator

$$\widetilde{P}_\varepsilon(x, hD_x; h) = U_\varepsilon^{-1} P_\varepsilon(x, hD_x; h) U_\varepsilon$$

is an h-pseudodifferential operator, defined microlocally near $\xi = 0$, whose symbol has a complete asymptotic expansion

$$\widetilde{P}_\varepsilon(x, \xi; h) \sim \widetilde{p}_0 + h\widetilde{p}_1 + \cdots \tag{5.3}$$

with all $\widetilde{p}_j = \widetilde{p}_j(x, \xi, \varepsilon)$ being smooth functions of $\varepsilon \in \mathrm{neigh}(0, \mathbb{R})$ with values in the space of holomorphic functions in a fixed complex neighborhood of $\xi = 0$ such that

$$\widetilde{p}_0(x, \xi, \varepsilon) = p(\xi) + i\varepsilon\widetilde{q}_0(x_1, \xi) + \mathcal{O}(\varepsilon^2) + \mathcal{O}\big(\varepsilon^{N+1} + \varepsilon(\xi_1 - f(\xi_2))^N\big). \tag{5.4}$$

Here the $\mathcal{O}(\varepsilon^2)$-term is independent of x_2 and has the properties described in Proposition 5.1. Furthermore, we still have $\widetilde{p}_1(x, \xi, \varepsilon) = \mathcal{O}(\varepsilon)$.

We shall now simplify the lower order terms $\widetilde{p}_j$, $j \geq 1$, in (5.3). To that end, let $A_\varepsilon(x, hD; h)$ be a classical analytic elliptic h-pseudodifferential operator of order 0, with symbol

$$A_\varepsilon(x, \xi; h) \sim a_0(x, \xi, \varepsilon) + ha_1(x, \xi, \varepsilon) + \cdots, \tag{5.5}$$

depending smoothly on $\varepsilon \in \mathrm{neigh}(0, \mathbb{R})$. Then

$$A_\varepsilon^{-1} U_\varepsilon^{-1} P_\varepsilon U_\varepsilon A_\varepsilon = A_\varepsilon^{-1} \widetilde{P}_\varepsilon A_\varepsilon =: \widehat{P}_\varepsilon(x, hD_x; h),$$

where

$$\widehat{P}_\varepsilon(x, \xi; h) \sim \widetilde{p}_0(x, \xi, \varepsilon) + h\widehat{p}_1(x, \xi, \varepsilon) + h^2\widehat{p}_2(x, \xi, \varepsilon) + \cdots \tag{5.6}$$

with

$$\widehat{p}_1 = \widetilde{p}_1 + \frac{1}{i} a_0^{-1} H_{\widetilde{p}_0} a_0 = \widetilde{p}_1 + \frac{1}{i} H_{\widetilde{p}_0} b_0 \tag{5.7}$$

if $b_0 = \ln a_0$, well-defined up to a constant. Thus, looking for b_0 in terms of a formal power series in ε and choosing the terms there suitably, we can arrange that

$$\widehat{p}_1(x, \xi, \varepsilon) = \widehat{p}_{1,0}(x_1, \xi) + \varepsilon\widehat{p}_{1,1}(x_1, \xi) + \cdots + \mathcal{O}\big(\varepsilon^{N+1} + (\xi_1 - f(\xi_2))^N\big),$$

where $\widehat{p}_{1,0}$ is any analytic function satisfying (4.20), (4.21) with q replaced by $\widetilde{p}_{1,\varepsilon=0}$, and inductively, $\widehat{p}_{1,k}$ is any analytic function satisfying (4.20), (4.21) with q replaced by a function depending on the previously chosen $\widehat{p}_{1,0}, \ldots, \widehat{p}_{1,k-1}$.

Iterating this procedure, by choosing also the lower order terms in the expansion of A_ε, we get the following result, giving a quantum secular normal form construction.

Proposition 5.2. *Let*

$$P_\varepsilon \sim p_0 + hp_1 + \cdots, \qquad (x,\xi) \in \mathrm{neigh}(\xi = 0, T^*\mathbb{T}^2),$$

be such that $p_0(x,\xi,\varepsilon)$ has the properties stated in Proposition 5.1. Let U_ε be an elliptic classical analytic h-Fourier integral operator of order 0, associated to the canonical transformation $\kappa_\varepsilon^{(N)}$ in Proposition 5.1. Then we can construct an elliptic classical analytic h-pseudodifferential operator of order 0 with symbol as in (5.5) such that microlocally near $\xi = 0$, we have

$$A_\varepsilon^{-1} U_\varepsilon^{-1} P_\varepsilon(x, hD; h) U_\varepsilon A_\varepsilon = \widehat{P}_\varepsilon(x, hD_x; h), \tag{5.8}$$

where $\widehat{P}_\varepsilon(x, hD_x; h)$ is of the form

$$\widehat{P}_\varepsilon(x,\xi;h) \sim \widetilde{p}_0(x,\xi,\varepsilon) + h\widehat{p}_1(x,\xi,\varepsilon) + h^2\widehat{p}_2(x,\xi,\varepsilon) + \cdots. \tag{5.9}$$

Here the leading term $\widetilde{p}_0$ is as in (5.4),

$$\widetilde{p}_0(x,\xi,\varepsilon) = p(\xi) + i\varepsilon\widetilde{q}_0(x_1,\xi) + \mathcal{O}(\varepsilon^2) + \mathcal{O}\big(\varepsilon^{N+1} + \varepsilon(\xi_1 - f(\xi_2))^N\big)$$

with the $\mathcal{O}(\varepsilon^2)$-term being independent of x_2. For $1 \le k \le N$, we have

$$\widehat{p}_k(x,\xi,\varepsilon) = \widehat{p}_{k,0}(x_1,\xi) + \varepsilon\widehat{p}_{k,1}(x_1,\xi) + \cdots + \mathcal{O}\big(\varepsilon^{N+1} + (\xi_1 - f(\xi_2))^N\big), \tag{5.10}$$

where $\widehat{p}_{k,\ell}$ is any function satisfying (4.20), (4.21) q replaced by $\widetilde{p}_{k,\ell}$,a function that depends on $\widehat{p}_{\widetilde{k},\widetilde{\ell}}$ for all $(\widetilde{k},\widetilde{\ell})$ such that either $\widetilde{k} < k$, or $\widetilde{k} = k$ and $\widetilde{\ell} < \ell$. In particular, we can choose $\widehat{p}_k$ independent of x_2 modulo $\mathcal{O}(\varepsilon^{N+1} + (\xi_1 - f(\xi_2))^N)$. We also have

$$\widehat{p}_1(x,\xi,\varepsilon) = \mathcal{O}(\varepsilon).$$

6. Harmonic approximation for non-selfadjoint operators

In the previous section, we have seen how to eliminate the x_2-dependence in the complete symbol of our operator, by means of successive averaging procedures, when working in a small neighborhood of the rational torus $\Lambda_0 = \{\xi = 0\} \subset T^*\mathbb{T}^2$. Following Proposition 5.2 and neglecting the remainder terms there, we shall now consider an operator of the form

$$P_\varepsilon = P_\varepsilon(x_1, hD_x; h), \tag{6.1}$$

defined microlocally near $\xi = 0$ in $T^*\mathbb{T}^2$ and acting on $L^2_\theta(\mathbb{T}^2)$, with a complete symbol independent of x_2. We assume that the leading symbol of P_ε is of the form

$$p_0(x_1,\xi,\varepsilon) = p(\xi) + i\varepsilon\widetilde{q}(x_1,\xi) + \mathcal{O}(\varepsilon^2), \qquad p(\xi) = p(f(\xi_2),\xi_2) + g(\xi)(\xi_1 - f(\xi_2))^2, \tag{6.2}$$

with $\widetilde{q}$ given in (4.22), and let us recall the assumption (2.28) implying that the function $\mathbb{T} \ni x_1 \mapsto \widetilde{q}(x_1, f(\xi_2), \xi_2)$ has a unique minimum which is non-degenerate for $\xi_2 \in \mathrm{neigh}(0,\mathbb{R})$. When discussing the spectral analysis of P_ε, it is natural, in view of its independence of x_2, to take a Fourier series expansion in x_2, thereby reducing the problem, at least formally, to a direct sum of one-dimensional operators

$$P_\varepsilon(x_1, hD_{x_1}, h(j-\theta_2); h), \quad j \in \mathbb{Z}, \quad \theta_2 = \frac{k_0(\alpha_2)}{4} + \frac{S_2}{2\pi h},$$

considered for those j for which $h(j-\theta_2) \in \mathrm{neigh}(0, \mathbb{R})$.

In what follows, we shall write $\xi_2 = h(j-\theta_2) \in \mathrm{neigh}(0, \mathbb{R})$, and concentrate our attention on the one-dimensional operator

$$P_\varepsilon(x_1, hD_{x_1}, \xi_2; h), \tag{6.3}$$

acting on $L^2_{\theta_1}(\mathbb{T})$. Modifying the Floquet conditions on $\mathbb{T}$, we may replace (6.3) by the conjugate operator

$$e^{-if(\xi_2)x_1/h} P_\varepsilon(x_1, hD_{x_1}, \xi_2; h) e^{if(\xi_2)x_1/h} = P_\varepsilon(x_1, f(\xi_2) + hD_{x_1}, \xi_2; h).$$

The full symbol of $P_\varepsilon(x_1, hD_{x_1} + f(\xi_2), \xi_2; h)$ is of the form

$$P_\varepsilon(x_1, \xi_1 + f(\xi_2), \xi_2; h) = \sum_{j=0}^{\infty} h^j p_{j,\varepsilon}(x_1, \xi). \tag{6.4}$$

Here

$$p_{0,\varepsilon}(x_1, \xi) = p(\xi_1 + f(\xi_2), \xi_2) + i\varepsilon \widetilde{q}(x_1, \xi_1 + f(\xi_2), \xi_2) + \mathcal{O}(\varepsilon^2), \tag{6.5}$$

and from Proposition 5.2 we recall that

$$p_{1,\varepsilon}(x_1, \xi) = \mathcal{O}(\varepsilon). \tag{6.6}$$

We can then write $p_{1,\varepsilon} = \varepsilon q_{1,\varepsilon}$ with $q_{1,\varepsilon} = \mathcal{O}(1)$.

Let us set

$$\widetilde{h} = h/\sqrt{\varepsilon}, \tag{6.7}$$

and assume that

$$\widetilde{h} \ll 1. \tag{6.8}$$

We have

$$\begin{aligned} &P_\varepsilon(x_1, \xi_1 + f(\xi_2), \xi_2; h) \\ &= p(\xi_1 + f(\xi_2), \xi_2) + i\varepsilon \widetilde{q}(x_1, \xi_1 + f(\xi_2), \xi_2) + \mathcal{O}(\varepsilon^2) + h\varepsilon q_{1,\varepsilon}(x_1, \xi) + \sum_{j=2}^{\infty} h^j p_{j,\varepsilon} \\ &= p(\xi_1 + f(\xi_2), \xi_2) + \varepsilon\Big(i\widetilde{q}(x_1, \xi_1 + f(\xi_2), \xi_2) + \mathcal{O}(\varepsilon) + \widetilde{h}\varepsilon^{1/2} q_{1,\varepsilon} + \sum_{j=2}^{\infty} \widetilde{h}^j \varepsilon^{j/2-1} p_{j,\varepsilon}\Big). \end{aligned}$$

Here, according to (6.2),

$$p(\xi_1 + f(\xi_2), \xi_2) = p(f(\xi_2), \xi_2) + g(\xi_1 + f(\xi_2), \xi_2)\xi_1^2, \quad g(0) > 0.$$

It follows that at the operator level, we have

$$P_\varepsilon(x_1, f(\xi_2) + hD_{x_1}, \xi_2; h) = p(f(\xi_2), \xi_2) + g(f(\xi_2) + hD_{x_1}, \xi_2)(hD_{x_1})^2 \\ + i\varepsilon\big(\widetilde{q}(x_1, f(\xi_2) + hD_{x_1}, \xi_2) + \mathcal{O}(\varepsilon + \widetilde{h}\varepsilon^{1/2} + \widetilde{h}^2)\big). \tag{6.9}$$

We shall be interested in computing eigenvalues of the one-dimensional operator $P_\varepsilon(x_1, f(\xi_2) + hD_{x_1}, \xi_2; h)$ in the region

$$|\mathrm{Re}\, z - p(f(\xi_2), \xi_2)| \le \mathcal{O}(\varepsilon\widetilde{h}),$$

and directly from (6.9), using cut-offs of the form $\chi(\xi_1/\sqrt{\varepsilon})$, we see that the corresponding eigenfunctions are microlocally concentrated to the region where $\xi_1 = \mathcal{O}(\sqrt{\varepsilon})$, provided that the smallness condition (6.8) is strengthened to

$$h/\sqrt{\varepsilon} \le h^\eta, \qquad \eta > 0. \tag{6.10}$$

It will then be convenient to perform a rescaling of the cotangent variable, corresponding to a suitable change of the semiclassical parameter. Let us write

$$hD_{x_1} = \sqrt{\varepsilon}\,\widetilde{h}D_{x_1},$$

and if $\xi_1, \widetilde{\xi}_1$ denote the cotangent variables corresponding to hD_{x_1} and $\widetilde{h}D_{x_1}$, respectively, we have

$$\xi_1 = \sqrt{\varepsilon}\,\widetilde{\xi}_1.$$

It follows that

$$\frac{1}{\varepsilon}P_\varepsilon(x_1, f(\xi_2) + hD_{x_1}, \xi_2; h) \tag{6.11}$$

can be viewed as an $\widetilde{h}$-pseudodifferential operator of the form

$$\frac{1}{\varepsilon}P_\varepsilon(x_1, f(\xi_2) + hD_{x_1}, \xi_2; h) = \frac{p(f(\xi_2), \xi_2)}{\varepsilon} \\ + g(f(\xi_2) + \sqrt{\varepsilon}\,\widetilde{h}D_{x_1}, \xi_2)(\widetilde{h}D_{x_1})^2 + i\widetilde{q}(x_1, f(\xi_2) + \sqrt{\varepsilon}\,\widetilde{h}D_{x_1}, \xi_2) + \mathcal{O}(\varepsilon) \\ + \widetilde{h}\mathcal{O}(\sqrt{\varepsilon}) + \mathcal{O}(\widetilde{h}^2). \tag{6.12}$$

Ignoring the constant term $p(f(\xi_2), \xi_2)/\varepsilon$ on the right hand side, we recognize here essentially a one-dimensional Schrödinger operator with a purely imaginary potential, and to be precise, we can write

$$\frac{1}{\varepsilon}P_\varepsilon(x_1, f(\xi_2) + hD_{x_1}, \xi_2; h) = \frac{p(f(\xi_2), \xi_2)}{\varepsilon} + A(x_1, \widetilde{h}D_{x_1}, \xi_2, \sqrt{\varepsilon}; \widetilde{h}),$$

where $A(x_1, \widetilde{h}D_{x_1}, \xi_2, \sqrt{\varepsilon}; \widetilde{h})$ is a well-behaved $\widetilde{h}$-pseudodifferential operator, depending smoothly on $\xi_2 \in \mathrm{neigh}(0, \mathbb{R})$ and $\sqrt{\varepsilon} \ge 0$, with leading symbol

$$g(f(\xi_2) + \sqrt{\varepsilon}\,\xi_1, \xi_2)\xi_1^2 + i\widetilde{q}(x_1, f(\xi_2) + \sqrt{\varepsilon}\,\xi_1, \xi_2) + \mathcal{O}(\varepsilon), \tag{6.13}$$

and with subprincipal symbol which is $\mathcal{O}(\sqrt{\varepsilon})$. Here we have dropped the tilde from the notation for the cotangent variable corresponding to $\widetilde{h}D_{x_1}$; also recall that the operator (6.12) is to be considered microlocally in the region where $\xi_1 = \mathcal{O}(1)$. The function g in (6.13) satisfies $g > 0$.

Remark. If (6.6) is no longer assumed, we can write, assuming that $h/\varepsilon \ll 1$,

$$\begin{aligned} P_\varepsilon(x_1, \xi_1 + f(\xi_2), \xi_2; h) &= p(\xi_1 + f(\xi_2), \xi_2) + i\varepsilon\widetilde{q}(x_1, \xi_1 + f(\xi_2), \xi_2) \\ &\quad + \mathcal{O}(\varepsilon^2) + hp_{1,\varepsilon}(x_1, \xi) + \sum_{j=2}^{\infty} h^j p_{j,\varepsilon}(x_1, \xi) \\ &= p(\xi_1 + f(\xi_2), \xi_2) + i\varepsilon\left(\widetilde{q}(x_1, \xi_1 + f(\xi_2), \xi_2) + \mathcal{O}\left(\varepsilon + \frac{h}{\varepsilon}\right) + \sum_{j=2}^{\infty} \widetilde{h}^j \varepsilon^{j/2-1} p_{j,\varepsilon}\right), \end{aligned}$$

and we can then view h/ε as an additional small parameter. As will be seen in Section 6, some pseudospectral considerations will force us to assume that ε/h should not be too large, and for that reason, in this work we make the assumption (2.29), leading to (6.6).

The discussion pursued in this section so far indicates that the spectral analysis of the original operator P_ε should reduce to that for a family of $\widetilde{h}$-pseudodifferential operators on $\mathbb{T}$, with leading symbols of the form (6.13). Letting $\varepsilon = 0$ in (6.13) for a while and suppressing the parameter ξ_2 altogether, we shall now pause to make a digression, in order to recall semiclassical asymptotics for the low-lying eigenvalues of non-selfadjoint h-pseudodifferential operators with double characteristics. In doing so, we shall follow the analysis of [10], which in turn follows [8] closely. Let us also remark that in the present one-dimensional case, the quadratic approximations along the double characteristics are elliptic, and consequently our discussion is considerably simplified compared with [10], [8].

Let $P_0(x, hD_x; h) : C^\infty(\mathbb{T}) \to C^\infty(\mathbb{T})$ be such that $P_0(x, hD_x; h) \in \mathrm{Op}_h^w(S(\langle\xi\rangle^2))$, and assume that the semiclassical leading symbol of P_0 is of the form

$$p_0(x, \xi) = \xi^2 + iV(x), \tag{6.14}$$

where $V \in C^\infty(\mathbb{T}; \mathbb{R})$. Assume also, for simplicity, that the subprincipal symbol of P_0 vanishes. Assume that if $a = \min V$ then $V^{-1}(a) = \{x_0\}$ with $V''(x_0) > 0$. We are interested in the eigenvalues of P_0 in an open disc $\{z \in \mathbb{C}; |z - ia| < Ch\}$ for some $C > 0$ fixed and all $h > 0$ small enough, and to that end we consider the operator

$$P(x, hD_x; h) = (1 - i)(P_0(x, hD_x; h) - ia), \tag{6.15}$$

whose leading symbol $p(x, \xi) = (1 - i)(p_0(x, \xi) - ia)$ is such that

$$\mathrm{Re}\, p(x, \xi) = \xi^2 + V(x) - a \geq 0$$

is elliptic for large ξ and vanishes precisely at the point $(x_0, 0) \in T^*\mathbb{T}$. In a neighborhood of $(x_0, 0)$ we have

$$p(x + x_0, \xi) = q(x, \xi) + \mathcal{O}((x, \xi)^3), \quad (x, \xi) \to (0, 0), \tag{6.16}$$

where q is a quadratic form such that $\mathrm{Re}\, q > 0$. When determining the eigenvalues of $P(x, hD_x; h)$ in an $\mathcal{O}(h)$-neighborhood of 0, naturally only the behavior of the operator in a small neighborhood of $(x_0, 0)$ matters, and by composing p with the inverse of the translation

$$\kappa : \mathrm{neigh}((x_0, 0), T^*\mathbb{T}) \to \mathrm{neigh}((0, 0), T^*\mathbb{R}), \qquad \kappa((x_0, 0)) = (0, 0), \tag{6.17}$$

we obtain an h-pseudodifferential operator

$$P(x, \xi; h) \sim \sum_{j=0}^{\infty} h^j p_j(x, \xi), \qquad p_1 = 0, \tag{6.18}$$

defined microlocally near $(0, 0) \in T^*\mathbb{R}$, such that the leading symbol $p_0 = p$ satisfies

$$p(x, \xi) = q(x, \xi) + \mathcal{O}((x, \xi)^3), \tag{6.19}$$

where q is quadratic with

$$\mathrm{Re}\, q > 0. \tag{6.20}$$

We extend $P(x, \xi; h)$ to be globally defined on $\mathbb{R}^2$ as an element of the symbol class $S(1)$, such that

$$\mathrm{Re}\, p(x, \xi) \geq 0, \qquad (\mathrm{Re}\, p)^{-1}(0) = \{(0, 0)\}, \tag{6.21}$$

$$\mathrm{Re}\, p(x, \xi) \geq 1/C, \qquad |(x, \xi)| \geq C > 0. \tag{6.22}$$

An application of [10, Theorem 1.1] allows us to conclude that the following result holds, which we state directly for the operator $P_0(x, hD_x; h)$. See also [9] for related results in the analytic case.

Theorem 6.1. *Let the operator $P_0(x, hD_x; h) : C^\infty(\mathbb{T}) \to C^\infty(\mathbb{T})$ have principal symbol of the form (6.14) and a vanishing subprincipal symbol, and assume that if $a = \min V$ then $V^{-1}(a) = \{x_0\}$ with $b := V''(x_0) > 0$. Let $C > 0$. Then there exists $h_0 > 0$ such that for all $0 < h \leq h_0$, the spectrum of $P_0(x, hD_x; h)$ in the open disc $D(ia, Ch)$ in the complex plane is given by the simple eigenvalues of the form*

$$z_k \sim ia + h(\lambda_{k,0} + h\lambda_{k,1} + h^2\lambda_{k,2} + \cdots). \tag{6.23}$$

Here $\lambda_{k,0}$ are the eigenvalues in $D(0, C)$ of the elliptic quadratic operator

$$q^w(x, D_x) = D_x^2 + i\frac{b}{2}x^2$$

acting on $L^2(\mathbb{R})$, which are given by

$$\lambda_{k,0} = (b/2)^{1/2} e^{i\pi/4}(2k + 1), \qquad k \in \mathbb{N}, \qquad k = \mathcal{O}(1).$$

Remark. Theorem 6.1 continues to be valid when the operator $P_0(x, hD_x; h)$ acts on an L^2-space of Floquet periodic functions on $\mathbb{T}$, and indeed the eigenvalues described in this result do not depend on the Floquet conditions modulo $\mathcal{O}(h^\infty)$.

Coming back to the operator in (6.12), with the leading symbol (6.13), we shall next have to extend the result of Theorem 6.1 to the parameter-dependent case, and to this end it will be convenient to recall briefly the main steps in the proof of Theorem 6.1. Let $P = P(x, hD_x; h)$ be an h-pseudodifferential operator on $\mathbb{R}$ satisfying (6.18)–(6.22). Following [10], let us recall that the proof of Theorem 6.1 proceeds by constructing a well-posed Grushin problem for the operator P, of the form

$$(P - hz)u + R_- u_- = v, \qquad R_+ u = v, \qquad z \in \mathrm{neigh}(\lambda_0, \mathbb{C}), \tag{6.24}$$

in the space $L^2(\mathbb{R}) \times \mathbb{C}$. Here λ_0 is an eigenvalue of $q^w(x, D_x)$ such that $|\lambda_0| < C$. The operators $R_- : \mathbb{C} \to L^2$ and $R_+ : L^2 \to \mathbb{C}$ are defined as follows:

$$R_- u_- = u_- e, \qquad R_+ u = (u, f)_{L^2}, \tag{6.25}$$

where e is an eigenfunction of $q^w(x, hD_x)$ corresponding to the eigenvalue $h\lambda_0$, and f is an eigenfunction of the adjoint operator $\overline{q}^w(x, hD_x)$ corresponding to the eigenvalue $h\overline{\lambda}_0$.

The verification of the well-posedness of (6.24) consists of two steps, both carried out after a metaplectic FBI transform

$$T : L^2(\mathbb{R}) \to H_{\Phi_0}(\mathbb{C}). \tag{6.26}$$

Here

$$H_{\Phi_0}(\mathbb{C}) = \mathrm{Hol}(\mathbb{C}) \cap L^2(\mathbb{C}; e^{-2\Phi_0/h} L(dx)),$$

and Φ_0 is a suitable strictly subharmonic quadratic form. In the first step, we concentrate on the region $|x| \le h^\rho$, $x \in \mathbb{C}$, for some $1/3 < \rho < 1/2$. Arguing as in [10], we obtain the following a priori estimate for the problem (6.24), based on the quadratic approximation of P near the origin (see [10, (3.25)])

$$\begin{aligned} &\|(h + |x|^2)^{1/2} \chi_0(x/h^\rho) u\| + h^{-1/2} |u_-| \\ &\quad \le C \|(h + |x|^2)^{-1/2} \chi_0(x/h^\rho) v\| + C \|(h + |x|^2)^{-1/2} \chi_0(x/h^\rho)(P - Q)u\| \\ &\qquad + \mathcal{O}(h^{1/2}) |v_+| + C\sqrt{h/h^{2\rho}} \|(h + |x|^2)^{1/2} 1_K(x/h^\rho) u\|. \end{aligned} \tag{6.27}$$

Here $u, v \in H_{\Phi_0}(\mathbb{C})$, the norms are taken in the space $L^2(\mathbb{C}; e^{-2\Phi_0/h} L(dx))$, and we have also written P for the conjugate operator TPT^{-1}. The function $\chi_0 \in C_0^\infty(\mathbb{C})$ is fixed, with $\chi_0 = 1$ near 0, and K is a fixed compact neighborhood of $\mathrm{supp}(\nabla \chi_0)$, $0 \notin K$. Furthermore, $Q = Tq^w(x, hD_x)T^{-1}$, and therefore, as explained in [10], we have

$$\|(h + |x|^2)^{-1/2} \chi_0(x/h^\rho)(P - Q)u\| = \mathcal{O}(h^{3\rho}/h^{1/2}) \|u\|. \tag{6.28}$$

Using (6.27) and (6.28), we obtain

$$
\begin{aligned}
&h\|\chi_0(x/h^\rho)u\|^2 + h^{-1}|u_-|^2\\
&\quad \le \frac{\mathcal{O}(1)}{h}\|v\|^2 + \mathcal{O}(h^{6\rho-1})\|u\|^2 + \mathcal{O}(h)|v_+|^2 + \mathcal{O}(h)\|1_K(x/h^\rho)u\|^2. \qquad (6.29)
\end{aligned}
$$

Notice that the lower bound $\rho > 1/3$ implies that here $h^{6\rho-1} \ll h$.

In the second step of the proof, we consider the exterior region $|x| \ge h^\rho$, and here we use

$$
\operatorname{Re} p\left(x, \frac{2}{i}\frac{\partial \Phi_0}{\partial x}(x)\right) \ge \frac{h^{2\rho}}{C}, \qquad C > 0.
$$

Exploiting the sharp Gårding inequality in the form of a quantization-multiplication formula, as explained in [10] (see also [36]), we obtain the following exterior a priori estimate for the problem (6.24):

$$
\begin{aligned}
h^{2\rho}\int \chi(x/h^\rho)|u(x)|^2 e^{-2\Phi_0(x)/h}\,L(dx) &\le \mathcal{O}(1)\|v\|\,\|u\|\\
&\quad + \mathcal{O}(h^\infty)|u_-|\,\|u\| + \mathcal{O}(h)\|u\|^2. \qquad (6.30)
\end{aligned}
$$

Here $\chi \in C_b^\infty(\mathbb{C}; [0,1])$ vanishes near $x = 0$ and $\chi = 1$ for large x. Assuming that $1/3 < \rho < 1/2$, the bounds (6.29) and (6.30) can be glued together, and we get the a priori estimate

$$
h\|u\| + |u_-| \le \mathcal{O}(1)(\|v\| + h|v_+|), \qquad (6.31)
$$

and the consequent well-posedness of the Grushin problem (6.24). Asymptotic expansions for the eigenvalues of P follow exactly as explained in [8], [10]. In the present one-dimensional situation, the eigenvalues are simple and only integer powers of h occur in the expansions (6.23).

Turning to the parameter-dependent case, let $P_\varepsilon = P_\varepsilon(x, hD_x; h) \in \operatorname{Op}_h^w(S(1))$, $\varepsilon \ge 0$, be an h-pseudodifferential operator depending smoothly on $\sqrt{\varepsilon}$ such that $P_{\varepsilon=0} = P$ satisfies (6.18)–(6.22). In particular, the leading symbol p_ε of P_ε satisfies

$$
p_\varepsilon(x,\xi) = p(x,\xi) + \mathcal{O}(\sqrt{\varepsilon}) \qquad (6.32)
$$

in the sense of symbols in $S(1)$, and the subprincipal symbol of P_ε is $\mathcal{O}(\sqrt{\varepsilon})$. Assume also that near $(0,0)$, (6.32) improves to

$$
p_\varepsilon(x,\xi) = p(x,\xi) + \mathcal{O}(\sqrt{\varepsilon}\,|\xi| + \varepsilon) \qquad (6.33)
$$

(see also (6.13)). We would like to conclude that the Grushin problem (6.24) with P replaced by P_ε remains well-posed, provided that $\varepsilon > 0$ is not too large, and to that end, we shall simply inspect the two steps above.

In the region $|x| \le h^\rho$, we argue as above with P replaced by P_ε, and using (6.33) together with the fact that the subprincipal symbol of P_ε is $\mathcal{O}(\sqrt{\varepsilon})$, we see that we get an additional term on the right hand side of (6.27) of the form

$$\|(h+|x|^2)^{-1/2}\chi_0(x/h^\rho)(P_\varepsilon - P)u\| = \mathcal{O}\left(\frac{\sqrt{\varepsilon}\,h^\rho + \varepsilon}{h^{1/2}}\right)\|u\|. \tag{6.34}$$

Here we also assume that we have chosen the FBI transform in (6.26) so that (6.33) holds on the transform side. As for the exterior region $|x| \geq h^\rho$, replacing P by P_ε we get an additional term on the right hand side of (6.30), given by

$$\mathcal{O}(1)\sqrt{\varepsilon}\,\|u\|^2. \tag{6.35}$$

It follows that to absorb the two extra terms (6.34), (6.35), we need to meet the following conditions:

$$\frac{\varepsilon^{1/2}h^\rho + \varepsilon}{h^{1/2}} \ll h^{1/2} \quad \text{and} \quad \sqrt{\varepsilon} \ll h^{2\rho}.$$

The first condition is satisfied provided that $\varepsilon \ll h^{2-2\rho}$, since $\rho < 1/2$, and the second one holds when $\varepsilon \ll h^{4\rho}$. We conclude that the Grushin problem (6.24) remains well-posed when P is replaced by P_ε provided that

$$\varepsilon \ll h^{4\rho}, \tag{6.36}$$

since $1/3 < \rho < 1/2$. Combining this observation with the standard perturbation theory for eigenvalues of multiplicity one [18], we obtain the following result.

Proposition 6.2. *Let $P_\varepsilon(x, hD_x; h)$, $\varepsilon \geq 0$, be a smooth function of $\sqrt{\varepsilon}$ with values in $\mathrm{Op}_h^w(S(1))$ such that when $\varepsilon = 0$, we have the properties (6.18)–(6.22). Assume that (6.33) holds. Then for $\varepsilon \leq h^{4/3+\eta}$, $\eta > 0$, the eigenvalues of $P_\varepsilon(x, hD_x; h)$ in the region $\{z \in \mathbb{C}; |z| < Ch\}$ are simple eigenvalues of the form*

$$z_k \sim h(\lambda_{k,0}(\sqrt{\varepsilon}) + h\lambda_{k,1}(\sqrt{\varepsilon}) + \cdots), \qquad k \in \mathbb{N}, \qquad k = \mathcal{O}(1),$$

where $\lambda_{k,j}(\sqrt{\varepsilon})$ are smooth functions of $\sqrt{\varepsilon} \geq 0$, $j \geq 0$, with $\lambda_{k,0}(0)$ being the eigenvalues of the quadratic operator $q^w(x, D_x)$ described explicitly in Theorem 6.1. When $z \in \mathbb{C}$ is such that $|z| < Ch$ and $\operatorname{dist}(z, \operatorname{Spec}(P_\varepsilon)) \geq h/\mathcal{O}(1)$, we have

$$(z - P_\varepsilon)^{-1} = \mathcal{O}(1)/h : L^2 \to L^2. \tag{6.37}$$

In our considerations (see (6.13)), when applying Proposition 6.2, we should replace the semiclassical parameter h by $\widetilde{h} = h/\sqrt{\varepsilon}$, which in view of (6.36) leads to the condition

$$\varepsilon \ll \widetilde{h}^{4\rho}, \qquad 1/3 < \rho < 1/2, \tag{6.38}$$

so that

$$\varepsilon \ll h^{4\rho/(2\rho+1)}.$$

When $\rho = 1/3$, the power on the right hand side is $4/5$, and it follows that we have the well-posedness of the Grushin problem provided that

$$\varepsilon \leq \mathcal{O}(h^{4/5+\eta}), \qquad \eta > 0. \tag{6.39}$$

Remark. In the proof of Proposition 6.2 above, the presence of the parameter $\sqrt{\varepsilon}$ was treated by a direct perturbation argument, leading to the upper bound (6.36). The purpose of this remark is to outline an alternative approach to the parameter-dependent case, leading to sharper bounds on ε. While sharpening the result of Proposition 6.2 below would not lead to an improvement in Theorem 2.1, which is the main result of this work, we believe that the alternative approach sketched below may be of some independent interest. Since its precise realization is likely to demand a greater technical investment, the argument developed in this remark will be quite brief and we hope to be able to develop it further in future work.

Let $P_\varepsilon(x, \xi; h)$ be a real analytic function of $\varepsilon \in \mathrm{neigh}(0, \mathbb{R})$ with values in the space of bounded holomorphic functions in a tubular neighborhood of $\mathbb{R}^2$, such that as $h \to 0^+$,

$$P_\varepsilon(\rho; h) \sim p_\varepsilon(\rho) + h p_{1,\varepsilon}(\rho) + \cdots, \qquad \rho = (x, \xi).$$

For $\varepsilon = 0$, assume that the leading symbol $p := p_0$ is such that $\mathrm{Re}\, p \geq 0$ is elliptic at infinity, vanishing precisely at $\rho = 0$. Assume furthermore that

$$p(\rho) = q(\rho) + \mathcal{O}(\rho^3), \qquad \rho \to 0,$$

where q is quadratic with $\mathrm{Re}\, q$ positive definite. In particular, $\rho = 0$ is a non-degenerate critical point for p, and an application of the implicit function theorem shows that for ε small, p_ε has a non-degenerate critical point $\rho(\varepsilon)$ in the complex domain, depending analytically on ε, with $\rho(\varepsilon) = \mathcal{O}(\varepsilon)$. Passing to the FBI transform side by means of a metaplectic FBI transform T, as in (6.26), let us continue to write $\rho(\varepsilon) = (x(\varepsilon), \xi(\varepsilon)) = \mathcal{O}(\varepsilon)$ for the image of the critical point $\rho(\varepsilon)$ under the complex linear canonical transformation κ_T associated to T.

We know from [29] that $\kappa_T(\mathbb{R}^2) = \Lambda_{\Phi_0} = \{(x, (2/i)\partial_x \Phi_0(x));\ x \in \mathbb{C}\}$, where Φ_0 is the strictly subharmonic quadratic form introduced in (6.26). We shall now discuss the problem of constructing a weight function $\Phi_\varepsilon \in C^\infty(\mathbb{C})$ such that

$$\Phi_\varepsilon = \Phi_0 + \mathcal{O}(h), \qquad |\nabla^2(\Phi_\varepsilon - \Phi_0)| \ll 1, \tag{6.40}$$

and with $\rho(\varepsilon) \in \Lambda_{\Phi_\varepsilon} = \{(x, (2/i)\partial_x \Phi_\varepsilon(x));\ x \in \mathbb{C}\}$. The function Φ_ε is then strictly subharmonic and if we set $H_{\Phi_\varepsilon}(\mathbb{C}) = \mathrm{Hol}(\mathbb{C}) \cap L^2(\mathbb{C};\ e^{-2\Phi_\varepsilon/h} L(dx))$, then $H_{\Phi_\varepsilon} = H_{\Phi_0}$, with uniformly equivalent norms. To get the complete asymptotic expansions of the eigenvalues of P_ε in $D(p_\varepsilon(\rho_\varepsilon), Ch)$, as in Proposition 6.2, one should then work with the operator P_ε acting on the space H_{Φ_ε}. We need

$$\xi(\varepsilon) = \frac{2}{i} \frac{\partial \Phi_\varepsilon}{\partial x}(x(\varepsilon)),$$

and notice that

$$\xi(\varepsilon) - \frac{2}{i} \frac{\partial \Phi_0}{\partial x}(x(\varepsilon)) = \mathcal{O}(\varepsilon).$$

With $\partial_x \Phi_\varepsilon(x(\varepsilon))$ already determined, we try

$$\begin{aligned}\Phi_\varepsilon(x) &= \Phi_0(x) + 2\,\mathrm{Re}\big((\partial_x \Phi_\varepsilon(x(\varepsilon)) - \partial_x \Phi_0(x(\varepsilon)))\cdot(x - x(\varepsilon))\big)\chi\left(\frac{x - x(\varepsilon)}{h^\alpha}\right)\\ &= \Phi_0(x) + h^\alpha(\ell_\varepsilon \chi)\left(\frac{x - x(\varepsilon)}{h^\alpha}\right).\end{aligned}$$

Here $\chi \in C_0^\infty(\mathbb{C})$ is a standard cut-off near 0 and

$$\ell_\varepsilon(y) = 2\,\mathrm{Re}\big((\partial_x \Phi_\varepsilon(x(\varepsilon)) - \partial_x \Phi_0(x(\varepsilon)))\cdot y\big)$$

is linear such that $\ell_\varepsilon = \mathcal{O}(\varepsilon)$ as a linear form. Then

$$\nabla^k(\Phi_\varepsilon - \Phi_0) = \mathcal{O}(\varepsilon) h^{\alpha - k\alpha}, \qquad k \geq 0,$$

and in view of (6.40), we need $\varepsilon h^\alpha \leq \mathcal{O}(h)$, $\varepsilon / h^\alpha \ll 1$. We get the conditions $\varepsilon \leq \mathcal{O}(h^{1-\alpha})$, $\varepsilon \ll \mathcal{O}(h^\alpha)$, and it follows that the optimal choice of α is $\alpha = 1/2$. This leads to the condition $\varepsilon \ll \mathcal{O}(\sqrt{h})$. In our applications, we should replace ε by $\sqrt{\varepsilon}$ and h by $\widetilde{h} = h/\sqrt{\varepsilon}$, leading to the condition $\varepsilon \ll \widetilde{h} = h/\sqrt{\varepsilon}$, so that we get

$$0 \leq \varepsilon \ll \mathcal{O}(h^{2/3}), \tag{6.41}$$

which is sharper than (6.39). One conjectures therefore that the result of Proposition 6.2 extends to this range of ε, and we hope to return to this observation in a future paper.

We shall finish this section by a formal application of Proposition 6.2 to the microlocally defined operator $P_\varepsilon(x_1, hD_{x_1}, \xi_2; h)$ in (6.3), acting on $L^2_{\theta_1}(\mathbb{T})$. Assume that $\varepsilon > 0$ is such that

$$\widetilde{h} = h/\sqrt{\varepsilon} \leq h^\eta, \qquad \eta > 0,$$

and that (6.39) holds. It follows that the eigenvalues of $P_\varepsilon(x_1, hD_{x_1}, \xi_2; h)$ in the region

$$|z - p(f(\xi_2), \xi_2) - i\varepsilon\langle q\rangle_2(x_1(\xi_2), f(\xi_2), \xi_2)| \leq \mathcal{O}(\sqrt{\varepsilon}\, h)$$

are given by

$$\begin{aligned} z_k = {} & p(f(\xi_2), \xi_2) + i\varepsilon\langle q\rangle_2(x_1(\xi_2), f(\xi_2), \xi_2)\\ & + \sqrt{\varepsilon}\, h(\lambda_{k,0} + \lambda_{k,1}\widetilde{h} + \lambda_{k,2}\widetilde{h}^2 + \cdots), \qquad \mathbb{N} \ni k \leq \mathcal{O}(1), \end{aligned} \tag{6.42}$$

where $\lambda_{k,j} = \lambda_{k,j}(\xi_2, \sqrt{\varepsilon})$, $j \geq 0$, is a smooth function of $\xi_2 \in \mathrm{neigh}(0, \mathbb{R})$, $\sqrt{\varepsilon} \geq 0$, with

$$\lambda_{k,0}(\xi_2, 0) = e^{i\pi/4}(\partial_{\xi_1}^2 p(f(\xi_2), \xi_2))^{1/2}(\partial_{x_1}^2\langle q\rangle_2(x_1(\xi_2), f(\xi_2), \xi_2))^{1/2}(k+1/2). \tag{6.43}$$

Here we recall from (2.28) that $x_1(\xi_2) \in \mathbb{T}$ is the unique minimum point of the function $x_1 \mapsto \langle q\rangle_2(x_1, f(\xi_2), \xi_2)$.

7. Pseudospectral bounds and the global Grushin problem

The discussion pursued in the previous section shows that we are able to determine the low-lying eigenvalues of suitable localized one-dimensional operators

$$P_\varepsilon(x_1, hD_{x_1}, \xi_2; h)$$

in (6.3), occurring in the normal form reduction, provided that the perturbative parameter ε satisfies

$$h^{2-\eta} \leq \varepsilon \leq h^{4/5+\eta}, \qquad \eta > 0. \tag{7.1}$$

The purpose of this section is to complete the proof of Theorem 2.1 by constructing a global well-posed Grushin problem for $P_\varepsilon - z$, leading to the description of the eigenvalues in the region described in Theorem 2.1. In doing so, we shall have to strengthen the bounds in (7.1), as a consequence of some precise pseudospectral analysis for the family of the one-dimensional non-selfadjoint operators $P_\varepsilon(x_1, hD_{x_1}, \xi_2; h)$, with ξ_2 playing the role of parameters.

Our first task is to give a global definition of the h-dependent weighted Hilbert space, where the Grushin problem will be studied. Similarly to [16], the weighted space in question will be associated to a globally defined IR-manifold $\Lambda \subset T^*\widetilde{M}$, which is $\mathcal{O}(\varepsilon)$-close to T^*M and agrees with it outside a compact set. Specifically, the manifold Λ will be obtained as an $\mathcal{O}(\varepsilon^2)$-perturbation of the IR-manifold $\Lambda_{\varepsilon G}$, introduced in (4.38), where the perturbative modification will only take place in a sufficiently small but fixed neighborhood of the rational torus Λ_0.

Let us recall therefore that in Section 4, we have shown that microlocally near the Lagrangian torus $\exp(i\varepsilon H_G)(\Lambda_0) \subset \Lambda_{\varepsilon G}$, the operator in (4.40) is unitarily equivalent to an analytic h-pseudodifferential operator P_ε, defined microlocally near $\xi = 0$ in $T^*\mathbb{T}^2$ and acting on $L^2_\theta(\mathbb{T}^2)$, such that the leading symbol of P_ε is of the form

$$p_0(x, \xi, \varepsilon) = p(\xi) + i\varepsilon\widetilde{q}(x_1, \xi) + \mathcal{O}(\varepsilon^2) + \varepsilon\mathcal{O}((\xi_1 - f(\xi_2))^N), \tag{7.2}$$

where $p(\xi)$ is given in (4.2) and $N \geq 2$ is arbitrarily large but fixed. See also (4.43) for an illustration of the unitary equivalence by means of a commutative diagram.

Let

$$\kappa_\varepsilon^{(N)} : \mathrm{neigh}(\xi = 0, T^*\widetilde{\mathbb{T}}^2) \to \mathrm{neigh}(\xi = 0, T^*\widetilde{\mathbb{T}}^2), \qquad \widetilde{\mathbb{T}}^2 = \mathbb{C}^2/2\pi\mathbb{Z}^2, \tag{7.3}$$

be the holomorphic canonical transformation introduced in Proposition 5.1. Considering the IR-manifold $\kappa_\varepsilon^{(N)}(T^*\mathbb{T}^2) \subset T^*\widetilde{\mathbb{T}}^2$, defined in a complex neighborhood of $\xi = 0$, we conclude, arguing as in [16, Section 5], that there exists a C^∞ strictly plurisubharmonic function $\Phi_\varepsilon(x)$, defined for $x \in \mathbb{C}^2/2\pi\mathbb{Z}^2$, $|\mathrm{Im}\, x| \leq 1/\mathcal{O}(1)$, such that in the C^∞-sense,

$$\Phi_\varepsilon(x) = \Phi_0(x) + \mathcal{O}(\varepsilon^2), \qquad \Phi_0(x) = \frac{1}{2}(\mathrm{Im}\, x)^2,$$

and the operator

$$P_\varepsilon = \mathcal{O}(1) : T^{-1}H_{\Phi_\varepsilon}(|\mathrm{Im}\, x| < 1/C) \to T^{-1}H_{\Phi_\varepsilon}(|\mathrm{Im}\, x| < 1/C) \tag{7.4}$$

is, microlocally near $\kappa_\varepsilon^{(N)}(\mathbb{T}^2 \times \{\xi = 0\})$, unitarily equivalent to an operator $\widetilde{P}_\varepsilon$, given in (5.3), (5.4), acting on $L^2_\theta(\mathbb{T}^2)$. Here

$$T : L^2(\mathbb{T}^2) \to H_{\Phi_0}(\mathbb{C}^2/2\pi\mathbb{Z}^2)$$

is the standard unitary FBI-Bargmann transform on the 2-torus associated to the quadratic phase function $i(x-y)^2/2$, as discussed in [14], and we have written

$$H_{\Phi_\varepsilon}(\Omega) = \mathrm{Hol}(\Omega) \cap L^2(\Omega, e^{-2\Phi_\varepsilon/h} L(dx))$$

for $\Omega \subset \mathbb{C}^2/2\pi\mathbb{Z}^2$ open, including the Floquet periodic versions of the spaces. Let us also point out that the unitary equivalence between the operators P_ε in (7.4) and $\widetilde{P}_\varepsilon$ is realized by means of a microlocally unitary h-Fourier integral operator U_ε in the complex domain, quantizing the canonical transformation in (7.3). Similarly to (4.43), we may illustrate it in a commutative diagram

$$\begin{array}{ccc} T^{-1}H_{\Phi_\varepsilon}(|\mathrm{Im}\, x| < 1/C) & \xrightarrow{P_\varepsilon} & T^{-1}H_{\Phi_\varepsilon}(|\mathrm{Im}\, x| < 1/C) \\ {\scriptstyle U_\varepsilon}\uparrow & & \uparrow{\scriptstyle U_\varepsilon} \\ L^2_\theta(\mathbb{T}^2) & \xrightarrow{\widetilde{P}_\varepsilon} & L^2_\theta(\mathbb{T}^2) \end{array} \tag{7.5}$$

In particular, according to Proposition 5.1, the leading symbol of $\widetilde{P}_\varepsilon$ is independent of x_2 modulo $\mathcal{O}(\varepsilon^{N+1} + \varepsilon(\xi_1 - f(\xi_2))^N)$. The subprincipal symbol of $\widetilde{P}_\varepsilon$ is $\mathcal{O}(\varepsilon)$.

Remark. From [16], we may recall that writing

$$\Lambda_{\Phi_\varepsilon} : \quad \xi = \frac{2}{i}\frac{\partial \Phi_\varepsilon}{\partial x}(x), \qquad |\mathrm{Im}\, x| \le \frac{1}{\mathcal{O}(1)},$$

we have $\Lambda_{\Phi_\varepsilon} = \kappa_T \circ \kappa_\varepsilon^{(N)}(T^*\mathbb{T}^2)$, where the canonical transformation κ_T associated to T is given by

$$T^*\widetilde{\mathbb{T}}^2 \ni (y, \eta) \mapsto (y - i\eta, \eta) = (x, \xi) \in T^*\widetilde{\mathbb{T}}^2.$$

Let us also remark that the writing (7.4) is somewhat informal, and a precise statement is obtained by considering the action of the conjugate operator $TP_\varepsilon T^{-1}$ on the space $H_{\Phi_\varepsilon}(|\mathrm{Im}\, x| < 1/C)$ (see also [16]).

We obtain a globally defined closed IR-manifold $\Lambda \subset T^*\widetilde{M}$, which is diffeomorphic to T^*M, ε-close to T^*M everywhere in the C^∞-sense, agrees with that set away from $p^{-1}(0)$, and in a complex neighborhood of Λ_0 it is obtained by replacing

$$\exp(i\varepsilon H_G) \circ \kappa_0^{-1}(T^*\mathbb{T}^2)$$

by

$$\exp(i\varepsilon H_G) \circ \kappa_0^{-1} \circ \kappa_\varepsilon^{(N)}(T^*\mathbb{T}^2), \tag{7.6}$$

which amounts to an $\mathcal{O}(\varepsilon^2)$-deformation $\Lambda_{\varepsilon G}$ in a neighborhood of Λ_0. Here we recall the holomorphic canonical transformation $\exp(i\varepsilon H_G)$, identifying $\Lambda_{\varepsilon G}$ and T^*M in a neighborhood of Λ_0, and the real analytic canonical transformation κ_0 in (4.1), given by the action-angle coordinates near Λ_0. The spectral analysis required in order to compute the extremal eigenvalues of P_ε in Theorem 2.1 will be carried out in the globally defined h-dependent Hilbert space $H(\Lambda)$, associated to the IR-manifold Λ by the FBI-Bargmann approach.

Recalling Proposition 4.2 and taking into account also Proposition 5.2, eliminating the x_2-dependence in the normal form by means of a pseudodifferential conjugation, we may summarize the discussion so far in the following result.

Proposition 7.1. *There exists a globally defined smooth IR-manifold $\Lambda \subset T^*\widetilde{M}$ and a C^∞-Lagrangian torus $\widehat{\Lambda}_0 \subset \Lambda$, which is an $\mathcal{O}(\varepsilon)$-perturbation of the rational torus Λ_0 in the C^∞-sense, such that when $\rho \in \Lambda$ is away from an ε^δ-neighborhood of $\widehat{\Lambda}_0$ in Λ and*

$$|\mathrm{Re}\, P_\varepsilon(\rho; h)| \leq \varepsilon^{2\delta}/C \tag{7.7}$$

for $C > 0$ large enough, then

$$\mathrm{Im}\, P_\varepsilon(\rho; h) \geq \varepsilon \inf Q_\infty(\Lambda_0) + \varepsilon^{2\delta+1}/\mathcal{O}(1). \tag{7.8}$$

Here $0 < \delta < 1/2$ is so small that $\varepsilon^\delta \gg \max(h^{1/2}, \varepsilon^{1/2})$. The manifold Λ is $\mathcal{O}(\varepsilon)$-close to T^*M and agrees with it away from a neighborhood of $p^{-1}(0) \cap T^*M$. We have

$$P_\varepsilon = \mathcal{O}(1) : H(\Lambda, m) \to H(\Lambda).$$

Furthermore, there exists an elliptic h-Fourier integral operator with a complex phase

$$U = \mathcal{O}(1) : H(\Lambda) \to L^2_\theta(\mathbb{T}^2)$$

such that microlocally near $\widehat{\Lambda}_0$, we have

$$U P_\varepsilon = (P(x_1, hD_x, \varepsilon; h) + R(x, hD_x, \varepsilon; h))U.$$

Here $P(x_1, hD_x, \varepsilon; h) + R(x, hD_x, \varepsilon; h)$ is defined microlocally near $\xi = 0$ in $T^*\mathbb{T}^2$, the full symbol of $P(x_1, hD_x, \varepsilon; h)$ is independent of x_2, and

$$R(x, \xi, \varepsilon; h) = \mathcal{O}\big(\varepsilon^{N+1} + (\xi_1 - f(\xi_2))^N + h^{N+1}\big), \quad f(0) = 0. \tag{7.9}$$

Here N is arbitrarily large but fixed. The leading symbol of $P(x_1, hD_x, \varepsilon; h)$ is of the form

$$p(\xi) + i\varepsilon\widetilde{q}(x_1, \xi) + \mathcal{O}(\varepsilon^2),$$

where

$$p(\xi) = p(f(\xi_2), \xi_2) + g(\xi)(\xi_1 - f(\xi_2))^2, \quad g(0) > 0, \quad f(0) = 0, \tag{7.10}$$

and $\mathbb{T} \ni x_1 \mapsto \widetilde{q}(x_1, f(\xi_2), \xi_2)$ has a unique minimum when $\xi_2 \in \mathrm{neigh}(0, \mathbb{R})$, which is also non-degenerate. The subprincipal symbol of $P(x_1, hD_x, \varepsilon; h)$ is $\mathcal{O}(\varepsilon)$.

Using Proposition 7.1, we shall now discuss a priori estimates for the equation

$$(P_\varepsilon - z)u = v \tag{7.11}$$

when $u \in H(\Lambda, m)$, $v \in H(\Lambda)$, and the spectral parameter $z \in \mathbb{C}$ is confined to the region

$$|\mathrm{Re}\, z| \leq \varepsilon^{2\delta}/\mathcal{O}(1), \qquad \mathrm{Im}\, z \leq \varepsilon \inf Q_\infty(\Lambda_0) + \mathcal{O}(\sqrt{\varepsilon}\, h). \tag{7.12}$$

When doing so, following [16], [15], we shall make use of a suitable partition of unity on the manifold Λ, defined using Proposition 7.1 and consisting of smooth functions satisfying slightly degenerate symbolic estimates. Indeed, the presence of such slightly exotic symbols is natural here, as we are dealing with methods based on the techniques of normal forms, introducing error terms vanishing to a high order along the invariant tori. See also [30]. When quantizing the corresponding symbols defined on Λ, in the case when $M = \mathbb{R}^2$ we follow [15] and reduce the quantization procedure to that of Weyl on the standard phase space $T^*\mathbb{R}^2$ by means of a C^∞-canonical transformation

$$\kappa : \mathrm{neigh}(p^{-1}(0), T^*\mathbb{R}^2) \to \mathrm{neigh}(p^{-1}(0), \Lambda)$$

such that

$$\kappa(X) = X + i\varepsilon H_G(X) + \mathcal{O}(\varepsilon^2),$$

and the corresponding unitary Fourier integral operator with a complex phase mapping $L^2(\mathbb{R}^2)$ to $H(\Lambda)$. When M is compact, we use the Toeplitz quantization, following [31].

Let us consider a smooth partition of unity on the manifold Λ,

$$1 = \chi + \psi_1 + \psi_2. \tag{7.13}$$

Here $\chi \in C_0^\infty(\Lambda)$, $\nabla^m \chi = \mathcal{O}(\varepsilon^{-2\delta m})$, $m \geq 0$, is a cut-off function supported in an ε^δ-neighborhood of $\widehat{\Lambda}_0$ intersected with the region where $|\mathrm{Re}\, P_\varepsilon| \leq \varepsilon^{2\delta}/C$. Specifically, we shall obtain χ by choosing a suitable function $\chi_0 \in C_0^\infty(T^*\mathbb{T}^2)$, $\partial^\alpha \chi_0 = \mathcal{O}(\varepsilon^{-2\delta|\alpha|})$, $|\alpha| \geq 0$, depending on ξ only, $\chi_0 = \chi_0(\xi)$, and conjugating the operator $\chi_0(hD_x)$ by the microlocal inverse of the operator U in Proposition 7.1. In particular, using the fact that the subprincipal symbol of $P_{\varepsilon=0}$ vanishes, we get

$$[P_\varepsilon, \chi] = \mathcal{O}(h^3/\varepsilon^{6\delta}) + \mathcal{O}(\varepsilon h/\varepsilon^{2\delta}) : H(\Lambda) \to H(\Lambda). \tag{7.14}$$

Here and in what follows, we use the same notation for the functions occurring in (7.13) and the corresponding h-Weyl quantizations.

The function $0 \leq \psi_1 \in C^\infty(\Lambda)$ in (7.13) satisfies

$$\nabla^m \psi_1 = \mathcal{O}_m(\varepsilon^{-2\delta m}), \qquad m \geq 0,$$

and is such that

$$|\mathrm{Re}\, P_\varepsilon(\rho; h)| \geq \varepsilon^{2\delta}/C \tag{7.15}$$

near the support of ψ_1. Finally, $0 \leq \psi_2 \in C_0^\infty(\Lambda)$ in (7.13) is such that

$$\nabla^m \psi_2 = \mathcal{O}_m(\varepsilon^{-2\delta m}), \qquad m \geq 0, \tag{7.16}$$

and furthermore ψ_2 is supported in a region invariant under the H_p-flow, where

$$\operatorname{Im} P_\varepsilon(\rho; h) \geq \varepsilon \inf Q_\infty(\Lambda_0) + \varepsilon^{1+2\delta}/\mathcal{O}(1). \tag{7.17}$$

We also arrange, as we may, that ψ_2 Poisson commutes with p, the leading symbol of $P_{\varepsilon=0}$ acting on $H(\Lambda)$, so that $\{\psi, p\} = 0$. We refer to [16], [15], [14], where partitions of unity similar to (7.13) are constructed and utilized.

Let us now return to the equation (7.11). Assume that $\delta \in (0, 1/2)$ is so small that

$$\varepsilon^{2\delta} \geq h^{1/2-\eta} \tag{7.18}$$

for some fixed $\eta > 0$. We can then follow the slightly degenerate parametrix construction for $P_\varepsilon - z$ near the support of ψ_1, described in detail in [15, Section 4], and obtain

$$\|\psi_1 u\| \leq \frac{\mathcal{O}(1)}{\varepsilon^{2\delta}}\|v\| + \mathcal{O}(h^\infty)\|u\|. \tag{7.19}$$

Here and in what follows, the norms are taken in the space $H(\Lambda)$.

When discussing estimates for $\psi_2 u$, let us notice that $\operatorname{Im} P_\varepsilon(\rho; h) = \mathcal{O}(\varepsilon)$ on Λ, and near $\operatorname{supp} \psi_2$ we have, in view of (7.17) and (7.12),

$$\operatorname{Im}(P_\varepsilon(\rho; h) - z) \geq \varepsilon^{1+2\delta}/\mathcal{O}(1) - \mathcal{O}(\sqrt{\varepsilon}\, h). \tag{7.20}$$

Therefore, with a new implicit constant, near the support of ψ_2, we get

$$\frac{1}{\varepsilon} \operatorname{Im}(P_\varepsilon(\rho; h) - z) \geq \varepsilon^{2\delta}/\mathcal{O}(1), \tag{7.21}$$

provided that

$$h^{2/(1+4\delta)} \ll \varepsilon. \tag{7.22}$$

The lower bound (7.22) is of the same form as (7.1). Using $h/\varepsilon^{4\delta}$ as the natural semiclassical parameter and applying the sharp Gårding inequality, as in [16, Section 5], we get, in view of (7.21),

$$\begin{aligned}\frac{1}{\varepsilon}\operatorname{Im}\left((P_\varepsilon - z)\psi_2 u, \psi_2 u\right) &\geq \left(\frac{\varepsilon^{2\delta}}{\mathcal{O}(1)} - \mathcal{O}(1)\frac{h}{\varepsilon^{4\delta}}\right)\|\psi_2 u\|^2 - \mathcal{O}(h^\infty)\|u\|^2 \\ &\geq \frac{\varepsilon^{2\delta}}{\mathcal{O}(1)}\|\psi_2 u\|^2 - \mathcal{O}(h^\infty)\|u\|^2, \end{aligned} \tag{7.23}$$

provided that we strengthen (7.18) by assuming that

$$h/\varepsilon^{6\delta} \leq h^\eta, \qquad \eta > 0. \tag{7.24}$$

It follows from (7.23) that

$$\frac{\varepsilon^{2\delta+1}}{\mathcal{O}(1)}\|\psi_2 u\|^2 \leq \mathcal{O}(1)\|v\|\,\|\psi_2 u\| + \operatorname{Im}\left([P_\varepsilon, \psi_2]\widetilde{\psi}_2 u, \psi_2 u\right) + \mathcal{O}(h^\infty)\|u\|^2. \tag{7.25}$$

Here $\widetilde{\psi}_2 \in C_0^\infty(\Lambda)$ has the same properties as ψ_2 and is such that $\widetilde{\psi}_2 = 1$ near $\mathrm{supp}(\psi_2)$. When estimating the commutator $[P_\varepsilon, \psi_2]$ in (7.25), by using the Weyl calculus and (7.16) together with the fact that the subprincipal symbol of $P_{\varepsilon=0}$ vanishes, and p and ψ_2 Poisson commute, we get

$$[P_\varepsilon, \psi_2] = [P_{\varepsilon=0}, \psi_2] + \mathcal{O}\left(\frac{\varepsilon h}{\varepsilon^{2\delta}}\right) = \mathcal{O}\left(\frac{h^3}{\varepsilon^{6\delta}}\right) + \mathcal{O}\left(\frac{\varepsilon h}{\varepsilon^{2\delta}}\right) = \mathcal{O}\left(\frac{\varepsilon h}{\varepsilon^{2\delta}}\right). \tag{7.26}$$

Here we have also used the fact that $h^2 \ll \varepsilon^{1+4\delta}$, in view of (7.22). Combining (7.25) and (7.26), we get

$$\frac{\varepsilon^{2\delta+1}}{\mathcal{O}(1)} \|\psi_2 u\|^2 \le \mathcal{O}(1)\|v\|\,\|\psi_2 u\| + \mathcal{O}\left(\frac{\varepsilon h}{\varepsilon^{2\delta}}\right)\|\widetilde{\psi}_2 u\|^2 + \mathcal{O}(h^\infty)\|u\|^2, \tag{7.27}$$

and therefore

$$\|\psi_2 u\|^2 \le \frac{\mathcal{O}(1)}{\varepsilon^{4\delta+2}}\|v\|^2 + \mathcal{O}\left(\frac{h}{\varepsilon^{4\delta}}\right)\|\widetilde{\psi}_2 u\|^2 + \mathcal{O}(h^\infty)\|u\|^2. \tag{7.28}$$

Combining (7.24), (7.28), and a standard iteration argument, we conclude that

$$\|\psi_2 u\| \le \frac{\mathcal{O}(1)}{\varepsilon^{1+2\delta}}\|v\| + \mathcal{O}(h^\infty)\|u\|. \tag{7.29}$$

Using (7.13), (7.19), and (7.29), we obtain the following a priori estimate for the problem (7.11), (7.12):

$$\|(1-\chi)u\| \le \frac{\mathcal{O}(1)}{\varepsilon^{1+2\delta}}\|v\| + \mathcal{O}(h^\infty)\|u\|, \tag{7.30}$$

which holds provided that $\delta \in (0, 1/2)$ and the conditions (7.22), (7.24) are fulfilled. In the subsequent analysis, we may therefore concentrate on the region $\mathrm{supp}(\chi)$ for the cut-off function χ in (7.13).

Let us recall that the function $\chi \in C_0^\infty(\Lambda)$, $\nabla^m \chi = \mathcal{O}(\varepsilon^{-2\delta m})$, $m \ge 0$, in (7.13) is supported in an ε^δ-neighborhood of $\widehat{\Lambda}_0$ intersected with the region where $|\mathrm{Re}\, P_\varepsilon| \le \varepsilon^{2\delta}/C$. Writing

$$(P_\varepsilon - z)\chi u = \chi v + [P_\varepsilon, \chi]u,$$

and applying the Fourier integral operator U of Proposition 7.1, we get

$$(P(x_1, hD_x, \varepsilon; h) - z)U\chi u = U\chi v + U[P_\varepsilon, \chi]u + Tu. \tag{7.31}$$

Here, using (7.9), we see that

$$T = \mathcal{O}(h^M) : H(\Lambda) \to L^2_\theta(\mathbb{T}^2), \tag{7.32}$$

where M can be taken as large as we wish, provided that the integer N in Proposition 7.1 is taken large enough. Furthermore, as discussed above, we may arrange that

$$U\chi = \chi_0 U + \mathcal{O}(h^\infty) : H(\Lambda) \to H(\Lambda),$$

where $\chi_0 = \chi_0(hD_x, \varepsilon)$ is of the form

$$\chi_0(\xi, \varepsilon) = \chi_1(\xi_1/\varepsilon^{\delta})\chi_1(\xi_2/\varepsilon^{2\delta}),$$

where $\chi_1 \in C_0^{\infty}(\mathbb{R})$ is a standard cut-off to a neighborhood of 0. In particular, using (7.10) we see that the support of χ_0 is contained in the region where

$$|\xi| = \mathcal{O}(\varepsilon^{\delta}), \quad |p(\xi)| \leq \mathcal{O}(\varepsilon^{2\delta}).$$

Modifying the operator T in (7.31) slightly, we get

$$(P(x_1, hD_x, \varepsilon; h) - z)\chi_1\left(\frac{hD_{x_1}}{\varepsilon^{\delta}}\right)\chi_2\left(\frac{hD_{x_2}}{\varepsilon^{2\delta}}\right)Uu = U\chi v + U[P_{\varepsilon}, \chi]u + Tu. \tag{7.33}$$

In the subsequent analysis we shall therefore be working on the cotangent space $T^*\mathbb{T}^2$ in the region where

$$\xi_1 = \mathcal{O}(\varepsilon^{\delta}), \tag{7.34}$$

while

$$\xi_2 = \mathcal{O}(\varepsilon^{2\delta}). \tag{7.35}$$

Taking a Fourier series expansion in x_2, we get a direct sum decomposition

$$P(x_1, hD_x, \varepsilon; h) = \bigoplus_{j\in\mathbb{Z}} P(x_1, hD_{x_1}, \xi_2, \varepsilon; h), \quad \xi_2 = h(j - \theta_2), \tag{7.36}$$

where, according to (7.35), the summation is restricted only to those $j \in \mathbb{Z}$ for which $\xi_2 = \mathcal{O}(\varepsilon^{2\delta})$. We shall consider the question of inverting the operator

$$P(x_1, hD_x, \varepsilon; h) - z = \bigoplus_{j} (P(x_1, hD_{x_1}, \xi_2, \varepsilon; h) - z), \tag{7.37}$$

where, compared to (7.12), as stated in Theorem 2.1, the real part of z will be localized further to the region

$$|\mathrm{Re}\, z| \leq \frac{h}{C\sqrt{\varepsilon}}, \tag{7.38}$$

where $C > 0$ is large enough but fixed. Since in Proposition 7.1 we have introduced errors that are $\mathcal{O}(h^M)$, $M \gg 1$ (see (7.32)), we would first like to show that the one-dimensional non-selfadjoint operator

$$P(x_1, hD_{x_1}, \xi_2, \varepsilon; h) - z : L^2_{\theta_1}(\mathbb{T}) \to L^2_{\theta_1}(\mathbb{T}) \tag{7.39}$$

is invertible, microlocally in the region where $\xi_1 = \mathcal{O}(\varepsilon^{\delta})$, with an inverse of temperate growth in $1/h$, when $\xi_2 = \mathcal{O}(\varepsilon^{2\delta})$ is such that $|\xi_2| \geq h/(C_1\sqrt{\varepsilon})$ for a suitable fixed C_1 satisfying $0 < C_1 < C$. In doing so, it will be convenient to distinguish two cases, depending on the sign of ξ_2.

Case 1. Let us assume first that $\xi_2 = \mathcal{O}(\varepsilon^{2\delta})$ is such that

$$\xi_2 \geq \frac{h}{C_1\sqrt{\varepsilon}}. \tag{7.40}$$

Then, after a unitary conjugation, we can write, at the level of operators,

$$\begin{aligned} & e^{-if(\xi_2)x_1/h}P(x_1,hD_{x_1},\xi_2,\varepsilon;h)e^{if(\xi_2)x_1/h}-z \\ & \quad = p(f(\xi_2),\xi_2)+g(f(\xi_2)+hD_{x_1},\xi_2)(hD_{x_1})^2+i\varepsilon\widetilde{q}(x_1,f(\xi_2)+hD_{x_1},\xi_2) \\ & \qquad +\mathcal{O}(\varepsilon^2)+h\mathcal{O}(\varepsilon)+\mathcal{O}(h^2)-z. \end{aligned} \tag{7.41}$$

Here the conjugate operator, acting on the space $L^2_{\theta_1+f(\xi_2)}(\mathbb{T})$ of Floquet periodic functions, is still considered microlocally in the region where $\xi_1=\mathcal{O}(\varepsilon^{\delta})$, since $f(\xi_2)=\mathcal{O}(\xi_2)=\mathcal{O}(\varepsilon^{2\delta})$. Recalling that the derivative of the function $\xi_2\mapsto p(f(\xi_2),\xi_2)$ is strictly positive near $\xi_2=0$, we conclude, using (7.38), (7.40), and the positivity of $g(f(\xi_2)+hD_{x_1},\xi_2)(hD_{x_1})^2$, that the real part of the operator in (7.41), which is of the form

$$p(f(\xi_2),\xi_2)+g(f(\xi_2)+hD_{x_1},\xi_2)(hD_{x_1})^2-\operatorname{Re}z+\mathcal{O}(\varepsilon^2+h^2)+h\mathcal{O}(\varepsilon),$$

is $\geq h/(\widetilde{C}\sqrt{\varepsilon})$ for some $\widetilde{C}>0$, and is therefore invertible, microlocally in the region $\xi_1=\mathcal{O}(\varepsilon^{\delta})$, with the norm of the inverse being $\mathcal{O}(\sqrt{\varepsilon}/h)$. Here we also use the fact that $\varepsilon^2\ll h/\sqrt{\varepsilon}$, in view of (7.1). It is therefore clear that the full operator in (7.41) is invertible, microlocally in the region $\xi_1=\mathcal{O}(\varepsilon^{\delta})$, with a microlocal inverse of norm $\mathcal{O}(\sqrt{\varepsilon}/h)$.

Case 2. We assume now that $\xi_2=\mathcal{O}(\varepsilon^{2\delta})$ is such that

$$\xi_2\leq-\frac{h}{C_1\sqrt{\varepsilon}}. \tag{7.42}$$

Similarly to (7.41), we write

$$\begin{aligned} & e^{-if(\xi_2)x_1/h}P(x_1,hD_{x_1},\xi_2,\varepsilon;h)e^{if(\xi_2)x_1/h}-z \\ & \quad = g(f(\xi_2)+hD_{x_1},\xi_2)(hD_{x_1})^2+i\varepsilon\widetilde{q}(x_1,f(\xi_2)+hD_{x_1},\xi_2) \\ & \qquad +\mathcal{O}(\varepsilon^2)+h\mathcal{O}(\varepsilon)+\mathcal{O}(h^2)-w, \end{aligned} \tag{7.43}$$

where

$$w=z-p(f(\xi_2),\xi_2) \tag{7.44}$$

satisfies

$$\operatorname{Re}w\geq\frac{h}{C_2\sqrt{\varepsilon}},\qquad \operatorname{Im}w\leq\varepsilon\inf Q_\infty(\Lambda_0)+\mathcal{O}(\sqrt{\varepsilon}\,h) \tag{7.45}$$

for a suitable $C_2>0$. In view of (7.35), we have

$$\widetilde{q}(x_1,f(\xi_2)+\xi_1,\xi_2)=\widetilde{q}(x_1,\xi_1,0)+\mathcal{O}(\varepsilon^{2\delta}), \tag{7.46}$$

and therefore at the operator level we obtain

$$\begin{aligned} & e^{-if(\xi_2)x_1/h}P(x_1,hD_{x_1},\xi_2,\varepsilon;h)e^{if(\xi_2)x_1/h}-z \\ & \quad = g(f(\xi_2)+hD_{x_1},\xi_2)(hD_{x_1})^2+i\varepsilon\widetilde{q}(x_1,hD_{x_1},0)+R-w, \end{aligned} \tag{7.47}$$

where

$$R = \mathcal{O}(\varepsilon^{1+2\delta} + \varepsilon h + \varepsilon^2 + h^2) : L^2_{\theta_1+f(\xi_2)}(\mathbb{T}) \to L^2_{\theta_1+f(\xi_2)}(\mathbb{T}). \tag{7.48}$$

We may also assume that in (7.47), the operator $\widetilde{q}(x_1, hD_{x_1}, 0)$ is given by the classical h-quantization. It follows from (7.33) that thanks to the presence of the cut-off $\chi_1(hD_x/\varepsilon^\delta)$, to invert the operator in (7.39) microlocally in the region where $\xi_1 = \mathcal{O}(\varepsilon^\delta)$, we should consider the equation

$$\left(g(f(\xi_2)+hD_{x_1}, \xi_2)(hD_{x_1})^2 + i\varepsilon\widehat{q}(x_1, hD_{x_1}) + R - w_1\right)\chi_1\left(\frac{hD_{x_1}}{\varepsilon^\delta}\right)u = v \tag{7.49}$$

for $u, v \in L^2_{\theta_1+f(\xi_2)}(\mathbb{T})$. Here $w_1 = w - i\varepsilon \inf Q_\infty(\Lambda_0)$ and

$$\widehat{q}(x_1, \xi_1) = \widehat{q}(x_1, 0) + k(x_1, \xi_1)\varphi(\xi_1/\varepsilon^\delta),$$

where

$$\widehat{q}(x_1, 0) = \langle q\rangle_2(x_1, 0) - \inf Q_\infty(\Lambda_0) \geq 0, \qquad k(x_1, \xi_1) = \xi_1 \int_0^1 (\partial_{\xi_1}\widetilde{q})(x_1, t\xi_1)\, dt$$

and $\varphi \in C_0^\infty(\mathbb{R})$ is such that $\varphi = 1$ near $\operatorname{supp}(\chi_1)$. In particular, the function $\widehat{q}(x_1, \xi_1)$ satisfies the assumptions for the function $\widetilde{V}$ in Proposition A.4 of the Appendix.

Let us set

$$A(x_1, hD_{x_1}) = g(f(\xi_2) + hD_{x_1}, \xi_2)(hD_{x_1})^2 + i\varepsilon\widehat{q}(x_1, hD_{x_1}).$$

We would like to invert the operator $A(x_1, hD_{x_1}) + R - w_1$ occurring on the left hand side of (7.49) by an application of Proposition A.4; to that end, we shall assume that

$$\varepsilon^{1+\delta/2} \ll h. \tag{7.50}$$

Write

$$A(x_1, hD_{x_1}) - w_1 = \varepsilon\left(\frac{A(x_1, hD_{x_1})}{\varepsilon} - w_2\right), \qquad w_2 = \frac{w_1}{\varepsilon} = \frac{w}{\varepsilon} - i \inf Q_\infty(\Lambda_0). \tag{7.51}$$

It follows from (7.45) that the spectral parameter w_2 satisfies

$$\operatorname{Re} w_2 \geq \frac{h}{C_2\varepsilon^{3/2}}, \qquad \operatorname{Im} w_2 \leq \mathcal{O}(\widetilde{h}). \tag{7.52}$$

Here we recall that $\widetilde{h} = h/\sqrt{\varepsilon}$. In order to be able to apply Proposition A.4 to (7.51) we finally have to impose the smallness condition

$$\widetilde{h}|w_2|^{1/2} \ll 1, \tag{7.53}$$

and using (7.35), (7.38), (7.44), and (7.51), we see that (7.53) holds provided that

$$\frac{h}{\varepsilon}\varepsilon^\delta \ll 1.$$

We shall therefore require that

$$h \ll \varepsilon^{1-\delta}. \tag{7.54}$$

Once (7.50) and (7.54) both hold, we are in a position to apply Proposition A.4 to (7.51), obtaining

$$(A(x_1, hD_{x_1}) - w_1)^{-1} = \varepsilon^{-1}\mathcal{O}(\widetilde{h}^{-2/3}|w_2|^{-1/3}) : L^2_{\theta_1+f(\xi_2)}(\mathbb{T}) \to L^2_{\theta_1+f(\xi_2)}(\mathbb{T}). \tag{7.55}$$

Using $|w_2| \geq h/(C_2\varepsilon^{3/2})$, we get

$$\begin{aligned}(A(x_1, hD_{x_1}) - w_1)^{-1} \\ = \varepsilon^{-1}\mathcal{O}(\widetilde{h}^{-1}\varepsilon^{1/3}) = \mathcal{O}(h^{-1}\varepsilon^{-1/6}) : L^2_{\theta_1+f(\xi_2)}(\mathbb{T}) \to L^2_{\theta_1+f(\xi_2)}(\mathbb{T}).\end{aligned} \tag{7.56}$$

Returning to the equation (7.49), we would like to use a standard Neumann series argument to invert the operator $A(x_1, hD_{x_1}) + R - w_1$ on the left hand side of (7.49), and according to (7.56) and (7.48), we know that this is possible provided that

$$h^{-1}\varepsilon^{-1/6}(\varepsilon^{2\delta+1} + \varepsilon h + \varepsilon^2 + h^2) \ll 1, \tag{7.57}$$

which, in view of (7.1) and (7.22), is equivalent to the condition

$$h^{-1}\varepsilon^{5/6+2\delta} \ll 1. \tag{7.58}$$

Comparing the upper bounds (7.58) and (7.50), we see that the latter is implied by the former provided that

$$0 < \delta < 1/9. \tag{7.59}$$

In what follows, we shall adopt the smallness condition (7.59). We arrive therefore at the following upper bound on ε:

$$\varepsilon \ll h^{6/(5+12\delta)}, \tag{7.60}$$

which is a strengthening of the upper bound in (7.1).

We shall now also examine the lower bounds on ε that we have imposed in the course of our argument in this section. Recall that the lower bounds have been introduced in (7.22), (7.24), and (7.54). Comparing first (7.22) and (7.54), we see that

$$h^{2/(1+4\delta)} \ll h^{1/(1-\delta)}$$

when (7.59) holds, and our lower bound on ε becomes

$$h^{1/(1-\delta)} \ll \varepsilon. \tag{7.61}$$

We should then check the validity of (7.24) when (7.61) holds, and to that end we observe that indeed,

$$\frac{h}{\varepsilon^{6\delta}} \leq \frac{h}{h^{6\delta/1-\delta}} \leq h^{\eta}, \qquad \eta > 0,$$

thanks to (7.59).

Combining the bounds (7.60) and (7.61), we get the permissible range

$$h^{1/(1-\delta)} \ll \varepsilon \ll h^{6/(5+12\delta)}, \tag{7.62}$$

where $\delta \in (0, 1/9)$. The range in (7.62) is non-empty for $\delta \in (0, 1/9)$ precisely when

$$\frac{1}{1-\delta} > \frac{6}{5+12\delta} \Leftrightarrow \delta > \frac{1}{18}.$$

Let us summarize the discussion above in the following result.

Proposition 7.2. *Consider the operator*

$$P(x_1, hD_x, \varepsilon; h) = \bigoplus_{j\in\mathbb{Z}} P(x_1, hD_{x_1}, \xi_2, \varepsilon; h), \qquad \xi_2 = h(j-\theta_2) = \mathcal{O}(\varepsilon^{2\delta}),$$

microlocally in the region $\xi_1 = \mathcal{O}(\varepsilon^{\delta})$, $\xi_2 = \mathcal{O}(\varepsilon^{2\delta})$, *where* $1/18 < \delta < 1/9$. *Assume that the spectral parameter* $z \in \mathbb{C}$ *is such that*

$$|\mathrm{Re}\, z| \leq \frac{h}{C\sqrt{\varepsilon}}, \qquad \mathrm{Im}\, z \leq \varepsilon \inf Q_\infty(\Lambda_0) + \mathcal{O}(\sqrt{\varepsilon}\, h), \tag{7.63}$$

for some constant $C > 0$. *Assume furthermore that*

$$h^{1/(1-\delta)} \ll \varepsilon \ll h^{6/(5+12\delta)}, \tag{7.64}$$

and the quantum numbers $\xi_2 = \mathcal{O}(\varepsilon^{2\delta})$ *satisfy*

$$|\xi_2| \geq \frac{h}{\mathcal{O}(1)\sqrt{\varepsilon}}.$$

Then there exists a family of operators $E(\xi_2, \varepsilon; h) = \mathcal{O}(\varepsilon^{-1/6}h^{-1}) : L^2_{\theta_1} \to L^2_{\theta_1}$ *such that*

$$\big(E(\xi_2, \varepsilon; h)(P(x_1, hD_{x_1}, \xi_2, \varepsilon; h) - z) - 1\big)\chi(hD_{x_1}/\varepsilon^{\delta}) = \mathcal{O}(h^\infty) : L^2_{\theta_1} \to L^2_{\theta_1}$$

for every $\chi \in C_0^\infty(\mathbb{R})$ *with support in a sufficiently small but fixed neighborhood of* 0.

Remark. Notice that to reach powers of h that are < 1 in (7.64), it suffices to take $\delta > 1/12$. To obtain the range in (7.64) that is as large as possible, we should choose $\delta \in (1/18, 1/9)$ to be close to $1/9$.

In what follows, we continue to assume that the spectral parameter $z \in \mathbb{C}$ is confined to the region (7.63), and we shall assume that (7.64) holds for some $\delta \in (1/18, 1/9)$. It follows therefore from Proposition 7.2 that in the orthogonal sum decomposition (7.37), we can restrict the attention to the quantum numbers $\xi_2 = h(j - \theta_2)$ such that

$$|\xi_2| \leq \frac{h}{C_1\sqrt{\varepsilon}} = \frac{\widetilde{h}}{C_1}, \qquad C_1 > 0. \tag{7.65}$$

Using this refined localization in the parameter ξ_2, we shall now proceed to show that the spectrum of the operator P_ε in the region (7.63) is contained in the union of the pairwise disjoint bands of the form

$$|p(f(\xi_2),\xi_2)-\operatorname{Re} z|\le C_0\sqrt{\varepsilon}\,h,\qquad \xi_2=h(j-\theta_2)=\mathcal{O}(\widetilde{h}), \tag{7.66}$$

where $C_0>1$ is large enough but fixed. When doing so, we shall proceed similarly to the arguments above, relying upon Proposition A.4 and treating the parameter ξ_2 in a perturbative way.

Let us assume that $z\in\mathbb{C}$ satisfies (7.63) and is such that for some sufficiently large fixed $C_0>1$, we have

$$|p(f(\xi_2),\xi_2)-\operatorname{Re} z|\ge C_0\sqrt{\varepsilon}\,h \tag{7.67}$$

for all $\xi_2=h(j-\theta_2)=\mathcal{O}(\widetilde{h})$. Similarly to (7.43), we write

$$\begin{aligned}
&e^{-if(\xi_2)x_1/h}P(x_1,hD_{x_1},\xi_2,\varepsilon;h)e^{if(\xi_2)x_1/h}-z\\
&\quad=g(f(\xi_2)+hD_{x_1},\xi_2)(hD_{x_1})^2+i\varepsilon\widetilde{q}(x_1,f(\xi_2)+hD_{x_1},\xi_2)+\mathcal{O}(\varepsilon^2)\\
&\qquad+h\mathcal{O}(\varepsilon)+\mathcal{O}(h^2)-w,
\end{aligned} \tag{7.68}$$

where $w=z-p(f(\xi_2),\xi_2)$ satisfies

$$|\operatorname{Re} w|\ge C_0\sqrt{\varepsilon}\,h,\qquad \operatorname{Im} w\le\varepsilon\inf Q_\infty(\Lambda_0)+\mathcal{O}(\sqrt{\varepsilon}\,h). \tag{7.69}$$

Now, in view of (7.65), we have

$$\widetilde{q}(x_1,f(\xi_2)+hD_{x_1},\xi_2)=\widetilde{q}(x_1,hD_{x_1},0)+\mathcal{O}(\widetilde{h}),$$

and arguing as in the discussion of Case 2 above, we see that we have to invert the problem

$$\big(g(f(\xi_2)+hD_{x_1},\xi_2)(hD_{x_1})^2+i\varepsilon\widehat{q}(x_1,hD_{x_1})+R-w_1\big)\chi_1(hD_{x_1}/\varepsilon^\delta)u=v, \tag{7.70}$$

where $\widehat{q}(x_1,hD_{x_1})$ satisfies the assumptions in Proposition A.4 and

$$R=\mathcal{O}(\varepsilon\widetilde{h}+\varepsilon^2+\varepsilon h+h^2):L^2_{\theta_1+f(\xi_2)}(\mathbb{T})\to L^2_{\theta_1+f(\xi_2)}(\mathbb{T}). \tag{7.71}$$

The spectral parameter w_1 in (7.70) satisfies, in view of (7.69),

$$\frac{1}{\varepsilon}|\operatorname{Re} w_1|\ge C_0\widetilde{h},\qquad \frac{1}{\varepsilon}\operatorname{Im} w_1\le\mathcal{O}(\widetilde{h}). \tag{7.72}$$

An application of Proposition A.4 gives, as before,

$$\begin{aligned}
&\big(g(f(\xi_2)+hD_{x_1},\xi_2)(hD_{x_1})^2+i\varepsilon\widehat{q}(x_1,hD_{x_1})-w_1\big)^{-1}\\
&\qquad=\varepsilon^{-1}\mathcal{O}(\widetilde{h}^{-2/3}|w_1|^{-1/3}\varepsilon^{1/3}):L^2_{\theta_1+f(\xi_2)}(\mathbb{T})\to L^2_{\theta_1+f(\xi_2)}(\mathbb{T}),
\end{aligned} \tag{7.73}$$

and using (7.72), we see that the bound on the operator norm in (7.73) does not exceed

$$\varepsilon^{-1}\mathcal{O}(C_0^{-1/3}\widetilde{h}^{-1}). \tag{7.74}$$

To invert the full operator

$$g(f(\xi_2)+hD_{x_1},\xi_2)(hD_{x_1})^2+i\varepsilon\widehat{q}(x_1,hD_{x_1})+R-w_1 \tag{7.75}$$

on the left hand side of (7.70), in view of (7.71) and (7.74) we have to check that

$$\varepsilon^{-1}\widetilde{h}^{-1}C_0^{-1/3}(\varepsilon\widetilde{h}+\varepsilon^2+h^2)\ll 1, \tag{7.76}$$

which is satisfied for $C_0>1$ large enough, since clearly $\varepsilon \ll \widetilde{h}$ in view of (7.64). The bound on the norm of the inverse of the operator in (7.75) is therefore also given by (7.74).

Combining Proposition 7.2 with the discussion above, including the estimates (7.73), (7.74), we conclude that if $z\in\mathbb{C}$ satisfies (7.63) and is such that (7.67) holds, then the operator $P(x_1,hD_x,\varepsilon;h)-z$ is invertible, microlocally in the region where $\xi_1=\mathcal{O}(\varepsilon^{\delta})$, $\xi_2=\mathcal{O}(\varepsilon^{2\delta})$, with a microlocal inverse

$$(P(x_1,hD_x,\varepsilon;h)-z)^{-1}=\mathcal{O}(\varepsilon^{-1/2}h^{-1}):L^2_\theta\to L^2_\theta. \tag{7.77}$$

Coming back to (7.33), we therefore obtain for such z's,

$$\|U\chi u\|\le\mathcal{O}(\varepsilon^{-1/2}h^{-1})\|v\|+\mathcal{O}(\varepsilon^{-1/2}h^{-1})\mathcal{O}(\varepsilon h/\varepsilon^{2\delta})\|u\|+\mathcal{O}(h^{M'})\|u\|, \tag{7.78}$$

where $M'\gg 1$. Here we have also used (7.14). Combining (7.78) and (7.30), we obtain the following result.

Proposition 7.3. *Assume that*

$$h^{1/(1-\delta)}\ll\varepsilon\ll h^{6/(5+12\delta)} \tag{7.79}$$

for some $\delta\in(1/18,1/9)$. *Then the spectrum of the operator* $P_\varepsilon: H(\Lambda,m)\to H(\Lambda)$ *in the region*

$$|\mathrm{Re}\, z|\le\frac{h}{C\sqrt{\varepsilon}},\qquad \mathrm{Im}\, z\le\varepsilon\inf Q_\infty(\Lambda_0)+\mathcal{O}(\sqrt{\varepsilon}\, h) \tag{7.80}$$

is contained in the disjoint union of the bands of the form

$$|p(f(\xi_2),\xi_2)-\mathrm{Re}\, z|\le C_0\sqrt{\varepsilon}\, h,\qquad \xi_2=h(j-\theta_2)=\mathcal{O}(\widetilde{h}), \tag{7.81}$$

where $C_0>1$ *is large enough but fixed.*

We shall finally obtain a precise description of the spectrum of P_ε in the region (7.81) for a given value of $j\in\mathbb{Z}$ such that $\xi_2=h(j-\theta_2)=\mathcal{O}(\widetilde{h})$. In doing so, in view of the localization for $\mathrm{Im}\, z$, we may assume that

$$|z-p(f(\xi_2),\xi_2)-i\varepsilon\langle q\rangle_2(x_1(\xi_2),f(\xi_2),\xi_2)|\le C_0\sqrt{\varepsilon}\, h, \tag{7.82}$$

where $\xi_2 = h(j - \theta_2) = \mathcal{O}(\widetilde{h})$, and we then know that only the operator

$$P(x_1, hD_{x_1}, \xi_2, \varepsilon; h) : L^2_{\theta_1} \to L^2_{\theta_1} \tag{7.83}$$

in (7.36) contributes to the spectrum in this region. Let us introduce the quadratic elliptic operator

$$Q(t, D_t; \xi_2) = g(f(\xi_2), \xi_2)D_t^2 + \frac{i}{2}\big(\partial^2_{x_1}\langle q\rangle_2(x_1(\xi_2), f(\xi_2), \xi_2)\big)t^2, \tag{7.84}$$

and let $e_{k,\xi_2} \in L^2(\mathbb{R})$, $k \in \mathbb{N}$, be eigenfunctions of $Q(t, D_t; \xi_2)$ corresponding to the eigenvalues $\lambda_k(\xi_2)$ given in (6.43). Let also f_{k,ξ_2} be eigenfunctions of the adjoint $Q^*(t, D_t; \xi_2)$ corresponding to the eigenvalues $\overline{\lambda_k(\xi_2)}$. An application of Proposition 6.2 allows us to conclude that if (7.82) holds and the rescaled spectral parameter

$$\frac{1}{\sqrt{\varepsilon}\, h}\big(z - p(f(\xi_2), \xi_2) - i\varepsilon\langle q\rangle_2(x_1(\xi_2), f(\xi_2), \xi_2)\big)$$

avoids a small but fixed neighborhood of the eigenvalues $\lambda_k(\xi_2)$ in the disc $|z| < C_0$, then z is not in the spectrum of the operator in (7.83), with

$$(P(x_1, hD_{x_1}, \varepsilon; h) - z)^{-1} = \mathcal{O}\left(\frac{1}{\varepsilon^{1/2}h}\right) : L^2_{\theta_1} \to L^2_{\theta_1}.$$

In view of the analysis above, we conclude that then $z \notin \mathrm{Spec}(P_\varepsilon)$. It remains therefore for us to discuss the setup of the global Grushin problem for P_ε when the spectral parameter $z \in \mathbb{C}$ is such that

$$\frac{1}{\sqrt{\varepsilon}\, h}\big(z - p(f(\xi_2), \xi_2) - i\varepsilon\langle q\rangle_2(x_1(\xi_2), f(\xi_2), \xi_2)\big) \in \mathrm{neigh}(\lambda_k(\xi_2), \mathbb{C}) \tag{7.85}$$

for some $k \in \mathbb{N}$ with $k = \mathcal{O}(1)$. Using the notation of Proposition 7.1, let us set

$$R_+ : H(\Lambda) \to \mathbb{C}, \qquad R_+u = R_+(\xi_2, k)(U\chi u, e_{\xi_2})_{L^2_{\theta_2}}, \tag{7.86}$$

where $e_{\xi_2}(x_2) = e^{i\xi_2 x_2/h}$ and $R_+(\xi_2, k)$ has been introduced in (6.25), using the eigenfunctions f_{k,ξ_2}. Define also

$$R_- : \mathbb{C} \to H(\Lambda), \qquad R_-u_- = U^{-1}(R_-(\xi_2, k)u_- \otimes e_{\xi_2}). \tag{7.87}$$

Here $R_-(\xi_2, k)$ has been introduced in (6.25), and U^{-1} is a microlocal inverse of U. Arguing as in [11, Section 6], we find that when (7.85) holds, the Grushin operator

$$\mathcal{P}(z) = \begin{pmatrix} (P_\varepsilon - z)/\varepsilon & R_- \\ R_+ & 0 \end{pmatrix} : H(\Lambda, m) \times \mathbb{C} \to H(\Lambda) \times \mathbb{C}$$

is invertible, and the corresponding effective Hamiltonian $E_{-+}(z) : \mathbb{C} \to \mathbb{C}$ vanishes precisely when z is of the form (6.42), (6.43). This completes the proof of Theorem 2.1.

8. Numerical illustrations of spectra

The purpose of this section is to present the results of numerical computations of the spectra of P_ε in the following situation, which is easily implemented: Let us consider

$$\begin{aligned} P_\varepsilon &= -h^2\Delta_{x,y} + i\varepsilon q(x,y;hD_x,hD_y), \\ q(x,y;hD_x,hD_y) &= q_0(x,y) + q_1(x,y)hD_x + q_2(x,y)hD_y, \end{aligned} \tag{8.1}$$

on the torus $M = \mathbb{T}^2_{x,y} = (\mathbb{R}/2\pi\mathbb{Z})^2$. Here q_0, q_1, q_2 are real trigonometric polynomials of degree $\leq F \in \{1,2,\dots\}$. We shall consider the spectrum of this operator near the energy $E_0 = 1$.

The general assumptions (2.7), (2.8), (2.10) are fulfilled, the operator $P_{\varepsilon=0}$ is selfadjoint, and the leading semiclassical symbol is of the form (2.11) with

$$p = \xi^2 + \eta^2, \qquad q(x,y;\xi,\eta) = q_0(x,y) + q_1(x,y)\xi + q_2(x,y)\eta. \tag{8.2}$$

We also have $dp \neq 0$ along $p^{-1}(1) \cap T^*M$.

The Hamilton flow of p is completely integrable and we have the decomposition (2.12), for $p^{-1}(1)$ rather than $p^{-1}(0)$, where

$$J = \bigcup_{(\xi,\eta)\in\mathbb{T}} \Lambda_{\xi,\eta}, \qquad \Lambda_{\xi,\eta} = \mathbb{T}^2_{x,y} \times \{(\xi,\eta)\}.$$

We have

$$q_\ell(x,y) = \sum_{|j|,\,|k|\leq F} \widehat{q}_\ell(j,k)e^{i(jx+ky)},$$

where the reality of q_ℓ is equivalent to the property

$$\widehat{q}_\ell(-j,-k) = \overline{\widehat{q}}_\ell(j,k).$$

Rather than taking some particular explicit choice of q, we generate $\widehat{q}_\ell$ at random by choosing

$$\widehat{q}_\ell(j,k) = \tfrac{1}{2}(A_\ell(j,k) + \overline{A}_\ell(-j,-k)), \qquad A_\ell(j,k) = e^{-\kappa(|j|+|k|)}\alpha^\ell_{j,k},$$

where $\alpha^\ell_{j,k} \sim \mathcal{N}(0,1)$ are independent Gaussian random variables. The parameter $\kappa > 0$ induces an off-diagonal exponential decay, corresponding to the assumption that $q(x,y;\xi,\eta)$ is analytic in (x,y). Then

$$q(x,y;\xi,\eta) = \sum_{(j,k)\in[-F,F]^2} \widehat{q}(j,k;\xi,\eta)e^{i(jx+ky)},$$

where

$$\widehat{q}(j,k;\xi,\eta) = \widehat{q}_0(j,k) + \widehat{q}_1(j,k)\xi + \widehat{q}_2(j,k)\eta.$$

Here and below it is understood that $[-F,F]$ is an interval in $\mathbb{Z}$.

Let $\Lambda_{\xi,\eta} \in J$ be a rational torus, so that $(\xi, \eta) \in \mathbb{T}$ and $\xi/\eta \in \mathbb{Q} \cup \{\infty\}$. The H_p-trajectories in $\Lambda_{\xi,\eta}$ are of the form

$$\gamma : \mathbb{R} \ni s \mapsto ((x_0, y_0) + 2s(\xi, \eta), (\xi, \eta)).$$

The restriction of q to such a trajectory is

$$q(\gamma(s); \xi, \eta) = \sum_{(j,k)\in[-F,F]^2} \widehat{q}(j, k; \xi, \eta)e^{i((x_0,y_0)+2s(\xi,\eta))\cdot(j,k)}. \tag{8.3}$$

For the corresponding limit of the trajectory average,

$$\langle q\rangle_\gamma = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} q(\gamma(s); \xi, \eta)\, ds,$$

the terms in (8.3) with $(\xi, \eta) \cdot (j, k) \neq 0$ give a zero contribution, and we get

$$\langle q\rangle_\gamma = \sum_{(j,k)\in[-F,F]^2\cap(\xi,\eta)^\perp} \widehat{q}(j, k; \xi, \eta)e^{i(x_0,y_0)\cdot(j,k)}.$$

If we write $(\xi, \eta) = (-n, m)/|(m, n)|$ with $(n, m) \in \mathbb{Z}^2$ and $\gcd(n, m) = 1$, then $\mathbb{Z}^2 \cap (\xi, \eta)^\perp = \mathbb{Z}(m, n)$ and the intersection of this set with $[-F, F]^2$ (viewed as a subset of $\mathbb{Z}^2$) is equal to

$$\{\mu(m, n);\ \mu \in \mathbb{Z},\ |\mu| \le F/\max(|m|, |n|)\}.$$

This gives

$$\langle q\rangle_\gamma = \sum_{\mu=-[F/\max(|m|,|n|)]}^{[F/\max(|m|,|n|)]} \widehat{q}(\mu(m, n); \xi, \eta)e^{i\mu t(\gamma)}, \tag{8.4}$$

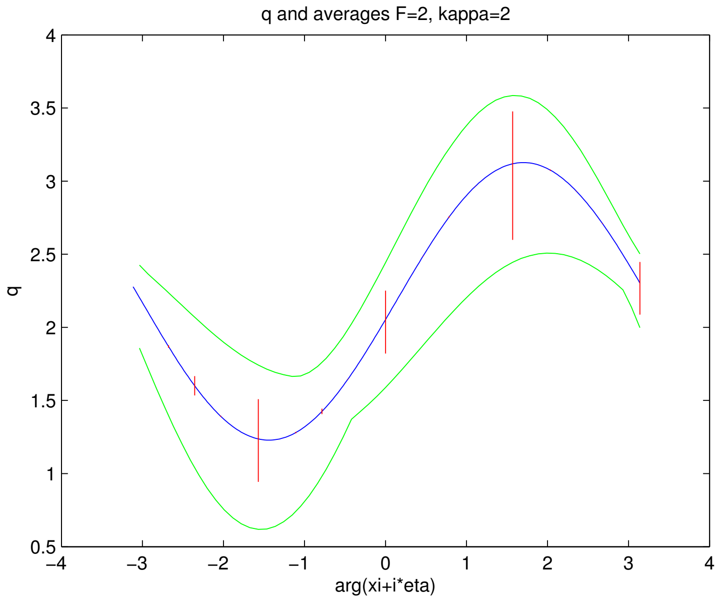
where $t(\gamma) = (x_0, y_0) \cdot (m, n)$ varies in $[0, 2\pi)$ and can take any value in that interval and $[\cdot]$ denotes the integer part. It follows that

$$Q_\infty(\Lambda_{\xi,\eta}) = \left\{ \sum_{\mu=-[F/\max(|m|,|n|)]}^{[F/\max(|m|,|n|)]} \widehat{q}(\mu(m, n); \xi, \eta)e^{i\mu t};\ 0 \le t \le 2\pi \right\}.$$

When $\max(|m|, |n|) > F$ this interval reduces to the torus average $\widehat{q}(0, 0; \xi, \eta) = \langle q\rangle_{\Lambda_{\xi,\eta}}$, so we get non-trivial intervals only for the finitely many values $(m, n) \in [-F, F]^2$ with $\gcd(m, n) = 1$.

We have written MatLab programs for the production of q and for the calculation of $\langle q\rangle_\Lambda$, $Q_\infty(\Lambda)$, as well as the supremum and infimum of q over each torus in J. For the graphics, we parametrize J by $\arg(\xi + i\eta)$ and the figure below shows:

- the torus average $\langle q\rangle_\Lambda$,
- the torus max and min of q,
- $Q_\infty(\Lambda)$ for each relevant rational torus.



By running the simulation several times we get a series of figures where quite a few exhibit the features above. In order to have a numerical illustration of the main result of this work it is important that some of the vertical segments (corresponding to $Q_\infty(\Lambda)$ for rational tori) reach above the supremum or below the infimum of the curve of torus averages. A larger F will produce a richer picture with more vertical segments, but it will also complicate the numerical calculations of the eigenvalues, so we settled for $F = 2$ as a reasonable choice. We also found that $\kappa = 2$ produces some — not too many — visible vertical segments.

Once an interesting q has been selected, we compute the spectrum numerically by working at the level of Fourier coefficients. Thus, if we are interested in the eigenvalues with real parts in $[E_1, E_2]$, where $E_1 < E_2$ are close to 1, we work with Fourier modes $e^{i(jx+ky)}$ for (j,k) in the set $\mathcal{E}$ of $(j,k) \in \mathbb{Z}^2$ satisfying $(hj)^2 + (hk)^2 \in [E_1, E_2]$, i.e.

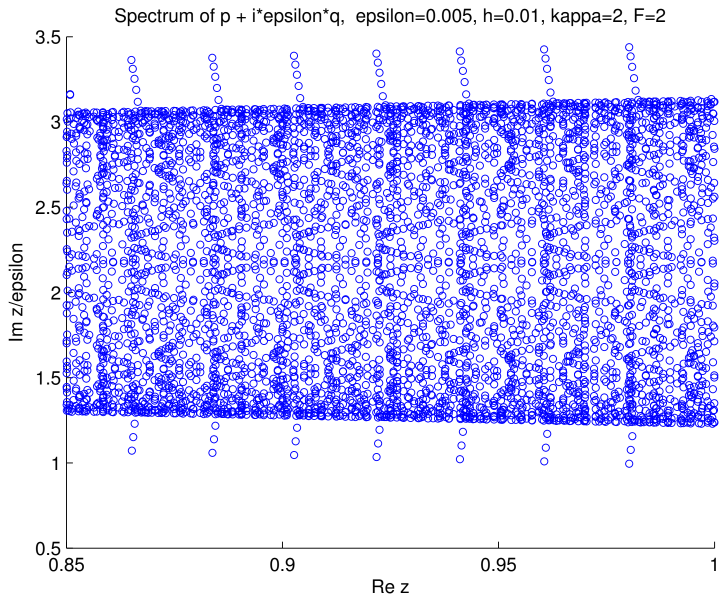
$$|(j,k)| \in \big[\sqrt{E_1}/h, \sqrt{E_2}/h\big].$$

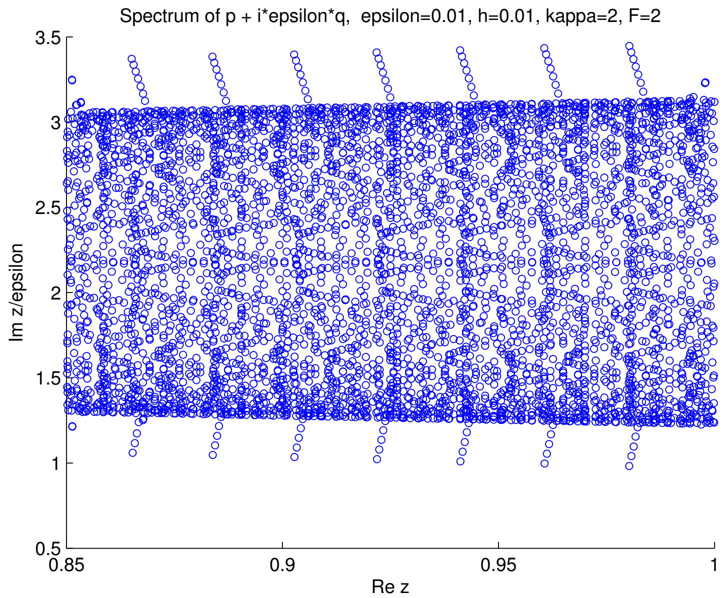
The number $\#\mathcal{E}$ of such modes is $\approx \pi(E_2 - E_1)/h^2$ and we ask MatLab to compute the spectrum of the $\mathcal{E} \times \mathcal{E}$-matrix $\mathcal{A}_\varepsilon = (a_\varepsilon(j,k;\widetilde{j},\widetilde{k}))_{(j,k),\,(\widetilde{j},\widetilde{k})\in\mathcal{E}}$ given by

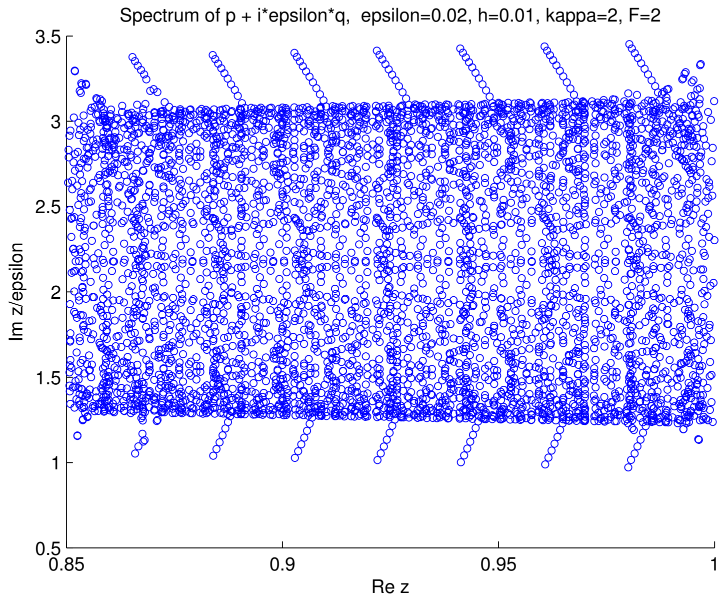
$$\begin{aligned} a_\varepsilon(j,k;\widetilde{j},\widetilde{k}) &= h^2(j^2+k^2)\delta_{(j,k),(\widetilde{j},\widetilde{k})} \\ &\quad + i\varepsilon\big(\widehat{q}_0(j-\widetilde{j},k-\widetilde{k}) + \widehat{q}_1(j-\widetilde{j},k-\widetilde{k})h\widetilde{j} + \widehat{q}_2(j-\widetilde{j},k-\widetilde{k})h\widetilde{k}\big). \end{aligned}$$

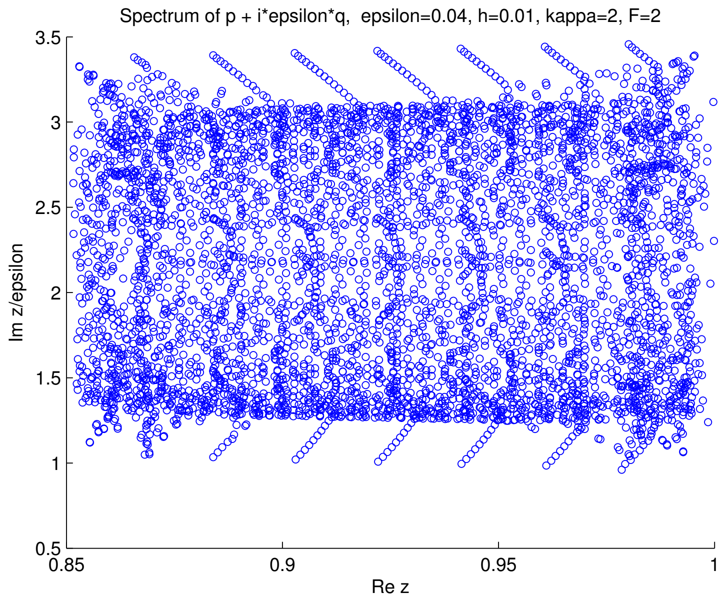
We cannot let $\#\mathcal{E}$ be larger than a few thousand and still we would like h to be small and the energy shell $[E_1, E_2]$ thick enough so that the eigenvalues with real part inside are not influenced by boundary effects. In the simulations below we have chosen the same q as the one in the figure above and we settled for $E_1 = 0.85$, $E_2 = 1$, $h = 1/100$, leading to

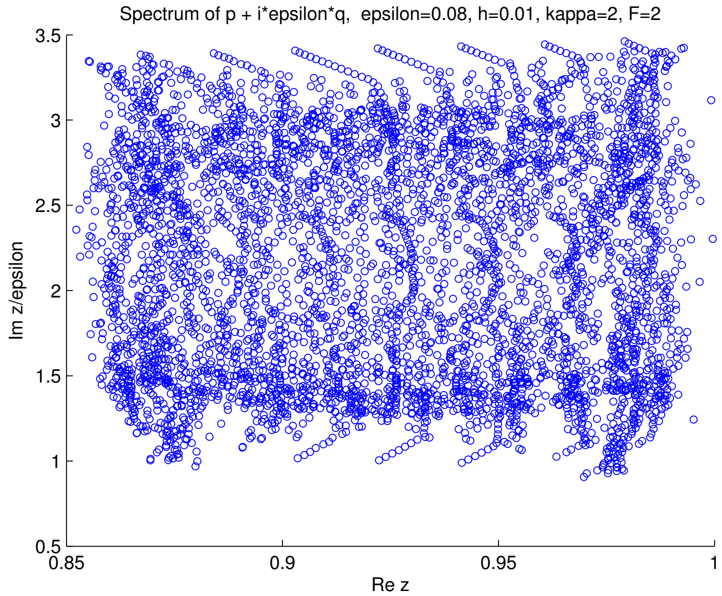
$\#\mathcal{E} \approx 5000$. Since the spectra are of width ε, we rescale the imaginary axis and represent graphically the set of $(\mathrm{Re}\, z, \mathrm{Im}\, z/\varepsilon)$ for z in the spectrum of P_ε. We let ε take the values $h/2$, h, $2h$, $4h$, $8h$, $16h$, in agreement with Theorem 2.1.

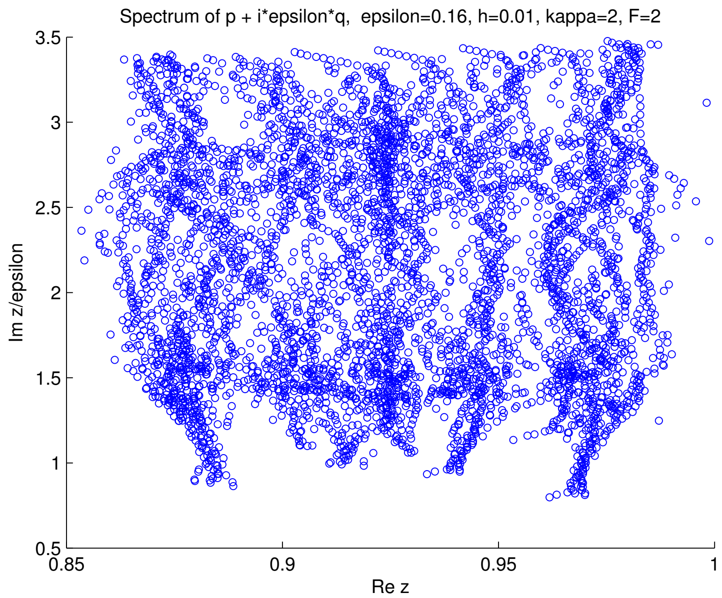





Spectrum of p + i*epsilon*q, epsilon=0.02, h=0.01, kappa=2, F=2
Im z/epsilon
Re z
3.5
3
2.5
2
1.5
1
0.5
0.85
0.9
0.95
1


Spectrum of p + i*epsilon*q, epsilon=0.04, h=0.01, kappa=2, F=2
Im z/epsilon
Re z
3.5
3
2.5
2
1.5
1
0.5
0.85
0.9
0.95
1





These eigenvalues form a kind of a centipede with legs sticking out from the main body. The majority of the eigenvalues are in the body whose position corresponds nicely to the range of the curve of torus averages on the first picture. The legs reach out to the supremum of the highest and the infimum of the lowest vertical segments corresponding to $Q_\infty(\Lambda)$ for rational tori Λ.

After undoing the scaling of $\operatorname{Im} z$, the inclination of the legs should theoretically be close to 45 degrees, and by measuring this for one of the legs on one of the figures we found some (but not excellent) agreement.

The main result of this work, Theorem 2.1, describes the individual eigenvalues near the extremities of the legs in terms of rational tori. A mathematical treatment of the eigenvalues further inside seems more difficult because of the pseudospectral effects that are likely to get stronger there.

By staring at the pictures directly from the pdf file and creating a movie by switching the pages, we see that most of the (rescaled) eigenvalues remain fixed, while those in the legs and some others move. The fixed ones probably correspond to irrational tori, and the moving ones to tori that are rational.

Appendix. Subelliptic estimates for Schrödinger type operators

The purpose of this appendix is to establish suitable resolvent estimates for some non-selfadjoint operators of Schrödinger type, instrumental in the pseudospectral analysis of Section 7. While in the considerations of Section 7, we are concerned with operators on the one-dimensional torus, it will be convenient to analyze the case of $\mathbb{R}$ first. See also [19] and [1].

Let

$$P_0 = g(hD_x)(hD_x)^2 + iV(x), \qquad V \in C^\infty(\mathbb{R};\mathbb{R}). \tag{A.1}$$

Assume that the function $g \in C^\infty(\mathbb{R};\mathbb{R})$ is such that

$$g - 1 \in C_0^\infty(\mathbb{R}) \tag{A.2}$$

with

$$g \geq 1, \qquad |\xi g'(\xi)| \ll 1. \tag{A.3}$$

We may notice that the conditions (A.3) are invariant under the scaling $g(\xi) \to g(\lambda\xi)$, $\lambda > 0$. We also assume that the potential V is such that

$$V \geq 0, \qquad \partial_x^j V \in L^\infty(\mathbb{R}), \qquad j \geq 2, \tag{A.4}$$

and let us make the ellipticity assumption,

$$V(x) \geq x^2/C, \qquad |x| \geq C, \tag{A.5}$$

for some constant $C > 0$. The semiclassical symbol of P_0, $p_0(x,\xi) = g(\xi)\xi^2 + iV(x)$, satisfies $p_0 \in S(m)$, where $m(x,\xi) = 1 + x^2 + \xi^2$, and it follows from (A.5) that when equipped with the domain $H(m)$, the natural Sobolev space associated to the order function m, the operator P_0 becomes closed densely defined on $L^2(\mathbb{R})$. The injection of $H(m)$ in $L^2(\mathbb{R})$ is compact, and therefore the spectrum of P_0 is discrete. Let us also notice that $\operatorname{Re} P_0 \geq 0$, $\operatorname{Re}(-iP_0) \geq 0$, and therefore

$$\operatorname{Spec}(P_0) \subset p_0(\mathbb{R}^2) = \{z \in \mathbb{C};\ \arg z \in [0, \pi/2]\}. \tag{A.6}$$

Let us make the basic assumption that

$$V^{-1}(0) = \{0\} \subset \mathbb{R} \tag{A.7}$$

and

$$V''(0) > 0. \tag{A.8}$$

We are interested in estimates for the resolvent of P_0,

$$(P_0 - z)^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R}),$$

when the spectral parameter $z \in \mathbb{C}$ is such that $0 \le \operatorname{Im} z \le \mathcal{O}(h)$ and $|z| \gg h$. When establishing those, we shall combine some of the results and techniques of [8] and [10].

In what follows, rather than working with P_0, it will be convenient to consider the operator

$$P = -iP_0 = V(x) - ig(hD_x)(hD_x)^2 \tag{A.9}$$

with symbol $p = p_1 + ip_2$, where

$$p_1(x,\xi) = V(x) = V_0(x) + \mathcal{O}(x^3), \qquad V_0(x) = \frac{1}{2}V''(0)x^2 > 0,$$
$$p_2(x,\xi) = -g(\xi)\xi^2.$$

It follows from (A.3) that $|\partial_\xi p_2| \sim |\xi|$, and therefore we obtain the fundamental property

$$V_0(x) + H_{p_2}^2 V_0(x,\xi) \sim |(x,\xi)|^2, \qquad (x,\xi) \in \mathbb{R}^2. \tag{A.10}$$

Following [8], let us set, writing $X = (x,\xi) \in \mathbb{R}^2$,

$$G_0(X;h) = h^{2/3}\frac{H_{p_2}V_0}{|X|^{4/3}}\psi\left(M\frac{V_0(x)}{(h|X|)^{2/3}}\right), \qquad |X| \ge h^{1/2}. \tag{A.11}$$

Here $M \ge 1$ is a constant to be taken large enough, and $\psi \in C_0^\infty(\mathbb{R};[0,1])$ is such that $\operatorname{supp}(\psi) \subset (-2,2)$ and $\psi = 1$ on $[-1,1]$. It is then straightforward to verify that in the region $|X| \ge h^{1/2}$, we have

$$G_0 = \mathcal{O}(h), \qquad H_{G_0} = \mathcal{O}(1)\frac{h^{2/3}}{|X|^{1/3}} = \mathcal{O}(h^{1/2}), \tag{A.12}$$

$$\partial^2 G_0 = \mathcal{O}(1)\left(\frac{h^{2/3}}{|X|^{4/3}} + \frac{h^{1/3}}{|X|^{2/3}} + \frac{h^{2/3}}{|X|^{1/3}}\right) = \mathcal{O}(1). \tag{A.13}$$

Indeed, the validity of (A.12) and (A.13) follows easily once we observe that in the region where $0 \le MV_0(x) \le 2(h|X|)^{2/3}$ and $|X| \ge h^{1/2}$, we have

$$H_{p_2}V_0 = \mathcal{O}(|X|h^{1/3}|X|^{1/3}), \qquad \nabla(H_{p_2}V_0) = \mathcal{O}(|X|), \qquad \nabla^2(H_{p_2}V_0) = \mathcal{O}(1+|X|),$$

and

$$\nabla^j\left(\psi\left(M\frac{V_0(x)}{(h|X|)^{2/3}}\right)\right) = \mathcal{O}(1)\frac{1}{h^{j/3}|X|^{j/3}}, \qquad j = 1,2.$$

Still working in the region $|X| \geq h^{1/2}$ and following [8] closely, let us obtain a lower bound for the function $V_0 + \varepsilon_0 H_{p_2} G_0$, where $\varepsilon_0 > 0$ is a constant to be chosen. We have, in view of (A.12),

$$H_{p_2} G_0 = \mathcal{O}(h^{2/3}|X|^{2/3}),$$

and therefore in the region $M V_0 \geq h^{2/3}|X|^{2/3}$ we get

$$V_0 + \varepsilon_0 H_{p_2} G_0 \geq \left(\frac{1}{M} - \mathcal{O}(\varepsilon_0)\right) h^{2/3}|X|^{2/3} \geq \frac{1}{\mathcal{O}(1)M} h^{2/3}|X|^{2/3},$$

if we choose $\varepsilon_0 > 0$ small enough. In the region $M V_0 < h^{2/3}|X|^{2/3}$, we have

$$G_0 = h^{2/3} \frac{H_{p_2} V_0}{|X|^{4/3}},$$

and therefore

$$V_0 + \varepsilon_0 H_{p_2} G_0 = V_0 + \varepsilon_0 h^{2/3} |X|^{-4/3} H_{p_2}^2 V_0 + R,$$

where

$$R = \varepsilon_0 h^{2/3} (H_{p_2} V_0) H_{p_2} |X|^{-4/3} = \mathcal{O}\left(\frac{\varepsilon_0 h}{M^{1/2}}\right) = \mathcal{O}\left(\frac{\varepsilon_0}{M^{1/2}}\right) h^{2/3}|X|^{2/3}.$$

Using (A.10), we see therefore that

$$V_0 + \varepsilon_0 H_{p_2} G_0 \geq \varepsilon_0 \frac{h^{2/3}|X|^{2/3}}{\mathcal{O}(1)} - \mathcal{O}\left(\frac{\varepsilon_0}{M^{1/2}}\right) h^{2/3}|X|^{2/3} \geq \varepsilon_0 \frac{h^{2/3}|X|^{2/3}}{\mathcal{O}(1)},$$

provided that we take M sufficiently large but fixed. It follows that in the entire region $|X| \geq h^{1/2}$, we get

$$V_0 + \varepsilon_0 H_{p_2} G_0 \geq \frac{h^{2/3}|X|^{2/3}}{\mathcal{O}(1)}. \tag{A.14}$$

We shall now extend the definition of G_0 to all of $\mathbb{R}^2$, and following [8], let us set

$$G(X; h) = \left(1 - \chi\left(\frac{X}{h^{1/2}}\right)\right) G_0(X; h). \tag{A.15}$$

Here $\chi \in C_0^\infty(\mathbb{R}^2; [0, 1])$ is such that $\chi = 1$ when $|X| \leq 1$. It follows from (A.12) and (A.13) that

$$G = \mathcal{O}(h), \qquad H_G = \mathcal{O}(h^{1/2}), \qquad \partial^2 G = \mathcal{O}(1).$$

Furthermore, using (A.14) we immediately check that on all of $\mathbb{R}^2$, we have

$$V_0 + \varepsilon_0 H_{p_2} G \geq h^{2/3}|X|^{2/3}/\mathcal{O}(1) - \mathcal{O}(h).$$

We may therefore summarize the discussion above by stating that there exist constants $\varepsilon_0, c_0 > 0$ such that for all $h > 0$ sufficiently small,

$$V_0(x) + \varepsilon_0 H_{p_2} G(X) + c_0 h \geq h^{2/3}|X|^{2/3}/\mathcal{O}(1), \qquad X = (x, \xi) \in \mathbb{R}^2. \tag{A.16}$$

Here the real-valued weight function $G = G(X, h) \in C^\infty(\mathbb{R}^2)$ has been defined in (A.11), (A.15).

It follows from (A.5), (A.7), (A.8) that $p_1(X) = V(x) \geq (1/C)V_0(x)$ for some constant $C > 1$, and therefore using (A.16) we get, with some $\varepsilon_1, c_1 > 0$,

$$\operatorname{Re} p(X) + \varepsilon_1 H_{\operatorname{Im} p} G(X) + c_1 h \geq \frac{1}{\mathcal{O}(1)} (h|X|)^{2/3}, \quad X \in \mathbb{R}^2. \tag{A.17}$$

The estimate (A.17) is analogous to [10, (4.26)], if we take $k_0 = 1$ there. See also [7]. Taking (A.17) as the starting point and arguing exactly as in that work, we find that everything works without any change, provided that the spectral parameter $z \in \mathbb{C}$ is such that for some fixed $C_0 > 1$,

$$\operatorname{Re} z \leq \mathcal{O}(1) h^{2/3} |z|^{1/3}, \quad Ch \leq |z| \leq C_0. \tag{A.18}$$

Here $C \gg 1$ is a constant large enough and the implicit constant in (A.18) does not depend on C. We therefore obtain the a priori estimate

$$h^{2/3} |z|^{1/3} \|u\|_{L^2} \leq \mathcal{O}(1) \|(P - z)u\|_{L^2}, \quad u \in \mathcal{S}(\mathbb{R}), \tag{A.19}$$

for z satisfying (A.18).

It therefore remains to discuss the case when $z \in \mathbb{C}$ is such that

$$\operatorname{Re} z \leq \mathcal{O}(1) h^{2/3} |z|^{1/3}, \quad |z| \geq C_0. \tag{A.20}$$

Continuing to follow the arguments of [10, Section 4], we deduce from [10, (4.34)] that there exist positive constants c_1, c_2 such that for $z \in \mathbb{C}$ satisfying (A.20), we have

$$\|(P - z)u\|_{L^2} \|u\|_{L^2} + c_1 h^{2/3} |z|^{1/3} (\varphi(|X|^2/|z|)^w u, u)_{L^2} \geq c_2 h^{2/3} |z|^{1/3} \|u\|^2_{L^2}. \tag{A.21}$$

Here $\varphi \in C_0^\infty(\mathbb{R}^2; [0, 1])$ is a cut-off near 0 such that

$$|p(X) - z| \geq |z|/2, \tag{A.22}$$

on the support of $\varphi(|X|^2/|z|)$. The spectral parameter z in (A.20) can be arbitrarily large and when estimating the scalar product in the left hand side of (A.21), we can apply [8, Lemma 8.2], exactly as it stands, to conclude that

$$(\varphi(|X|^2/|z|)^w u, u)_{L^2} \leq \frac{\mathcal{O}(1)}{|z|^2} \|(P - z)u\|^2_{L^2} + \mathcal{O}(1) h \|u\|^2_{L^2}. \tag{A.23}$$

Indeed, it is easily seen that the proof of [8, Lemma 8.2] applies in the present situation, in view of the ellipticity property (A.22) and the fact that the symbol p satisfies

$$|p(X)| \leq \mathcal{O}(1)|X|^2, \quad X \in \mathbb{R}^2, \quad \partial^\alpha p \in L^\infty(\mathbb{R}^2), \quad |\alpha| \geq 2. \tag{A.24}$$

Combining (A.21) and (A.23), we get

$$Z \|u\|^2_{L^2} \leq \mathcal{O}(1) \frac{Z}{|z|^2} \|(P - z)u\|^2_{L^2} + \mathcal{O}(1) \|(P - z)u\|_{L^2} \|u\|_{L^2}. \tag{A.25}$$

Here we have written $Z = h^{2/3}|z|^{1/3}$ for brevity. It follows that

$$Z\|u\|^2 \leq \mathcal{O}(1)\left(\frac{1}{Z} + \frac{Z}{|z|^2}\right)\|(P - z)u\|_{L^2}^2, \tag{A.26}$$

and using also $Z \leq |z|$, we get

$$h^{2/3}|z|^{1/3}\|u\|_{L^2} \leq \mathcal{O}(1)\|(P - z)u\|_{L^2}. \tag{A.27}$$

We summarize the discussion above in the following result.

Theorem A.1. *Let $P_0 = g(hD_x)(hD_x)^2 + iV(x)$ be such that (A.2)–(A.5) and (A.7)–(A.8) hold. There exist constants $c_0, c_1 > 0$ such that for every $C > 1$ large enough, and for*

$$\operatorname{Im} z \leq c_1 h^{2/3}|z|^{1/3}, \qquad |z| \geq Ch, \tag{A.28}$$

we have

$$h^{2/3}|z|^{1/3}\|u\|_{L^2} \leq c_1\|(P_0 - z)u\|_{L^2}, \qquad u \in H(m). \tag{A.29}$$

It follows that if z satisfies (A.28) then it is in the resolvent set of P_0 and we get the resolvent estimate

$$(P_0 - z)^{-1} = \mathcal{O}(h^{-2/3}|z|^{-1/3}) : L^2(\mathbb{R}) \to L^2(\mathbb{R}). \tag{A.30}$$

Remark. The discussion above and the result of Theorem A.1 extend to the case of operators on $\mathbb{R}^n$.

In what follows, the result of Theorem A.1 will only be applied when $\operatorname{Im} z = \mathcal{O}(h)$. Furthermore, in the considerations in Section 7, we are concerned with operators on the one-dimensional torus $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, and our next task is therefore to adapt Theorem A.1 to this setting. Let us therefore consider

$$P = g(hD_x)(hD_x)^2 + iV(x), \tag{A.31}$$

where $g \in C^\infty(\mathbb{R}; \mathbb{R})$ satisfies (A.2)–(A.3), and let $0 \leq V \in C^\infty(\mathbb{T})$ be such that $V^{-1}(0) = \{x_0\}$ with $V''(x_0) > 0$. We may write

$$P = (hD_x)^2 + \varphi(hD_x) + iV(x), \qquad \varphi \in C_0^\infty(\mathbb{R}; \mathbb{R}). \tag{A.32}$$

Let $\chi \in C^\infty(\mathbb{T}^n; [0, 1])$ be supported in a small neighborhood of x_0, and such that $\chi = 1$ near x_0. On the support of χ, the result of Theorem A.1 can be applied, and we conclude, using the pseudolocality of P, that

$$h^{2/3}|z|^{1/3}\|\chi u\|_{L^2} \leq \mathcal{O}(1)\|(P - z)\chi u\|_{L^2} + \mathcal{O}(h^\infty)\|u\|_{L^2}, \qquad u \in C^\infty(\mathbb{T}), \tag{A.33}$$

provided that

$$|z| \geq Ch, \qquad \operatorname{Im} z \leq \mathcal{O}(h). \tag{A.34}$$

Since on the support of $1-\chi$, the potential V is bounded from below and $\operatorname{Im} z = \mathcal{O}(h)$, we see that for all $h > 0$ small enough,

$$\operatorname{Im}((P-z)(1-\chi)u, (1-\chi)u)_{L^2} \geq \frac{1}{\mathcal{O}(1)} \|(1-\chi)u\|_{L^2}^2, \tag{A.35}$$

and therefore

$$\|(1-\chi)u\|_{L^2} \leq C\|(P-z)(1-\chi)u\|_{L^2}. \tag{A.36}$$

Combining the estimates (A.33) and (A.36), we see that for all $h > 0$ small enough,

$$Z\|\chi u\|_{L^2} + \|(1-\chi)u\|_{L^2} \leq \mathcal{O}(1)\|(P-z)u\|_{L^2} + \mathcal{O}(1)\|[P,\chi]u\|_{L^2}. \tag{A.37}$$

Here we continue to write $Z = h^{2/3}|z|^{1/3}$ and we notice that $h \ll Z$. We would like to estimate the commutator term on the right hand side of (A.37); to that end we write, using (A.32),

$$\mathcal{O}(1)\|[P,\chi]u\|_{L^2} \leq \mathcal{O}(h)\|u\|_{L^2} + \mathcal{O}(h)\|hu'\|_{L^2}. \tag{A.38}$$

The first term on the right hand side of (A.38) can be absorbed into the left hand side of (A.37), and we only have to estimate $\mathcal{O}(h)\|hu'\|_{L^2}$. Now

$$(\varphi(hD_x)u, u)_{L^2} + \|hu'\|_{L^2}^2 = \operatorname{Re}((P-z)u, u)_{L^2} + \operatorname{Re} z\|u\|_{L^2}^2,$$

and therefore

$$\|hu'\|_{L^2} \leq \|(P-z)u\|_{L^2}^{1/2}\|u\|_{L^2}^{1/2} + |z|^{1/2}\|u\|_{L^2} + \mathcal{O}(1)\|u\|_{L^2}. \tag{A.39}$$

Combining (A.37)–(A.39), we obtain

$$\begin{aligned} &Z\|\chi u\|_{L^2} + \|(1-\chi)u\|_{L^2} \\ &\quad \leq \mathcal{O}(1)\|(P-z)u\| + \mathcal{O}(h)|z|^{1/2}\|u\|_{L^2} + \mathcal{O}(h)\|(P-z)u\|_{L^2}^{1/2}\|u\|_{L^2}^{1/2}, \end{aligned} \tag{A.40}$$

so that

$$Z\|\chi u\|_{L^2} + \|(1-\chi)u\|_{L^2} \leq \mathcal{O}(1)\|(P-z)u\|_{L^2} + \mathcal{O}(Z^{3/2})\|u\|_{L^2}. \tag{A.41}$$

Assuming that $Z = h^{2/3}|z|^{1/3} \ll 1$, we may absorb the second term on the right hand side of (A.41) into the left hand side, obtaining

$$h^{2/3}|z|^{1/3}\|u\|_{L^2} \leq \mathcal{O}(1)\|(P-z)u\|_{L^2}.$$

We may summarize the discussion above in the following proposition.

Proposition A.2. *Let $P = g(hD_x)(hD_x)^2 + iV(x)$ be such that $g - 1 \in C_0^\infty(\mathbb{R};\mathbb{R})$ satisfies $g \geq 1$ and $|\xi g'(\xi)| \ll 1$. Assume furthermore that $0 \leq V \in C^\infty(\mathbb{T})$ with $V^{-1}(0) = \{x_0\}$ and $V''(x_0) > 0$. Then for every $C > 1$ large enough, and for*

$$\operatorname{Im} z = \mathcal{O}(h), \quad |z| \geq Ch,$$

satisfying

$$h|z|^{1/2} \ll 1, \tag{A.42}$$

we have

$$h^{2/3}|z|^{1/3}\|u\|_{L^2} \leq \mathcal{O}(1)\|(P-z)u\|_{L^2}. \tag{A.43}$$

The a priori estimate (A.43) is equivalent to the corresponding estimate for the resolvent of P, which provides a resolvent bound in the model case, fundamental for the analysis in Section 7. Now the operators that one encounters there are somewhat more general than the Schrödinger type operator in (A.31), in that the potential $V(x)$ should be replaced by a more general h-pseudodifferential operator, which furthermore is multiplied by a small coupling constant. We shall now proceed to analyze this more general case, essentially by reducing it to the model situation treated above.

Let us first consider the following operator on $\mathbb{R}$:

$$P_\varepsilon(x, hD_x) = g(hD_x)(hD_x)^2 + i\varepsilon \widetilde{V}^w(x, hD_x), \tag{A.44}$$

where g satisfies (A.2)–(A.3), and following (7.1) we assume that

$$h^2 \ll \varepsilon \ll h^{4/5}. \tag{A.45}$$

The function $\widetilde{V} \in C^\infty(\mathbb{R}^2)$ in (A.44) is of the form

$$\widetilde{V}(x, \xi) = V(x) + k(x, \xi)\varphi(\xi/\varepsilon^\delta), \qquad \delta \in (0, 1/2), \tag{A.46}$$

where V is assumed to satisfy (A.4), (A.5), (A.7), and (A.8), and $\varphi \in C_0^\infty(\mathbb{R}^n)$ is a standard cut-off function near $\xi = 0$. The function k is such that

$$\partial^\alpha k \in L^\infty(\mathbb{R}^2), \quad \alpha \in \mathbb{N}^2, \quad k(x, 0) = 0. \tag{A.47}$$

We would like to extend Theorem A.1 to the operator $P_\varepsilon(x, hD_x)$; to that end we shall simply inspect the arguments above. Writing

$$\frac{1}{i\varepsilon} P_\varepsilon(x, hD_x) = -ig(\sqrt{\varepsilon}\widetilde{h}D_x)(\widetilde{h}D_x)^2 + \widetilde{V}^w(x, \sqrt{\varepsilon}\widetilde{h}D_x), \quad \widetilde{h} = \frac{h}{\sqrt{\varepsilon}} \ll 1, \tag{A.48}$$

we shall view $(1/(i\varepsilon))P_\varepsilon$ as an $\widetilde{h}$-pseudodifferential operator with symbol

$$p(x, \xi) = p_1 + ip_2, \tag{A.49}$$

where

$$p_1(x, \xi) = \widetilde{V}(x, \sqrt{\varepsilon}\,\xi), \qquad p_2(x, \xi) = -g(\sqrt{\varepsilon}\,\xi)\xi^2. \tag{A.50}$$

Writing

$$p_1(x, 0) = V(x) = V_0(x) + \mathcal{O}(x^3), \qquad V_0(x) = \frac{1}{2}V''(0)x^2,$$

we see that uniformly in ε,

$$V_0 + H_{p_2}^2 V_0 \sim |X|^2, \qquad X \in \mathbb{R}^2. \tag{A.51}$$

Here we also notice that $\partial^\alpha p \in L^\infty(\mathbb{R}^2)$ for all $\alpha \in \mathbb{N}^2$ with $|\alpha| \geq 2$, uniformly in ε. Arguing as in (A.11), (A.16), (A.17), we conclude that there exists a real-valued weight function $G \in C^\infty(\mathbb{R}^2)$ with

$$G(X) = \mathcal{O}(\widetilde{h})$$

such that for some constants $\delta_1, c_1 > 0$ and $\widetilde{h} > 0$ small enough, we have

$$\operatorname{Re} p(x,0) + \delta_1 H_{\operatorname{Im} p} G(X) + c_1 \widetilde{h} \geq \frac{1}{C} \widetilde{h}^{2/3} |X|^{2/3}, \quad X = (x,\xi) \in \mathbb{R}^2. \tag{A.52}$$

Using (A.46) and (A.52), we obtain

$$\operatorname{Re} p(X) + \delta_1 H_{\operatorname{Im} p} G(X) + c_1 \widetilde{h} \geq \frac{1}{C} \widetilde{h}^{2/3} |X|^{2/3} - 1_{\sqrt{\varepsilon}|\xi| \leq \mathcal{O}(\varepsilon^{\delta})} \mathcal{O}(\sqrt{\varepsilon}\,|\xi|). \tag{A.53}$$

Here we have used the fact that

$$k(x,\xi) = \mathcal{O}(|\xi|). \tag{A.54}$$

When estimating the right hand side in (A.53), we notice that

$$\begin{aligned} \frac{1}{2C} \widetilde{h}^{2/3} |X|^{2/3} - 1_{\sqrt{\varepsilon}|\xi| \leq \mathcal{O}(\varepsilon^{\delta})} \mathcal{O}(\sqrt{\varepsilon}\,|\xi|) &\geq \frac{1}{2C} |\xi|^{2/3} \big(\widetilde{h}^{2/3} - 1_{\sqrt{\varepsilon}|\xi| \leq \mathcal{O}(\varepsilon^{\delta})} \mathcal{O}(\sqrt{\varepsilon}\,|\xi|^{1/3})\big) \\ &\geq \frac{1}{2C} |\xi|^{2/3} (\widetilde{h}^{2/3} - \mathcal{O}(\sqrt{\varepsilon}\varepsilon^{\delta/3 - 1/6})) \geq 0, \end{aligned} \tag{A.55}$$

provided that

$$\varepsilon^{\delta/3 + 1/2 - 1/6} \ll \widetilde{h}^{2/3} = h^{2/3}/\varepsilon^{1/3}. \tag{A.56}$$

The latter condition is equivalent to

$$\varepsilon^{1+\delta/2} \ll h. \tag{A.57}$$

Assuming that (A.57) holds, we conclude that

$$\operatorname{Re} p(X) + \delta_1 H_{\operatorname{Im} p} G(X) + c_1 \widetilde{h} \geq \frac{1}{\mathcal{O}(1)} \widetilde{h}^{2/3} |X|^{2/3}, \quad X \in \mathbb{R}^2. \tag{A.58}$$

In order to be able to apply the general arguments of [10] and [8] to the operator $P_\varepsilon/(i\varepsilon)$, similarly to the discussion leading to Theorem A.1 above, we should also observe that for $|z| \gg \widetilde{h}$, in the region $|X|^2 \leq |z|/C$ we have the ellipticity condition

$$|p(X) - z| \geq |z|/C_1,$$

uniformly in ε. Indeed, this follows from (A.46), (A.49), (A.50), (A.54), as well as the fact that $\varepsilon \ll \widetilde{h} \ll |z|$. It is then straightforward to check that the arguments at the beginning of the Appendix apply, and we obtain the following result.

Theorem A.3. *Let $P_\varepsilon = g(hD_x)(hD_x)^2 + i\varepsilon \widetilde{V}^w(x, hD_x)$, where $h^2 \ll \varepsilon \ll h^{4/5}$. Assume that $g \in C^\infty(\mathbb{R};\mathbb{R})$ satisfies (A.2)–(A.3), $\widetilde{V}$ is of the form (A.46), (A.47), and suppose that (A.4), (A.5), (A.7), (A.8) hold. Assume that*

$$\varepsilon^{1+\delta/2} \ll h.$$

Then, writing $\widetilde{h} = h/\sqrt{\varepsilon}$, we have

$$(P_\varepsilon/\varepsilon - z)^{-1} = \mathcal{O}(\widetilde{h}^{-2/3} |z|^{-1/3}) : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \tag{A.59}$$

provided that $|z| \geq C\widetilde{h}$ for $C > 1$ sufficiently large, and $\operatorname{Im} z \leq \mathcal{O}(\widetilde{h})$.

Repeating the arguments leading to Proposition A.2, with Theorem A.1 replaced by Theorem A.3 and with an estimate of the form (A.35) obtained by an application of Gårding's inequality, we next obtain an adaptation of Theorem A.3 to the setting of the torus.

Proposition A.4. *Let $P_\varepsilon = g(hD_x)(hD_x)^2 + i\varepsilon \widetilde{V}^w(x, hD_x)$, where $h^2 \ll \varepsilon \ll h^{4/5}$ and $g \in C^\infty(\mathbb{R}; \mathbb{R})$ is such that $g - 1 \in C_0^\infty(\mathbb{R})$ with*

$$g \geq 1, \quad |\xi g'(\xi)| \ll 1.$$

Assume that

$$\widetilde{V}(x, \xi) = V(x) + k(x, \xi)\varphi(\xi/\varepsilon^\delta), \quad \delta \in (0, 1/2).$$

Here $0 \leq V \in C^\infty(\mathbb{T})$, $V^{-1}(0) = \{x_0\}$, $V''(x_0) > 0$, $k \in S(T^*\mathbb{T}, 1)$, $k(x, 0) = 0$, and $\varphi \in C_0^\infty(\mathbb{R})$. Assume that $\varepsilon^{1+\delta/2} \ll h$ and set

$$\widetilde{h} = h/\sqrt{\varepsilon} \ll 1.$$

Let $z \in \mathbb{C}$ be such that $\operatorname{Im} z = \mathcal{O}(\widetilde{h})$, $|z| \geq C\widetilde{h}$, for $C > 1$ sufficiently large, and

$$\widetilde{h}|z|^{1/2} \ll 1.$$

Then

$$(P_\varepsilon/\varepsilon - z)^{-1} = \mathcal{O}(\widetilde{h}^{-2/3}|z|^{-1/3}) : L^2(\mathbb{T}) \to L^2(\mathbb{T}). \tag{A.60}$$

Acknowledgments. We are grateful to Michael Hall and San Vũ Ngọc for interesting discussions concerning numerical computations of eigenvalues and for showing us promising numerical results for a damped spherical oscillator. The second author is supported by the project NOSEVOL ANR 2011 BS 01019 01. We would also like to thank the referees for numerous helpful comments and suggestions.

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