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On localization sequences in the algebraic K -theory of ring spectra

Received April 14, 2015

Abstract. We identify the K -theoretic fiber of a localization of ring spectra in terms of the K -theory of the endomorphism algebra spectrum of a Koszul-type complex. Using this identification, we provide a negative answer to a question of Rognes for $n > 1$ by comparing the traces of the fiber of the map $K(\mathrm{BP}\langle n \rangle) \rightarrow K(E(n))$ and of $K(\mathrm{BP}\langle n-1 \rangle)$ in rational topological Hochschild homology.

Keywords. Algebraic K -theory, structured ring spectra, trace methods

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0. Introduction

This paper is about the algebraic K -theory of structured ring spectra, or \mathbb{E}_1 -rings, occurring in chromatic homotopy theory and the Ausoni–Rognes program for computing the K -theory of the sphere spectrum. The two ring spectra of interest, the truncated Brown–Peterson spectrum $\mathrm{BP}\langle n \rangle$ and the Johnson–Wilson theory $E(n)$ ($n \geq 0$), are constructed

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Mathematics Subject Classification (2010): Primary [19D55](#), [55P43](#); Secondary [16E40](#), [18E30](#), [19D10](#)

using the complex cobordism spectrum MU and exist for any prime p ; their homotopy rings are

$$\pi_*BP\langle n \rangle \cong \mathbb{Z}_{(p)}[v_1, \dots, v_n] \quad \text{and} \quad \pi_*E(n) = \mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}, v_n^{\pm 1}],$$

where v_i has degree $2p^i - 2$.

The following well-known question¹ of Rognes first appears in [5, 0.1]; see also [8, Example 5.15], [10, Example 5.15], [7, Example 11.17], and the introduction to [14].

Question 0.1 (Rognes). Is the sequence

$$K(BP\langle n - 1 \rangle_p) \rightarrow K(BP\langle n \rangle_p) \rightarrow K(E(n)_p)$$

of connective algebraic K -theory spectra a fiber sequence of connective spectra?

The map $K(BP\langle n \rangle_p) \rightarrow K(E(n)_p)$ is induced by the map $BP\langle n \rangle \rightarrow E(n)$ which inverts v_n and p -completion, while $K(BP\langle n - 1 \rangle_p) \rightarrow K(BP\langle n \rangle_p)$ is the *transfer* map, obtained by viewing $BP\langle n - 1 \rangle_p$, the cofiber of multiplication by v_n , as a compact $BP\langle n \rangle_p$ -module.

When $n = 0$, this is a special case of a theorem of Quillen [31, Theorem 5], saying that there is a fiber sequence $K(\mathbb{F}_p) \rightarrow K(\mathbb{Z}_p) \rightarrow K(\mathbb{Q}_p)$. In this case it is common to let $v_0 = p$ in \mathbb{Z}_p and $BP\langle -1 \rangle = H\mathbb{F}_p$. When $n = 1$, the sequence was conjectured by Rognes and proved by Blumberg and Mandell in [14]. In fact, both Quillen and Blumberg–Mandell prove p -local and integral versions of these statements.

The backdrop of the question of Rognes is the Ausoni–Rognes program to compute the algebraic K -theory of the sphere spectrum while keeping control of chromatic phenomenon. The layers in the chromatic tower are closely related to $K(E_n)$, and it is expected that $K(E_n)$ should behave in a similar way to $K(E(n))$. However, there is a fundamental problem with computing $K(E_n)$ and $K(E(n))$: they are nonconnective ring spectra. There are no general methods for computing the K -groups of nonconnective ring spectra, and it is in general even difficult to produce candidate elements. The little success that has been had here is to relate the K -theory of the nonconnective ring spectrum to the K -theory of connective ring spectra, where there are a variety of methods of computation, such as using determinants and traces or studying $BGL(R)^+$.

One reason to study $K(BP\langle n \rangle_p)$ in its own right is that it is expected to exhibit redshift, a phenomenon visible for small values of n in which the K -theory of a ring spectrum related to chromatic height n carries chromatic height $n + 1$ information [5, p. 7]. For example, v_0 acts trivially on \mathbb{F}_p , whereas $K(\mathbb{F}_p)_p \simeq H\mathbb{Z}_p$ and hence $K(\mathbb{F}_p)$ carries a highly non-trivial v_0 -self map: multiplication by p .

We give a negative answer to Rognes’ question in all of the remaining cases.

Theorem 0.2. *For $n > 1$, the sequence*

$$K(BP\langle n - 1 \rangle_p) \rightarrow K(BP\langle n \rangle_p) \rightarrow K(E(n)_p)$$

is not a fiber sequence of connective spectra.

¹ The authors of [5] refer to this statement as an expectation. It has come to be known in the literature as a conjecture, especially in the work of Barwick and Blumberg–Mandell.

When $n = 0, 1$, both Quillen and Blumberg–Mandell relate the fiber of $\mathbb{K}(\mathrm{BP}\langle n \rangle_p) \rightarrow \mathbb{K}(\mathrm{E}(n)_p)$ to $\mathbb{K}(\mathrm{BP}\langle n - 1 \rangle_p)$ using a dévissage argument, and this perspective has been used by Barwick and Lawson to prove some other examples of this localization behavior [9]. Most attempts to answer Rognes’ question for $n > 1$, such as the approach outlined by Barwick [8], focus on conjectural dévissage arguments. Our theorem shows that in some sense these cannot work in general. As a corollary of Theorem 0.2, the ∞ -category $Z(f_\star)$ of [8, Example 5.15] is not weakly contractible.

Our approach is Morita-theoretic. As motivation, consider the major result of Thomason–Trobrough [39, Theorem 7.4], the localization theorem. It states that if X is a quasi-compact and quasi-separated scheme, and if $U \subseteq X$ is a quasi-compact Zariski open with complement Z , then there is a fiber sequence of nonconnective algebraic K -theory spectra

$$\mathbb{K}(X \text{ on } Z) \rightarrow \mathbb{K}(X) \rightarrow \mathbb{K}(U),$$

where $\mathbb{K}(X \text{ on } Z)$ is the K -theory of perfect complexes on X that are acyclic on U .

In general $\mathbb{K}(X \text{ on } Z)$ is *not* equivalent to $\mathbb{K}(Z)$, the K -theory of the closed subscheme Z . The main examples when $\mathbb{K}(X \text{ on } Z)$ is equivalent to $\mathbb{K}(Z)$ occur when X and Z are regular and noetherian, and the argument passes through G -theory via dévissage. However, Bökstedt and Neeman [20, Proposition 6.1] showed that nevertheless $\mathbb{K}(X \text{ on } Z)$ is the K -theory of a ring spectrum. More specifically, they showed that at the level of derived categories, the kernel of the localization

$$D_{\mathrm{qc}}(X) \rightarrow D_{\mathrm{qc}}(U)$$

is generated by a single compact object K . If $A = \mathrm{End}_X(K)^{\mathrm{op}}$ denotes the opposite of the dg-algebra of endomorphisms of K , then

$$\mathbb{K}(A) \simeq \mathbb{K}(X \text{ on } Z),$$

so we can write our localization fiber sequence as

$$\mathbb{K}(A) \rightarrow \mathbb{K}(X) \rightarrow \mathbb{K}(U). \tag{1}$$

Antieau and Gepner [3, Proposition 6.9] proved the analogue of this result for localizations of spectral schemes, which motivated our approach here. However, the truncated Brown–Peterson spectra are not known to admit \mathbb{E}_∞ -ring structures, a necessary input in [3]. Noncommutative localization sequences have been studied extensively in the K -theory of ordinary rings (see Neeman and Ranicki [30] and the references there). We prove a spectral noncommutative analogue of (1), which will be strong enough for our application to Rognes’ question, and gives a partial generalization of Neeman–Ranicki. To state it, let R be an \mathbb{E}_1 -ring and let $r \in \pi_*R$ be a homogeneous element such that $\{1, r, r^2, \dots\}$ satisfies the right Ore condition. By Proposition 1.10, there is an \mathbb{E}_1 -ring $R[r^{-1}]$ and an \mathbb{E}_1 -ring map $R \rightarrow R[r^{-1}]$ inducing an isomorphism $(\pi_*R)[r^{-1}] \cong \pi_*(R[r^{-1}])$.

Theorem 0.3. *For R and $r \in \pi_*R$ as above, there is a fiber sequence*

$$\mathbb{K}(A) \rightarrow \mathbb{K}(R) \rightarrow \mathbb{K}(R[r^{-1}])$$

of spectra, where $A = \mathrm{End}_R(R/r)^{\mathrm{op}}$.

The fiber of the map $\mathbb{K}(\mathrm{BP}\langle n \rangle) \rightarrow \mathbb{K}(\mathrm{E}(n))$ has been considered before in Barwick’s work [7, Example 11.16], where it is observed that the fiber is the K -theory of v_n -nilpotent $\mathrm{BP}\langle n \rangle$ -modules. One of the main contributions of this paper is to identify the fiber as the K -theory of an \mathbb{E}_1 -ring. This is the special case of Theorem 0.3 when $R = \mathrm{BP}\langle n \rangle$ and $r = v_n$. In this case, we write $A\langle n - 1 \rangle = \mathrm{End}_{\mathrm{BP}\langle n \rangle}(\mathrm{BP}\langle n \rangle/v_n)^{\mathrm{op}}$; in particular, we have a natural \mathbb{E}_1 -ring map $\mathrm{BP}\langle n - 1 \rangle \simeq \mathrm{BP}\langle n \rangle/v_n \rightarrow A\langle n - 1 \rangle$.

Theorem 0.4. *For all $n \geq 0$, there is an \mathbb{E}_1 -ring $A\langle n - 1 \rangle$ and a fiber sequence*

$$\mathbb{K}(A\langle n - 1 \rangle) \rightarrow \mathbb{K}(\mathrm{BP}\langle n \rangle) \rightarrow \mathbb{K}(\mathrm{E}(n))$$

of spectra. Moreover, the transfer map $\mathbb{K}(\mathrm{BP}\langle n - 1 \rangle) \rightarrow \mathbb{K}(\mathrm{BP}\langle n \rangle)$ factors through the map $\mathbb{K}(\mathrm{BP}\langle n - 1 \rangle) \rightarrow \mathbb{K}(A\langle n - 1 \rangle)$ induced by $\mathrm{BP}\langle n - 1 \rangle \rightarrow A\langle n - 1 \rangle$.

Here is an outline of how we use Theorem 0.4 to prove Theorem 0.2. To begin, we show that the homotopy ring of $A\langle n - 1 \rangle$ for $n > 0$ is

$$\pi_* A\langle n - 1 \rangle \cong \mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}] \otimes \Lambda_{\mathbb{Z}_{(p)}}\langle \varepsilon_{1-2p^n} \rangle,$$

where ε_{1-2p^n} has degree $1 - 2p^n$. Moreover, we show that if the question of Rognes has a positive answer, then the natural map

$$\mathbb{K}(\mathrm{BP}\langle n - 1 \rangle) \rightarrow \mathbb{K}(A\langle n - 1 \rangle)$$

must be an equivalence.

We use a rational trace argument to compare the K -theories of $\mathrm{BP}\langle n - 1 \rangle$ and $A\langle n - 1 \rangle$. After rationalization, we show that both $\mathrm{BP}\langle n - 1 \rangle_{\mathbb{Q}} = \mathrm{H}\mathbb{Q} \otimes_{\mathbb{S}} \mathrm{BP}\langle n - 1 \rangle$ and $A\langle n - 1 \rangle_{\mathbb{Q}} = \mathrm{H}\mathbb{Q} \otimes_{\mathbb{S}} A\langle n - 1 \rangle$ admit \mathbb{E}_{∞} -ring structures. Despite the dearth of computational techniques for the K -theory of nonconnective ring spectra, the fact that $\pi_* \mathrm{BP}\langle n - 1 \rangle_{\mathbb{Q}}$ and $\pi_* A\langle n - 1 \rangle_{\mathbb{Q}}$ are both graded-commutative polynomial algebras allows us to use trace methods to construct many classes in positive degree in $\mathbb{K}(A\langle n - 1 \rangle)$ that cannot come from $\mathbb{K}(\mathrm{BP}\langle n - 1 \rangle)$.

To construct these classes, we study the commutative diagram

$$\begin{array}{ccccc} \mathrm{BGL}_1(\mathrm{BP}\langle n - 1 \rangle) & \longrightarrow & \Omega^{\infty} \mathbb{K}(\mathrm{BP}\langle n - 1 \rangle) & \longrightarrow & \Omega^{\infty} \mathrm{HH}^{\mathrm{H}\mathbb{Q}}(\mathrm{BP}\langle n - 1 \rangle_{\mathbb{Q}}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{BGL}_1(A\langle n - 1 \rangle) & \longrightarrow & \Omega^{\infty} \mathbb{K}(A\langle n - 1 \rangle) & \longrightarrow & \Omega^{\infty} \mathrm{HH}^{\mathrm{H}\mathbb{Q}}(A\langle n - 1 \rangle_{\mathbb{Q}}) \end{array}$$

of units and trace maps. A Hochschild–Kostant–Rosenberg-type isomorphism yields the identification

$$\mathrm{HH}_*^{\mathrm{H}\mathbb{Q}}(A\langle n - 1 \rangle_{\mathbb{Q}}) \cong \mathbb{Q}[v_1, \dots, v_{n-1}, \delta_{2-2p^n}] \otimes \Lambda_{\mathbb{Q}}\langle \sigma_1, \dots, \sigma_{n-1}, \varepsilon_{1-2p^n} \rangle, \quad (2)$$

where the degree of σ_i is $2p^i - 1$. Moreover, $\mathrm{HH}_*^{\mathrm{H}\mathbb{Q}}(\mathrm{BP}\langle n - 1 \rangle_{\mathbb{Q}}) \rightarrow \mathrm{HH}_*^{\mathrm{H}\mathbb{Q}}(A\langle n - 1 \rangle_{\mathbb{Q}})$ is the inclusion of the subalgebra generated by the v_i and σ_i classes. Finally, we compute the effect in homotopy of the compositions $\mathrm{BGL}_1(\mathrm{BP}\langle n - 1 \rangle) \rightarrow \mathrm{HH}^{\mathrm{H}\mathbb{Q}}(\mathrm{BP}\langle n - 1 \rangle_{\mathbb{Q}})$ and $\mathrm{BGL}_1(A\langle n - 1 \rangle) \rightarrow \mathrm{HH}^{\mathrm{H}\mathbb{Q}}(A\langle n - 1 \rangle_{\mathbb{Q}})$ to prove the following result.

Theorem 0.5. *If $x = v_1^{a_1} \cdots v_{n-1}^{a_{n-1}} \varepsilon_{1-2p^n}$ is a monomial in $\pi_*\mathbb{A}\langle n-1 \rangle$ of positive total degree, then the class*

$$v_1^{a_1} \cdots v_{n-1}^{a_{n-1}} \delta_{2-2p^n} + \sum_{i=1}^{n-1} a_i v_1^{a_1} \cdots v_i^{a_i-1} \cdots v_{n-1}^{a_{n-1}} \sigma_i \varepsilon_{1-2p^n}$$

is in the image of the map $\mathbb{K}(\mathbb{A}\langle n-1 \rangle) \rightarrow \mathrm{HH}^{\mathrm{H}\mathbb{Q}}(\mathbb{A}\langle n-1 \rangle_{\mathbb{Q}})$ and not in the image of $\mathrm{HH}^{\mathrm{H}\mathbb{Q}}(\mathrm{BP}\langle n-1 \rangle_{\mathbb{Q}}) \rightarrow \mathrm{HH}^{\mathrm{H}\mathbb{Q}}(\mathbb{A}\langle n-1 \rangle_{\mathbb{Q}})$.

From this fact we immediately obtain Theorem 0.2. It is also clear why this method does not contradict the known cases $n = 0$ and $n = 1$ of Rognes’ question. Indeed, in those cases there are no such monomials of positive total degree.

Remark 0.6. Building on work of Rognes [32, 33], Blumberg and Mandell [17, 18] also give another approach to the computation of the K -groups of the sphere, which completely determines the homotopy type of the fiber of $\mathbb{K}(\mathbb{S}) \rightarrow \mathbb{K}(\mathbb{Z})$ in terms of the K -groups of \mathbb{Z} , the homotopy groups of $\mathbb{C}P_{-1}^{\infty}$, and the stable homotopy groups of spheres.

Outline. Sections 1 and 2 contain our theorem on localization sequences arising from inverting elements in \mathbb{E}_1 -rings and the trace machinery we will use. Section 3 provides a concrete method for computing the trace map involving Kähler differentials, in some cases. The \mathbb{E}_1 -ring structures on $\mathrm{BP}\langle n \rangle$ are described briefly in Section 4. In Section 5 we construct the \mathbb{E}_{∞} -ring structures on $\mathrm{BP}\langle n-1 \rangle_{\mathbb{Q}}$ and $\mathbb{A}\langle n-1 \rangle_{\mathbb{Q}}$. Finally, in Section 6, we give the proof of Theorem 0.2, resolving in the negative Rognes’ question for $n > 1$.

Notation. As a matter of convention, and unless noted otherwise, we will use ∞ -categories throughout, following Lurie’s approach to stable homotopy theory developed in [27]. We will speak of \mathbb{E}_n -rings, as opposed to \mathbb{E}_n -ring spectra, of \mathbb{E}_1 -algebras over \mathbb{E}_n -rings for $n > 1$, and of right modules, as opposed to right module spectra. If \mathcal{C} is an ∞ -category, we will write either $\mathcal{C}(x, y)$ or $\mathrm{map}_{\mathcal{C}}(x, y)$ for the space of maps between two objects $x, y \in \mathcal{C}$. If \mathcal{C} is in addition stable, we will write $\mathrm{Map}_{\mathcal{C}}(x, y)$ for the mapping spectrum. In the important case where $\mathcal{C} = \mathrm{Mod}_A$, the stable ∞ -category of right A -modules for an \mathbb{E}_1 -ring A , we write $\mathrm{map}_A(x, y)$ and $\mathrm{Map}_A(x, y)$ for the mapping space and spectrum.

1. The K -theory fiber of a localization of rings

In this section, we introduce algebraic K -theory and prove a theorem which describes the fiber in K -theory of a localization of an \mathbb{E}_1 -ring in certain cases. Note that we follow Lurie [27] in terminology wherever possible. In particular, using [12], we will view connective algebraic K -theory (denoted \mathbb{K}) and nonconnective algebraic K -theory (denoted $\mathbb{K}\mathbb{K}$) as a functor defined on small stable ∞ -categories. There is no substantive difference between this approach and the approach via Waldhausen categories: see [12, Section 7.2].

1.1. K -theory

We start by introducing some terminology about small stable ∞ -categories.

- Definition 1.1.** 1. A small stable ∞ -category \mathcal{C} is *idempotent complete* if it is closed under summands. The ∞ -category of small stable idempotent complete ∞ -categories and exact functors between them is denoted by $\text{Cat}_\infty^{\text{perf}}$.
2. A sequence $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{g} \mathcal{E}$ in $\text{Cat}_\infty^{\text{perf}}$ is *exact* if the composite $\mathcal{C} \rightarrow \mathcal{E}$ is zero, $\mathcal{C} \rightarrow \mathcal{D}$ is fully faithful, and $\mathcal{D}/\mathcal{C} \rightarrow \mathcal{E}$ is an equivalence. Note that the cofiber is taken in $\text{Cat}_\infty^{\text{perf}}$ and is the idempotent completion of the usual Verdier quotient.
 3. Such a sequence is *split-exact* if moreover there exist right adjoints $f_\rho: \mathcal{D} \rightarrow \mathcal{C}$ and $g_\rho: \mathcal{E} \rightarrow \mathcal{D}$ such that $f_\rho \circ f \simeq \text{id}_{\mathcal{C}}$ and $g \circ g_\rho \simeq \text{id}_{\mathcal{E}}$.
 4. Let Sp denote the ∞ -category of spectra, as defined in [27, Section 1.4]. An *additive invariant* of small stable ∞ -categories is a functor $F: \text{Cat}_\infty^{\text{perf}} \rightarrow \text{Sp}$ that takes split-exact sequences to split fiber sequences of spectra.
 5. A *localizing invariant* of small stable ∞ -categories is a functor $F: \text{Cat}_\infty^{\text{perf}} \rightarrow \text{Sp}$ that takes exact sequences to fiber sequences of spectra.

To connect exact sequences in $\text{Cat}_\infty^{\text{perf}}$ to localization, let $\text{Mod}_{\mathcal{C}} = \text{Fun}^{\text{ex}}(\mathcal{C}^{\text{op}}, \text{Sp})$, the stable presentable ∞ -category of right \mathcal{C} -modules in spectra. An exact sequence in $\text{Cat}_\infty^{\text{perf}}$ then gives rise by left Kan extensions to an exact sequence

$$\text{Mod}_{\mathcal{C}} \rightarrow \text{Mod}_{\mathcal{D}} \rightarrow \text{Mod}_{\mathcal{E}},$$

and the functor $\text{Mod}_{\mathcal{D}} \rightarrow \text{Mod}_{\mathcal{E}}$ is a localization, in the sense that its right adjoint is fully faithful. Moreover, if $L: \text{Mod}_{\mathcal{D}} \rightarrow \text{Mod}_{\mathcal{E}}$ is a localization such that the kernel is generated by the ∞ -category \mathcal{C} of objects x of \mathcal{D} such that $L(x) \simeq 0$, then

$$\mathcal{C} \rightarrow \mathcal{D} \rightarrow \mathcal{E}$$

is an exact sequence in $\text{Cat}_\infty^{\text{perf}}$.

Additive, or connective, K-theory is an additive invariant $\mathbb{K}: \text{Cat}_\infty^{\text{perf}} \rightarrow \text{Sp}$ (see [12, Section 7] and [13, Section 2]). When \mathcal{C} is the ∞ -category associated to a Waldhausen category by hammock localization, $\mathbb{K}(\mathcal{C})$ agrees with Waldhausen K -theory by [13, Theorem 2.5].

Localizing, or nonconnective, K-theory is a localizing invariant $\mathbb{K}: \text{Cat}_\infty^{\text{perf}} \rightarrow \text{Sp}$. It is defined in [12, Section 9]. The idea goes back to Bass. It agrees with the K -theory of Thomason and Trobaugh for the relevant cases of schemes. By construction, there is a natural equivalence $\mathbb{K}(\mathcal{C}) \xrightarrow{\sim} \tau_{\geq 0}\mathbb{K}(\mathcal{C})$, where $\tau_{\geq 0}\mathbb{K}(\mathcal{C})$ is the connective cover.

The question of Rognes is as stated about additive K -theory, but it will prove easier to pass first through nonconnective K -theory, essentially for the results about localizations of \mathbb{E}_1 -rings, to which we now turn.

Definition 1.2. If A is an \mathbb{E}_1 -ring, then $\mathbb{K}(A)$ and $\mathbb{K}(A)$ are defined as the connective and nonconnective K -theories of Mod_A^{q} , the small stable ∞ -category of compact right A -modules.

Definition 1.3. Let \mathcal{C} be a small stable idempotent complete ∞ -category. We will say that a compact object M of \mathcal{C} *generates* \mathcal{C} if $\text{Map}_{\text{Mod}_{\mathcal{C}}}(M, N) \simeq 0$ implies that $N \simeq 0$ for N in $\text{Mod}_{\mathcal{C}}$. Here $\text{Map}_{\text{Mod}_{\mathcal{C}}}(M, N)$ denotes the mapping spectrum from M to N .

The following theorem is the main theorem of Morita theory for small stable ∞ -categories.

Theorem 1.4 (Schwede–Shipley [35, Theorem 3.3]). *Suppose that \mathcal{C} is a small idempotent complete stable ∞ -category generated by an object M . If $A = \text{End}_{\mathcal{C}}(M)^{\text{op}}$, then $\text{Mod}_A \simeq \text{Mod}_{\mathcal{C}}$ and hence $\text{Mod}_A^{\omega} \simeq \mathcal{C}$.*

In this situation, we will also say that M is a *compact generator* of $\text{Mod}_{\mathcal{C}}$ (remembering that $\mathcal{C} \simeq \text{Mod}_{\mathcal{C}}^{\omega}$).

Corollary 1.5. *In the situation of the theorem, $K(A) \simeq K(\mathcal{C})$ and $\mathbb{K}(A) \simeq \mathbb{K}(\mathcal{C})$.*

We will heavily use a map from $\text{BGL}_1(A)$ to $\Omega^{\infty} K(A) \simeq \Omega^{\infty} \mathbb{K}(A)$, the algebraic space of A .

Definition 1.6. The group of units of an \mathbb{E}_1 -ring A is defined as the homotopy pullback in the square

$$\begin{array}{ccc} \text{GL}_1(A) & \longrightarrow & \Omega^{\infty} A \\ \downarrow & & \downarrow \\ \pi_0 A^{\times} & \longrightarrow & \pi_0 A \end{array}$$

Multiplication induces a grouplike \mathbb{E}_1 -algebra structure on the space $\text{GL}_1(A)$, which we can deloop to obtain $\text{BGL}_1(A)$.

Unwinding the approach of [12, Section 7], we see that there is an equivalence

$$\Omega^{\infty} K(A) \simeq \Omega |S_{\bullet} \text{Mod}_A^{\omega}|^{\text{eq}},$$

where $S_{\bullet} \text{Mod}_A^{\omega}$ is the simplicial ∞ -category whose ∞ -category of n -simplices $S_n \text{Mod}_A^{\omega}$ is equivalent to the functor category $\text{Fun}([n - 1], \text{Mod}_A^{\omega})$, and where Mod_A^{ω} denotes the ∞ -category of compact objects in Mod_A . Here, $[n - 1]$ denotes the nerve of the graph $0 \rightarrow 1 \rightarrow \dots \rightarrow n - 1$, and $[-1] = \emptyset$.

Definition 1.7. The *units map* $\text{BGL}_1(A) \rightarrow \Omega^{\infty} K(A)$ is induced by the map from $\text{BGL}_1(A)$ into the 1-skeleton $\text{map}([0], \text{Mod}_A^{\omega})$ of $S_{\bullet} \text{Mod}_A^{\omega}$ given by the subspace of functors $[0] \rightarrow \text{Mod}_A^{\omega}$ sending 0 to A .

In fact, similar reasoning defines a map from $\text{BAut}_A(P)$ to $\Omega^{\infty} K(A)$ for any compact right A -module P . There is an induced map

$$\coprod_{P \text{ f.g. projective}} \text{BAut}_A(P) \rightarrow \Omega^{\infty} K(A)$$

of \mathbb{E}_{∞} -spaces, where the \mathbb{E}_{∞} -space structure on the left-hand side is induced by direct sum of finitely generated projective modules.² Since $\Omega^{\infty} K(A)$ is grouplike, the map factors through the group completion map $\coprod_{P \text{ f.g. projective}} \text{BAut}_A(P) \rightarrow \Omega \text{B}(\coprod_{P \text{ f.g. projective}} \text{BAut}_A(P))$.

² Just as in ordinary algebra, a projective A -module is a retract of a free A -module A^I for some set I . See [27, Section 7.2.2].

Theorem 1.8 ([22, Theorem VI.7.1]). *If A is connective, the induced map*

$$\Omega\mathbb{B}\left(\coprod_{P \text{ f.g. projective}} \mathbb{B}\text{Aut}_A(P)\right) \rightarrow \Omega^\infty \mathbb{K}(A)$$

is an equivalence.

Note that the presentation of [22] uses only finitely generated free A -modules. The proof of the result stated here is no different, and has the correct \mathbb{K}_0 group.

1.2. Localizations of \mathbb{E}_1 -rings

We are interested in identifying the fiber of $\mathbb{K}(A) \rightarrow \mathbb{K}(B)$, where A is an \mathbb{E}_1 -ring and B is a localization of A obtained by inverting some elements of π_*A . It is not always possible to invert elements in noncommutative rings. One way to ensure that it is possible is to impose the following Ore condition.

Definition 1.9. Let A be an \mathbb{E}_1 -ring, and let $S \subset \pi_*A$ be a multiplicatively closed set of homogeneous elements. Then S satisfies the *right Ore condition* if (1) for every pair of elements $x \in \pi_*A$ and $s \in S$ there exist $y \in \pi_*A$ and $t \in S$ such that $xt = sy$, and (2) if $sx = 0$ for some $x \in \pi_*A$ and $s \in S$, then there exists $t \in S$ such that $xt = 0$.

The conditions are particularly easy to verify when π_*A is in fact a graded-commutative ring. The following proposition collects several facts due to Lurie about the localization of an \mathbb{E}_1 -ring at a multiplicative set satisfying the right Ore condition.

Proposition 1.10 (Lurie). *Let A be an \mathbb{E}_1 -ring, and let $S \subseteq \pi_*A$ be a multiplicatively closed set satisfying the right Ore condition. Then:*

- (1) *there is an \mathbb{E}_1 -ring $S^{-1}A$ and an \mathbb{E}_1 -ring map $A \rightarrow S^{-1}A$, where $\pi_*S^{-1}A \cong S^{-1}\pi_*A$;*
- (2) *the induced functor $\text{Mod}_A \rightarrow \text{Mod}_{S^{-1}A}$ is a localization;*
- (3) *the induced functor $\text{Mod}_A \rightarrow \text{Mod}_{S^{-1}A}$ has as fiber the full subcategory $\text{Mod}_A^{\text{Nil}(S)}$ of S -nilpotent A -modules, i.e., those right A -modules M such that for every homogeneous element $x \in \pi_*M$, there exists $s \in S$ such that $xs = 0$.*

Proof. For (1), see [27, Proposition 7.2.4.20 and Remark 7.2.4.26]. This identifies $S^{-1}A$ with an \mathbb{E}_1 -ring $A[S^{-1}]$, which is by definition the endomorphism algebra of the image of A under a functor $\text{Mod}_A \rightarrow \text{Mod}_A^{\text{Loc}(S)}$ and serves as a compact generator for $\text{Mod}_A^{\text{Loc}(S)}$. This functor is a localization with kernel the S -nilpotent A -modules by [27, Proposition 7.2.4.17]. This proves (2) and (3). \square

The result of this section identifies the fiber in \mathbb{K} -theory of an Ore localization generated by a single element. The role of S -nilpotent modules has also been studied by Barwick [7, Proposition 11.15]. Our identification below of a special S -nilpotent module can be viewed as a partial generalization of the work of Neeman and Ranicki [30] to noncommutative \mathbb{E}_1 -rings.

Theorem 1.11. *Let A be an \mathbb{E}_1 -ring, and let $S = \{1, r, r^2, \dots\}$ be a multiplicatively closed subset of π_*A satisfying the right Ore condition and generated by a single homogeneous element $r \in \pi_d A$. If M denotes the cofiber of the map $\Sigma^d A \xrightarrow{r} A$ of right A -modules, then there is a fiber sequence*

$$\mathbb{K}(\text{End}_A(M)^{\text{op}}) \rightarrow \mathbb{K}(A) \rightarrow \mathbb{K}(S^{-1}A)$$

of spectra.

Proof. By Proposition 1.10, there is a localization sequence

$$\text{Mod}_A^{\text{Nil}(S)} \rightarrow \text{Mod}_A \rightarrow \text{Mod}_{S^{-1}A}.$$

It follows that after taking compact objects we obtain an exact sequence

$$\text{Mod}_A^{\text{Nil}(S), \omega} \rightarrow \text{Mod}_A^\omega \rightarrow \text{Mod}_{S^{-1}A}^\omega$$

of small stable ∞ -categories, where $\text{Mod}_A^{\text{Nil}(S), \omega}$ is equivalent to the stable ∞ -category of S -nilpotent compact right A -modules. By definition of localizing K -theory and by Corollary 1.5, it suffices to show that M is a compact generator of $\text{Mod}_A^{\text{Nil}(S)}$. Since M is built in finitely many steps from A by taking cofibers, it is an object of Mod_A^ω . Moreover, the localization of M is clearly zero since r is a unit in $S^{-1}A$.

By [27, Proposition 7.2.4.14], an A -module N is S -local if and only if $\text{Map}_A(A/s, N) \simeq 0$ for all $s \in S$, where A/s denotes the cofiber of $\Sigma^{|s|} A \xrightarrow{s} A$. Under the hypothesis above, we can replace this condition by the condition that $\text{Map}_A(A/r, N) \simeq 0$. With $M = A/r$, we will now argue that if $\text{Map}_A(M, N) \simeq 0$ and N is S -nilpotent, then $N \simeq 0$. This will show that M is a compact generator of $\text{Mod}_A^{\text{Nil}(S)}$. Consider the cofiber sequence $\Sigma^d A \xrightarrow{r} A \rightarrow M$ of right A -modules. Under the assumption that $\text{Map}_A(M, N) \simeq 0$, the induced map

$$N \simeq \text{Map}_A(A, N) \xrightarrow{r} \text{Map}_A(\Sigma^d A, N) \simeq \Sigma^{-d} N$$

of left A -modules is an equivalence. Since N is S -nilpotent, requirement (1) of the Ore condition implies that every homogeneous element of $\pi_* N$ is annihilated by multiplication on the left by a power of r . Hence, $N \simeq 0$. \square

The localization at the heart of Rognes' question is the map $\text{BP}\langle n \rangle \rightarrow E(n)$, where $E(n)$ is obtained from $\text{BP}\langle n \rangle$ by inverting v_n . Since $\pi_* \text{BP}\langle n \rangle$ is in fact a commutative ring, the right Ore condition is trivially satisfied. It follows from the theorem that if we can compute the endomorphism ring of the cofiber of $v_n: \Sigma^{2p^n-2} \text{BP}\langle n \rangle \rightarrow \text{BP}\langle n \rangle$, then we obtain an \mathbb{E}_1 -ring whose K -theory can be analyzed and compared to the K -theory of $\text{BP}\langle n-1 \rangle$. We will return to the truncated Brown–Peterson spectra after introducing the computational techniques we will use.

We briefly discuss two different directions in which one can generalize the theorem.

First of all one can consider localizations $A \rightarrow S^{-1}A$ where S is generated by a left-regular sequence (r_1, \dots, r_n) of homogeneous elements. We give the statement for $n = 2$. Suppose that $\text{deg}(r_i) = d_i$ for $i = 1, 2$ and suppose that r_1 and r_2 commute up to

homotopy when acting by multiplication on the left. That is, suppose that in $\text{Ho}(\text{Mod}_A)$ the left-hand square in

$$\begin{array}{ccccccc}
 \Sigma^{d_1+d_2} A & \xrightarrow{r_1} & \Sigma^{d_2} A & \longrightarrow & \Sigma^{d_2} M_1 & \longrightarrow & \Sigma^{d_1+d_2+1} A \\
 \downarrow r_2 & & \downarrow r_2 & & \downarrow r_2 & & \downarrow \Sigma r_2 \\
 \Sigma^{d_1} A & \xrightarrow{r_1} & A & \longrightarrow & M_1 & \longrightarrow & \Sigma^{d_1+1} A
 \end{array}$$

commutes, where M_1 is the cofiber of r_1 acting by multiplication on the left as a right A -module map. In this case, the triangulated category axioms assert that there is an extension of this diagram to the right by a map $r_2: \Sigma^{d_2} M_1 \rightarrow M_1$. The map r_2 is an example of a Toda bracket and need not be unique. We will say that (r_1, r_2) is a *left-regular sequence* if the multiplicatively closed set generated by r_1 and r_2 satisfies the right Ore condition, if r_1 acts injectively on the homotopy of A , if the square above commutes, and if for *some* choice of fill $r_2: \Sigma^{d_2} M_1 \rightarrow M_1$ the map in homotopy is injective. Note that a priori (r_1, r_2) may be a left-regular sequence while (r_2, r_1) is not.

Proposition 1.12. *Suppose that A is an \mathbb{E}_1 -ring and S is the multiplicative set generated by a left-regular sequence (r_1, r_2) . Let M_2 be the cofiber of $\Sigma^{d_2} M_1 \xrightarrow{r_2} M_1$ for some choice of map r_2 as above which is injective in homotopy. Then there is a fiber sequence*

$$\mathbb{K}(\text{End}_A(M_2)^{\text{op}}) \rightarrow \mathbb{K}(A) \rightarrow \mathbb{K}(S^{-1}A)$$

of spectra.

Proof. We prove that M_2 is a compact generator of the S -nilpotent spectra. Suppose that N is S -nilpotent and that $\text{Map}_A(M_2, N) \simeq 0$. As above one then sees that r_2 induces an equivalence

$$\text{Map}_A(M_1, N) \rightarrow \text{Map}_A(\Sigma^{d_2} M_1, N).$$

On the other hand, r_2 acts nilpotently on the homotopy groups of N . Consider the Ext spectral sequence

$$E_2^{s,t} = \text{Ext}_{\pi_* A}^s(\pi_* M_1, \pi_* N)_t \Rightarrow \pi_{t-s} \text{Map}_A(M_1, N).$$

By the right Ore condition and the left-regularity of r_1 , it follows that the right $\pi_* A$ -module $\pi_* M_1$ has graded projective dimension 1. Therefore, the filtration on the abutment is finite. But, r_2 must then act nilpotently on $\pi_* \text{Map}_A(M_1, N)$, which implies that $\text{Map}_A(M_1, N) \simeq 0$. The argument concludes as in the proof of the theorem. \square

An appropriate similar hypothesis can be imposed on localizations of more elements to ensure the existence of fiber sequences in K -theory with fiber given by the K -theory of an endomorphism algebra.

In the second direction we can consider localizations of connective \mathbb{E}_∞ -rings or more generally of derived schemes (see [3]).

Theorem 1.13. *Let X be a quasi-compact and quasi-separated derived scheme and $U \subseteq X$ a quasi-compact Zariski open subscheme with complement Z . Then there is a fiber sequence*

$$\mathbb{K}(A) \rightarrow \mathbb{K}(X) \rightarrow \mathbb{K}(U),$$

where $\mathbb{K}(A) \simeq \mathbb{K}(X \text{ on } Z)$ is the K -theory of an \mathbb{E}_1 -ring spectrum $A = \text{End}_X(M)^{\text{op}}$.

Proof. In this situation, Antieau and Gepner [3, Proposition 6.9] proved that there is a single compact object M that generates the fiber of $\text{Mod}_X \rightarrow \text{Mod}_U$. This completes the proof. \square

Typically, M can be taken to be a kind of generalized Koszul complex. For example, when $X = \text{Spec } R$ is affine, and when U is the open set defined by inverting r_1, \dots, r_n all of degree zero, then M can be taken as

$$\bigotimes_{i=1}^n \text{cofib}(R \xrightarrow{r_i} R),$$

which is precisely the Koszul complex when R is an Eilenberg–MacLane spectrum and the r_i form a regular sequence.

2. Hochschild homology and trace

Throughout this section, R will denote an \mathbb{E}_∞ -ring.

2.1. Hochschild homology

Definition 2.1. Let A be an \mathbb{E}_1 - R -algebra and M an A -bimodule in Mod_R . The Hochschild homology $\text{HH}^R(A, M)$ of A over R with coefficients in M is the geometric realization of the cyclic bar construction $\mathbf{B}_\bullet^{\text{cyc}}(A/R, M)$, the simplicial spectrum with level n the R -module

$$\mathbf{B}_n^{\text{cyc}}(A/R, M) = M \otimes_R A^{\otimes n}.$$

The face and degeneracy maps are given by the same formulas as in ordinary Hochschild homology (see for instance [36, Definition 4.1.2]). If $M = A$ with its natural bimodule structure then we typically write $\mathbf{B}_\bullet^{\text{cyc}} A$ for the simplicial spectrum and $\text{HH}^R(A)$ for its geometric realization, omitting reference to R when no confusion can occur. When $R = \mathbf{H}\pi_0 R$ is an Eilenberg–MacLane spectrum, our (implicit) use of the derived tensor product implies that this definition can disagree with ordinary Hochschild homology even when M and A are themselves Eilenberg–MacLane. It follows from the Tor spectral sequence that no discrepancy occurs if their homotopy groups are flat over $\pi_0 R$.

We will need the following Hochschild–Kostant–Rosenberg theorem for our later computations. Our argument follows the proof of a similar theorem of McCarthy and Minasian [28, Theorem 6.1]. Let CAlg_R denote the ∞ -category of \mathbb{E}_∞ - R -algebras, and let $\text{Sym}_R: \text{Mod}_R \rightarrow \text{CAlg}_R$ denote the symmetric algebra functor, which gives the free

\mathbb{E}_∞ - R -algebra on an R -module M . It is left adjoint to the forgetful functor $\text{Mod}_R \leftarrow \text{CAlg}_R$. Note also that if $R \rightarrow S$ is a map of \mathbb{E}_∞ -rings, then there is a natural equivalence $S \otimes_R \text{Sym}_R(M) \simeq \text{Sym}_S(S \otimes_R M)$.

Theorem 2.2. *Let R be an \mathbb{E}_∞ -ring, and let M be an R -module. The Hochschild homology of the free \mathbb{E}_∞ - R -algebra $S = \text{Sym}_R M$ over R is given by*

$$\text{HH}^R(S) \simeq \text{Sym}_S(S \otimes_R \Sigma M)$$

as an \mathbb{E}_∞ - S -algebra, where $S \otimes_R \Sigma M$ is the tensor product of R -modules.

Proof. By McClure–Schwänzl–Vogt [29], there is an equivalence

$$\text{HH}^R(S) \simeq S^1 \otimes_R S = S^1 \otimes_R \text{Sym}_R M,$$

where $S^1 \otimes_R S$ refers to the simplicial structure on the category of \mathbb{E}_∞ - R -algebras. As the symmetric algebra functor $\text{Sym}_R: \text{Mod}_R \rightarrow \text{CAlg}_R$ is a left adjoint, it commutes with all all small colimits. Since the symmetric monoidal structure on CAlg_R is the cocartesian symmetric monoidal structure, Sym_R is symmetric monoidal for the cocartesian monoidal structure on Mod_R . Thus, there is a natural equivalence $\text{Sym}_R(M \oplus \Sigma M) \simeq \text{Sym}_R(M) \otimes_R \text{Sym}_R(\Sigma M)$. It follows that

$$\begin{aligned} S^1 \otimes_R \text{Sym}_R M &\simeq \text{Sym}_R(S^1 \otimes_R M) \simeq \text{Sym}_R(M \oplus \Sigma M) \\ &\simeq \text{Sym}_R(M) \otimes_R \text{Sym}_R(\Sigma M) \simeq \text{Sym}_{\text{Sym}_R M}(\text{Sym}_R M \otimes_R \Sigma M), \end{aligned}$$

as desired. □

We will only apply this HKR-type theorem in the case where the base ring R is $\mathbb{H}\mathbb{Q}$ and M is a compact $\mathbb{H}\mathbb{Q}$ -module. In this case, the result is especially simple and the reader can compare this result to the main result [24].

Corollary 2.3. *Let M be a compact $\mathbb{H}\mathbb{Q}$ -module with a basis for the even homology given by homogeneous elements x_1, \dots, x_m and for odd homology by homogeneous elements y_1, \dots, y_n . Let $S = \text{Sym}_{\mathbb{H}\mathbb{Q}} M$, so that $\pi_* S \cong \mathbb{Q}[x_1, \dots, x_m] \otimes \Lambda_{\mathbb{Q}}(y_1, \dots, y_n)$. Then*

$$\pi_* \text{HH}^{\mathbb{H}\mathbb{Q}}(S) \cong \mathbb{Q}[x_1, \dots, x_m, \delta(y_1), \dots, \delta(y_n)] \otimes \Lambda_{\mathbb{Q}}(y_1, \dots, y_n, \delta(x_1), \dots, \delta(x_m)),$$

where the $\delta(y_i)$ and $\delta(x_i)$ are elements in degrees $|y_i| + 1$ and $|x_i| + 1$, respectively, induced from a map $\Sigma M \rightarrow \text{HH}^{\mathbb{H}\mathbb{Q}}(S)$.

2.2. The trace map

Bökstedt introduced a natural transformation $\text{K} \rightarrow \text{HH}^{\mathbb{S}}$ from additive K -theory to Hochschild homology over the sphere called the topological Dennis trace map, generalizing earlier work of Waldhausen [40] in the special case of K -theory of spaces. We will refer to this simply as the trace map. Note that $\text{HH}^{\mathbb{S}}$ is precisely what is usually termed topological Hochschild homology and denoted by THH . This map and its refinements

(such as topological cyclic homology) play a central role in the contemporary approach to computations of K -groups. See Hesselholt and Madsen’s work [23] for an exemplary case.

There is another approach which rests on a definition of $\mathrm{HH}^{\mathbb{S}}$ for arbitrary small spectral categories or stable ∞ -categories. The idea goes back to Dundas and McCarthy [21]; see [16, Definition 3.4] for the definition for spectral categories. Blumberg, Gepner, and Tabuada show that this definition defines a localizing invariant of small idempotent complete ∞ -categories. It follows that $\mathrm{HH}_0^{\mathbb{S}}(\mathbb{S})$ is naturally bijective to the group of natural transformations $\mathbb{K} \rightarrow \mathrm{HH}^{\mathbb{S}}$ of localizing invariants. They give a conceptual identification of Waldhausen’s trace, showing in [12, Theorem 10.6] that it is the unique such natural transformation with the property that the composition

$$\mathbb{S} \rightarrow \mathbb{K}(\mathbb{S}) \rightarrow \mathrm{HH}^{\mathbb{S}}(\mathbb{S}) \xrightarrow{\sim} \mathbb{S}$$

is homotopic to the identity, where $\mathbb{S} \rightarrow \mathbb{K}(\mathbb{S})$ is the unit map in K -theory, and $\mathrm{HH}^{\mathbb{S}}(\mathbb{S}) \xrightarrow{\sim} \mathbb{S}$ is the augmentation, i.e., the inverse to the unit map $\mathbb{S} \xrightarrow{\sim} \mathrm{HH}^{\mathbb{S}}(\mathbb{S})$ in Hochschild homology.

Definition 2.4. If G is a grouplike \mathbb{E}_1 -algebra in spaces, we let $\mathbf{B}_{\bullet}^{\mathrm{cyc}} G$ be the simplicial space with $\mathbf{B}_n^{\mathrm{cyc}} = G^{n+1}$ and the usual face and degeneracy maps. Writing $\mathbf{B}^{\mathrm{cyc}} G$ for the geometric realization of $\mathbf{B}_{\bullet}^{\mathrm{cyc}} G$, we obtain by [25, Theorem 6.2] a model for the free loop space LBG of BG . There is a natural fibration sequence $G \rightarrow \mathbf{B}^{\mathrm{cyc}} G \rightarrow BG$, and there is a section on the right $BG \rightarrow \mathbf{B}^{\mathrm{cyc}} G$ given by including BG as the constant loops. This map in fact exists even before geometric realizations as a map $c: \mathbf{B}_{\bullet} G \rightarrow \mathbf{B}_{\bullet}^{\mathrm{cyc}} G$ which on level n is described by

$$c(g_1, \dots, g_n) = (g_n^{-1} \cdots g_1^{-1}, g_1, \dots, g_n).$$

See [26, Proposition 7.3.4, Theorem 7.3.11]. This concrete cycle-level description will be important below.

The following proposition is well-known and appears in similar forms in work of Bökstedt–Hsiang–Madsen [19] and Schlichtkrull [34, Section 4.4].

Proposition 2.5. *Let A be an \mathbb{E}_1 -ring. The composition*

$$\Sigma^{\infty} \mathrm{BGL}_1(A)_+ \rightarrow \mathbf{K}(A) \rightarrow \mathbb{K}(A) \rightarrow \mathrm{HH}^{\mathbb{S}}(A)$$

is equivalent to

$$\Sigma^{\infty} (\mathrm{BGL}_1(A))_+ \xrightarrow{c} \Sigma^{\infty} (\mathbf{B}^{\mathrm{cyc}} \mathrm{GL}_1(A))_+ \simeq \mathbf{B}^{\mathrm{cyc}} \Sigma^{\infty} \mathrm{GL}_1(A)_+ \rightarrow \mathbf{B}^{\mathrm{cyc}} A = \mathrm{HH}^{\mathbb{S}}(A),$$

where $\mathbf{B}^{\mathrm{cyc}} \Sigma^{\infty} \mathrm{GL}_1(A)_+ \rightarrow \mathbf{B}^{\mathrm{cyc}} A$ is induced by the map $\Sigma^{\infty} \mathrm{GL}_1(A)_+ \rightarrow A$ of \mathbb{E}_1 -rings adjoint to the inclusion of units.

Proof. Given an \mathbb{E}_1 -ring A , there is a natural diagram of spectra

$$\begin{array}{ccccccc} \Sigma^\infty \mathrm{BGL}_1(\tau_{\geq 0}A)_+ & \longrightarrow & \mathrm{K}(\tau_{\geq 0}A) & \longrightarrow & \mathbb{K}(\tau_{\geq 0}A) & \longrightarrow & \mathrm{HH}^{\mathbb{S}}(\tau_{\geq 0}A) \\ \downarrow \simeq & & \downarrow & & \downarrow & & \downarrow \\ \Sigma^\infty \mathrm{BGL}_1(A)_+ & \longrightarrow & \mathrm{K}(A) & \longrightarrow & \mathbb{K}(A) & \longrightarrow & \mathrm{HH}^{\mathbb{S}}(A) \end{array}$$

where $\tau_{\geq 0}A$ denotes the connective cover of A . In particular, to compute the restriction of the trace $\mathbb{K}(A) \rightarrow \mathrm{HH}^{\mathbb{S}}(A)$ to the classifying space of the units of A , we can assume that A is connective. For connective rings, Waldhausen defined the trace map in the following way [19, Section 5]. For each finitely generated projective right A -module P , look at the map

$$\mathrm{BAut}_A(P) \xrightarrow{c} \mathrm{B}^{\mathrm{cyc}} \mathrm{Aut}_A(P) \rightarrow \Omega^\infty \mathrm{B}^{\mathrm{cyc}} \mathrm{End}_A(P) \rightarrow \Omega^\infty \mathrm{HH}^{\mathbb{S}}(A),$$

where $\mathrm{End}_A(P)$ is the endomorphism algebra spectrum of P , and where the right-hand map is the usual trace map in Hochschild homology (defined as in [41, Section 9.5] or [16, Theorem 5.12]). The right-hand map is an equivalence if P is a faithful projective right A -module by Morita theory. In any case, these maps are compatible with direct sum, and hence they induce a well-defined map

$$\coprod_{P \text{ f.g. projective}} \mathrm{BAut}_A(P) \rightarrow \Omega^\infty \mathrm{HH}^{\mathbb{S}}(A)$$

which is compatible with the direct sum structure on the left. Therefore, they induce a map from the group completion, which is equivalent to $\Omega^\infty \mathrm{K}(A)$ by Theorem 1.8, to $\Omega^\infty \mathrm{HH}^{\mathbb{S}}(A)$. Using the adjunction between Ω^∞ and Σ_+^∞ completes the proof. \square

We will use homotopy classes in $\mathrm{BGL}_1(A)$ and prove that they are nonzero in $\mathrm{HH}^{\mathbb{S}}(A)$. This will have the consequence that they map to nonzero K -theory classes. More specifically, we will construct nonzero homotopy classes in the 1-skeleton of $\mathrm{BGL}_1(A)$, and we will need to know that they survive to nonzero classes in $\mathrm{HH}^{\mathbb{S}}(A)$. For this, we will need to understand the Bökstedt spectral sequence converging to $\mathrm{HH}^{\mathbb{S}}$, namely

$$E_{s,t}^2 = H_s(\Delta^{\mathrm{op}}, \pi_t \mathrm{B}_\bullet^{\mathrm{cyc}} A) \Rightarrow \mathrm{HH}_{s+t}^{\mathbb{S}}(A).$$

This is nothing other than the homotopy colimit spectral sequence for the geometric realization of a simplicial spectrum. Moreover, there are natural isomorphisms

$$H_s(\Delta^{\mathrm{op}}, \pi_t \mathrm{B}_\bullet^{\mathrm{cyc}} A) \cong H_s(C_*(\pi_t \mathrm{B}_\bullet^{\mathrm{cyc}} A))$$

for all t , where $C_*(\pi_t \mathrm{B}_\bullet^{\mathrm{cyc}} A)$ is the unnormalized chain complex associated to the simplicial abelian group $\pi_t \mathrm{B}_\bullet^{\mathrm{cyc}} A$.

In particular, we would like to understand the map $\pi_t \mathrm{B}_\bullet \mathrm{GL}_1(A) \rightarrow \pi_t \mathrm{B}_\bullet^{\mathrm{cyc}} A$ of simplicial abelian groups in terms of elements of these groups. The next proposition provides the necessary formula, but we need to note a couple of easy lemmas first.

Lemma 2.6. *If G is a grouplike \mathbb{E}_1 -space, then the inverse map acts as -1 on $\pi_t G$ for $t \geq 1$.*

Proof. This follows from the Eckmann–Hilton argument. \square

Given two based spaces X and Y , consider the Tor spectral sequence

$$E_2^{s,t} = \text{Tor}_s^{\pi_*\mathbb{S}}(\pi_*\Sigma^\infty X_+, \pi_*\Sigma^\infty Y_+)_t \Rightarrow \pi_{s+t}(\Sigma^\infty X_+ \otimes \Sigma^\infty Y_+),$$

with differentials d_r of bidegree $(-r, r - 1)$. When $s = 0$, we have $\pi_*\Sigma^\infty X_+ \otimes_{\pi_*\mathbb{S}} \pi_*\Sigma^\infty Y_+$, and this contributes to the lowest part of the filtration on the abutment because every differential vanishes on $E_r^{0,t}$. In particular, given $a \in \pi_t X$ and $b \in \pi_t Y$, the classes $a \otimes 1$ and $1 \otimes b$ determine homotopy classes in $\pi_t(\Sigma^\infty X_+ \otimes \Sigma^\infty Y_+)$.

Lemma 2.7. *Let X and Y be based spaces. Then the map*

$$\pi_t(X \times Y) \rightarrow \pi_t^s(X \times Y)_+ \rightarrow \pi_t(\Sigma^\infty X_+ \otimes \Sigma^\infty Y_+)$$

sends (a, b) to $a \otimes 1 + 1 \otimes b$.

Proof. If either a or b is zero, this is obvious. But $(a, b) = (a, 0) + (0, b)$. □

Similar considerations apply in the following proposition. For the sake of clarity, write u for the natural map $\text{GL}_1(A) \rightarrow \Omega^\infty A$, as well as the corresponding map $\pi_t \text{GL}_1(A) \rightarrow \pi_t A$, which is an isomorphism for $t \geq 1$.

Proposition 2.8. *The composition*

$$\pi_t \text{B}_n \text{GL}_1(A) \rightarrow \pi_t \text{B}_n^{\text{cyc}} \text{GL}_1(A) \rightarrow \pi_t \text{B}_n^{\text{cyc}} \Sigma^\infty \text{GL}_1(A)_+ \rightarrow \pi_t \text{B}_n^{\text{cyc}} A$$

sends (g_1, \dots, g_n) to

$$(1 \otimes u(g_1) \otimes 1 \otimes \dots \otimes 1) + \dots + (1 \otimes \dots \otimes 1 \otimes u(g_n)) - (u(g_1 + \dots + g_n) \otimes 1 \otimes \dots \otimes 1)$$

for $t \geq 1$.

Proof. The first map sends (g_1, \dots, g_n) to $(-g_1 - \dots - g_n, g_1, \dots, g_n)$ by Lemma 2.6 and the description of the inclusion of the constant loops at the simplicial level. The rest of the description then follows from Lemma 2.7 and the fact that $u: \text{GL}_1(A) \rightarrow \Omega^\infty A$ factors as $\text{GL}_1(A) \rightarrow \Omega^\infty \Sigma^\infty \text{GL}_1(A)_+ \rightarrow \Omega^\infty A$. □

3. Kähler differentials

In this section we study the composition

$$\text{Tr}: \text{BGL}_1(A) \rightarrow \Omega^\infty K(A) \rightarrow \Omega^\infty \text{HH}^S(A) \rightarrow \Omega^\infty \text{HH}^{\text{H}\mathbb{Q}}(A_{\mathbb{Q}})$$

in the special case where $A_{\mathbb{Q}} = \text{H}\mathbb{Q} \otimes_{\mathbb{S}} A$ admits an \mathbb{E}_∞ -ring structure. We can then assume by [37, Theorem 1.2] that $A_{\mathbb{Q}}$ is a rational commutative differential graded algebra, which we do in this section.

Recall (for example from [26, Paragraph 5.42] in the connective case) that for a rational commutative dga A the dg module $\Omega_{A/\mathbb{Q}}$ of Kähler differentials can be defined as the free

A -module generated by symbols da for the homogeneous elements a of A modulo the relations

$$d(ab) = ad(b) + (-1)^{|a||b|}bd(a),$$

where the map $d : A \rightarrow \Omega_{A/\mathbb{Q}}$ is the universal derivation of A over \mathbb{Q} . We should remark that by using the cotangent complex one may obtain more information, but this is not necessary in the present paper.

Proposition 3.1. *Suppose that A is an \mathbb{E}_1 -ring such that $A_{\mathbb{Q}}$ admits an \mathbb{E}_{∞} -ring structure (compatible with the \mathbb{E}_1 -ring structure). Then there is a natural map*

$$D : \mathrm{HH}^{\mathrm{H}\mathbb{Q}}(A_{\mathbb{Q}}) \rightarrow \Sigma\Omega_{A_{\mathbb{Q}}/\mathbb{Q}}$$

such that the composition

$$D \circ \mathrm{Tr} : \pi_{*-1} \mathrm{GL}_1(A) \rightarrow \mathrm{K}_*(A) \rightarrow \mathrm{HH}_*^{\mathbb{S}}(A) \rightarrow \mathrm{HH}_*^{\mathrm{H}\mathbb{Q}}(A_{\mathbb{Q}}) \rightarrow \pi_{*-1}\Omega_{A_{\mathbb{Q}}/\mathbb{Q}}$$

agrees with the composition of rationalization $A \rightarrow A_{\mathbb{Q}}$ and the universal derivation $d : A_{\mathbb{Q}} \rightarrow \Omega_{A_{\mathbb{Q}}/\mathbb{Q}}$ in homotopy groups in positive degrees.

Proof. The Hochschild complex

$$\dots \rightarrow A_{\mathbb{Q}}^{\otimes 3} \xrightarrow{\partial_0 - \partial_1 + \partial_2} A_{\mathbb{Q}}^{\otimes 2} \xrightarrow{\partial_0 - \partial_1} A_{\mathbb{Q}} \rightarrow 0$$

can be considered as a chain complex in chain complexes (or a double complex by changing the vertical differentials using the vertical sign trick). Since $A_{\mathbb{Q}}$ is graded-commutative, $\partial_0 - \partial_1 = 0$. Similarly,

$$\begin{aligned} (\partial_0 - \partial_1 + \partial_2)(a_0 \otimes a_1 \otimes a_2) &= a_0a_1 \otimes a_2 - a_0 \otimes a_1a_2 + (-1)^{|a_2|(|a_0|+|a_1|)}a_2a_0 \otimes a_1 \\ &= a_0(a_1 \otimes a_2 - 1 \otimes a_1a_2 + (-1)^{|a_2||a_1|}a_2 \otimes a_1), \end{aligned}$$

where the sign conventions are the usual ones for the Hochschild complex of a differential graded algebra. See the material in [42, Section 9.9.1] on cyclic homology.

Hence, the map $D : A_{\mathbb{Q}}^{\otimes 2} \rightarrow \Omega_{A_{\mathbb{Q}}/\mathbb{Q}}$ defined by

$$D(a_0 \otimes a_1) = a_0d(a_1)$$

defines a quasi-isomorphism

$$A_{\mathbb{Q}}^{\otimes 2} / \mathrm{im}(\partial_0 - \partial_1 + \partial_2) \rightarrow \Omega_{A_{\mathbb{Q}}/\mathbb{Q}},$$

and we write D also for the prolongation by zero to the entire Hochschild complex.

Since Tr is defined at the simplicial level, we look at

$$\mathrm{Tr} : \pi_t \mathrm{B}_1 \mathrm{GL}_1(A) \rightarrow \pi_t \mathrm{B}_1^{\mathrm{cyc}} A,$$

which is given by

$$\mathrm{Tr}(a) = -a \otimes 1 + 1 \otimes a$$

by Proposition 2.8. Applying D , we get $D \circ \mathrm{Tr}(a) = d(a)$ since $d(1) = 0$. □

Write $\delta : \Sigma A \rightarrow \Sigma\Omega_{A/\mathbb{Q}}$ for the suspension of d .

Corollary 3.2. *Let M be a compact $\mathbb{H}\mathbb{Q}$ -module, and let $A = \text{Sym}_{\mathbb{H}\mathbb{Q}} M$. There exists a section*

$$s: \Sigma\Omega_{A/\mathbb{Q}} \rightarrow \text{HH}^{\mathbb{H}\mathbb{Q}}(A)$$

such that Tr factors at the level of homotopy groups as in the following commutative diagram:

$$\begin{array}{ccc}
 \pi_{*-1} \text{GL}_1(A) & \longrightarrow & \pi_{*-1} A \\
 & \searrow \text{Tr} & \downarrow \delta \\
 & & \pi_{*-1} \Omega_{A/\mathbb{Q}} \\
 & & \downarrow s \\
 & & \text{HH}_*^{\mathbb{H}\mathbb{Q}}(A)
 \end{array}$$

Proof. We can and do assume that M has zero differential, so that A is a formal rational dga. Then $\Omega_{A/\mathbb{Q}}$ is a formal A -module, equivalent to $A \otimes_{\mathbb{H}\mathbb{Q}} M$. In particular, to give the map s , we just have to specify where to map generators of the homology of M . Since the map $\pi_* D: \text{HH}_*^{\mathbb{H}\mathbb{Q}}(A) \rightarrow \pi_* \Sigma\Omega_{A/\mathbb{Q}}$ is surjective, such a section s exists, and we simply pick one.

Note that by our choice of s and Proposition 3.1, the maps $D \circ \text{Tr}$ and $D \circ s \circ \delta$ do agree at the level of homotopy groups. It follows that to prove the corollary we need to show that the image of Tr is contained in the image of s .

Consider the Bökstedt spectral sequence computing $\text{HH}^{\mathbb{H}\mathbb{Q}}(A)$. There is a map of the homotopy colimit spectral sequences for $\text{BGL}_1(A)$ and $\text{HH}^{\mathbb{H}\mathbb{Q}}(A)$ (the Bökstedt spectral sequence) induced by Tr , which on E_2 terms is given by

$$H_s(C_*(\pi_t \mathbf{B}_\bullet \text{GL}_1(A))) \rightarrow H_s(C_*(\pi_t \mathbf{B}_\bullet^{\text{cyc}} A)).$$

The left-hand side is concentrated in the terms $E_{1,t}^2$, and it follows that the image of Tr is contained in $E_{1,t}^\infty$. But the proof of Proposition 3.1 implies that

$$E_{1,t}^2 = E_{1,t}^\infty \cong \pi_t \Omega_{A/\mathbb{Q}}.$$

Therefore, Tr factors through the image of s , as desired. □

It follows that the classes $\delta(x_i)$ and $\delta(y_i)$ appearing in Corollary 2.3 can be chosen to be the suspensions of $d(x_i)$ and $d(y_i)$.

4. The truncated Brown–Peterson spectra as algebras

Fix a prime p . Let BP denote the \mathbb{E}_4 -ring constructed by Basterra and Mandell [11, Theorem 1.1] as an \mathbb{E}_4 -algebra summand of $\text{MU}_{(p)}$, the p -local complex cobordism spectrum. The homotopy ring of BP is

$$\pi_* \text{BP} = \mathbb{Z}_{(p)}[v_1, v_2, \dots],$$

where v_i has degree $2p^i - 2$. By convention, we set $v_0 = p \in \pi_0 \text{BP}$. We will work everywhere with \mathbb{E}_1 -algebras over BP .

Lemma 4.1. *For any v_i , BP/v_i admits \mathbb{E}_1 -algebra structures over BP .*

Proof. By Strickland [38, Corollary 3.3], there exist products on BP/v_i . Moreover, by [38, Proposition 3.1], these are homotopy associative. It follows that we get A_3 -structures on BP/v_i . By Angeltveit [2, Corollary 3.7], which Angeltveit remarks in the introduction to that section applies equally well over any \mathbb{E}_∞ -ring, these extend to A_∞ -structures over $\mathrm{MU}_{(p)}$. Giving an A_∞ -structure over $\mathrm{MU}_{(p)}$ is equivalent to making BP/v_i an \mathbb{E}_1 -ring over $\mathrm{MU}_{(p)}$. Using the \mathbb{E}_4 -ring map $\mathrm{BP} \rightarrow \mathrm{MU}_{(p)}$ of [11, Section 5], we obtain \mathbb{E}_1 -algebra structures on BP/v_i over BP by restriction. \square

In particular, since v_i is not a zero-divisor and by the lemma, the cofiber BP/v_i has the expected homotopy ring, namely

$$\pi_*\mathrm{BP}/v_i \cong \mathbb{Z}_{(p)}[v_1, \dots, \widehat{v_i}, \dots],$$

the quotient of $\pi_*\mathrm{BP}$ by the ideal generated by v_i .

We define $\mathrm{BP}\langle n \rangle$ as the iterated cofiber

$$\operatorname{colim}_{i>n} \mathrm{BP}/v_{n+1} \otimes_{\mathrm{BP}} \cdots \otimes_{\mathrm{BP}} \mathrm{BP}/v_i.$$

Lemma 4.2. *The truncated Brown–Peterson spectra $\mathrm{BP}\langle n \rangle$ admit \mathbb{E}_1 -algebra structures over BP .*

Proof. Since the forgetful functor $\mathrm{Alg}_{\mathrm{BP}} \rightarrow \mathrm{Mod}_{\mathrm{BP}}$ preserves filtered colimits, the underlying BP -module of the colimit $\operatorname{colim}_{i>n} \mathrm{BP}/v_{n+1} \otimes_{\mathrm{BP}} \cdots \otimes_{\mathrm{BP}} \mathrm{BP}/v_i$ in $\mathrm{Alg}_{\mathrm{BP}}$ is $\mathrm{BP}\langle n \rangle$. \square

The proof shows that for any choice of \mathbb{E}_1 -algebra structure on $\mathrm{BP}\langle n \rangle$ and BP/v_n over BP , we obtain a BP -algebra structure on $\mathrm{BP}\langle n-1 \rangle$. Just as above, the homotopy rings of the truncated Brown–Peterson spectra are

$$\pi_*\mathrm{BP}\langle n \rangle \cong \mathbb{Z}_{(p)}[v_1, \dots, v_n]$$

for $n \geq 0$. Additionally, by Proposition 1.10, given any \mathbb{E}_1 -algebra structure on $\mathrm{BP}\langle n \rangle$ over BP , there is an \mathbb{E}_1 -algebra structure on $\mathrm{E}(n) = \mathrm{BP}\langle n \rangle[v_n^{-1}]$ over BP obtained by inverting v_n .

Lemma 4.3. *For $n \geq 1$ and any \mathbb{E}_1 -algebra structures on $\mathrm{BP}\langle n \rangle$ and BP/v_n over BP , the natural map $\mathrm{BP}\langle n \rangle \rightarrow \mathrm{BP}\langle n-1 \rangle$ is a map of \mathbb{E}_1 -algebras over BP .*

Proof. The map in question is the tensor product of $\mathrm{BP}\langle n \rangle$ with the map $\mathrm{BP} \rightarrow \mathrm{BP}/v_n$ in $\mathrm{Alg}_{\mathrm{BP}}$. \square

Remark 4.4. At the moment, it is not obvious to us that the different algebra structures on $\mathrm{BP}\langle n \rangle$ all result in the same K-theories. Hence, we pick once and for all BP -algebra structures on $\mathrm{BP}\langle n-1 \rangle$ and $\mathrm{BP}\langle n \rangle$ so that $\mathrm{BP}\langle n \rangle \rightarrow \mathrm{BP}\langle n-1 \rangle$ is a map of BP -algebras. Our proofs work regardless of these choices, so there is no harm in them.

The fiber sequence $\Sigma^{2p^n-2}\mathbf{BP}\langle n \rangle \xrightarrow{v_n} \mathbf{BP}\langle n \rangle \rightarrow \mathbf{BP}\langle n-1 \rangle$ exhibits $\mathbf{BP}\langle n-1 \rangle$ as a perfect right $\mathbf{BP}\langle n \rangle$ -module. The \mathbb{E}_1 -algebra

$$A\langle n-1 \rangle = \mathrm{End}_{\mathbf{BP}\langle n \rangle}(\mathbf{BP}\langle n-1 \rangle)^{\mathrm{op}}$$

will play a central role in this paper. Using the forgetful functor $\mathrm{Mod}_{\mathbf{BP}\langle n-1 \rangle} \rightarrow \mathrm{Mod}_{\mathbf{BP}\langle n \rangle}$, we get an \mathbb{E}_1 -algebra map $\mathbf{BP}\langle n-1 \rangle \rightarrow A\langle n-1 \rangle$ over \mathbf{BP} .

Lemma 4.5. *As a left $\mathbf{BP}\langle n-1 \rangle$ -module, $A\langle n-1 \rangle$ is equivalent to*

$$\mathbf{BP}\langle n-1 \rangle \oplus \Sigma^{1-2p^n}\mathbf{BP}\langle n-1 \rangle.$$

Proof. The defining sequence given by multiplication by v_n on the left, $\Sigma^{2p^n-2}\mathbf{BP}\langle n \rangle \rightarrow \mathbf{BP}\langle n \rangle \rightarrow \mathbf{BP}\langle n-1 \rangle$, is a cofiber sequence of right $\mathbf{BP}\langle n \rangle$ -modules. Dualizing, we obtain a cofiber sequence of left $\mathbf{BP}\langle n \rangle$ -modules

$$\mathrm{Map}_{\mathbf{BP}\langle n \rangle}(\mathbf{BP}\langle n-1 \rangle, \mathbf{BP}\langle n \rangle) \rightarrow \mathbf{BP}\langle n \rangle \xrightarrow{v_n} \Sigma^{2-2p^n}\mathbf{BP}\langle n \rangle.$$

Tensoring on the left over $\mathbf{BP}\langle n \rangle$ with the $(\mathbf{BP}\langle n-1 \rangle, \mathbf{BP}\langle n \rangle)$ -bimodule $\mathbf{BP}\langle n-1 \rangle$, we obtain a fiber sequence

$$A\langle n-1 \rangle \rightarrow \mathbf{BP}\langle n-1 \rangle \xrightarrow{v_n} \Sigma^{2-2p^n}\mathbf{BP}\langle n-1 \rangle$$

of left $\mathbf{BP}\langle n-1 \rangle$ -modules. It suffices now to show that v_n is nullhomotopic as a self-map of $\mathbf{BP}\langle n-1 \rangle$. But v_n is zero in the homotopy ring by definition of $\mathbf{BP}\langle n-1 \rangle$. As $\mathbf{BP}\langle n-1 \rangle$ is an algebra, v_n is nullhomotopic. \square

We will return to give a closer analysis of $A\langle n-1 \rangle$ in the next section.

5. Rational \mathbb{E}_∞ -structures

The goal of this section is to show that the $\mathbb{H}\mathbb{Q}$ -algebras $\mathbf{BP}\langle n-1 \rangle_{\mathbb{Q}}$ and $A\langle n-1 \rangle_{\mathbb{Q}}$ admit \mathbb{E}_∞ -ring structures and that $\mathbf{BP}\langle n-1 \rangle_{\mathbb{Q}} \rightarrow A\langle n-1 \rangle_{\mathbb{Q}}$ is an \mathbb{E}_∞ -ring map. Our arguments for the rational \mathbb{E}_∞ -ring structure on $A\langle n-1 \rangle_{\mathbb{Q}}$ uses explicit rational commutative differential graded rings.

Remark 5.1. In order to construct the \mathbb{E}_∞ -ring structures we are interested in we have to replace $\mathbf{BP}\langle n \rangle_{\mathbb{Q}}$ and $A\langle n \rangle_{\mathbb{Q}}$ by weakly equivalent models. This however does not affect the rest of the argument, as all functors considered in this paper are homotopy invariant.

Proposition 5.2. *For $n \geq 0$, $\mathbf{BP}\langle n \rangle_{\mathbb{Q}}$ is an \mathbb{E}_∞ - $\mathbb{H}\mathbb{Q}$ -algebra.*

Proof. Fix a prime p . We begin by noting that $\mathbf{BP}_{\mathbb{Q}}$ admits a natural \mathbb{E}_∞ -ring structure arising from $\mathrm{MU}_{\mathbb{Q}}$. Indeed, recall from [1] that $\pi_*\mathrm{MU}_{\mathbb{Q}} \cong \mathbb{Q}[m_1, \dots]$, where m_i is a degree $2i$ class represented by a rational multiple of the cobordism class of $\mathbb{C}\mathbb{P}^i$. To construct the spectrum $\mathbf{BP}_{\mathbb{Q}}$ from $\mathrm{MU}_{\mathbb{Q}}$, one kills each m_i where $i+1$ is not a p -power. The choice of $m_i \in \pi_{2i}\mathrm{MU}_{\mathbb{Q}}$ determines by definition a map

$$\mathrm{Sym}_{\mathbb{H}\mathbb{Q}} \Sigma^{2i}\mathbb{H}\mathbb{Q} \rightarrow \mathrm{MU}_{\mathbb{Q}}$$

of \mathbb{E}_∞ -algebras over $H\mathbb{Q}$. The \mathbb{E}_∞ - $H\mathbb{Q}$ -algebra $MU_{\mathbb{Q}}/m_i$ is the pushout

$$\begin{array}{ccc} \mathrm{Sym}_{H\mathbb{Q}} \Sigma^{2i} H\mathbb{Q} & \longrightarrow & MU_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ H\mathbb{Q} & \longrightarrow & MU_{\mathbb{Q}}/m_i \end{array}$$

in \mathbb{E}_∞ - $H\mathbb{Q}$ -algebras, and it follows from the cofiber sequence

$$\Sigma^{2i} \mathrm{Sym}_{H\mathbb{Q}} \Sigma^{2i} H\mathbb{Q} \rightarrow \mathrm{Sym}_{H\mathbb{Q}} \Sigma^{2i} H\mathbb{Q} \rightarrow H\mathbb{Q},$$

where the right-hand map is the map of \mathbb{E}_∞ - $H\mathbb{Q}$ -algebras defined by sending a generator to $0 \in \pi_{2i} H\mathbb{Q}$, that

$$\pi_* MU_{\mathbb{Q}}/m_i \cong \mathbb{Q}[m_1, \dots, m_{i-1}, \widehat{m}_i, m_{i+1}, \dots].$$

In other words, $MU_{\mathbb{Q}}/m_i$ is the cofiber of multiplication by m_i on $MU_{\mathbb{Q}}$ as a module. Just as in the previous section, we now find that $BP_{\mathbb{Q}}$ has an \mathbb{E}_∞ - $H\mathbb{Q}$ -ring structure obtained by taking the colimit

$$\mathrm{colim}_{i \neq p^j - 1} MU_{\mathbb{Q}}/m_i \xrightarrow{\sim} BP_{\mathbb{Q}}$$

in $\mathrm{CAlg}_{MU_{\mathbb{Q}}}$, the category of \mathbb{E}_∞ -algebras over $MU_{\mathbb{Q}}$. The same process works to reduce from $BP_{\mathbb{Q}}$ to $BP\langle n \rangle_{\mathbb{Q}}$. \square

It would be convenient to view $A\langle n - 1 \rangle_{\mathbb{Q}}$ as an \mathbb{E}_1 - $BP\langle n - 1 \rangle_{\mathbb{Q}}$ -algebra now that we know that $BP\langle n - 1 \rangle_{\mathbb{Q}}$ is an \mathbb{E}_∞ -ring. However, simply having a map of \mathbb{E}_1 -algebras $BP\langle n - 1 \rangle_{\mathbb{Q}} \rightarrow A\langle n - 1 \rangle_{\mathbb{Q}}$ is not enough to guarantee this.

Proposition 5.3. *The map*

$$BP\langle n - 1 \rangle_{\mathbb{Q}} \rightarrow A\langle n - 1 \rangle_{\mathbb{Q}}$$

makes $A\langle n - 1 \rangle_{\mathbb{Q}}$ into an \mathbb{E}_1 - $BP\langle n - 1 \rangle_{\mathbb{Q}}$ -algebra.

Proof. As rationalization is a localization, $A\langle n - 1 \rangle_{\mathbb{Q}}$ is equivalently the endomorphism algebra of $BP\langle n - 1 \rangle_{\mathbb{Q}}$ over $BP\langle n \rangle_{\mathbb{Q}}$, i.e., $A\langle n - 1 \rangle_{\mathbb{Q}} \simeq \mathrm{End}_{BP\langle n \rangle_{\mathbb{Q}}}(BP\langle n - 1 \rangle_{\mathbb{Q}})$. Since $BP\langle n \rangle_{\mathbb{Q}}$ is an \mathbb{E}_∞ -ring, $A\langle n - 1 \rangle_{\mathbb{Q}}$ is automatically equipped with the structure of a $BP\langle n \rangle_{\mathbb{Q}}$ -algebra. Over the rationals, we have an \mathbb{E}_∞ -ring map $BP\langle n - 1 \rangle_{\mathbb{Q}} \rightarrow BP\langle n \rangle_{\mathbb{Q}}$, and this makes $A\langle n - 1 \rangle_{\mathbb{Q}}$ into a $BP\langle n - 1 \rangle_{\mathbb{Q}}$ -algebra. To see that this map agrees with the map $BP\langle n - 1 \rangle_{\mathbb{Q}} \rightarrow A\langle n - 1 \rangle_{\mathbb{Q}}$ coming from Morita theory note that both maps restrict to equivalent maps

$$\bigoplus_{i=1}^{n-1} \Sigma^{2p^i - 2} H\mathbb{Q} \rightarrow A\langle n - 1 \rangle_{\mathbb{Q}},$$

which is enough to conclude since $BP\langle n - 1 \rangle_{\mathbb{Q}}$ is a free \mathbb{E}_∞ -algebra. \square

In order to prove that $A\langle n - 1 \rangle_{\mathbb{Q}}$ admits an \mathbb{E}_∞ -ring structure, we need to know the homotopy ring of $A\langle n - 1 \rangle$, which we determine now. Unfortunately, we do not say much about the ring structure over \mathbb{S} , but we do find $A\langle n - 1 \rangle_{\mathbb{Q}}$ up to weak equivalence.

Lemma 5.4. *The homotopy ring of $A\langle n - 1 \rangle$ is*

$$\pi_*A\langle n - 1 \rangle \cong \Lambda_{\pi_*BP\langle n-1 \rangle} \langle \varepsilon_{1-2p^n} \rangle,$$

*the graded exterior algebra over $\pi_*BP\langle n - 1 \rangle$ on one generator ε_{1-2p^n} in degree $1 - 2p^n$.*

Proof. Under the splitting of Lemma 4.5, we let $\varepsilon = \varepsilon_{1-2p^n}$ denote the class of the map $\Sigma^{1-2p^n}BP\langle n - 1 \rangle \rightarrow A\langle n - 1 \rangle$. Because of degree considerations, $\varepsilon^2 = 0$ in the homotopy ring of $A\langle n - 1 \rangle$. The description of $\pi_*A\langle n - 1 \rangle$ is correct as a left $\pi_*BP\langle n - 1 \rangle$ -module by Lemma 4.5. The only question is whether $v_i\varepsilon = \varepsilon v_i$ for $1 \leq i \leq n - 1$. Since $\pi_*A\langle n - 1 \rangle \rightarrow \pi_*A\langle n - 1 \rangle_{\mathbb{Q}}$ is injective, it is enough to prove this statement rationally. By Proposition 5.2, $BP\langle n \rangle_{\mathbb{Q}}$ is a formal rational dga on v_1, \dots, v_n . For the remainder of the proof, we work entirely with $BP\langle n \rangle_{\mathbb{Q}}$ as a formal rational dga, and we use dg-modules over $BP\langle n \rangle_{\mathbb{Q}}$. Let $X = \text{cone}(v_n)$, where $v_n : \Sigma^{2p^n-2}BP\langle n \rangle_{\mathbb{Q}} \rightarrow BP\langle n \rangle_{\mathbb{Q}}$. Thus, X is a dg $BP\langle n \rangle_{\mathbb{Q}}$ -module with $X_k = \pi_k BP\langle n \rangle_{\mathbb{Q}} \oplus \pi_{k-1}(\Sigma^{2p^n-2}BP\langle n \rangle_{\mathbb{Q}})$, and with the differential $X_k \rightarrow X_{k-1}$ given by

$$\begin{pmatrix} 0 & v_n \\ 0 & 0 \end{pmatrix}.$$

Recall that $A\langle n - 1 \rangle_{\mathbb{Q}} \simeq \text{End}_{BP\langle n \rangle_{\mathbb{Q}}}(X)$. Each v_i for $1 \leq i \leq n - 1$ acts as an endomorphism of X in the obvious way, with matrix representation

$$\begin{pmatrix} v_i & 0 \\ 0 & v_i \end{pmatrix}.$$

The element ε can be represented as well. Let

$$\sigma : \pi_{*-1}(\Sigma^{2p^n-2}BP\langle n \rangle_{\mathbb{Q}}) \rightarrow \pi_{1-2p^n+*}BP\langle n \rangle_{\mathbb{Q}}$$

be a fixed $\pi_*BP\langle n \rangle_{\mathbb{Q}}$ -module isomorphism. Then

$$\varepsilon = \begin{pmatrix} 0 & \sigma \\ 0 & 0 \end{pmatrix}$$

from $\Sigma^{1-2p^n}X$ to X is a map of $BP\langle n \rangle_{\mathbb{Q}}$ -modules. For degree reasons, this map is unique up to the choice of σ , which in turn is unique up to multiplying by a nonzero rational number. Hence, we can assume that this ε represents ε_{1-2p^n} above, as X is cofibrant as a dg $BP\langle n \rangle_{\mathbb{Q}}$ -module. Now, we see that $v_i\varepsilon = \varepsilon v_i$ for $1 \leq i \leq n - 1$. \square

Similar ideas allow us to prove that $A\langle n - 1 \rangle_{\mathbb{Q}}$ is equivalent to a commutative rational dga.

Proposition 5.5. *The algebra $A\langle n - 1 \rangle_{\mathbb{Q}}$ admits the structure of a commutative rational differential graded algebra.*

Proof. For simplicity, let $d = 2p^n - 2$. As in the proof above, we can assume that $A\langle n - 1 \rangle_{\mathbb{Q}}$ is the endomorphism dga of the cone X of v_n . As a graded algebra,

$$A\langle n - 1 \rangle_{\mathbb{Q}} = \bigoplus_k \text{Hom}_{\pi_*BP\langle n \rangle_{\mathbb{Q}}}^k(X, X),$$

where the degree k part consists of homogeneous degree k maps of graded $\pi_*\mathbf{BP}\langle n \rangle_{\mathbb{Q}}$ -modules. Since X is isomorphic to $\pi_*\mathbf{BP}\langle n \rangle_{\mathbb{Q}} \oplus \pi_*\Sigma^{d+1}\mathbf{BP}\langle n \rangle_{\mathbb{Q}}$ as a graded module, it follows that $A\langle n-1 \rangle_{\mathbb{Q}}$ as a graded algebra is isomorphic to the graded matrix ring $M_2(\pi_*\mathbf{BP}\langle n \rangle_{\mathbb{Q}})$, determined by letting

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

be in degree $-d-1 = 1-2p^n$. So, an element of $A\langle n-1 \rangle_{\mathbb{Q}}$ of degree k is represented by an element

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $a, d \in \pi_k\mathbf{BP}\langle n \rangle_{\mathbb{Q}}$, the element b is in $\pi_{k+d+1}\mathbf{BP}\langle n \rangle_{\mathbb{Q}}$, and $c \in \pi_{k-d-1}\mathbf{BP}\langle n \rangle_{\mathbb{Q}}$. The differential on $A\langle n-1 \rangle_{\mathbb{Q}}$ is defined by the equation

$$d(f) = d_X \circ f - (-1)^k f \circ d_X$$

if f is homogeneous of degree k . With this convention,

$$d\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} v_n c & v_n d \\ 0 & 0 \end{pmatrix} - (-1)^k \begin{pmatrix} 0 & a v_n \\ 0 & c v_n \end{pmatrix} = \begin{pmatrix} v_n c & v_n d - (-1)^k a v_n \\ 0 & -(-1)^k c v_n \end{pmatrix}$$

for a homogeneous element f of degree k , written as an element of $M_2(\pi_*\mathbf{BP}\langle n \rangle_{\mathbb{Q}})$.

Since $\pi_*\mathbf{BP}\langle n \rangle_{\mathbb{Q}}$ is commutative and v_n is a regular homogeneous element, the cycles of degree k in $A\langle n-1 \rangle_{\mathbb{Q}}$ are all of the form

$$\begin{pmatrix} a & b \\ 0 & (-1)^k a \end{pmatrix}.$$

Let Z be the graded subalgebra of $A\langle n-1 \rangle_{\mathbb{Q}}$ given by cycles such that v_n does not divide either a or b . This graded submodule when equipped with the trivial differential is a dg-submodule (over \mathbb{Q} or $\mathbf{BP}\langle n-1 \rangle_{\mathbb{Q}}$), and the inclusion respects multiplication and is unital; that is, it is a dg-subalgebra of $A\langle n-1 \rangle_{\mathbb{Q}}$. Because of the fact that $\pi_*\mathbf{BP}\langle n \rangle_{\mathbb{Q}}$ is concentrated in even degrees, and as $d+1$ is odd, a and b can never be simultaneously nonzero. It follows from this that Z is in fact a commutative differential graded algebra. Now, we claim that the inclusion $Z \rightarrow A\langle n-1 \rangle_{\mathbb{Q}}$ is a quasi-isomorphism. Indeed, the map in homology is clearly injective, since all boundaries in $A\langle n-1 \rangle_{\mathbb{Q}}$ contain a v_n -term, and it is surjective because all cycles in which a v_n -term appears must be boundaries (using again the fact that a and b in the notation above are not simultaneously nonzero). As Z is a commutative dga, the proposition follows. \square

Remark 5.6. An alternative way of proving the results in this section is via Goerss–Hopkins obstruction theory. The obstruction groups for the existence of \mathbb{E}_{∞} -ring structures on $A\langle n-1 \rangle_{\mathbb{Q}}$ lifting the square-zero structure on coefficients vanish, while the groups parameterizing choices of lifts are zero except at $p = 2$.

6. Rognes' question

In this section we give a negative answer to the question of Rognes for $n > 1$ at all primes p . Note that while we work with the non- p -completed Brown–Peterson and Johnson–Wilson spectra for notational simplicity, no alterations are needed in the argument to handle the p -complete case.

Question 6.1 (Non- p -complete Rognes question). For $n > 0$, is the sequence

$$K(\mathbf{BP}\langle n - 1 \rangle) \rightarrow K(\mathbf{BP}\langle n \rangle) \rightarrow K(\mathbf{E}(n)) \tag{3}$$

a fiber sequence of connective spectra, where $K(\mathbf{BP}\langle n - 1 \rangle) \rightarrow K(\mathbf{BP}\langle n \rangle)$ is the transfer map?

An affirmative answer would identify the fiber of a localization map in K -theory. Our earlier results allow us to do this unconditionally when the localization comes from a reasonable localization of \mathbb{E}_1 -rings. In the case of truncated Brown–Peterson spectra, we get the following result.

Theorem 6.2. Fix $n > 0$, and let $A\langle n - 1 \rangle = \text{End}_{\mathbf{BP}\langle n \rangle}(\mathbf{BP}\langle n - 1 \rangle)^{\text{op}}$. There is a fiber sequence

$$\mathbb{K}(A\langle n - 1 \rangle) \rightarrow \mathbb{K}(\mathbf{BP}\langle n \rangle) \rightarrow \mathbb{K}(\mathbf{E}(n))$$

of nonconnective K -theory spectra.

Proof. Since the homotopy ring $\pi_*\mathbf{BP}\langle n \rangle \simeq \mathbb{Z}_{(p)}[v_1, \dots, v_n]$ is graded-commutative, the multiplicative subset generated by v_n trivially satisfies the right Ore condition. The localization is nothing other than the Johnson–Wilson spectrum $\mathbf{E}(n)$, and the cofiber of $\Sigma^{2p^n-2}\mathbf{BP}\langle n \rangle \xrightarrow{v_n} \mathbf{BP}\langle n \rangle$ is the spectrum $\mathbf{BP}\langle n - 1 \rangle$, viewed as a right $\mathbf{BP}\langle n \rangle$ -module. The theorem follows now from Theorem 1.11. \square

The transfer map in the statement of the question is obtained by viewing $\mathbf{BP}\langle n - 1 \rangle$ as a perfect right $\mathbf{BP}\langle n \rangle$ -module.

Lemma 6.3. There is a commutative diagram

$$\begin{array}{ccccc} \mathbb{K}(A\langle n - 1 \rangle) & \longrightarrow & \mathbb{K}(\mathbf{BP}\langle n \rangle) & \longrightarrow & \mathbb{K}(\mathbf{E}(n)) \\ \uparrow & & \nearrow & & \\ \mathbb{K}(\mathbf{BP}\langle n - 1 \rangle) & & & & \end{array}$$

where the diagonal arrow is the transfer map and the vertical map is induced from the algebra map $\mathbf{BP}\langle n - 1 \rangle \rightarrow A\langle n - 1 \rangle$.

Proof. There is a commutative diagram

$$\begin{array}{ccc}
 \text{Mod}_{A\langle n-1 \rangle} & \longrightarrow & \text{Mod}_{\text{BP}\langle n \rangle} \\
 \uparrow & \nearrow & \\
 \text{Mod}_{\text{BP}\langle n-1 \rangle} & &
 \end{array}$$

Here the horizontal arrow is the fully faithful functor arising from the equivalence of v_n -nilpotent $\text{BP}\langle n \rangle$ -modules and right $A\langle n-1 \rangle$ -modules, the diagonal arrow is the forgetful functor along the map of \mathbb{E}_1 -rings $\text{BP}\langle n \rangle \rightarrow \text{BP}\langle n-1 \rangle$, and the vertical map is induced since the diagonal map lands in the subcategory of v_n -nilpotent $\text{BP}\langle n \rangle$ -modules. All three functors preserve compact objects, and the maps in K -theory in the statement of the lemma are those induced by these three functors. \square

The next result is a trivial consequence of the lemma, but the observation is at the heart of our approach to the question of Rognes.

Lemma 6.4. *Suppose that (3) is a fiber sequence of connective spectra. Then $\mathbb{K}(\text{BP}\langle n-1 \rangle) \rightarrow \mathbb{K}(A\langle n-1 \rangle)$ is an equivalence.*

By the theorem of Blumberg and Mandell [14], when $n = 1$ the question has a positive answer, and hence the theorem applies. We state the analogous result for complex K -theory. Let $A = \text{End}_{\text{ku}}(\mathbb{H}\mathbb{Z})$. Then the methods above give a fiber sequence

$$\mathbb{K}(A) \rightarrow \mathbb{K}(\text{ku}) \rightarrow \mathbb{K}(\text{KU}).$$

On the other hand, Blumberg and Mandell [14] showed that at the level of connective K -theory, one has a fiber sequence

$$\mathbb{K}(\mathbb{H}\mathbb{Z}) \rightarrow \mathbb{K}(\text{ku}) \rightarrow \mathbb{K}(\text{KU})$$

of connective spectra. It follows that $\mathbb{K}(\mathbb{H}\mathbb{Z}) \rightarrow \mathbb{K}(A)$ is an equivalence. In this case, A has nonzero homotopy groups $\pi_0 A \cong \mathbb{Z}$ and $\pi_{-3} A \cong \mathbb{Z} \cdot \varepsilon_{-3}$.

It is not difficult using group completion techniques to show that when $n > 1$ the map

$$\mathbb{K}_i(\text{BP}\langle n-1 \rangle) \rightarrow \mathbb{K}_i(\tau_{\geq 0} A\langle n-1 \rangle)$$

is not an isomorphism for general $i > 0$. However, because it seems difficult to analyze the map $\mathbb{K}(\tau_{\geq 0} A\langle n-1 \rangle) \rightarrow \mathbb{K}(A\langle n-1 \rangle)$, this does not directly solve Rognes’ question. The strategy of the main theorem will be to compute classes in the image of the trace map to topological Hochschild homology in order to conclude that there are positive degree classes in $\mathbb{K}(A\langle n-1 \rangle)$ not in the image of $\mathbb{K}(\text{BP}\langle n-1 \rangle) \rightarrow \mathbb{K}(A\langle n-1 \rangle)$. This of course implies that the question of Rognes (including the p -complete versions) has a negative answer for $n > 1$.

Now we come to the main theorem of the paper.

Theorem 6.5. *The transfer map $\mathbb{K}(\text{BP}\langle n-1 \rangle) \rightarrow \mathbb{K}(A\langle n-1 \rangle)$ is not an equivalence when $n > 1$. In particular, for $n > 1$,*

$$\mathbb{K}(\text{BP}\langle n-1 \rangle) \rightarrow \mathbb{K}(\text{BP}\langle n \rangle) \rightarrow \mathbb{K}(E(n))$$

is not a fiber sequence of connective spectra.

Proof. In view of the commutative diagram

$$\begin{array}{ccc} K(\mathbf{BP}\langle n-1 \rangle) & \longrightarrow & \mathrm{HH}^{\mathrm{H}\mathbb{Q}}(\mathbf{BP}\langle n-1 \rangle_{\mathbb{Q}}) \\ \downarrow & & \downarrow \\ K(\mathbf{A}\langle n-1 \rangle) & \longrightarrow & \mathrm{HH}^{\mathrm{H}\mathbb{Q}}(\mathbf{A}\langle n-1 \rangle_{\mathbb{Q}}) \end{array}$$

this is an immediate consequence of the next lemma. □

By Proposition 5.5, $\mathbf{A}\langle n-1 \rangle_{\mathbb{Q}}$ admits the structure of an \mathbb{E}_{∞} - $\mathrm{H}\mathbb{Q}$ -algebra. Since the homotopy ring $\pi_*\mathbf{A}\langle n-1 \rangle_{\mathbb{Q}}$ is a free graded-commutative \mathbb{Q} -algebra, it follows that $\mathbf{A}\langle n-1 \rangle_{\mathbb{Q}}$ is equivalent to a free \mathbb{E}_{∞} - $\mathrm{H}\mathbb{Q}$ -algebra, so that Corollary 2.3 applies and gives

$$\mathrm{HH}_*^{\mathrm{H}\mathbb{Q}}(\mathbf{A}\langle n-1 \rangle_{\mathbb{Q}}) \cong \mathbb{Q}[v_1, \dots, v_{n-1}, \delta_{2-2p^n}] \otimes \Lambda_{\mathbb{Q}}\langle \sigma_1, \dots, \sigma_{n-1}, \varepsilon_{1-2p^n} \rangle, \quad (4)$$

where the degree of σ_i is $2p^i - 1$ and the degree of ε_{1-2p^n} is $1 - 2p^n$.

Lemma 6.6. *If $x = v_1^{a_1} \cdots v_{n-1}^{a_{n-1}} \varepsilon_{1-2p^n}$ is a monomial in $\pi_*\mathbf{A}\langle n-1 \rangle$ of positive total degree, i.e., $\sum_{i=1}^{n-1} a_i(p^i - 1) \geq p^n$, then the class*

$$v_1^{a_1} \cdots v_{n-1}^{a_{n-1}} \delta_{2-2p^n} + \sum_{i=1}^{n-1} a_i v_1^{a_1} \cdots v_i^{a_i-1} \cdots v_{n-1}^{a_{n-1}} \sigma_i \varepsilon_{1-2p^n}$$

is in the image of the map $K(\mathbf{A}\langle n-1 \rangle) \rightarrow \mathrm{HH}^{\mathrm{H}\mathbb{Q}}(\mathbf{A}\langle n-1 \rangle_{\mathbb{Q}})$ and not in the image of $\mathrm{HH}^{\mathrm{H}\mathbb{Q}}(\mathbf{BP}\langle n-1 \rangle_{\mathbb{Q}}) \rightarrow \mathrm{HH}^{\mathrm{H}\mathbb{Q}}(\mathbf{A}\langle n-1 \rangle_{\mathbb{Q}})$.

Proof. Consider the commutative diagram

$$\begin{array}{ccccc} \Sigma^{\infty} \mathrm{BGL}_1(\mathbf{BP}\langle n-1 \rangle_+) & \longrightarrow & K(\mathbf{BP}\langle n-1 \rangle) & \longrightarrow & \mathrm{HH}^{\mathrm{H}\mathbb{Q}}(\mathbf{BP}\langle n-1 \rangle_{\mathbb{Q}}) \\ \downarrow & & \downarrow & & \downarrow \\ \Sigma^{\infty} \mathrm{BGL}_1(\mathbf{A}\langle n-1 \rangle_+) & \longrightarrow & K(\mathbf{A}\langle n-1 \rangle) & \longrightarrow & \mathrm{HH}^{\mathrm{H}\mathbb{Q}}(\mathbf{A}\langle n-1 \rangle_{\mathbb{Q}}) \end{array}$$

Using Corollary 2.3, we see that the right-hand vertical map is in fact an inclusion of algebras

$$\begin{aligned} &\mathbb{Q}[v_1, \dots, v_{n-1}] \otimes \Lambda_{\mathbb{Q}}\langle \sigma_1, \dots, \sigma_{n-1} \rangle \\ &\quad \rightarrow \mathbb{Q}[v_1, \dots, v_{n-1}, \delta_{2-2p^n}] \otimes \Lambda_{\mathbb{Q}}\langle \sigma_1, \dots, \sigma_{n-1}, \varepsilon_{1-2p^n} \rangle. \end{aligned}$$

If x has positive degree d , let y denote the class associated to x in

$$\pi_{d+1} \mathrm{BGL}_1(\mathbf{A}\langle n-1 \rangle) \cong \pi_d \mathbf{A}\langle n-1 \rangle.$$

By Corollary 3.2, the class y maps via the trace map and rationalization to the nonzero element

$$v_1^{a_1} \cdots v_{n-1}^{a_{n-1}} \delta_{2-2p^n} + \sum_{i=1}^{n-1} a_i v_1^{a_1} \cdots v_i^{a_i-1} \cdots v_{n-1}^{a_{n-1}} \sigma_i \varepsilon_{1-2p^n}$$

of $\mathrm{HH}_{d+1}^{\mathrm{H}\mathbb{Q}}(\mathbb{A}\langle n-1\rangle_{\mathbb{Q}})$. Because of the monomials involving ε_{1-2p^n} and δ_{2-2p^n} , this class is not in the image of $\mathrm{HH}^{\mathrm{H}\mathbb{Q}}(\mathrm{BP}\langle n-1\rangle_{\mathbb{Q}})$. Of course, since its image in $\mathrm{HH}_{d+1}^{\mathrm{H}\mathbb{Q}}(\mathbb{A}\langle n-1\rangle_{\mathbb{Q}})$ is nonzero, the class y must map to a nonzero class in $\mathrm{K}_{d+1}(\mathbb{A}\langle n-1\rangle)$. This class cannot be in the image of $\mathrm{K}(\mathrm{BP}\langle n-1\rangle) \rightarrow \mathrm{K}(\mathbb{A}\langle n-1\rangle)$. \square

The proof of the theorem requires both the negative degree class ε_{1-2p^n} and the positive degree class v_1 in the homotopy groups of $\mathbb{A}\langle n-1\rangle$. When $n = 0, 1$, there is no v_1 , which is why this method does not contradict the earlier theorems of Quillen and Blumberg–Mandell.

Remark 6.7. Recent work of Blumberg and Mandell [15] provides a different approach to the homotopy groups of $\mathrm{K}(\mathbb{S})$ in the spirit of the Ausoni–Rognes program, in which analyzing the K -theory of $E(n)$ is skipped in favor of looking at E_n directly. Blumberg and Mandell prove as an extension of their earlier work in [14] that there is a fiber sequence

$$\mathrm{K}(\mathbb{W}[\![u_1, \dots, u_{n-1}]\!]) \rightarrow \mathrm{K}(\mathrm{BP}_n) \rightarrow \mathrm{K}(E_n) \tag{5}$$

of connective spectra for all $n > 0$, where BP_n denotes the connective cover of E_n , \mathbb{W} is the p -typical Witt ring, and the u_i are in degree 0. Note, however, that the Ausoni–Rognes program in principle allows a computation of $\mathrm{K}(\mathrm{BP}\langle n\rangle)$, whereas $\mathrm{K}(\mathrm{BP}_n)$ is more difficult to compute using their techniques.

In the end, the successful dévissage-type results of Quillen [31], Blumberg–Mandell [14, 15], and Barwick–Lawson [9] can all be expressed in terms of Barwick’s theorem of the heart [6, Theorem 8.7]. For example, consider the \mathbb{E}_{∞} -ring spectrum ku and its localization $\mathrm{KU} = \mathrm{ku}[\beta^{-1}]$. In the notation of Section 1.2, there is an exact sequence $\mathrm{Mod}_{\mathrm{ku}}^{\mathrm{Nil}(S),\omega} \rightarrow \mathrm{Mod}_{\mathrm{ku}}^{\omega} \rightarrow \mathrm{Mod}_{\mathrm{KU}}^{\omega}$, where $S = \{1, \beta, \beta^2, \dots\}$. The natural Postnikov t -structure on $\mathrm{Mod}_{\mathrm{ku}}$ is *not* bounded, but it restricts to a bounded t -structure on $\mathrm{Mod}_{\mathrm{ku}}^{\mathrm{Nil}(S),\omega}$ with heart the category of finitely generated abelian groups.

Barwick’s theorem of the heart says that the connective K -theory of a stable ∞ -category with a bounded t -structure is equivalent to the connective K -theory of the heart. Thus, we obtain the fiber sequence $\mathrm{K}(\mathbb{Z}) \rightarrow \mathrm{K}(\mathrm{ku}) \rightarrow \mathrm{K}(\mathrm{KU})$ of connective spectra. The same argument establishes the fiber sequences in (5). In fact, Barwick’s theorem was recently extended in [4] to negative K -theory when the heart is noetherian. Thus, there are fiber sequences

$$\mathbb{K}(\mathbb{Z}) \rightarrow \mathbb{K}(\mathrm{ku}) \rightarrow \mathbb{K}(\mathrm{KU})$$

and

$$\mathbb{K}(\mathbb{W}[\![u_1, \dots, u_{n-1}]\!]) \rightarrow \mathbb{K}(\mathrm{BP}_n) \rightarrow \mathbb{K}(E_n)$$

of *nonconnective* K -theory spectra. This amounts to proving that the negative K -theory of KU and E_n vanish. On the other hand, the following problem is open.

Question 6.8. Do the negative K -groups of $E(n)$ vanish for $n \geq 2$?

Since $\pi_0\mathrm{BP}\langle n \rangle \cong \mathbb{Z}_{(p)}$ is regular and noetherian, $K_{-i}(\mathrm{BP}\langle n \rangle) = 0$ for $i \geq 1$ by [12, Theorem 9.53]. So, we may ask well as about the negative K -theory of $A\langle n-1 \rangle$. The question of Rognes can be viewed as asking for some structure on $\mathrm{Mod}_{A\langle n-1 \rangle}^\omega$ that generalizes that of a t -structure, with generalized heart equivalent to $\mathrm{Mod}_{\mathrm{BP}\langle n-1 \rangle}^\omega$, as well as for a generalization of the theorem of the heart. The main theorem of this paper shows that this is too much to hope for.

Acknowledgments. We would like to thank Andy Baker, Bjørn Dundas, John Greenlees, Owen Gwilliam, Mike Hopkins, Ayelet Lindenstrauss, Charles Rezk, Christian Schlichtkrull, and Sean Tilson for conversations about this work. This project emerged while all three authors participated in the 2014 Algebraic Topology Program at MSRI, and we thank both the institute for its hospitality and the organizers of the program for creating such a stimulating environment. The first named author would like to give special thanks to Yankı Lekili who asked him an apparently unrelated question which eventually led to the discovery of a sign problem obstructing progress. Special thanks are reserved for Clark Barwick, Andrew Blumberg, Mike Mandell, and John Rognes for their comments on an early draft. Finally, we thank the referees for their careful reading of the text.

Benjamin Antieau was supported by NSF Grant DMS-1461847.

David Gepner was supported by NSF Grant DMS-1406529.

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