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Noncommutative Riesz transforms— dimension free bounds and Fourier multipliers

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Abstract. We obtain dimension free estimates for noncommutative Riesz transforms associated to conditionally negative length functions on group von Neumann algebras. This includes Poisson semigroups, beyond Bakry's results in the commutative setting. Our proof is inspired by Pisier's method and a new Khintchine inequality for crossed products. New estimates include Riesz transforms associated to fractional laplacians in $\mathbb{R}^n$ (where Meyer's conjecture fails) or to the word length of free groups. Lust-Piquard's work for discrete laplacians on LCA groups is also generalized in several ways. In the context of Fourier multipliers, we will prove that Hörmander–Mikhlin multipliers are Littlewood-Paley averages of our Riesz transforms. This is highly surprising in the Euclidean and (most notably) noncommutative settings. As application we provide new Sobolev/Besov type smoothness conditions. The Sobolev-type condition we give refines the classical one and yields dimension free constants. Our results hold for arbitrary unimodular groups.

Keywords. Riesz transform, group von Neumann algebra, dimension free estimates, Hörmander–Mikhlin multiplier

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Introduction

The classical Riesz transforms $R_j f = \partial_j(-\Delta)^{-1/2} f$ are higher-dimensional forms of the Hilbert transform in $\mathbb{R}$. Dimension free estimates for the associated square functions $\mathcal{R}f = |\nabla(-\Delta)^{-1/2} f|$ were first proved by Gundy/Varopoulos [28] and shortly after by Stein [71], who pointed out the significance of a dimension free formulation of Euclidean harmonic analysis. The aim of this paper is to provide dimension free estimates for a much broader class of Riesz transforms and apply them for further insight in Fourier multiplier L_p-theory. Our approach is surprisingly simple and it is valid in the general context of group von Neumann algebras.

A relevant generalization appeared in the work of P. A. Meyer [50], continued by Bakry, Gundy and Pisier [3, 4, 27, 55] among others. The probabilistic approach consists in replacing $-\Delta$ by the infinitesimal generator A of a nice semigroup acting on a probability space (Ω, μ). The gradient form $\langle \nabla f_1, \nabla f_2 \rangle$ is also replaced by the so-called "carré du champs" $\Gamma_A(f_1, f_2) = \frac{1}{2}(\overline{A(f_1)} f_2 + \overline{f_1} A(f_2) - A(\overline{f_1} f_2))$, and *Meyer's problem* for (Ω, μ, A) consists in determining whether

$$\|\Gamma_A(f, f)^{1/2}\|_p \sim_{c(p)} \|A^{1/2} f\|_p \quad (1 < p < \infty) \tag{MP}$$

holds on a dense subspace of dom A. As usual, $A \sim B$ means $\delta \leq A/B \leq M$ for some absolute constants $M, \delta > 0$. We write $A \sim_c B$ when $\max\{M, 1/\delta\} \leq c$. Meyer proved this for the Ornstein–Uhlenbeck semigroup, while Bakry considered other diffusion semigroups assuming the $\Gamma^2 \geq 0$ condition, which yields in turn a lower bound for the Ricci curvature in Riemannian manifolds [4, 43]. Clifford algebras were considered by Lust-Piquard [46, 47], and other topics concerning optimal linear estimates can be found in [12, 22] and the references therein. Further dimension free estimates for maximal functions appear in [6, 10, 51, 70].

Contrary to what might be expected, (MP) *fails* for the Poisson semigroup in $\mathbb{R}^n$ when $1 < p \leq 2n/(n+1)$, even allowing constants depending on n (see Appendix D for details). Bakry's argument heavily uses commutative diffusion properties, and hence the failure of (MP) for subordinated processes and $p < 2$ does not contradict his work. Moreover, besides the heat semigroup, convolution processes have not been studied systematically. Lust-Piquard's theorem on discrete laplacians for LCA groups [48] seems to be the only exception. Our first goal is to fill this gap and study Meyer's problem for Markov convolution semigroups in the Euclidean case and other group algebras. In this paper, we introduce a new form of (MP) which holds in much larger generality. As we shall see, this requires following the tradition of noncommutative Khintchine inequalities [49, 60] which involves considering an infimum over decompositions into two terms when

$p < 2$. In the terminology of noncommutative geometry, our decomposition takes place in the space of differential forms of order 1 (see Appendix, Lemma C1). Indeed, the deeper understanding of derivations in noncommutative analysis provides a better understanding of Riesz transforms, even for classical semigroups of convolution type.

Let us first consider a simple model. Given a discrete abelian group G, let $(\Omega, \mu) = (\widehat{\mathrm{G}}, \nu)$ be the compact dual group with its normalized Haar measure and construct the group characters $\chi_g : \Omega \to \mathbb{T}$. By Schoenberg's theorem [64] a given convolution semigroup $\mathcal{S}_{\psi,t} : \chi_g \mapsto e^{-t\psi(g)}\chi_g$ is Markovian in Ω iff $\psi(e) = 0$ for the identity e, $\psi(g) = \psi(g^{-1})$, and $\sum_g a_g = 0 \Rightarrow \sum_{g,h} \overline{a_g} a_h \psi(g^{-1}h) \le 0$. Any such function ψ is called a *conditionally negative length*. $A_\psi(\chi_g) = \psi(g)\chi_g$ is the generator, which determines the gradient form Γ_ψ. Does (MP) or a generalization of it hold for arbitrary pairs (G, ψ)? To answer this question we first widen the scope of the problem and consider its formulation for nonabelian discrete groups G. The rôle of $L_\infty(\Omega, \mu)$ is now played by the group von Neumann algebra $\mathcal{L}(\mathrm{G})$, widely studied in noncommutative geometry and operator algebras [9, 18, 20].

Let G be a discrete group with left regular representation $\lambda : \mathrm{G} \to \mathcal{B}(\ell_2(\mathrm{G}))$ given by $\lambda(g)\delta_h = \delta_{gh}$, where the δ_g's form the unit vector basis of $\ell_2(\mathrm{G})$. Write $\mathcal{L}(\mathrm{G})$ for its group von Neumann algebra, the weak operator closure of the linear span of $\lambda(\mathrm{G})$ in $\mathcal{B}(\ell_2(\mathrm{G}))$. Consider the standard trace $\tau(\lambda(g)) = \delta_{g=e}$ where e denotes the identity of G. Any $f \in \mathcal{L}(\mathrm{G})$ has a Fourier series expansion

$$\sum_{g \in \mathrm{G}} \widehat{f}(g)\lambda(g) \quad \text{with} \quad \tau(f) = \widehat{f}(e).$$

Define the L_p space over the noncommutative measure space $(\mathcal{L}(\mathrm{G}), \tau)$ as

$$L_p(\widehat{\mathbf{G}}) = L_p(\mathcal{L}(\mathrm{G}), \tau) \equiv \text{closure of } \mathcal{L}(\mathrm{G}) \text{ in the norm } \|f\|_{L_p(\widehat{\mathbf{G}})} = (\tau|f|^p)^{1/p}.$$

In general, the (unbounded) operator $|f|^p$ is obtained from functional calculus on the Hilbert space $\ell_2(\mathrm{G})$ (see [60] or Appendix B for further details). It turns out that

$$L_p(\widehat{\mathbf{G}}) \simeq L_p(\widehat{\mathrm{G}}) = L_p(\Omega, \mu)$$

for abelian G. Indeed, the map $\lambda(g) \in (\mathcal{L}(\mathrm{G}), \tau) \to \chi_g \in L_\infty(\widehat{\mathrm{G}}, \nu)$ extends to a trace preserving $*$-homomorphism, hence to an L_p isometry for $p \ge 1$. This means that we can identify Fourier series in both spaces sending $\lambda(g)$ to the group character χ_g and

$$\Big\| \sum_{g \in \mathrm{G}} \widehat{f}(g)\lambda(g) \Big\|_{L_p(\widehat{\mathbf{G}})} = \Big\| \sum_{g \in \mathrm{G}} \widehat{f}(g)\chi_g \Big\|_{L_p(\widehat{\mathrm{G}})}.$$

Harmonic analysis on $\mathcal{L}(\mathrm{G})$ places the group on the frequency side. This approach is partly inspired by the remarkable results of Cowling/Haagerup [19, 29] on approximation properties and Fourier multipliers on group algebras. This paper is part of an effort [34, 35, 53] to extend modern harmonic analysis to the unexplored context of group von

Neumann algebras. Markovian semigroups acting on $\mathcal{L}(\mathrm{G})$ are composed of self-adjoint, completely positive and unital maps. Schoenberg's theorem is still valid and

$$\mathcal{S}_{\psi,t} f = \sum_{g \in \mathrm{G}} e^{-t\psi(g)} \widehat{f}(g) \lambda(g)$$

will be Markovian if and only if $\psi : \mathrm{G} \to \mathbb{R}_+$ is a conditionally negative length.

Riesz transforms should look like $R_{\psi,j} f = \partial_{\psi,j} A_\psi^{-1/2} f$ where the laplacian is replaced by $A_\psi(\lambda(g)) = \psi(g)\lambda(g)$ and $\partial_{\psi,j}$ is a certain differential operator playing the role of a directional derivative. Unlike for $\mathbb{R}^n$, there is no standard differential structure for an arbitrary discrete G. The additional structure comes from the length ψ, which allows a broader interpretation of tangent space in terms of the associated cocycle. Namely, conditionally negative lengths are in one-to-one correspondence with affine representations $(\mathcal{H}_\psi, \alpha_\psi, b_\psi)$, where $\alpha_\psi : \mathrm{G} \to O(\mathcal{H}_\psi)$ is an orthogonal representation over a real Hilbert space $\mathcal{H}_\psi$ and $b_\psi : \mathrm{G} \to \mathcal{H}_\psi$ is a mapping satisfying the cocycle law (see Appendix B for further details)

$$b_\psi(gh) = \alpha_{\psi,g}(b_\psi(h)) + b_\psi(g) \quad \text{and} \quad \|b_\psi(g)\|^2_{\mathcal{H}_\psi} = \psi(g).$$

Since $\partial_j(\exp(2\pi i \langle x, \cdot \rangle)) = 2\pi i x_j \exp(2\pi i \langle x, \cdot \rangle)$, it is natural to define

$$R_{\psi,j} f = \partial_{\psi,j} A_\psi^{-1/2} f = 2\pi i \sum_{g \in \mathrm{G}} \frac{\langle b_\psi(g), e_j \rangle_{\mathcal{H}_\psi}}{\sqrt{\psi(g)}} \widehat{f}(g) \lambda(g)$$

for some orthonormal basis $(e_j)_{j \ge 1}$ of $\mathcal{H}_\psi$. Recalling that $b_\psi(g)/\sqrt{\psi(g)}$ is always a normalized vector, we recover the usual symbol of R_j as a Fourier multiplier. Note also that classical Riesz transforms can be seen from this viewpoint. Namely, de Leeuw's theorem [21] allows us to replace $\mathbb{R}^n$ by its Bohr compactification, whose L_p spaces come from the group von Neumann algebra $\mathcal{L}(\mathbb{R}^n_{\mathrm{disc}})$ of $\mathbb{R}^n$ equipped with the discrete topology. Then the classical Riesz transforms arise from the standard cocycle where $\psi(\xi) = |\xi|^2$ (generating the heat semigroup) and $\mathcal{H}_\psi = \mathbb{R}^n$ with the trivial action α_ψ and the identity map b_ψ on $\mathbb{R}^n$. Moreover, the classical Riesz transforms vanish on the L_p-functions fixed by the heat semigroup: the constant functions on the n-torus and the zero function on the Euclidean space. This is also the case here and $R_{\psi,j}$ will be properly defined on

$$L_p^\circ(\widehat{\mathbf{G}}) = \{ f \in L_p(\widehat{\mathbf{G}}) \mid \widehat{f}(g) = 0 \text{ whenever } b_\psi(g) = 0 \}.$$

An elementary calculation shows that

$$\Gamma_\psi(A_\psi^{-1/2} f, A_\psi^{-1/2} f) = \sum_{j \ge 1} |R_{\psi,j} f|^2.$$

By Khintchine's inequality, (MP) in the commutative setting is equivalent to

$$\Big\| \sum_{j \ge 1} \gamma_j R_{\psi,j} f \Big\|_{L_p(\Omega \times \widehat{\mathrm{G}})} \sim_{c(p)} \|f\|_{L_p(\widehat{\mathrm{G}})}$$

for any family $(\gamma_j)_{j\geq 1}$ of centered independent gaussians in Ω. We have pointed out that this fails when $A = (-\Delta)^{1/2}$ is the generator of the Poisson semigroup. According to the standard gaussian measure space construction (see below), we may construct a canonical action $\beta_\psi : \mathrm{G} \curvearrowright L_\infty(\Omega)$ determined by ψ. As we shall justify in this paper, a natural version of Meyer's problem (MP) is to ask whether

$$\Big\| \sum_{j\geq 1} \gamma_j \rtimes R_{\psi,j} f \Big\|_{L_p(L_\infty(\Omega)\rtimes \widehat{\mathrm{G}})} \sim_{c(p)} \|f\|_{L_p(\widehat{\mathrm{G}})}.$$

Our first result claims that this form of (MP) holds for all Markov convolution semigroups on group von Neumann algebras, including the Poisson semigroup in the Euclidean space. The Riesz transforms above were introduced in [34] under the additional assumption that $\dim \mathcal{H}_\psi < \infty$. Our dimension free estimates below—in cocycle form, see Theorem A2 for a Meyer-type formulation—allow one to consider Riesz transforms associated to infinite-dimensional cocycles.

Theorem A1. *Let* G *be a discrete group,* $f \in L_p^\circ(\widehat{\mathbf{G}})$ *and* $1 < p < \infty$.

(i) *If* $1 < p \leq 2$*, then*

$$\|f\|_{L_p(\widehat{\mathbf{G}})} \sim_{c(p)} \inf_{R_{\psi,j}f = a_j + b_j} \Big\| \Big(\sum_{j\geq 1} a_j^* a_j \Big)^{1/2} \Big\|_{L_p(\widehat{\mathbf{G}})} + \Big\| \Big(\sum_{j\geq 1} \widetilde{b}_j \widetilde{b}_j^* \Big)^{1/2} \Big\|_{L_p(\widehat{\mathbf{G}})}.$$

(ii) *If* $2 \leq p < \infty$*, then*

$$\|f\|_{L_p(\widehat{\mathbf{G}})} \sim_{c(p)} \max\Big\{ \Big\| \Big(\sum_{j\geq 1} |R_{\psi,j} f|^2 \Big)^{1/2} \Big\|_{L_p(\widehat{\mathbf{G}})}, \Big\| \Big(\sum_{j\geq 1} |R_{\psi,j} f^*|^2 \Big)^{1/2} \Big\|_{L_p(\widehat{\mathbf{G}})} \Big\}.$$

Theorem A1 and most of our main results below also hold in fact for arbitrary unimodular groups. The introduction of the corresponding group algebras as well as the proofs of these results will be postponed (for clarity of exposition) to Appendix A. The infimum in Theorem A1(i) runs over all possible decompositions $R_{\psi,j} f = a_j + b_j$ in the tangent module, the noncommutative analogue of the module of differential forms of order one. A more precise description will be possible after the statement of Theorem A2. A crucial aspect comes from the $\widetilde{b}_j$'s, twisted forms of b_j's that will be rigorously defined. The failure of Theorem A1 for b_j's in place of their twisted forms goes back to [48, Proposition 2.9]. It shows certain 'intrinsic noncommutativity' in the problem, since the statement for $p < 2$ does not simplify for G abelian unless the action α_ψ is trivial!

A great variety of new and known estimates for Riesz transforms and other Fourier multipliers arise from Theorem A1, by considering different lengths. All conditionally negative length functions appear as deformations of the canonical inner cocycle for the left regular representation. Namely, if we consider the space Π_0 of trigonometric polynomials in $\mathcal{L}(\mathrm{G})$ whose Fourier coefficients have vanishing sum—finite sums $\sum_g a_g \lambda(g)$ with $\sum_g a_g = 0$—then $\psi : \mathrm{G} \to \mathbb{R}_+$ is conditionally negative iff $\psi(g) = \tau_\psi(2\lambda(e) - \lambda(g) - \lambda(g^{-1}))$ for some positive linear functional $\tau_\psi : \Pi_0 \to \mathbb{C}$. This characterization will be useful along the paper; it is proved in Appendix B. Having identified the exact form of conditionally negative lengths, let us now illustrate Theorem A1 with a few examples which will be analysed in the body of the paper:

(a) **Fractional laplacians in $\mathbb{R}^n$.** Recall that Theorem A1 also holds for group algebras over arbitrary unimodular groups. In the particular case $\mathrm{G} = \mathbb{R}^n$ we may consider conditionally negative lengths of the form

$$\psi(\xi) = 2\int_{\mathbb{R}^n} (1 - \cos(2\pi\langle x, \xi\rangle))\, d\mu_\psi(x)$$

for a positive Borel measure μ_ψ satisfying $\psi(\xi) < \infty$ for all $\xi \in \mathbb{R}^n$. If $d\mu_\psi(x) = dx/|x|^{n+2\beta}$ for any $0 < \beta < 1$, we get $\psi(\xi) = \mathrm{k}_n(\beta)|\xi|^{2\beta}$. This will provide us dimension free estimates for Riesz transforms associated to fractional laplacians, which are new. The estimates predicted by Meyer fail for $\beta = 1/2$ (see Appendix D). In contrast to the case $\beta = 1$, we find highly nontrivial cocycles. The vast family of measures μ_ψ are explored in further generality in the second part of this paper.

(b) **Discrete laplacians in LCA groups.** Let Γ_0 be a locally compact abelian group and $s_0 \in \Gamma_0$ be torsion free. If $\partial_j f(\gamma) = f(\gamma) - f(\gamma_1, \ldots, s_0\gamma_j, \ldots, \gamma_n)$ stand for discrete directional derivatives in $\Gamma = \Gamma_0 \times \cdots \times \Gamma_0$, we may consider the laplacian $L = \sum_j \partial_j^* \partial_j$ and $R_j = \partial_j L^{-1/2}$. Lust-Piquard [48] provided dimension free estimates for these Riesz transforms. If we set $\sigma_j = (0, \ldots, 0, s_0, 0, \ldots, 0)$ with s_0 in the j-th entry, consider the sum of point-masses $\mu_\psi = \sum_j \delta_{\sigma_j}$. Then we shall recover Lust-Piquard's theorem via Theorem A1 taking

$$\psi(g) = \widehat{L}(g) = \int_\Gamma (2 - \chi_g - \chi_{g^{-1}})(\gamma)\, d\mu_\psi(\gamma) \quad \text{for } g \in \mathrm{G} = \widehat{\Gamma}.$$

The advantage is that we do not need to require s_0 to be torsion free. Moreover, our formulation holds for any finite sum of point-masses, so that we may allow the shift s_0 to depend on the entry j or even the group Γ not to be given in a direct product form... This solves the problem of discrete laplacians of a very general form; continuous analogues can also be given.

(c) **Word-length laplacians.** Consider a finitely generated group G and write $|g|$ to denote the word length of g, its distance to e in the Cayley graph. If it is conditionally negative—like for free, cyclic, Coxeter groups—a natural laplacian is $A_{|\,|}(\lambda(g)) = |g|\lambda(g)$, and the Riesz transforms

$$R_{|\,|,j} f = \partial_{|\,|,j} A_{|\,|}^{-1/2} f = 2\pi i \sum_{g \in \mathrm{G}} \frac{\langle b_{|\,|}(g), e_j\rangle_{\mathcal{H}_{|\,|}}}{\sqrt{|g|}} \widehat{f}(g)\lambda(g)$$

satisfy Theorem A1. Many other transforms arise from other conditionally negative lengths. The natural example given above is out of the scope of [34]. It yields new interesting inequalities; here are two examples in the (simpler) case $p \geq 2$. When $\mathrm{G} = \mathbb{Z}_{2m}$,

$$\Big\| \sum_{j \in \mathbb{Z}_{2m}} \widehat{f}(j) e^{2\pi i \frac{j}{2m}\cdot} \Big\|_p \sim_{c(p)} \Big\| \Big(\sum_{k \in \mathbb{Z}_{2m}} \Big| \sum_{j \in \Lambda_k} \frac{\widehat{f}(j)}{\sqrt{j \wedge (2m - j)}} e^{2\pi i \frac{j}{2m}\cdot} \Big|^2 \Big)^{1/2} \Big\|_p$$

for $\Lambda_k = \{ j \in \mathbb{Z}_{2m} \mid j - k \equiv s \bmod 2m \text{ with } 0 \leq s \leq m-1 \}$. When $\mathrm{G} = \mathbb{F}_n$,

$$\|f\|_p \sim_{c(p)} \Big\| \Big(\sum_{h \neq e} \Big| \sum_{g \geq h} \frac{1}{\sqrt{|g|}} \widehat{f}(g)\lambda(g) \Big|^2 + \Big| \sum_{g \geq h} \frac{1}{\sqrt{|g|}} \overline{\widehat{f}(g^{-1})}\lambda(g) \Big|^2 \Big)^{1/2} \Big\|_p.$$

Let us now go back to Meyer's problem (MP) for convolution Markov semigroups. In the Euclidean case, integrating by parts we get $-\Delta = \nabla^* \circ \nabla$. According to Sauvageot's theorem [63], a similar factorization takes place for Markovian semigroups. Namely, there exists a Hilbert $\mathcal{L}(\mathrm{G})$-bimodule M_ψ and a densely defined closable symmetric derivation

$$\delta_\psi : L_2(\widehat{\mathbf{G}}) \to \mathsf{M}_\psi \quad \text{such that} \quad A_\psi = \delta_\psi^* \delta_\psi.$$

If $B : \mathcal{H}_\psi \ni e_j \mapsto \gamma_j \in L_2(\Omega, \Sigma, \mu)$ denotes the standard gaussian measure space construction, we will find in our case that $\mathsf{M}_\psi = L_\infty(\Omega, \Sigma, \mu) \rtimes_{\beta_\psi} \mathrm{G}$ where G acts via the cocycle action as $\beta_{\psi,g}(B(h)) = B(\alpha_{\psi,g}(h))$. The derivation is

$$\delta_\psi : \lambda(g) \mapsto B(b_\psi(g)) \rtimes \lambda(g), \qquad \delta_\psi^* : \rho \rtimes \lambda(g) \mapsto \langle \rho, B(b_\psi(g)) \rangle \lambda(g).$$

If we consider the conditional expectation onto $\mathcal{L}(\mathrm{G})$,

$$\mathsf{E}_{\mathcal{L}(\mathrm{G})} : \mathsf{M}_\psi \ni \sum_{g \in \mathrm{G}} \rho_g \rtimes \lambda(g) \mapsto \sum_{g \in \mathrm{G}} \Big(\int_\Omega \rho_g \, d\mu \Big) \lambda(g) \in \mathcal{L}(\mathrm{G}),$$

and recall the identity

$$\Gamma_\psi(f_1, f_2) = \mathsf{E}_{\mathcal{L}(\mathrm{G})}((\delta_\psi f_1)^* \delta_\psi f_2)$$

from Remark 1.3, we obtain the following solution to (MP) for (G, ψ).

Theorem A2. *The following norm equivalences hold for* G *discrete:*

(i) *If* $1 < p \le 2$*, then*

$$\|A_\psi^{1/2} f\|_{L_p(\widehat{\mathbf{G}})} \sim_{c(p)} \inf_{\delta_\psi f = \phi_1 + \phi_2} \|\mathsf{E}_{\mathcal{L}(\mathrm{G})}(\phi_1^* \phi_1)^{1/2}\|_{L_p(\widehat{\mathbf{G}})} + \|\mathsf{E}_{\mathcal{L}(\mathrm{G})}(\phi_2 \phi_2^*)^{1/2}\|_{L_p(\widehat{\mathbf{G}})}.$$

(ii) *If* $2 \le p < \infty$*, then*

$$\|A_\psi^{1/2} f\|_{L_p(\widehat{\mathbf{G}})} \sim_{c(p)} \max\{\|\Gamma_\psi(f, f)^{1/2}\|_{L_p(\widehat{\mathbf{G}})}, \|\Gamma_\psi(f^*, f^*)^{1/2}\|_{L_p(\widehat{\mathbf{G}})}\}.$$

We are now ready to describe the families of operators along which we allow our decompositions in Theorems A1(i) and A2(i) to run over. Recall that $\phi \in G_p(\mathbb{C}) \rtimes \mathrm{G}$ does not imply that $\mathsf{E}_{\mathcal{L}(\mathrm{G})}(\phi^* \phi)^{1/2}$ or $\mathsf{E}_{\mathcal{L}(\mathrm{G})}(\phi \phi^*)^{1/2}$ lie in L_p when $p < 2$. This is crucial in Appendix D. Consider the subspace $G_p(\mathbb{C}) \rtimes \mathrm{G}$ of $L_p(L_\infty(\Omega) \rtimes \mathrm{G})$ formed by the operators of the form

$$\phi = \sum_{g \in \mathrm{G}} \underbrace{\sum_{j \ge 1} \phi_{g,j} B(e_j)}_{\phi_g} \rtimes \lambda(g)$$

with $\phi_{g,j} \in \mathbb{C}$ and $\phi_g \in L_p(\Omega)$. The infimum in Theorem A2 is taken over all possible decompositions $\delta_\psi f = \phi_1 + \phi_2$ with $\phi_1, \phi_2 \in G_p(\mathbb{C}) \rtimes \mathrm{G}$. On the other hand, to describe the infimum in Theorem A1 we introduce the maps

$$u_j : G_p(\mathbb{C}) \rtimes \mathrm{G} \ni \phi \mapsto \sum_{g \in \mathrm{G}} \langle \phi_g, B(e_j) \rangle_{L_2(\Omega)} \lambda(g) \in L_p(\widehat{\mathbf{G}}).$$

Then $R_{\psi,j} f = a_j + b_j$ runs over $(a_j, b_j) = (u_j(\phi_a), u_j(\phi_b))$ for $\phi_a, \phi_b \in G_p(\mathbb{C}) \rtimes \mathrm{G}$.

As in Theorem A1, we recover Meyer's inequalities (MP) when G is abelian and the cocycle action is trivial; the general case is more involved. The infimum cannot be reduced to decompositions $f = f_1 + f_2$ (see Remark 1.4). The main result in [33] provides lower estimates for $p \geq 2$ and regular Markov semigroups satisfying $\Gamma^2 \geq 0$. In the context of group algebras, Theorem A2 goes much further. We refer to Remark 1.5 for a brief analysis of optimal constants.

Theorem A1 follows from Theorem A2 by standard manipulations. The proof of the latter is inspired by a crossed product extension of Pisier's method [55], which ultimately relies on a Khintchine-type inequality of independent interest. The key point in Pisier's argument is to identify the Riesz transform as a combination of the transferred Hilbert transform

$$Hf(x,y) = \mathrm{p.v.}\, \frac{1}{\pi} \int_{\mathbb{R}} \beta_t f(x,y) \, \frac{dt}{t} \quad \text{where} \quad \beta_t f(x,y) = f(x+ty)$$

and the gaussian projection

$$Q : L_p(\mathbb{R}^n, \gamma) \to \overline{L_p\text{-span}}\{B(\xi) \mid \xi \in \mathbb{R}^n\}.$$

Here the gaussian variables are given by $B(\xi)(y) = \langle \xi, y \rangle$, homogeneous polynomials of degree 1. The following identity can be found in [55] for any smooth $f : \mathbb{R}^n \to \mathbb{C}$:

$$\sqrt{2/\pi}\, \delta(-\Delta)^{-1/2} f = (\mathrm{id}_{L_\infty(\mathbb{R}^n)} \otimes Q)\Big(\mathrm{p.v.}\, \frac{1}{\pi} \int_{\mathbb{R}} \beta_t f \, \frac{dt}{t}\Big), \tag{RI}$$

where $\delta : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ is the derivation

$$\delta(f)(x,y) = \sum_{k=1}^n \frac{\partial f}{\partial x_k} y_k = \langle \nabla f(x), y \rangle.$$

Our Khintchine inequality allows us to generalize this formula to pairs (G, ψ). It seems fair to say that for the Euclidean case, this kind of formula has its roots in the work of Duoandikoetxea and Rubio de Francia [24] through the use of Calderón's method of rotations. Pisier's main motivation was to establish similar identities involving Riesz transforms for the Ornstein–Ulhenbeck semigroup.

Our class of ψ-Riesz transforms becomes very large when we vary ψ. This yields a fresh perspective in Fourier multiplier theory, mainly around Hörmander–Mikhlin smoothness conditions in terms of Sobolev and (limiting) Besov norms. We refer to [34] for a more in-depth discussion of smoothness conditions for Fourier multipliers defined on discrete groups. The main idea is that this smoothness may be measured through the use of cocycles, via lifting multipliers $\widetilde{m}$ living in the cocycle Hilbert space (identified with $\mathbb{R}^n$ for some $n \geq 1$), so that $m = \widetilde{m} \circ b_\psi$.

Let $\mathsf{M}_p(\mathrm{G})$ be the space of multipliers $m : \mathrm{G} \to \mathbb{C}$ equipped with the $p \to p$ norm of the map $\lambda(g) \mapsto m(g)\lambda(g)$. Consider the classical differential operators in $\mathbb{R}^n$ given by

$$\mathsf{D}_\alpha = (-\Delta)^{\alpha/2} \quad \text{so that} \quad \widehat{\mathsf{D}_\alpha f}(\xi) = |\xi|^\alpha \widehat{f}(\xi) \quad \text{for } \xi \in \mathbb{R}^n = \mathcal{H}_\psi.$$

We shall also need the fractional laplacian lengths

$$\psi_\varepsilon(\xi) = 2\int_{\mathbb{R}^n} \big(1-\cos(2\pi\langle \xi, x\rangle)\big)\,\frac{dx}{|x|^{n+2\varepsilon}} = \mathrm{k}_n(\varepsilon)|\xi|^{2\varepsilon}.$$

Our next result provides new Sobolev conditions for the lifting multiplier.

Theorem B1. *Let* (G,ψ) *be a discrete group equipped with a conditionally negative length giving rise to an* n*-dimensional cocycle* $(\mathcal{H}_\psi,\alpha_\psi,b_\psi)$. *Let* $(\varphi_j)_{j\in\mathbb{Z}}$ *denote a standard radial Littlewood–Paley partition of unity in* $\mathbb{R}^n$. *If* $1<p<\infty$ *and* $\varepsilon>0$, *then*

$$\|m\|_{\mathsf{M}_p(\mathrm{G})} \lesssim_{c(p,n)} |m(e)| + \inf_{m=\widetilde{m}\circ b_\psi}\Big\{\sup_{j\in\mathbb{Z}}\big\|\mathsf{D}_{n/2+\varepsilon}\big(\sqrt{\psi_\varepsilon}\,\varphi_j\widetilde{m}\big)\big\|_{L_2(\mathbb{R}^n)}\Big\}.$$

The infimum runs over all lifting multipliers $\widetilde{m}:\mathcal{H}_\psi\to\mathbb{C}$ *such that* $m=\widetilde{m}\circ b_\psi$.

Our Sobolev-type condition in Theorem B1 is formally less demanding than the standard one (see below), and our argument is also completely different from the classical approach used in [34]. As a crucial novelty, we will show that every Hörmander–Mikhlin type multiplier (those for which the term on the right hand side is finite, in particular the classical ones) is in fact a Littlewood–Paley average of Riesz transforms associated to a single infinite-dimensional cocycle! The magic formula comes from an isometric isomorphism between the Sobolev-type norm in Theorem B1 and mean-zero elements of $L_2(\mathbb{R}^n,\mu_\varepsilon)$ with $d\mu_\varepsilon(x) = |x|^{-(n+2\varepsilon)}dx$. In other words, if $b_\varepsilon:\mathbb{R}^n\to L_2(\mathbb{R}^n,\mu_\varepsilon)$ denotes the cocycle map associated to ψ_ε then $\widetilde{m}:\mathbb{R}^n\to\mathbb{C}$ satisfies

$$\big\|\mathsf{D}_{n/2+\varepsilon}\big(\sqrt{\psi_\varepsilon}\,\widetilde{m}\big)\big\|_{L_2(\mathbb{R}^n)} < \infty$$

iff there exists a mean-zero $h\in L_2(\mathbb{R}^n,\mu_\varepsilon)$ such that

$$\widetilde{m}(\xi) = \frac{\langle h, b_\varepsilon(\xi)\rangle_{\mu_\varepsilon}}{\sqrt{\psi_\varepsilon(\xi)}} \quad \text{and} \quad \|h\|_{L_2(\mathbb{R}^n,\mu_\varepsilon)} = \big\|\mathsf{D}_{n/2+\varepsilon}\big(\sqrt{\psi_\varepsilon}\,\widetilde{m}\big)\big\|_{L_2(\mathbb{R}^n)}$$

(see Lemma 2.5 for further details). A few remarks are in order:

- Theorem B1 holds for unimodular ADS groups (see Appendix A).
- Our bound is majorized by the classical one

$$\sup_{j\in\mathbb{Z}}\|(1+|\ |^2)^{n/4+\varepsilon/2}(\varphi_0\widetilde{m}(2^j\,\cdot))^\wedge\|_{L_2(\mathbb{R}^n)}.$$

  A crucial fact is that our Sobolev norm is dilation invariant; more details in Corollary 2.7. Moreover, our bound is more appropriate in terms of dimensional behavior of the constants (see Remark 2.8).
- Our result is stronger than the main result in [34] in two respects. First we obtain Sobolev-type conditions, which are way more flexible than the Mikhlin assumptions

$$\sup_{\xi\in\mathbb{R}^n\setminus\{0\}}\ \sup_{|\beta|\le[n/2]+1} |\xi|^{|\beta|}|\partial_\beta\widetilde{m}(\xi)| < \infty.$$

Second, we avoid the modularity restriction in [34]. Namely, there we needed a simultaneous control of left/right cocycles for nonabelian discrete groups, when there is no spectral gap. In the lack of that, we could also work only with the left cocycle at the price of extra decay in the smoothness condition. In Theorem B1, it suffices to satisfy our Sobolev-type conditions for the left cocycle. On the other hand, the approach in [34] is still necessary. First, it explains the connection between Hörmander–Mikhlin multipliers and Calderón–Zygmund theory for group von Neumann algebras. Second, the Littlewood–Paley estimates in [34, Theorem C] are crucial for this paper and [53]. Third, our approach here does not give $L_\infty \to \mathrm{BMO}$ bounds.

The dimension dependence in the constants of Theorem B1 has its roots in the use of certain Littlewood–Paley inequalities on $\mathcal{L}(\mathrm{G})$, but not on the Sobolev-type norm itself. This yields a form of the Hörmander–Mikhlin condition with dimension free constants, replacing the compactly supported smooth functions φ_j by a certain class $\mathcal{J}$ of analytic functions which arises from Cowling/McIntosh holomorphic functional calculus, the simplest of which is $x \mapsto xe^{-x}$ that already appears in the work of Stein [68]. Our result is the following.

Theorem B2. *Let* G *be a discrete group and* Λ_{G} *the set of conditionally negative lengths* $\psi : \mathrm{G} \to \mathbb{R}_+$ *giving rise to a finite-dimensional cocycle* $(\mathcal{H}_\psi, \alpha_\psi, b_\psi)$. *Let* $\varphi : \mathbb{R}_+ \to \mathbb{C}$ *be an analytic function in the class* $\mathcal{J}$. *If* $1 < p < \infty$ *and* $\varepsilon > 0$, *then*

$$\|m\|_{\mathsf{M}_p(\mathrm{G})} \lesssim_{c(p)} |m(e)| + \inf_{\substack{\psi \in \Lambda_{\mathrm{G}} \\ m = \widetilde{m} \circ b_\psi}} \Big\{ \operatorname*{ess\,sup}_{s>0} \Big\| \mathsf{D}_{(\dim \mathcal{H}_\psi)/2+\varepsilon} \big(\sqrt{\psi_\varepsilon}\, \varphi(s|\cdot|^2) \widetilde{m} \big) \Big\|_{L_2(\mathcal{H}_\psi)} \Big\}.$$

The infimum runs over all $\psi \in \Lambda_{\mathrm{G}}$ *and all* $\widetilde{m} : \mathcal{H}_\psi \to \mathbb{C}$ *such that* $m = \widetilde{m} \circ b_\psi$.

Theorem B2 also holds for unimodular groups (see Appendix A). Taking the trivial cocycle in $\mathbb{R}^n$ whose associated length function is $|\xi|^2$, we find a Sobolev condition which works up to dimension free constants; we do not know whether this statement is known in the Euclidean setting. The versatility of Theorems B1 and B2 for general groups is an illustration of what can be done using other conditionally negative lengths to start with. Replacing for instance the fractional laplacian lengths by some others associated to limiting measures when $\varepsilon \to 0$, we may improve the Besov-type conditions à la Baernstein/Sawyer [2] (see also the related work of Seeger [66, 67] and [11, 44, 65]). The main idea is to replace the measures $d\mu_\varepsilon(x) = |x|^{-(n+2\varepsilon)} dx$, used to prove Theorem B1, by the limiting measure $d\nu(x) = u(x)dx$ with

$$u(x) = \frac{1}{|x|^n} \Big(1_{\mathrm{B}_1(0)}(x) + \frac{1}{1 + \log^2 |x|} 1_{\mathbb{R}^n \setminus \mathrm{B}_1(0)}(x) \Big).$$

Let us also consider the associated length

$$\ell(\xi) = 2 \int_{\mathbb{R}^n} (1 - \cos(2\pi \langle \xi, x \rangle)) u(x)\, dx.$$

Then, if $1 < p < \infty$ and $\dim \mathcal{H}_\psi = n$, we prove in Theorem 2.15 that

$$\|m\|_{\mathsf{M}_p(\mathrm{G})} \lesssim_{c(p,n)} |m(e)| + \inf_{m = \widetilde{m} \circ b_\psi} \Big\{ \sup_{j \in \mathbb{Z}} \Big(\sum_{k \in \mathbb{Z}} 2^{nk} \mathrm{w}_k \big\| \widehat{\varphi}_k * \big(\sqrt{\ell}\, \varphi_j \widetilde{m} \big) \big\|_{L_2(\mathbb{R}^n)}^2 \Big)^{1/2} \Big\},$$

where $(\varphi_j)_{j\in\mathbb{Z}}$ is a standard radial Littlewood–Paley partition of unity in $\mathbb{R}^n$ and the weights w_k are of the form $\delta_{k\leq 0}+k^2\delta_{k>0}$ for $k\in\mathbb{Z}$. A more detailed analysis of this result will be given in Section 2.4. In fact, an even more general construction is possible which relates Riesz transforms to 'Sobolev-type norms' directly constructed in group von Neumann algebras. We will not explore this direction here; further details in Remarks 2.11 and 2.12.

Let us now consider a given branch in the Cayley graph of $\mathbb{F}_\infty$, the free group with infinitely many generators. Of particular interest are two applications we have found for operators (frequency) supported by such a branch. If we fix a branch B of $\mathbb{F}_\infty$ let us set

$$L_p(\widehat{\mathbf{B}})=\{f\in L_p(\mathcal{L}(\mathbb{F}_\infty))\mid \widehat{f}(g)=0 \text{ for all } g\notin \mathrm{B}\}.$$

As usual, we shall write $|\ |$ for the word length of the free group $\mathbb{F}_\infty$.

Theorem C. *Given any branch* B *of* $\mathbb{F}_\infty$*, we have*

(i) (Hörmander–Mikhlin multipliers) *If* $m:\mathbb{Z}_+\to\mathbb{C}$*, then*

$$\big\|\widetilde{m}\circ|\ |\big\|_{\mathsf{M}_p(\mathrm{B})}\lesssim_{c(p)}\sup_{j\geq 1}\big(|\widetilde{m}(j)|+j|\widetilde{m}(j)-\widetilde{m}(j-1)|\big),$$

where $\mathsf{M}_p(\mathrm{B})$ *denotes the space of* $L_p(\widehat{\mathbf{B}})$*-bounded Fourier multipliers.*

(ii) (Twisted Littlewood–Paley estimates) *Consider a standard Littlewood–Paley partition of unity* $(\varphi_j)_{j\geq 1}$ *in* $\mathbb{R}_+$*, generated by dilations of a function* ϕ *with* $\sqrt{\phi}$ *Lipschitz. Let* $\Lambda_j:\lambda(g)\mapsto\sqrt{\varphi_j(|g|)}\,\lambda(g)$ *denote the corresponding radial multipliers in* $\mathcal{L}(\mathbb{F}_\infty)$*. Then, for any* $f\in L_p(\widehat{\mathbf{B}})$ *and* $1<p<2$*,*

$$\inf_{\Lambda_j f=a_j+b_j}\Big\|\Big(\sum_{j\geq 1}(a_j^*a_j+\widetilde{b}_j\widetilde{b}_j^*)\Big)^{1/2}\Big\|_{L_p(\widehat{\mathbf{B}})}\lesssim_{c(p)}\|f\|_{L_p(\widehat{\mathbf{B}})},$$

$$\|f\|_{L_p(\widehat{\mathbf{B}})}\lesssim_{c(p)}\inf_{\Lambda_j f=a_j+b_j}\Big\|\Big(\sum_{j\geq 1}(a_j^*a_j+b_jb_j^*)\Big)^{1/2}\Big\|_{L_p(\widehat{\mathbf{B}})}.$$

In analogy with Theorem A1, the first infimum runs over all decompositions with $(a_j,b_j)=(v_j(\phi_a),v_j(\phi_b))$ where $\phi_a,\phi_b\in G_p(\mathbb{C})\rtimes \mathrm{G}$ and $v_j(\phi)=\sum_g\langle\phi_g,B(h_j)\rangle\lambda(g)$ for certain $h_j\in\mathcal{H}_{|\ |}$ to be defined later. The second infimum runs over $(a_j,b_j)\in C_p(L_p)\times R_p(L_p)$, the largest space where it is meaningful. Here $C_p(L_p)$ and $R_p(L_p)$ denote the closures of the finite sequences $(u_j)_j$ in $L_p(\widehat{\mathbf{B}})$ in the norms $\|(\sum_j u_j^*u_j)^{1/2}\|_p$ and $\|(\sum_j u_ju_j^*)^{1/2}\|_p$ respectively. Theorem C shows that Hörmander–Mikhlin multipliers on branches of $\mathbb{F}_\infty$ behave as in the 1-dimensional groups $\mathbb{Z}$ or $\mathbb{R}$. However, general branches have no group structure and L_p-norms admit less elementary combinatorics ($p\in 2\mathbb{Z}_+$) than the trivial ones $g,g^2,g^3,\ldots$ with g a generator. The key idea for our Littlewood–Paley inequalities is to realize the Littlewood–Paley partition of unity as a family of Riesz transforms. The crucial difference from Theorems A1 and A2 is that this family does not arise from an orthonormal basis, but from a quasi-orthonormal incomplete system. It is hence very likely that norm equivalences do not hold for nontrivial branches. On the contrary, our result shows that the untwisted square function is greater than the twisted one, and both coincide when the cocycle action is trivial and the product commutes. This is the case for trivial branches (associated to subgroups isomorphic to $\mathbb{Z}$) since we may replace the word length by the one coming from the heat semigroup on $\mathbb{T}$, which yields a trivial cocycle action. We may also obtain lower estimates for $p > 2$ (see Corollary 3.2). At the time of this writing, we do not know of an appropriate upper bound for $\|f\|_p$ ($p > 2$) since standard duality fails due to the twisted nature of square functions. Bożejko–Fendler's theorem [8] indicates that sharp truncations might not work for all values of $1 < p < \infty$.

Our approach requires some background on noncommutative L_p-spaces, group von Neumann algebras, crossed products and geometric group theory. A brief survey of the main notions/results needed for this paper is given in Appendix B for the nonexpert reader. Appendix C contains a geometric analysis of our results in terms of the tangent module associated to the infinitesimal generator A_ψ.

1. Riesz transforms

In this section we shall focus on our dimension free estimates for noncommutative Riesz transforms. More specifically, we will prove Theorems A1 and A2. We shall also illustrate our results with a few examples which provide new estimates both in the commutative and in the noncommutative settings.

1.1. Khintchine inequalities

Our results rely on Pisier's method [55] and a modified version of Lust-Piquard/Pisier's noncommutative Khintchine inequalities [45, 49]. Given a noncommutative measure space $(\mathcal{M}, \varphi)$, we let $RC_p(\mathcal{M})$ be the closure of the finite sequences in $L_p(\mathcal{M})$ equipped with the norm

$$\|(f_k)\|_{RC_p(\mathcal{M})} = \begin{cases} \inf\limits_{f_k = g_k + h_k} \Big\| \Big(\sum_k g_k^* g_k \Big)^{1/2} \Big\|_p + \Big\| \Big(\sum_k h_k h_k^* \Big)^{1/2} \Big\|_p & \text{if } 1 \le p \le 2, \\ \max \Big\{ \Big\| \Big(\sum_k f_k^* f_k \Big)^{1/2} \Big\|_p, \Big\| \Big(\sum_k f_k f_k^* \Big)^{1/2} \Big\|_p \Big\} & \text{if } 2 \le p < \infty. \end{cases}$$

The noncommutative Khintchine inequality reads as $G_p(\mathcal{M}) = RC_p(\mathcal{M})$, where $G_p(\mathcal{M})$ denotes the closed span in $L_p(\Omega, \mu; L_p(\mathcal{M}))$ of a family (γ_k) of centered independent gaussian variables on (Ω, μ). The specific statement for $1 \le p < \infty$ is

$$\Big(\int_\Omega \Big\| \sum_k \gamma_k(w) f_k \Big\|_{L_p(\mathcal{M})}^p d\mu(w) \Big)^{1/p} \sim_{c(p)} \|(f_k)\|_{RC_p(\mathcal{M})}.$$

Our goal is to prove a similar result with a group action added to the picture. Let $\mathcal{H}$ be a separable real Hilbert space. Choosing an orthonormal basis $(e_j)_{j \ge 1}$, we consider the linear map $B : \mathcal{H} \to L_2(\Omega, \mu)$ given by $B(e_j) = \gamma_j$. Let Σ stand for the smallest

σ-algebra making all the $B(h)$'s measurable. Then the well known gaussian measure space construction [14] tells us that, for every real unitary α in $O(\mathcal{H})$, we can construct a measure preserving automorphism β on $L_2(\Omega, \Sigma, \mu)$ such that $\beta(B(h)) = B(\alpha(h))$.

Now, assume that a discrete group G acts by real unitaries on $\mathcal{H}$ and isometrically on some finite von Neumann algebra $\mathcal{M}$. In particular, G acts isometrically on $L_\infty(\Omega, \Sigma, \mu) \bar{\otimes} \mathcal{M}$ and we may consider the space $G_p(\mathcal{M}) \rtimes \mathrm{G}$ of operators of the form

$$\sum_{g \in \mathrm{G}} \underbrace{\sum_{j \geq 1} (B(e_j) \otimes f_{g,j})}_{f_g} \rtimes \lambda(g) \in L_p(\mathcal{A})$$

with $\mathcal{A} = (L_\infty(\Omega, \Sigma, \mu) \bar{\otimes} \mathcal{M}) \rtimes \mathrm{G}$ and $f_g \in G_p(\mathcal{M})$. We will also need the conditional expectation $\mathsf{E}_{\mathcal{M} \rtimes \mathrm{G}}(f_g \rtimes \lambda(g)) = (\int_\Omega f_g \, d\mu) \rtimes \lambda(g)$, which takes $L_p(\mathcal{A})$ contractively to $L_p(\mathcal{M} \rtimes \mathrm{G})$. The conditional L_p norms

$$L_p^{rc}(\mathsf{E}_{\mathcal{M} \rtimes \mathrm{G}}) = \begin{cases} L_p^r(\mathsf{E}_{\mathcal{M} \rtimes \mathrm{G}}) + L_p^c(\mathsf{E}_{\mathcal{M} \rtimes \mathrm{G}}) & \text{if } 1 \leq p \leq 2, \\ L_p^r(\mathsf{E}_{\mathcal{M} \rtimes \mathrm{G}}) \cap L_p^c(\mathsf{E}_{\mathcal{M} \rtimes \mathrm{G}}) & \text{if } 2 \leq p < \infty, \end{cases}$$

are determined by

$$\|f\|_{L_p^r(\mathsf{E}_{\mathcal{M} \rtimes \mathrm{G}})} = \|\mathsf{E}_{\mathcal{M} \rtimes \mathrm{G}}(ff^*)^{1/2}\|_p \quad \text{and} \quad \|f\|_{L_p^c(\mathsf{E}_{\mathcal{M} \rtimes \mathrm{G}})} = \|\mathsf{E}_{\mathcal{M} \rtimes \mathrm{G}}(f^*f)^{1/2}\|_p$$

(see [30, Section 2] for a precise analysis of these spaces). In particular, if $1 < p < \infty$ and $1/p + 1/q = 1$, we recall that $L_p^r(\mathsf{E}_{\mathcal{M} \rtimes \mathrm{G}})^* = L_q^r(\mathsf{E}_{\mathcal{M} \rtimes \mathrm{G}})$ using the anti-linear duality bracket. The same holds for column spaces. Define $RC_p(\mathcal{M}) \rtimes \mathrm{G}$ as the gaussian space $G_p(\mathcal{M}) \rtimes \mathrm{G}$, with the norm inherited from $L_p^{rc}(\mathsf{E}_{\mathcal{M} \rtimes \mathrm{G}})$. Then we may generalize the noncommutative Khintchine inequality as follows.

Theorem 1.1. *Let* G *be a discrete group. If* $1 < p < \infty$, *then*

$$C_1 \sqrt{\frac{p-1}{p}} \, \|f\|_{RC_p(\mathcal{M}) \rtimes \mathrm{G}} \leq \|f\|_{G_p(\mathcal{M}) \rtimes \mathrm{G}} \leq C_2 \sqrt{p} \, \|f\|_{RC_p(\mathcal{M}) \rtimes \mathrm{G}}.$$

$G_p(\mathcal{M}) \rtimes \mathrm{G}$ *is complemented in*

$$L_p(L_\infty(\Omega, \Sigma, \mu) \bar{\otimes} \mathcal{M} \rtimes \mathrm{G})$$

and the norm of the corresponding projection $\widehat{Q}$ *is* $\sim \sqrt{p^2/(p-1)}$.

Proof. Let us first assume $p > 2$; the case $p = 2$ is trivial. Then the lower estimate holds with constant 1 from the continuity of the conditional expectation on $L_{p/2}$. The upper estimate relies on a suitable application of the central limit theorem. Indeed, assume first that f is a finite sum $\sum_{g,h} (B(h) \otimes f_{g,h}) \rtimes \lambda(g)$. Fix $m \geq 1$, use the diagonal action (copying the original action on $\mathcal{H}$ entrywise) on $\ell_2^m(\mathcal{H})$ and repeat the gaussian measure space construction on the larger Hilbert space resulting in a map $B_m : \ell_2^m(\mathcal{H}) \to$

$L_2(\Omega_m, \Sigma_m, \mu_m)$. Let $\phi_m : \mathcal{H} \to \ell_2^m(\mathcal{H})$ denote the isometric diagonal embedding $h \mapsto m^{-1/2} \sum_{j \le m} h \otimes e_j$ and let $F_1, \ldots, F_k$ be bounded functions on $\mathbb{R}$. Then

$$\pi\big(F_1(B(h_1)) \cdots F_k(B(h_k))\big) = F_1(B_m(\phi_m(h_1))) \cdots F_k(B_m(\phi_m(h_k)))$$

extends to a measure preserving $*$-homomorphism $L_\infty(\Omega, \Sigma, \mu) \to L_\infty(\Omega_m, \Sigma_m, \mu_m)$ which is in addition G-equivariant, i.e. $\pi(\beta_g(f)) = \beta_g^m(\pi(f))$. The action here is given by $\beta_g(B(h)) = B(\alpha_g(h))$ with α the cross-product action. Thus, we obtain a trace preserving isomorphism $\pi_{\mathrm{G}} = (\pi \otimes \mathrm{id}_{\mathcal{M}}) \rtimes \mathrm{id}_{\mathrm{G}}$ from $(L_\infty(\Omega, \Sigma, \mu) \bar{\otimes} \mathcal{M}) \rtimes \mathrm{G}$ to the larger space $(L_\infty(\Omega_m, \Sigma_m, \mu_m) \bar{\otimes} \mathcal{M}) \rtimes \mathrm{G}$. This implies

$$\|f\|_{G_p(\mathcal{M}) \rtimes \mathrm{G}} = \Big\| \frac{1}{\sqrt{m}} \sum_{j=1}^m \sum_{g,h} (B_m(h \otimes e_j) \otimes f_{g,h}) \rtimes \lambda(g) \Big\|_p.$$

The random variables

$$f_j = \sum_{g,h} (B_m(h \otimes e_j) \otimes f_{g,h}) \rtimes \lambda(g)$$

are mean-zero and independent over $\mathsf{E}_{\mathcal{M} \rtimes \mathrm{G}}$ (see Appendix B for precise definitions). Hence, the noncommutative Rosenthal inequality—(B.1) in the Appendix—yields

$$\begin{aligned} &\|f\|_{G_p(\mathcal{M}) \rtimes \mathrm{G}} \\ &\le \frac{Cp}{\sqrt{m}} \Big[\Big(\sum_{j=1}^m \|f_j\|_p^p \Big)^{1/p} + \Big\| \Big(\sum_{j=1}^m \mathsf{E}_{\mathcal{M} \rtimes \mathrm{G}}(f_j^* f_j) \Big)^{1/2} \Big\|_p + \Big\| \Big(\sum_{j=1}^m \mathsf{E}_{\mathcal{M} \rtimes \mathrm{G}}(f_j f_j^*) \Big)^{1/2} \Big\|_p \Big]. \end{aligned}$$

Note that $\mathsf{E}_{\mathcal{M} \rtimes \mathrm{G}}(f_j f_j^*) = \mathsf{E}_{\mathcal{M} \rtimes \mathrm{G}}(f f^*)$ and $\mathsf{E}_{\mathcal{M} \rtimes \mathrm{G}}(f_j^* f_j) = \mathsf{E}_{\mathcal{M} \rtimes \mathrm{G}}(f^* f)$ for all j. Moreover, $\|f_j\|_p = \|f\|_p$. Therefore, the second inequality with constant $O(p)$ follows by letting $m \to \infty$. An improved Rosenthal inequality [40] actually yields

$$\|f\|_{G_p(\mathcal{M}) \rtimes \mathrm{G}} \le C\sqrt{p}\, \|f\|_{RC_p(\mathcal{M}) \rtimes \mathrm{G}},$$

which provides the correct order of the constant in our Khintchine inequality.

Let us now consider the case $1 < p < 2$. We will proceed by duality as follows. Define the gaussian projection by

$$Q(f) = \sum_k \Big(\int_\Omega f \gamma_k \, d\mu \Big) \gamma_k,$$

which is independent of the choice of the basis. Let $\widehat{Q} = (Q \otimes \mathrm{id}_{\mathcal{M}}) \rtimes \mathrm{id}_{\mathrm{G}}$ be the amplified gaussian projection on $L_p(L_\infty(\Omega, \Sigma, \mu) \bar{\otimes} \mathcal{M} \rtimes \mathrm{G})$. It is clear that $G_p(\mathcal{M}) \rtimes \mathrm{G}$ is the image of this L_p-space under the gaussian projection. Similarly $RC_p(\mathcal{M}) \rtimes \mathrm{G}$ is the image of $L_p^{rc}(\mathsf{E}_{\mathcal{M} \rtimes \mathrm{G}})$. Note that

$$\widehat{Q} : L_p^{rc}(\mathsf{E}_{\mathcal{M} \rtimes \mathrm{G}}) \to RC_p(\mathcal{M}) \rtimes \mathrm{G}$$

is a contraction. Indeed, we have

$$\mathsf{E}_{\mathcal{M} \rtimes \mathrm{G}}(f f^*) = \mathsf{E}_{\mathcal{M} \rtimes \mathrm{G}}(\widehat{Q} f \widehat{Q} f^*) + \mathsf{E}_{\mathcal{M} \rtimes \mathrm{G}}(\widehat{Q}^\perp f \widehat{Q}^\perp f^*) \ge \mathsf{E}_{\mathcal{M} \rtimes \mathrm{G}}(\widehat{Q} f \widehat{Q} f^*)$$

by orthogonality, and the same holds for the column case. By the duality between $L_p^{rc}(\mathsf{E}_{\mathcal{M}\rtimes \mathrm{G}})$ and $L_q^{rc}(\mathsf{E}_{\mathcal{M}\rtimes \mathrm{G}})$ and also between $L_p(\mathcal{A})$ and $L_q(\mathcal{A})$, we obtain

$$\begin{aligned}\|f\|_{RC_p(\mathcal{M})\rtimes \mathrm{G}} &= \sup_{\|g\|_{L_q^{rc}}\le 1} |\mathrm{tr}(fg)| = \sup_{\|g\|_{L_q^{rc}}\le 1} |\mathrm{tr}(f\widehat{Q}g)| \\ &= \sup_{\|g\|_{RC_q}\le 1} |\mathrm{tr}(fg)| \le \Big(\sup_{\|g\|_{RC_q}\le 1} \|g\|_{G_q(\mathcal{M})\rtimes \mathrm{G}}\Big)\|f\|_{G_p(\mathcal{M})\rtimes \mathrm{G}}\end{aligned}$$

with $1/p+1/q=1$. In conjunction with our estimates for $q\ge 2$, this proves the lower estimate for $p\le 2$. The upper estimate is a consequence of the continuous inclusion $L_p^{rc}(\mathsf{E}_{\mathcal{M}\rtimes \mathrm{G}})\to L_p$ for $p\le 2$ [38, Theorem 7.1]. Indeed,

$$\|f\|_{G_p(\mathcal{M})\rtimes \mathrm{G}} = \big\||f|^2\big\|_{L_{p/2}(\mathcal{A})}^{1/2} \le 2^{1/(2p)}\|\mathsf{E}_{\mathcal{M}\rtimes \mathrm{G}}(|f|^2)\|_{L_{p/2}(\mathcal{A})}^{1/2} = 2^{1/(2p)}\|f\|_{L_p^c(\mathsf{E}_{\mathcal{M}\rtimes \mathrm{G}})}.$$

It remains to prove the complementation result. Since the gaussian projection is self-adjoint, we may assume $p\le 2$. Moreover, the upper estimate in the first assertion together with the contractivity of the gaussian projection on $L_p^{rc}(\mathsf{E}_{\mathcal{M}\rtimes \mathrm{G}})$ give rise to

$$\begin{aligned}\|\widehat{Q}f\|_{G_p(\mathcal{M})\rtimes \mathrm{G}} \lesssim \|\widehat{Q}f\|_{RC_p(\mathcal{M})\rtimes \mathrm{G}} &= \sup_{\|g\|_{RC_q}\le 1} |\mathrm{tr}((\widehat{Q}f)g)| = \sup_{\|g\|_{RC_q}\le 1} |\mathrm{tr}(f\widehat{Q}g)| \\ &\le \sup_{\|g\|_{RC_q}\le 1} \|\widehat{Q}g\|_{G_q(\mathcal{M})\rtimes \mathrm{G}}\|f\|_{L_p(\mathcal{A})} \le C_2\sqrt{q}\|f\|_{L_p(\mathcal{A})}.\end{aligned}$$

The last inequality follows from the first assertion for q. □

1.2. Riesz transforms in Meyer form

Denote by λ the Lebesgue measure and write γ for the normalized gaussian measure in $\mathbb{R}^n$. With this choice, the maps $\beta_t f(x,y) = f(x+ty)$ are measure preserving $*$-homomorphisms from $L_\infty(\mathbb{R}^n,\lambda)$ to $L_\infty(\mathbb{R}^n\times\mathbb{R}^n,\lambda\times\gamma)$. We may also replace λ by the Haar measure ν on the Bohr compactification $\mathbb{R}^n_{\mathrm{bohr}}$ of $\mathbb{R}^n$, this latter case including $n=\infty$. Moreover, if G acts on $\mathbb{R}^n$ then β_t commutes with the diagonal action. As we already recalled in the Introduction, (RI) takes the form

$$\sqrt{2/\pi}\,\delta(-\Delta)^{-1/2} f = (\mathrm{id}_{L_\infty(\mathbb{R}^n,\lambda)}\otimes Q)\Big(\mathrm{p.v.}\,\frac{1}{\pi}\int_{\mathbb{R}} \beta_t f\,\frac{dt}{t}\Big)$$

with Q the gaussian projection and $\delta : C^\infty(\mathbb{R}^n)\to C^\infty(\mathbb{R}^n\times\mathbb{R}^n)$ the derivation

$$\delta(f)(x,y) = \sum_{j=1}^n \frac{\partial f}{\partial x_j} y_j = \langle \nabla f(x), y\rangle = B(\nabla f(x))(y).$$

Lemma 1.2. *Let* G *be a discrete group acting on* $L_\infty(\mathbb{R}^n_{\mathrm{bohr}},\nu)$*. If* $1<p<\infty$*, then*

$$\delta(-\Delta)^{-1/2}\rtimes \mathrm{id}_{\mathrm{G}} : L_p(L_\infty(\mathbb{R}^n_{\mathrm{bohr}},\nu)\rtimes \mathrm{G}) \to L_p(L_\infty(\mathbb{R}^n_{\mathrm{bohr}}\times\mathbb{R}^n,\nu\times\gamma)\rtimes \mathrm{G})$$

with norm bounded by $Cp^3/(p-1)^{3/2}$*. Moreover, the same holds when* $n=\infty$*.*

Proof. The cross-product extension of (RI) reads

$$\sqrt{2/\pi}\,(\delta(-\Delta)^{-1/2} \rtimes \mathrm{id_G})f = \widehat{Q}\Big(\text{p.v.}\,\frac{1}{\pi}\int_{\mathbb{R}}(\beta_t \rtimes \mathrm{id_G})f\,\frac{dt}{t}\Big).$$

This gives $\delta(-\Delta)^{-1/2} \rtimes \mathrm{id_G} = \sqrt{\pi/2}\,\widehat{Q}(H \rtimes \mathrm{id_G})$ where H is the transferred Hilbert transform

$$Hf(x,y) = \text{p.v.}\,\frac{1}{\pi}\int_{\mathbb{R}}\beta_t f(x,y)\,\frac{dt}{t}.$$

By de Leeuw's theorem [21, Corollary 2.5], the Hilbert transform is bounded on $L_p(\mathbb{R}_{\mathrm{bohr}},\nu)$ with the same constants as in $L_p(\mathbb{R},\lambda)$. The operator above can be seen as a directional Hilbert transform at x in the direction of y, which also preserves the same constants for y fixed. In particular, a Fubini argument combined with a gaussian average easily gives

$$H : L_p(\mathbb{R}^n_{\mathrm{bohr}},\nu) \to L_p(\mathbb{R}^n_{\mathrm{bohr}} \times \mathbb{R}^n, \nu \times \gamma)$$

again with the classical constants, even for $n = \infty$.

To analyze the crossed product $H \rtimes \mathrm{id_G}$ we note that $\beta_t \rtimes \mathrm{id_G}$ is a trace preserving $*$-automorphism

$$L_\infty(\mathbb{R}^n_{\mathrm{bohr}},\nu) \rtimes \mathrm{G} \to L_\infty(\mathbb{R}^n_{\mathrm{bohr}} \times \mathbb{R}^n, \nu \times \gamma) \rtimes \mathrm{G}.$$

According to the Coifman–Weiss transference principle [17, Theorem 2.4] and the fact that H is G-equivariant, we see that $H \rtimes \mathrm{id_G}$ extends to a bounded map on L_p with constant $c(p) \sim p^2/(p-1)$. Indeed, it is straightforward that the proof of the transference principle for one-parameter automorphisms translates verbatim to the present setting. Then the assertion follows from the complementation result in Theorem 1.1 for $\mathcal{M} = L_\infty(\mathbb{R}^n_{\mathrm{bohr}},\nu)$. □

Given a length $\psi : \mathrm{G} \to \mathbb{R}_+$, consider its cocycle map $b_\psi : \mathrm{G} \to \mathbb{R}^n$ and the crossed product $L_\infty(\mathbb{R}^n,\gamma) \rtimes \mathrm{G}$ defined via the cocycle action α_ψ. The derivation $\delta_\psi : \mathcal{L}(\mathrm{G}) \to L_\infty(\mathbb{R}^n,\gamma) \rtimes \mathrm{G}$ is determined by $\delta_\psi(\lambda(g)) = B(b_\psi(g)) \rtimes \lambda(g)$. We include the case $n = \infty$, so that any length function/cocycle is admissible. The cocycle law yields the Leibniz rule

$$\delta_\psi(\lambda(gh)) = \delta_\psi(\lambda(g))\lambda(h) + \lambda(g)\delta_\psi(\lambda(h)).$$

An alternative argument to the proof given below can be found in Appendix A.

Proof of Theorem A2. Given the infinitesimal generator $A_\psi(\lambda(g)) = \psi(g)\lambda(g)$ and according to the definition of the norm in $RC_p(\mathbb{C}) \rtimes \mathrm{G}$, it suffices to prove that

$$\|A_\psi^{1/2} f\|_{L_p(\widehat{\mathbf{G}})} \sim_{c_1(p)} \|\delta_\psi f\|_{G_p(\mathbb{C})\rtimes \mathrm{G}} \sim_{c_2(p)} \|\delta_\psi f\|_{RC_p(\mathbb{C})\rtimes \mathrm{G}}.$$

The second norm equivalence follows from the Khintchine inequality in Theorem 1.1 with constant $c_2(p) \sim \sqrt{p^2/(p-1)}$; let us prove the first one. Since b_ψ is a cocycle, the map

$$\pi : \mathcal{L}(\mathrm{G}) \ni \lambda(g) \mapsto \exp(2\pi i \langle b_\psi(g), \cdot\rangle) \rtimes \lambda(g) \in L_\infty(\mathbb{R}^n_{\mathrm{bohr}},\nu) \rtimes \mathrm{G}$$

is a trace preserving $*$-homomorphism which satisfies

$$(\delta(-\Delta)^{-1/2} \rtimes \mathrm{id}_{\mathrm{G}}) \circ \pi = i(\mathrm{id}_{L_\infty(\mathbb{R}^n,\gamma)} \rtimes \pi) \circ \delta_\psi A_\psi^{-1/2}.$$

Indeed, if we let the left hand side act on $\lambda(g)$ we obtain

$$\begin{aligned}(\delta(-\Delta)^{-1/2} \rtimes \mathrm{id}_{\mathrm{G}}) \circ \pi(\lambda(g)) &= \delta(-\Delta)^{-1/2}\big(\exp(2\pi i\langle b_\psi(g),\cdot\rangle)\big) \rtimes \lambda(g) \\ &= \frac{1}{2\pi\|b_\psi(g)\|_{\mathcal{H}_\psi}}\delta\big(\exp(2\pi i\langle b_\psi(g),\cdot\rangle)\big) \rtimes \lambda(g) \\ &= \frac{i}{\sqrt{\psi(g)}}\exp(2\pi i\langle b_\psi(g),\cdot\rangle) \otimes B(b_\psi(g)) \rtimes \lambda(g),\end{aligned}$$

which coincides with $i(\mathrm{id}_{L_\infty(\mathbb{R}^n,\gamma)} \rtimes \pi) \circ \delta_\psi A_\psi^{-1/2}(\lambda(g))$. By Lemma 1.2, both sides in this intertwining identity are bounded $L_p(\mathcal{L}(\mathrm{G})) \to L_p(L_\infty(\mathbb{R}^n_{\mathrm{bohr}} \times \mathbb{R}^n, \nu\times\gamma) \rtimes \mathrm{G})$ for $1 < p < \infty$. In particular,

$$\sqrt{2/\pi}\, i(\mathrm{id}_{L_\infty(\mathbb{R}^n,\gamma)} \rtimes \pi) \circ \delta_\psi A_\psi^{-1/2} f = \widehat{Q}\Big(\mathrm{p.v.}\,\frac{1}{\pi}\int_{\mathbb{R}} (\beta_t \rtimes \mathrm{id}_{\mathrm{G}})\pi f\,\frac{dt}{t}\Big).$$

Since $\mathrm{id}_{L_\infty(\mathbb{R}^n,\gamma)} \rtimes \pi$ is also a trace preserving $*$-homomorphism, this yields

$$\begin{aligned}\|\delta_\psi f\|_{L_p(L_\infty(\mathbb{R}^n,\gamma)\rtimes\mathrm{G})} &= \|(\mathrm{id}_{L_\infty(\mathbb{R}^n,\gamma)} \rtimes \pi)\delta_\psi A_\psi^{-1/2}(A_\psi^{1/2} f)\|_{L_p(L_\infty(\mathbb{R}^n_{\mathrm{bohr}}\times\mathbb{R}^n,\nu\times\gamma)\rtimes\mathrm{G})} \\ &\lesssim \frac{p^3}{(p-1)^{3/2}}\|A_\psi^{1/2} f\|_{L_p(\widehat{\mathbf{G}})}.\end{aligned}$$

The constant above also follows from Lemma 1.2. The reverse estimate follows with the same constant from a duality argument. Indeed, if we fix f to be a trigonometric polynomial, there exists another trigonometric polynomial f' with $\|f'\|_{p'} = 1$ such that

$$(1-\varepsilon)\|A_\psi^{1/2} f\|_p \leq \tau(f'^* A_\psi^{1/2} f).$$

Note that $A_\psi^{-1/2}$ is only well-defined on $f'' = \sum_{\psi(g)\neq 0} \widehat{f'}(g)\lambda(g)$. However, since $\mathrm{G}_0 = \{g \in \mathrm{G} \mid \psi(g) = 0\}$ is a subgroup, we may consider the associated conditional expectation E_{G_0} on $\mathcal{L}(\mathrm{G})$ and obtain $f'' = f' - E_{\mathrm{G}_0} f'$ so that $\|f''\|_{p'} \leq 2$. On the other hand, we note the crucial identity

$$\begin{aligned}&\mathrm{tr}_{L_\infty(\Omega)\rtimes\mathrm{G}}((\delta_\psi f_1)^*\delta_\psi f_2) \\ &\quad= \sum_{g,h\in\mathrm{G}} \overline{\widehat{f_1}(g)}\widehat{f_2}(h)\Big(\int_\Omega \alpha_{g^{-1}}\big(\overline{B(b_\psi(g))}B(b_\psi(h))\big)\,d\mu\Big)\tau(\lambda(g^{-1}h)) \\ &\quad= \sum_{g\in\mathrm{G}} \overline{\widehat{f_1}(g)}\widehat{f_2}(g)\langle b_\psi(g), b_\psi(g)\rangle_\psi = \sum_{g\in\mathrm{G}} \overline{\widehat{f_1}(g)}\widehat{f_2}(g)\psi(g) = \tau((A_\psi^{1/2} f_1)^* A_\psi^{1/2} f_2).\end{aligned}$$

Combining both results we get

$$\begin{aligned}\|A_\psi^{1/2} f\|_p &\le \frac{1}{1-\varepsilon}\tau(f'^* A_\psi^{1/2} f) = \frac{1}{1-\varepsilon}\tau(f''^* A_\psi^{1/2} f)\\ &= \frac{1}{1-\varepsilon}\,\mathrm{tr}_{L_\infty(\Omega)\rtimes \mathrm{G}}\,((\delta_\psi A_\psi^{-1/2} f'')^* \delta_\psi f)\\ &\le \frac{1}{1-\varepsilon}\|\delta_\psi A_\psi^{-1/2} f''\|_{p'}\|\delta_\psi f\|_p \;\lesssim\; \frac{p^3}{(p-1)^{3/2}}\|\delta_\psi f\|_p.\end{aligned}$$

The last estimate was already obtained in the first part of this proof, it also follows from Lemma 1.2 and yields $c_1(p) \lesssim p^3/(p-1)^{3/2}$. This completes the proof. □

Remark 1.3. Let

$$\Gamma_\psi(f_1, f_2) = \tfrac{1}{2}\big(A_\psi(f_1^*)f_2 + f_1^* A_\psi(f_2) - A_\psi(f_1^* f_2)\big).$$

In the Introduction we related Theorem A2 to Meyer's formulation in terms of Γ_ψ via the identity $\mathsf{E}_{\mathcal{L}(\mathrm{G})}((\delta_\psi f_1)^* \delta_\psi f_2) = \Gamma_\psi(f_1, f_2)$. The proof follows by arguing as above, and we find

$$\mathsf{E}_{\mathcal{L}(\mathrm{G})}((\delta_\psi f_1)^* \delta_\psi f_2) = \sum_{g,h} \overline{\widehat{f_1}(g)}\,\widehat{f_2}(h)\langle b_\psi(g), b_\psi(h)\rangle_\psi \lambda(g^{-1}h) = \Gamma_\psi(f_1, f_2)$$

since $\langle b_\psi(g), b_\psi(h)\rangle_{\mathcal{H}_\psi} = \frac{1}{2}(\psi(g) + \psi(h) - \psi(g^{-1}h))$, as explained in Appendix B.

Remark 1.4. When $1 < p < 2$, we may consider decompositions $f = f_1 + f_2$ so that $\delta_\psi f = \phi_1 + \phi_2$ with $\phi_j = \delta_\psi f_j$ in our result. These particular decompositions give rise to

$$\|A_\psi^{1/2} f\|_p \le c(p) \inf_{f=f_1+f_2}\big(\|\Gamma_\psi(f_1, f_1)\|_p + \|\Gamma_\psi(f_2^*, f_2^*)\|_p\big).$$

Somewhat surprisingly, the reverse inequality does not hold. Indeed, using the arguments in the next section it would imply that Theorem A1 holds with the untwisted operators (b_j), but this was already disproved by F. Lust-Piquard [48].

Remark 1.5. Our constants grow like $p^{3/2}$ as $p \to \infty$ with a dual behavior as $p \to 1$. According to the results in the literature, one might expect a linear growth of the constant $\sim p$. It is however not clear to us whether this is true in our context since we admit semigroups which are not diffusion semigroups in the sense of Bakry, like the Poisson semigroup. It is an interesting problem to determine the optimal behavior of dimension free constants for (say) the Riesz transform associated to the Poisson semigroup $(e^{-t\sqrt{-\Delta}})$ in $\mathbb{R}^n$. On the other hand, we are not aware of any dimension free estimates with constants better than $p^{3/2}$ for matrix-valued functions in $\mathbb{R}^n$, even for the heat semigroup $(e^{t\Delta})$.

1.3. Riesz transforms in cocycle form

We are now ready to prove Theorem A1. The main ingredient comes from a factorization of the conditional expectation $\mathsf{E}_{\mathcal{L}(\mathrm{G})} : L_\infty(\Omega,\mu) \rtimes \mathrm{G} \to \mathcal{L}(\mathrm{G})$ in terms of a certain right $\mathcal{L}(\mathrm{G})$-module map. As predicted by Hilbert module theory [30], this factorization is always possible. In our case, when $\phi_1, \phi_2 \in G_p(\mathbb{C}) \rtimes \mathrm{G}$, it takes the form

$$\mathsf{E}_{\mathcal{L}(\mathrm{G})}(\phi_1^*\phi_2) = u(\phi_1)^* u(\phi_2),$$

where $u : G_p(\mathbb{C}) \rtimes \mathrm{G} \to C_p(L_p(\widehat{\mathbf{G}}))$ is defined as follows:

$$u(\phi) = u\Big(\sum_{g\in\mathrm{G}} \phi_g \rtimes \lambda(g)\Big) = \sum_{j\ge 1} \underbrace{\Big(\sum_{g\in\mathrm{G}} \langle \phi_g, B(e_j)\rangle_{L_2(\Omega,\mu)} \lambda(g)\Big)}_{u_j(\phi)} \otimes e_{j1}.$$

Before proving Theorem A1, it is convenient to explain where the infimum is taken and how we define the twisted form of (b_j). The infimum $R_{\psi,j} f = a_j + b_j$ runs over all possible families of the form $a_j = u_j(\phi_1)$ and $b_j = u_j(\phi_2)$ for some $\phi_1, \phi_2 \in G_p(\mathbb{C}) \rtimes \mathrm{G}$. This is equivalent to requiring that $\sum_j B(e_j) \rtimes a_j \in G_p(\mathbb{C}) \rtimes \mathrm{G}$, and the same for (b_j). Once this is settled, if we note that

$$b_j = \sum_{g\in\mathrm{G}} \Big\langle \sum_{k\ge 1} \widehat{b_k}(g) e_k, e_j \Big\rangle_{\mathcal{H}_\psi} \lambda(g),$$

then the twisted form of the family $(b_j)_{j\ge 1}$ is determined by the formula

$$\widetilde{b_j} = \sum_{g\in\mathrm{G}} \Big\langle \sum_{k\ge 1} \widehat{b_k}(g) e_k, \alpha_{\psi,g}(e_j) \Big\rangle_{\mathcal{H}_\psi} \lambda(g).$$

Proof of Theorem A1. If $f \in L_p^\circ(\widehat{\mathbf{G}})$, we have

$$u(\delta_\psi A_\psi^{-1/2} f) = \sum_{j\ge 1} \Big(\sum_{g\in\mathrm{G}} \frac{\langle b_\psi(g), e_j\rangle_{\mathcal{H}_\psi}}{\sqrt{\psi(g)}} \widehat{f}(g)\lambda(g) \Big) \otimes e_{j1} = \sum_{j\ge 1} R_{\psi,j} f \otimes e_{j1}.$$

When $p \ge 2$, the assertion follows directly from Theorem A2 since

$$\begin{aligned}\Gamma_\psi(A_\psi^{-1/2} f, A_\psi^{-1/2} f) &= \mathsf{E}_{\mathcal{L}(\mathrm{G})}((\delta_\psi A_\psi^{-1/2} f)^* \delta_\psi A_\psi^{-1/2} f) \\ &= |u(\delta_\psi A_\psi^{-1/2} f)|^2 = \sum_{j\ge 1} |R_{\psi,j} f|^2,\end{aligned}$$

and similarly with f replaced by f^*. When $1 < p \le 2$ we first observe that

$$\begin{aligned}\|f\|_p &= \|A_\psi^{1/2} A_\psi^{-1/2} f\|_p \\ &\sim \inf_{\delta_\psi A_\psi^{-1/2} f = \phi_1 + \phi_2} \big(\|\mathsf{E}_{\mathcal{L}(\mathrm{G})}(\phi_1^*\phi_1)^{1/2}\|_p + \|\mathsf{E}_{\mathcal{L}(\mathrm{G})}(\phi_2\phi_2^*)^{1/2}\|_p \big)\end{aligned}$$

by Theorem A2. By the factorization of $\mathsf{E}_{\mathcal{L}(\mathrm{G})}$, we also recall the identities

$$\mathsf{E}_{\mathcal{L}(\mathrm{G})}(\phi_1^*\phi_1)^{1/2} = |u(\phi_1)| = \Big(\sum_{j\geq 1} u_j(\phi_1)^* u_j(\phi_1)\Big)^{1/2},$$
$$\mathsf{E}_{\mathcal{L}(\mathrm{G})}(\phi_2\phi_2^*)^{1/2} = |u(\phi_2^*)| = \Big(\sum_{j\geq 1} u_j(\phi_2^*)^* u_j(\phi_2^*)\Big)^{1/2}.$$

The injectivity of u implies that $\|f\|_p$ is comparable to the norm

$$\inf_{R_{\psi,j}f=u_j(\phi_1)+u_j(\phi_2)} \Big[\Big\|\Big(\sum_{j\geq 1} u_j(\phi_1)^* u_j(\phi_1)\Big)^{1/2}\Big\|_p + \Big\|\Big(\sum_{j\geq 1} u_j(\phi_2^*)^* u_j(\phi_2^*)\Big)^{1/2}\Big\|_p\Big].$$

Therefore, if $a_j = u_j(\phi_1)$ and $b_j = u_j(\phi_2)$ it suffices to see that $\widetilde{b_j} = u_j(\phi_2^*)^*$ to settle the case $1 < p \leq 2$. Since u is injective and $\phi_2 = u^{-1}(\sum_k b_k \otimes e_{k1})$, we may prove such an identity as follows:

$$\begin{aligned}
u_j(\phi_2^*)^* &= \Big(u_j\Big[\Big(\sum_{g\in \mathrm{G}}\Big(\sum_{k\geq 1}\widehat{b}_k(g)B(e_k)\Big) \rtimes \lambda(g)\Big)^*\Big]\Big)^* \\
&= \Big(u_j\Big[\sum_{g\in \mathrm{G}}\Big(\sum_{k\geq 1}\overline{\widehat{b}_k(g)}B(\alpha_{\psi,g^{-1}}(e_k))\Big) \rtimes \lambda(g^{-1})\Big]\Big)^* \\
&= \Big[\sum_{g\in \mathrm{G}}\Big\langle\sum_{k\geq 1}\overline{\widehat{b}_k(g)}B(\alpha_{\psi,g^{-1}}(e_k)),\, B(e_j)\Big\rangle\lambda(g^{-1})\Big]^* \\
&= \sum_{g\in \mathrm{G}}\Big\langle\alpha_{\psi,g^{-1}}\Big(\sum_{k\geq 1}\widehat{b}_k(g)e_k\Big), e_j\Big\rangle_{\mathcal{H}_\psi}\lambda(g) = \widetilde{b}_j.
\end{aligned}$$

Note that we have implicitly used the fact that $\mathcal{H}_\psi$ and $L_2(\Omega,\mu)$ are real Hilbert spaces. □

Remark 1.6. Let A be the infinitesimal generator of a Markovian semigroup $\mathcal{S} = (\mathcal{S}_t)_{t\geq 0}$ acting on $(\mathcal{M},\tau)$. Sauvageot's theorem [63] provides a factorization $A = \delta^*\delta$ in terms of a certain symmetric derivation $\delta : L_2(\mathcal{M}) \to \mathsf{M}$ taking values in some Hilbert $\mathcal{M}$-bimodule M. As a consequence of our results, it is the nature of the tangent module M and not of $\mathcal{M}$ itself which dictates the behavior of Riesz transforms on $L_p(\mathcal{M},\tau)$, in the sense that we find noncommutative phenomena as long as M is not abelian regardless of the nature of $\mathcal{M}$.

Remark 1.7. Theorems A1 and A2 are formulated for left cocycles, although an alternative form is possible for right cocycles or both together (see the precise definitions in Appendix B). The only (cosmetic) change appears in the statement of Theorem A1 for $p \geq 2$. The row version of $R_{\psi,j}$ is

$$R'_{\psi,j}f = 2\pi i \sum_{g\in \mathrm{G}} \frac{\langle b_\psi(g^{-1}), e_j\rangle_{\mathcal{H}_\psi}}{\sqrt{\psi(g)}}\widehat{f}(g)\lambda(g).$$

Note that $R_{\psi,j}(f^*) = -R'_{\psi,j}(f)^*$ so that $\sum_j (R_{\psi,j} f^*)^*(R_{\psi,j} f^*)$ can be written as a row square function in terms of the row Riesz transforms $R'_{\psi,j}$. Although these two formulations become identical for G abelian (left and right cocycles coincide), the statement does not simplify for nontrivial cocycle actions and $1 < p < 2$.

Remark 1.8. On the other hand, it is interesting to note that Theorems A1 and A2 hold in the category of operator spaces [58]. In other words, the same inequalities are valid with matrix Fourier coefficients when the operators involved act trivially on the matrix amplification.

1.4. Examples, commutative or not

In order to illustrate Theorems A1 and A2, it will be instrumental to present conditionally negative lengths in a more analytic way. As will be justified in Appendix B (Theorem B.4), these lengths are all deformations of the standard inner cocycle which acts by left multiplication. Namely, $\psi : \mathrm{G} \to \mathbb{R}_+$ is conditionally negative iff it can be written as

$$\psi(g) = \tau_\psi\big(2\lambda(e) - \lambda(g) - \lambda(g^{-1})\big) \tag{CN}$$

for a positive linear functional $\tau_\psi : \Pi_0 \to \mathbb{C}$ defined on the space Π_0 of trigonometric polynomials in $\mathcal{L}(\mathrm{G})$ whose Fourier coefficients have vanishing sum. Having this in mind, let us consider some examples illustrating Theorems A1 and A2.

A. Fractional laplacians in $\mathbb{R}^n$**.** The Riesz potentials $f \mapsto (-\Delta)^{-\beta/2} f$ are classical operators in Euclidean harmonic analysis [69]. It seems however that dimension free estimates for associated Riesz transforms in $\mathbb{R}^n$ are unexplored. If we let our infinitesimal generator be $A_\beta = (-\Delta)^\beta$ and $p \geq 2$ (for simplicity), the problem in $\mathbb{R}^n$ consists in showing that

$$\Big\| \Big(\sum_{j\geq 1} |\partial_{\beta,j} A_\beta^{-1/2} f|^2 \Big)^{1/2} \Big\|_{L_p(\mathbb{R}^n)} \sim_{c(p)} \|f\|_{L_p(\mathbb{R}^n)}$$

for some differential operators $\partial_{\beta,j}$ and constants $c(p)$ independent of the dimension n. A_β generates a Markov semigroup for $0 \leq \beta \leq 1$. Indeed, the nonelementary cases $0 < \beta < 1$ require to know that the length $\psi'_\beta(\xi) = |\xi|^{2\beta}$ is conditionally negative. Since (CN) holds for $\mathrm{G} = \mathbb{R}^n$, the claim follows from the simple identity

$$\psi'_\beta(\xi) = |\xi|^{2\beta} = \frac{1}{\mathrm{k}_n(\beta)} \int_{\mathbb{R}^n} (2 - e^{2\pi i \langle x,\xi\rangle} - e^{-2\pi i\langle x,\xi\rangle})\, d\mu_\beta(x) = \frac{1}{\mathrm{k}_n(\beta)} \psi_\beta(\xi)$$

with $d\mu_\beta(x) = dx/|x|^{n+2\beta}$ and

$$\mathrm{k}_n(\beta) = 2\int_{\mathbb{R}^n} \big(1 - \cos(2\pi\langle \xi/|\xi|, s\rangle)\big) \frac{ds}{|s|^{n+2\beta}} \sim \frac{\pi^{n/2}}{\Gamma(n/2)} \max\Big\{ \frac{1}{\beta}, \frac{1}{1-\beta} \Big\}.$$

The constant $\mathrm{k}_n(\beta)$ only makes sense for $0 < \beta < 1$ and is independent of ξ. The last equivalence follows from the fact that the integral on the left hand side is comparable to

$$\int_{\mathrm{B}_1(0)} \frac{|\langle s, \xi/|\xi|\rangle|^2}{|s|^{n+2\beta}}\, ds + \int_{\mathbb{R}^n\setminus \mathrm{B}_1(0)} \frac{1}{|s|^{n+2\beta}}\, ds,$$

and the equivalence follows by using polar coordinates. The associated cocycle $(\mathcal{H}_\beta, \alpha_\beta, b_\beta)$ is given by the action $\alpha_{\beta,\xi}(f) = \exp(2\pi i \langle \cdot, \xi\rangle) f$ and the cocycle map $\xi \mapsto 1 - \exp(2\pi i \langle \cdot, \xi\rangle) \in \mathcal{H}_\beta = L_2(\mathbb{R}^n, \mu_\beta / \mathrm{k}_n(\beta))$. Of course, we may regard $\mathcal{H}_\beta$ as a real Hilbert space by identifying $\exp(2\pi i \langle x, \xi\rangle) \in \mathbb{C}$ with $(\cos(2\pi\langle x, \xi\rangle), \sin(2\pi\langle x, \xi\rangle))$ in $\mathbb{R}^2$ and the product $\exp(2\pi i \langle x, \xi\rangle) \cdot [\ ]$ with a rotation by $2\pi\langle x, \xi\rangle$. In contrast to the standard Riesz transforms for $\beta = 1$, for which $\mathcal{H}_\beta = \mathbb{R}^n$ and the cocycle is trivial, we need to represent $\mathbb{R}^n$ in infinitely many dimensions to obtain the right differential operators

$$\widehat{R_{\beta,j} f}(\xi) = \partial_{\beta,j} \widehat{A_\beta^{-1/2}} f(\xi) = \frac{\langle b_\beta(\xi), e_j\rangle_{\mathcal{H}_\beta}}{|\xi|^\beta} \widehat{f}(\xi).$$

In particular, Theorem A1 (or its extension to unimodular groups in Appendix A) gives norm equivalences for all $1 < p < \infty$ which differ from the classical statement when $1 < p < 2$, since the cocycle action is not trivial anymore for $0 < \beta < 1$. It is also interesting to look at Theorem A2. Taking into account that $\widehat{A_\beta f}(\xi) = |\xi|^{2\beta} \widehat{f}(\xi)$ and the definition of the associated gradient form Γ_β, we obtain

$$\|(-\Delta)^{\beta/2} f\|_p \sim \Big\| \Big(\int_{\mathbb{R}^n} M_{f,\beta}(\cdot, \xi) e^{2\pi i \langle \cdot, \xi\rangle} \, d\xi \Big)^{1/2} \Big\|_p$$

for f smooth enough and

$$M_{f,\beta}(x, \xi) = \tfrac{1}{2}\big(f(x)\overline{\widehat{f}(-\xi)} + \overline{f(x)}\widehat{f}(\xi) - \widehat{|f|^2}(\xi)\big)|\xi|^{2\beta}.$$

More applications to Euclidean L_p multipliers will be analysed in the next section.

B. Discrete laplacians in LCA groups. Let Γ_0 be a locally compact abelian group and $s_0 \in \Gamma_0$ be torsion free. If $\partial_j \varphi(\gamma) = \varphi(\gamma) - \varphi(\gamma_1, \ldots, s_0\gamma_j, \ldots, \gamma_n)$ stand for discrete directional derivatives in $\Gamma = \Gamma_0 \times \cdots \times \Gamma_0$, we may construct the laplacian $L = \sum \partial_j^* \partial_j$. Lust-Piquard's [48] main result establishes dimension free estimates in this context for the family of discrete Riesz transforms given by $R_j = \partial_j L^{-1/2}$ and $R_j^* = L^{-1/2}\partial_j^*$. If $p \geq 2$, she obtained

$$\Big\| \Big(\sum_{j=1}^n (|R_j\varphi|^2 + |R_j^*\varphi|^2) \Big)^{1/2} \Big\|_{L_p(\Gamma)} \sim_{c(p)} \|\varphi\|_{L_p(\Gamma)}.$$

It is not difficult to recover and generalize Lust-Piquard's theorem from Theorem A1 for Γ LCA, justified in Appendix A. Indeed, let Γ be any LCA group equipped with a positive measure μ_ψ. If G denotes the dual group of Γ, let us write $\chi_g : \Gamma \to \mathbb{T}$ for the associated characters and ν for the Haar measure on G. Define

$$A_\psi \varphi = \int_{\mathrm{G}} \widehat{\varphi}(g) A_\psi(\chi_g) \, d\nu(g) = \int_{\mathrm{G}} \widehat{\varphi}(g) \underbrace{\Big[\int_\Gamma (2\chi_e - \chi_g - \chi_{g^{-1}})(\gamma) \, d\mu_\psi(\gamma) \Big]}_{\psi(g) = \|1-\chi_g\|^2_{L_2(\Gamma,\mu_\psi)}} \chi_g \, d\nu(g).$$

If $\psi : \mathrm{G} \to \mathbb{R}_+$, then it is a conditionally negative length which may be represented by the cocycle

$$(\mathcal{H}_\psi, \alpha_{\psi,g}, b_\psi(g)) = \big(L_2(\Gamma, \mu_\psi), \chi_g \cdot [\ ], 1 - \chi_g \big).$$

In other words, we have $\psi(g) = \langle b_\psi(g), b_\psi(g)\rangle_{\mathcal{H}_\psi}$. Again, we may regard $\mathcal{H}_\psi$ as a real Hilbert space by identifying $\mathbb{C} \ni \chi_g(\gamma) \mapsto (\mathrm{Re}(\chi_g(\gamma)), \mathrm{Im}(\chi_g(\gamma))) \in \mathbb{R}^2$ and the product $\chi_g(\gamma) \cdot [\;]$ with a rotation by $\arg(\chi_g(\gamma))$. Set

$$R^1_\gamma \varphi = \int_{\mathrm{G}} \frac{\langle b_\psi(g), e^1_\gamma\rangle_{\mathcal{H}_\psi}}{\sqrt{\psi(g)}} \widehat{\varphi}(g)\chi_g \, d\nu(g) = \int_{\mathrm{G}} \frac{\mathrm{Re}\, b_\psi(g,\gamma)}{\sqrt{\psi(g)}} \widehat{\varphi}(g)\chi_g \, d\nu(g),$$
$$R^2_\gamma \varphi = \int_{\mathrm{G}} \frac{\langle b_\psi(g), e^2_\gamma\rangle_{\mathcal{H}_\psi}}{\sqrt{\psi(g)}} \widehat{\varphi}(g)\chi_g \, d\nu(g) = \int_{\mathrm{G}} \frac{\mathrm{Im}\, b_\psi(g,\gamma)}{\sqrt{\psi(g)}} \widehat{\varphi}(g)\chi_g \, d\nu(g).$$

Then Theorem A1 takes the following form for LCA groups and $p \geq 2$:

$$\left\| \left(\int_\Gamma (|R^1_\gamma \varphi|^2 + |R^2_\gamma \varphi|^2) \, d\mu_\psi(\gamma) \right)^{1/2} \right\|_{L_p(\Gamma)} \sim_{c(p)} \|\varphi\|_{L_p(\Gamma)}.$$

Now, let us go back to Lust-Piquard's setting $\Gamma = \Gamma_0 \times \cdots \times \Gamma_0$ and write $\sigma_j = (0, \ldots, 0, s_0, 0, \ldots, 0)$ with s_0 in the j-th entry. If we pick our measure μ_ψ to be the sum $\sum_j \delta_{\sigma_j}$ of point-masses and recall the simple identities

$$R_j = \partial_j L^{-1/2} = R^1_{\sigma_j} - i R^2_{\sigma_j}, \qquad R^*_j = L^{-1/2}\partial^*_j = R^1_{\sigma_j} + i R^2_{\sigma_j},$$

then we deduce Lust-Piquard's theorem as a particular case of Theorem A1 for LCA groups. We may also recover her inequalities for $1 < p < 2$—which have a more intricate statement—from our result; we leave the details to the reader. The advantage of our formulation is that it is much more flexible. For instance, we do not need to require s_0 to be torsion free. We may also consider other point-masses giving rise to other forms of discrete laplacians. In particular, we may allow the shift s_0 to depend on j, or even Γ not to be given in direct product form. Many other (not necessarily finitely supported) measures can also be considered. We will not make an exhaustive analysis of this here.

C. Word-length laplacians. We may work with many other discrete groups equipped with more or less standard lengths. Let us consider one of the most canonical examples, word length. Given a finitely generated discrete group G, the word length $|g|$ is the distance from g to e in the Cayley graph of G. It is not always conditionally negative, but when this is the case we may represent it via the cocycle $(\mathcal{H}_{|\,|}, \alpha_{|\,|}, b_{|\,|})$, where $\mathcal{H}_{|\,|}$ is the closure of the pre-Hilbert space defined in Π_0 by

$$\langle f_1, f_2\rangle_{|\,|} = -\frac{1}{2} \sum_{g,h \in \mathrm{G}} \overline{\widehat{f_1}(g)} \widehat{f_2}(h) |g^{-1}h|$$

and $(\alpha_{|\,|,g}(f), b_{|\,|}(g)) = (\lambda(g) f, \lambda(e) - \lambda(g))$, as usual. The identification of this Hilbert space as a real one requires taking real and imaginary parts as in the previous examples. The Riesz transforms

$$R_{|\,|,j} f = \partial_{|\,|,j} A^{-1/2}_{|\,|} f = 2\pi i \sum_{g \in \mathrm{G}} \frac{\langle b_{|\,|}(g), e_j\rangle_{\mathcal{H}_{|\,|}}}{\sqrt{|g|}} \widehat{f}(g)\lambda(g)$$

satisfy Theorem A1 for any orthonormal basis $(e_j)_{j\geq 1}$ of $\mathcal{H}_{|\,|}$. Theorem A2 also applies with $A_{|\,|}(\lambda(g)) = |g|\lambda(g)$ and the associated gradient form $\Gamma_{|\,|}$. Further Riesz transforms arise from other positive linear functionals τ_ψ on Π_0.

Let us recall a few well-known groups for which $|\,|$ is conditionally negative:

(i) **Free groups.** The conditional negativity of the length function for the free group $\mathbb{F}_n$ with n generators was established by Haagerup [29]. A concrete description of the associated cocycle yields an interesting application of Theorem A1. The Gromov form is

$$K(g,h) = \langle b_{|\,|}(g), b_{|\,|}(h)\rangle_{\mathcal{H}_{|\,|}} = \frac{|g|+|h|-|g^{-1}h|}{2} = |\min(g,h)|$$

where $\min(g,h)$ is the longest word inside the common branch of g and h in the Cayley graph. If g and h do not share a branch, we let $\min(g,h)$ be the empty word. Let $\mathbb{R}[\mathbb{F}_n]$ stand for the algebra of finitely supported real valued functions on $\mathbb{F}_n$ and consider the bracket $\langle \delta_g, \delta_h\rangle = K(g,h)$. If N denotes its null space, let $\mathcal{H}$ be the completion of $\mathbb{R}[\mathbb{F}_n]/N$. Then, writing g^- for the word which results after deleting the last generator on the right of g, the system $\xi_g = \delta_g - \delta_{g^-} + N$ for all $g \in \mathbb{F}_n \setminus \{e\}$ is orthonormal and generates $\mathcal{H}$. Indeed, it is obvious that δ_e belongs to N and we may write

$$\phi = \sum_{g\in\mathbb{F}_n} a_g \delta_g = \sum_{g\in\mathbb{F}_n} \Big(\sum_{h\geq g} a_h\Big)\xi_g$$

for any $\phi \in \mathbb{R}[\mathbb{F}_n]$. Here $h \geq g$ iff g belongs to the (unique) path from e to h in the Cayley graph and we set $\xi_e = \delta_e$. This shows that $N = \mathbb{R}\delta_e$ and $\dim \mathcal{H} = \infty$. It yields a cocycle with $\alpha = \lambda$ and

$$b : \mathbb{F}_n \ni g \mapsto \delta_g + \mathbb{R}\delta_e \in \mathbb{R}[\mathbb{F}_n]/\mathbb{R}\delta_e.$$

Considering the ONB $(\xi_h)_{h\neq e}$ we see that $\langle b(g), \xi_h\rangle = \delta_{g\geq h}$, and the operators

$$R_{|\,|,h} f = \sum_{g\geq h} \frac{1}{\sqrt{|g|}} \widehat{f}(g)\lambda(g)$$

form a natural family of Riesz transforms in $\mathcal{L}(\mathbb{F}_n)$ with respect to word length. If $p \geq 2$, Theorem A1 gives rise to the inequalities below in the free group algebra with constants $c(p)$ only depending on p:

$$\|f\|_{L_p(\widehat{\mathbf{F}}_n)} \sim_{c(p)} \Big\| \Big(\sum_{h\neq e} \Big| \sum_{g\geq h} \frac{\widehat{f}(g)}{\sqrt{|g|}} \lambda(g)\Big|^2 + \Big| \sum_{g\geq h} \frac{\overline{\widehat{f}(g^{-1})}}{\sqrt{|g|}} \lambda(g)\Big|^2 \Big)^{1/2} \Big\|_{L_p(\widehat{\mathbf{F}}_n)}.$$

The case $1 < p < 2$ can be formulated similarly according to Theorem A1. Note that the L_p-boundedness of the Riesz transforms $R_{|\,|,h}$ is out of the scope of Haagerup's inequality [29], since they are unbounded on L_∞.

(ii) **Finite cyclic groups.** The group $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ with the counting measure is a central example. Despite its simplicity it may be difficult to provide precise bounds for Fourier multipliers (see for instance [35] for a discussion on optimal hypercontractivity bounds). The conditional negativity for the word length $|k| = \min(k, n-k)$ in $\mathbb{Z}_n$ was justified in that paper. For simplicity we will assume that $n = 2m$ is an even integer. Its Gromov form

$$K(j,k) = \frac{(j \wedge (2m-j)) + (k \wedge (2m-k)) - ((k-j) \wedge (2m-k+j))}{2}$$

can be written as follows for $0 \leq j \leq k \leq 2m-1$:

$$K(j,k) = \langle \delta_j, \delta_k \rangle = j \wedge (2m-k) \wedge (m-k+j)_+$$

with $s_+ = 0 \vee s$; the details are left to the reader. Note that $K(j,j) = \psi(j)$ as it should be. Using the above formula, we may consider the associated bracket in $\Pi_0 \simeq \mathbb{R}[\mathbb{Z}_{2m}] \ominus \mathbb{R}\delta_0$ and deduce, after rearrangement,

$$\Big\langle \sum_{j=1}^{2m-1} a_j \delta_j, \sum_{j=1}^{2m-1} b_j \delta_j \Big\rangle = \sum_{k=1}^{m} \Big(\sum_{j=k}^{k+m-1} a_j \Big) \Big(\sum_{j=k}^{k+m-1} b_j \Big) = \sum_{k=1}^{m} \alpha_k \beta_k.$$

This shows that the null space N of the bracket above is the space of elements $\sum_j a_j \delta_j$ in $\mathbb{R}[\mathbb{Z}_{2m}] \ominus \mathbb{R}\delta_0$ with vanishing coordinates α_k. If we quotient out this subspace we end up with a Hilbert space $\mathcal{H}$ of dimension $(2m-1)-(m-1) = m$. Our discussion establishes an isometric isomorphism

$$\Phi : \mathcal{H} \ni \sum_{j=1}^{2m-1} a_j \delta_j \mapsto \sum_{k=1}^{m} \alpha_k e_k \in \mathbb{R}^m,$$

where $\alpha_k = \langle \sum_{j \leq 2m-1} a_j \delta_j, 1_{\Lambda_k} \rangle$ with

$$\Lambda_k = \{ j \in \mathbb{Z}_{2m} \mid j - k \equiv s \bmod 2m \ \text{ for some } \ 0 \leq s \leq m-1 \},$$

so that $1_{\Lambda_k} = \Phi^{-1}(e_k)$. Consider the ONB of $\mathcal{H}$ given by 1_{Λ_k} for $1 \leq k \leq m$. We are ready to construct explicit Riesz transforms associated to the word length in $\mathbb{Z}_{2m}$. Namely, let $\widehat{\mathbb{Z}}_{2m} = \mathbb{T}_{2m}$ be the group of $2m$-roots of unity with the normalized counting measure. Then using the ONB above we set

$$R_{|\,|,k} f = \sum_{j \in \Lambda_k} \frac{1}{\sqrt{j \wedge (2m-j)}} \widehat{f}(j) \chi_j \quad \text{with} \quad \chi_j(z) = z^j.$$

If $p \geq 2$, Theorem A1 establishes the following equivalence in $L_p(\widehat{\mathbb{Z}}_{2m})$:

$$\Big\| \sum_{j \in \mathbb{Z}_{2m}} \widehat{f}(j) \chi_j \Big\|_{L_p(\widehat{\mathbb{Z}}_{2m})} \sim_{c(p)} \Big\| \Big(\sum_{k=1}^{m} \Big| \sum_{j \in \Lambda_k} \frac{\widehat{f}(j)}{\sqrt{j \wedge (2m-j)}} \chi_j \Big|^2 \Big)^{1/2} \Big\|_{L_p(\widehat{\mathbb{Z}}_{2m})}$$

with constants independent of m. Similar computations can be performed for $n = 2m+1$ odd, and Theorem A1 can be reformulated in this case. We leave the details to the reader.

(iii) **Infinite Coxeter groups.** Any group presented by

$$\mathrm{G} = \langle g_1, \ldots, g_m \mid (g_j g_k)^{s_{jk}} = e \rangle$$

with $s_{jj} = 1$ and $s_{jk} \geq 2$ for $j \neq k$ is called a *Coxeter group*. Bożejko [7] proved that word length is conditionally negative for any infinite Coxeter group. The Cayley graph of these groups is more involved and we will not construct here a natural ONB for the associated cocycle; we invite the reader to do it.

Other interesting examples include discrete Heisenberg groups or symmetric groups.

2. Hörmander–Mikhlin multipliers

In this section we shall prove Theorems B1 and B2. Before that we obtain some preliminary Littlewood–Paley type inequalities. Afterwards we shall also study a few Besov limiting conditions in the spirit of Seeger for group von Neumann algebras that follow from Riesz transforms estimates.

2.1. Littlewood–Paley estimates

Let (G, ψ) be a discrete group equipped with a conditionally negative length and associated cocycle $(\mathcal{H}_\psi, \alpha_\psi, b_\psi)$. If $h \in \mathcal{H}_\psi$ we shall write $R_{\psi,h}$ for the h-directional Riesz transform

$$R_{\psi,h} f = 2\pi i \sum_{g \in \mathrm{G}} \frac{\langle b_\psi(g), h \rangle_{\mathcal{H}_\psi}}{\sqrt{\psi(g)}} \widehat{f}(g) \lambda(g).$$

We begin with a consequence of Theorem A1 which also generalizes it.

Lemma 2.1. *Given $(h_j)_{j\geq 1}$ in $\mathcal{H}_\psi$ and $p \geq 2$,*

$$\Big\| \Big(\sum_{j\geq 1} |R_{\psi,h_j} f_j|^2 \Big)^{1/2} \Big\|_{L_p(\widehat{\mathbf{G}})} \lesssim_{c(p)} \Big(\sup_{j\geq 1} \|h_j\|_{\mathcal{H}_\psi} \Big) \Big\| \Big(\sum_{j\geq 1} |f_j|^2 \Big)^{1/2} \Big\|_{L_p(\widehat{\mathbf{G}})}.$$

Proof. Given an ONB $(e_k)_{k\geq 1}$ of $\mathcal{H}_\psi$, we have

$$\begin{aligned} &\Big\| \Big(\sum_{j\geq 1} |R_{\psi,h_j} f_j|^2 \Big)^{1/2} \Big\|_{L_p(\widehat{\mathbf{G}})} \\ &\quad = \Big\| \sum_{j\geq 1} \Big(\sum_{g\in \mathrm{G}} \sum_{k\geq 1} \frac{\langle h_j, e_k \rangle_{\mathcal{H}_\psi} \langle b_\psi(g), e_k \rangle_{\mathcal{H}_\psi}}{\sqrt{\psi(g)}} \widehat{f_j}(g) \lambda(g) \Big) \otimes e_{j,1} \Big\|_{S_p(L_p)} \\ &\quad = \Big\| \sum_{j,k\geq 1} R_{\psi,k} f_j \otimes \langle h_j, e_k \rangle_{\mathcal{H}_\psi} e_{j,1} \Big\|_{S_p(L_p)} = \Big\| \sum_{j,k\geq 1} R_{\psi,k} f_j \otimes \Lambda(e_{(j,k),1}) \Big\|_{S_p(L_p)}, \end{aligned}$$

where $\Lambda : \ell_2(\mathbb{N}\times\mathbb{N}) \ni \delta_{(j,k)} \mapsto \langle h_j, e_k \rangle_{\mathcal{H}_\psi} \delta_j \in \ell_2(\mathbb{N})$. Since $\ell_2^c := \mathcal{B}(\mathbb{C}, \ell_2)$ is a homogeneous Hilbertian operator space [58], the cb-norm of Λ coincides with its norm, which in turn equals $\sup_j \|h_j\|_{\mathcal{H}_\psi}$. Altogether, we deduce

$$\begin{aligned}
\Big\| \Big(\sum_{j\geq 1} |R_{\psi,h_j} f_j|^2 \Big)^{1/2} \Big\|_{L_p(\widehat{\mathbf{G}})} &\leq \Big(\sup_{j\geq 1} \|h_j\|_{\mathcal{H}_\psi} \Big) \Big\| \sum_{j,k\geq 1} R_{\psi,k} f_j \otimes e_{(j,k),1} \Big\|_{S_p(L_p)} \\
&= \Big(\sup_{j\geq 1} \|h_j\|_{\mathcal{H}_\psi} \Big) \Big\| \Big(\sum_{j,k\geq 1} |R_{\psi,k} f_j|^2 \Big)^{1/2} \Big\|_{L_p(\widehat{\mathbf{G}})} \\
&= \Big(\sup_{j\geq 1} \|h_j\|_{\mathcal{H}_\psi} \Big) \Big\| \Big(\sum_{k\geq 1} |\widetilde{R}_{\psi,k} f|^2 \Big)^{1/2} \Big\|_{S_p(L_p)}
\end{aligned}$$

for $\widetilde{R}_{\psi,k} = R_{\psi,k} \otimes \mathrm{id}_{\mathcal{B}(\ell_2)}$ and $f = \sum_j f_j \otimes e_{j,1}$. Now, since Theorem A1 also holds in the category of operator spaces, the last term on the right hand side is dominated by $c(p)\|f\|_{S_p(L_p)}$, which yields the inequality we are looking for. □

We need to fix some standard terminology for our next result. Consider a sequence of functions $\varphi_j : \mathbb{R}_+ \to \mathbb{C}$ in $\mathcal{C}^{\mathrm{k}_n}(\mathbb{R}_+ \setminus \{0\})$ for $\mathrm{k}_n = [n/2] + 1$ such that

$$\Big(\sum_j \Big| \frac{d}{d\xi^k} \varphi_j(\xi) \Big|^2 \Big)^{1/2} \lesssim |\xi|^{-k} \quad \text{for all } 0 \leq k \leq \mathrm{k}_n.$$

Let $\mathsf{M}_p(\mathrm{G})$ stand for the space of symbols $m : \mathrm{G} \to \mathbb{C}$ associated to L_p-bounded Fourier multipliers in the group von Neumann algebra $\mathcal{L}(\mathrm{G})$; equip any such symbol m with the $p \to p$ norm of the multipler $\lambda(g) \mapsto m(g)\lambda(g)$. Now, given any conditionally negative length $\psi : \mathrm{G} \to \mathbb{R}_+$ and h in the associated cocycle Hilbert space $\mathcal{H}_\psi$, let us consider the symbols

$$m_{\psi,h}(g) = \langle b_\psi(g), h \rangle_\psi / \sqrt{\psi(g)} \quad \text{so that} \quad R_{\psi,h}(\lambda(g)) = 2\pi i m_{\psi,h}(g)\lambda(g).$$

Then, we may combine families of these symbols into a single Fourier multiplier patching them via the Littlewood–Paley decompositions provided by the families (φ_j) and finite-dimensional cocycles on G. The result is the following.

Lemma 2.2. *Let* G *be a discrete group equipped with two n-dimensional cocycles with associated length functions ψ_1, ψ_2 and an arbitrary cocycle with associated length function ψ_3. Let (φ_{1j}), (φ_{2j}) be Littlewood–Paley decompositions satisfying the assumptions above and $(h_j)_{j\geq 1}$ in $\mathcal{H}_{\psi_3}$. Then, for any $1 < p < \infty$,*

$$\Big\| \sum_{j\geq 1} \varphi_{1j}(\psi_1(\cdot))\varphi_{2j}(\psi_2(\cdot)) m_{\psi_3,h_j} \Big\|_{\mathsf{M}_p(\mathrm{G})} \lesssim_{c(p,n)} \sup_{j\geq 1} \|h_j\|_{\mathcal{H}_{\psi_3}}.$$

Proof. Consider the multipliers

$$\Lambda_{\psi,\varphi} : \lambda(g) \mapsto \varphi(\psi(g))\lambda(g).$$

When $p \geq 2$, let $f \in L_p(\widehat{\mathbf{G}})$ and $\widetilde{f} \in L_q(\widehat{\mathbf{G}})$ with $1/p + 1/q = 1$ so that

$$\begin{aligned}
\Big| \tau \Big(\widetilde{f}^* \sum_j \Lambda_{\psi_1,\varphi_{1j}} \Lambda_{\psi_2,\varphi_{2j}} R_{\psi_3,h_j}(f) \Big) \Big| \\
&= \Big| \sum_j \tau \big((\Lambda_{\psi_1,\varphi_{1j}} \widetilde{f})^* R_{\psi_3,h_j}(\Lambda_{\psi_2,\varphi_{2j}} f) \big) \Big| \\
&\leq \Big\| \sum_j \Lambda_{\psi_1,\varphi_{1j}} \widetilde{f} \otimes \delta_j \Big\|_{RC_q(\mathcal{L}(\mathrm{G}))} \Big\| \sum_j R_{\psi_3,h_j}(\Lambda_{\psi_2,\varphi_{2j}} f) \otimes \delta_j \Big\|_{RC_p(\mathcal{L}(\mathrm{G}))}
\end{aligned}$$

by anti-linear duality. On the other hand, Lemma 2.1 trivially extends to RC_p-spaces. Indeed, the row inequality follows from the column one applied to f_j^*'s together with Remark 1.7 and the fact that the maps R'_{ψ,h_j} satisfy the same estimates. We apply it to the last term of the right hand side. Combining that with the Littlewood–Paley inequality in [34, Theorem C] we obtain

$$\Big| \tau \Big(\widetilde{f}^* \sum_{j \geq 1} \Lambda_{\psi_1,\varphi_{1j}} \Lambda_{\psi_2,\varphi_{2j}} R_{\psi_3,h_j}(f) \Big) \Big| \lesssim_{c(p,n)} \Big(\sup_{j\geq 1} \|h_j\|_{\mathcal{H}_{\psi_3}} \Big) \|f\|_{L_p(\widehat{\mathbf{G}})} \|\widetilde{f}\|_{L_q(\widehat{\mathbf{G}})}.$$

The result follows by taking supremums over $\widetilde{f}$ running over the unit ball of $L_q(\widehat{\mathbf{G}})$. On the other hand, when $1 < p < 2$ the result follows easily by duality from the case $p > 2$ since the multipliers involved are $\mathbb{R}$-valued and therefore self-adjoint. □

The only drawback of Lemma 2.2 is that we need finite-dimensional cocycles to apply our Littlewood–Paley estimates from [34]. We may ignore that requirement at the price of using other square function inequalities from [32], where the former φ_j are now dilations of a fixed function. Namely, given $1 < p < \infty$ and $\varphi : \mathbb{R}_+ \to \mathbb{C}$ belonging to a certain class $\mathcal{J}_p$ of analytic functions, it turns our that the column Hardy norm

$$\|f\|_{H_p^c(\varphi,\psi)} = \Big\| \Big(\int_{\mathbb{R}_+} |\varphi(sA_\psi)f|^2 \frac{ds}{s} \Big)^{1/2} \Big\|_{L_p(\widehat{\mathbf{G}})}$$

does not depend on the chosen function $\varphi \in \mathcal{J}_p$. We know from [32] that

$$L_p(\widehat{\mathbf{G}}) \subset \begin{cases} H_p^r(\varphi,\psi) + H_p^c(\varphi,\psi) & \text{if } 1 < p \leq 2, \\ H_p^r(\varphi,\psi) \cap H_p^c(\varphi,\psi) & \text{if } 2 \leq p < \infty, \end{cases}$$

for any $\varphi \in \mathcal{J} = \bigcap_{1<p<\infty} \mathcal{J}_p \neq \emptyset$ and any conditionally negative length $\psi : \mathrm{G} \to \mathbb{R}_+$.

Lemma 2.3. *Let* G *be a discrete group equipped with three arbitrary cocycles with associated length functions* ψ_1, ψ_2, ψ_3. *Let* $\varphi_1, \varphi_2 \in \mathcal{J}$ *and* $(h_s)_{s>0}$ *in* $\mathcal{H}_{\psi_3}$. *Then for any* $1 < p < \infty$,

$$\Big\| \int_{\mathbb{R}_+} \varphi_1(s\psi_1(\cdot))\varphi_2(s\psi_2(\cdot)) m_{\psi_3,h_s} \frac{ds}{s} \Big\|_{\mathrm{M}_p(\mathrm{G})} \lesssim_{c(p)} \operatorname*{ess\,sup}_{s>0} \|h_s\|_{\mathcal{H}_{\psi_3}}.$$

Proof. Let us write $L_p(\widehat{\mathbf{G}}; L_2^{rc}(\mathbb{R}_+))$ for the space $RC_p(\mathcal{L}(\mathrm{G}))$ in which we replace discrete sums over $\mathbb{Z}_+$ by integrals on $\mathbb{R}_+$ with the harmonic measure. Arguing as in Lemma 2.1, it is not difficult to show that Theorem A1 implies

$$\Big\| \int_{\mathbb{R}_+}^{\oplus} R_{\psi_3,h_s} \frac{ds}{s} : L_p(\widehat{\mathbf{G}}; L_2^{rc}(\mathbb{R}_+)) \to L_p(\widehat{\mathbf{G}}; L_2^{rc}(\mathbb{R}_+)) \Big\| \lesssim_{c(p)} \operatorname*{ess\,sup}_{s>0} \|h_s\|_{\mathcal{H}_{\psi_3}},$$

for $p \geq 2$. According to [32, Theorem 7.6], the maps

$$\begin{aligned} \Phi_1 &: L_p(\widehat{\mathbf{G}}) \ni f \mapsto (\varphi_1(\cdot\, A_{\psi_1}) f)_{s>0} \in L_p(\widehat{\mathbf{G}}; L_2^{rc}(\mathbb{R}_+)), \\ \Phi_2 &: L_{p'}(\widehat{\mathbf{G}}) \ni f \mapsto (\varphi_2(\cdot\, A_{\psi_2}) f)_{s>0} \in L_{p'}(\widehat{\mathbf{G}}; L_2^{rc}(\mathbb{R}_+)), \end{aligned}$$

will be bounded as long as A_{ψ_j} are sectorial of type $\omega \in (0,\pi)$ on $L_p(\widehat{\mathbf{G}})$ and admit a bounded $H^\infty(\Sigma_\theta)$ functional calculus for some $\theta \in (\omega_p, \pi)$ with $\omega_p = \pi|1/p - 1/2|$. Sectoriality is confirmed by [32, Theorem 5.6], whereas the $H^\infty(\Sigma_\theta)$ calculus follows from the existence of a dilation as proved in [32, Proposition 3.12]. The fact that $\mathcal{S}_{\psi,t}$ admits a dilation is a consequence of the main result in [61] (see also [37]). The combination of these arguments is explained for the Poisson semigroup on the free group in [32, Theorem 10.12]. Therefore, the result follows for $p \geq 2$ by noticing that

$$\int_{\mathbb{R}_+} \varphi_1(sA_{\psi_1})\varphi_2(sA_{\psi_2}) R_{\psi_3,h_j} \frac{ds}{s} = \Phi_2^* \circ \Big(\int_{\mathbb{R}_+}^{\oplus} R_{\psi_3,h_s} \frac{ds}{s} \Big) \circ \Phi_1.$$

The case $1 < p < 2$ then follows by self-duality exactly as in Lemma 2.2. □

The dimension free estimate in Lemma 2.3 will be useful in Section 2.3 below.

Remark 2.4. Slight modifications also give:

(i) If $p \geq 2$, then

$$\Big\| \Big(\sum_{j \geq 1} |R_{\psi,h_j} f|^2 \Big)^{1/2} \Big\|_{L_p(\widehat{\mathbf{G}})} \lesssim_{c(p)} \|(\langle h_j, h_k \rangle)\|_{\mathcal{B}(\ell_2)}^{1/2} \|f\|_{L_p(\widehat{\mathbf{G}})}.$$

(ii) If $1 < p < \infty$, then

$$\Big\| \sum_{j \geq 1} \Lambda_{\psi_1,\varphi_j} R_{\psi_3,h_j} f \otimes \delta_j \Big\|_{RC_p(\mathcal{L}(\mathrm{G}))} \lesssim_{c(p,n)} \Big(\sup_{j \geq 1} \|h_j\|_{\mathcal{H}_{\psi_3}} \Big) \|f\|_{L_p(\widehat{\mathbf{G}})}.$$

It becomes an equivalence when $\sum_j |\varphi_j|^2 \equiv 1$ *and* (h_j) *is an ONB of* $\mathcal{H}_{\psi_2}$.

(iii) If $1 < p < \infty$ and $0 < \theta < 2/p \wedge 2/p'$, then

$$\Big\| \int_{\mathbb{R}_+} \varphi_1(s\psi_1(\cdot))\varphi_2(s\psi_2(\cdot)) m_s \frac{ds}{s} \Big\|_{\mathsf{M}_p(\mathrm{G})} \lesssim_{c(p)} \operatorname*{ess\,sup}_{s>0} \|m_s\|_{[\mathrm{X}_{\psi_3}, \ell_\infty(\mathrm{G})]^\theta},$$

where $\sqrt{\psi_3(g)}\, m_s(g) = \langle h_s, b_{\psi_3}(g) \rangle_{\mathcal{H}_{\psi_3}}$ for all g and $\|m_s\|_{\mathrm{X}_{\psi_3}} = \|h_s\|_{\mathcal{H}_{\psi_3}}$.

Assertion (i) follows as in Lemma 2.1 with $\Lambda' : \ell_2(\mathbb{N}) \ni \delta_k \mapsto \sum_j \langle h_j, e_k \rangle_{\mathcal{H}_\psi} \delta_j \in \ell_2(\mathbb{N})$ instead of the map Λ used there. Note that $\Lambda'(\Lambda')^* = (\langle h_j, h_k \rangle)_{j,k}$. Moreover, by duality we also find the following inequality for $1 < p \le 2$:

$$\text{(iv)} \quad \Big\| \sum_{j \ge 1} R_{\psi, h_j} f_j \Big\|_{L_p(\widehat{\mathbf{G}})} \lesssim_{c(p)} \|(\langle h_j, h_k \rangle)^{1/2}\|_{\mathcal{B}(\ell_2)} \Big\| \Big(\sum_{j \ge 1} |f_j|^2 \Big)^{1/2} \Big\|_{L_p(\widehat{\mathbf{G}})}.$$

The same holds with the row square function on the right hand side. Indeed, applying (i) for f^* in conjunction with Remark 1.7 and duality, we obtain (iv) with row square functions and R'_{ψ,h_j} instead of R_{ψ,h_j}. However, all our results hold equally well for $R_{\psi,j}$ and $R'_{\psi,j}$, so we deduce the assertion.

Estimate (ii) follows from the argument in Lemma 2.2. According to Theorem A1 and [34, Theorem C], the equivalence holds when the sequence $|\varphi_j|^2$ forms a Littlewood–Paley partition of unity and the h_j form an ONB of $\mathcal{H}_{\psi_3}$.

Finally, (iii) is just an improvement of Lemma 2.3 by interpolation. Namely, assuming (without loss of generality) that $b_{\psi_3}(g)$ span $\mathcal{H}_{\psi_3}$ when g runs over G, it is clear that h_s is univocally determined by m_s and the norm in X_{ψ_3} is well-defined. Once this is settled, we know from Lemma 2.3 that (iii) holds for $(p,\theta) = (q,0)$ with any $q < \infty$ and also for $(2,1)$. Interpolation of the maps $L_\infty(\mathbb{R}_+; \mathrm{X}_{\psi_3}) \to \mathsf{M}_q(\mathrm{G})$ and $L_\infty(\mathbb{R}_+; \ell_\infty(\mathrm{G})) \to \mathsf{M}_2(\mathrm{G})$—the latter since $\varphi_j \in \mathcal{J} \subset L_2(\mathbb{R}_+, ds/s)$—with parameters $1/p = (1-\theta)/q + \theta/2$ yields the expected inequality. Note that

$$[L_\infty(\mathbb{R}_+; \mathrm{X}_{\psi_3}), L_\infty(\mathbb{R}_+; \ell_\infty(\mathrm{G}))]^\theta = L_\infty(\mathbb{R}_+; [\mathrm{X}_{\psi_3}, \ell_\infty(\mathrm{G})]^\theta).$$

2.2. *A refined Sobolev condition*

In order to prove Theorem B1, we start with a basic inequality for Euclidean Fourier multipliers which is apparently new. Recall the notation $\mathsf{D}_\alpha = (-\Delta)^{\alpha/2}$ and the fractional laplacian lengths $\psi_\varepsilon(\xi) = \mathrm{k}_n(\varepsilon)|\xi|^{2\varepsilon}$ from the Introduction. Here $\mathcal{H}_\varepsilon = L_2(\mathbb{R}^n, \mu_\varepsilon)$ with $d\mu_\varepsilon(x)|x|^{n+2\varepsilon} = dx$ and $(b_\varepsilon, \alpha_\varepsilon)$ stand for the corresponding cocycle map and cocycle action. The result below shows how large is the family of Riesz transforms of convolution type in $\mathbb{R}^n$ when we allow infinite-dimensional cocycles.

Lemma 2.5. *If* $1 < p < \infty$ *and* $\varepsilon > 0$, *then*

$$\|m\|_{\mathsf{M}_p(\mathbb{R}^n)} \lesssim_{c(p)} \big\| \mathsf{D}_{n/2+\varepsilon}\big(\sqrt{\psi_\varepsilon}\, m\big) \big\|_{L_2(\mathbb{R}^n)}.$$

In fact, when the right hand side is finite we have

$$m(\xi) = m_{\psi_\varepsilon, h}(\xi) = \langle h, b_\varepsilon(\xi) \rangle_{\mathcal{H}_\varepsilon} / \sqrt{\psi_\varepsilon(\xi)}$$

for some $h \in L_2^\circ(\mathbb{R}^n, \mu_\varepsilon)$ *with*

$$\|h\|_{L_2(\mathbb{R}^n, \mu_\varepsilon)} = \big\| \mathsf{D}_{n/2+\varepsilon}\big(\sqrt{\psi_\varepsilon}\, m\big) \big\|_{L_2(\mathbb{R}^n)}.$$

Moreover, the constants in the above inequality are independent of ε *and* n.

Proof. If we consider the Sobolev-type space

$$\mathsf{W}^2_{(n/2,\varepsilon)}(\mathbb{R}^n) = \big\{ m : \mathbb{R}^n \to \mathbb{C} \mid m \text{ measurable}, \ \big\|\mathsf{D}_{n/2+\varepsilon}\big(\sqrt{\psi_\varepsilon}\, m\big)\big\|_{L_2(\mathbb{R}^n)} < \infty \big\},$$

and $L_2^\circ(\mathbb{R}^n, \mu_\varepsilon)$ is the mean-zero subspace of $L_2(\mathbb{R}^n, \mu_\varepsilon)$, then we claim that

$$\Lambda : L_2^\circ(\mathbb{R}^n, \mu_\varepsilon) \ni h \mapsto m_h \in \mathsf{W}^2_{(n/2,\varepsilon)}(\mathbb{R}^n),$$
$$m_h(\xi) = \frac{1}{\sqrt{\psi_\varepsilon(\xi)}} \int_{\mathbb{R}^n} h(x)(e^{2\pi i \langle \xi, x\rangle} - 1)\, d\mu_\varepsilon(x),$$

extends to a surjective isometry. Indeed, let h be a Schwartz function in $L_2^\circ(\mathbb{R}^n, \mu_\varepsilon)$ with compact support away from 0 and write $\omega_\varepsilon(x) = 1/|x|^{n+2\varepsilon}$ for the density $d\mu_\varepsilon(x)/dx$. In that case, the function $h\omega_\varepsilon$ is a mean-zero Schwartz function in $L_2(\mathbb{R}^n)$ and

$$\sqrt{\psi_\varepsilon}\, m_h = \int_{\mathbb{R}^n} h(x) e^{2\pi i \langle \cdot, x\rangle}\, d\mu_\varepsilon(x) = (h\omega_\varepsilon)^\vee.$$

In particular, this implies that

$$\begin{aligned} \|\Lambda(h)\|_{\mathsf{W}^2_{(n/2,\varepsilon)}(\mathbb{R}^n)} &= \big\|\mathsf{D}_{n/2+\varepsilon}\big(\sqrt{\psi_\varepsilon}\, m_h\big)\big\|_{L_2(\mathbb{R}^n)} \\ &= \Big\| \frac{1}{\sqrt{\omega_\varepsilon}} (h\omega_\varepsilon)^{\vee\wedge} \Big\|_{L_2(\mathbb{R}^n)} = \|h\|_{L_2^\circ(\mathbb{R}^n, \mu_\varepsilon)}. \end{aligned}$$

Therefore, since smooth mean-zero compactly supported functions away from 0 are clearly dense in $L_2^\circ(\mathbb{R}^n, \mu_\varepsilon)$, we may extend Λ to an isometry. Now, to show surjectivity we observe from elementary facts on Sobolev spaces [1] that the class of Schwartz functions is dense in our W-space. Therefore, it suffices to show that $\Lambda^{-1}(m)$ exists for any such Schwartz function m. By Plancherel theorem, we find

$$\big\|\mathsf{D}_{n/2+\varepsilon}\big(\sqrt{\psi_\varepsilon}\, m\big)\big\|_{L_2(\mathbb{R}^n)} = \Big\| \frac{1}{\sqrt{\omega_\varepsilon}} \widehat{\sqrt{\psi_\varepsilon} m} \Big\|_{L_2(\mathbb{R}^n)} = \Big\| \frac{1}{w_\varepsilon} \widehat{\sqrt{\psi_\varepsilon} m} \Big\|_{L_2(\mathbb{R}^n, \mu_\varepsilon)}.$$

Denoting $h = \Lambda^{-1}(m) = \frac{1}{\omega_\varepsilon} \widehat{\sqrt{\psi_\varepsilon} m}$, h is also a Schwartz function. We may write

$$\begin{aligned} \sqrt{\psi_\varepsilon(\xi)} m(\xi) &= \int_{\mathbb{R}^n} h(x)\omega_\varepsilon(x) e^{2\pi i \langle \xi, x\rangle} dx \\ &= \int_{\mathbb{R}^n} h(x)(e^{2\pi i \langle \xi, x\rangle} - 1)\, d\mu_\varepsilon(x) + \int_{\mathbb{R}^n} h(x)\omega_\varepsilon(x)\, dx = \mathrm{A}(\xi) + \mathrm{B}. \end{aligned}$$

The function $\mathrm{A}(\xi)$ is well-defined since both h and $e^{2\pi i \langle \xi, x\rangle} - 1$ are elements of the Hilbert space $L_2(\mathbb{R}^n, \mu_\varepsilon)$, so its product is absolutely integrable. Moreover, since $h\omega_\varepsilon \in \mathcal{S}(\mathbb{R}^n)$ we see that $\mathrm{A}(0) = 0$ and conclude $\mathrm{B} = \sqrt{\psi_\varepsilon(0)}\, m(0) = 0$. This means that $h = \Lambda^{-1}(m)$ is a mean-zero function in $L_2(\mathbb{R}^n, \mu_\varepsilon)$, as desired. In other words, every $m \in \mathsf{W}^2_{(n/2,\varepsilon)}(\mathbb{R}^n)$ satisfies

$$m(\xi) = \frac{\langle h, e^{2\pi i \langle \xi, \cdot\rangle} - 1\rangle_{\mu_\varepsilon}}{\sqrt{\psi_\varepsilon(\xi)}} \quad \text{and} \quad \big\|\mathsf{D}_{n/2+\varepsilon}\big(\sqrt{\psi_\varepsilon} m\big)\big\|_2 = \|h\|_{L_2(\mathbb{R}^n, \mu_\varepsilon)}.$$

This shows that every m in our Sobolev-type space is a ψ_ε-Riesz transform and its norm coincides with that of its symbol h. The only drawback is that we use a complex Hilbert space for our cocycle, so the inner product is not the right one. Consider the cocycle $(\mathcal{H}_\varepsilon, \alpha_\varepsilon, b_\varepsilon)$ with

$$\begin{aligned}
\mathcal{H}_\varepsilon &= L_2(\mathbb{R}^n, \mu_\varepsilon; \mathbb{R}^2) \simeq L_2(\mathbb{R}^n, \mu_\varepsilon; \mathbb{C}),\\
b_\varepsilon(\xi) &= \big(\cos(2\pi\langle \xi, \cdot\rangle) - 1, \sin(2\pi\langle \xi, \cdot\rangle)\big) \simeq e^{2\pi i\langle \xi, \cdot\rangle} - 1,\\
\alpha_{\varepsilon,\xi}(f) &= \begin{pmatrix} \cos(2\pi\langle \xi, \cdot\rangle) & -\sin(2\pi\langle \xi, \cdot\rangle)\\ \sin(2\pi\langle \xi, \cdot\rangle) & \cos(2\pi\langle \xi, \cdot\rangle)\end{pmatrix}\begin{pmatrix} f_1\\ f_2\end{pmatrix} \simeq e^{2\pi i\langle \xi, \cdot\rangle} f.
\end{aligned}$$

Then it is easily checked that:

- If h is $\mathbb{R}$-valued and odd, then $\langle h, e^{2\pi i\langle \xi, \cdot\rangle} - 1\rangle_{\mu_\varepsilon} = i\big\langle \binom{0}{h}, b_\varepsilon(\xi)\big\rangle_{\mathcal{H}_\varepsilon}$.
- If h is $\mathbb{R}$-valued and even, then $\langle h, e^{2\pi i\langle \xi, \cdot\rangle} - 1\rangle_{\mu_\varepsilon} = \big\langle \binom{h}{0}, b_\varepsilon(\xi)\big\rangle_{\mathcal{H}_\varepsilon}$.

Therefore, decomposing

$$m = \big(\mathrm{Re}(m_{\mathrm{odd}}) + \mathrm{Re}(m_{\mathrm{even}})\big) + i\big(\mathrm{Im}(m_{\mathrm{odd}}) + \mathrm{Im}(m_{\mathrm{even}})\big)$$

and noticing the elementary inequalities

$$\|\mathrm{Re}/\mathrm{Im}(m_{\mathrm{odd/even}})\|_{\mathsf{W}^2_{(n/2,\varepsilon)}(\mathbb{R}^n)} \le \|m\|_{\mathsf{W}^2_{(n/2,\varepsilon)}(\mathbb{R}^n)}$$

we see that every element in our Sobolev-type space decomposes as a sum of four Riesz transforms whose M_p-norms are all dominated by $c(p)\|m\|_{\mathsf{W}^2_{(n/2,\varepsilon)}(\mathbb{R}^n)}$. □

Lemma 2.6. *Let* (G, ψ) *be a discrete group equipped with a conditionally negative length giving rise to an n-dimensional cocycle* $(\mathcal{H}_\psi, \alpha_\psi, b_\psi)$. *If* $1 < p < \infty$ *and* $\varepsilon > 0$, *then*

$$\|m\|_{\mathsf{M}_p(\mathrm{G})} \lesssim_{c(p)} \inf_{m = \widetilde{m}\circ b_\psi} \|\mathsf{D}_{n/2+\varepsilon}(\sqrt{\psi_\varepsilon}\,\widetilde{m})\|_{L_2(\mathbb{R}^n)}.$$

Proof. According to Lemma 2.5,

$$m(g) = \sum_{j=1}^4 \frac{\langle h_j, b_\varepsilon \circ b_\psi(g)\rangle_{\mathcal{H}_\varepsilon}}{\sqrt{\psi_\varepsilon(b_\psi(g))}} = \frac{1}{\sqrt{\psi_\varepsilon(b_\psi(g))}}\Big\langle \sum_{j=1}^4 h_j, b_\varepsilon \circ b_\psi(g)\Big\rangle_{\mathcal{H}_\varepsilon}$$

for any $\widetilde{m} \in \mathsf{W}^2_{(n/2,\varepsilon)}(\mathbb{R}^n)$ satisfying $m = \widetilde{m}\circ b_\psi$ and certain $h_j \in L_2(\mathbb{R}^n, \mu_\varepsilon)$. Next we observe that each of the four summands above can still be regarded as the Riesz transform on $\mathcal{L}(\mathrm{G})$ with respect to the following cocycle:

$$\begin{aligned}
\mathcal{H}_{\psi,\varepsilon} &= L_2(\mathbb{R}^n, \mu_\varepsilon; \mathbb{R}^2),\\
b_{\psi,\varepsilon}(g) &= \big(\cos(2\pi\langle b_\psi(g), \cdot\rangle) - 1, \sin(2\pi\langle b_\psi(g), \cdot\rangle)\big),\\
\alpha_{\psi,\varepsilon,g}(f) &= \begin{pmatrix} \cos(2\pi\langle b_\psi(g), \cdot\rangle) & -\sin(2\pi\langle b_\psi(g), \cdot\rangle)\\ \sin(2\pi\langle b_\psi(g), \cdot\rangle) & \cos(2\pi\langle b_\psi(g), \cdot\rangle)\end{pmatrix}\begin{pmatrix} f_1 \circ \alpha_{\psi,g^{-1}}(\cdot)\\ f_2 \circ \alpha_{\psi,g^{-1}}(\cdot)\end{pmatrix}.
\end{aligned}$$

We conclude by noticing that

$$\|m\|_{\mathsf{M}_p(\mathrm{G})} \le c(p) \Big\| \sum_{j=1}^{4} h_j \Big\|_{\mathcal{H}_\varepsilon} \lesssim c(p) \|\widetilde{m}\|_{\mathsf{W}^2_{(n/2,\varepsilon)}(\mathbb{R}^n)}.$$

Namely, since $\psi_\varepsilon \circ b_\psi$ is a conditionally negative length (associated to the cocycle above), the first inequality follows from Lemma 2.1 for $j = 1$ when $p \ge 2$ and from Remark 2.4(iv) when $1 < p < 2$. On the other hand, the second inequality follows from Lemma 2.5. □

Proof of Theorem B1. Let $\mathrm{G}_{\psi,0} = \{g \in \mathrm{G} \mid \psi(g) = 0\}$ denote the subgroup of elements with vanishing ψ-length, which trivializes for injective cocycles. According to $m = \widetilde{m} \circ b_\psi$, the multiplier m is constant on $\mathrm{G}_{\psi,0}$ and takes the value $m(e)$. This means that the Fourier multiplier associated to $m_0 = m(e) 1_{\mathrm{G}_{\psi,0}}$ is nothing but $m(e)$ times the conditional expectation onto $\mathcal{L}(\mathrm{G}_{\psi,0})$, so

$$\|m\|_{\mathsf{M}_p(\mathrm{G})} \le |m(e)| + \|m - m_0\|_{\mathsf{M}_p(\mathrm{G})}.$$

Since $m - m_0 = (\widetilde{m} - m(e)\delta_0) \circ b_\psi$ and $\widetilde{m} = \widetilde{m} - m(e)\delta_0$ almost everywhere, we may just proceed by assuming that $m(g) = 0$ for all $g \in \mathrm{G}_{\psi,0}$. Let η be a radially decreasing smooth function on $\mathbb{R}$ with $1_{(-1,1)} \le \eta \le 1_{(-2,2)}$ and set $\phi(\xi) = \eta(\xi) - \eta(2\xi)$, so that we may construct a standard Littlewood–Paley partition of unity $\phi_j(\xi) = \phi(2^{-j}\xi)$ for $j \in \mathbb{Z}$. Note that

$$\sum_{j\in\mathbb{Z}} \phi_j(\xi) = \begin{cases} 1 & \text{if } \xi \neq 0, \\ 0 & \text{if } \xi = 0. \end{cases}$$

Since $\operatorname{supp}\phi_j \cap \operatorname{supp}\phi_k = \emptyset$ for $|j-k| \ge 2$, we see that $\rho_j := \frac{1}{3}(\phi_{j-1} + \phi_j + \phi_{j+1})$ forms a partition of unity satisfying $\rho_j \equiv 1/3$ on the support of ϕ_j. We shall be working with the radial Littlewood–Paley partition of unity in $\mathbb{R}^n$ given by $\varphi_j = \rho_j \circ |\ |^2$. On the other hand, if we set $\varphi_{1j} = \varphi_{2j} = \sqrt{\phi_j}$ we may write m as follows (recall that m vanishes where ψ does):

$$\begin{aligned} m &= \sum_{j\in\mathbb{Z}} (\phi_j \circ \psi) m = 3 \sum_{j\in\mathbb{Z}} (\phi_j \circ \psi)(\rho_j \circ \psi) m \\ &= 3 \sum_{j\in\mathbb{Z}} (\varphi_{1j} \circ \psi)(\varphi_{2j} \circ \psi)(\rho_j \circ \psi) m = 3 \sum_{j\in\mathbb{Z}} (\varphi_{1j} \circ \psi)(\varphi_{2j} \circ \psi) m_j. \end{aligned}$$

Since

$$m_j(g) = \rho_j(\psi(g)) m(g) = \rho_j(|b_\psi(g)|^2) \widetilde{m}(b_\psi(g)) = (\varphi_j \widetilde{m})(b_\psi(g)),$$

we deduce that $m_j = \widetilde{m}_j \circ b_\psi$ with $\widetilde{m}_j = \varphi_j \widetilde{m}$. We know by assumption that

$$\sup_{j\in\mathbb{Z}} \big\| \mathsf{D}_{n/2+\varepsilon}\big(\sqrt{\psi_\varepsilon}\, \varphi_j \widetilde{m}\big) \big\|_2 < \infty.$$

According to Lemma 2.5 and the proof of Lemma 2.6, this implies that we may write m_j as a Riesz transform m_{γ,h_j} with respect to the length function $\psi_\varepsilon \circ b_\psi$, whose associated infinite-dimensional cocycle was also described in the proof of Lemma 2.6. Since the families $\varphi_{1j}, \varphi_{2j}$ satisfy the assumptions of Lemma 2.2, we may combine Lemma 2.2 with Lemma 2.5 to obtain

$$\begin{aligned}\|m\|_{\mathsf{M}_p(\mathrm{G})} &= \Big\| \sum_{j\in\mathbb{Z}} \varphi_{1j}(\psi_1(\cdot))\varphi_{2j}(\psi_2(\cdot)) m_{\gamma,h_j} \Big\|_{\mathsf{M}_p(\mathrm{G})} \\ &\leq c(p,n) \sup_{j\in\mathbb{Z}} \|h_j\|_{\mathcal{H}_\gamma} \ \leq \ c(p,n) \sup_{j\in\mathbb{Z}} \big\| \mathsf{D}_{n/2+\varepsilon}\big(\sqrt{\psi_\varepsilon}\varphi_j \widetilde{m}\big)\big\|_2.\end{aligned}$$

The dependence on n of $c(p,n)$ comes from the Littlewood–Paley inequalities. □

In the following result, we use the standard notation

$$\begin{aligned}\mathsf{H}^2_\alpha(\Omega) &= \{ f \text{ supported by } \Omega \mid \|(1+|\ |^2)^{\alpha/2}\widehat{f}\|_{L_2(\mathbb{R}^n)} < \infty \}, \\ \mathring{\mathsf{H}}^2_\alpha(\Omega) &= \Big\{ f \text{ supported by } \Omega \ \Big| \ \|\mathsf{D}_\alpha f\|_{L_2(\mathbb{R}^n)} = \big\| |\ |^\alpha \widehat{f} \big\|_{L_2(\mathbb{R}^n)} < \infty \Big\}.\end{aligned}$$

Corollary 2.7. *If* $1 < p < \infty$ *and* $0 < \varepsilon < [n/2] + 1 - n/2$*, then*

$$\|m\|_{\mathsf{M}_p(\mathrm{G})} \lesssim_{c(p,n,\varepsilon)} |m(e)| + \inf_{m=\widetilde{m}\circ b_\psi} \Big\{ \sup_{j\in\mathbb{Z}} \|\varphi_0 \widetilde{m}(2^j\ \cdot)\|_{\mathsf{H}^2_{n/2+\varepsilon}(\mathbb{R}^n)} \Big\}$$

for any pair (G,ψ) *which gives rise to an n-dimensional cocycle* $(\mathcal{H}_\psi, \alpha_\psi, b_\psi)$.

Proof. We have

$$\begin{aligned}\|\mathsf{D}_{n/2+\varepsilon}(\sqrt{\psi_\varepsilon}\,\varphi_j\widetilde{m})\|_2 &= \sqrt{\mathsf{k}_n(\varepsilon)}\, \|\mathsf{D}_{n/2+\varepsilon}(|\ |^\varepsilon \varphi_j \widetilde{m})\|_2 \\ &= \sqrt{\mathsf{k}_n(\varepsilon)}\, \|\mathsf{D}_{n/2+\varepsilon}(|\ |^\varepsilon \varphi_0 \widetilde{m}(2^j\ \cdot))\|_2\end{aligned}$$

with $\mathsf{k}_n(\varepsilon) \sim \frac{\pi^{n/2}}{\Gamma(n/2)}\frac{1}{\varepsilon}$. Indeed, it is easy to check that $\mathsf{W}^2_{(n/2,\varepsilon)}(\mathbb{R}^n)$ has a dilation invariant norm. It therefore suffices to show that, up to a constant $c(n,\varepsilon)$,

$$\|\mathsf{D}_{n/2+\varepsilon}(|\ |^\varepsilon f)\|_2 \lesssim \|(1+|\ |^2)^{(n/2+\varepsilon)/2}\widehat{f}\|_2$$

for functions supported by (say) the corona $\Omega = \mathrm{B}_2(0) \setminus \mathrm{B}_1(0)$, which is the form of the support of φ_0. In other words, we need to show that

$$\Lambda_\varepsilon : \mathsf{H}^2_{n/2+\varepsilon}(\Omega) \ni f \mapsto |\ |^\varepsilon f \in \mathring{\mathsf{H}}^2_{n/2+\varepsilon}(\Omega)$$

defines a bounded operator. These two families of spaces satisfy the expected interpolation identities in the variable $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$ with respect to the complex method. To prove that Λ_ε is bounded we use Stein's interpolation. Assume by homogeneity that f is in the unit ball of $\mathsf{H}^2_{n/2+\varepsilon}(\Omega)$ and let $\theta = 2\varepsilon$ for n odd and $\theta = \varepsilon$ for n even. Thus, there exists F analytic on the strip satisfying $F(\theta) = f$ and

$$\max\Big\{ \sup_{t\in\mathbb{R}} \|F(it)\|_{\mathsf{H}^2_{n/2}(\Omega)}, \sup_{t\in\mathbb{R}} \|F(1+it)\|_{\mathsf{H}^2_{[n/2]+1}(\Omega)} \Big\} \leq 1.$$

Given $0 < \delta \leq 1$, define the analytic family of operators

$$\mathcal{L}_z(F) = \exp(\delta(z-\theta)^2)|\ |^{\varepsilon z/\theta} F(z),$$

so that $\mathcal{L}_\theta(F) = \Lambda_\varepsilon(f)$. We claim that

$$\begin{aligned} \|\mathcal{L}_{it}(F)\|_{\mathring{\mathsf{H}}^2_{n/2}}(\Omega) &\leq c(n)(1+|t|)^{n/2} e^{-\delta t^2} e^{\delta\theta^2}, \\ \|\mathcal{L}_{1+it}(F)\|_{\mathring{\mathsf{H}}^2_{[n/2]+1}}(\Omega) &\leq c(n)(1+|t|)^{[n/2]+1} e^{-\delta t^2} e^{\delta(1-\theta)^2}. \end{aligned}$$

Then the trivial estimate $e^{-\delta t^2}(1+|t|)^\beta \lesssim \|e^{-t^2}(\sqrt{\delta}+|t|)^\beta\|_\infty \delta^{-\beta/2}$ in conjunction with the three lines lemma imply the statement of the corollary with the constant

$$c(n)\sqrt{\mathrm{k}_n(\varepsilon)}\frac{e^{\delta((1-\theta)\theta^2+\theta(1-\theta)^2)}}{\delta^{n/4+\varepsilon/2}}.$$

If $\alpha = \frac{\varepsilon}{\theta}(1+it)$ and $u_\alpha(\xi) = |\xi|^\alpha$, our second claim follows from the simple inequality

$$\begin{aligned} &\|\mathcal{L}_{1+it}(F)\|_{\mathring{\mathsf{H}}^2_{[n/2]+1}}(\Omega) \\ &\qquad \lesssim c(n)|e^{\delta(1+it-\theta)^2}| \sup_{0\leq|\beta|\leq[n/2]+1} \|\partial_\beta u_\alpha\|_{L_\infty(\Omega)} \|F(1+it)\|_{\mathsf{H}^2_{[n/2]+1}(\Omega)}. \end{aligned}$$

Indeed, since $[n/2]+1 \in \mathbb{Z}$, the Sobolev norm above can be computed using ordinary derivatives and the given estimate arises from the Leibniz rule and the triangle inequality. A similar argument shows that the map $f \mapsto |\ |^{it} f$ is contractive on $L_2(\Omega)$ and bounded on $\mathsf{H}^2_{[n/2]+1}(\Omega)$ up to a constant $c(n)(1+|t|)^{[n/2]+1}$. By complex interpolation,

$$\big\||\ |^{it} f\big\|_{\mathring{\mathsf{H}}^2_{n/2}}(\Omega) \leq \big\||\ |^{it} f\big\|_{\mathsf{H}^2_{n/2}(\Omega)} \leq c(n)(1+|t|)^{n/2}\|f\|_{\mathsf{H}^2_{n/2}(\Omega)}.$$

Since $\varepsilon/\theta \in \{1/2, 1\}$, our first claim follows by taking $f = F(it)$. □

Remark 2.8. On the other hand, a quick look at the constant we obtain in the proof of Corollary 2.7 shows that our Sobolev-type norm is more appropriate than the classical one in its dimensional behavior. Namely, it is easily checked that the constant $c(n)$ above grows linearly with n since it arises from applying the Leibniz rule $[n/2]+1$ times. In particular, we obtain a constant

$$c(n)\sqrt{\mathrm{k}_n(\varepsilon)} \sim \frac{n\pi^{n/4}}{\sqrt{\Gamma(n/2)}}\frac{1}{\sqrt{\varepsilon}}$$

which decreases to 0 very fast with n. We pay a price for small ε though. This could be rephrased by saying that our Sobolev-type norm is "dimension free" (see Section 2.3 below) and encodes the dependence on $\varepsilon > 0$. Indeed, the constant $c(p,n)$ in Theorem B1 is independent of ε.

Remark 2.9. The Coifman–Rubio de Francia–Semmes theorem [16] shows that functions in $\mathbb{R}$ of bounded 2-variation define L_p-bounded Fourier multipliers for $1 < p < \infty$. In this 1-dimensional setting it can be proved that our abstract Sobolev condition implies bounded 2-variation. In summary, if we set $\mathsf{HM}_{\mathbb{R}}$ for the class of Hörmander–Mikhlin multipliers in $\mathbb{R}$, ψ-$\mathsf{Riesz}_{\mathbb{R}}$ for the multipliers in $\mathbb{R}$ satisfying the hypotheses of Theorem B1, and $\mathsf{CRS}_{\mathbb{R}}$ for the Coifman–Rubio de Francia–Semmes class, Corollary 2.7 and the comment above yield

$$\mathsf{HM}_{\mathbb{R}} \subset \psi\text{-}\mathsf{Riesz}_{\mathbb{R}} \subset \mathsf{CRS}_{\mathbb{R}}.$$

In higher dimensions, Xu extended the notion of q-variation to generalize the CRS theorem [73]. Although we do not know how to compare our condition in Theorem B1 with Xu's, ours seems easier to check in many cases.

Remark 2.10. Corollary 2.7 implies, for $m = \widetilde{m} \circ b_\psi$,

$$\|m\|_{\mathsf{M}_p(\mathrm{G})} \lesssim_{c(p,n)} |m(e)| + \sup_{0 \le |\beta| \le [n/2]+1} \big\| |\ |^{|\beta|} \partial_\beta \widetilde{m} \big\|_\infty.$$

Indeed, this follows from the well-known relation between Mikhlin smoothness and Sobolev–Hörmander smoothness (see e.g. [23, Chapter 8]). This already improves the main result in [34] for $1 < p < \infty$. In the case of unimodular groups whose Haar measure does not have an atom at e, the term $|m(e)|$ can be removed from Theorem B1, Corollary 2.7 and the inequality above (see Appendix A).

2.3. A dimension free formulation

A quick look at our argument for Theorem B1 shows that the only dependence of the constant we get on the dimension of the cocycle comes from the use of our Littlewood–Paley inequalities in Lemma 2.2. The proof of Theorem B2 just requires replacing that result by Lemma 2.3.

Proof of Theorem B2. As in the statement, let ψ be a conditionally negative length whose associated cocycle $(\mathcal{H}_\psi, \alpha_\psi, b_\psi)$ is finite-dimensional; let $\widetilde{m}$ be a lifting multiplier for that cocycle, so that $m = \widetilde{m} \circ b_\psi$; and let $\varphi : \mathbb{R}_+ \to \mathbb{C}$ be an analytic function in the class $\mathcal{J}$ considered in Section 2.1. Arguing as in Theorem B1 we may assume with no loss of generality that m vanishes where ψ does. In other words, certain noninjective cocycles may be used as long as m is constant on a nontrivial subgroup of G. Since ds/s is dilation invariant,

$$m(g) = \mathrm{k}_\varphi \int_{\mathbb{R}_+} \varphi(s\psi(g))^3 m(g) \, \frac{ds}{s} = \mathrm{k}_\varphi \int_{\mathbb{R}_+} \varphi(s\psi(g))^2 m_s(g) \, \frac{ds}{s}$$

for some constant $\mathrm{k}_\varphi \neq 0$. Note that

$$m_s(g) = \varphi(s\psi(g)) m(g) \ \Rightarrow \ m_s = \widetilde{m}_s \circ b_\psi \text{ with } \widetilde{m}_s = \varphi(s|\cdot|^2)\widetilde{m}.$$

Then we proceed again as in Theorem B1. Indeed, we know by assumption that

$$\operatorname*{ess\,sup}_{s>0} \big\| \mathsf{D}_{(\dim \mathcal{H}_\psi)/2+\varepsilon} \big(\sqrt{\psi_\varepsilon} \, \widetilde{m}_s \big) \big\|_2 < \infty.$$

According to the proof of Lemma 2.6, this implies that we may write m_s as a Riesz transform m_{γ,h_s} with respect to the length function $\gamma = \psi_\varepsilon \circ b_\psi$ for almost every $s > 0$. Since the families $\varphi_1, \varphi_2 = \varphi$ satisfy the assumptions of Lemma 2.3, we may combine this result with Lemma 2.5 to obtain

$$\begin{aligned}\|m\|_{\mathsf{M}_p(\mathrm{G})} &= \mathrm{k}_\varphi \Big\| \int_{\mathbb{R}_+} \varphi(s\psi(\cdot))^2 m_{\gamma,h_s} \frac{ds}{s} \Big\|_{\mathsf{M}_p(\mathrm{G})} \\ &\lesssim c(p) \operatorname*{ess\,sup}_{s>0} \|h_s\|_{\mathcal{H}_\gamma} \le c(p) \operatorname*{ess\,sup}_{s>0} \big\| \mathsf{D}_{(\dim \mathcal{H}_\psi)/2+\varepsilon} \big(\sqrt{\psi_\varepsilon}\, \varphi(s|\cdot|^2)\widetilde{m}\big)\big\|_2. \qquad \square\end{aligned}$$

Remark 2.11. Lemma 2.5 admits generalizations for any conditionally negative length $\ell : \mathbb{R}^n \to \mathbb{R}_+$ whose associated measure ν is absolutely continuous with respect to the Lebesgue measure and such that $1 - \cos(2\pi\langle \xi, \cdot\rangle) \in L_1(\mathbb{R}^n, \nu)$ for all $\xi \in \mathbb{R}^n$. Indeed, if we set $d\nu(x) = u(x)dx$ we also have

$$\|m\|_{\mathsf{M}_p(\mathbb{R}^n)} \lesssim_{c(p)} \Big\| \frac{1}{\sqrt{u}} \widehat{\sqrt{\ell}\, m} \Big\|_{L_2(\mathbb{R}^n)}$$

as long as the space determined by the right hand side admits a dense subspace of functions m for which $\sqrt{\ell}\, m$ satisfies the Fourier inversion theorem. The Schwartz class was enough for our choice $(\ell, \nu) = (\psi_\varepsilon, \mu_\varepsilon)$. Lemma 2.6 can also be extended when ν is invariant under the action α_ψ of the chosen finite-dimensional cocycle $(\mathcal{H}_\psi, \alpha_\psi, b_\psi)$. This invariance is necessary to make sure that the construction in the proof of Lemma 2.6 yields a well-defined cocycle out of ℓ and ψ.

Remark 2.12. Although the above mentioned applications are of independent interest, it is perhaps more significant to read our approach as a way to relate certain kernel reproducing formulas to some differential operators/Sobolev norms in von Neumann algebras. Namely, consider any length

$$\psi(g) = \tau_\psi\big((2\lambda(e) - \lambda(g) - \lambda(g^{-1}))\big) \quad \text{with} \quad \tau_\psi(f) = \tau(f\omega_\psi)$$

for some positive invertible density ω_ψ, and construct the spaces

$$\begin{aligned} L_2(\widehat{\mathbf{G}}, \tau_\psi) &= \{ h \in L_0(\widehat{\mathbf{G}}) \mid \|h\|_{2,\psi} = \tau_\psi(|h|^2)^{1/2} < \infty \}, \\ \mathsf{W}^2_\psi(\widehat{\mathbf{G}}, \tau) &= \Big\{ m \in \ell_\infty(\mathrm{G}) \ \Big| \ \|m\|_{\mathsf{W},\psi} = \Big\| \lambda\big(\sqrt{\psi}\, m\big) \frac{1}{\sqrt{\omega_\psi}} \Big\|_{L_2(\widehat{\mathbf{G}})} < \infty \Big\}. \end{aligned}$$

Here λ stands for the left regular representation, playing the role of the Fourier transform. This is the analogue of what we do in the Euclidean case. Then, there exists a linear isometry

$$\Lambda_\psi : L_2^\circ(\widehat{\mathbf{G}}, \tau_\psi) \ni h \mapsto \frac{\tau_\psi(b_\psi(\cdot)h)}{\sqrt{\psi(\cdot)}} \in \mathsf{W}^2_\psi(\widehat{\mathbf{G}}, \tau),$$

where $L_2^\circ(\widehat{\mathbf{G}}, \tau_\psi)$ denotes the subspace of mean-zero elements. The surjectivity of Λ_ψ depends as above on the existence of a dense subspace of our Sobolev-type space admitting Fourier inversion in $\mathcal{L}(\mathrm{G})$ (in fact, this can be used in the opposite direction to identify nice Schwartz-type classes in group algebras). Whenever the map Λ_ψ is surjective, it relates the "Sobolev norm" of a Riesz transform to the L_2-norm of its symbol, which in turn dominates its multiplier norm up to $c(p)$. We have not explored applications in this general setting.

Remark 2.13. Using Laplace transforms in the spirit of Stein [68], we can prove Littlewood–Paley estimates in discrete time and also L_p bounds for smooth Fourier multipliers even for infinite-dimensional cocycles.

2.4. Limiting Besov-type conditions

Let η be a radially decreasing smooth function on $\mathbb{R}^n$ with $1_{\mathrm{B}_1(0)} \le \eta \le 1_{\mathrm{B}_2(0)}$ and set $\varphi(\xi) = \eta(\xi) - \eta(2\xi)$, so that me may construct a standard Littlewood–Paley partition of unity $\varphi_k(\xi) = \varphi(2^{-k}\xi)$ for $k \in \mathbb{Z}$. Consider the function $\phi = 1 - \sum_{j\ge 1}\varphi_j$ and let $\alpha \in \mathbb{R}$ and $1 \le p, q \le \infty$. Then the Besov space $B^p_{\alpha q}(\mathbb{R}^n)$ and its homogeneous analogue are defined as subspaces of tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ in the following way:

$$\mathring{B}^p_{\alpha q}(\mathbb{R}^n) = \Big\{ f \in \mathcal{S}'(\mathbb{R}^n) \ \Big| \ |||f|||^p_{\alpha q} = \Big(\sum_{k\in\mathbb{Z}} 2^{kq\alpha} \|\widehat{\varphi}_k * f\|_p^q \Big)^{1/q} < \infty \Big\},$$

$$B^p_{\alpha q}(\mathbb{R}^n) = \Big\{ f \in \mathcal{S}'(\mathbb{R}^n) \ \Big| \ \|f\|^p_{\alpha q} = \|\widehat{\phi} * f\|_p + \Big(\sum_{k\ge 1} 2^{kq\alpha} \|\widehat{\varphi}_k * f\|_p^q \Big)^{1/q} < \infty \Big\}.$$

Note that $\| \ \|^p_{\alpha q}$ is a norm while $||| \ |||^p_{\alpha q}$ is a seminorm. Besov spaces refine Sobolev spaces in an obvious way. For instance, it is straightforward to show that

$$B^2_{\alpha 2}(\mathbb{R}^n) \simeq \mathsf{H}^2_\alpha(\mathbb{R}^n) \quad \text{and} \quad \mathring{B}^2_{\alpha 2}(\mathbb{R}^n) \simeq \mathring{\mathsf{H}}^2_\alpha(\mathbb{R}^n)$$

with constants depending on the dimension n. It is a very natural question to study how we can modify the Sobolev $(n/2+\varepsilon)$-condition in the Hörmander–Mikhlin theorem when ε approaches 0. This problem has been studied notably by Seeger [66, 67] (see also [11, 44, 65] and the references therein). In terms of L_p-bounded Fourier multipliers for $1 < p < \infty$ the best known result is

$$\|m\|_{\mathsf{M}_p(\mathbb{R}^n)} \lesssim_{c(p,n)} \sup_{j\in\mathbb{Z}} \|\varphi_0 m(2^j \cdot)\|^2_{n/2,1},$$

where $\varphi_0(\xi) = \varphi(\xi) = \eta(\xi) - \eta(2\xi)$ as defined above. Of course, we cannot expect to replace the Besov space on the right hand side by the bigger one $B^2_{n/2,2}(\mathbb{R}^n)$ or its homogeneous analogue

$$\begin{aligned} \sup_{j\in\mathbb{Z}} |||\varphi_j m|||^2_{n/2,2} &= \sup_{j\in\mathbb{Z}} \Big(\sum_{k\in\mathbb{Z}} 2^{nk} \|\widehat{\varphi}_k * (\varphi_j m)\|_2^2 \Big)^{1/2} \\ &= \sup_{j\in\mathbb{Z}} \Big(\sum_{k\in\mathbb{Z}} 2^{nk} \|\widehat{\varphi}_k * (\varphi_0 m(2^j \cdot))\|_2^2 \Big)^{1/2} \end{aligned}$$

which holds by the dilation invariance of the norm we use.

In the following theorem we show that a log-weighted form of this space sits in $\mathsf{M}_p(\mathbb{R}^n)$. Similar results for Euclidean multipliers were already found by Baernstein/Sawyer [2], the main difference being that they impose a more demanding ℓ_1-Besov condition. Our argument is also very different. We formulate our result in the context of discrete group von Neumann algebras, although it also holds for arbitrary unimodular groups. The main idea is to replace the measures $d\mu_\varepsilon(x) = \omega_\varepsilon(x)dx$ with $\omega_\varepsilon(x) = |x|^{-(n+2\varepsilon)}$, used to prove Theorem B1, by the limiting measure $d\nu(x) = u(x)dx$ with

$$u(x) = \frac{1}{|x|^n}\Big(1_{\mathrm{B}_1(0)}(x) + \frac{1}{1+\log^2|x|}1_{\mathbb{R}^n\setminus\mathrm{B}_1(0)}(x)\Big).$$

Let us also consider the associated length

$$\ell(\xi) = 2\int_{\mathbb{R}^n}(1-\cos(2\pi\langle\xi,x\rangle))u(x)\,dx.$$

After Theorem 2.15 we give more examples and a comparison with Seeger's results.

Lemma 2.14. *The length above satisfies*

$$\ell(\xi) \sim \frac{1}{1+\big|\log|\xi|\big|}1_{\mathrm{B}_1(0)}(\xi) + \big(1+\big|\log|\xi|\big|\big)1_{\mathbb{R}^n\setminus\mathrm{B}_1(0)}(\xi).$$

Sketch of the proof. We have

$$\begin{aligned}\frac{\ell(\xi)}{2} &= \int_{\mathrm{B}_1(0)}(1-\cos(2\pi\langle x,\xi\rangle))\frac{dx}{|x|^n} + \int_{\mathbb{R}^n\setminus\mathrm{B}_1(0)}\frac{1-\cos(2\pi\langle x,\xi\rangle)}{1+\log^2|x|}\frac{dx}{|x|^n}\\ &= A(\xi)+B(\xi).\end{aligned}$$

By dilation invariance of $|x|^{-n}dx$, we can write

$$A(\xi) = \int_{\mathrm{B}_{|\xi|}(0)}\Big(1-\cos\Big(2\pi\Big\langle x,\frac{\xi}{|\xi|}\Big\rangle\Big)\Big)\frac{dx}{|x|^n}.$$

By symmetry the direction of ξ is irrelevant and using polar coordinates we find

- If $|\xi| < 1/2$, then
$$A(\xi) \sim \int_{\mathrm{B}_{|\xi|}(0)}\frac{|\langle x,e_1\rangle|^2}{|x|^n}\,dx \sim c(n)|\xi|^2.$$
- If $|\xi| \geq 1/2$, then
$$A(\xi) \sim \int_{\mathrm{B}_{1/2}(0)}\frac{|\langle x,e_1\rangle|^2}{|x|^n}\,dx + \int_{\mathrm{B}_{|\xi|}(0)\setminus\mathrm{B}_{1/2}(0)}\frac{dx}{|x|^n} \sim c(n)\big(1+\big|\log|\xi|\big|\big).$$

On the other hand, using spherical symmetry we may also write

$$B(\xi) = \int_{\mathbb{R}^n\setminus\mathrm{B}_1(0)}\frac{1-\cos(2\pi\langle x|\xi|,e_1\rangle)}{1+\log^2|x|}\frac{dx}{|x|^n}.$$

- When $|\xi| < 1/2$, we find
$$B(\xi) \sim \int_{\mathbb{R}^n \setminus \mathrm{B}_{1/(2|\xi|)}(0)} \frac{1}{1+\log^2|x|} \frac{dx}{|x|^n} + \int_{\mathrm{B}_{1/(2|\xi|)}(0) \setminus \mathrm{B}_1(0)} \frac{|\xi|^2 |\langle x, e_1 \rangle|^2}{1+\log^2|x|} \frac{dx}{|x|^n}.$$
By using polar coordinates it is easy to see that $B(\xi) \sim c(n)/\big(1+\big|\log|\xi|\big|\big)$.
- When $|\xi| \geq 1/2$ we just get
$$B(\xi) \sim \int_{\mathbb{R}^n \setminus \mathrm{B}_1(0)} \frac{dx}{(1+\log^2|x|)|x|^n} \sim c(n).$$

Combining the estimates above we get the equivalence in the statement with $\mathrm{B}_1(0)$ replaced by $\mathrm{B}_{1/2}(0)$. However, since $1+\big|\log|\xi|\big| \sim 1$ when $|\xi| \in [1/2, 1]$, this is equivalent to the right hand side in the statement. □

Theorem 2.15. *Let* (G, ψ) *be a discrete group with a conditionally negative length giving rise to an* n*-dimensional cocycle* $(\mathcal{H}_\psi, \alpha_\psi, b_\psi)$. *Let* $(\varphi_j)_{j \in \mathbb{Z}}$ *denote a standard radial Littlewood–Paley partition of unity in* $\mathbb{R}^n$. *If* $1 < p < \infty$
$$\begin{aligned}\|m\|_{\mathrm{M}_p(\mathrm{G})} &\lesssim_{c(p,n)} |m(e)| + \inf_{m=\widetilde{m}\circ b_\psi} \Big\{ \sup_{j\in\mathbb{Z}} \Big\| \frac{1}{\sqrt{u}} \big(\sqrt{\ell}\varphi_j \widetilde{m}\big)^\wedge \Big\|_{L_2(\mathbb{R}^n)} \Big\} \\ &\sim_{c(p,n)} |m(e)| + \inf_{m=\widetilde{m}\circ b_\psi} \Big\{ \sup_{j\in\mathbb{Z}} \Big(\sum_{k\in\mathbb{Z}} 2^{nk} \mathrm{w}_k \big\| \widehat{\varphi}_k * \big(\sqrt{\ell}\,\varphi_j \widetilde{m}\big) \big\|_2^2 \Big)^{1/2} \Big\},\end{aligned}$$
where u, ℓ *are as above and the weights* w_k *are of the form* $\delta_{k\leq 0} + k^2 \delta_{k>0}$ *for* $k \in \mathbb{Z}$.

Proof. Arguing as in Theorem B1 we may assume with no loss of generality that m vanishes where ψ does, so that $m(e) = 0$. According to Remark 2.11 and Lemma 2.14, Lemmas 2.5 and 2.6 apply in our setting (ℓ, ν, u). Moreover, according to our argument for Theorem B1, we find
$$m(g) = 3 \sum_{j\in\mathbb{Z}} \phi_j(\psi(g))(\varphi_j \widetilde{m})(b_\psi(g))$$
for a certain smooth partition of unity ϕ_j. If
$$\sup_{j\in\mathbb{Z}} \Big\| \frac{1}{\sqrt{u}} \big(\sqrt{\ell}\,\varphi_j \widetilde{m}\big)^\wedge \Big\|_2 < \infty,$$
the (generalized) proofs of Lemmas 2.5 and 2.6 imply that $m_j = (\varphi_j \widetilde{m}) \circ b_\psi$ is a Riesz transform m_{ζ, h_j} with respect to the length function $\zeta = \ell \circ b_\psi$, which comes from a composite cocycle as in Lemma 2.6 thanks to the orthogonal invariance of ν (radial density). Since the families $\varphi_{1j} = \varphi_{2j} = \phi_j$ satisfy the assumptions of Lemma 2.2, we combine this with Lemma 2.5 to obtain
$$\begin{aligned}\|m\|_{\mathrm{M}_p(\mathrm{G})} &= \Big\| \sum_{j\in\mathbb{Z}} \varphi_{1j}(\psi(\cdot))\varphi_{2j}(\psi(\cdot)) m_{\zeta, h_j} \Big\|_{\mathrm{M}_p(\mathrm{G})} \\ &\leq c(p,n) \sup_{j\in\mathbb{Z}} \|h_j\|_{\mathcal{H}_\zeta} \leq c(p,n) \sup_{j\in\mathbb{Z}} \Big\| \frac{1}{\sqrt{u}} \big(\sqrt{\ell}\,\varphi_j \widetilde{m}\big)^\wedge \Big\|_2.\end{aligned}$$

This proves the first estimate of the theorem. The second follows since

$$\Big\| \frac{1}{\sqrt{u}} \widehat{f} \Big\|_2^2 = \sum_{k\in\mathbb{Z}} \Big\| \frac{1}{\sqrt{u}} \varphi_k \widehat{f} \Big\|_2^2 \sim \sum_{k\in\mathbb{Z}} \frac{1}{u(2^k)} \|\widehat{\varphi}_k * f\|_2^2 \sim \sum_{k\in\mathbb{Z}} 2^{nk} \mathrm{w}_k \|\widehat{\varphi}_k * f\|_2^2. \qquad \square$$

Remark 2.16. According to Remark 2.11, other limiting measures ν apply as long as we know that $1 - \cos(2\pi\langle \xi, \cdot\rangle)$ belongs to $L_1(\mathbb{R}^n, \nu)$ and the measure ν is invariant under the cocycle action α_ψ. In particular, any radial measure $d\nu(x) = u(x)\, dx$ such that $u(s)(s^2 1_{(0,1)}(s) + 1_{(1,\infty)}(s)) \in L_1(\mathbb{R}_+, ds)$ satisfies these conditions. Note that any such measure will provide an associated length of polynomial growth, so that the associated Sobolev-type space has the Schwartz class as a dense subspace satisfying the Fourier inversion formula, as demanded by Remark 2.11. If fact, it is conceivable that for slow-increasing lengths ℓ,

$$\big\|\widehat{\varphi}_k * \big(\sqrt{\ell}\, \varphi_j \widetilde{m}\big)\big\|_2^2 \sim \ell(2^j) \|\widehat{\varphi}_k * (\varphi_j \widetilde{m})\|_2^2.$$

Therefore, under this assumption we would finally get

$$\|m\|_{\mathsf{M}_p(\mathrm{G})} \lesssim_{c(p,n)} |m(e)| + \inf_{m = \widetilde{m} \circ b_\psi} \Big\{ \sup_{j\in\mathbb{Z}} \Big(\sum_{k\in\mathbb{Z}} 2^{nk} \frac{\ell(2^j)}{u(2^k)} \|\widehat{\varphi}_k * (\varphi_j \widetilde{m})\|_2^2 \Big)^{1/2} \Big\}.$$

Remark 2.17. The Besov space $B^2_{n/2,1}(\mathbb{R}^n)$ used in Seeger's result is not dilation invariant. Thus, in order to compare his estimates with ours, we first need to dilate from $\varphi_0 m(2^j\, \cdot)$ to $\varphi_j m$. Using the elementary identity

$$\|\widehat{\rho} * f(2^j\, \cdot)\|_{L_2(\mathbb{R}^n)} = 2^{-jn/2} \|\widehat{\rho(2^j\, \cdot)} * f\|_{L_2(\mathbb{R}^n)}$$

and easy calculations, we obtain

$$\begin{aligned} \|\varphi_0 m(2^j\, \cdot)\|_{B^2_{n/2,1}(\mathbb{R}^n)} &= \|\widehat{\phi} * (\varphi_0 m(2^j\, \cdot))\|_2 + \sum_{k\geq 1} 2^{nk/2} \|\widehat{\varphi}_k * (\varphi_0 m(2^j\, \cdot))\|_2 \\ &\sim 2^{-jn/2} \Big(\sum_{k+j\leq 0} \|\widehat{\varphi}_k * (\varphi_j m)\|_2^2 \Big)^{1/2} + \sum_{k+j\geq 1} 2^{nk/2} \|\widehat{\varphi}_k * (\varphi_j m)\|_2. \end{aligned}$$

On the other hand, our estimate in Theorem 2.15 gives

$$\Big(\sum_{k\leq 0} 2^{nk} \big\|\widehat{\varphi}_k * \big(\sqrt{\ell}\, \varphi_j m\big)\big\|_2^2 \Big)^{1/2} + \Big(\sum_{k\geq 1} 2^{nk} k^2 \big\|\widehat{\varphi}_k * \big(\sqrt{\ell}\, \varphi_j m\big)\big\|_2^2 \Big)^{1/2}.$$

It seems we get better estimates for $k \leq \min(0, -j)$ and worse for $k > \max(0, -j)$.

3. Analysis in free group branches

In this section we prove Theorem C. We shall use the same terminology as in Section 1.4 for the natural cocycle of $\mathbb{F}_\infty$ associated to the word length $|\ |$. In particular, recall that the Hilbert space is $\mathcal{H}_{|\ |} = \mathbb{R}[\mathbb{F}_\infty]/\mathbb{R}\delta_e$, with $\alpha_{|\ |} = \lambda$ and $b_{|\ |}(g) = \delta_g + \mathbb{R}\delta_e$. The system $\xi_g = \delta_g - \delta_{g^-}$ with $g \neq e$ forms an orthonormal basis of $\mathcal{H}_{|\ |}$, where g^- is the word which results after deleting the last generator on the right of g. By a *branch* of $\mathbb{F}_\infty$, we mean a subset $\mathrm{B} = (g_k)_{k\geq 1}$ with $g_k = g_{k+1}^-$. We will say that g_1 is the *root* of B.

Proof of Theorem C(i). Given $t > 0$, let

$$\widetilde{m}_t(j) = tje^{-tj}\widetilde{m}(j).$$

Our hypotheses on $\widetilde{m}$ imply that

$$|\widetilde{m}_t(j) - \widetilde{m}_t(j-1)| \lesssim te^{-tj} + te^{-tj/2} \lesssim te^{-tj/2}.$$

In particular, we find that

$$\begin{aligned}
&\sup_{t>0} \sum_{g\in \mathrm{B}} \big|\widetilde{m}_t(|g|)\sqrt{|g|} - \widetilde{m}_t(|g^-|)\sqrt{|g^-|}\big|^2 \\
&\qquad \lesssim \sup_{t>0} \sum_{j\geq 1} \Big(|\widetilde{m}_t(j) - \widetilde{m}_t(j-1)|^2 j + |\widetilde{m}_t(j-1)|^2 \frac{1}{j} \Big) \\
&\qquad \lesssim \sup_{t>0} \Big(\int_{\mathbb{R}_+} t^2 e^{-ts} s \, ds + \int_{\mathbb{R}_+} (ts)^2 e^{-2ts} \frac{ds}{s} \Big) = \int_{\mathbb{R}_+} x(e^{-x} + e^{-2x}) \, dx < \infty.
\end{aligned}$$

Therefore, if we define $h_t = \sum_{g\in \mathrm{B}} \langle h_t, \xi_g \rangle_{\mathcal{H}_{|\,|}} \xi_g$ with

$$\langle h_t, \xi_g \rangle_{\mathcal{H}_{|\,|}} = \widetilde{m}_t(|g|)\sqrt{|g|} - \widetilde{m}_t(|g^-|)\sqrt{|g^-|} \quad \text{for } g \in \mathrm{B},$$

it turns out that $(h_t)_{t>0}$ is uniformly bounded in $\mathcal{H}_{|\,|}$. Moreover

$$\delta_g = \sum_{h\leq g} \xi_h = b_{|\,|}(g) \ \Rightarrow\ \widetilde{m}_t(|g|) = \frac{\langle b_{|\,|}(g), h_t \rangle_{\mathcal{H}_{|\,|}}}{\sqrt{|g|}} \quad \text{for all } g \in \mathrm{B}.$$

In other words, $\widetilde{m}_t \circ |\,|$ coincides on B with the Fourier symbol of the Riesz transform $R_{|\,|,h_t}$ associated to the word length $|\,|$ in the direction of h_t. Assume (without loss of generality) $p \geq 2$. Now fix $f \in L_p(\mathcal{L}(\mathbb{F}_\infty))$ with vanishing Fourier coefficients outside B. Recall that $T_{\widetilde{m}\circ|\,|}(\lambda(g)) = \widetilde{m}(|g|)\lambda(g)$ and $A_{|\,|}$ generates a 'noncommutative diffusion semigroup' as defined in [32, Chapter 5], which satisfies the assumptions of [32, Corollary 7.7] by [32, Proposition 5.4 and Theorem 10.12]. One side of [32, Corollary 7.7] applied to $x = T_{\widetilde{m}\circ|\,|}f$ and $x = T_{\widetilde{m}\circ|\,|}f^*$ with $F(z) = z^2e^{-2z}$ implies

$$\begin{aligned}
&\|T_{\widetilde{m}\circ|\,|}f\|_p \\
&\quad \lesssim_{c(p)} \Big\| \Big(\int_{\mathbb{R}_+} \big(|(tA_{|\,|})^2 e^{-2tA_{|\,|}} T_{\widetilde{m}\circ|\,|}f|^2 + |(tA_{|\,|})^2 e^{-2tA_{|\,|}} T_{\widetilde{m}\circ|\,|}f^*|^2 \big) \frac{dt}{t} \Big)^{1/2} \Big\|_p \\
&\quad = \Big\| \Big(\int_{\mathbb{R}_+} \big(|R_{|\,|,h_t}(tA_{|\,|}e^{-tA_{|\,|}})f|^2 + |R_{|\,|,h_t}(tA_{|\,|}e^{-tA_{|\,|}})f^*|^2 \big) \frac{dt}{t} \Big)^{1/2} \Big\|_p
\end{aligned}$$

since $T_{\widetilde{m}_t\circ|\,|} = R_{|\,|,h_t}$ on B. By the integral version of Lemma 2.1 for $p \geq 2$, together with the uniform boundedness of $(h_t)_{t>0}$,

$$\|T_{\widetilde{m}\circ|\,|}f\|_p \lesssim_{c(p)} \Big\| \Big(\int_{\mathbb{R}_+} \big(|G(tA_{|\,|})f|^2 + |G(tA_{|\,|})f^*|^2 \big) \frac{dt}{t} \Big)^{1/2} \Big\|_p$$

for $G(z) = ze^{-z}$. By the other side of [32, Corollary 7.7] applied to $x = f$ and $x = f^*$, the right hand side is dominated by $\|f\|_p$. □

We will say that a family $\mathcal{T} = \{\mathrm{B}_k : k \geq 1\}$ of branches forms a partition of the free group when $\mathbb{F}_\infty = \{e\} \cup \bigcup_k \mathrm{B}_k$ and the B_k's are pairwise disjoint. Given a branch $\mathrm{B} \in \mathcal{T}$ let us write $g_{\mathrm{B},1}$ for its root. We will say that B is a *principal branch* when its root satisfies $|g_{\mathrm{B},1}| = 1$. If B is not a principal branch, there is a unique branch B^- in $\mathcal{T}$ which contains $g_{\mathrm{B},1}^-$. Given $g \in \mathbb{F}_\infty$, define $\Pi_{\mathrm{B}} g$ to be the biggest element in B which is smaller than or equal to g. If there is no such element, set $\Pi_{\mathrm{B}} g = e$. Now let us fix a standard Littlewood–Paley partition of unity in $\mathbb{R}_+$. That is, given a smooth decreasing function $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ with $1_{(0,1)} \leq \eta \leq 1_{(0,2)}$, set $\phi(\xi) = \eta(\xi) - \eta(2\xi)$ and $\phi_k(\xi) = \phi(2^{-k}\xi)$ for $k \in \mathbb{Z}$. Assume in addition that $\sqrt{\phi}$ is Lipschitz (as we assume in Theorem C). Then construct

$$\varphi_1 = \sum_{j \leq 1} \phi_j \quad \text{and} \quad \varphi_j = \phi_j \quad \text{for } j \geq 2,$$

so that $\sum_{j \geq 1} \varphi_j = 1$. Define $\Lambda_j : \lambda(g) \mapsto \sqrt{\varphi_j(|g|)}\, \lambda(g)$ and

$$\Lambda_{j,k} : \lambda(g) \mapsto \delta_{\Pi_{\mathrm{B}_k} g \neq e} \frac{\sqrt{\varphi_j(|\Pi_{\mathrm{B}_k} g|)|\Pi_{\mathrm{B}_k} g|} - \sqrt{\varphi_j(|\Pi_{\mathrm{B}_k^-} g|)|\Pi_{\mathrm{B}_k^-} g|}}{\sqrt{|g|}} \lambda(g)$$

for any $g \in \mathbb{F}_\infty$, with the convention that $\Pi_{\mathrm{B}_k^-} g = e$ if B_k is a principal branch.

Lemma 3.1. *If* $1 < p < 2$ *and* $f \in L_p(\mathcal{L}(\mathbb{F}_\infty))$*, then*

$$\inf_{\Lambda_{j,k} f = a_{j,k} + b_{j,k}} \Big\| \Big(\sum_{j,k} (a_{j,k}^* a_{j,k} + \widetilde{b}_{j,k} \widetilde{b}_{j,k}^*) \Big)^{1/2} \Big\|_p \lesssim_{c(p)} \|f\|_p.$$

Proof. Define $h_{j,k} = \sum_{g \in \mathrm{B}_k} \langle h_{j,k}, \xi_g \rangle_{\mathcal{H}_{|\,|}} \xi_g$ with

$$\langle h_{j,k}, \xi_g \rangle = \sqrt{\varphi_j(|g|)|g|} - \sqrt{\varphi_j(|g^-|)|g^-|} \quad \text{for } g \in \mathrm{B}_k.$$

To show that $h_{j,k} \in \mathcal{H}_{|\,|}$, we note that φ_1 and φ_j $(j \geq 2)$ are supported by $[0,4]$ and $[2^{j-1}, 2^{j+1}]$ respectively. Therefore, arguing as we did in the proof of Theorem C(i), we obtain (using the fact that $\sqrt{\phi}$ is Lipschitz)

$$\begin{aligned}
\|h_{j,k}\|_{\mathcal{H}_{|\,|}}^2 &\lesssim \sum_{g \in \mathrm{B}_k} \Big(\big| \sqrt{\varphi_j(|g|)} - \sqrt{\varphi_j(|g^-|)} \big|^2 |g| + |\varphi_j(|g^-|)| \big(\sqrt{|g|} - \sqrt{|g^-|} \big)^2 \Big) \\
&\lesssim \sum_{2^{j-1} \leq i \leq 2^{j+1}+1} \Big(\big| \sqrt{\varphi_j(i)} - \sqrt{\varphi_j(i-1)} \big|^2 i + |\varphi_j(i-1)| \frac{1}{i} \Big) \\
&\lesssim \sum_{2^{j-1} \leq i \leq 2^{j+1}+1} \Big(\frac{i}{4^j} + \frac{1}{i} \Big) \lesssim 1.
\end{aligned}$$

In particular, the family $(h_{j,k})$ is uniformly bounded in $\mathcal{H}_{|\,|}$ and

$$\delta_g = \sum_{h \leq g} \xi_h = b_{|\,|}(g) \ \Rightarrow\ \langle b_{|\,|}(g), h_{j,k} \rangle_{\mathcal{H}_{|\,|}} = \sum_{e \neq h \leq \Pi_{\mathrm{B}_k} g} \langle \xi_h, h_{j,k} \rangle_{\mathcal{H}_{|\,|}}.$$

By cancellation, the latter sum is

$$\sqrt{\varphi_j(\Pi_{\mathrm{B}_k} g)|\Pi_{\mathrm{B}_k} g|} - \sqrt{\varphi_j(\Pi_{\mathrm{B}_k^-} g)|\Pi_{\mathrm{B}_k^-} g|} \quad \text{when } \Pi_{\mathrm{B}_k} g \neq e.$$

Hence

$$\Lambda_{j,k} f = \sum_{g \in \mathbb{F}_\infty} \frac{\langle b_{|\,|}(g), h_{j,k}\rangle_{\mathcal{H}_{|\,|}}}{\sqrt{|g|}} \widehat{f}(g)\lambda(g) = R_{|\,|,h_{j,k}} f.$$

Now, according to the definition of $h_{j,k}$ it is easily checked that $\langle h_{j,k}, h_{j',k'}\rangle_{\mathcal{H}_{|\,|}}$ vanishes unless $k = k'$ and $|j - j'| \leq 1$. This implies that each of the subsystems $(h_{2j,k})$ and $(h_{2j+1,k})$ is orthogonal and uniformly bounded. Once this is known and splitting into two systems, the assertion follows from Theorem A1 by standard considerations. □

Proof of Theorem C(ii). It clearly suffices to prove the result for B a principal branch. Let $\mathcal{T} = \{\mathrm{B}_k : k \geq 1\}$ form a partition of $\mathbb{F}_\infty$ which contains $\mathrm{B}_1 = \mathrm{B}$ as a principal branch. Given $f \in L_p(\mathcal{L}(\mathbb{F}_\infty))$ with vanishing Fourier coefficients outside B, it is then easily checked that $\Lambda_j f = \Lambda_{j,1} f$ because B is a principal branch and $\Lambda_{j,k} f = 0$ for other values of k. Therefore, the first estimate follows from Lemma 3.1. For the second estimate, we use the fact that φ_j is a partition of unity together with the inequality in Remark 2.4(iv). Namely, we obtain

$$\begin{aligned} \|f\|_{L_p(\widehat{\mathbf{B}})} &= \Big\| \sum_{j \geq 1} \Lambda_j^2 f \Big\|_{L_p(\widehat{\mathbf{B}})} = \Big\| \sum_{j \geq 1} R_{|\,|,h_{j,1}}(a_j + b_j) \Big\|_{L_p(\widehat{\mathbf{B}})} \\ &\leq \|(\langle h_{j,1}, h_{k,1}\rangle)\|_{\mathcal{B}(\ell_2)}^{1/2} \Big\| \Big(\sum_{j \geq 1} (a_j^* a_j + b_j b_j^*) \Big)^{1/2} \Big\|_{L_p(\mathcal{L}(\mathbb{F}_\infty))}. \end{aligned}$$

Now observe that $\langle h_{j,1}, h_{k,1}\rangle$ vanishes when $|j - k| > 1$, and it is bounded above otherwise. This shows that the matrix above is bounded since it is a band diagonal matrix of width 3 with uniformly bounded entries. We are done. □

Corollary 3.2. *If* B *is any branch of* $\mathbb{F}_\infty$ *and* $2 < p < \infty$, *then*

$$\Big\| \Big(\sum_{j \geq 1} |\Lambda_j f|^2 \Big)^{1/2} \Big\|_{L_p(\widehat{\mathbf{B}})} \lesssim_{c(p)} \|f\|_{L_p(\widehat{\mathbf{B}})}$$

with the multipliers $\Lambda_j : \lambda(g) \mapsto \sqrt{\varphi_j(|g|)}\,\lambda(g)$ *defined as in Theorem* C(ii).

Proof. This easily follows from the identity $\Lambda_j = R_{|\,|,h_{j,1}}$ and Remark 2.4(i). □

Remark 3.3. Bożejko–Fendler's theorem [8] shows that Fourier summability fails in $L_p(\mathcal{L}(\mathbb{F}_n))$ when $|1/2 - 1/p| > 1/6$ and the partial sums are chosen to lie in a sequence of increasing balls with respect to word length. This may be regarded as some sort of Fefferman's disc multiplier theorem [26] for the free group algebra, although discreteness might allow some room for Fourier summability near L_2 in the spirit of Bochner–Riesz multipliers. This result indicates that we should not expect Littlewood–Paley estimates for nontrivial branches arising from sharp (nonsmooth) truncations in our partitions of unity, as it holds for $\mathbb{R}$ or $\mathbb{Z}$.

Appendix A. Unimodular groups

Let (G, ν) be a locally compact unimodular group with its Haar measure. Write $\lambda : \mathrm{G} \to \mathcal{B}(L_2(\mathrm{G}))$ for the left regular representation determined by $\lambda(g)(\rho)(h) = \rho(g^{-1}h)$ for any $\rho \in L_2(\mathrm{G})$. Recall in passing the definition of the convolution in G:

$$\rho * \eta(g) = \int_{\mathrm{G}} \rho(h)\eta(h^{-1}g)\, d\nu(h).$$

We say that $\rho \in L_2(\mathrm{G})$ is *left bounded* if the map $\mathcal{C}_c(\mathrm{G}) \ni \eta \mapsto \rho * \eta \in L_2(\mathrm{G})$ extends to a bounded operator on $L_2(\mathrm{G})$, denoted by $\lambda(\rho)$. This operator defines the Fourier transform of ρ. The weak operator closure of the linear span of $\lambda(\mathrm{G})$ defines the group von Neumann algebra $\mathcal{L}(\mathrm{G})$. It can also be described as the weak closure in $\mathcal{B}(L_2(\mathrm{G}))$ of the set of operators of the form

$$f = \lambda(\widehat{f}) = \int_{\mathrm{G}} \widehat{f}(g)\lambda(g)\, d\mu(g) \quad \text{with } \widehat{f} \in \mathcal{C}_c(\mathrm{G}).$$

The *Plancherel weight* $\tau : \mathcal{L}(\mathrm{G})_+ \to [0, \infty]$ is given by

$$\tau(f^* f) = \int_{\mathrm{G}} |\widehat{f}(g)|^2\, d\mu(g)$$

when $f = \lambda(\widehat{f})$ for some left bounded $\widehat{f} \in L_2(\mathrm{G})$, and $\tau(f^* f) = \infty$ for any other $f \in \mathcal{L}(\mathrm{G})$. After breaking into positive parts, this extends to a weight on a weak-$*$ dense domain within the von Neumann algebra $\mathcal{L}(\mathrm{G})$. It is instrumental to observe that the standard identity

$$\tau(f) = \widehat{f}(e)$$

holds for $\widehat{f} \in \mathcal{C}_c(\mathrm{G}) * \mathcal{C}_c(\mathrm{G})$ (see [54, Section 7.2] and also [72, Section VII.3] for a detailed construction of the Plancherel weight). Note that τ is tracial precisely due to the unimodularity of G, and it coincides with the finite trace $\tau(f) = \langle f\delta_e, \delta_e\rangle$ for G discrete. The pair $(\mathcal{L}(\mathrm{G}), \tau)$ is a semifinite von Neumann algebra and we may construct the noncommutative L_p-spaces

$$L_p(\mathcal{L}(\mathrm{G}), \tau) = L_p(\widehat{\mathbf{G}}) = \begin{cases} \overline{\lambda(\mathcal{C}_c(\mathrm{G}) * \mathcal{C}_c(\mathrm{G}))}^{\|\ \|_p} & \text{for } 1 \le p < 2, \\ \overline{\lambda(\mathcal{C}_c(\mathrm{G}))}^{\|\ \|_p} & \text{for } 2 \le p < \infty, \end{cases}$$

where the norm is given by $\|f\|_p = \tau(|f|^p)^{1/p}$ and the p-th power is calculated by functional calculus applied to the (possibly unbounded) operator f (see Appendix B below for more details on the construction of noncommutative L_p-spaces). On the other hand, since left bounded functions are dense in $L_2(\mathrm{G})$, the map $\lambda : \rho \mapsto \lambda(\rho)$ extends to an isometry from $L_2(\mathrm{G})$ to $L_2(\widehat{\mathbf{G}})$.

Given $m : \mathrm{G} \to \mathbb{C}$, set

$$T_m f = \int_{\mathrm{G}} m(g)\widehat{f}(g)\lambda(g)\, d\mu(g) \quad \text{for } \widehat{f} \in \mathcal{C}_c(\mathrm{G}) * \mathcal{C}_c(\mathrm{G}).$$

T_m is called an *L_p-Fourier multiplier* if it extends to a bounded map on $L_p(\widehat{\mathbf{G}})$.

Our goal in this appendix is to study the validity of our main results in this paper—Theorems A1, A2, B1 and B2—in the context of not necessarily discrete unimodular groups. More precisely, given G unimodular we shall be working with continuous conditionally negative lengths $\psi : \mathrm{G} \to \mathbb{R}_+$ which are in one-to-one correspondence with continuous affine representations $(\mathcal{H}_\psi, \alpha_\psi, b_\psi)$, as follows from Schoenberg's theorem. Precise definitions of these notions as well as the construction of crossed products of von Neumann algebras with locally compact unimodular groups will be given in Appendix B below. Most of the time, our results in the discrete case must be modified by the simple replacement of sums over G (discrete case) with integrals over G (unimodular case) with respect to the Haar measure ν. We should also replace finite sums (trigonometric polynomials) by elements in $\lambda(\mathcal{C}_c(\mathrm{G}))$ or $\lambda(\mathcal{C}_c(\mathrm{G}) * \mathcal{C}_c(\mathrm{G}))$ depending on whether $p \geq 2$ or $p < 2$.

Notice that the proof of Theorem A1 for discrete groups rests on the validity of Theorem A2 for this class of groups. In the unimodular case, we may follow verbatim the proof (replacing sums by integrals as indicated above) and conclude that Theorem A1 is valid for arbitrary unimodular groups as long as the same holds for its 'Meyer form' in Theorem A2. Thus, we shall only prove this latter result for unimodular groups. The crucial difference from the original argument is that the group homomorphism $\mathrm{G} \ni g \mapsto b_\psi(g) \rtimes g \in \mathcal{H}_\psi \rtimes \mathrm{G}$ is no longer continuous when we take the discrete topology on $\mathcal{H}_\psi$ and the product topology on the semidirect product $\mathcal{H}_\psi \rtimes \mathrm{G}$. Recall that the discrete topology was crucial in our argument to obtain a trace preserving $*$-homomorphism

$$\pi : \mathcal{L}(\mathrm{G}) \ni \lambda(g) \mapsto \lambda_\rtimes(b_\psi(g) \rtimes g) \in \mathcal{L}(\mathcal{H}_\psi \rtimes \mathrm{G}) \simeq \mathcal{L}(\mathcal{H}_\psi) \rtimes \mathrm{G}.$$

Here we use the isometric isomorphism $\lambda_\rtimes(b_\psi(g) \rtimes g) \mapsto \exp(2\pi i \langle b_\psi(g), \cdot \rangle_{\mathcal{H}_\psi}) \rtimes \lambda(g)$ between $\mathcal{L}(\mathcal{H}_\psi \rtimes \mathrm{G})$ and $\mathcal{L}(\mathcal{H}_\psi) \rtimes \mathrm{G}$. Thus, our argument at this point requires an explanation or a modification. We would like to thank the referee for the proof presented below. Although in the same line as ours (even longer), it is a bit more natural, it avoids this subtle point and improves certain estimates obtained along our argument. Of course, according to Theorem 1.1—whose proof extends trivially to unimodular groups—and as we did in our proof of Theorem A2, it suffices to show that

$$\|f\|_{L_p^\circ(\widehat{\mathbf{G}})} \sim_{c(p)} \|\delta_\psi A_\psi^{-1/2} f\|_{G_p(\mathbb{C}) \rtimes \mathrm{G}}. \tag{A.1}$$

Proof of (A.1). Let γ denote the standard gaussian measure in the real Hilbert space $\mathcal{H}_\psi$ (recall that $\dim \mathcal{H}_\psi = \infty$ is admissible). Let $\mathcal{M} := L_\infty(\mathcal{H}_\psi, \gamma) \rtimes_{\alpha_\psi} \mathrm{G}$ and consider the map

$$J_\psi : \mathcal{L}(\mathrm{G}) \ni \lambda(g) \mapsto \mathbf{1} \rtimes \lambda(g) \in \mathcal{M},$$

which extends trivially to a $*$-homomorphism. Let $\mathsf{E}_{\mathcal{L}(\mathrm{G})} : \mathcal{M} \to \mathcal{L}(\mathrm{G})$ stand for the corresponding conditional expectation. Construct the $*$-derivation D_ψ, densely defined in the weak-$*$ topology of $\mathcal{M}$ as

$$D_\psi \Bigg(\underbrace{\int_{\mathrm{G}} f_g \rtimes \lambda(g) \, d\nu(g)}_{F} \Bigg) = 2\pi i \int_{\mathrm{G}} f_g \langle b_\psi(g), \cdot \rangle \rtimes \lambda(g) \, d\nu(g)$$

where the function $g \mapsto f_g$ is continuous and compactly supported on G with values in $L_\infty(\mathcal{H}_\psi, \gamma)$, so that $D_\psi J_\psi = \delta_\psi$. If $\mathcal{R} = \delta_\psi A_\psi^{-1/2}$, the crucial identity is

$$\mathcal{R}f = \frac{-i}{\sqrt{2\pi}} \widehat{Q}\Big(\text{p.v.} \int_{\mathbb{R}} e^{tD_\psi} \circ J_\psi(f)\, \frac{dt}{t}\Big) \quad \text{for } f \in L_2^\circ(\widehat{\mathbf{G}}) \cap \lambda(\mathcal{C}_c(\mathrm{G})), \tag{A.2}$$

where $\widehat{Q}$ is defined as in Theorem 1.1. Before justifying it, we shall complete the proof of (A.1). We claim that

$$\|\mathcal{U}(F)\|_{L_p(\mathcal{M})} := \Big\|\text{p.v.} \int_{\mathbb{R}} e^{tD_\psi}(F)\, \frac{dt}{t}\Big\|_{L_p(\mathcal{M})} \lesssim \frac{p^2}{p-1}\|F\|_{L_p(\mathcal{M})} \tag{A.3}$$

for $1 < p < \infty$. (A.1) follows from (A.2) & (A.3). Indeed, let $f \in L_p^\circ(\widehat{\mathbf{G}}) \cap \lambda(\mathcal{C}_c(\mathrm{G}))$, which is admissible by density. Since $\delta_\psi^* \delta_\psi = A_\psi$, we deduce $\mathcal{R}^*\mathcal{R} = \mathrm{id}_{L_2^\circ(\widehat{\mathbf{G}})}$ and obtain

$$f = \mathcal{R}^*\mathcal{R}f = \frac{1}{2\pi} J_\psi^* \circ \mathcal{U}^* \circ \widehat{Q}^* \circ \widehat{Q} \circ \mathcal{U} \circ J_\psi(f) = \frac{1}{2\pi} \mathsf{E}_{\mathcal{L}(\mathrm{G})} \circ \mathcal{U}^* \circ \mathcal{R}(f).$$

This yields the upper estimate in (A.1) with constant $\sim p^2/(p-1)$ (which improves the one obtained in our former proof of Theorem A2). The lower estimate is a trivial consequence of (A.2) and (A.3), and the constant behaves like $p^3/(p-1)^{3/2}$ as found in our former proof. It remains to justify (A.2) and (A.3). To prove (A.2), we start with the simple identity

$$\begin{aligned}
\mathcal{R}f &= \delta_\psi\Big(\int_{\mathrm{G}} \frac{\widehat{f}(g)}{\sqrt{\psi(g)}} \lambda(g)\, d\nu(g)\Big) \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathrm{G}} \widehat{f}(g)\Big(\int_{\mathbb{R}} e^{-\frac{1}{2}t^2|b_\psi(g)|^2}\, dt\Big) \langle b_\psi(g), \cdot\rangle \rtimes \lambda(g)\, d\nu(g) \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathrm{G}} \widehat{f}(g) \int_{\mathbb{R}}\Big(\int_{\mathcal{H}_\psi} e^{it\langle b_\psi(g), y\rangle}\, d\gamma(y)\Big) \langle b_\psi(g), \cdot\rangle \rtimes \lambda(g)\, dt\, d\nu(g)
\end{aligned}$$

for $f \in L_2^\circ(\widehat{\mathbf{G}}) \cap \lambda(\mathcal{C}_c(\mathrm{G}))$. Integrating by parts coordinatewise yields

$$\begin{aligned}
\Big(\int_{\mathcal{H}_\psi} e^{it\langle b_\psi(g), y\rangle}\, d\gamma(y)\Big)\langle b_\psi(g), \cdot\rangle &= \sum_{j\geq 1}\Big(\int_{\mathcal{H}_\psi} \langle b_\psi(g), e_j\rangle e^{it\langle b_\psi(g), y\rangle}\, d\gamma(y)\Big)\langle e_j, \cdot\rangle \\
&= \frac{1}{it}\sum_{j\geq 1}\Big(\int_{\mathcal{H}_\psi} \frac{\partial}{\partial y_j} e^{it\langle b_\psi(g), y\rangle}\, d\gamma(y)\Big)\langle e_j, \cdot\rangle \\
&= \frac{1}{it}\sum_{j\geq 1}\Big(\int_{\mathcal{H}_\psi} e^{it\langle b_\psi(g), y\rangle} y_j\, d\gamma(y)\Big)\langle e_j, \cdot\rangle \\
&= \frac{1}{it}\int_{\mathcal{H}_\psi} e^{it\langle b_\psi(g), y\rangle}\langle y, \cdot\rangle\, d\gamma(y) = \frac{1}{it} Q(e^{it\langle b_\psi(g), \cdot\rangle}).
\end{aligned}$$

Combining this identity with our expression above for $\mathcal{R}f$, we obtain

$$\mathcal{R}f = \frac{-i}{\sqrt{2\pi}} \int_{\mathrm{G}} \widehat{f}(g) \Big[\int_{\mathbb{R}} \Big(\int_{\mathcal{H}_\psi} e^{it\langle b_\psi(g), y\rangle} \langle y, \cdot\rangle \, d\gamma(y) \Big) \frac{dt}{t} \Big] \rtimes \lambda(g) \, d\nu(g).$$

Truncating the integral over $\mathbb{R}$ to the compact set $\Omega_{\mathrm{N},\varepsilon} = [-\mathrm{N}, \mathrm{N}] \setminus (-\varepsilon, \varepsilon)$, it is clear that we can apply Fubini. In particular, we may rewrite the term in square brackets above as

$$\lim_{\substack{\varepsilon\to 0\\ \mathrm{N}\to\infty}} \int_{\mathcal{H}_\psi} A_{\mathrm{N},\varepsilon}(g, y)\langle y, \cdot\rangle \, d\gamma(y) \quad \text{with} \quad A_{\mathrm{N},\varepsilon}(g, y) = \int_{\Omega_{\mathrm{N},\varepsilon}} e^{it\langle b_\psi(g), y\rangle} \frac{dt}{t}.$$

By the symmetry of $\Omega_{\mathrm{N},\varepsilon}$, we may replace the imaginary exponential in $A_{\mathrm{N},\varepsilon}(g, y)$ by $\sin(t\langle b_\psi(g), y\rangle)$. Thus, $A_{\mathrm{N},\varepsilon}(g, y)$ is uniformly bounded in N, ε for g fixed and $\langle b_\psi(g), y\rangle \neq 0$. In particular, by the dominated convergence theorem,

$$\lim_{\substack{\varepsilon\to 0\\ \mathrm{N}\to\infty}} \int_{\mathcal{H}_\psi} A_{\mathrm{N},\varepsilon}(g, y)\langle y, \cdot\rangle \, d\gamma(y) = \int_{\mathcal{H}_\psi} \Big(\text{p.v.} \int_{\mathbb{R}} e^{it\langle b_\psi(g), y\rangle} \frac{dt}{t} \Big) \langle y, \cdot\rangle \, d\gamma(y).$$

Now, since $\widehat{f} \in \mathcal{C}_c(\mathrm{G})$, we obtain

$$\begin{aligned}
\mathcal{R}f &= \frac{-i}{\sqrt{2\pi}} \int_{\mathrm{G}} \widehat{f}(g) \Big[\int_{\mathcal{H}_\psi} \Big(\text{p.v.} \int_{\mathbb{R}} e^{it\langle b_\psi(g), y\rangle} \frac{dt}{t} \Big) \langle y, \cdot\rangle \, d\gamma(y) \Big] \rtimes \lambda(g) \, d\nu(g) \\
&= \frac{-i}{\sqrt{2\pi}} \int_{\mathrm{G}} \widehat{f}(g) \mathcal{Q} \Big(\text{p.v.} \int_{\mathbb{R}} e^{it\langle b_\psi(g), \cdot\rangle} \frac{dt}{t} \Big) \rtimes \lambda(g) \, d\nu(g) \\
&= \frac{-i}{\sqrt{2\pi}} \widehat{\mathcal{Q}} \Big[\int_{\mathrm{G}} \widehat{f}(g) \, \text{p.v.} \int_{\mathbb{R}} e^{it\langle b_\psi(g), \cdot\rangle} \frac{dt}{t} \rtimes \lambda(g) \, d\nu(g) \Big].
\end{aligned}$$

This reduces the proof of (A.2) to showing that the term in square brackets is $\mathcal{U} \circ J_\psi(f)$. This follows by applying Fubini, which in turn can be justified as above. Note that $e^{it\langle b_\psi(g), \cdot\rangle} \rtimes \lambda(g) = e^{tD_\psi} J_\psi(\lambda(g))$ follows from $D_\psi = \mathrm{id}_{L_\infty(\mathcal{H}_\psi, \gamma)} \rtimes \delta_\psi$ and the fact that δ_ψ is a derivation because $(\mathcal{H}_\psi, \alpha_\psi, b_\psi)$ is a cocycle. (A.3) follows from the fact that $(\exp(tD_\psi))_{t\in\mathbb{R}}$ is a one-parameter group of isometries of $L_p(\mathcal{M})$. Indeed, each of the maps $\exp(tD_\psi)$ is a $*$-automorphism since $(\mathcal{H}_\psi, \alpha_\psi, b_\psi)$ is a cocycle. On the other hand, both $\exp(tD_\psi)$ and $\exp(-tD_\psi)$ are trace preserving, hence isometries of $L_1(\mathcal{M})$. By interpolation, we obtain a one-parameter group of isometries of $L_p(\mathcal{M})$ for $1 \leq p \leq \infty$. According to [5, Theorems 5.12 and 5.16], we deduce (A.3) as a consequence of the fact that $L_p(\mathcal{M})$ is UMD for $1 < p < \infty$. □

It remains to study Theorems B1 and B2 for unimodular groups. As we shall see, Theorem B2 holds for arbitrary unimodular groups, whereas Theorem B1 will be proved under the additional assumption that G is ADS (see below for a precise definition of ADS group). The validity of Theorem B1 for arbitrary unimodular groups is left as an open problem for the interested reader. Namely, assume for simplicity that $b_\psi : \mathrm{G} \to \mathbb{R}_+$ is injective and $\nu\{e\} = 0$, so that we do not have to worry about the value $m(e)$ as we did in Theorem B1. Then, a careful reading of the proof of Theorem B1 shows that it holds for a given unimodular group G as long as Lemmas 2.2 and 2.6 hold for G. Lemma 2.6 only uses Theorem A1 and Remark 2.4(iv). However, the latter is dual to Remark 2.4(i) which follows just as Lemma 2.1. Finally, this lemma is again a consequence of Theorem A1. Therefore, it turns out that Lemma 2.6 holds for arbitrary unimodular groups because we have already shown that Theorem A1 does. On the other hand, the proof of Lemma 2.2 works as well for a given unimodular group G as long as [34, Theorem 4.3] does. The latter result is the Littlewood–Paley estimate

$$\Big\| \sum_{j=1}^{\infty} \Lambda_{\psi,\varphi_j} f \otimes \delta_j \Big\|_{RC_p(\mathcal{L}(\mathrm{G}))} \lesssim_{c(p,\dim \mathcal{H}_\psi)} \|f\|_{L_p(\widehat{\mathbf{G}})}. \tag{A.4}$$

The problem here is again that we use the map

$$\pi : \mathcal{L}(\mathrm{G}) \ni \lambda(g) \mapsto \lambda_{\rtimes}(b_\psi(g), g) \in \mathcal{L}(\mathcal{H}_\psi \rtimes \mathrm{G}) \simeq \mathcal{L}(\mathcal{H}_\psi) \rtimes \mathrm{G}$$

(where $\mathcal{H}_\psi$ is some $\mathbb{R}^n_{\mathrm{disc}}$ and G is discrete) in a crucial way for [34, Theorem 4.3]. Since the argument used above for Theorem A2 seems to be very specific to the Riesz transform, we need an alternative approach. The strategy is to use the fact that (A.4) holds for arbitrary discrete groups, as proved in [34]. In particular, if we let $\mathrm{G}_{\mathrm{disc}}$ denote the group G equipped with the discrete topology and consider the linear map

$$L_\psi f = \sum_{j=1}^{\infty} \Lambda_{\psi,\varphi_j} f \otimes \delta_j,$$

our goal is to prove the following inequality for $1 < p < \infty$:

$$\|L_\psi : L_p(\widehat{\mathbf{G}}) \to RC_p(\mathcal{L}(\mathrm{G}))\| \lesssim \|L_\psi : L_p(\widehat{\mathbf{G}}_{\mathrm{disc}}) \to RC_p(\mathcal{L}(\mathrm{G}_{\mathrm{disc}}))\|. \tag{A.5}$$

This can be regarded as a Hilbert space valued version of the noncommutative de Leeuw compactification theorem [13, Theorem D(i)]. We will show that (A.5) and therefore (A.4) holds for every unimodular ADS group.

A unimodular group G is called *approximable by discrete subgroups* (ADS) when there exists a family $(\Gamma_k)_{k\geq 1}$ of lattices in G and associated fundamental domains X_k which form a neighborhood basis of the identity. It is worth noting that every nilpotent Lie group is ADS; we refer to [13] for more details and a discussion of the limitations to going beyond ADS groups for restriction and compactification theorems.

Proof of (A.5). We clearly have

$$L_p(\widehat{\Gamma}_k) \subset L_p(\widehat{\mathbf{G}}_{\mathrm{disc}})$$

isometrically for all $k \geq 1$. In particular, this immediately yields

$$\sup_{k\geq 1} \|L_{\psi}{}_{|_{\widehat{\Gamma}_k}} : L_p(\widehat{\Gamma}_k) \to RC_p(\mathcal{L}(\Gamma_k))\| \leq \|L_\psi : L_p(\widehat{\mathbf{G}}_{\mathrm{disc}}) \to RC_p(\mathcal{L}(\mathrm{G}_{\mathrm{disc}}))\|,$$
$$\sup_{k\geq 1} \|L^*_{\psi}{}_{|_{\widehat{\Gamma}_k}} : RC_p(\mathcal{L}(\Gamma_k)) \to L_p(\widehat{\Gamma}_k)\| \leq \|L^*_\psi : RC_p(\mathcal{L}(\mathrm{G}_{\mathrm{disc}})) \to L_p(\widehat{\mathbf{G}}_{\mathrm{disc}})\|.$$

Indeed, both inequalities are obvious for $p > 2$, and the case $1 < p < 2$ follows by duality. Since the case $p = 2$ is clear, using duality one more time it suffices to show for

$1 < p < 2$ that

$$\|L_\psi : L_p(\widehat{\mathbf{G}}) \to RC_p(\mathcal{L}(\mathrm{G}))\| \le \sup_{k\ge 1} \|L_{\psi_{|\widehat{\Gamma}_k}} : L_p(\widehat{\Gamma}_k) \to RC_p(\mathcal{L}(\Gamma_k))\|,$$
$$\|L_\psi^* : RC_p(\mathcal{L}(\mathrm{G})) \to L_p(\widehat{\mathbf{G}})\| \le \sup_{k\ge 1} \|L^*_{\psi_{|\widehat{\Gamma}_k}} : RC_p(\mathcal{L}(\Gamma_k)) \to L_p(\widehat{\Gamma}_k)\|.$$

This is a Hilbert space valued form of the noncommutative extension of Igari's lattice approximation theorem [13, Theorem C]. Following the notation used in the proof of [13, Theorem C], define

$$S_{jk}^\psi = \Phi_k^p \circ \Lambda_{\psi,\varphi_j}{}_{|\Gamma_k} \circ \Psi_k^p \quad \text{where} \quad \Psi_k^p = (\Phi_k^{p'})^*.$$

It is worth mentioning that although the maps Φ_k^p and Ψ_k^p depend on p, the operator S_{jk}^ψ does not. This will be relevant below. We shall also use the operators

$$A_{\psi,k} f = \sum_{j=1}^\infty S_{jk}^\psi f \otimes \delta_j \quad \text{and} \quad B_{\psi,k}\Big(\sum_{j=1}^\infty f_j \otimes \delta_j\Big) = \sum_{j=1}^\infty S_{jk}^\psi f_j.$$

Since $\Phi_k^p : L_p(\widehat{\Gamma}_k) \to L_p(\widehat{\mathbf{G}})$ is a complete contraction [13], we deduce

$$\|A_{\psi,k} : L_p(\widehat{\mathbf{G}}) \to RC_p(\mathcal{L}(\mathrm{G}))\| \le \sup_{k\ge 1} \|L_{\psi_{|\widehat{\Gamma}_k}} : L_p(\widehat{\Gamma}_k) \to RC_p(\mathcal{L}(\Gamma_k))\|,$$
$$\|B_{\psi,k} : RC_p(\mathcal{L}(\mathrm{G})) \to L_p(\widehat{\mathbf{G}})\| \le \sup_{k\ge 1} \|L^*_{\psi_{|\widehat{\Gamma}_k}} : RC_p(\mathcal{L}(\Gamma_k)) \to L_p(\widehat{\Gamma}_k)\|.$$

Therefore, the assertion will follow if we can show that

$$L_\psi f = \text{w-}RC_p(\mathcal{L}(\mathrm{G}))\text{-}\lim_{k\to\infty} A_{\psi,k} f, \quad L_\psi^* \mathbf{f} = \text{w-}L_p(\widehat{\mathbf{G}})\text{-}\lim_{k\to\infty} B_{\psi,k}\mathbf{f},$$

on a dense class in $L_p(\widehat{\mathbf{G}})$, $RC_p(\mathcal{L}(\mathrm{G}))$ respectively. To justify the first limit, we argue as in [13, Theorem C] and reduce it to proving strong L_2-convergence for $f = \lambda(\widehat{f})$ with $\widehat{f} \in \mathcal{C}_c(\mathrm{G}) * \mathcal{C}_c(\mathrm{G})$. Indeed, if $(q_j)_{j\ge 1}$ is a sequence of projections in $\ell_r(L_r(\widehat{\mathbf{G}}))$ with $1/p = 1/2 + 1/r$, then

$$\begin{aligned}\Big\|\sum_{j=1}^\infty q_j(S_{jk}^\psi f - \Lambda_{\psi,\varphi_j} f) \otimes \delta_j\Big\|_{RC_p(\mathcal{L}(\mathrm{G}))} &\le \sum_{j=1}^\infty \|q_j(S_{jk}^\psi f - \Lambda_{\psi,\varphi_j} f)\|_{L_p(\widehat{\mathbf{G}})} \\ &\le \Big(\sum_{j=1}^\infty \|q_j\|_{L_r(\widehat{\mathbf{G}})}^r\Big)^{1/r} \Big(\sum_{j=1}^\infty \|S_{jk}^\psi f - \Lambda_{\psi,\varphi_j} f\|_{L_2(\widehat{\mathbf{G}})}^2\Big)^{1/2}.\end{aligned}$$

Since $\widehat{f}$ is compactly supported, only finitely many elements in $(S_{jk}^\psi f - \Lambda_{\psi,\varphi_j} f)_{j\ge 1}$ do not vanish. As in [13, Theorem C], this implies that strong L_2-convergence implies weak L_p-convergence. On the other hand, since only finitely many j's give nonzero terms, it suffices to show that

$$L_2\text{-}\lim_{k\to\infty} S_{jk}^\psi f = \Lambda_{\psi,\varphi_j} f \tag{A.6}$$

for each $j \ge 1$ and every $f \in \lambda(\mathcal{C}_c(\mathrm{G}) * \mathcal{C}_c(\mathrm{G}))$. The proof of this is exactly the same as in [13, Theorem C] and we shall not reproduce it here. This justifies the first limit.

The second one is very similar. Again we may reduce it to proving strong L_2-convergence, this time with the exact same argument as in [13]. Moreover, we may pick $\mathbf{f} = \sum_j f_j \otimes \delta_j$ with $f_j \neq 0$ for finitely many j's. Then the problem reduces once more to justifying (A.6) as indicated above. □

Finally, we conclude by analyzing Theorem B2. In this case, the proof is completely parallel to that of Theorem B1 with the only difference that we use Lemma 2.3 instead of Lemma 2.2. The Littlewood–Paley estimate used in that lemma follows from [32] and holds for any semifinite von Neumann algebra $\mathcal{M}$. In particular, it holds with $\mathcal{M} = \mathcal{L}(\mathrm{G})$ for every unimodular group G.

Appendix B. Operator-algebraic tools

Along this paper we have used some concepts from noncommutative integration which include noncommutative L_p-spaces and sums of independent noncommuting random variables. In the context of group von Neumann algebras, we have also used crossed products, length functions and cocycles. In this section we briefly review these notions for the readers who are not familiar with them.

Noncommutative integration. Part of von Neumann algebra theory has evolved as the noncommutative form of measure theory and integration. A *von Neumann algebra* [41, 72] is a unital weak-operator closed C*-algebra; and, according to the Gelfand–Naimark–Segal theorem, any such algebra $\mathcal{M}$ embeds in the algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$. We write $\mathbf{1}_{\mathcal{M}}$ for the unit. The positive cone $\mathcal{M}_+$ is the set of positive operators in $\mathcal{M}$, and a *trace* $\tau : \mathcal{M}_+ \to [0, \infty]$ is a linear map satisfying

$$\tau(f^* f) = \tau(f f^*).$$

It is *normal* if $\sup_\alpha \tau(f_\alpha) = \tau(\sup_\alpha f_\alpha)$ for bounded increasing nets (f_α) in $\mathcal{M}_+$; it is *semifinite* if for any nonzero $f \in \mathcal{M}_+$ there exists $0 < f' \leq f$ such that $\tau(f') < \infty$; and it is *faithful* if $\tau(f) = 0$ implies that $f = 0$. The trace τ plays the rôle of the integral in the classical case. A von Neumann algebra is *semifinite* when it admits a normal semifinite faithful (n.s.f. in short) trace τ. Any operator f is a linear combination $f_1 - f_2 + i f_3 - i f_4$ of four positive operators. Thus, we can extend τ to the whole algebra $\mathcal{M}$ and the tracial property can be restated in the familiar form $\tau(fg) = \tau(gf)$. Unless explicitly stated, $(\mathcal{M}, \tau)$ will denote a semifinite von Neumann algebra equipped with a n.s.f. trace. We will refer to it as a *noncommutative measure space*. Note that commutative von Neumann algebras correspond to classical L_∞-spaces.

According to the GNS construction, the noncommutative analog of measurable sets (characteristic functions) are orthogonal projections. Given $f \in \mathcal{M}_+$, the *support* of f is the least projection q in $\mathcal{M}$ such that $qf = f = fq$; it is denoted by supp f. Let $\mathcal{S}_{\mathcal{M}}^+$ be the set of all $f \in \mathcal{M}_+$ such that $\tau(\operatorname{supp} f) < \infty$ and set $\mathcal{S}_{\mathcal{M}}$ to be the linear span of $\mathcal{S}_{\mathcal{M}}^+$. If we write $|f| = \sqrt{f^* f}$, we can use the spectral measure $d\gamma : \mathbb{R}_+ \to \mathcal{B}(\mathcal{H})$ of $|f|$ to define

$$|f|^p = \int_{\mathbb{R}_+} s^p \, d\gamma(s) \quad \text{for } 0 < p < \infty.$$

We have $f \in \mathcal{S}_\mathcal{M} \Rightarrow |f|^p \in \mathcal{S}_\mathcal{M}^+ \Rightarrow \tau(|f|^p) < \infty$. If we set $\|f\|_p = \tau(|f|^p)^{1/p}$, we obtain a norm in $\mathcal{S}_\mathcal{M}$ for $1 \leq p < \infty$ and a p-norm for $0 < p < 1$. Since $\mathcal{S}_\mathcal{M}$ is an involutive strongly dense ideal of $\mathcal{M}$, we can define the *noncommutative* L_p*-space* $L_p(\mathcal{M})$ associated to the pair $(\mathcal{M}, \tau)$ as the completion of $(\mathcal{S}_\mathcal{M}, \| \, \|_p)$. On the other hand, we set $L_\infty(\mathcal{M}) = \mathcal{M}$ equipped with the operator norm. Many fundamental properties of classical L_p-spaces like duality, real and complex interpolation, Hölder inequalities, etc. hold in this setting. Elements of $L_p(\mathcal{M})$ can also be described as measurable operators affiliated to $(\mathcal{M}, \tau)$; we refer to Pisier/Xu's survey [60] for more information and historical references. Note that classical L_p-spaces are denoted in the noncommutative terminology as $L_p(\Omega, \mu) = L_p(\mathcal{M})$ where $\mathcal{M}$ is the commutative von Neumann algebra $L_\infty(\Omega, \mu)$.

A unital, weakly closed $*$-subalgebra is called a *von Neumann subalgebra*. A *conditional expectation* $\mathsf{E} : \mathcal{M} \to \mathcal{N}$ from a von Neumann algebra $\mathcal{M}$ onto a von Neumann subalgebra $\mathcal{N}$ is a positive contractive projection. It is called *normal* if the adjoint map E^* sends $L_1(\mathcal{M})$ to $L_1(\mathcal{N})$. In this case, the restriction map $\mathsf{E}_1 = \mathsf{E}^*|_{L_1(\mathcal{M})}$ satisfies $\mathsf{E}_1^* = \mathsf{E}$. Note that such normal conditional expectation exists if and only if the restriction of τ to the von Neumann subalgebra $\mathcal{N}$ remains semifinite (see [72] for further details). Any such conditional expectation is *trace preserving*: $\tau \circ \mathsf{E} = \tau$, and satisfies the *bimodule property*

$$\mathsf{E}(a_1 b a_2) = a_1 \mathsf{E}(b) a_2 \quad \text{for all } a_1, a_2 \in \mathcal{N} \text{ and } b \in \mathcal{M}.$$

Given von Neumann algebras $\mathcal{N} \subset \mathcal{A}, \mathcal{B} \subset \mathcal{M}$, we will say that $\mathcal{A}, \mathcal{B}$ are *independent* over E whenever $\mathsf{E}(ab) = \mathsf{E}(a)\mathsf{E}(b)$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Similarly, we will say that a family $(f_j)_{j \in \mathcal{J}}$ of random variables in $\mathcal{M}$ is *fully independent* over E if the von Neumann algebras generated by any two disjoint subsets of $(f_j)_{j \in \mathcal{J}}$ are independent over E. The noncommutative analog of the Rosenthal inequality [62] was obtained in [39] and reads as follows for $p \geq 2$. If the random variables $(f_j)_{j \in \mathcal{J}} \subset L_p(\mathcal{M})$ satisfy $\mathsf{E}(f_j) = 0$ and are fully independent over E, then

$$\frac{1}{p} \Big\| \sum_{j \in \mathcal{J}} f_j \Big\|_p \sim \Big(\sum_{j \in \mathcal{J}} \|f_j\|_p^p \Big)^{1/p} + \Big\| \Big(\sum_{j \in \mathcal{J}} \mathsf{E}(f_j^* f_j) \Big)^{1/2} \Big\|_p + \Big\| \Big(\sum_{j \in \mathcal{J}} \mathsf{E}(f_j f_j^*) \Big)^{1/2} \Big\|_p. \tag{B.1}$$

Group von Neumann algebras. Let G be a discrete group with left regular representation $\lambda : \mathrm{G} \to \mathcal{B}(\ell_2(\mathrm{G}))$ given by $\lambda(g)\delta_h = \delta_{gh}$, where the δ_g's form the unit vector basis of $\ell_2(\mathrm{G})$. Write $\mathcal{L}(\mathrm{G})$ for its *group von Neumann algebra*, the weak operator closure of the linear span of $\lambda(\mathrm{G})$ in $\mathcal{B}(\ell_2(\mathrm{G}))$. Consider the standard trace $\tau(\lambda(g)) = \delta_{g=e}$ where e denotes the identity of G. Any $f \in \mathcal{L}(\mathrm{G})$ has a Fourier series expansion of the form

$$\sum_{g \in \mathrm{G}} \widehat{f}(g) \lambda(g) \quad \text{with} \quad \tau(f) = \widehat{f}(e).$$

Define

$$L_p(\widehat{\mathbf{G}}) = L_p(\mathcal{L}(\mathrm{G}), \tau) \equiv \text{closure of } \mathcal{L}(\mathrm{G}) \text{ in the norm } \|f\|_{L_p(\widehat{\mathbf{G}})} = (\tau|f|^p)^{1/p},$$

the natural L_p-space over the noncommutative measure space $(\mathcal{L}(\mathrm{G}), \tau)$. Note that when G is abelian we get the L_p-space on the dual group equipped with its normalized Haar measure, after identifying $\lambda(g)$ with the character χ_g. The group von Neumann algebra $\mathcal{L}(\mathrm{G})$ associated to a locally compact unimodular group G is defined similarly; we refer to Appendix A above for the details.

Let G be a locally compact unimodular group. Given another noncommutative measure space $(\mathcal{M}, \nu)$ with $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ assume there exists a trace preserving continuous action $\alpha : \mathrm{G} \to \mathrm{Aut}(\mathcal{M})$. Define the *crossed product algebra* $\mathcal{M} \rtimes_\alpha \mathrm{G}$ as the weak operator closure of the $*$-algebra generated by $\mathbf{1}_{\mathcal{M}} \otimes \lambda(\mathrm{G})$ and $\rho(\mathcal{M})$ in $\mathcal{B}(L_2(\mathrm{G}; \mathcal{H}))$. The $*$-representation $\rho : \mathcal{M} \to \mathcal{B}(L_2(\mathrm{G}; \mathcal{H}))$ is determined by the identity

$$[\rho(x)](\varphi)(g) := \alpha_{g^{-1}}(x)(\varphi(g)).$$

When G is discrete, the operator $\rho(x)$ takes the form

$$\rho(x) = \sum_{h \in \mathrm{G}} \alpha_{h^{-1}}(x) \otimes e_{h,h}$$

with $e_{g,h}$ the matrix units in $\mathcal{B}(\ell_2(\mathrm{G}))$. A generic element of $\mathcal{M} \rtimes_\alpha \mathrm{G}$ has the form $\sum_g f_g \rtimes_\alpha \lambda(g)$ with $f_g \in \mathcal{M}$. By playing with λ and ρ, it is easy to see that $\mathcal{M} \rtimes_\alpha \mathrm{G}$ sits in $\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2(\mathrm{G}))$:

$$\begin{aligned} \sum_g f_g \rtimes_\alpha \lambda(g) &= \sum_g \rho(f_g)(\mathbf{1}_{\mathcal{M}} \otimes \lambda(g)) = \sum_{g,h,h'} (\alpha_{h^{-1}}(f_g) \otimes e_{h,h})(\mathbf{1}_{\mathcal{M}} \otimes e_{gh',h'}) \\ &= \sum_{g,h} \alpha_{h^{-1}}(f_g) \otimes e_{h,g^{-1}h} = \sum_{g,h} \alpha_{g^{-1}}(f_{gh^{-1}}) \otimes e_{g,h}. \end{aligned}$$

When G is unimodular, the expression for $\rho(x)$ is replaced by a direct integral with respect to the Haar measure on G, and a generic element in $\mathcal{M} \rtimes_\alpha \mathrm{G}$ has the form $\int_{\mathrm{G}} f_g \rtimes \lambda(g)\, d\mu(g)$. Similar computations lead to the following formulae for the basic operations in the crossed product algebra:

- $(f \rtimes_\alpha \lambda(g))^* = \alpha_{g^{-1}}(f^*) \rtimes_\alpha \lambda(g^{-1})$,
- $(f \rtimes_\alpha \lambda(g))(f' \rtimes_\alpha \lambda(g')) = f\alpha_g(f') \rtimes_\alpha \lambda(gg')$,

Moreover, if ν denotes the trace in $\mathcal{M}$ we consider the trace

$$\nu \rtimes_\alpha \tau \bigg(\int_{\mathrm{G}} f_g \rtimes_\alpha \lambda(g)\, d\mu(g) \bigg) = \nu(f_e).$$

Since α will be fixed, we relax the notation and write $\int_{\mathrm{G}} f_g \rtimes \lambda(g)\, d\mu(g) \in \mathcal{M} \rtimes \mathrm{G}$.

Let us now consider semigroups $\mathcal{S}_\psi = (\mathcal{S}_{\psi,t})_{t \ge 0}$ of operators on $\mathcal{L}(\mathrm{G})$ which act diagonally on the trigonometric system. In other words, $\mathcal{S}_{\psi,t} : \lambda(g) \mapsto e^{-t\psi(g)}\lambda(g)$ for some function $\psi : \mathrm{G} \to \mathbb{R}_+$. The semigroup $\mathcal{S}_\psi = (\mathcal{S}_{\psi,t})_{t \ge 0}$ defines a *noncommutative Markov semigroup* when:

(i) $\mathcal{S}_{\psi,t}(\mathbf{1}_{\mathcal{L}(\mathrm{G})}) = \mathbf{1}_{\mathcal{L}(\mathrm{G})}$ for all $t \ge 0$.
(ii) Each $\mathcal{S}_{\psi,t}$ is normal and completely positive on $\mathcal{L}(\mathrm{G})$.
(iii) Each $\mathcal{S}_{\psi,t}$ is self-adjoint, i.e. $\tau((\mathcal{S}_{\psi,t} f)^* g) = \tau(f^*(\mathcal{S}_{\psi,t} g))$ for $f, g \in \mathcal{L}(\mathrm{G})$.
(iv) $\mathcal{S}_{\psi,t} f \to f$ as $t \to 0^+$ in the weak-$*$ topology of $\mathcal{L}(\mathrm{G})$.

These conditions are reminiscent of Stein's notion of diffusion semigroup [68]. They imply that $\mathcal{S}_{\psi,t}$ is completely contractive, trace preserving and also extends to a semigroup of contractions on $L_p(\mathcal{L}(\mathrm{G}))$ for any $1 \le p \le \infty$. As in the classical case, $\mathcal{S}_\psi$ always admits an infinitesimal generator

$$-A_\psi = \lim_{t\to 0} \frac{\mathcal{S}_{\psi,t} - \mathrm{id}_{\mathcal{L}(\mathrm{G})}}{t} \quad \text{with} \quad \mathcal{S}_{\psi,t} = \exp(-tA_\psi).$$

In the L_2 setting, A_ψ is an unbounded operator defined on

$$\mathrm{dom}_2(A_\psi) = \Big\{ f \in L_2(\widehat{\mathbf{G}}) \ \Big| \ \lim_{t\to 0} \frac{\mathcal{S}_{\psi,t} f - f}{t} \in L_2(\widehat{\mathbf{G}}) \Big\}.$$

As an operator in $L_2(\widehat{\mathbf{G}})$, A_ψ is positive and so we may define the subordinated Poisson semigroup $\mathcal{P}_\psi = (\mathcal{P}_{\psi,t})_{t\ge 0}$ by $\mathcal{P}_{\psi,t} = \exp(-t\sqrt{A_\psi})$. This is again a Markov semigroup. Note that P_t is chosen so that $(\partial_t^2 - A_\psi)\mathcal{P}_{\psi,t} = 0$. In general, we let $-A_{\psi,p}$ denote the generator of the realization of $\mathcal{S}_\psi = (\mathcal{S}_{\psi,t})_{t\ge 0}$ on $L_p(\mathcal{L}(\mathrm{G}))$. It should be noticed that $\ker A_{\psi,p}$ is a complemented subspace of $L_p(\mathcal{L}(\mathrm{G}))$. Let E_p denote the corresponding projection and $J_p = \mathrm{id}_{L_p(\widehat{\mathbf{G}})} - E_p$. Consider the complemented subspaces

$$L_p^\circ(\widehat{\mathbf{G}}) = J_p(L_p(\widehat{\mathbf{G}})) = \Big\{ f \in L_p(\widehat{\mathbf{G}}) \ \Big| \ \lim_{t\to\infty} \mathcal{S}_{\psi,t} f = 0 \Big\}.$$

The associated gradient form or "carré du champs" is defined as

$$\Gamma_\psi(f_1, f_2) = \tfrac{1}{2}\big(A_\psi(f_1^*)f_2 + f_1^* A_\psi(f_2) - A_\psi(f_1^* f_2)\big).$$

Since $\mathcal{S}_\psi = (\mathcal{S}_{\psi,t})_{t\ge 0}$ is a Fourier multiplier, we get $A_\psi(\lambda(g)) = \psi(g)\lambda(g)$ and

$$\Gamma_\psi(f_1, f_2) = \int_{\mathrm{G}\times\mathrm{G}} \overline{\widehat{f_1}(g)}\,\widehat{f_2}(h) \frac{\psi(g^{-1}) + \psi(h) - \psi(g^{-1}h)}{2} \lambda(g^{-1}h)\, d\mu(g)\, d\mu(h).$$

The crucial condition $\Gamma_\psi(f, f) \ge 0$ is characterized in the following subsection.

Length functions and cocycles. A *left cocycle* $(\mathcal{H}, \alpha, b)$ for the unimodular group G is a triple given by a Hilbert space $\mathcal{H}$, a continuous isometric action $\alpha : \mathrm{G} \to \mathrm{Aut}(\mathcal{H})$ and a continuous map $b : \mathrm{G} \to \mathcal{H}$ such that

$$\alpha_g(b(h)) = b(gh) - b(g).$$

A *right cocycle* satisfies the relation $\alpha_g(b(h)) = b(hg^{-1}) - b(g^{-1})$ instead. In this paper, we say that $\psi : \mathrm{G} \to \mathbb{R}_+$ is a *length function* if it vanishes at the identity e, $\psi(g) = \psi(g^{-1})$ and

$$\sum_g \beta_g = 0 \ \Rightarrow \ \sum_{g,h} \overline{\beta}_g \beta_h \psi(g^{-1}h) \le 0$$

for any finite family of coefficients β_g. Functions satisfying the last condition are called *conditionally negative*. It is straightforward to show that length functions take values in $\mathbb{R}_+$. In what follows, we only consider cocycles with values in real Hilbert spaces. Any cocycle $(\mathcal{H}, \alpha, b)$ gives rise to an associated length function $\psi_b(g) = \langle b(g), b(g)\rangle_{\mathcal{H}}$, as can be checked by the reader. Conversely, any length function ψ gives rise to a left and a right cocycle. This is a standard application of the ideas around Schoenberg's theorem [64], which states that $\psi : \mathrm{G} \to \mathbb{R}_+$ is a length function if and only if the associated semigroup $\mathcal{S}_\psi = (\mathcal{S}_{\psi,t})_{t\geq 0}$ given by $S_{\psi,t} : \lambda(g) \mapsto \exp(-t\psi(g))\lambda(g)$ is Markovian on $\mathcal{L}(\mathrm{G})$. Let us collect these well-known results.

Lemma B.1. *If $\psi : \mathrm{G} \to \mathbb{R}_+$ is a continuous length, then:*

(i) *The Gromov forms*

$$K_\psi^1(g,h) = \frac{\psi(g) + \psi(h) - \psi(g^{-1}h)}{2}, \qquad K_\psi^2(g,h) = \frac{\psi(g) + \psi(h) - \psi(gh^{-1})}{2}$$

define positive matrices on $\mathrm{G} \times \mathrm{G}$ *and lead to*

$$\langle f_1, f_2\rangle_{\psi,j} = \sum_{g,h} \widehat{f_1}(g) K_\psi^j(g,h) \widehat{f_2}(h)$$

on the group subalgebra $\mathbb{R}[\mathrm{G}]$ *of* $\mathcal{L}(\mathrm{G})$ *given by*

$$\mathbb{R}[\mathrm{G}] = \{\lambda(\widehat{f}) \mid \widehat{f} : \mathrm{G} \to \mathbb{R} \text{ finitely supported}\}.$$

(ii) *Let $\mathcal{H}_\psi^j$ be the Hilbert space completion of*

$$(\mathbb{R}[\mathrm{G}]/N_\psi^j, \langle\cdot,\cdot\rangle_{\psi,j}) \quad \text{with} \quad N_\psi^j = \text{null space of } \langle\cdot,\cdot\rangle_{\psi,j}.$$

The mappings $b_\psi^j : \mathrm{G} \ni g \mapsto \lambda(g) - \lambda(e) + N_\psi^j \in \mathcal{H}_\psi^j$ form left/right cocycles together with

$$\alpha_{\psi,g}^1(u + N_\psi^1) = \lambda(g)u + N_\psi^1, \qquad \alpha_{\psi,g}^2(u + N_\psi^2) = u\lambda(g^{-1}) + N_\psi^2,$$

which determine isometric actions $\alpha_\psi^j : \mathrm{G} \to \mathrm{Aut}(\mathcal{H}_\psi^j)$ of G *on $\mathcal{H}_\psi^j$.*

(iii) *If $\mathcal{H}_\psi^j$ is endowed with the discrete topology, then the semidirect product $\mathrm{G}_\psi^j = \mathcal{H}_\psi^j \rtimes \mathrm{G}$ becomes a unimodular group and we find the group homomorphisms*

$$\pi_\psi^1 : \mathrm{G} \ni g \mapsto b_\psi^1(g) \rtimes g \in \mathrm{G}_\psi^1, \qquad \pi_\psi^2 : \mathrm{G} \ni g \mapsto b_\psi^2(g^{-1}) \rtimes g \in \mathrm{G}_\psi^2.$$

The lemma allows one to introduce two pseudo-metrics on the unimodular group G in terms of the length function ψ. Indeed, a short calculation leads to the crucial identities

$$\psi(g^{-1}h) = \langle b_\psi^1(g) - b_\psi^1(h), b_\psi^1(g) - b_\psi^1(h)\rangle_{\psi,1} = \|b_\psi^1(g) - b_\psi^1(h)\|_{\mathcal{H}_\psi^1}^2,$$

$$\psi(gh^{-1}) = \langle b_\psi^2(g) - b_\psi^2(h), b_\psi^2(g) - b_\psi^2(h)\rangle_{\psi,2} = \|b_\psi^2(g) - b_\psi^2(h)\|_{\mathcal{H}_\psi^2}^2.$$

In particular,

$$\mathrm{dist}_1(g,h) = \sqrt{\psi(g^{-1}h)} = \|b_\psi^1(g) - b_\psi^1(h)\|_{\mathcal{H}_\psi^1}$$

defines a pseudo-metric on G, which becomes a metric when the cocycle map is injective. Similarly, we may work with $\mathrm{dist}_2(g,h) = \sqrt{\psi(gh^{-1})}$. When the cocycle map is not injective, the inverse image of 0,

$$\mathrm{G}_0 = \{g \in \mathrm{G} \mid \psi(g) = 0\},$$

is a subgroup. The following elementary observation is relevant.

Remark B.2. Let $(\mathcal{H}_1, \alpha_1, b_1)$ and $(\mathcal{H}_2, \alpha_2, b_2)$ be a left and a right cocycle on G. Assume that the associated length functions ψ_{b_1} and ψ_{b_2} coincide. Then we find an isometric isomorphism

$$\Lambda_{12} : \mathcal{H}_1 \ni b_1(g) \mapsto b_2(g^{-1}) \in \mathcal{H}_2.$$

In particular, given a length function ψ we see that $\mathcal{H}_\psi^1 \simeq \mathcal{H}_\psi^2$ via $b_\psi^1(g) \mapsto b_\psi^2(g^{-1})$.

Remark B.3. According to Schoenberg's theorem, Markov semigroups of Fourier multipliers in $\mathcal{L}(\mathrm{G})$ are in one-to-one correspondence with conditionally negative length functions $\psi : \mathrm{G} \to \mathbb{R}_+$. Lemma B.1 automatically gives

$$\begin{aligned}\Gamma_\psi(f,f) &= \tfrac{1}{2}\big(A_\psi(f^*)f + f^*A_\psi(f) - A_\psi(f^*f)\big)\\ &= \int_{\mathrm{G}\times\mathrm{G}} \overline{\widehat{f}(g)}\widehat{f}(h) K_\psi(g,h)\lambda(g^{-1}h)\, d\mu(g)\, d\mu(h) \geq 0.\end{aligned}$$

Theorem B.4. *Let Π_0 denote the space of trigonometric polynomials in $\mathcal{L}(\mathrm{G})$ whose Fourier coefficients have vanishing sum, as defined in the Introduction. A given function $\psi : \mathrm{G} \to \mathbb{R}_+$ defines a conditionally negative length if and only if there exists a positive linear functional $\tau_\psi : \Pi_0 \to \mathbb{C}$ satisfying the identity*

$$\psi(g) = \tau_\psi\big(2\lambda(e) - \lambda(g) - \lambda(g^{-1})\big).$$

Proof. Assume first that $\psi : \mathrm{G} \to \mathbb{R}_+$ satisfies the given identity for some positive linear functional $\tau_\psi : \Pi_0 \to \mathbb{C}$. To show that ψ is a conditionally negative length it suffices to construct a cocycle $(\mathcal{H}_\psi, \alpha_\psi, b_\psi)$ so that $\psi(g) = \langle b_\psi(g), b_\psi(g)\rangle_{\mathcal{H}_\psi}$. Since Π_0 is a $*$-subalgebra of $\mathcal{L}(\mathrm{G})$, $\langle f_1, f_2\rangle_{\mathcal{H}_\psi} = \tau_\psi(f_1^* f_2)$ is well-defined on $\Pi_0 \times \Pi_0$. If we quotient out the null space of this bracket, we may define $\mathcal{H}_\psi$ as the completion of such a quotient. As usual, we interpret $\mathcal{H}_\psi$ as a real Hilbert space by decomposing every element in Π_0 into its real and imaginary parts. If N_ψ denotes the null space, let

$$\alpha_{\psi,g}(f + N_\psi) = \lambda(g)f + N_\psi \quad \text{and} \quad b_\psi(g) = \lambda(g) - \lambda(e) + N_\psi.$$

It is easily checked that $(\mathcal{H}_\psi, \alpha_\psi, b_\psi)$ defines a left cocycle on G. Moreover, since $2\lambda(e) - \lambda(g) - \lambda(g^{-1}) = |\lambda(g) - \lambda(e)|^2$, our assumption can be rewritten in the form $\psi(g) = \langle b_\psi(g), b_\psi(g)\rangle_{\mathcal{H}_\psi}$ as expected.

Let us now prove the converse. Assume that $\psi : \mathrm{G} \to \mathbb{R}_+$ defines a conditionally negative length and define

$$\tau_\psi(\lambda(g) - \lambda(e)) = -\tfrac{1}{2}\psi(g).$$

Since the polynomials $\lambda(g) - \lambda(e)$ span Π_0, τ_ψ extends to a linear functional on Π_0 which satisfies $\tau_\psi(2\lambda(e) - \lambda(g) - \lambda(g^{-1}) = \frac{1}{2}(\psi(g) + \psi(g^{-1})) = \psi(g)$. Therefore, it just remains to show that $\tau_\psi : \Pi_0 \to \mathbb{C}$ is positive. Let $f = \sum_g a_g \lambda(g) \in \Pi_0$ so that $\sum_g a_g = 0$. In particular, we also have $f = \sum_g a_g(\lambda(g) - \lambda(e))$. By the conditional negativity of ψ we find

$$\begin{aligned}\tau_\psi(|f|^2) &= \sum_{g,h\in \mathrm{G}} \overline{a_g} a_h \tau_\psi\big((\lambda(g) - \lambda(e))^*(\lambda(h) - \lambda(e))\big)\\ &= \sum_{g,h\in \mathrm{G}} \overline{a_g} a_h \tau_\psi\big(\lambda(g^{-1}h) - \lambda(g^{-1}) - \lambda(h) + \lambda(e)\big)\\ &= \sum_{g,h\in \mathrm{G}} \overline{a_g} a_h \frac{\psi(g^{-1}) + \psi(h) - \psi(g^{-1}h)}{2} = -\frac{1}{2} \sum_{g,h\in \mathrm{G}} \overline{a_g} a_h \psi(g^{-1}h) \geq 0.\end{aligned}$$

This shows that our functional $\tau_\psi : \Pi_0 \to \mathbb{C}$ is positive. □

Appendix C. A geometric perspective

In this appendix we will describe tangent modules associated with a given length function, and how they can be combined with Riesz transform estimates. Recall that a *Hilbert module* over an algebra $\mathcal{A}$ is a vector space X with a bilinear map $m : \mathrm{X} \times \mathcal{A} \ni (\rho, a) \mapsto \rho a \in \mathrm{X}$ and a sesquilinear form $\langle\ ,\ \rangle : \mathrm{X} \times \mathrm{X} \to \mathcal{A}$ such that $\langle \rho, \eta a\rangle = \langle \rho, \eta\rangle a$, $\langle \rho a, \eta\rangle = a^*\langle \rho, \eta\rangle$ and $\langle \rho, \rho\rangle \geq 0$. We refer to Lance's book [42] for more information. Define $\mathcal{L}(\mathrm{X})$ as the C*-algebra of right-module maps T which admit an adjoint. That is, there exists a linear map $S : \mathrm{X} \to \mathrm{X}$ such that $\langle S\rho, \eta\rangle = \langle \rho, T\eta\rangle$. A *Hilbert bimodule* is additionally equipped with a $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{L}(\mathrm{X})$, and a *derivation* $\delta : \mathcal{A} \to \mathrm{X}$ is a linear map which satisfies the Leibniz rule

$$\delta(ab) = \pi(a)\delta(b) + \delta(a)b.$$

A typical example for such a derivation is given by an inclusion $\mathcal{A} \subset \mathcal{M}$ into some von Neumann algebra $\mathcal{M}$, a conditional expectation $E : \mathcal{M} \to \mathcal{A}''$, and a vector ρ such that $\delta(a) = a\rho - \rho a$. We have seen above that for a conditionally negative length function ψ, we can construct an associated (left) cocycle $(\mathcal{H}_\psi, \alpha_\psi, b_\psi)$. In the following, we will assume that the $\mathbb{R}$-linear span of $b_\psi(\mathrm{G})$ is $\mathcal{H}_\psi$.

We recall the Brownian functor $B : \mathcal{H}_\psi \to L_2(\Omega)$ which comes with an extended action α of G on $L_\infty(\Omega)$. This construction is usually called the *gaussian measure space construction*. Here the derivation is given by

$$\delta_\psi : \lambda(g) \mapsto B(b_\psi(g)) \rtimes \lambda(g).$$

We have already encountered the corresponding bimodule in the context of the Khintchine inequality. The proof of the following lemma is obvious. In fact, here left and right actions are induced by the actions on $\mathcal{M} = L_\infty(\Omega) \rtimes \mathrm{G}$.

Lemma C.1. *Let* G *be discrete. The Hilbert bimodule*

$$\Omega_\psi(\mathrm{G}) = \delta_\psi(\mathbb{C}[\mathrm{G}])(\mathbf{1} \rtimes \lambda(\mathrm{G})) \quad \textit{with} \quad \langle \rho, \eta \rangle = \mathsf{E}_{\mathcal{L}(\mathrm{G})}(\rho^* \eta)$$

is exactly given by the vector space $\mathrm{X}_\psi = \{\sum_g B(\xi_g) \rtimes \lambda(g) \mid \xi_g \in \mathcal{H}_\psi\}$.

Proof. For $\xi = b_\psi(g)$ we consider $\delta_\psi(\lambda(g))(\mathbf{1} \rtimes \lambda(g^{-1})) = B(\xi)$. Since $\mathcal{H}_\psi$ is the real linear span of such ξ's, we deduce that X_ψ is contained in $\delta_\psi(\mathbb{C}[\mathrm{G}])(\mathbf{1} \rtimes \lambda(\mathrm{G}))$. The converse is obvious. Moreover, since δ_ψ is a derivation, it is easy to see that the space $\delta(\mathbb{C}[\mathrm{G}])(\mathbf{1} \rtimes \lambda(\mathrm{G}))$ is invariant under the left action. □

Of course, the gaussian functor B is not really necessary to describe the bimodule $\Omega_\psi(\mathrm{G}) \simeq \mathcal{H}_\psi \rtimes \mathrm{G}$. Note that the product of two elements ρ, η in $L_2(L_\infty(\Omega) \rtimes \mathrm{G})$ is well-defined as an element of $L_1(L_\infty(\Omega) \rtimes \mathrm{G})$, so $\mathsf{E}_{\mathcal{L}(\mathrm{G})}(\rho^*\eta) = \langle \rho, \eta \rangle$ makes perfect sense. The following proposition shows that our previous results extend to the tangent module and not only to differential forms with 'constant' coefficients given by elements in $\mathcal{H}_\psi \subset \Omega_\psi(\mathrm{G})$. Given $\rho = \sum_{h \in \mathrm{G}} B(\xi_h) \rtimes \lambda(h) \in \Omega_\psi(\mathrm{G})$, define the extended Riesz transform in the direction of ρ as follows:

$$\begin{aligned} \mathbf{R}_{\psi,\rho} f &= \sum_{h \in \mathrm{G}} \lambda(h^{-1}) R_{\psi,\xi_h} f = 2\pi i \sum_{g,h \in \mathrm{G}} \frac{\langle b_\psi(g), \xi_h \rangle_{\mathcal{H}_\psi}}{\sqrt{\psi(g)}} \widehat{f}(g) \lambda(h^{-1}g) \\ &= 2\pi i \mathsf{E}_{\mathcal{L}(\mathrm{G})} \Big[\Big(\sum_{h \in \mathrm{G}} B(\xi_h) \rtimes \lambda(h) \Big)^* \Big(\sum_{g \in \mathrm{G}} \frac{\widehat{f}(g)}{\sqrt{\psi(g)}} B(b_\psi(g)) \rtimes \lambda(g) \Big) \Big] \\ &= 2\pi i \mathsf{E}_{\mathcal{L}(\mathrm{G})}(\rho^* \delta_\psi A_\psi^{-1/2} f) \end{aligned} \tag{C.1}$$

since

$$\int_\Omega B(\xi) B(\xi')\, d\mu = \langle \xi, \xi' \rangle_{\mathcal{H}_\psi}.$$

Note that we recover the Riesz transforms $R_{\psi,h}$ for $\rho = B(h) \rtimes \lambda(e)$.

Proposition C.2. *Given* $\rho, \rho_j \in \Omega_\psi(\mathrm{G})$, *and* $2 \le p < \infty$*:*

(i) $\|\mathbf{R}_{\psi,\rho} : L_p(\widehat{\mathbf{G}}) \to L_p(\widehat{\mathbf{G}})\| \lesssim_{c(p)} \|\rho\|_{\Omega_\psi(\mathrm{G})}$,

(ii) $\Big\| \Big(\sum_{j \ge 1} |\mathbf{R}_{\psi,\rho_j}(f)|^2 \Big)^{1/2} \Big\|_p \lesssim_{c(p)} \|(\langle \rho_j, \rho_k \rangle)\|^{1/2}_{\mathcal{B}(\ell_2)\bar{\otimes}\mathcal{L}(\mathrm{G})} \|f\|_p$,

(iii) $\Big\| \Big(\sum_{j \ge 1} |\mathbf{R}_{\psi,\rho_j}(f_j)|^2 \Big)^{1/2} \Big\|_p \lesssim_{c(p)} \sup_{j \ge 1} \|\mathsf{E}_{\mathcal{L}(\mathrm{G})}(\rho_j^* \rho_j)\|^{1/2}_{\mathcal{L}(\mathrm{G})} \Big\| \Big(\sum_{j \ge 1} |f_j|^2 \Big)^{1/2} \Big\|_p$.

Proof. Assertion (i) follows from (ii) or (iii). The second and third assertions follow from the well-known Cauchy–Schwarz inequality for conditional expectations, whose noncommutative form follows from Hilbert module theory [30]:

$$\|\mathsf{E}_{\mathcal{L}(\mathrm{G})} \otimes \mathrm{id}_{\mathcal{B}(\ell_2)}(xy)\|_p \le \|\mathsf{E}_{\mathcal{L}(\mathrm{G})} \otimes \mathrm{id}_{\mathcal{B}(\ell_2)}(xx^*)\|_\infty^{1/2} \|(\mathsf{E}_{\mathcal{L}(\mathrm{G})} \otimes \mathrm{id}_{\mathcal{B}(\ell_2)}(y^*y))^{1/2}\|_p.$$

Indeed, for (ii) we use (C.1) and pick

$$x = \sum_{j \geq 1} \rho_j^* \otimes e_{j1} \quad \text{and} \quad y = \delta_\psi A_\psi^{-1/2} f \otimes e_{11}.$$

Now, the inequality follows from Theorem A2. On the other hand, for (iii) we take

$$x = \sum_{j \geq 1} \rho_j^* \otimes e_{jj} \quad \text{and} \quad y = \sum_{j \geq 1} \delta_\psi A_\psi^{-1/2} f_j \otimes e_{j1}.$$

Then the result follows from the cb-extension of Theorem A2 in Remark 1.8. $\square$

Remark C.3. The 'adjoint' of $\mathbf{R}_{\psi,\rho}$ given by

$$\mathbf{R}_{\psi,\rho}^\dagger(f) = \mathbf{R}_{\psi,\rho}(f^*)^*$$

can be written as $\mathbf{R}_{\psi,\rho}^\dagger(f) = \mathsf{E}_{\mathcal{L}(\mathrm{G})}(\rho^* \delta_\psi A_\psi^{-1/2}(f^*))^* = -\mathsf{E}_{\mathcal{L}(\mathrm{G})}(\delta_\psi A_\psi^{-1/2}(f)\rho)$. Hence

$$\Big\| \Big(\sum_{j \geq 1} |\mathbf{R}_{\psi,\rho_j}^\dagger(f_j)^*|^2 \Big) \Big\|_p \lesssim_{c(p)} \sup_{j \geq 1} \|\mathsf{E}_{\mathcal{L}(\mathrm{G})}(\rho_j^* \rho_j)\|_{\mathcal{L}(\mathrm{G})}^{1/2} \Big\| \Big(\sum_{j \geq 1} |f_j^*|^2 \Big)^{1/2} \Big\|_p.$$

It is however, in general, difficult to find an element η such that $\mathbf{R}_{\psi,\eta}^\dagger(f) = \mathbf{R}_{\psi,\rho}(f)$ unless G is commutative and the action is trivial. This is a particular challenge if we want to extend the results from above literally to $p < 2$, because then we need both a row and a column bound to accommodate the decomposition $R(f) = a + b$ in the tangent module $\Omega_\psi(\mathrm{G})$.

Let us now indicate how to construct the corresponding real spectral triple. We first recall that $\Omega_\psi(\mathrm{G})$ is a quotient of the universal object $\Omega_\bullet(\mathrm{G}) \subset \mathbb{C}[\mathrm{G}] \otimes \mathbb{C}[\mathrm{G}]$ spanned by $\delta_\bullet(a)b$, where the universal derivation is $\delta_\bullet(a) = a \otimes \mathbf{1} - \mathbf{1} \otimes a$. In view of $J(a \otimes b) = b^* \otimes a^*$, the universal object becomes a real bimodule. In other words, the left and right representations $\pi_\ell(a)(\rho) = (a \otimes \mathbf{1})\rho$ and $\pi_r(a)(\rho) = \rho(\mathbf{1} \otimes a)$ are related via $J\pi_r(a^*)J = \pi_\ell(a)$. This implies in particular that $[a, JbJ] = 0$, and hence we find a real spectral triple. In our concrete situation, we have a natural isometry $J(x) = x^*$ on $\mathrm{M} = L_\infty(\Omega) \rtimes \mathrm{G}$, which leaves the subspace $\Omega_\psi(\mathrm{G}) \subset L_2(\mathrm{M})$ invariant. Hence $\Omega_\psi(\mathrm{G})$ is a quotient of $\Omega_\bullet(\mathrm{G})$. The Dirac operator for this spectral triple is easy to construct. The underlying Hilbert space is

$$\mathcal{H} = \overline{\Omega_\psi(\mathrm{G})} \oplus L_2(\widehat{\mathbf{G}})$$

where $\overline{\Omega_\psi(\mathrm{G})}$ denotes the closure of $\Omega_\psi(\mathrm{G})$ in $L_2(\mathrm{M})$, and

$$D_\psi = \begin{pmatrix} 0 & \delta_\psi \\ \delta_\psi^* & 0 \end{pmatrix}.$$

Note that $\delta_\psi^*(B(\xi) \rtimes \lambda(g)) = \langle \xi, b_\psi(g) \rangle \lambda(g)$ is densely defined. Using the diagonal representation for $\mathbb{C}[\mathrm{G}]$, we find that $T_g = [\delta_\psi, \lambda(g)]$ is a right module map from $L_2(\widehat{\mathbf{G}})$ to $\Omega_\psi(\mathrm{G})$ such that $T_g(x) = b_\psi(g) \rtimes (\lambda(g)x)$, and hence

$$\langle x, T_h^* T_g y \rangle = \langle b_\psi(h), b_\psi(g) \rangle \langle x, \lambda(h^{-1}g) y \rangle = (x \Gamma_\psi(\lambda(h), \lambda(g)) y)$$

for all $x, y \in L_2(\widehat{\mathbf{G}})$. It then follows that

$$\left\| \left[D_\psi, \begin{pmatrix} \pi(f) & 0 \\ 0 & f \end{pmatrix} \right] \right\|_{\mathcal{B}(\mathcal{H})} = \max\{\|\Gamma_\psi(f, f)\|_{\mathcal{L}(\mathrm{G})}^{1/2}, \|\Gamma_\psi(f^*, f^*)\|_{\mathcal{L}(\mathrm{G})}^{1/2}\}.$$

Here $\pi(\lambda(g))\delta(\lambda(h)) = \delta(\lambda(g)\lambda(h)) - \delta(\lambda(g))\lambda(h)$. This gives precisely the Lip-norm considered in [33]. The drawback, however, is that we have replaced the natural candidate $L_2(L_\infty(\Omega) \rtimes \mathrm{G})$ by the much 'smaller' module $\mathcal{H}$.

Replacing the gaussian construction by the corresponding free analogue, it is possible to work with a larger object. As in the gaussian category, given any Hilbert space $\mathcal{K}$, we have a function

$$s : \mathcal{K} \to \Gamma_0(\mathcal{K})$$

into the von Neumann algebra generated by free semicircular random variables and a representation $\alpha : O(\mathcal{K}) \to \mathrm{Aut}(\Gamma_0(\mathcal{K}))$ with $s(o(h)) = \alpha_o(s(h))$. This allows us to define $\delta_{\mathrm{free}}^\psi(\lambda(g)) = s(b_\psi(g)) \rtimes \lambda(g) \in \Gamma_0(\mathcal{H}_\psi) \rtimes \mathrm{G}$. The boundedness of the corresponding Riesz transforms

$$f \mapsto \delta_{\mathrm{free}}^\psi A_\psi^{-1/2} f$$

follows from the corresponding Khintchine inequality for $x = \sum_{\xi,g} s(\xi) \rtimes \lambda(g)$. Namely, we find

$$\|x\|_p \sim_C \max\{\|\mathsf{E}_{\mathcal{L}(\mathrm{G})}(x^*x)^{1/2}\|_p, \|\mathsf{E}_{\mathcal{L}(\mathrm{G})}(xx^*)^{1/2}\|_p\}$$

for $2 \le p < \infty$, and

$$\|x\|_{p'} \sim_C \inf_{x=x_1+x_2} \big(\|\mathsf{E}_{\mathcal{L}(\mathrm{G})}(x_1^*x_1)^{1/2}\|_p + \|\mathsf{E}_{\mathcal{L}(\mathrm{G})}(x_2x_2^*)^{1/2}\|_p\big).$$

In fact, it turns out that

$$\mathsf{E}_{\mathcal{L}(\mathrm{G})}(|x_{\mathrm{free}}|^2) = \mathsf{E}_{\mathcal{L}(\mathrm{G})}(|x_{\mathrm{gauss}}|^2)$$

for $x_{\mathrm{free}} = \sum_{\xi,g} s(\xi) \rtimes \lambda(g)$ and $x_{\mathrm{gauss}} = \sum_{\xi,g} B(\xi) \rtimes \lambda(g)$. This means the bimodule X_ψ can be realized either with independent gaussian or free semicircular variables, where $\mathrm{X}_\psi^{\mathrm{free}} = s(\mathcal{H}_\psi) \rtimes \mathrm{G} \subset \Gamma_0(\mathcal{H}_\psi) \rtimes \mathrm{G}$. Recall the natural inclusion of $\Gamma_0(\mathcal{H}_\psi) \rtimes \mathrm{G}$ into the Hilbert space $L_2(\Gamma_0(\mathcal{H}_\psi) \rtimes \mathrm{G}) \simeq L_2(\Gamma_0(\mathcal{H}_\psi)) \otimes \ell_2(\mathrm{G})$. More formally, we may denote by 1_τ the separating vector in the GNS construction and then find $\lambda(g)1_\tau = e_g$. The map D_{free}^ψ is densely defined on $L_2(\Gamma_0(\mathcal{H}_\psi) \rtimes \mathrm{G})$ as follows:

$$D_{\mathrm{free}}^\psi(a \otimes e_g) = -s(b_\psi(g^{-1}))a \otimes e_g.$$

Proposition C.4. *The tuple*

$$(\mathbb{C}[\mathrm{G}], L_2(\Gamma_0(\mathcal{H}_\psi) \rtimes \mathrm{G}), D_{\mathrm{free}}^\psi, J)$$

is a real spectral triple satisfying the following identities for $f = \sum_g \widehat{f}(g)\lambda(g)$ *in* $\mathbb{C}[\mathrm{G}] \subset \Gamma_0(\mathcal{H}_\psi) \rtimes \mathrm{G}$*:*

$$[D_{\mathrm{free}}^\psi, \lambda(g)] = s(b_\psi(g)) \rtimes \lambda(g),$$
$$\|[D_{\mathrm{free}}^\psi, f]\|_{\Gamma_0(\mathcal{H}_\psi) \rtimes \mathrm{G}} \sim \max\{\|\Gamma_\psi(f, f)\|_{\mathcal{L}(\mathrm{G})}^{1/2}, \|\Gamma_\psi(f^*, f^*)\|_{\mathcal{L}(\mathrm{G})}^{1/2}\}.$$

Proof. If $g \in \mathrm{G}$, we find

$$\begin{aligned}[D^{\psi}_{\mathrm{free}}, \lambda(g)](a \otimes e_h) &= D^{\psi}_{\mathrm{free}}(a \otimes e_{gh}) - \lambda(g)(-s(b_{\psi}(h^{-1})a \otimes e_h) \\ &= -s(b_{\psi}((gh)^{-1}))(a \otimes e_{gh}) + s(b_{\psi}(h^{-1}))(a \otimes e_{gh}) \\ &= \alpha^{-1}_{\psi,gh}(s(b_{\psi}(g)))a \otimes e_{gh} = \rho(s(b_{\psi}(g))) \circ \lambda(g)(a \otimes e_h),\end{aligned}$$

where $\rho(b)(c \otimes e_g) = \alpha^{-1}_{\psi,g}(b)c \otimes e_g$ on the tensor product. After the corresponding identifications in the inclusion $\Gamma_0(\mathcal{H}_{\psi}) \otimes \mathcal{L}(\mathrm{G}) \subset L_2(\Gamma_0(\mathcal{H}_{\psi}) \rtimes \mathrm{G})$, this implies $[D^{\psi}_{\mathrm{free}}, \lambda(g)] = \delta^{\psi}_{\mathrm{free}}(\lambda(g))$. The operation J is the adjoint for the crossed product, and hence $[\lambda(g), J(\lambda(h))J] = 0$ shows that we have obtained a real spectral triple (we ignore further compatibility properties for $D^{\psi}_{\mathrm{free}}, J$ at this point). By linearity we deduce that

$$[D^{\psi}_{\mathrm{free}}, f] = \delta^{\psi}_{\mathrm{free}}(f).$$

Now, we use a central limit procedure. Consider the crossed product $\Gamma_0(\ell_2^m(H)) \rtimes \mathrm{G}$. Then the copies $\pi_j(\Gamma_0(H) \rtimes \mathrm{G})$ given by the j-th coordinate are freely independent over $\mathcal{L}(\mathrm{G})$. Thus Voiculescu's inequality from [31] applies and yields, for any $\omega = \sum_{\xi,g} a_{\xi,g} s(\xi) \rtimes \lambda(g)$ and the sum of independent copies,

$$\begin{aligned}&\Big\| \sum_{j=1}^m a_{\xi,g} s(\xi \otimes e_j) \rtimes \lambda(g) \Big\|_{\Gamma_0 \rtimes \mathrm{G}} \\ &\qquad \le \|\omega\|_{\Gamma_0 \rtimes \mathrm{G}} + \sqrt{m}\, \|\mathsf{E}_{\mathcal{L}(\mathrm{G})}(\omega^*\omega)\|^{1/2}_{\mathcal{L}(\mathrm{G})} + \sqrt{m}\, \|\mathsf{E}_{\mathcal{L}(\mathrm{G})}(\omega\omega^*)\|^{1/2}_{\mathcal{L}(\mathrm{G})}.\end{aligned}$$

Dividing by $\sqrt{m}$ and observing that ω and $m^{-1/2} \sum_{j=1}^m a_{\xi,g} s(\xi \otimes e_j) \rtimes \lambda(g)$ are equal in distribution, we find indeed, letting $m \to \infty$,

$$\|\omega\|_{\Gamma_0 \rtimes \mathrm{G}} \sim \max\{\|\mathsf{E}_{\mathcal{L}(\mathrm{G})}(\omega^*\omega)\|^{1/2}_{\mathcal{L}(\mathrm{G})}, \|\mathsf{E}_{\mathcal{L}(\mathrm{G})}(\omega\omega^*)\|^{1/2}_{\mathcal{L}(\mathrm{G})}\}.$$

Thus for a differential form $\omega \in \mathrm{X}^{\mathrm{free}}_{\psi}(\mathrm{G})$, we get

$$\|\omega\|_{\mathrm{X}^{\mathrm{free}}_{\psi}} \sim \max\{\|\omega\|_{\mathrm{X}_{\psi}}, \|\omega^*\|_{\mathrm{X}_{\psi}}\}.$$

In particular, we conclude that

$$\|\delta^{\psi}_{\mathrm{free}}(f)\|_{\Gamma_0 \rtimes \mathrm{G}} \sim \max\{\|\Gamma_{\psi}(f, f)\|^{1/2}_{\mathcal{L}(\mathrm{G})}, \|\Gamma_{\psi}(f^*, f^*)\|^{1/2}_{\mathcal{L}(\mathrm{G})}\}.$$

This expression is certainly finite for $f \in \mathbb{C}[\mathrm{G}]$, and the proof is complete. □

It turns out that in the free case D^{ψ}_{free} cannot be extended to a global derivation on $\Gamma_0(\mathcal{H}_{\psi}) \rtimes \mathrm{G}$. On the other hand, for the gaussian case $\delta_{\psi}(\lambda(g)) = B(b_{\psi}(g)) \rtimes \lambda(g)$ does not belong to $L_{\infty}(\Omega) \rtimes \mathrm{G}$, and hence both models for generalized tangent spaces have their advantages and disadvantages.

Let us now return to the gaussian spectral triple on $\mathcal{H}$. As in [15], we have to deal with the fact that this spectral triple might have some degenerate parts, but in many calculations of the ζ-function of $|D_\psi|$ the kernel is usually ignored. We recall that for a self-adjoint operator D, the signature is defined as $\mathrm{sgn}(D) = D|D|^{-1}$. In our particular case, if $A_\psi^{-1}(e_g) = (1/\psi(g))e_g$ is a compact operator and $F = D_\psi|D_\psi|^{-1}$ is the corresponding signature, then it is well-known [15] that

$$\left[F, \begin{pmatrix} \pi(a) & 0 \\ 0 & a \end{pmatrix}\right]$$

is compact for all $a \in \mathbb{C}[\mathrm{G}]$. This follows from the boundedness of

$$\left[D_\psi, \begin{pmatrix} \pi(a) & 0 \\ 0 & a \end{pmatrix}\right]$$

[15, Proposition 2.4], and then [15, Proposition 2.7] applies. In our situation, $\delta_\psi = R_\psi A_\psi^{1/2}$, and hence δ_ψ vanishes on $\mathcal{H}_0 = \mathrm{span}\{e_g \mid \psi(g) = 0\}$. Clearly, $\delta_\psi^* \delta_\psi = A_\psi$ is the generator of our semigroup which also vanishes on $\mathcal{H}_0$. On the other hand,

$$\delta_\psi \delta_\psi^* = R_\psi A_\psi R_\psi^*,$$

and hence the range of $\delta_\psi \delta_\psi^*$ is given by the first Hodge projection $\Pi_{Hdg} = R_\psi R_\psi^*$. This can be described explicitly. Indeed, for g with $\psi(g) \neq 0$ we denote by Q_g the projection onto the span of $B(b_\psi(g)) \in L_2(\Omega)$ and get

$$R_\psi R_\psi^* = \sum_g Q_g \otimes e_{gg} \ \Rightarrow \ F = \begin{pmatrix} 0 & R_\psi \\ R_\psi^* & 0 \end{pmatrix}.$$

Problem C.5. *Show that* $F : \overline{\Omega_\psi(\mathrm{G}) + \mathcal{L}(\mathrm{G})} \to L_p(L_\infty(\Omega) \rtimes \mathrm{G})$*, where the closure is taken in* L_p*, admits dimension free estimates.*

Appendix D. Meyer's problem for Poisson

Let $\Delta = \partial_x^2$ be the laplacian operator on $\mathbb{R}^n$. The classical theory of semigroups of operators shows that the fractional laplacians $(-\Delta)^\beta$ with $0 < \beta < 1$ are closed densely defined operators on $L_p(\mathbb{R}^n)$ [74, Chapter 9, Section 11], and regarding them as convolution operators we see that the Schwartz class lies in the domain of any of them. Moreover, they generate Markov semigroups on $L_\infty(\mathbb{R}^n)$. When $\beta = 1/2$, we get the Poisson semigroups $P_t = \exp(-t\sqrt{-\Delta})$.

In this appendix we shall show that Meyer's problem (MP) fails for this generator when $p \leq 2n/(n+1)$. Recall that Theorem A2 confirms that (MP) holds for $p \geq 2$ and provides a substitute for $1 < p < 2$. Let us first give a formula for the corresponding carré du champs $\Gamma_{1/2}$.

Lemma D.1. *For any Schwartz function* f*, we have*

$$\Gamma_{1/2}(f, f) = \int_0^\infty P_t |\nabla P_t f|^2 \, dt$$

where $\nabla g(x,t) = (\partial_{x_1} g, \ldots, \partial_{x_n} g, \partial_t g)$ *includes spatial and time variables.*

Proof. Let $\varphi_t = |P_t f|^2$ and $F_t = (\partial_t P_t)(\varphi_t) - P_t(\partial_t \varphi_t)$. Since $\partial_t^2 P_t + \Delta P_t = 0$,

$$\partial_t F_t = (\partial_t^2 P_t)(\varphi_t) - P_t(\partial_t^2 \varphi_t) = -\Delta P_t(\varphi_t) - P_t(\partial_t^2 \varphi_t).$$

On the other hand, we may calculate

$$\begin{aligned}\partial_t^2 \varphi_t &= 2|\partial_t P_t f|^2 + (P_t f^*)(\partial_t^2 P_t f) + (\partial_t^2 P_t f^*)(P_t f) \\ &= 2|\partial_t P_t f|^2 - (P_t f^*)(\Delta P_t f) - (\Delta P_t f^*)(P_t f).\end{aligned}$$

Therefore, we get

$$\partial_t F_t = -2P_t(|\nabla P_t f|^2).$$

Note that $F_0 = \lim_{t\to 0} F_t = 2\Gamma_{1/2}(f, f)$ by the definition of carré du champ, and $F_t \to 0$ as $t \to \infty$. We get

$$2\Gamma_{1/2}(f, f) = \int_0^\infty -\partial_t F_t \, dt = 2\int_0^\infty P_t |\nabla P_t f|^2 \, dt. \qquad \square$$

Proposition D.2. *The equivalence* (MP) *fails for the Poisson semigroup* $P_t = e^{-t\sqrt{-\Delta}}$ *on* $\mathbb{R}^n$ *for any* $1 < p \le 2n/(n+1)$ *with* $n \ge 2$. *More precisely, for any nonzero Schwartz function* f *we have*

$$\Gamma_{1/2}(f, f)^{1/2} \notin L_p(\mathbb{R}^n) \quad \textit{for any } p \le \frac{2n}{n+1}.$$

Proof. We follow an argument from [25]. Fix a nonzero $f \in L_p(\mathbb{R}^n)$ and $|x| > 4$. Then

$$\begin{aligned}\Gamma_{1/2}(f, f)(x) &= \int_0^\infty P_t(|\nabla P_t f|^2)(x) \, dt \\ &= \int_0^\infty \int_{\mathbb{R}^n} c_n \frac{t}{(|x-y|^2 + t^2)^{(n+1)/2}} |\nabla P_t f(y)|^2 \, dy \, dt \\ &\ge c_n \int_1^2 \int_{|y|<1} \frac{1}{|x|^{n+1}} |\nabla P_t f(y)|^2 \, dy \, dt = c_n c_f \frac{1}{|x|^{n+1}}\end{aligned}$$

for $c_f = \int_1^2 \int_{|y|<1} |\nabla P_t f(y)|^2 \, dy \, dt > 0$ (since $f \neq 0$) and any $|x| > 4$. Then

$$\Gamma_{1/2}(f, f)^{1/2} \in L_p(\mathbb{R}^n) \;\Rightarrow\; \frac{p}{2}(n+1) > n \;\Rightarrow\; p > \frac{2n}{n+1}.$$

We then conclude that

$$\|\Gamma_{1/2}(f, f)^{1/2}\|_p = \infty \quad \text{while} \quad \|(-\Delta)^{1/4} f\|_p < \infty$$

for any nonzero Schwartz function f with $p \le 2n/(n+1)$. Therefore, (MP) fails. $\square$

Thus, our revision of (MP) in this paper is needed even for commutative semigroups.

Remark D.3. According to the proof of Proposition D.2, there exists $f \in \mathcal{S}(\mathbb{R}^n)$ such that $(-\Delta)^{1/4} f \in L_p(\mathbb{R}^n)$ but $\Gamma_{1/2}(f, f) = \mathsf{E}_{\mathcal{L}(\mathbb{R}^n)}(\delta_{1/2} f^* \delta_{1/2} f)$ does not belong to $L_{p/2}(\mathbb{R}^n)$. This should be compared with Theorem A2 for $\mathrm{G} = \mathbb{R}^n$, which states that there is a decomposition $\delta_{1/2} f = \phi_1 + \phi_2$ with $\phi_1, \phi_2 \in L_p(L_\infty(\Omega) \rtimes \mathbb{R}^n)$ and such that $\mathsf{E}_{\mathcal{L}(\mathbb{R}^n)}(\phi_1^* \phi_1)$ and $\mathsf{E}_{\mathcal{L}(\mathbb{R}^n)}(\phi_2 \phi_2^*)$ belong to $L_{p/2}(\mathbb{R}^n)$. On the other hand, by [30, Proposition 2.8] we know that

$$\|\Gamma_{1/2}(f, f)^{1/2}\|_p \le \|\mathsf{E}_{\mathcal{L}(\mathbb{R}^n)}(\phi_1^* \phi_1)^{1/2}\|_p + \|\mathsf{E}_{\mathcal{L}(\mathbb{R}^n)}(\phi_2^* \phi_2)^{1/2}\|_p.$$

This implies that

$$\mathsf{E}_{\mathcal{L}(\mathbb{R}^n)}(\phi_2^* \phi_2) \notin L_{p/2}(\mathbb{R}^n),$$

even if we know that $\phi_2 \in L_p(L_\infty(\Omega) \rtimes \mathbb{R}^n)$. We recover the known fact: for $p/2 < 1$,

$$\|\mathsf{E}_{\mathcal{L}(\mathbb{R}^n)}(\phi_2^* \phi_2)\|_{L_{p/2}(\mathbb{R}^n)} \not\lesssim \|\phi_2^* \phi_2\|_{L_{p/2}(L_\infty(\Omega) \rtimes \mathbb{R}^n)} = \|\phi_2\|^2_{L_p(L_\infty(\Omega) \rtimes \mathbb{R}^n)}.$$

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