J. Eur. Math. Soc. 20, 657–745

DOI 10.4171/JEMS/775

Thomas Alazard · Pietro Baldi · Daniel Han-Kwan

Control of water waves

Received April 15, 2015

Abstract. We prove local exact controllability in arbitrarily short time of the two-dimensional incompressible Euler equation with free surface, in the case with surface tension. This proves that one can generate arbitrarily small amplitude periodic gravity-capillary water waves by blowing on a localized portion of the free surface of a liquid.

Keywords. Controllability, water waves, capillarity (surface tension), Ingham inequality, paradifferential calculus

Contents

T. Alazard: CNRS et Département de Mathématiques et Applications UMR 8553, École Normale Supérieure, 45 rue d'Ulm, Paris F-75005, France; e-mail: thomas.alazard@cmla.ens-cachan.fr

P. Baldi: Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Università di Napoli Federico II, Via Cintia, 80126 Napoli, Italy; e-mail: pietro.baldi@unina.it

D. Han-Kwan: CNRS et Centre de Mathématiques Laurent Schwartz UMR 7640, École Polytechnique, 91128 Palaiseau Cedex, France; e-mail: daniel.han-kwan@polytechnique.edu

Mathematics Subject Classification (2010): 93B05; 76B45; 35S50

1. Introduction

Water waves are disturbances of the free surface of a liquid. They are, in general, produced by the immersion of a solid body, the oscillation of a solid portion of the boundary or impulsive pressures applied on the free surface. The question we address in this paper is the following: which waves can be generated from the rest position by a localized pressure distribution applied on the free surface? This question is strictly related to the generation of waves in a pneumatic wave maker (see [47, §21], [16]). Our main result asserts that, in arbitrarily small time, one can generate any small amplitude, two-dimensional, gravity-capillary water waves. This is a result from control theory. More precisely, this article is devoted to the study of the local exact controllability of the incompressible Euler equation with free surface.

There are many known control results for linear or nonlinear equations (see the book of Coron [17]), including equations describing water waves in some asymptotic regimes, like Benjamin–Ono [36, 33], KdV [44, 34] or nonlinear Schrödinger equation [20]. In this paper, instead, we consider the full model, that is, the incompressible Euler equation with free surface. Two key properties of this equation are that it is quasi-linear (instead of semilinear as Benjamin–Ono, KdV or NLS) and it is not a partial differential equation but instead a pseudo-differential equation, involving the Dirichlet–Neumann operator which is nonlocal and also depends nonlinearly on the unknown. As we explain later in this introduction, this requires introducing new tools to prove the controllability.

To our knowledge, this is the first control result for a quasi-linear wave equation relying on propagation of energy. In particular, using dispersive properties of gravity-capillary water waves (namely the infinite speed of propagation), we prove that, for any control domain, one can control the equation in arbitrarily small time intervals.

1.1. Main result

We consider the dynamics of an incompressible fluid moving under the force of gravitation and surface tension. At time t, the fluid domain $\Omega(t)$ has a rigid bottom and a free surface described by the equation $y = \eta(t, x)$, so that

$$\Omega(t) = \{(x, y) \in \mathbb{R}^2 \,;\, -b < y < \eta(t, x)\},$$

for some positive constant b (our result also holds in infinite depth, for $b = \infty$). The Eulerian velocity field v is assumed to be irrotational. It follows that $v = \nabla_{x,y}\phi$ for some time-dependent potential ϕ satisfying

$$\Delta_{x,y}\phi = 0, \qquad \partial_t\phi + \tfrac{1}{2}|\nabla_{x,y}\phi|^2 + gy = P, \qquad \partial_y\phi|_{y=-b} = 0, \tag{1.1}$$

where $g > 0$ is the gravity acceleration, $-P$ is the pressure (we prefer to change the sign for notational convenience), $\nabla_{x,y} = (\partial_x, \partial_y)$ and $\Delta_{x,y} = \partial_x^2 + \partial_y^2$. The water waves equations are given by two boundary conditions on the free surface: firstly,

$$\partial_t\eta = \sqrt{1 + (\partial_x\eta)^2}\,\partial_n\phi|_{y=\eta}$$

where ∂_n is the outward normal derivative, so $\sqrt{1+(\partial_x\eta)^2}\,\partial_n\phi = \partial_y\phi - (\partial_x\eta)\partial_x\phi$. Secondly, the balance of forces across the free surface reads

$$P|_{y=\eta} = \kappa H(\eta) + P_{\text{ext}}(t,x)$$

where κ is a positive constant, P_{ext} is an external source term and $H(\eta)$ is the curvature:

$$H(\eta) := \partial_x\left(\frac{\partial_x\eta}{\sqrt{1+(\partial_x\eta)^2}}\right) = \frac{\partial_x^2\eta}{(1+(\partial_x\eta)^2)^{3/2}}.$$

Following Zakharov [50] and Craig and Sulem [19], it is equivalent to work with the trace of ϕ at the free boundary

$$\psi(t,x) = \phi(t,x,\eta(t,x)),$$

and introduce the Dirichlet–Neumann operator $G(\eta)$ that relates ψ to the normal derivative $\partial_n\phi$ of the potential by

$$(G(\eta)\psi)(t,x) = \sqrt{1+(\partial_x\eta)^2}\,\partial_n\phi|_{y=\eta(t,x)}.$$

Hereafter the surface tension coefficient κ is taken to be 1. Then (η,ψ) solves (see [19]) the system

$$\begin{cases} \partial_t\eta = G(\eta)\psi, \\ \partial_t\psi + g\eta + \dfrac{1}{2}(\partial_x\psi)^2 - \dfrac{1}{2}\dfrac{\big(G(\eta)\psi + (\partial_x\eta)(\partial_x\psi)\big)^2}{1+(\partial_x\eta)^2} = H(\eta) + P_{\text{ext}}. \end{cases} \tag{1.2}$$

This system is augmented with initial data

$$\eta|_{t=0} = \eta_{\text{in}}, \qquad \psi|_{t=0} = \psi_{\text{in}}. \tag{1.3}$$

We consider the case when η and ψ are 2π-periodic in the space variable x and we set $\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$. Recall that the mean value of η is conserved in time and can be taken to be 0 without loss of generality. We thus introduce the Sobolev spaces $H_0^\sigma(\mathbb{T})$ of functions with mean value 0. Our main result asserts that, given any control domain ω and an arbitrary control time $T > 0$, equation (1.2) is controllable in time T for small enough data.

Theorem 1.1. *There exists $\sigma > 0$ such that the following holds. Let $T > 0$ and consider a nonempty open subset $\omega \subset \mathbb{T}$. There exists a positive constant M_0 small enough such that, for any $(\eta_{\text{in}},\psi_{\text{in}})$, $(\eta_{\text{final}},\psi_{\text{final}}) \in H_0^{\sigma+1/2}(\mathbb{T}) \times H^\sigma(\mathbb{T})$ satisfying*

$$\|\eta_{\text{in}}\|_{H^{\sigma+1/2}} + \|\psi_{\text{in}}\|_{H^\sigma} < M_0, \qquad \|\eta_{\text{final}}\|_{H^{\sigma+1/2}} + \|\psi_{\text{final}}\|_{H^\sigma} < M_0,$$

there exists P_{ext} in $C^0([0,T];H^\sigma(\mathbb{T}))$, supported in $[0,T]\times\omega$, that is,

$$\operatorname{supp} P_{\text{ext}}(t,\cdot) \subset \omega, \qquad \forall t \in [0,T],$$

such that the Cauchy problem (1.2)–(1.3) has a unique solution

$$(\eta,\psi) \in C^0([0,T];H_0^{\sigma+1/2}(\mathbb{T}) \times H^\sigma(\mathbb{T})),$$

and the solution (η,ψ) satisfies $(\eta|_{t=T},\psi|_{t=T}) = (\eta_{\text{final}},\psi_{\text{final}})$.

Remark 1.2. (i) This result holds for any $T > 0$ and not only for T large enough. Compared to the Cauchy problem, for the control problem it is more difficult to work on short time intervals than on large time intervals.

(ii) This result also holds in the infinite depth case (it suffices to replace $\tanh(b|\xi|)$ by 1 in the proof). In finite depth, the noncavitation assumption $\eta(t,x) > -b$ holds automatically for small enough solutions.

1.2. Strategy of the proof

We conclude this introduction by explaining the strategy of the proof and the difficulties one has to cope with.

Remarks about the linearized equation. We use in an essential way the fact that the water waves equation is a dispersive equation. This is used to obtain a control result which holds on arbitrarily small time intervals. To explain this, as well as to introduce the control problem, we begin with the analysis of the linearized equation around the null solution. Recall that $G(0)$ is the Fourier multiplier $|D_x|\tanh(b|D_x|)$. After removing quadratic and higher order terms in the equation, system (1.2) becomes

$$\begin{cases} \partial_t \eta = G(0)\psi, \\ \partial_t \psi + g\eta - \partial_x^2 \eta = P_{\text{ext}}. \end{cases}$$

Introduce the Fourier multiplier (of order $3/2$)

$$L := ((g - \partial_x^2)G(0))^{1/2}.$$

The operator $G(0)^{-1}$ is well-defined on periodic functions with mean value zero. Then $u = \psi - iLG(0)^{-1}\eta$ satisfies the dispersive equation

$$\partial_t u + iLu = P_{\text{ext}}.$$

To our knowledge, the first control result for this linear equation is due to Reid [45] who proved a result with a distributed control. He proved that one can steer any initial data to zero in finite time using a control of the form $P_{\text{ext}}(t,x) = g(x)U(t)$ (g is given and U is unknown). His proof is based on the characterization of Riesz basis and a variant of Ingham's inequality (see (1.12)). In this paper we are interested in localized control, satisfying $P_{\text{ext}}(t,x) = \mathbf{1}_\omega P_{\text{ext}}(t,x)$ where $\omega \subset \mathbb{T}$ is a given open subset. However, using the same Ingham inequality (1.12) and the HUM method, one obtains a variant of Reid's control result where the control is localized. We also refer the reader to the articles by Miller [41] and Lissy [37] for other control results about dispersive equations involving a fractional Laplacian.

Step 1: Reduction to a dispersive equation. The proof of Theorem 1.1 relies on various tools and various previous results. Firstly, Theorem 1.1 is related to the study of the Cauchy problem. The literature on the subject goes back to the pioneering works of

Nalimov [43], Yosihara [49] and Craig [18]. There are many results and we quote only some of them starting with the well-posedness of the Cauchy problem without smallness assumption, which was first proved by Wu [48] and Beyer–Günther [12] for the case with surface tension. For some recent results about gravity-capillary waves, we refer to Christianson–Hur–Staffilani [15], Germain–Masmoudi–Shatah [21], Iguchi [24], Ifrim–Tataru [23], Ionescu–Pusateri [25, 26], Mésognon-Gireau [38] and Ming–Rousset–Tzvetkov [42].

Our study is based on the analysis of the Eulerian formulation of the water waves equations by means of microlocal analysis. In this respect it is influenced by Lannes [30] as well as [5, 2]. More precisely, we use a paradifferential approach in order to paralinearize the water waves equations and then to symmetrize the resulting equations. We refer the reader to the appendix for the definition of paradifferential operators T_a.

It is proved in [2] that one can reduce the water waves equations to a single dispersive wave equation that is similar to the linearized equation. Namely, it is proved there that there are symbols $p = p(t, x, \xi)$ and $q = q(t, x, \xi)$, with p of order 0 in ξ and q of order 1/2, such that $u = T_p \psi + i T_q \eta$ satisfies an equation of the form

$$P(u)u = P_{\text{ext}} \quad \text{with} \quad P(u) := \partial_t + T_{V(u)}\partial_x + i L^{1/2}(T_{c(u)} L^{1/2}\,\cdot),$$

where $L^{1/2} = ((g - \partial_x^2)G(0))^{1/4}$, and $T_{V(u)}$ and $T_{c(u)}$ are paraproducts. Here V, c depend on the unknown u with $V(0) = 0$ and $c(0) = 1$, and hence $P(0) = \partial_t + iL$ is the linearized operator around the null solution. We have oversimplified the result (neglecting remainder terms and simplifying the dependence of V, c on u) and we refer to Proposition 2.5 for the full statement.

We complement the analysis of [2] in two directions. Firstly, using elementary arguments (Neumann series and the implicit function theorem), we prove that one can invert the mapping $(\eta, \psi) \mapsto u$. Secondly, we prove that, up to modifying the subprincipal symbols of p and q, one can further require that

$$\int_{\mathbb{T}} \operatorname{Im} u(t, x)\, dt = 0. \tag{1.4}$$

Step 2: Quasi-linear scheme. Since the water waves system (1.2) is quasi-linear, one cannot deduce the controllability of the nonlinear equation from the one of $P(0)$. Instead of using a fixed point argument, we use a quasi-linear scheme and seek P_{ext} as the limit of *real-valued* functions P_n determined by means of approximate control problems. To guarantee that P_{ext} will be real-valued we seek P_n as the real part of some function. To ensure that $\operatorname{supp} P_n \subset \omega$ we seek P_n of the form

$$P_n = \chi_\omega \operatorname{Re} f_n.$$

Hereafter, we fix ω, a nonempty open subset of $\mathbb{T}$, and a C^∞ cut-off function χ_ω, supported on ω, such that $\chi_\omega(x) = 1$ for all x in some open interval $\omega_1 \subset \omega$.

The approximate control problems are defined by induction as follows: we choose f_{n+1} by requiring that the unique solution u_{n+1} of the Cauchy problem

$$P(u_n)u_{n+1} = \chi_\omega \operatorname{Re} f_{n+1}, \qquad u_{n+1}|_{t=0} = u_{\text{in}},$$

satisfies $u(T) = u_{\text{final}}$. Our goal is to prove that

- this scheme is well-defined (that is, one has to prove a controllability result for $P(u_n)$);
- the sequences (f_n) and (u_n) are bounded in $C^0([0,T]; H^\sigma(\mathbb{T}))$;
- the series $\sum(f_{n+1} - f_n)$ and $\sum(u_{n+1} - u_n)$ converge in $C^0([0,T]; H^{\sigma-3/2}(\mathbb{T}))$.

It follows that (f_n) and (u_n) are Cauchy sequences in $C^0([0,T]; H^{\sigma-3/2}(\mathbb{T}))$ (and in fact, by interpolation, in $C^0([0,T]; H^{\sigma'}(\mathbb{T}))$ for any $\sigma' < \sigma$).

To use the quasi-linear scheme, we need to study a sequence of linear approximate control problems. The key point is to study the control problem for the linear operator $P(\underline{u})$ for some given function $\underline{u}$. Our goal is to prove the following result.

Proposition 1.3. *Let $T > 0$. There exists s_0 such that if $\|\underline{u}\|_{C^0([0,T];H^{s_0})}$ is small enough, depending on T, then the following properties hold.*

(i) (Controllability) *For all $\sigma \geq s_0$ and all*

$$u_{\text{in}}, u_{\text{final}} \in \tilde{H}^\sigma(\mathbb{T}) := \left\{ w \in H^\sigma(\mathbb{T})\,;\ \operatorname{Im} \int_{\mathbb{T}} w(x)\,dx = 0 \right\},$$

there exists f satisfying $\|f\|_{C^0([0,T];H^\sigma)} \leq K(\sigma, T)(\|u_{\text{in}}\|_{H^\sigma} + \|u_{\text{final}}\|_{H^\sigma})$ such that the unique solution u to

$$P(\underline{u})u = \chi_\omega \operatorname{Re} f, \qquad u|_{t=0} = u_{\text{in}},$$

satisfies $u(T) = u_{\text{final}}$.

(ii) (Stability) *Consider another state $\underline{u}'$ with $\|\underline{u}'\|_{C^0([0,T];H^{s_0})}$ small enough and denote by f' the control associated to $\underline{u}'$. Then*

$$\|f - f'\|_{C^0([0,T];H^{\sigma-3/2})} \leq K'(\sigma, T)(\|u_{\text{in}}\|_{H^\sigma} + \|u_{\text{final}}\|_{H^\sigma})\|\underline{u} - \underline{u}'\|_{C^0([0,T];H^{s_0})}.$$

Remark 1.4. (i) We oversimplified the assumptions and refer the reader to Section 9 for the full statement.

(ii) Notice that the smallness assumption on $\underline{u}$ involves only some H^{s_0}-norm, while the result holds for all initial data in H^σ with $\sigma \geq s_0$. This is possible because we consider a paradifferential equation. This plays a key role in the analysis to overcome losses of derivatives with respect to the coefficients.

Step 3: Reduction to a regularized problem. We next reduce the analysis by proving that it is sufficient

- to consider a *classical* equation instead of a *paradifferential* equation;
- to prove an L^2 result instead of a Sobolev result.

This is obtained by conjugating $P(\underline{u})$ with some well-chosen elliptic operator $\Lambda_{h,s}$ of order s with

$$s = \sigma - 3/2$$

and depending on a small parameter h (the reason to introduce h is explained below). In particular $\Lambda_{h,s}$ is chosen so that the operator

$$\widetilde{P}(\underline{u}) := \Lambda_{h,s} P(\underline{u}) \Lambda_{h,s}^{-1}$$

satisfies

$$\widetilde{P}(\underline{u}) = P(\underline{u}) + R(\underline{u}) \tag{1.5}$$

where $R(\underline{u})$ is a remainder term of order 0. For instance, if $s = 3m$ with $m \in \mathbb{N}$, set

$$\Lambda_{h,s} = I + h^s \mathcal{L}^{2s/3} \quad \text{where} \quad \mathcal{L} := L^{1/2}(T_c L^{1/2} \cdot).$$

With this choice one has $[\Lambda_{h,s}, \mathcal{L}] = 0$, so (1.5) holds with $R(\underline{u}) = [\Lambda_{h,s}, T_{V(\underline{u})}]\Lambda_{h,s}^{-1}$. It follows from symbolic calculus that $\|R(\underline{u})\|_{\mathcal{L}(L^2)} \lesssim \|V\|_{W^{1,\infty}}$ uniformly in h.

Moreover, since $V(\underline{u})$ and $c(\underline{u})$ are continuous in time with values in $H^{s_0}(\mathbb{T})$ with s_0 large, one can replace paraproducts by usual products, up to remainder terms in $C^0([0,T]; \mathcal{L}(L^2))$. We have

$$\widetilde{P}(\underline{u}) = \partial_t + V(\underline{u})\partial_x + iL^{1/2}(c(\underline{u})L^{1/2} \cdot) + R_2(\underline{u})$$

where

$$R_2(\underline{u}) := R(\underline{u}) + (T_{V(\underline{u})} - V(\underline{u}))\partial_x + iL^{1/2}\big((T_{c(\underline{u})} - c(\underline{u}))L^{1/2} \cdot\big).$$

The remainder $R_2(\underline{u})$ belongs to $C^0([0,T]; \mathcal{L}(L^2))$ uniformly in h. On the other hand,

$$\|[\Lambda_{h,s}, \chi_\omega]\Lambda_{h,s}^{-1}\|_{\mathcal{L}(L^2)} = O(h), \tag{1.6}$$

which is the reason to introduce the parameter h. The key point is that one can reduce the proof of Proposition 1.3 to the proof of the following result.

Proposition 1.5. *Let $T > 0$. There exists s_0 such that if $\|\underline{u}\|_{C^0([0,T]; H^{s_0})}$ is small enough, then the following properties hold.*

(i) (Controllability) *For all $v_{\text{in}} \in L^2(\mathbb{T})$ there exists f with $\|f\|_{C^0([0,T]; L^2)} \le K(T)\|v_{\text{in}}\|_{L^2}$ such that the unique solution v to $\widetilde{P}(\underline{u})v = \chi_\omega \operatorname{Re} f$, $v|_{t=0} = v_{\text{in}}$, is such that $v(T)$ is an imaginary constant:*

$$\exists b \in \mathbb{R}\ \forall x \in \mathbb{T}, \qquad v(T,x) = ib.$$

(ii) (Regularity) $\|f\|_{C^0([0,T]; H^{3/2})} \le K(T)\|v_{\text{in}}\|_{H^{3/2}}$.

(iii) (Stability) *Consider another state $\underline{u}'$ with $\|\underline{u}'\|_{C^0([0,T]; H^{s_0})}$ small enough and denote by f' the control associated to $\underline{u}'$. Then*

$$\|f - f'\|_{C^0([0,T]; L^2)} \le K'(T)\|v_{\text{in}}\|_{H^{3/2}}\|\underline{u} - \underline{u}'\|_{C^0([0,T]; H^{s_0})}.$$

Let us explain how to deduce Proposition 1.3 from the latter proposition. Consider $u_{\text{in}}, u_{\text{final}}$ in $\tilde{H}^\sigma(\mathbb{T})$ and seek $f \in C^0([0,T]; H^\sigma(\mathbb{T}))$ such that

$$[P(\underline{u})u = \chi_\omega \operatorname{Re} f,\ u(0) = u_{\text{in}}] \ \Rightarrow\ u(T) = u_{\text{final}}.$$

Since the equation is reversible in time, one can exchange initial and final states, and hence it is sufficient to consider the case where $u_{\text{final}} = 0$. Now, to deduce this result from Proposition 1.5, the main difficulty is that conjugation with $\Lambda_{h,s}$ introduces a nonlocal term: indeed, $\Lambda_{h,s}^{-1}(\chi_\omega f)$ is not compactly supported in general. This is a possible source of difficulty since we seek a localized control term. We overcome this by considering the control problem for $\widetilde{P}(\underline{u})$ associated to some well-chosen initial data v_{in}. Proposition 1.5 asserts that for all $v_{\text{in}} \in H^{3/2}(\mathbb{T})$ there is $\tilde{f} \in C^0([0,T]; H^{3/2}(\mathbb{T}))$ such that

$$[\widetilde{P}(\underline{u})v_1 = \chi_\omega \operatorname{Re} \tilde{f},\ v_1|_{t=0} = v_{\text{in}}] \ \Rightarrow\ v_1(T,x) = ib,\ b \in \mathbb{R}.$$

Define $\mathcal{K}v_{\text{in}} = v_2(0)$ where v_2 is the solution to

$$\widetilde{P}(\underline{u})v_2 = [\Lambda_{h,s}, \chi_\omega]\Lambda_{h,s}^{-1} \operatorname{Re} \tilde{f}, \qquad v_2|_{t=T} = 0.$$

Using (1.6) one can prove that the $\mathcal{L}(H^{3/2})$-norm of $\mathcal{K}$ is $O(h)$ and hence $I + \mathcal{K}$ is invertible for h small. So, v_{in} can be so chosen that $v_{\text{in}} + \mathcal{K}v_{\text{in}} = \Lambda_{h,s}u_{\text{in}}$. Then, setting $f := \Lambda_{h,s}^{-1}\tilde{f}$ and $u := \Lambda_{h,s}^{-1}(v_1 + v_2)$, one checks that

$$P(\underline{u})u = \chi_\omega \operatorname{Re} f, \qquad u(0) = u_{\text{in}}, \qquad u(T,x) = ib,\ b \in \mathbb{R}.$$

It remains to prove that $u(T)$ is not only an imaginary constant, but it is 0. This follows from the property (1.4). Indeed, P can be so defined that if $P(\underline{u})u$ is a real-valued function, then $\frac{d}{dt}\int_{\mathbb{T}} \operatorname{Im} u(t,x)\,dx = 0$. Since $\int_{\mathbb{T}} \operatorname{Im} u(0,x)\,dx = 0$ by assumption, one deduces that $\int_{\mathbb{T}} \operatorname{Im} u(T,x)\,dx = 0$ and hence $u(T) = 0$.

Step 4: Reduction to a constant coefficient equation. The controllability of $\widetilde{P}(\underline{u})$ will be deduced from the classical HUM method. A key step consists in proving that some bilinear mapping is coercive. To determine the bilinear mapping, we follow an idea introduced in [1] and conjugate $\widetilde{P}(\underline{u})$ to a constant coefficient operator modulo a remainder term of order 0.

To do so, we use a change of variables and a pseudo-differential change of unknowns to find an operator $M(\underline{u})$ such that

$$M(\underline{u})\widetilde{P}(\underline{u})M(\underline{u})^{-1} = \partial_t + iL + \mathcal{R}(\underline{u}),$$

where $\|\mathcal{R}(\underline{u})\|_{\mathcal{L}(L^2)} \lesssim \|\underline{u}\|_{H^{s_0}}$ (and hence $\mathcal{R}(\underline{u})$ is a *small* perturbation of order 0).

To find $M(\underline{u})$, we begin by considering three changes of variables of the form

$$(1 + \partial_x\kappa(t,x))^{1/2}h(t, x + \kappa(t,x)), \qquad h(a(t), x), \qquad h(t, x - b(t)), \tag{1.7}$$

to replace $\widetilde{P}(\underline{u})$ with

$$Q(\underline{u}) = \partial_t + W\partial_x + iL + R_3, \tag{1.8}$$

where $W = W(t,x)$ satisfies $\int_{\mathbb{T}} W(t,x)\,dx = 0$, $\|W\|_{C^0([0,T];H^{s_0-d})} \lesssim \|\underline{u}\|_{C^0([0,T];H^{s_0})}$ where $d > 0$ is a universal constant, and R_3 is of order zero. This is not trivial since the equation is nonlocal and also because this exhibits a cancellation of a term of order $1/2$. Indeed, in general the conjugation of $L^{1/2}(c(\underline{u})L^{1/2}\,\cdot)$ and a change of variables generates also a term of order $3/2 - 1$. This term disappears here since we consider transformations which preserve the $L^2(dx)$ scalar product. Then we use the Egorov theorem to estimate the remainder terms (see Remark 5.2 and also [8], [9]).

We next seek an operator A such that $i[A, |D_x|^{3/2}] + W\partial_x A$ is a zero order operator. This leads us to consider a pseudo-differential operator $A = \mathrm{Op}(a)$ for some symbol $a = a(x,\xi)$ in the Hörmander class $S^0_{\rho,\rho}$ with $\rho = 1/2$, namely $a = \exp(i|\xi|^{1/2}\beta(t,x))$ for some function β depending on W (see Proposition 5.8 for a complete statement that also includes a zero order amplitude). Here we follow [1]. To keep the paper self-contained (and since some modifications are needed), we recall the strategy of the proof in Section 5.

Concerning the latter transformation, let us compare the equation $P(u)u = 0$ with the Benjamin–Ono equation

$$\partial_t w + w\partial_x w + \mathcal{H}\partial_x^2 w = 0, \tag{1.9}$$

where $\mathcal{H}$ is the Hilbert transform. The control problem for this equation has been studied through elaborate techniques (see for instance the recent paper [33]) that are specific to this equation and cannot be applied to the water waves equations.[1] On the other hand, let us discuss one difference which appears when applying to (1.9) the strategy previously described. Given a function $W = W(t,x)$ with zero mean in x, let us seek an operator B such that the leading order term in $[B, \mathcal{H}\partial_x^2] + W\partial_x B$ vanishes. This requires (see [7]) introducing a classical pseudo-differential operator $B = \mathrm{Op}(b)$ with $b \in S^0_{1,0}$. Then the key difference between the two cases could be explained as follows: For r large enough,

- the mapping $W \mapsto B$ is Lipschitz from H^r into $\mathcal{L}(L^2)$;
- the mapping $W \mapsto A$ is only continuous from H^r into $\mathcal{L}(L^2)$ (indeed, if $\|W\|_{H^r} = O(\delta)$ then we merely have $\|A - I\|_{\mathcal{L}(L^2;H^{-1/2})} = O(\delta)$).

This is another reason why one cannot use a fixed point argument based on a contraction estimate to deduce the existence of the control.

Step 5: Observability. Next, we establish an observability inequality. That is, we prove in Proposition 7.1 that there exists $\varepsilon > 0$ such that for any initial data v_0 whose mean value $\langle v_0\rangle = (2\pi)^{-1}\int_{\mathbb{T}} v_0(x)\,dx$ satisfies

$$|\mathrm{Re}\,\langle v_0\rangle| \geq \tfrac{1}{2}|\langle v_0\rangle| - \varepsilon\|v_0\|_{L^2}, \tag{1.10}$$

the solution v of

$$\partial_t v + iLv = 0, \qquad v(0) = v_0,$$

satisfies

$$\int_0^T \int_\omega |\mathrm{Re}\,(Av)(t,x)|^2\,dx\,dt \geq K\int_{\mathbb{T}} |v_0(x)|^2\,dx. \tag{1.11}$$

This inequality with the real part on the left-hand side allows one to prove the existence of a real-valued control function; a similar property is proved for systems of wave equations by Burq and Lebeau [14].

The observability inequality is deduced using a variant of Ingham's inequality (see Section 6). Recall that Ingham's inequality is an inequality for the L^2-norm of a sum of oscillatory functions which generalizes Parseval's inequality (it applies to pseudo-periodic

[1] This can be seen at the level of the Cauchy problem: for the Euler equation with free surface, the well-posedness of the Cauchy problem in the energy space is entirely open.

functions and not only to periodic functions; see for instance [29]). For example, one such result asserts that for any $T > 0$ there exist positive constants $C_1 = C_1(T)$ and $C_2 = C_2(T)$ such that

$$C_1 \sum_{n\in\mathbb{Z}} |w_n|^2 \le \int_0^T \Big|\sum_{n\in\mathbb{Z}} w_n e^{in|n|^{1/2}t}\Big|^2 \,dt \le C_2 \sum_{n\in\mathbb{Z}} |w_n|^2 \tag{1.12}$$

for all sequences (w_n) in $\ell^2(\mathbb{C})$. That this holds for any $T > 0$ (and not only for T large enough) is a consequence of a general result due to Kahane [28] on lacunary series.

Note that since the original problem is quasi-linear, we are forced to prove an Ingham type inequality for sums of oscillatory functions whose phases differ from the phase of the linearized equation. For our purposes, we need to consider phases that do not depend linearly on t, of the form

$$\operatorname{sign}(n)[\ell(n)^{3/2}t + \beta(t,x)|n|^{1/2}], \qquad \ell(n) := \big((g+|n|^2)|n|\tanh(b|n|)\big)^{1/2},$$

where x plays the role of a parameter. Though it is a subprincipal term, taking into account the perturbation $\beta(t,x)|n|^{1/2}$ requires some care since $e^{i\beta(t,x)|n|^{1/2}} - 1$ is not small. In particular we need to prove upper bounds for expressions in which we allow some amplitude depending on time (and whose derivatives in time of order k can grow as $|n|^{k/2}$).

Step 6: HUM method. Inverting A, we deduce from (1.11) an observability result for the adjoint operator $Q(\underline{u})^*$ ($Q(\underline{u})$ is as in (1.8)). Then the controllability will be deduced from the classical HUM method (we refer to Section 8 for a version that makes it possible to consider a real-valued control). The idea is that the observability property implies that some bilinear form is coercive, and hence the existence of the control follows from the Riesz theorem and a duality argument. A possible difficulty is that the control P_{ext} acts only on the equation for ψ. To explain this, consider the case where $(\eta_{\text{final}}, \psi_{\text{final}}) = (0,0)$. Since the HUM method is based on orthogonality arguments, the control not acting on both equations means for our problem that the final state is orthogonal to a codimension 1 space. That this final state can be chosen to be 0 will be shown by choosing this codimension 1 space in an appropriate way, introducing an auxiliary function $M = M(x)$ to be specified later on.

Consider any real function $M = M(x)$ with $M - 1$ small enough, and introduce

$$L^2_M := \Big\{\varphi \in L^2(\mathbb{T};\mathbb{C})\,;\ \operatorname{Im}\int_{\mathbb{T}} M(x)\varphi(x)\,dx = 0\Big\}.$$

Notice that L^2_M is an $\mathbb{R}$-Hilbert space. Also, for any $v_0 \in L^2_M$, the condition (1.10) holds. Then, using a variant of the HUM method in this space, one deduces that for all $v_{\text{in}} \in L^2$ (not necessarily in L^2_M) there is $f \in C^0([0,T];L^2)$ such that if

$$Q(\underline{u})w = \partial_t w + W\partial_x w + iLw + R_3 w = \chi_\omega \operatorname{Re} f, \qquad w(0) = w_{\text{in}},$$

then $w(T,x) = ibM(x)$ for some constant $b \in \mathbb{R}$. Now

$$Q(\underline{u}) = \Phi(\underline{u})^{-1}\widetilde{P}(\underline{u})\Phi(\underline{u}),$$

where $\Phi(\underline{u})$ is the composition of the transformations in (1.6). Since $\Phi(\underline{u})$ and $\Phi(\underline{u})^{-1}$ are local operators, one easily deduces a controllability result for $\widetilde{P}(\underline{u})$ from the one proved for $Q(\underline{u})$. Now, choosing $M = \Phi(\underline{u}(T,\cdot))(1)$ where 1 is the constant function 1, we deduce from $w(T,x) = ibM(x)$ that $u(T,x)$ is an imaginary constant, as asserted in Proposition 1.3(i). Concerning M, notice that $M \neq 1$ because of the factor $(1 + \partial_x \kappa(t,x))^{1/2}$ multiplying $h(t, x + \kappa(t,x))$ in (1.7).

Step 7: Convergence of the scheme. Let us discuss the proof of the convergence of the sequence (f_n) of approximate controls to the desired control P_{ext}. This part requires new stability estimates in order to prove that (f_n) and (u_n) are Cauchy sequences. This is where we need Proposition 1.3(ii), to estimate the difference of two controls associated with different coefficients. To prove this stability estimate we shall introduce an auxiliary control problem which, loosely speaking, interpolates the two control problems. Since the original nonlinear problem is quasi-linear, there is a loss of derivative (this reflects the fact that the flow map is expected to be merely continuous and not Lipschitz on Sobolev spaces). We overcome this loss by proving and using a regularity property of the control Proposition 1.5(iii). This regularity result is proved by adapting an argument used by Dehman–Lebeau [20] and Laurent [32]. We also need to study how the control depends on T or on the function M.

1.3. Outline of the paper

In Section 2 we recall how to use paradifferential analysis to symmetrize the water waves equations. As mentioned above, the control problem for the water waves equations is studied by means of a nonlinear scheme. This requires solving a linear control problem at each step. We introduce this linear equation in Section 3 and state the main result about it. In Section 4, we conjugate the equations with a well-chosen elliptic operator to obtain a regularized problem. Once this step is achieved, in Section 5 we further transform the equations by means of a change of variables and by conjugating the equation with some pseudo-differential operator. Ingham's type inequalities are proved in Section 6 and then used in Section 7 to deduce an observability result which in turn is used in Section 8 to obtain a controllability result. In Section 8 we also study the way in which the control depends on the coefficients, which requires introducing several auxiliary control problems. Eventually, in Sections 9 and 10 we use the previous control results for linear equations to deduce our main result, Theorem 1.1, by means of a quasi-linear scheme.

To keep the paper self-contained, we add an appendix which contains two sections about paradifferential calculus and Sobolev energy estimates for classical or paradifferential evolution equations. The appendix also contains the analysis of various changes of variables which are used to conjugate the equations to a simpler form.

2. Symmetrization of the water waves equations

Consider the system

$$\begin{cases} \partial_t \eta = G(\eta)\psi, \\ \partial_t \psi + g\eta + \dfrac{1}{2}(\partial_x \psi)^2 - \dfrac{1}{2}\dfrac{(G(\eta)\psi + (\partial_x \eta)(\partial_x \psi))^2}{1+(\partial_x \eta)^2} = H(\eta) + P_{\text{ext}}(t,x). \end{cases} \tag{2.1}$$

In this section, following [2, 5] we recall how to use paradifferential analysis to rewrite the above system as a wave type equation for some new unknown u. This analysis is performed in §2.2. In §2.1 and §2.3, we complement the analysis of [2, 5] by proving that all the coefficients can be expressed in terms of u only.

We refer the reader to the appendix for the definitions and the main results of paradifferential calculus.

2.1. Properties of the Dirichlet–Neumann operator

We begin by recalling that if η is in $W^{1,\infty}(\mathbb{T})$ and ψ is in $H^{1/2}(\mathbb{T})$, then $G(\eta)\psi$ is well-defined and belongs to $H^{-1/2}(\mathbb{T})$. Moreover, if (η,ψ) belongs to $H^s(\mathbb{T})\times H^s(\mathbb{T})$ for some $s > 3/2$, then $G(\eta)\psi$ belongs to $H^{s-1}(\mathbb{T})$ together with the estimate (see [31, Thm. 3.15])

$$\|G(\eta)\psi\|_{H^{s-1}} \le C(\|\eta\|_{H^s})\|\psi\|_{H^s}. \tag{2.2}$$

Following [5, 2], the analysis is based on the so-called good unknown of Alinhac defined in the next lemma and denoted by ω (the same letter is applied for the control domain, but the two notations will not be used simultaneously). For comments and explanations why this unknown plays a crucial role, we refer to [5, §3] and [4, pp. 8–9].

Lemma 2.1. *Let $s > 3/2$ and (η,ψ) in $H^s(\mathbb{T})\times H^s(\mathbb{T})$. Then the functions*

$$\begin{aligned} &B(\eta)\psi := \frac{G(\eta)\psi + (\partial_x\eta)(\partial_x\psi)}{1+(\partial_x\eta)^2}, \qquad V(\eta)\psi := \partial_x\psi - (B(\eta)\psi)\partial_x\eta, \\ &\omega(\eta)\psi := \psi - T_{B(\eta)\psi}\eta \end{aligned} \tag{2.3}$$

belong, respectively, to $H^{s-1}(\mathbb{T})$, $H^{s-1}(\mathbb{T})$, $H^s(\mathbb{T})$ and satisfy

$$\|B(\eta)\psi\|_{H^{s-1}} + \|V(\eta)\psi\|_{H^{s-1}} + \|\omega(\eta)\psi\|_{H^s} \le C(\|\eta\|_{H^s})\|\psi\|_{H^s}. \tag{2.4}$$

Proof. The estimates for $B(\eta)\psi$ and $V(\eta)\psi$ follow from (2.2), by applying the usual nonlinear estimates in Sobolev spaces (see (A.18) and (A.16)). The Sobolev embedding then implies that $B(\eta)\psi \in L^\infty(\mathbb{T})$. As a paraproduct with an L^∞-function acts on any Sobolev space (see (A.10)), we deduce that

$$\|T_{B(\eta)\psi}\eta\|_{H^s} \lesssim \|B(\eta)\psi\|_{L^\infty}\|\eta\|_{H^s} \le C(\|\eta\|_{H^s})\|\psi\|_{H^s}\|\eta\|_{H^s}. \tag{2.5}$$

This immediately implies the estimate for $\omega(\eta)\psi$ in (2.4). □

Consider a Banach space X and an operator A whose operator norm is strictly smaller than 1. Then it is well-known that $I - A$ is invertible. Now write $\omega(\eta)\psi$ as $(I-A)\psi$ with $A\psi = T_{B(\eta)\psi}\eta$. By applying the previous argument, (2.5) yields the following result.

Lemma 2.2. *Let $s > 3/2$. There exists $\varepsilon_0 > 0$ such that the following holds. If $\|\eta\|_{H^s} < \varepsilon_0$, then there exists a linear operator $\Psi(\eta)$ such that:*

(i) *for any ψ in $H^{1/2}(\mathbb{T})$,*

$$\Psi(\eta)\omega(\eta)\psi = \psi;$$

(ii) *if $\omega \in H^s(\mathbb{T})$ then $\Psi(\eta)\omega \in H^s(\mathbb{T})$ and*

$$\|\Psi(\eta)\omega\|_{H^s} \le C(\|\eta\|_{H^s})\|\omega\|_{H^s}. \tag{2.6}$$

Notation 2.3. Hereafter, we often simply write B, V, ω instead of $B(\eta)\psi$, $V(\eta)\psi$, $\omega(\eta)\psi$. It follows from the above lemma that if η is small enough in $H^s(\mathbb{T})$, then B and V can be expressed in terms of η and ω:

$$B = B(\eta)\Psi(\eta)\omega, \qquad V = V(\eta)\Psi(\eta)\omega.$$

We also record the following corollary of the analysis in [5, 2].

Proposition 2.4. *Let $s \ge s_0$ with s_0 large enough. Then there exists $\theta \in (0,1]$ such that*

$$G(\eta)\psi = G(0)\omega - \partial_x(T_V\eta) + F(\eta)\psi, \tag{2.7}$$

where

$$\|F(\eta)\psi\|_{H^{s+1/2}} \le C(\|\eta\|_{H^s})\|\eta\|_{H^s}^{\theta}\|\psi\|_{H^s}. \tag{2.8}$$

Proof. We prove that

$$\|F(\eta)\psi\|_{H^{s+1}} \le C(\|\eta\|_{H^s})\|\psi\|_{H^s}, \tag{2.9}$$

$$\|F(\eta)\psi\|_{H^{s-2}} \le C(\|\eta\|_{H^s})\|\eta\|_{H^s}\|\psi\|_{H^s}. \tag{2.10}$$

The estimate (2.8) then follows by interpolation in Sobolev spaces.

Let us prove (2.9). In [5, 2] it is proved that, for any N, when s is large enough, $G(\eta)\psi = |D_x|\omega - \partial_x(T_V\eta) + \tilde{F}(\eta)\psi$ where $\|\tilde{F}(\eta)\psi\|_{H^{s+N}} \le C(\|\eta\|_{H^s})\|\psi\|_{H^s}$. Notice that (2.7) holds with $F(\eta)\psi = (|D_x| - G(0))\omega + \tilde{F}(\eta)\psi$. Since $G(0) = |D_x|\tanh(b|D_x|)$, the difference $|D_x| - G(0)$ is a smoothing operator. So using the estimate (2.4) for ω, we find that $\|F(\eta)\psi\|_{H^{s+N}}$ is bounded by the right-hand side of (2.9). Taking $N = 1$ gives the desired result.

We now prove (2.10). As for (2.5), by the paraproduct rule (A.10) and (2.4),

$$\|\omega - \psi\|_{H^s} + \|\partial_x(T_V\eta)\|_{H^{s-1}} \lesssim (\|B\|_{L^\infty} + \|V\|_{L^\infty})\|\eta\|_{H^s} \le C(\|\eta\|_{H^s})\|\eta\|_{H^s}\|\psi\|_{H^s},$$

hence it is sufficient to prove that $\|G(\eta)\psi - G(0)\psi\|_{H^{s-2}}$ is bounded by the rhs of (2.10). This in turn will be deduced from an estimate of $\|\varphi'(\tau)\|_{H^{s-2}}$ where $\varphi(\tau) = G(\tau\eta)\psi$. Set $B_\tau = B(\tau\eta)\psi$ and $V_\tau = V(\tau\eta)\psi$. It follows from the computation of the shape derivative of the Dirichlet–Neumann operator [30] that $\varphi'(\tau) = -G(\tau\eta)(B_\tau\eta) - \partial_x(V_\tau\eta)$. Now (2.4) implies $\|\varphi'(\tau)\|_{H^{s-2}} \le C(\|\eta\|_{H^s})\|\eta\|_{H^s}\|\psi\|_{H^s}$. Integrating over τ we complete the proof. ∎

2.2. Symmetrization

As already mentioned, the linearized equations are

$$\begin{cases} \partial_t \eta = G(0)\psi, \\ \partial_t \psi + g\eta - \partial_x^2 \eta = P_{\text{ext}}, \end{cases}$$

where $G(0) = |D_x| \tanh(b|D_x|)$. Introducing the Fourier multiplier (of order 3/2)

$$L := ((g - \partial_x^2)G(0))^{1/2}$$

with symbol

$$\ell(\xi) := ((g + |\xi|^2)\lambda(\xi))^{1/2} \quad \text{where} \quad \lambda(\xi) := |\xi| \tanh(b|\xi|) \tag{2.11}$$

(so that $L = \ell(D_x)$), and considering $u = \psi - iLG(0)^{-1}\eta$, one obtains the equation

$$\partial_t u + iLu = P_{\text{ext}}.$$

The following proposition contains a similar diagonalization of system (2.1).

Proposition 2.5. *Let σ, σ_0 be such that $\sigma \geq \sigma_0$ with σ_0 large enough. Consider a solution (η, ψ) of (2.1) on the time interval $[0, T]$ with $0 < T < \infty$ such that*

$$(\eta, \psi) \in C^0([0, T]; H_0^{\sigma+1/2}(\mathbb{T}) \times H^\sigma(\mathbb{T})).$$

Introduce a function $c = c(x)$ and two symbols $p = p(x, \xi)$, $q = q(x, \xi)$ such that

$$\begin{aligned} &c := (1 + (\partial_x \eta)^2)^{-3/4}, \\ &p := c^{-1/3} + \frac{5}{18i} \frac{\chi(\xi)\partial_\xi \ell(\xi)}{\ell(\xi)} c^{-4/3} \partial_x c, \qquad q = \chi(\xi)\left(c^{2/3} \frac{\ell(\xi)}{\lambda(\xi)} + (\partial_x c^{2/3}) \frac{\ell(\xi)}{i\xi\lambda(\xi)}\right), \end{aligned} \tag{2.12}$$

where ℓ, λ are as in (2.11), $\chi \in C^\infty$ satisfies $\chi(\xi) = 1$ for $|\xi| \geq 2/3$ and $\chi(\xi) = 0$ for $|\xi| \leq 1/2$. Then

$$u := T_p \omega - iT_q \eta$$

satisfies

$$\partial_t u + T_V \partial_x u + iL^{1/2}(T_c L^{1/2} u) + R(\eta, \psi) = T_p P_{\text{ext}} \tag{2.13}$$

for some remainder $R(\eta, \psi) = R_1(\eta)\psi + R_2(\eta)\eta$ with

$$\begin{aligned} &\|R_1(\eta)\psi\|_{H^\sigma} \leq C(\|\eta\|_{H^{\sigma+1/2}})\|\eta\|_{H^{\sigma+1/2}}^\theta \|\psi\|_{H^\sigma}, \\ &\|R_2(\eta_1)\eta_2\|_{H^\sigma} \leq C(\|\eta_1\|_{H^{\sigma+1/2}})\|\eta_1\|_{H^{\sigma+1/2}}^\theta \|\eta_2\|_{H^{\sigma+1/2}}, \end{aligned} \tag{2.14}$$

for $\theta \in (0, 1]$ as in Proposition 2.4.

Remark 2.6. Compared to a similar result proved in [2], there are two differences. We here obtain a superlinear remainder term (see (2.14)), and secondly q can be so chosen that $T_q = \partial_x T_Q$ for some symbol Q; namely,

$$T_q = \partial_x T_Q \quad \text{with} \quad Q := \chi(\xi) c^{2/3} \frac{\ell(\xi)}{\lambda(\xi) i\xi}. \tag{2.15}$$

This will be used to deduce that $\int T_q \eta \, dx = 0$. Since it is not a trivial task to obtain these additional properties, we shall recall the strategy of the proof from [2] and give a detailed analysis of the required modifications.

Proof of Proposition 2.5. The first step consists in paralinearizing the equation. We use in particular the paralinearization of the Dirichlet–Neumann operator (see (2.7)). Then, by using the paralinearization formula for products (replacing ab by $T_a b + T_b a + \mathcal{R}(a, b)$), it follows from direct computations [2] that

$$\begin{cases} \partial_t \eta + \partial_x (T_V \eta) - G(0)\omega = F^1, \\ \partial_t \omega + T_V \partial_x \omega + T_a \eta - H(\eta) = F^2 + P_{\text{ext}}, \end{cases} \tag{2.16}$$

where a denotes the Taylor coefficient, which is

$$a = g + \partial_t B + V \partial_x B,$$

and F^1 and F^2 are given by (see (A.12) for the definition of $\mathcal{R}(a, b)$)

$$\begin{aligned} F^1 &= F(\eta)\psi, \\ F^2 &= (T_V T_{\partial_x \eta} - T_{V \partial_x \eta}) B + (T_{V \partial_x B} - T_V T_{\partial_x B})\eta \\ &\quad + \tfrac{1}{2}\mathcal{R}(B, B) - \tfrac{1}{2}\mathcal{R}(V, V) + T_V \mathcal{R}(B, \partial_x \eta) - \mathcal{R}(B, V \partial_x \eta). \end{aligned}$$

On the other hand, the paralinearization estimate (A.14) applied with $\alpha = \sigma - 1/2$ implies that

$$\frac{\partial_x \eta}{\sqrt{1 + (\partial_x \eta)^2}} = T_r \partial_x \eta + \tilde{f}, \qquad r := (1 + (\partial_x \eta)^2)^{-3/2},$$

where $\tilde{f} \in L^\infty(0, T; H^{2\sigma - 3/2})$ is such that

$$\|\tilde{f}\|_{H^{2\sigma-3/2}} \le C(\|\eta\|_{H^{\sigma+1/2}}) \|\eta\|^2_{H^{\sigma+1/2}}$$

for some nondecreasing function C. Hence, directly from (2.16), we obtain

$$\begin{cases} \partial_t \eta + T_V \partial_x \eta - G(0)\omega = f^1, \\ \partial_t \omega + T_V \partial_x \omega + g\eta - \partial_x (T_r \partial_x \eta) = f^2 + P_{\text{ext}}, \end{cases}$$

where

$$f^1 := F^1 - T_{\partial_x V} \eta, \qquad f^2 := F^2 + \partial_x \tilde{f} + T_{g-a} \eta.$$

Set $\zeta := T_q\eta$ and $\theta := T_p\omega$. Then

$$\begin{cases} \partial_t\zeta + T_V\partial_x\zeta - T_qG(0)\omega = \tilde{f}^1, \\ \partial_t\theta + T_V\partial_x\theta + T_p(g\eta - \partial_x(T_r\partial_x\eta)) = \tilde{f}^2 + T_pP_{\text{ext}}, \end{cases} \tag{2.17}$$

where

$$\tilde{f}^1 := T_qf^1 + T_{\partial_t q}\eta + [T_V\partial_x, T_q]\eta,$$
$$\tilde{f}^2 := T_pf^2 + T_{\partial_t p}\omega + [T_V\partial_x, T_p]\omega.$$

Assuming that q and p are as in the statement of the proposition, it easily follows from (2.8) and the paradifferential rules (A.4), (A.10) and (A.7) (applied with $\rho = 1$ to bound the operator norm of the commutators $[T_V\partial_x, T_q]$ and $[T_V\partial_x, T_p]$) that

$$\|(f^1, f^2)\|_{H^\sigma} \le C(\|\eta\|_{H^{\sigma+1/2}})\|\eta\|^{\theta}_{H^{\sigma+1/2}}\{\|\psi\|_{H^\sigma} + \|\eta\|_{H^{\sigma+1/2}}\}.$$

It remains to compute $T_qG(0)\omega$ and $T_p(g\eta - \partial_x(T_r\partial_x\eta))$. More precisely, it remains to establish that

$$\begin{aligned} &\|T_qG(0)\omega - L^{1/2}T_cL^{1/2}T_p\omega\|_{H^\sigma} \le C(\|\eta\|_{H^{\sigma+1/2}})\|\eta\|^{\theta}_{H^{\sigma+1/2}}\|\omega\|_{H^\sigma}, \\ &\|T_p(g\eta - \partial_x(T_r\partial_x\eta)) - L^{1/2}T_cL^{1/2}T_q\eta\|_{H^\sigma} \\ &\qquad\qquad \le C(\|\eta\|_{H^{\sigma+1/2}})\|\eta\|^{\theta}_{H^{\sigma+1/2}}\|\eta\|_{H^{\sigma+1/2}}. \end{aligned} \tag{2.18}$$

(We prove these estimates below with $\theta = 1$.) Then the estimates (2.14) follow from (2.4) which gives a bound for $\|\omega\|_{H^\sigma}$ in terms of $\|\psi\|_{H^\sigma}$.

To prove (2.18), it is convenient to introduce the following notation: Given two operators, the notation $A \sim B$ means that, for any $\mu \in \mathbb{R}$ there is a constant $C(\|\eta\|_{H^{\sigma+1/2}})$ such that

$$\|(A - B)u\|_{H^\mu} \le C(\|\eta\|_{H^{\sigma+1/2}})\|\eta\|_{H^{\sigma+1/2}}\|u\|_{H^\mu}.$$

In words, $A \sim B$ means that A equals B modulo a remainder which is of order 0 and quadratic.

For instance consider real numbers m, m' with $m + m' = 2$ and two operators $A = T_{a^{(m)}+a^{(m-1)}}$ and $B = T_{b^{(m')}+b^{(m'-1)}}$ where

$$a^{(m)} \in \Gamma^m_2, \quad a^{(m-1)} \in \Gamma^{m-1}_1, \quad b^{(m')} \in \Gamma^{m'}_2, \quad b^{(m'-1)} \in \Gamma^{m'-1}_1$$

(see Definition A.2) with (see (A.1))

$$M^{m'}_2(b^{(m')}) + M^{m'-1}_1(b^{(m'-1)}) \le C(\|\eta\|_{H^{\sigma+1/2}}),$$
$$M^m_2(a^{(m)}) + M^{m-1}_1(a^{(m-1)}) \le C(\|\eta\|_{H^{\sigma+1/2}})\|\eta\|_{H^{\sigma+1/2}}.$$

By applying (A.6) with $\rho = 2$ and (A.7) with $\rho = 1$, we obtain

$$T_{a^{(m)}}T_{b^{(m')}} \sim T_{a^{(m)}b^{(m')} + \frac{1}{i}\partial_\xi a^{(m)}\partial_x b^{(m')}}, \qquad T_{a^{(m)}}T_{b^{(m'-1)}} \sim T_{a^{(m)}b^{(m'-1)}},$$
$$T_{a^{(m-1)}}T_{b^{(m')}} \sim T_{a^{(m-1)}b^{(m')}}, \qquad T_{a^{(m-1)}}T_{b^{(m'-1)}} \sim 0,$$

so

$$AB \sim T_{a^{(m)}b^{(m')}+\frac{1}{i}\partial_\xi a^{(m)}\partial_x b^{(m')}+a^{(m)}b^{(m'-1)}+a^{(m-1)}b^{(m')}}. \tag{2.19}$$

Using the previous notation, to prove (2.18) we have to prove that

$$\begin{aligned} &T_q G(0) \sim L^{1/2}T_c L^{1/2}\chi(D_x)T_p,\\ &T_p(gI - \partial_x(T_r\partial_x\,\cdot))\chi(D_x) \sim L^{1/2}T_c L^{1/2}\chi(D_x)T_q. \end{aligned} \tag{2.20}$$

Notice that $\chi(D_x)\eta = \eta$ and $L^{1/2}\chi(D_x)u = L^{1/2}u$ for any periodic function u. This is why we can introduce the cut-off function χ in the calculations. It is used to handle symbols which are not smooth at $\xi = 0$.

We remark that, by definition of paradifferential operators, we have

$$T_q G(0) = T_{q\lambda(\xi)}, \qquad gI - \partial_x(T_r\partial_x\,\cdot) = T_{g+r\xi^2-(\partial_x r)(i\xi)}.$$

Study of the first relation in (2.20). It follows from symbolic calculus (see (A.6)) that

$$L^{1/2}T_c L^{1/2}\chi(D_x) \sim T_\gamma \quad \text{with} \quad \gamma = \chi c\ell + \frac{\chi}{i}(\partial_\xi\sqrt{\ell})\sqrt{\ell}\,\partial_x c. \tag{2.21}$$

Now we seek q of the form $q = q^{(1/2)} + q^{(-1/2)}$ where $q^{(1/2)}$ is of order $1/2$ in ξ (more precisely, $q \in \Gamma^{1/2}_2$) and $q^{(-1/2)}$ is of order $-1/2$ (in $\Gamma^{-1/2}_1$). Similarly, we seek $p = p^{(0)} + p^{(-1)}$ with $p \in \Gamma^0_2$ and $p^{(-1)} \in \Gamma^{-1}_1$.

Also, it follows from (2.19) that $L^{1/2}T_c L^{1/2}\chi(D_x)T_p \sim T_\gamma T_p \sim T_{\wp_1}$ with

$$\wp_1 = \gamma p^{(0)} + \chi c\ell p^{(-1)} + \frac{1}{i}\chi c(\partial_\xi\ell)\partial_x p^{(0)}$$

(the contribution of $(\partial_\xi\chi)\partial_x p^{(0)}$ is in the remainder term). The first identity in (2.20) will be satisfied if

$$q^{(1/2)} := \chi c p^{(0)}\frac{\ell(\xi)}{\lambda(\xi)}, \qquad q^{(-1/2)} = \frac{\chi}{i}\frac{\partial_\xi\ell(\xi)}{\lambda(\xi)}\left[\tfrac{1}{2}(\partial_x c)p^{(0)} + c\partial_x p^{(0)}\right] + \chi c p^{(-1)}\frac{\ell(\xi)}{\lambda(\xi)}.$$

Study of the second relation in (2.20). As above, it follows from symbolic calculus that $L^{1/2}T_c L^{1/2}\chi(D_x)T_q \sim T_\gamma T_q \sim T_{\wp_2}$ with (see (2.19))

$$\wp_2 = \gamma q^{(1/2)} + \frac{1}{i}\chi c(\partial_\xi\ell)\partial_x q^{(1/2)} + \chi c\ell q^{(-1/2)}.$$

With $q^{(1/2)}$ and $q^{(-1/2)}$ as given above, we compute that

$$\wp_2 = \chi\left\{c\ell q^{(1/2)} + \frac{\chi}{i}\frac{\partial_\xi\ell^2}{\lambda(\xi)}(cp^{(0)}\partial_x c + c^2\partial_x p^{(0)}) + \chi c^2 p^{(-1)}\frac{\ell(\xi)^2}{\lambda(\xi)}\right\}.$$

Moreover, by definition of $\ell(\xi)$ one has

$$\frac{\partial_\xi\ell^2}{\lambda(\xi)} = 3\xi + r_1, \qquad \frac{\ell(\xi)^2}{\lambda(\xi)} = \xi^2 + r_2, \qquad r_1, r_2 \text{ of order } 0.$$

Notice that the contribution of the term $r_1(cp^{(0)}\partial_x c + c^2\partial_x p^{(0)})$ to $T_{\wp_2}$ and the one of $r_2 c^2 p^{(-1)}$ can be handled as remainder terms and hence

$$L^{1/2}T_c L^{1/2}\chi(D_x)T_q \sim T_{\tilde{\wp}_2}$$

with

$$\tilde{\wp}_2 = \chi\left\{c\ell q^{(1/2)} + \frac{3\chi}{i}\xi(cp^{(0)}\partial_x c + c^2\partial_x p^{(0)}) + \chi c^2 p^{(-1)}\xi^2\right\}.$$

On the other hand,

$$T_p(gI - \partial_x(T_r\partial_x\,\cdot))\chi(D_x) \sim T_{\chi p(g+r\xi^2-(\partial_x r)(i\xi))}.$$

By definition of $c, \ell, q^{(1/2)}$, recall that $r = c^2$ and $\ell^2 = (g+\xi^2)\lambda(\xi)$ and hence

$$\begin{aligned}p(g + r\xi^2 - (\partial_x r)(i\xi)) &= pc^2\xi^2 + gp - p(\partial_x r)(i\xi)\\ &= pc^2\frac{\ell(\xi)^2}{\lambda(\xi)} - gpc^2 + gp - p(\partial_x r)(i\xi).\end{aligned}$$

Since $q^{(1/2)} := \chi cp^{(0)}\frac{\ell(\xi)}{\lambda(\xi)}$, we deduce that

$$p\big(g + r\xi^2 - (\partial_x r)(i\xi)\big)\chi = c\ell q^{(1/2)} + \chi\{p^{(-1)}c^2(g+\xi^2) + gp(1-c^2) - ip(\partial_x r)\xi\}.$$

Since $1-c^2$ and $\partial_x r$ depend at least linearly on η, and since p and $p^{(-1)}\xi$ are symbols of order 0, it follows from the estimate (A.4) for the operator norm of a paradifferential operator that

$$\begin{aligned}&\|T_{p(1-c^2)}u\|_{H^\mu} \le C(\|\eta\|_{H^{\sigma+1/2}})\|\eta\|^2_{H^{\sigma+1/2}}\|u\|_{H^\mu},\\ &\|T_{p^{(-1)}(\partial_x r)\xi}u\|_{H^\mu} \le C(\|\eta\|_{H^{\sigma+1/2}})\|\eta\|_{H^{\sigma+1/2}}\|u\|_{H^\mu}.\end{aligned}$$

Similarly, assuming that $p^{(-1)}$ is a symbol of order -1 depending linearly on η (this will be true, see (2.12)), we have

$$\|T_{p^{(-1)}c^2 g}u\|_{H^\mu} \le C(\|\eta\|_{H^{\sigma+1/2}})\|\eta\|_{H^{\sigma+1/2}}\|u\|_{H^\mu}.$$

Therefore,

$$T_p(gI - \partial_x(T_r\partial_x\,\cdot))\chi(D_x) \sim T_{c\ell q^{(1/2)} + \chi p^{(-1)}c^2\xi^2 - \chi p^{(0)}(\partial_x r)(i\xi)}.$$

Now since $r = c^2$, with $p^{(0)} = c^{-1/3}$, we have

$$-p^{(0)}(\partial_x r)(i\xi) = +\frac{3}{i}\xi(cp^{(0)}\partial_x c + c^2\partial_x p^{(0)}),$$

as can be verified by a direct calculation, so the second identity in (2.20) holds.

It remains to compute q. We have

$$q = \chi\left\{cp^{(0)}\frac{\ell(\xi)}{\lambda(\xi)} + \frac{1}{i}\frac{\partial_\xi\ell(\xi)}{\lambda(\xi)}\left[\tfrac{1}{2}(\partial_x c)p^{(0)} + c\partial_x p^{(0)}\right] + cp^{(-1)}\frac{\ell(\xi)}{\lambda(\xi)}\right\}.$$

Observe that

$$\tfrac{1}{2}(\partial_x c)p^{(0)} + c\partial_x p^{(0)} = \tfrac{1}{6}c^{-1/3}\partial_x c.$$

We now seek $p^{(-1)}$ such that

$$cp^{(-1)}\frac{\ell(\xi)}{\lambda(\xi)} = \alpha\frac{1}{i}\frac{\partial_\xi \ell(\xi)}{\lambda(\xi)}c^{-1/3}\partial_x c$$

for some constant α to be determined. We thus set

$$p^{(-1)} := \alpha\frac{\chi(\xi)}{i}\frac{\partial_\xi \ell(\xi)}{\ell(\xi)}c^{-4/3}\partial_x c.$$

Then (replacing χ^2 by χ, at the cost of adding a smoothing operator in the remainder), we have

$$q := \chi\left\{c^{2/3}\frac{\ell(\xi)}{\lambda(\xi)} + \frac{1}{i}\frac{\partial_\xi \ell(\xi)}{\lambda(\xi)}\left[\left(\alpha + \frac{1}{6}\right)c^{-1/3}\partial_x c\right]\right\}.$$

Since $\chi(\xi)\xi\partial_\xi \ell = \frac{3}{2}\chi\ell + \tau(\xi)$ with $\tau(\xi)$ a smooth symbol of order $1/2$, we have

$$\frac{2}{3}\chi(\xi)\frac{\ell}{\lambda(\xi)i\xi} = \frac{4}{9i}\chi(\xi)\frac{\partial_\xi \ell}{\lambda(\xi)} + r'$$

where r' is of order $-3/2$. Then, choosing α such that $\alpha + 1/6 = 4/9$, we find that

$$q = c^{2/3}\frac{\ell(\xi)}{\lambda(\xi)} + (\partial_x c^{2/3})\frac{\ell(\xi)}{i\xi\lambda(\xi)} + \tilde{r}$$

where $\tilde{r}$ is such that

$$\|T_{\tilde{r}}u\|_{H^{\mu+3/2}} \le C(\|\eta\|_{H^{\sigma+1/2}})\|\eta\|_{H^{\sigma+1/2}}\|u\|_{H^\mu}.$$

In particular, the contribution of $\tilde{r}$ can be handled as a remainder term and the same results hold when q is replaced by the same expression without $\tilde{r}$, yielding (2.12). This completes the proof of (2.20) and hence the proof of the proposition. □

2.3. Invertibility of the change of unknowns

We have thus obtained an equation of the form

$$\partial_t u + T_V\partial_x u + iL^{1/2}(T_c L^{1/2}u) + R(\eta,\psi) = T_p P_{\text{ext}},$$

where the coefficients V and c depend on the original unknowns (η,ψ). We conclude this section by proving that V and c can be expressed in terms of u only. We have already seen in Lemma 2.2 that these coefficients can be expressed in terms of η and ω. So it remains to express (η,ω) in terms of u.

In this subsection, time is seen as a parameter and we skip it.

Notation 2.7. Let $\tilde{H}^\sigma(\mathbb{T};\mathbb{C})$ be the space of complex-valued functions u satisfying

$$\int_{\mathbb{T}} \operatorname{Im} u(x)\, dx = 0.$$

Recall (see (A.4)) that a paradifferential operator with symbol in Γ_0^m is bounded from any Sobolev space $H^\mu(\mathbb{T})$ to $H^{\mu-m}(\mathbb{T})$. Recall also that $\omega \in H^\sigma(\mathbb{T})$ whenever $(\eta,\psi) \in H_0^{\sigma+1/2}(\mathbb{T}) \times H^\sigma(\mathbb{T})$. Since, as already mentioned, $T_q\eta = T_{q\chi}\eta$ where χ is as defined after (2.12) and since $q\chi \in \Gamma_0^{1/2}$, we deduce that $u \in H^\sigma(\mathbb{T})$. Moreover, it follows from (2.15) that $T_p\omega - iT_q\eta \in \tilde{H}^\sigma(\mathbb{T};\mathbb{C})$.

We now define a mapping $U\colon H_0^{\sigma+1/2}(\mathbb{T}) \times H^\sigma(\mathbb{T}) \to \tilde{H}^\sigma(\mathbb{T};\mathbb{C})$ by

$$U(\eta,\psi) := T_p\omega - iT_q\eta.$$

The following result shows that this nonlinear mapping can be inverted.

Lemma 2.8. *Let $\sigma_0 > 5/2$. There exist $\varepsilon_0 > 0$ and K such that the following holds. If $\|\eta\|_{H^{\sigma_0}} < \varepsilon_0$, then there exists*

$$Y\colon \tilde{H}^{\sigma_0}(\mathbb{T};\mathbb{C}) \to H_0^{\sigma_0+1/2}(\mathbb{T}) \times H^{\sigma_0}(\mathbb{T})$$

such that $Y(u) = (\eta,\psi)$ with $u = U(\eta,\psi)$. Moreover, for any $\sigma > 5/2$,

$$\|\eta\|_{H^{\sigma+1/2}} \le 2\|u\|_{H^\sigma}, \qquad \|\psi\|_{H^\sigma} \le 2\|u\|_{H^\sigma}. \tag{2.22}$$

Proof. Set $u = U(\eta,\psi) := T_p\omega - iT_q\eta$. Then $T_q\eta = -\operatorname{Im} u$ and $T_p\omega = \operatorname{Re} u$, where q and p depend on η. The only difficulty is to express η in terms of $\operatorname{Im} u$. Once this is done, to invert the equation $T_p\omega = \operatorname{Re} u$ we use the fact that T_p is a small bounded perturbation of the identity so that T_p is invertible; indeed (recalling that $M_\rho^m(a)$ is defined by (A.1)),

$$\|T_p - I\|_{\mathcal{L}(H^\sigma)} \lesssim M_0^0(p-1) \le C(\|\eta\|_{H^{\sigma+1/2}})\|\eta\|_{H^{\sigma+1/2}}.$$

Now to solve the equation $T_q\eta = -\operatorname{Im} u$, we use the Banach fixed point theorem. Denote by Q the Fourier multiplier with symbol $Q(\xi) := \chi(\xi)\ell(\xi)/\lambda(\xi) = \chi(\xi)\sqrt{g+\xi^2}/\sqrt{\lambda(\xi)}$. The reason to introduce this symbol is that, with q given by (2.12),

$$M_0^{1/2}(q(x,\xi) - Q(\xi)) \le C(\|\eta\|_{H^{\sigma+1/2}})\|\eta\|_{H^{\sigma+1/2}}, \tag{2.23}$$

which is obtained by considering separately the principal and subprincipal terms in the definition of q. Then seek η in $H_0^{\sigma+1/2}(\mathbb{T})$ such that $\Phi(\eta) = \eta$ with

$$\Phi(\eta) := -(g-\partial_x^2)^{-1/2}G(0)^{1/2}((T_q - Q)\eta + \operatorname{Im} u).$$

It is easily verified that if $\Phi(\eta) = \eta$ then $T_q\eta = -\operatorname{Im} u$ and also that Φ maps $H_0^{s+1/2}(\mathbb{T})$ into itself. To see that Φ is a contraction, we use (2.23) to obtain

$$\begin{aligned}
\|\Phi(\eta_1) - \Phi(\eta_2)\|_{H^{\sigma+1/2}} &\lesssim \|(T_{q_1} - Q)(\eta_1-\eta_2)\|_{H^\sigma} + \|(T_{q_1} - T_{q_2})\eta_2\|_{H^\sigma} \\
&\lesssim M_0^{1/2}(q_1 - Q)\|\eta_1-\eta_2\|_{H^{\sigma+1/2}} + M_0^{1/2}(q_1-q_2)\|\eta_2\|_{H^{\sigma+1/2}} \\
&\le C(M)M\|\eta_1-\eta_2\|_{H^{\sigma+1/2}},
\end{aligned}$$

where $M := \|\eta_1\|_{H^{\sigma+1/2}} + \|\eta_2\|_{H^{\sigma+1/2}}$. If M is small enough, then Φ is a contraction. □

3. The linear equation

As mentioned in the introduction, we shall study the control problem for the water waves equations by means of a nonlinear scheme. This requires solving a linear control problem at each step. In this section we introduce the linear equation we are going to study until Section 10, emphasize one key property of this equation and state the main result we want to prove.

We have seen in the previous section that one can express $V = V(\eta)\psi$ in terms of u only. To simplify notation, we write $V = V(u)$, and similarly we write $c = c(u)$. Also, one can write the remainder $R(\eta, \psi)$ in the form $R(u)u$ where, for any $\underline{u}$, the mapping $u \mapsto R(\underline{u})u$ is linear.

We have proved that, for σ large enough and a solution (η, ψ) of (2.1) on the time interval $[0, T]$ satisfying

$$(\eta, \psi) \in C^0([0, T]; H_0^{\sigma+1/2}(\mathbb{T}) \times H^\sigma(\mathbb{T})),$$

the new unknown u satisfies $u \in C^0([0, T]; \tilde{H}^\sigma(\mathbb{T}; \mathbb{C}))$ (where $\tilde{H}^\sigma(\mathbb{T}; \mathbb{C})$ is defined in Notation 2.7) and

$$\partial_t u + T_{V(u)}\partial_x u + iL^{1/2}(T_{c(u)}L^{1/2}u) + R(u)u = T_{p(u)}P_{\text{ext}}.$$

We now fix $\underline{u} \in C^0([0, T]; \tilde{H}^\sigma(\mathbb{T}; \mathbb{C}))$, set

$$V = V(\underline{u}), \qquad c = c(\underline{u}), \qquad R = R(\underline{u}), \qquad p = p(\underline{u}), \tag{3.1}$$

and consider the linear operator

$$P = \partial_t + T_V\partial_x + iL^{1/2}(T_cL^{1/2}\cdot) + R.$$

Except for the second condition in Assumption 3.1 below, we shall not use the way in which the coefficients depend on $\underline{u}$, and hence we shall state all the assumptions on V, c, p, R forgetting their dependence on $\underline{u}$ through (3.1).

Assumption 3.1. (i) *Consider two real-valued functions V, c in $C^0([0, T]; H^{s_0}(\mathbb{T}))$ for some s_0 large enough, with c bounded from below by $1/2$. The symbol p is given by $p := c^{-1/3} + \frac{5}{18i}\frac{\chi(\xi)\partial_\xi\ell(\xi)}{\ell(\xi)}c^{-4/3}\partial_x c$ with χ as in (2.12). It is always assumed that the $W^{3/2,\infty}$-norm of $c - 1$ is small enough.*

(ii) *If Pu is a real-valued function then*

$$\frac{d}{dt}\int_{\mathbb{T}} \operatorname{Im} u(t, x)\, dx = 0.$$

Fix an open domain $\omega \subset \mathbb{T}$ and denote by χ_ω a C^∞ cut-off function such that $\chi_\omega(x) = 1$ for $x \in \omega$. We want to study the following control problem: given an initial data v_{in} find f such that the unique solution to

$$Pv = T_p\chi_\omega \operatorname{Re} f, \qquad v|_{t=0} = v_{\text{in}}, \tag{3.2}$$

satisfies $v|_{t=T} = 0$. The fact that the Cauchy problem (3.2) admits a unique solution is proved in the appendix (Proposition B.1).

Our main goal until Section 10 will be to prove the following control result.

Proposition 3.2. *There exists s_0 large enough such that, for all $T \in (0,1]$ and all $s \geq s_0$, if Assumption 3.1 holds then there exist positive constants $\tilde{\delta} = \tilde{\delta}(T,s)$ and $K = K(T,s)$ such that if*

$$\begin{aligned} &\|V\|_{C^0([0,T];H^{s_0})} + \|c-1\|_{C^0([0,T];H^{s_0})} \leq \tilde{\delta}, \\ &\|\partial_t^k V\|_{C^0([0,T];H^1)} + \|\partial_t^k c\|_{C^0([0,T];H^1)} \leq \tilde{\delta} \quad (1 \leq k \leq 3), \\ &\|R\|_{C^0([0,T];\mathcal{L}(H^s))} \leq \tilde{\delta}, \end{aligned} \tag{3.3}$$

then for any initial data $v_{\text{in}} \in \tilde{H}^s(\mathbb{T};\mathbb{C})$ there exists $f \in C^0([0,T];H^s(\mathbb{T}))$ such that:

(1) *the unique solution v to $Pv = T_p\chi_\omega \operatorname{Re} f$, $v|_{t=0} = v_{\text{in}}$, satisfies $v(T) = 0$;*
(2) $\|f\|_{C^0([0,T];H^s)} \leq K\|v_{\text{in}}\|_{H^s}$.

Remark 3.3. Notice that the smallness assumption on V and c involves only some H^{s_0}-norm, while the result holds for initial data in H^s with $s \geq s_0$. We shall use this property with $s_0 = s - 2$ in the analysis of the quasi-linear scheme. This is possible only because we consider a paradifferential equation.

We conclude this section by proving that the second condition in Assumption 3.1 holds when V, c, p, R are given by (3.1).

Lemma 3.4. *Consider $\underline{u} \in C^0([0,T];\tilde{H}^{s_0}(\mathbb{T};\mathbb{C}))$ with s_0 large enough and assume that V, c, p, R are given by (3.1). If Pu is a real-valued function, then*

$$\frac{d}{dt}\int_{\mathbb{T}} \operatorname{Im} u(t,x)\,dx = 0.$$

Proof. Set $\zeta = -\operatorname{Im} u$. It follows from (2.17) that

$$\begin{aligned} &\partial_t\zeta + T_V\partial_x\zeta - T_q G(0)\omega = \tilde{f}^1, \\ &\tilde{f}^1 = T_q(F(\eta)\psi - T_{\partial_x V}\eta) + T_{\partial_t q}\eta + [T_V\partial_x, T_q]\eta, \end{aligned}$$

where $F(\eta)\psi$ is given by (2.7). One can write this equation in the form

$$\partial_t\zeta + T_q(\partial_x(T_V\eta)) - T_q G(0)\omega = T_q F(\eta)\psi + T_{\partial_t q}\eta. \tag{3.4}$$

Notice that $T_q G(0)\omega$ and $T_q F(\eta)\psi$ are well-defined since $\widehat{G(0)\omega}(0) = 0 = \widehat{F(\eta)\psi}(0)$ (this follows from the definition (2.7) and the fact that the mean values of $G(\eta)\psi$, $G(0)\omega$ and $\partial_x(T_V\eta)$ are all 0). Using (2.15), one finds that $\int_{\mathbb{T}} T_q v\,dx = 0 = \int_{\mathbb{T}} T_{\partial_t q} v\,dx$ for any function v. So integrating (3.4) we obtain the desired result. □

4. Reduction to a regularized equation

In this section, we reduce the proof of Proposition 3.2 to that of a simpler result. We shall prove that:

- it is enough to consider a classical equation instead of a paradifferential equation (this observation will be used below to simplify the computation of a change of variable);
- it is enough to prove an L^2-result instead of a result in higher order Sobolev spaces (this plays a crucial role).

As explained in the introduction, the idea is to conjugate the equation with an elliptic semiclassical operator $\Lambda_{h,s}$ of order s. The key point is to prove that $\Lambda_{h,s}$ can be so chosen that it satisfies the following commutator estimates:

$$\|[\Lambda_{h,s}, P]\Lambda_{h,s}^{-1}\|_{\mathcal{L}(L^2)} = O(1), \qquad \|[\Lambda_{h,s}, \chi_\omega]\Lambda_{h,s}^{-1}\|_{\mathcal{L}(L^2)} = O(h),$$

which is the reason to introduce the small parameter h. Some care is required to do so, and we introduce

$$\Lambda_{h,s} = I + h^s T_{c^{2s/3}} L^{2s/3}. \tag{4.1}$$

Lemma 4.1. (i) *Assume that the $L^\infty_{t,x}$-norm of $c - 1$ is small enough. Then $\Lambda_{h,s}$ is invertible from H^s to L^2.*

(ii) *Moreover, for any $s' \in [0, s]$, $h^{s'}\Lambda_{h,s}^{-1}$ is uniformly bounded from L^2 to $H^{s'}$: there is $K > 0$ such that for any $h \in (0, 1]$ and any $u \in L^2(\mathbb{T})$,*

$$\|h^{s'}\Lambda_{h,s}^{-1}u\|_{H^{s'}} \leq K\|u\|_{L^2}. \tag{4.2}$$

Proof. Set $r = 2s/3$. Statement (i) is obtained by writing $\Lambda_{h,s}$ as $(I + B)(I + h^s L^r)$ where B is a bounded operator from L^2 into itself. To do so, write

$$\Lambda_{h,s} = I + h^s T_{c^r} L^r = I + h^s L^r + h^s T_{c^r - 1} L^r,$$

to obtain the desired result with $B := h^s T_{c^r-1} L^r (I + h^s L^r)^{-1}$. We now claim that B is a bounded operator on L^2, with operator norm $O(\|c - 1\|_{L^\infty})$. This follows easily from (A.10) (which implies that T_{c^r-1} is of order 0 with operator norm $O(\|c - 1\|_{L^\infty})$) and, on the other hand, from the fact that $h^s L^r (I + h^s L^r)^{-1}$ is bounded on L^2 uniformly in h (as can be verified using the Fourier transform).

Now for $\|c - 1\|_{L^\infty}$ small enough, one has $\|B\|_{\mathcal{L}(L^2)} \leq 1/2$ and one can invert $I + B$ to obtain

$$\Lambda_{h,s}^{-1} = (I + h^s L^r)^{-1}(I + B)^{-1}, \tag{4.3}$$

and statement (ii) follows from the fact that $h^{s'}(I + h^s L^r)^{-1}$ is uniformly bounded in $\mathcal{L}(L^2, H^{s'})$ for $0 \leq s' \leq s$. □

The key property is that one has good estimates for the commutators of $\Lambda_{h,s}$ and the various operators appearing in the equation.

Lemma 4.2. *Assume that the $W^{3/2,\infty}$-norm of $c-1$ is small enough. Then there is $K > 0$ such that for any $h \in (0, 1]$ and any $u \in L^2(\mathbb{T})$,*

$$\|[\Lambda_{h,s}, T_V \partial_x]\Lambda_{h,s}^{-1}u\|_{L^2} \leq K\|V\|_{W^{1,\infty}}\|u\|_{L^2}, \tag{4.4}$$

$$\|[\Lambda_{h,s}, \chi_\omega]\Lambda_{h,s}^{-1}u\|_{L^2} \leq Kh\|\chi_\omega\|_{H^{s+1}}\|u\|_{L^2}, \tag{4.5}$$

$$\|[\Lambda_{h,s}, L^{1/2}(T_c L^{1/2}\cdot)]\Lambda_{h,s}^{-1}u\|_{L^2} \leq K\|u\|_{L^2}. \tag{4.6}$$

Proof. Write
$$[\Lambda_{h,s}, T_V \partial_x]\Lambda_{h,s}^{-1} = [T_{(c\ell)^{2s/3}}, T_V \partial_x] h^s \Lambda_{h,s}^{-1}$$
to obtain
$$\|[\Lambda_{h,s}, T_V \partial_x]\Lambda_{h,s}^{-1} u\|_{L^2} \leq K \|[T_{(c\ell)^{2s/3}}, T_V \partial_x]\|_{\mathcal{L}(H^s, L^2)} \|h^s \Lambda_{h,s}^{-1}\|_{\mathcal{L}(L^2, H^s)} \|u\|_{L^2}.$$
It follows from (4.2) that $\|h^s \Lambda_{h,s}^{-1}\|_{\mathcal{L}(L^2,H^s)}$ is uniformly bounded in h. On the other hand, the commutator estimate (A.9) implies that
$$\|[T_{(c\ell)^{2s/3}}, T_V \partial_x]\|_{\mathcal{L}(H^s, L^2)} \leq K \|V\|_{W^{1,\infty}},$$
where K depends on $\|c\|_{W^{3/2,\infty}}$ (which by assumption can be bounded by 2).

To estimate the second commutator, we begin by establishing that
$$\|[\Lambda_{h,s}, T_{\chi_\omega}]\Lambda_{h,s}^{-1} u\|_{L^2} \leq K h \|\chi\|_{H^s} \|u\|_{L^2}. \tag{4.7}$$
To see this, write
$$[\Lambda_{h,s}, T_{\chi_\omega}]\Lambda_{h,s}^{-1} = h[T_{(c\ell)^{2s/3}}, T_{\chi_\omega}] h^{s-1} \Lambda_{h,s}^{-1}.$$
Then we notice that, as above,
$$\|[T_{(c\ell)^{2s/3}}, T_{\chi_\omega}]\|_{\mathcal{L}(H^{s-1}, L^2)} \leq K \|\chi_\omega\|_{W^{1,\infty}},$$
and we use the fact that, thanks to (4.2), $h^{s-1}\Lambda_{h,s}^{-1}$ is uniformly bounded from L^2 to H^{s-1}.

Now it remains to estimate $[\Lambda_{h,s}, \chi_\omega - T_{\chi_\omega}]$. It follows from Proposition A.8 (applied with $(r, \mu, \gamma) = (s+1, 0, s)$) that
$$\begin{aligned}\|h^s T_{(c\ell)^{2s/3}}(\chi_\omega - T_{\chi_\omega})\Lambda_{h,s}^{-1} u\|_{L^2} &\lesssim h^s \|(\chi_\omega - T_{\chi_\omega})\Lambda_{h,s}^{-1} u\|_{H^s} \\ &\lesssim h^s \|\chi_\omega\|_{H^{s+1}} \|\Lambda_{h,s}^{-1} u\|_{L^2} \lesssim h^s \|\chi_\omega\|_{H^{s+1}} \|u\|_{L^2},\end{aligned}$$
and similarly
$$\|(\chi_\omega - T_{\chi_\omega}) h^s T_{(c\ell)^{2s/3}} \Lambda_{h,s}^{-1} u\|_{L^2} \leq K h^s \|\chi_\omega\|_{H^{s+1}} \|u\|_{L^2}.$$
By combining these two estimates, we find that
$$\|[\Lambda_{h,s}, (\chi_\omega - T_{\chi_\omega})]\Lambda_{h,s}^{-1} u\|_{L^2} \leq K h^s \|\chi_\omega\|_{H^{s+1}} \|u\|_{L^2}. \tag{4.8}$$
From (4.7) and (4.8) we deduce (4.5).

We now prove the last property (4.6). Write $L^{1/2}(T_c L^{1/2}\,\cdot) = T_{c\ell} + T_\wp + R$ where R is of order 0 and $\wp = i^{-1}\sqrt{\ell}(\partial_\xi \sqrt{\ell})(\partial_x c)$. Since $\Lambda_{h,s} = I + h^s T_{(c\ell)^{2s/3}}$, by definition, $[\Lambda_{h,s}, L^{1/2}(T_c L^{1/2}\,\cdot)]\Lambda_{h,s}^{-1}$ can be written as $(I) + (II) + (III)$ with
$$\begin{aligned}&(I) := [T_{(c\ell)^{2s/3}}, T_{c\ell}] h^s \Lambda_{h,s}^{-1}, \qquad (II) := [T_{(c\ell)^{2s/3}}, T_\wp] h^s \Lambda_{h,s}^{-1}, \\ &(III) := [T_{(c\ell)^{2s/3}}, R] h^s \Lambda_{h,s}^{-1}.\end{aligned}$$

Since $h^s \Lambda_{h,s}^{-1}$ belongs to $\mathcal{L}(L^2, H^s)$ uniformly in h, we need only estimate

$$\| [T_{(c\ell)^{2s/3}}, T_{c\ell}] \|_{\mathcal{L}(H^s,L^2)}, \qquad \| [T_{(c\ell)^{2s/3}}, T_{\wp}] \|_{\mathcal{L}(H^s,L^2)}.$$

The second term is estimated by means of (A.6) applied with $\rho = 1/2$. To estimate the first term we notice that the Poisson bracket of the symbols vanishes:

$$\{(c\ell)^{2s/3}, c\ell\} = \frac{1}{i}\big((\partial_\xi (c\ell)^{2s/3}) \partial_x (c\ell) - (\partial_x (c\ell)^{2s/3}) \partial_\xi^\alpha (c\ell) \big) = 0.$$

Since $\|c\|_{W^{3/2,\infty}} \leq 2$ by assumption, it follows from (A.6) applied with $\rho = 3/2$ that

$$\| [T_{(c\ell)^{2s/3}}, T_{c\ell}] \|_{\mathcal{L}(H^s,L^2)} \lesssim 1. \qquad \square$$

Next we conjugate P with $\Lambda_{h,s}$: set

$$\widetilde{P}_h := \Lambda_{h,s} P \Lambda_{h,s}^{-1}.$$

Then

$$\widetilde{P}_h = \partial_t + T_V \partial_x + i L^{1/2}(T_c L^{1/2} \cdot) + R_1 \quad \text{where}$$
$$R_1^h := \Lambda_{h,s} R \Lambda_{h,s}^{-1} + [\Lambda_{h,s}, \partial_t]\Lambda_{h,s}^{-1} + [\Lambda_{h,s}, T_V \partial_x]\Lambda_{h,s}^{-1} + i[\Lambda_{h,s}, L^{1/2}(T_c L^{1/2} \cdot)]\Lambda_{h,s}^{-1}.$$

Lemma 4.3. *Assume that the $W^{3/2,\infty}$-norm of $c - 1$ is small enough. Then*

$$\|R_1^h u\|_{L^2} \leq K(\|V\|_{W^{1,\infty}} + \|\partial_t c\|_{L^\infty} + h^{-s}\|R\|_{\mathcal{L}(H^s)})\|u\|_{L^2} \tag{4.9}$$

for some constant K independent of h.

Remark 4.4. The constant h^{-s} is harmless, since at the end of this section, h will be fixed depending only on T.

Proof of Lemma 4.3. We have

$$\|\Lambda_{h,s} R \Lambda_{h,s}^{-1}\|_{\mathcal{L}(L^2)} \leq \|\Lambda_{h,s}\|_{\mathcal{L}(H^s;L^2)} \|R\|_{\mathcal{L}(H^s;H^s)} \|\Lambda_{h,s}^{-1}\|_{\mathcal{L}(L^2,H^s)} \leq K h^{-s} \|R\|_{\mathcal{L}(H^s,H^s)},$$

since $\|\Lambda_{h,s}\|_{\mathcal{L}(H^s,L^2)} \lesssim 1$ and $\|\Lambda_{h,s}^{-1}\|_{\mathcal{L}(L^2,H^s)} \lesssim h^{-s}$.

On the other hand, $[\Lambda_{h,s}, L^{1/2}(T_c L^{1/2} \cdot)]\Lambda_{h,s}^{-1}$ and $[\Lambda_{h,s}, T_V \partial_x]\Lambda_{h,s}^{-1}$ are estimated by means of Lemma 4.2, and $[\Lambda_{h,s}, \partial_t]\Lambda_{h,s}^{-1}$ is estimated by similar arguments. $\square$

We further transform the equation by replacing $T_V \partial_x$ and $L^{1/2}(T_c L^{1/2} \cdot)$ by $V \partial_x$ and $L^{1/2}(c L^{1/2} \cdot)$ modulo remainder terms. Namely, write

$$\widetilde{P}_h := \partial_t + V \partial_x + i L^{1/2}(c L^{1/2} \cdot) + R_2^h \tag{4.10}$$

where c stands for the operator of multiplication by c and

$$R_2^h u = R_1^h u + T_V \partial_x u - V \partial_x u + i(L^{1/2} T_c L^{1/2} u - L^{1/2}(c L^{1/2} u)).$$

Lemma 4.5. *Let $s_0 > 2$ and assume that the $W^{3/2,\infty}$-norm of $c-1$ is small enough. Then*

$$\|R_2^h u\|_{L^2} \le K(\|V\|_{H^{s_0}} + \|c-1\|_{H^{s_0}} + \|\partial_t c\|_{H^1} + h^{-s}\|R\|_{\mathcal{L}(H^s)})\|u\|_{L^2} \tag{4.11}$$

for some constant K independent of h.

Proof. We have already estimated R_1^h, and the right-hand side of (4.9) is less than the one of (4.11) provided that $s_0 > 3/2$. To estimate $T_V \partial_x u - V\partial_x u$, we apply Proposition A.8 with $(r,\mu,\gamma) = (s_0, -1, 0)$ (and $s_0 > 3/2$) to obtain

$$\|T_V \partial_x u - V\partial_x u\|_{L^2} \lesssim \|V\|_{H^{s_0}}\|\partial_x u\|_{H^{-1}} \le \|V\|_{H^{s_0}}\|u\|_{L^2}.$$

The estimate for $\mathcal{L} - L^{1/2}(cL^{1/2}\cdot) = L^{1/2}((T_c - cI)L^{1/2}\cdot)$ follows in the same way, assuming that $s_0 > 2$. □

We are now ready to give the main reduction. Our goal in this section is to prove that one can deduce Proposition 3.2 from the following proposition.

Proposition 4.6. *Consider an operator of the form*

$$\widetilde{P} := \partial_t + V\partial_x + iL^{1/2}(cL^{1/2}\cdot) + R_2. \tag{4.12}$$

Let $T \in (0,1]$ and consider an open subset $\omega \subset \mathbb{T}$. There exist an integer s_0 large enough and positive constants $\delta = \delta(T)$ and $K = K(T)$ such that if

$$\begin{aligned} &\|V\|_{C^0([0,T];H^{s_0})} + \|c-1\|_{C^0([0,T];H^{s_0})} \le \delta, \\ &\|\partial_t^k V\|_{C^0([0,T];H^1)} + \|\partial_t^k c\|_{C^0([0,T];H^1)} \le \delta \quad (1 \le k \le 3), \\ &\|R_2\|_{C^0([0,T];\mathcal{L}(L^2))} \le \delta, \end{aligned} \tag{4.13}$$

then for any initial data $v_{\text{in}} \in L^2(\mathbb{T})$ there exists $f \in C^0([0,T];L^2(\mathbb{T}))$ such that:

(1) *the unique solution v to $\widetilde{P}v = \chi_\omega \operatorname{Re} f$, $v|_{t=0} = v_{\text{in}}$, is such that $v(T)$ is an imaginary constant:*

$$\exists b \in \mathbb{R}\ \forall x \in \mathbb{T}, \qquad v(T,x) = ib;$$

(2) $\|f\|_{C^0([0,T];L^2)} \le K\|v_{\text{in}}\|_{L^2}$.

Remark 4.7. Notice that the final state $v(T)$ is not 0 but an imaginary constant.

This result will be proved later. Granting it, we now prove Proposition 3.2.

Proof of Proposition 3.2 given Proposition 4.6. Proposition 4.6 holds for any $\widetilde{P}$ of the form (4.12). In particular, in view of (4.10), it holds for $\widetilde{P}$ replaced by $\widetilde{P}_h := \Lambda_{h,s} P \Lambda_{h,s}^{-1}$. Let us mention that h will be fixed at the end of the proof by asking that $K'(T)h < 1/4$ where $K'(T)$ depends only on T.

The idea is to apply the control property for $\widetilde{P}_h$ associated with an unknown initial data to be determined.

We shall prove that Proposition 4.6 implies that Proposition 3.2 holds with conclusion (1) replaced by $v(T) \in i\mathbb{R}$. Then one deduces that $v(T) = 0$ by using Assumption 3.1(ii) and the fact that $\int_{\mathbb{T}} v_{\text{in}}(x)\, dx = 0$.

Assume that $\tilde{\delta} \leq h^s \delta$ where $\tilde{\delta}$ appears in the statement of Proposition 3.2 and δ is given by Proposition 4.6. Then $h^{-s}\tilde{\delta} \leq \delta$. Therefore, if the smallness condition (3.3) holds, then Lemma 4.5 implies that $\|R_2^h\|_{C^0([0,T];\mathcal{L}(L^2))}$ is small, and hence (4.13) holds. This explains why one may apply the conclusion of Proposition 4.6 under the assumption of Proposition 3.2.

By Proposition 4.6, for any $y \in L^2(\mathbb{T})$ there is $\tilde{f} \in C^0([0,T]; L^2(\mathbb{T}))$ satisfying

$$\|\tilde{f}\|_{C^0([0,T];L^2)} \leq K(T)\|y\|_{L^2} \tag{4.14}$$

and such that the unique solution u_1 to

$$\widetilde{P}_h u_1 = \chi_\omega \operatorname{Re} \tilde{f}, \qquad u_1|_{t=0} = y,$$

is such that $u_1(T,x) = ib$ for some $b \in \mathbb{R}$ and all $x \in \mathbb{T}$.

Now let u_2 be the unique solution of the Cauchy problem (with data at time T)

$$\widetilde{P}_h u_2 = (\Lambda_{h,s} T_p \chi_\omega \Lambda_{h,s}^{-1} - \chi_\omega) \operatorname{Re} \tilde{f}, \qquad u_2(T) = 0.$$

Again, this Cauchy problem has a unique solution by Proposition B.1. One can then define a linear operator $\mathcal{K}$ by

$$\mathcal{K}y = u_2(0). \tag{4.15}$$

The reason to introduce u_2 and $\mathcal{K}$ is that the function $u := u_1 + u_2$ satisfies

$$\widetilde{P}_h u = \Lambda_{h,s}(T_p \chi_\omega \Lambda_{h,s}^{-1} \operatorname{Re} \tilde{f}), \qquad u(T) = ib, \qquad u|_{t=0} = y + \mathcal{K}y.$$

Now, assume that $I + \mathcal{K}$ is invertible with $(I + \mathcal{K})^{-1} \in \mathcal{L}(L^2)$. Then y can be so chosen that $y + \mathcal{K}y = \Lambda_{h,s} v_{\text{in}}$. Since $\Lambda_{h,s} b = b$ and hence $\Lambda_{h,s}^{-1} b = b$ for any constant b, it follows that, with $f := \Lambda_{h,s}^{-1}\tilde{f}$ and $v := \Lambda_{h,s}^{-1} u$,

$$Pv = T_p \chi_\omega \operatorname{Re} f, \qquad v(T) = ib, \qquad v(0) = v_{\text{in}},$$

where P is the original operator, so that $\widetilde{P}_h = \Lambda_{h,s} P \Lambda_{h,s}^{-1}$. Moreover, it follows from Proposition 4.6(2) that $\|\tilde{f}\|_{C^0([0,T];L^2)} \leq K\|y\|_{L^2}$, which yields $\|f\|_{C^0([0,T];H^s)} \leq K(h)\|y\|_{H^s}$. The fact that the last constant depends on h is not a problem since h is fixed, depending on T. Now to see that Proposition 3.2 holds, it remains to check that $v(T) = 0$. As already mentioned, this follows from the fact that $v(T) = ib$ together with Assumption 3.1(ii) and the fact that $\int_{\mathbb{T}} v_{\text{in}}(x)\, dx = 0$.

Thus it remains to prove that $I + \mathcal{K}$ is a bijection from L^2 into itself. To see this, it is sufficient to show that $\mathcal{K}$ is a bounded operator whose operator norm in $\mathcal{L}(L^2)$ is < 1. In this direction, we first use the energy estimate (B.3) for the operator $\widetilde{P}_h$:

$$\|u_2(t)\|_{L^2} \leq e^{CT}\left(\|u_2(T)\|_{L^2} + \int_0^T \|\widetilde{P}_h u_2\|_{L^2}\, dt'\right)$$

for some constant C depending only on

$$M_{s_0} := \sup_{t' \in [0,T]} \{\|V(t')\|_{H^{s_0}} + \|c(t') - 1\|_{H^{s_0}} + \|R_2(t')\|_{\mathcal{L}(L^2)}\}.$$

Since $u_2(T)=0$, this implies that

$$\|u_2\|_{C^0([0,T];L^2)} \le e^{CT}\int_0^T \|(\Lambda_{h,s}T_p\chi_\omega\Lambda_{h,s}^{-1}-\chi_\omega)\tilde{f}(t')\|_{L^2}\,dt'.$$

To estimate the term $(\Lambda_{h,s}T_p\chi_\omega\Lambda_{h,s}^{-1}-\chi_\omega)\tilde{f}$ we write it as

$$[\Lambda_{h,s},\chi_\omega]\Lambda_{h,s}^{-1}\tilde{f}+\Lambda_{h,s}(T_p-I)\chi_\omega\Lambda_{h,s}^{-1}\tilde{f}.$$

It follows from (4.5) that

$$\|[\Lambda_{h,s},\chi_\omega]\Lambda_{h,s}^{-1}\tilde{f}\|_{L^2}\le Kh\|\chi_\omega\|_{H^{s+1}}\|\tilde{f}\|_{L^2}.$$

It remains to estimate $\Lambda_{h,s}(T_p-I)\chi_\omega\Lambda_{h,s}^{-1}\tilde{f}$. To do so, we write $\Lambda_{h,s}=I+h^sT_{c^{2s/3}}L^{2s/3}$ to split this term as

$$(T_p-I)\chi_\omega\Lambda_{h,s}^{-1}\tilde{f}+T_{c^{2s/3}}L^{2s/3}(T_p-I)\chi_\omega(h^s\Lambda_{h,s}^{-1})\tilde{f}.$$

For the first term we have (using (A.4) and (4.2) with $s'=0$)

$$\|(T_p-I)\chi_\omega\Lambda_{h,s}^{-1}\tilde{f}\|_{L^2}\lesssim M_0^0(p-1)\|\chi_\omega\|_{L^\infty}\|\tilde{f}\|_{L^2}.$$

For the second term write (using (A.4), (A.17) and (4.2) with $s'=s$)

$$\begin{aligned}\|T_{c^{2s/3}}L^{2s/3}(T_p-I)\chi_\omega(h^s\Lambda_{h,s}^{-1})\tilde{f}\|_{L^2}&\lesssim \|(T_p-I)\chi_\omega(h^s\Lambda_{h,s}^{-1})\tilde{f}\|_{H^s}\\ &\lesssim \|T_{p-1}\|_{\mathcal{L}(H^s)}\|\chi_\omega\|_{H^s}\|h^s\Lambda_{h,s}^{-1}\tilde{f}\|_{H^s}\lesssim M_0^0(p-1)\|\chi_\omega\|_{H^s}\|\tilde{f}\|_{L^2}.\end{aligned}$$

We find that

$$\|\Lambda_{h,s}(T_p-I)\chi_\omega\Lambda_{h,s}^{-1}\tilde{f}\|_{L^2}\lesssim(\|c-1\|_{L^\infty}+\|\partial_x c\|_{L^\infty})\|\chi_\omega\|_{H^s}\|\tilde{f}\|_{L^2}\lesssim\tilde{\delta}\|\tilde{f}\|_{L^2}.$$

This yields

$$\|u_2\|_{C^0([0,T];L^2)}\lesssim(h+\tilde{\delta})e^{CT}\int_0^T\|\tilde{f}\|_{L^2}\,dt'.$$

In view of (4.14), we conclude that

$$\|u_2\|_{C^0([0,T];L^2)}\le K'(T)(h+\tilde{\delta})\|y\|_{L^2}$$

for some constant $K'(T)$. Then choose $h,\tilde{\delta}$ such that $K'(T)h, K'(T)\tilde{\delta}<1/4$. We conclude that

$$\forall t\in[0,T],\qquad \|u_2(t)\|_{L^2}\le\tfrac{1}{2}\|y\|_{L^2}. \tag{4.16}$$

By applying this inequality with $t=0$, one obtains $\|\mathcal{K}y\|_{L^2}\le\frac{1}{2}\|y\|_{L^2}$ which proves that $I+\mathcal{K}$ is invertible in $\mathcal{L}(L^2)$. This completes the proof of Proposition 3.2. □

5. Further reductions

Recall that until now we have reduced the study of the control problem in Sobolev spaces for $P := \partial_t + T_V \partial_x + iL^{1/2}T_c L^{1/2} + R$ to the one of the control problem in L^2 for $\widetilde{P} = \partial_t + V\partial_x + iL^{1/2}(cL^{1/2}\cdot) + R_2$.

5.1. Change of variables

The goal of this subsection is to reduce the analysis to an equation where $L^{1/2}(cL^{1/2}\cdot)$ is replaced by an operator with constant coefficients. To do so, we use three changes of variable which preserve the $L^2(dx)$ scalar product. This allows us to conjugate $\widetilde{P}$ to an operator of the form

$$\partial_t + W\partial_x + iL + R$$

where R is of order 0 and $W = W(t,x)$ satisfies $\int_{\mathbb{T}} W(t,x)\,dx = 0$.

Proposition 5.1. *There exist universal constants $\delta_0 \in (0,1)$, $r \geq 2$, $C > 0$ such that the following holds. Assume that c, V, R_2 satisfy*

$$\|c-1\|_{C^0([0,T];L^\infty)} < \delta_0, \quad \mathcal{N}_0 \leq 1, \tag{5.1}$$

where

$$\mathcal{N}_0 := \|c-1\|_{C^0([0,T];H^r)} + \|V\|_{C^0([0,T];H^1)} + \|\partial_t c\|_{C^0([0,T];H^1)} + \|R_2\|_{C^0([0,T];\mathcal{L}(L^2))}.$$

Then there exist a constant $T_1 > 0$ and a bounded, invertible linear map

$$\Phi\colon C^0([0,T];L^2(\mathbb{T})) \to C^0([0,T_1];L^2(\mathbb{T}))$$

with bounded inverse Φ^{-1} such that

$$\widetilde{P}u = m\Phi^{-1}(\widetilde{P}_3(\Phi u)),$$

where $m = m(t)$ is a function of time only, defined for $t \in [0,T]$, and

$$\widetilde{P}_3 = \partial_t + W\partial_x + iL + R_3.$$

The function $W = W(t,x)$ is defined for $t \in [0,T_1]$, it satisfies $\int_{\mathbb{T}} W(t,x)\,dx = 0$, and

$$\|W\|_{C^0([0,T_1];H^2)} \leq C\big(\|(c-1,V)\|_{C^0([0,T];H^2)} + \|\partial_t c\|_{C^0([0,T];H^1)}\big). \tag{5.2}$$

The operator R_3 maps $C^0([0,T_1];L^2(\mathbb{T}))$ into itself with

$$\|R_3\|_{C^0([0,T_1];\mathcal{L}(L^2))} \leq C\mathcal{N}_0. \tag{5.3}$$

The constant T_1 and the function m satisfy

$$|T_1/T - 1| + \|m-1\|_{C^0([0,T])} \leq C\|c-1\|_{C^0([0,T];L^\infty)}.$$

The map Φ is the composition $\varphi_*^{-1}\psi_*^{-1}\Psi_1$ of three local transformations, where

$$\begin{aligned}(\Psi_1 h)(t,x) &:= (1+\partial_x\tilde{\beta}_1(t,x))^{1/2}h(t,x+\tilde{\beta}_1(t,x)),\\ (\psi_*^{-1}h)(t,x) &:= h(\psi^{-1}(t),x), \qquad (\varphi_*^{-1}h)(t,x) := h(t,x-p(t)),\end{aligned} \tag{5.4}$$

with $\tilde{\beta}_1$, ψ, p given by (C.2), (C.33), (C.34) and (C.36) in Appendix C.

Proof. This is proved in Appendix C. □

Remark 5.2. (i) The proof is based on computations similar to the ones in [1]. However, the analysis in [1] used some special properties of the Hilbert transform which cannot be applied in the present setting. Instead, we shall rely on the Egorov theorem. Moreover, adapting an argument used in [8] and, with Egorov analysis, in [9], it is convenient to introduce a change of variables which preserves the skew-symmetric structure of the operator $iL^{1/2}(cL^{1/2}\,\cdot)$. This allows us to prove that some operator of order $1/2$ vanishes, which plays an essential role below. This in turn forces us to revisit the analysis of changes of variables, which explains why the proof is done in detail in Appendix C.

(ii) In sharp contrast with other transformations to be performed below, a change of variable is a local transformation, hence transforms a localized control into another localized control (this is used below to prove Lemma 9.2).

In addition to Proposition 5.1, higher regularity and stability estimates are given in Proposition C.2.

5.2. Conjugation

To study the control problem for the new equation

$$\partial_t + W\partial_x + iL + R_3$$

we will use the HUM method. A key point is then to prove an observability inequality for solutions of the dual equation, which reads

$$(-\partial_t - \partial_x(W\,\cdot) - iL + R_3^*)w = 0.$$

This equation can be written as $\mathcal{P}w = 0$ with

$$\mathcal{P}w := \partial_t w + W\partial_x w + iLw + R_4 w,$$

where

$$R_4 w := -R_3^* w + (\partial_x W)w. \tag{5.5}$$

The observability inequality will be proved later. As a preparation, in this section we prove that $\mathcal{P}$ is conjugate to a simpler operator where $\partial_t w + W\partial_x w$ is replaced by $\partial_t w$. To do so, we use the analysis in [1]. For the sake of completeness, we recall the strategy and the main steps of the proof.

Below we often use the following notation: given a function f with zero mean, $\partial_x^{-1} f$ is the zero-mean primitive of f, defined by

$$\partial_x^{-1} f = \sum_{j\neq 0} \frac{f_j}{ij} e^{ijx}, \qquad f(x) = \sum_{j\neq 0} f_j e^{ijx}.$$

We seek an operator A such that

$$(\partial_t + W\partial_x + iL + R_4)A = A(\partial_t + iL + R_5),$$

where R_5 is a remainder term of order 0. By definition

$$R_5 := A^{-1}([\partial_t, A] + R_4 A + W\partial_x A + i[L, A]). \tag{5.6}$$

Seeking A as a pseudo-differential operator, and trying to cancel the leading order terms (that is, $W\partial_x A + i[L, A]$), it is natural to introduce A as follows. Let

$$\phi(t, x, \xi) := \xi x + \beta(t, x)|\xi|^{1/2}$$

for some function β to be determined. Consider also an amplitude $q(t, x, \xi)$ to be determined. Then define the operator $A(t)$ by setting

$$Au(t, x) = \sum_{\xi\in\mathbb{Z}} \hat{u}_\xi(t)\, q(t, x, \xi) e^{i\phi(t,x,\xi)}, \tag{5.7}$$

for periodic functions u, where $\hat{u}_\xi(t) = (2\pi)^{-1} \int e^{-ix\xi} u(t, x)\, dx$ are the Fourier coefficients of u, so that $u(t, x) = \sum_{\xi\in\mathbb{Z}} \hat{u}_\xi(t) e^{ix\xi}$.

Below t is seen as a parameter and we omit it in most expressions. Given a symbol $a = a(x, \xi)$ periodic in x, we denote by $\mathrm{Op}(a)$ the pseudo-differential operator defined by

$$\mathrm{Op}(a)u(x) = \sum_{\xi\in\mathbb{Z}} a(x, \xi)\hat{u}_\xi e^{ix\xi}.$$

Assumption 5.3. *Set*

$$\mathcal{N} := \|V\|_{C^0([0,T];H^{s_0})} + \|c - 1\|_{C^0([0,T];H^{s_0})} + \|\partial_t c\|_{C^0([0,T];H^1)} + \|R_2\|_{C^0([0,T];\mathcal{L}(L^2))},$$

where s_0 is some fixed large enough integer. In this section, we always assume that $\mathcal{N}$ is small enough without recalling this assumption in all statements.

Hereafter, s_0 always refers to an index large enough whose value may vary from one statement to another.

Lemma 5.4 ([1, Lemma 12.9]). *There exists a universal constant $\delta > 0$ with the following properties.*

(i) *Consider the case when the amplitude q is a perturbation of* 1,

$$q(x, \xi) = 1 + b(x, \xi).$$

Denote $|b|_s := \sup_{\xi\in\mathbb{Z}} \|b(\cdot, \xi)\|_{H^s(\mathbb{T})}$. If

$$\|\beta\|_{H^3} + |b|_3 \le \delta,$$

then A and A^* are invertible from $L^2(\mathbb{T})$ onto itself, with

$$\|Au\|_{L^2} + \|A^{-1}u\|_{L^2} + \|A^* u\|_{L^2} + \|(A^*)^{-1}u\|_{L^2} \le C\|u\|_{L^2},$$

where $C > 0$ is a universal constant.

(ii) *Consider the case when the amplitude q is small, namely*

$$\|\beta\|_{H^3} + |q|_3 \le \delta.$$

Then

$$\|Au\|_{L^2} \le C\delta\|u\|_{L^2},$$

where $C > 0$ is a universal constant.

Proposition 5.5 ([1, Lemma 12.10]). *Assume that* $\|\beta\|_{W^{1,\infty}} \le 1/4$ *and* $\|\beta\|_{H^2} \le 1/2$. *Let*

$$r, m, s_0 \in \mathbb{R}, \qquad m \ge 0, \qquad s_0 > 1/2, \qquad M \in \mathbb{N}, \qquad M \ge 2(m+r+1)+s_0.$$

Then

$$|D_x|^r Au = \sum_{\alpha=0}^{M-1} \mathrm{Op}\Big(\frac{1}{i^\alpha \alpha!}(\partial_\xi^\alpha |\xi|^r)\partial_x^\alpha\big(q(x,\xi)e^{i|\xi|^{1/2}\beta(x)}\big)\Big)u + R_M u,$$

where, for every $s \ge s_0$, *the remainder satisfies*

$$\|R_M |D_x|^m u\|_{H^s} \le C(s)\{\mathcal{K}_{2(m+r+s_0+1)}\|u\|_{H^s} + \mathcal{K}_{s+M+m+2}\|u\|_{H^{s_0}}\} \tag{5.8}$$

with $\mathcal{K}_\mu := |q-1|_\mu + |q|_1\|\beta\|_{H^{\mu+1}}$ *and* $|q|_\mu := \sup_t \sup_{\xi\in\mathbb{Z}} \|q(t,\cdot,\xi)\|_{H^\mu}$.

We now deduce the following result (which is a variant of a result proved in [1], more precisely in the proof of Lemma 9.3 there, with a slightly different estimate for the remainder).

Corollary 5.6. *There exists a universal constant* $\delta > 0$ *with the following property. Assume that*

$$|q-1|_{14} + \|\beta\|_{H^{14}} \le \delta,$$

and let $A := \mathrm{Op}(q(x,\xi)e^{i|\xi|^{1/2}\beta(x)})$. *For any* u *in* L^2,

$$\begin{aligned} &i[|D_x|^{3/2}, A]u \\ &= \frac{3}{2}(\partial_x\beta)\partial_x(Au) + \mathrm{Op}\Big(\Big(\frac{3}{2}\frac{\xi}{|\xi|}\partial_x q - \frac{9i}{8}(\partial_x\beta)^2 q\Big)|\xi|^{1/2}e^{i|\xi|^{1/2}\beta}\Big)u + R_A u, \end{aligned} \tag{5.9}$$

where

$$\|R_A u\|_{L^2} \le C\delta\|u\|_{L^2}.$$

Proof. Denote $p = q(x,\xi)e^{i|\xi|^{1/2}\beta(x)}$. Set $M = 8$ and write

$$i|D_x|^{3/2}A = \mathrm{Op}\Big(\sum_{\alpha=0}^{2} \frac{i}{i^\alpha \alpha!}\partial_\xi^\alpha|\xi|^{3/2}\,\partial_x^\alpha p\Big) + R_0 + R_M, \tag{5.10}$$

where

$$R_0 := \mathrm{Op}\Big(\sum_{\alpha=3}^{M-1} \frac{i}{i^\alpha \alpha!}\partial_\xi^\alpha|\xi|^{3/2}\,\partial_x^\alpha p\Big).$$

For any $3 \le \alpha \le M-1$, the symbol $\partial_\xi^\alpha|\xi|^{3/2}\,\partial_x^\alpha p$ is a linear combination of terms of the form $m(x,\xi)b(x)e^{i|\xi|^{1/2}\beta}$ where m is of order 0 (that is, $\partial_\xi^l m(x,\xi) \lesssim |\xi|^{-l}$) and $b(x)$ is of the form $(\partial_x^{\alpha_0}q)(\partial_x^{\alpha_1}\beta)\cdots(\partial_x^{\alpha_m}\beta)$. It follows from Lemma 5.4(ii) that R_0 is of order 0 with

$$\|R_0 u\|_{L^2} \le C(\delta)\delta\|u\|_{L^2}.$$

We now estimate the operator norm of R_M. If $s = s_0 = 1$, $m = 1$ and $M = 8$, then the inequality (5.8) implies that

$$\forall u \in L^2(\mathbb{T}), \qquad \| R_M \, |D_x| \, u \|_{H^1} \le C(1)\mathcal{K}_{12} \|u\|_{H^1}.$$

Now we estimate the L^2-norm of $R_M v$ for v in L^2. We can assume without loss of generality that v has zero mean (since $R_M C = 0$ for any constant C) and set $u = |D_x|^{-1} v$. The previous inequality yields

$$\|R_M v\|_{L^2} \le \|R_M v\|_{H^1} = \|R_M \, |D_x| \, u\|_{H^1} \le C(\delta)\delta \|v\|_{L^2}.$$

Therefore

$$\|(R_0 + R_M)u\|_{L^2} \le C(\delta)\delta\|u\|_{L^2}.$$

It remains to study the sum for $0 \le |\alpha| \le 2$ on the right-hand side of (5.10). One can split this sum into two symbols such that the contribution of the first symbol is the two terms on the right-hand side of (5.9), while the other symbol is of the form $Q(x,\xi)e^{i|\xi|^{1/2}\beta}$ with Q of order 0. Therefore the contribution of the second symbol can be estimated by means of Lemma 5.4, so it can be added to $R_0 + R_M$ to obtain an operator R_A satisfying the estimate in the statement of the lemma. □

Notation 5.7. Set

$$\mathcal{N}' := \|W\|_{C^0([0,T];H^{s_0-d})} + \|R_3\|_{C^0([0,T];\mathcal{L}(L^2))},$$

where s_0 is the large enough integer which appears in the definition of $\mathcal{N}$ (see Assumption 5.3) and d is an absolute number independent of s_0 (as in the statement of Proposition 5.1).

We now choose β of the form $\beta_0(t) + \beta_1(t,x)$ for some function coefficient $\beta_0(t)$ to be determined later and with $\beta_1 = \frac{2}{3}\partial_x^{-1} W$. Then

$$\tfrac{3}{2}\partial_x \beta = \tfrac{3}{2}\partial_x \beta_1 = W.$$

Recall from (5.6) that

$$R_5 = A^{-1}([\partial_t, A] + R_4 A + W\partial_x A + i[L, A]). \tag{5.11}$$

Now we split $i[L, A]$ as $i[|D_x|^{3/2}, A] + i[L - |D_x|^{3/2}, A]$. Then it follows from Corollary 5.6 that

$$R_5 = A^{-1}\bigg([\partial_t, A] - \mathrm{Op}\bigg(\bigg(\frac{3}{2}\frac{\xi}{|\xi|}\partial_x q - \frac{9i}{8}(\partial_x\beta)^2 q\bigg)|\xi|^{1/2}e^{i|\xi|^{1/2}\beta}\bigg) \\ + R_4 A + i[L - |D_x|^{3/2}, A] - R_A\bigg), \tag{5.12}$$

where R_A is as given by Corollary 5.6. Recall that R_4 is an operator of order 0. On the other hand,

$$[\partial_t, A] = \mathrm{Op}\big((\partial_t q + i|\xi|^{1/2}(\partial_t\beta)q)e^{i|\xi|^{1/2}\beta}\big).$$

So one can write $R_5 = R_5^{(1/2)} + R_5^{(0)}$ where $R_5^{(1/2)}$ (resp. $R_5^{(0)}$) is of order $1/2$ (resp. 0),

$$R_5^{(1/2)} := A^{-1}\operatorname{Op}\Big(i|\xi|^{1/2}\Big(\partial_t\beta + \frac{9}{8}(\partial_x\beta)^2\Big)p - \frac{3}{2}\frac{\xi}{|\xi|}(\partial_x q)|\xi|^{1/2}e^{i|\xi|^{1/2}\beta}\Big),$$
$$R_5^{(0)} := A^{-1}\big(R_4 A - R_A + i[L - |D_x|^{3/2}, A] + \operatorname{Op}((\partial_t q)e^{i|\xi|^{1/2}\beta})\big).$$

We claim that

$$\|R_5^{(0)}\|_{C^0([0,T];\mathcal{L}(L^2))} \lesssim \mathcal{N}'. \tag{5.13}$$

Indeed, R_A has already been estimated, and directly from (5.5), the Sobolev embedding $\|\partial_x W\|_{L^\infty} \le \|W\|_{H^2}$ and (5.3), one has $\|R_4\|_{C^0([0,T];\mathcal{L}(L^2))} \lesssim \mathcal{N}'$. The last term is estimated by means of Lemma 5.4, and to estimate $[A, L - |D_x|^{3/2}]$ we notice that $L - |D_x|^{3/2}$ is a smoothing operator.

Now, from (5.2) and (5.3) one has $\mathcal{N}' \lesssim \mathcal{N}$, and hence $\|R_5^{(0)}\|_{C^0([0,T];\mathcal{L}(L^2))} \lesssim \mathcal{N}$.

It remains to prove that β and q can be so chosen that $R_5^{(1/2)} = 0$. To do so, we first fix $\beta_0(t)$ such that

$$2\pi\,\partial_t\beta_0 = -\int_{\mathbb{T}}\Big(\partial_t\beta_1 + \frac{9}{8}(\partial_x\beta_1)^2\Big)(t,x)\,dx, \tag{5.14}$$

where recall that $\beta_1 = -\frac{2}{3}\partial_x^{-1}W$, so that

$$\int_{\mathbb{T}}\Big(\partial_t\beta + \frac{9}{8}(\partial_x\beta)^2\Big)(t,x)\,dx = 0.$$

Now define q as $q = e^\gamma$ where γ is such that

$$\gamma = \frac{2}{3}i\frac{\xi}{|\xi|}\partial_x^{-1}\Big(\partial_t\beta + \frac{9}{8}(\partial_x\beta)^2\Big). \tag{5.15}$$

(Notice that the previous cancellation for the mean implies that γ is periodic in x.) With this choice one has $R_5^{(1/2)} = 0$.

By combining the previous results, we end up with the following proposition.

Proposition 5.8. *Assume that s_0 is large enough. Consider the operator*

$$A := \operatorname{Op}\big(q(t,x,\xi)e^{i\beta(t,x)|\xi|^{1/2}}\big) \quad \textit{with} \quad \beta = \beta_0(t) + \tfrac{2}{3}\partial_x^{-1}W,$$

where β_0 is determined by (5.14), and $q = e^\gamma$ where γ is given by (5.15). Then

$$(\partial_t + W\partial_x + iL + R_4)A = A(\partial_t + iL + R_5) \quad \textit{with} \quad \|R_5\|_{C^0([0,T];\mathcal{L}(L^2))} \lesssim \mathcal{N},$$

where $\mathcal{N}$ is as in Assumption 5.3.

6. Ingham type inequalities

As already mentioned, the controllability of the linearized equation around the null solution is based on (a modification of) Ingham's inequality: for every $T > 0$ there exist positive constants $C_1 = C_1(T)$ and $C_2 = C_2(T)$ such that, for all $(w_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z};\mathbb{C})$,

$$C_1 \sum_{n\in\mathbb{Z}} |w_n|^2 \le \int_0^T \Big|\sum_{n\in\mathbb{Z}} w_n e^{in|n|^{1/2}t}\Big|^2 dt \le C_2 \sum_{n\in\mathbb{Z}} |w_n|^2.$$

Hereafter, $(w_n)_{n\in\mathbb{Z}}$ always refers to an arbitrary complex-valued sequence in $\ell^2(\mathbb{Z})$.

For our purposes, we need to consider more general phases that do not depend linearly on t. For a given real-valued function $\beta \in C^3(\mathbb{R})$, set

$$\mu_n(t) = \operatorname{sign}(n)[\ell(n)t + \beta(t)|n|^{1/2}], \qquad \ell(n) = (g+n^2)^{1/2}|n|^{1/2}\tanh^{1/2}(b|n|),$$

with $\mu_0 = 0$ and $\operatorname{sign}(n) = n/|n|$ for $n \neq 0$. We recall that ℓ is the symbol of the linear operator $L = (g - \partial_x^2)^{1/2} G(0)^{1/2}$ obtained by linearizing the water waves system around the null solution (see Section 2.2). We begin by proving a lower bound which holds for any $T > 0$ provided that the functions contain only large enough frequencies.

Proposition 6.1 (High frequencies). *Let $T > 0$. Then there exists $N_0 \ge 0$ such that, for all $N \ge N_0$, the following holds. If*

$$|\partial_t \beta| \le \tfrac{1}{2}\tanh^{1/2}(b) \quad \textit{and} \quad |\partial_t^2 \beta| \le 1 \quad \textit{for all } t \in [0,T],$$

then

$$\frac{T}{2} \sum_{\substack{n\in\mathbb{Z}\\ |n|\ge N}} |w_n|^2 \le \int_0^T \Big|\sum_{\substack{n\in\mathbb{Z}\\ |n|\ge N}} w_n e^{i\mu_n(t)}\Big|^2 dt. \tag{6.1}$$

Remark 6.2. (i) For T small, one can take $N_0 = CT^{-2-\varepsilon}$ for some $\varepsilon > 0$. See (6.8) for more details on this estimate.

(ii) For $\|\partial_t^2\beta\|_{L^\infty}$ small enough and T large enough, the result holds with $N_0 = 0$.

Proof of Proposition 6.1. Splitting the sum into $n = m$ and $n \neq m$, we write

$$\int_0^T \Big|\sum_{\substack{n\in\mathbb{Z}\\ |n|\ge N}} w_n e^{i\mu_n(t)}\Big|^2 dt \ge T \sum_{\substack{n\in\mathbb{Z}\\ |n|\ge N}} |w_n|^2 + \sum_{\substack{n\neq m\\ |m|,|n|\ge N}} w_n \overline{w_m} \int_0^T e^{i(\mu_n(t)-\mu_m(t))}\, dt.$$

We have to estimate

$$K(n,m) := \int_0^T e^{i(\mu_n(t)-\mu_m(t))}\, dt.$$

Integrating by parts yields

$$K(n,m) = \left[\frac{e^{i(\mu_n(t)-\mu_m(t))}}{i(\mu_n'(t)-\mu_m'(t))}\right]_{t=0}^{t=T} + \int_0^T e^{i(\mu_n(t)-\mu_m(t))} \frac{\mu_n''-\mu_m''}{i(\mu_n'-\mu_m')^2}\, dt,$$

and therefore

$$|K(n,m)| \le \kappa(n,m) := \left\|\frac{2}{\mu_n'-\mu_m'}\right\|_{L^\infty([0,T])} + \int_0^T \frac{|\mu_n''-\mu_m''|}{|\mu_n'-\mu_m'|^2}\, dt.$$

Since $\kappa(n,m)=\kappa(m,n)$, we have

$$\Big|\sum_{n\neq m} w_n\overline{w_m}\,K(n,m)\Big| \leq \frac{1}{2}\sum_{n\neq m}(|w_n|^2+|w_m|^2)\kappa(n,m) \leq \sum_{n\neq m}|w_n|^2\kappa(n,m).$$

Hence

$$\int_0^T \Big|\sum_{\substack{n\in\mathbb{Z}\\|n|\geq N}} w_n e^{i\mu_n(t)}\Big|^2\,dt \geq \sum_{\substack{n\in\mathbb{Z}\\|n|\geq N}}\Big(T-\sum_{\substack{m\in\mathbb{Z}\setminus\{n\}\\|m|\geq N}}\kappa(n,m)\Big)|w_n|^2.$$

We have to prove that N can be so chosen that

$$T-\sum_{\substack{m\in\mathbb{Z}\setminus\{n\}\\|m|\geq N}}\kappa(n,m)\geq\frac{T}{2}. \tag{6.2}$$

To do so, we use the following lemma.

Lemma 6.3. *Assume that $|\partial_t\beta(t)|\leq 1/2\tanh^{1/2}(b)$ for all t. Let $\varepsilon\in(0,1/2)$.*

(i) *There exists a positive constant K_ε such that, for all integers $N\geq 0$ and all $n\in\mathbb{Z}$ with $|n|\geq N$,*

$$\sum_{\substack{m\in\mathbb{Z}\setminus\{n\}\\|m|\geq N}}\Big\|\frac{1}{\mu_n'-\mu_m'}\Big\|_{L^\infty([0,T])}\leq\frac{K_\varepsilon}{(1+N)^{1/2-\varepsilon}}. \tag{6.3}$$

(ii) *For all integers n,m with $n\neq m$, and all t,*

$$\frac{|\mu_n''-\mu_m''|}{|\mu_n'-\mu_m'|}\leq 2\tanh^{-1/2}(b)|\partial_t^2\beta|. \tag{6.4}$$

Proof. Let us prove (i). Since $\kappa(-n,m)=\kappa(n,-m)$, we can assume that $n\geq 0$. Let $|\partial_t\beta|\leq\frac{1}{2}\tanh^{1/2}(b)$, and note that $\tanh(b)<1\leq 1+g$. Then for all $n\geq 0$,

$$\tanh^{1/2}(b)n^{3/2}\leq\ell(n)\leq(1+g)^{1/2}n^{3/2},\qquad \tfrac{1}{2}\tanh^{1/2}(b)n^{3/2}\leq\mu_n'(t)\leq\tfrac{3}{2}(1+g)^{1/2}n^{3/2}.$$

For $m\leq 0$, $m\neq n$, one has

$$|\mu_n'-\mu_m'|=\mu_n'+\mu_{-m}'\geq\tfrac{1}{2}\tanh^{1/2}(b)\,(n^{3/2}+|m|^{3/2})\geq\tfrac{1}{4}\tanh^{1/2}(b)\,(1+|m|^{3/2}),$$

and therefore

$$\sum_{m\leq -N}\Big\|\frac{1}{\mu_n'-\mu_m'}\Big\|_{L^\infty([0,T])}\leq\sum_{m\leq -N}\frac{C}{1+|m|^{3/2}}\leq\frac{C'}{\sqrt{1+N}}$$

for some constant $C'>0$. We now consider the case $m>0$ and split the sum into two pieces. For $m\geq An$ with $A:=(36(1+g)/\tanh(b))^{1/3}$ one has $\mu_m'\geq 2\mu_n'$, and

$$|\mu_n'-\mu_m'|\geq\mu_m'|1-\mu_n'/\mu_m'|\geq\tfrac{1}{2}\mu_m'\geq\tfrac{1}{4}\tanh^{1/2}(b)m^{3/2},$$

which again leads to a convergent series

$$\sum_{\substack{m>0\\m\geq An,\,m\geq N}}\Big\|\frac{1}{\mu_n'-\mu_m'}\Big\|_{L^\infty([0,T])}\leq\frac{C}{\sqrt{1+N}}.$$

It remains to consider the sum over all $m > 0$ such that $N \leq m < An$. Denote $\sigma(n) := \sqrt{(g+n^2)n}$. Then

$$\frac{\sigma(m)-\sigma(n)}{m-n} = \frac{\sigma(m)^2-\sigma(n)^2}{(m-n)(\sigma(m)+\sigma(n))} = \frac{m^2+n^2+nm+g}{\sigma(m)+\sigma(n)}.$$

Using the elementary inequality $ab \leq \frac{1}{2}(a^2+b^2)$, one has

$$(\sigma(n)+\sigma(m))\sqrt{n} = \sqrt{n^2+g}\,n + \sqrt{m^2+g}\sqrt{nm} \leq m^2+n^2+nm+g$$

for all $m, n \geq 0$. Therefore

$$|\sigma(m)-\sigma(n)| \geq \sqrt{\max\{n,m\}}\,|m-n| \tag{6.5}$$

for all $m, n \geq 0$. Now suppose that $m \leq n$ with $m \geq 0$, $n \geq 1$. Then

$$\begin{aligned}
\mu_n' - \mu_m' &= (\sigma(n)-\sigma(m))\tanh^{1/2}(bn) + \sigma(m)(\tanh^{1/2}(bn) - \tanh^{1/2}(bm)) \\
&\quad + (n^{1/2}-m^{1/2})\partial_t\beta \\
&\geq (\sigma(n)-\sigma(m))\tanh^{1/2}(bn) - (n^{1/2}-m^{1/2})|\partial_t\beta| \\
&\geq n^{1/2}(n-m)\tanh^{1/2}(b)\left(1 - \frac{|\partial_t\beta|}{\sqrt{n}(\sqrt{n}+\sqrt{m})\sqrt{\tanh(b)}}\right) \\
&\geq \tfrac{1}{2}\tanh^{1/2}(b)\sqrt{n}(n-m)
\end{aligned}$$

if $|\partial_t\beta| \leq \frac{1}{2}\sqrt{\tanh(b)}$. We deduce that

$$|\mu_n' - \mu_m'| \geq C\sqrt{\max\{n,m\}}\,|n-m| \tag{6.6}$$

for all $m, n \geq 0$, with $C = \frac{1}{2}\tanh^{1/2}(b)$. Now, for $n \geq 1$, we obtain

$$\sum_{\substack{m>0,\, m\neq n \\ N\leq m<An}} \left\| \frac{1}{\mu_n' - \mu_m'} \right\|_{L^\infty([0,T])} \leq \frac{1}{C\sqrt{n}} \sum_{\substack{m>0,\, m\neq n \\ N\leq m<An}} \frac{1}{|n-m|} \leq \frac{c\log(cn)}{\sqrt{n}} \tag{6.7}$$

for some $c > 0$. For $n \geq 1$ and $n \geq N$, one has $n \geq \frac{1}{2}(1+N)$, and

$$c\log(cn)n^{-1/2} \leq C_\varepsilon n^{-1/2+\varepsilon} \leq C_\varepsilon 2^{1/2-\varepsilon}(1+N)^{-1/2+\varepsilon}$$

for $\varepsilon \in (0, 1/2)$, with some $C_\varepsilon > 0$. On the other hand, for $n = 0$ the first sum in (6.7) is zero because it has no terms. Thus the first sum in (6.7) is $\leq C_\varepsilon(1+N)^{-1/2+\varepsilon}$ for any $n \geq 0$. This completes the proof of (i). Statement (ii) is proved by using (6.6). □

The previous lemma and the definition of $\kappa(n,m)$ imply that

$$\sum_{\substack{m\in\mathbb{Z}\setminus\{n\} \\ |m|\geq N}} \kappa(n,m) \leq \frac{2K_\varepsilon}{(1+N)^{1/2-\varepsilon}}(1+T\|\partial_t^2\beta\|_{L^\infty}).$$

Hence (6.2) is satisfied provided that

$$\frac{4K_\varepsilon}{T}(1+T\|\partial_t^2\beta\|_{L^\infty}) \le (1+N)^{1/2-\varepsilon}, \tag{6.8}$$

and Proposition 6.1 is proved. □

From (6.6) applied with $\beta=0$, we deduce that

$$|\ell(n)-\ell(m)| \ge C\sqrt{\max\{n,m\}}\,|n-m| \tag{6.9}$$

for all $m,n\ge 0$, with $C=\frac{1}{2}\tanh^{1/2}(b)$.

We now prove upper bounds. By contrast with the previous proposition, we shall see that these estimates hold for any function (not only for high frequencies). Also, a key point for our later purpose is that one can add some amplitudes ζ_n depending on time (and whose k-th order time derivatives can grow with n as $|n|^{k/2}$).

Proposition 6.4. *There exists $C>0$ with the following property. Let $T>0$. Let $|\partial_t\beta|\le \frac{1}{2}\tanh^{1/2}(b)$, and $|\partial_t^k\beta|\le 1$, $k=2,3$, on $[0,T]$. Then, for all $(w_n)\in\ell^2(\mathbb{Z};\mathbb{C})$,*

$$\int_0^T\Big|\sum_{n\in\mathbb{Z}} w_n\zeta_n(t)e^{i\mu_n(t)}\Big|^2\,dt \le CM(\zeta)^2(1+T)\sum_{n\in\mathbb{Z}}|w_n|^2, \tag{6.10}$$

where

$$M(\zeta):=\sup_{n\in\mathbb{Z}}\|\zeta_n\|_{L^\infty}+\sup_{n\in\mathbb{Z}}\frac{\|\partial_t\zeta_n\|_{L^\infty}}{\sqrt{1+|n|}}+\sup_{n\in\mathbb{Z}}\frac{\|\partial_t^2\zeta_n\|_{L^\infty}}{1+|n|}. \tag{6.11}$$

Proof. Splitting the sum into $n=m$ and $n\ne m$, we write

$$\int_0^T\Big|\sum_{n\in\mathbb{Z}} w_n\zeta_n(t)e^{i\mu_n(t)}\Big|^2\,dt=\sum_{n\in\mathbb{Z}}\Big(\int_0^T|\zeta_n(t)|^2\,dt\Big)|w_n|^2+\sum_{n\ne m}w_n\overline{w_m}\,E(n,m)$$

with

$$E(n,m):=\int_0^T\zeta_n(t)\overline{\zeta_m(t)}\,e^{i(\mu_n(t)-\mu_m(t))}\,dt.$$

The first sum on the right-hand side is easily estimated. It remains to bound the sum for $n\ne m$. Integrating by parts twice, one has

$$\begin{aligned}E(n,m)=\int_0^T fe^{ih}\,dt&=[e^{ih}(-ifp+f'p^2-fh''p^3)]_0^T\\&\quad+\int_0^T e^{ih}(f''p^2-3f'h''p^3+3fh''^2p^4-fh'''p^3)\,dt\end{aligned}$$

with

$$f:=\zeta_n\overline{\zeta_m},\qquad h:=\mu_n-\mu_m,\qquad p:=\frac{1}{\mu_n'-\mu_m'}.$$

Thus $|E(n,m)| \le e(n,m)$, where

$$e(n,m) := 2\|fp\|_{L^\infty} + 2\|f'p^2\|_{L^\infty} + 2\|fh''p^3\|_{L^\infty} \\ + T(\|f''p^2\|_{L^\infty} + 3\|f'h''p^3\|_{L^\infty} + 3\|fh''^2p^4\|_{L^\infty} + \|fh'''p^3\|_{L^\infty}). \tag{6.12}$$

We have to estimate the sum $\sum_{m\in\mathbb{Z}\setminus\{n\}} e(n,m)$, uniformly in n. First, we note that

$$\|\partial_t^k(\zeta_n\overline{\zeta_m})\|_{L^\infty} = \|\partial_t^k f\|_{L^\infty} \le \{(1+|n|)^{1/2} + (1+|m|)^{1/2}\}^k M(\zeta)^2, \qquad k = 0,1,2.$$

We have already seen in (6.4) that $|h''p| \le 2|\partial_t^2\beta|$. Similarly, $|h'''p| \le 2|\partial_t^3\beta|$. Also, applying (6.3) with $N = 0$, $\varepsilon = 1/4$, we deduce that $\sum_{m\in\mathbb{Z}\setminus\{n\}} \|p\|_{L^\infty} \le C$ for some absolute constant C. Therefore the sum of the first, the third and the last two terms in (6.12) (i.e. those with f) is bounded by $CM(\zeta)^2(1+T)$. The remaining three terms are also bounded by $CM(\zeta)^2(1+T)$ provided that

$$\sum_{m\in\mathbb{Z}\setminus\{n\}} \left\| \frac{|n|+|m|}{(\mu_n'-\mu_m')^2} \right\|_{L^\infty} \le C \tag{6.13}$$

for all $n \in \mathbb{Z}$, for some C independent of n. The bound (6.13) is proved by using the same splitting and estimates as in the proof of Lemma 6.3. □

By combining the last two propositions with an induction argument (following [10, 22, 40]), we now deduce the following result.

Proposition 6.5 (Sharp Ingham type inequality). *Let $T > 0$. Then there exist positive constants $C(T)$ and $\delta(T)$ such that if*

$$\|\beta\|_X := \sup_{t\in[0,T]} |(\partial_t\beta, \partial_t^2\beta, \partial_t^3\beta)| \le \delta(T), \tag{6.14}$$

then, for all $(w_n) \in \ell^2(\mathbb{Z};\mathbb{C})$,

$$C(T)\sum_{n\in\mathbb{Z}} |w_n|^2 \le \int_0^T \Big|\sum_{n\in\mathbb{Z}} w_n e^{i\mu_n(t)}\Big|^2\,dt.$$

Proof. This proposition will be deduced from Proposition 6.1, the following claim and an immediate induction argument (with a finite number of steps).

Claim 6.6. *Consider two subsets $\mathcal{A}$, $\mathcal{A}'$ of $\mathbb{Z}$ with $\mathcal{A}' = \mathcal{A}\cup\{N\}$ for some $N \in \mathbb{Z}$, and with $|n| \ge |N|$ for all n in $\mathcal{A}$. Assume that for every $T > 0$ there exist positive constants $\delta(T)$ and $K(T)$ such that*

$$\|\beta\|_X \le \delta(T) \;\Rightarrow\; K(T)\sum_{n\in\mathcal{A}} |w_n|^2 \le \int_0^T \Big|\sum_{n\in\mathcal{A}} w_n e^{i\mu_n(t)}\Big|^2\,dt. \tag{6.15}$$

Then for every $T > 0$ there exist positive constants $\delta'(T)$ and $K'(T)$ such that

$$\|\beta\|_X \le \delta'(T) \;\Rightarrow\; K'(T)\sum_{n\in\mathcal{A}'} |w_n|^2 \le \int_0^T \Big|\sum_{n\in\mathcal{A}'} w_n e^{i\mu_n(t)}\Big|^2\,dt. \tag{6.16}$$

To prove the claim, we introduce

$$f(t) := \sum_{n\in\mathcal{A}} w_n e^{i\mu_n(t)}, \quad f'(t) := \sum_{n\in\mathcal{A}'} w_n e^{i\mu_n(t)}, \quad f_1(t) := \sum_{n\in\mathcal{A}'} w_n e^{i\mu_n(t)-i\mu_N(t)},$$

so that $f' = f + w_N e^{i\mu_N}$, $f_1 = e^{-i\mu_N} f' = f e^{-i\mu_N} + w_N$, and

$$\int_0^T |f_1(t)|^2\,dt = \int_0^T |f'(t)|^2\,dt = \int_0^T \Big|\sum_{n\in\mathcal{A}'} w_n e^{i\mu_n(t)}\Big|^2\,dt.$$

We prove that there exist constants C_1, C_2 (both depending on T) such that

$$C_1 \sum_{n\in\mathcal{A}} |w_n|^2 \le \int_0^T |f'(t)|^2\,dt, \qquad C_2|w_N|^2 \le \int_0^T |f'(t)|^2\,dt. \tag{6.17}$$

Then (6.17) implies the second inequality of (6.16) with $K'(T) := \frac{1}{2}\min\{C_1, C_2\}$. Let us begin with the first inequality of (6.17). Let $\tau := \frac{1}{2}\min\{1, T\}$, and remark that

$$\int_0^\tau (f_1(t+\eta) - f_1(t))\,d\eta = e^{-i\mu_N(t)} \sum_{n\in\mathcal{A}} w_n e^{i\mu_n(t)}\theta_n(t) \tag{6.18}$$

(notice that the sum is over $\mathcal{A}$ and not $\mathcal{A}'$) with

$$\theta_n(t) = \int_0^\tau (e^{i(\mu_n(t+\eta)-\mu_n(t)-\mu_N(t+\eta)+\mu_N(t))} - 1)\,d\eta.$$

Assume that n, N are positive. We split $\theta_n = c_n + \zeta_n$, where c_n is a constant, independent of time (such that $c_n = \theta_n$ for $\beta = 0$), and ζ_n is defined to be the difference, namely

$$c_n := \int_0^\tau (e^{i[\ell(n)-\ell(N)]\eta} - 1)\,d\eta = \frac{e^{i[\ell(n)-\ell(N)]\tau} - 1}{i[\ell(n)-\ell(N)]} - \tau,$$
$$\zeta_n := \int_0^\tau e^{i[\ell(n)-\ell(N)]\eta}(e^{i[\beta(t+\eta)-\beta(t)](\sqrt{n}-\sqrt{N})} - 1)\,d\eta.$$

Now we use the following elementary inequality: there exists an absolute constant $c_0 > 0$ such that, for all $\vartheta \in \mathbb{R}$,

$$|e^{i\vartheta} - 1 - i\vartheta|^2 \ge c_0 \min\{\vartheta^2, \vartheta^4\}.$$

This holds because $|e^{i\vartheta} - 1 - i\vartheta|^2 = (1-\cos\vartheta)^2 + (\vartheta - \sin\vartheta)^2$ is positive for all $\vartheta \neq 0$ and it has asymptotic expansion $\vartheta^2 + o(\vartheta^2)$ for $|\vartheta| \to \infty$, and $\frac{1}{4}\vartheta^4 + o(\vartheta^4)$ for $\vartheta \to 0$. We apply this inequality with $\vartheta = [\ell(n) - \ell(N)]\tau$, and, using (6.9), we get

$$|c_n|^2 \ge c\tau^4$$

for some $c > 0$ (note that $\min\{\tau^2, \tau^4\} = \tau^4$ because, by assumption, $\tau < 1$).

It remains to estimate ζ_n and its derivatives. From the definition,

$$|\zeta_n| \le 2\tau, \quad |\partial_t\zeta_n| \le 2\|\partial_t\beta\|_{L^\infty}\tau\sqrt{n}, \quad |\partial_t^2\zeta_n| \le 4(\|\partial_t\beta\|_{L^\infty}^2 + \|\partial_t^2\beta\|_{L^\infty})\tau n.$$

However, we need a sharper bound on ζ_n which shows that ζ_n is small when β is small. Such a bound could be easily obtained by estimating $|e^{if} - 1| \leq |f|$, but this would produce an extra factor $\sqrt{n}$. Instead, we integrate by parts to obtain

$$\zeta_n(t) = \frac{e^{i[\ell(n)-\ell(N)]\tau}}{i[\ell(n)-\ell(N)]}\left(e^{i[\beta(t+\tau)-\beta(t)](\sqrt{n}-\sqrt{N})} - 1\right) - \int_0^\tau \frac{e^{i[\ell(n)-\ell(N)]\eta}}{i[\ell(n)-\ell(N)]}\,\partial_\eta\left(e^{i[\beta(t+\eta)-\beta(t)](\sqrt{n}-\sqrt{N})} - 1\right) d\eta,$$

and it is easily checked, using (6.9) and the bound $|\beta(t+\tau)-\beta(t)| \leq \tau\|\partial_t\beta\|_{L^\infty}$, that $|\zeta_n| \leq C\tau\|\partial_t\beta\|_{L^\infty}$. By combining the previous estimates, we have $M(\zeta) \leq C\tau\|\beta\|_X$ where $M(\zeta)$ is given by (6.11), and C is independent of T, τ.

Set $F(t) := \sum_{n\in\mathcal{A}} w_n e^{i\mu_n(t)}\theta_n(t)$ and split $F = F_1 + F_2$ with

$$F_1(t) := \sum_{n\in\mathcal{A}} w_n e^{i\mu_n(t)} c_n, \qquad F_2(t) := \sum_{n\in\mathcal{A}} w_n e^{i\mu_n(t)} \zeta_n(t).$$

Since $|c_n|^2 \geq c\tau^4$, the assumption (6.15) implies that if $\|\beta\|_X \leq \delta(T-\tau)$, then

$$c\tau^4 K(T-\tau)\sum_{n\in\mathcal{A}} |w_n|^2 \leq K(T-\tau)\sum_{n\in\mathcal{A}} |w_n c_n|^2 \leq \int_0^{T-\tau} |F_1(t)|^2\,dt.$$

On the other hand, Proposition 6.4 applied with $M(\zeta) \leq C\tau\|\beta\|_X$ implies that if $\|\beta\|_X \leq \frac{1}{2}\tanh^{1/2}(b)$, then

$$\int_0^{T-\tau} |F_2(t)|^2\,dt \leq C_0\tau^2\|\beta\|_X^2(1+T-\tau)\sum_{n\in\mathcal{A}} |w_n|^2,$$

where C_0 is independent of T, τ. Therefore, if

$$4C_0\tau^2\|\beta\|_X^2(1+T-\tau) \leq c\tau^4 K(T-\tau), \tag{6.19}$$

then $\int_0^{T-\tau} |F_2|^2\,dt \leq \frac{1}{4}\int_0^{T-\tau} |F_1|^2\,dt$, whence $\int_0^{T-\tau} |F|^2\,dt \geq \frac{1}{4}\int_0^{T-\tau} |F_1|^2\,dt$. By (6.18), this implies that

$$\frac{1}{4}c\tau^4 K(T-\tau)\sum_{n\in\mathcal{A}} |w_n|^2 \leq \int_0^{T-\tau} |F(t)|^2\,dt \leq \int_0^{T-\tau}\left|\int_0^\tau (f_1(t+\eta) - f_1(t))\,d\eta\right|^2 dt.$$

The condition (6.19) holds if

$$\|\beta\|_X \leq \frac{\tau\sqrt{c\,K(T-\tau)}}{2\sqrt{C_0(1+T)}}, \tag{6.20}$$

and we set $\delta'(T)$ to be the minimum of $\frac{1}{2}\tanh^{1/2}(b)$, $\delta(T)$, and the constant on the right in (6.20). Moreover,

$$\begin{aligned}\int_0^{T-\tau}\left|\int_0^\tau (f_1(t+\eta)-f_1(t))\,d\eta\right|^2 dt &\le \int_0^{T-\tau}\tau\int_0^\tau |(f_1(t+\eta)-f_1(t))|^2\,d\eta\,dt\\ &\le 2\tau\int_0^{T-\tau}\int_0^\tau |f_1(t+\eta)|^2\,d\eta\,dt+2\tau\int_0^{T-\tau}\int_0^\tau |f_1(t)|^2\,d\eta\,dt\\ &\le 2T\tau\int_0^T |f_1(t)|^2\,dt = 2T\tau\int_0^T |f'(t)|^2\,dt,\end{aligned}$$

and we infer that the first inequality in (6.17) holds with $C_1=\frac{1}{8}c\tau^3K(T-\tau)T^{-1}$.

Now we prove the second inequality in (6.17). We have $|w_N|^2=|f'(t)-f(t)|^2$ for any t, and so

$$|w_N|^2=\frac{1}{T}\int_0^T |f'(t)-f(t)|^2\,dt\le\frac{2}{T}\left(\int_0^T |f'(t)|^2\,dt+\int_0^T |f(t)|^2\,dt\right).$$

It follows from Proposition 6.4 (applied with $\zeta_n=1$) that

$$\int_0^T |f(t)|^2\,dt\le(1+T)C\sum_{n\in\mathcal{A}}|w_n|^2.$$

Using the first inequality of (6.17), we deduce that

$$\int_0^T |f(t)|^2\,dt\le\frac{(1+T)C}{C_1}\int_0^T |f'(t)|^2\,dt,$$

where C is the constant of Proposition 6.4 and C_1 has been found above. Consequently, the second inequality in (6.17) holds with $C_2=\frac{1}{2}TC_1[C_1+(1+T)C]^{-1}$. We set $K'(T)=\frac{1}{2}\min\{C_1,C_2\}$ and obtain (6.16). This completes the proof of the claim for n, N positive. The other cases are analogous. □

7. Observability

We now use the previous inequalities for sums of oscillatory functions to prove an observability property. In particular, we prove that it is sufficient to control the real part of the solution to bound the initial data.

Proposition 7.1 (Observability). *Let $T>0$. Consider an open subset $\omega\subset\mathbb{T}$ and a constant $0<c\le 1$. Then there exist positive constants K,ε_1 such that the following holds. Consider a pseudo-differential operator A_0 with symbol $\exp(i\beta(t,x)|\xi|^{1/2})$ for some function β satisfying*

$$\sup_{t\in[0,T]}\sup_{x\in[0,2\pi]}|(\partial_t\beta(t,x),\partial_t^2\beta(t,x),\partial_t^3\beta(t,x))|\le\delta(T),$$

where $\delta(T)$ is the constant in Proposition 6.5. Then for every initial data $v_0 \in L^2(\mathbb{T})$ whose mean value $\langle v_0 \rangle = (2\pi)^{-1} \int_{\mathbb{T}} v_0(x)\, dx$ satisfies

$$|\mathrm{Re}\, \langle v_0 \rangle| \geq c|\langle v_0 \rangle| - \varepsilon_1 \|v_0\|_{L^2}, \tag{7.1}$$

the solution v of

$$\partial_t v + iLv = 0, \qquad v(0) = v_0, \tag{7.2}$$

satisfies

$$\int_0^T \int_\omega |\mathrm{Re}\, (A_0 v)(t,x)|^2 \, dx\, dt \geq K \int_0^{2\pi} |v_0(x)|^2 \, dx. \tag{7.3}$$

Remark 7.2. The condition (7.1) cannot be eliminated. To see this, consider the simplest case $\beta = 0$, so $A_0 = I$, and consider a constant solution $v(t,x) = C$ of (7.2). Then (7.3) holds for some K if and only if the real part of C is nonzero. This suggests assuming that

$$|\mathrm{Re}\, \langle v_0 \rangle| \geq c|\langle v_0 \rangle|. \tag{7.4}$$

In fact, it is sufficient to consider the weaker assumption (7.1). The advantage of assuming (7.1) instead of (7.4) is used below (see (7.14)).

Proof of Proposition 7.1. Write

$$v(t,x) = \sum_{n \in \mathbb{Z}} a_n e^{inx} e^{i\ell(n)t}, \qquad a_n = \frac{1}{2\pi} \int_0^{2\pi} e^{-inx} v_0(x)\, dx,$$

where $\ell(n) = (g + n^2)^{1/2}(|n| \tanh(b|n|))^{1/2}$ is the symbol of L. Then set $w = A_0 v$, given by

$$w(t,x) = \sum_{n \in \mathbb{Z}} a_n e^{inx} e^{i(\ell(n)t + \beta(t,x)|n|^{1/2})}.$$

For $n \in \mathbb{Z}$, set

$$\lambda_n = \ell(n)t + \beta(t,x)|n|^{1/2}, \qquad \mu_n = \mathrm{sign}(n)\lambda_n, \qquad c_n(x) = a_n e^{inx}.$$

Since $\mu_n = \mathrm{sign}(n)\lambda_n$ and $\mu_{-n} = -\mu_n$, we write

$$\begin{aligned} 2\,\mathrm{Re}\, w &= 2\,\mathrm{Re}\, a_0 + \sum_{n>0} c_n e^{i\lambda_n} + \sum_{n>0} \overline{c_n} e^{-i\lambda_n} + \sum_{n<0} c_n e^{i\lambda_n} + \sum_{n<0} \overline{c_n} e^{-i\lambda_n} \\ &= 2\,\mathrm{Re}\, c_0 + \sum_{n>0} c_n e^{i\mu_n} + \sum_{n<0} \overline{c_{-n}} e^{i\mu_n} + \sum_{n>0} c_{-n} e^{i\mu_n} + \sum_{n<0} \overline{c_n} e^{i\mu_n} \end{aligned}$$

to obtain

$$2\,\mathrm{Re}\, w = \sum_{n \in \mathbb{Z}} \gamma_n e^{i\mu_n} \quad \text{with} \quad \gamma_n = \begin{cases} c_n + c_{-n} & \text{for } n > 0, \\ 2\,\mathrm{Re}\, c_0 & \text{for } n = 0, \\ \overline{c_n} + \overline{c_{-n}} & \text{for } n < 0. \end{cases}$$

Consider an interval $\omega_0 = [a, b] \subset \omega$. By Proposition 6.5,

$$\int_0^T \int_\omega |\mathrm{Re}(w(t,x))|^2 \, dx \, dt \geq \int_{\omega_0} \int_0^T |\mathrm{Re}(w(t,x))|^2 \, dt \, dx$$
$$\geq \frac{C(T)}{4} \int_{\omega_0} \sum_{n \in \mathbb{Z}} |\gamma_n(x)|^2 \, dx, \tag{7.5}$$

where $C(T)$ is the constant given in Proposition 6.5. For $n \neq 0$ we write

$$|\gamma_n(x)|^2 = |a_n|^2 + |a_{-n}|^2 + a_n \overline{a_{-n}} \, e^{2inx} + \overline{a_n} \, a_{-n} e^{-2inx},$$

so that

$$\int_{\omega_0} |\gamma_n(x)|^2 \, dx \geq |\omega_0| \{ |a_n|^2 + |a_{-n}|^2 \} - |a_n| \, |a_{-n}| \left(\left| \int_{\omega_0} e^{2inx} \, dx \right| + \left| \int_{\omega_0} e^{-2inx} \, dx \right| \right).$$

Now

$$\left| \int_{\omega_0} e^{2inx} \, dx \right| = \left| \int_{\omega_0} e^{-2inx} \, dx \right| = \left| \frac{\sin(n(b-a))}{n} \right|.$$

Moreover there is a small universal constant $\delta_0 > 0$ such that, for all $\delta \in (0, \delta_0)$,

$$\forall |x| \geq \delta, \qquad \left| \frac{\sin(x)}{x} \right| \leq \frac{\sin(\delta)}{\delta}.$$

We can assume that $0 < b - a \leq \delta_0$, so that

$$\forall n \in \mathbb{Z}^*, \qquad (b-a) - \left| \frac{\sin(n(b-a))}{n} \right| \geq (b-a) - \sin(b-a).$$

As a consequence, for all $n \neq 0$,

$$\int_{\omega_0} |\gamma_n(x)|^2 \, dx \geq c' (|a_n|^2 + |a_{-n}|^2),$$

where $c' := (b-a) - \sin(b-a) > 0$. Then, recalling that $\gamma_0 = 2 \, \mathrm{Re} \, a_0$, it follows from (7.5) that

$$\int_0^T \int_\omega |\mathrm{Re}(w(t,x))|^2 \, dx \, dt \geq C(T) \left[(b-a) |\mathrm{Re} \, a_0|^2 + \frac{c'}{2} \sum_{n \in \mathbb{Z} \setminus \{0\}} |a_n|^2 \right].$$

Now, using $(x+y)^2 \geq \frac{1}{2} x^2 - y^2$ and (7.1), one has $|\mathrm{Re} \, \langle v_0 \rangle|^2 \geq \frac{1}{2} c^2 |\langle v_0 \rangle|^2 - \varepsilon_1^2 \| v_0 \|_{L^2}^2$, namely

$$|\mathrm{Re} \, a_0|^2 \geq \frac{c^2}{2} |a_0|^2 - 2\pi \varepsilon_1^2 \sum_{n \in \mathbb{Z}} |a_n|^2,$$

and therefore

$$\int_0^T \int_\omega |\mathrm{Re}(w(t,x))|^2 \, dx \, dt \geq K \sum_{n \in \mathbb{Z}} |a_n|^2$$

with $K = C(T) \min \left\{ (b-a) \left(\frac{1}{2} c^2 - 2\pi \varepsilon_1^2 \right), \frac{1}{2} c' - (b-a) 2\pi \varepsilon_1^2 \right\}$. If ε_1 is small enough, then $K > 0$, which completes the proof. □

Corollary 7.3. *Let $T > 0$, let $\omega \subset \mathbb{T}$ be an open subset and let $0 < c \leq 1$. Then there exist positive constants $\varepsilon_0, \varepsilon_1, r, K$ such that the following holds. Assume that $\langle W(t) \rangle = 0$ for all $t \in [0, T]$ and*

$$\sup_{t\in[0,T]} \sum_{1\leq k\leq 3} \|\partial_t^k W(t)\|_{H^1} + \sup_{t\in[0,T]} \|W(t)\|_{H^r} \leq \varepsilon_0,$$

and consider the pseudo-differential operator A, given by Proposition 5.8, with symbol $q(t,x,\xi)\exp(i\beta(t,x)|\xi|^{1/2})$. Then for every initial data $v_0 \in L^2(\mathbb{T})$ whose mean value satisfies

$$|\mathrm{Re}\,\langle v_0\rangle| \geq c|\langle v_0\rangle| - \varepsilon_1 \|v_0\|_{L^2}, \tag{7.6}$$

the solution v of

$$\partial_t v + iLv = 0, \qquad v(0) = v_0, \tag{7.7}$$

satisfies

$$\int_0^T \int_\omega |\mathrm{Re}(Av)(t,x)|^2 \, dx \, dt \geq K \int_0^{2\pi} |v_0(x)|^2 \, dx. \tag{7.8}$$

(*The constants $\varepsilon_0, \varepsilon_1, K$ depend on T, c, while r is a universal constant.*)

Proof. Split A as $A_0 + A_1$ with

$$A_0 := \mathrm{Op}\big(\exp(i\beta(t,x)|\xi|^{1/2})\big), \qquad A_1 := \mathrm{Op}\big((q(t,x,\xi)-1)\exp(i\beta(t,x)|\xi|^{1/2})\big).$$

The contribution due to A_0 is estimated by Proposition 7.1. Notice that, for ε_0 small enough, the smallness assumption on β in Proposition 7.1 is satisfied because

$$\sup_{t\in[0,T]} \sup_{x\in[0,2\pi]} |(\partial_t\beta(t,x), \partial_t^2\beta(t,x), \partial_t^3\beta(t,x))| \lesssim \sup_{t\in[0,T]} \sum_{1\leq k\leq 3} \|\partial_t^k W(t)\|_{H^1} \lesssim \varepsilon_0.$$

On the other hand, it follows from the definition of q and β and the estimate given by Lemma 5.4(ii) that A_1 is bounded from L^2 onto itself, with operator norm of size $O(\|W\|_{H^r}) = O(\varepsilon_0)$. Then

$$\int_0^T \int_\omega |\mathrm{Re}\,(A_1 v)(t,x)|^2 \, dx \, dt \leq \int_0^T \|A_1 v(t)\|_{L^2}^2 \, dt \lesssim \int_0^T \varepsilon_0^2 \|v(t)\|_{L^2}^2 \, dt.$$

Since $\|v(t)\|_{L^2} = \|v(0)\|_{L^2}$, by taking ε_0 small enough the desired estimate follows from the triangle inequality. □

We now want to deduce an observability result for equations of the form

$$\partial_t w + W\partial_x w + iLw + \mathcal{R}w = 0,$$

where $\mathcal{R}$ is an operator of order 0. In the appendix we prove that the Cauchy problem for this equation is well-posed (see Lemma B.3).

Corollary 7.4. *Let $T > 0$, let $\omega \subset \mathbb{T}$ be a nonempty open domain and let $0 < c \leq 1$. Then there exist positive constants $\varepsilon_2, \varepsilon_3, r, K$ such that the following holds. Assume that $\langle W(t)\rangle = 0$ for all $t \in [0, T]$ and*

$$\sup_{t\in[0,T]} \sum_{1\leq k\leq 3} \|\partial_t^k W(t)\|_{H^1} + \sup_{t\in[0,T]} \|W(t)\|_{H^r} + \sup_{t\in[0,T]} \|\mathcal{R}(t)\|_{\mathcal{L}(L^2)} \leq \varepsilon_2. \tag{7.9}$$

Then for every initial data $w_0 \in L^2(\mathbb{T})$ whose mean value satisfies

$$|\mathrm{Re}\,\langle w_0\rangle| \geq c|\langle w_0\rangle| - \varepsilon_3\|w_0\|_{L^2}, \tag{7.10}$$

the solution w of

$$\partial_t w + W\partial_x w + iLw + \mathcal{R}w = 0, \qquad w(0) = w_0, \tag{7.11}$$

satisfies

$$\int_0^T \int_\omega |\mathrm{Re}\, w|^2 \, dx\, dt \geq K \int_0^{2\pi} |w_0(x)|^2 \, dx. \tag{7.12}$$

Remark 7.5. Corollary 7.4 also holds for data at time T, that is: If $w_0 \in L^2(\mathbb{T})$ satisfies (7.10), then the solution w of

$$\partial_t w + W\partial_x w + iLw + \mathcal{R}w = 0, \qquad w(T) = w_0, \tag{7.13}$$

also satisfies (7.12). Note that the data in (7.13) is at time T instead of 0. To prove it, notice that the function $\widetilde{w}(t,x) := w(T-t,x)$ satisfies

$$-\partial_t \widetilde{w} + \widetilde{W}\partial_x \widetilde{w} + iL\widetilde{w} + \widetilde{\mathcal{R}}\widetilde{w} = 0,$$

where $\widetilde{W}(t), \widetilde{\mathcal{R}}(t)$ stand for $W(T-t), \mathcal{R}(T-t)$. Since $\widetilde{W}$ and $\widetilde{\mathcal{R}}$ satisfy the same assumptions as $W, \mathcal{R}$, one can apply (7.12) with w replaced by $\widetilde{w}$, noticing that

$$\int_0^T \int_\omega |\mathrm{Re}\, w|^2 \, dx\, dt = \int_0^T \int_\omega |\mathrm{Re}\, \widetilde{w}|^2 \, dx\, dt.$$

Proof of Corollary 7.4. It follows from Proposition 5.8 that there is a change of unknown $w = Av$ such that v satisfies an equation of the form

$$\partial_t v + iLv + \mathfrak{R}v = 0$$

for some operator $\mathfrak{R}$ of order 0 satisfying $\|\mathfrak{R}(t)v\|_{L^2} \leq C\varepsilon_2\|v\|_{L^2}$ for all $t \in [0, T]$. By a perturbation argument, we shall deduce observability for this equation from observability for the equation without $\mathfrak{R}$. To do so, split v as $v_1 + v_2$ where v_1 and v_2 are given by the Cauchy problems

$$\begin{cases} \partial_t v_1 + iLv_1 = 0, \\ v_1(0) = v_0, \end{cases} \qquad \begin{cases} \partial_t v_2 + iLv_2 + \mathfrak{R}v_2 = -\mathfrak{R}v_1, \\ v_2(0) = 0, \end{cases}$$

and $v_0 := v(0) = (A^{-1}w)(0)$. We begin by estimating v_1, claiming that its initial data v_0 satisfies the hypothesis (7.6) of Corollary 7.3, which is

$$|\mathrm{Re}\,\langle v_0\rangle| \geq c|\langle v_0\rangle| - \varepsilon_1\|v_0\|_{L^2} \tag{7.14}$$

(where ε_1 is given in Corollary 7.3). To prove (7.14), we write $v = w + (I - A)v$ to obtain, at time $t = 0$,

$$|\mathrm{Re}\,\langle v_0\rangle| = |\mathrm{Re}\,\langle w_0\rangle + \mathrm{Re}\,\langle (I-A)v_0\rangle| \geq |\mathrm{Re}\,\langle w_0\rangle| - |\langle (I-A)v_0\rangle|.$$

Thus, by the assumption (7.10),

$$|\mathrm{Re}\,\langle v_0\rangle| \geq c|\langle w_0\rangle| - \varepsilon_3\|w_0\|_{L^2} - |\langle (I-A)v_0\rangle|.$$

Since $w = v + (A - I)v$, we have $\langle w_0\rangle = \langle v_0\rangle + \langle (A-I)v_0\rangle$, and

$$|\mathrm{Re}\,\langle v_0\rangle| \geq c|\langle v_0\rangle| - (c+1)|\langle (A-I)v_0\rangle| - \varepsilon_3\|w_0\|_{L^2}.$$

By (7.9), $|\langle (A-I)v_0\rangle| \leq C\varepsilon_2\|v_0\|_{L^2}$ (see Lemma 7.6 below). Also, $\|w_0\|_{L^2} \leq C\|v_0\|_{L^2}$ because A is bounded on L^2 (see Lemma 5.4). Thus

$$|\mathrm{Re}\,\langle v_0\rangle| \geq c|\langle v_0\rangle| - \big((c+1)C\varepsilon_2 + C\varepsilon_3\big)\|v_0\|_{L^2},$$

and the claim is satisfied if $\varepsilon_2, \varepsilon_3$ are small enough. As a consequence, from Corollary 7.3 we deduce that

$$\int_0^T \int_\omega |\mathrm{Re}(Av_1)|^2\,dx\,dt \geq K \int_0^{2\pi} |v_0(x)|^2\,dx. \tag{7.15}$$

On the other hand, it follows from (B.11) (applied with $V = 0$, $c = 1$ and $R = \mathfrak{R}$) that

$$\|v_2\|_{C^0([0,T];L^2)} \leq C\|\mathfrak{R}v_1\|_{L^1([0,T];L^2)}.$$

Since $\|\mathfrak{R}(t)v\|_{L^2} \leq C\varepsilon_2\|v\|_{L^2}$, by using (7.9) we find that the last quantity is bounded by $C\varepsilon_2 T\|v_0\|_{L^2}$. Since A is bounded on L^2, we deduce that

$$\begin{aligned}\int_0^T \int_\omega |\mathrm{Re}(Av_2)|^2\,dx\,dt &\leq \int_0^T \|Av_2(t)\|_{L^2}^2\,dt \leq T \sup_{[0,T]} \|Av_2(t)\|_{L^2}^2 \\ &\leq CT\|v_2\|_{C^0([0,T];L^2)}^2 \leq CT^3\varepsilon_2^2\|v_0\|_{L^2}^2.\end{aligned} \tag{7.16}$$

Using the elementary inequality $(x+y)^2 \geq \frac{1}{2}x^2 - y^2$, for ε_2 small enough we get

$$\int_0^T \int_\omega |\mathrm{Re}(Av)|^2\,dx\,dt \geq \frac{K}{4} \int_0^{2\pi} |v_0(x)|^2\,dx.$$

Since $Av = w$ and $\|w_0\|_{L^2} = \|Av_0\|_{L^2} \leq C\|v_0\|_{L^2}$, we obtain

$$\int_0^T \int_\omega |\mathrm{Re}\,w|^2\,dx\,dt \geq \frac{K}{4} \int_0^{2\pi} |v_0(x)|^2\,dx \geq K' \int_0^{2\pi} |w_0(x)|^2\,dx,$$

which completes the proof. □

Now we prove a technical result used in the proof above.

Lemma 7.6. *Let A be a pseudo-differential operator with symbol $q(x,\xi)e^{i\beta(x)|\xi|^{1/2}}$. There exist universal positive constants δ, C such that if $\|\beta\|_{H^3} + |q-1|_3 \le \delta$, then $|\langle (A-I)u\rangle| \le C\delta\|u\|_{L^2}$ for all $u \in L^2(\mathbb{T})$.*

Proof. As in the proof of Corollary 7.3, we split $A = A_0 + A_1$, with

$$A_0 := \mathrm{Op}\big(\exp(i\beta(x)|\xi|^{1/2})\big), \qquad A_1 := \mathrm{Op}\big((q(x,\xi)-1)\exp(i\beta(x)|\xi|^{1/2})\big).$$

Directly from Lemma 5.4(ii) we have $\|A_1\|_{\mathcal{L}(L^2)} \le C\delta$, whence $|\langle A_1 u\rangle| \le C\delta\|u\|_{L^2}$. To estimate $A_0 - I$, let $u(x) = \sum_{n\in\mathbb{Z}} u_n e^{inx}$, and calculate

$$\int_{\mathbb{T}} (A_0 - I)u\,dx = \sum_{n\neq 0} u_n c_n, \qquad c_n = \int_{\mathbb{T}} e^{i(nx+|n|^{1/2}\beta(x))}\,dx.$$

Integrating by parts gives

$$c_n = \int_{\mathbb{T}} \frac{\partial_x\{e^{i(nx+|n|^{1/2}\beta(x))}\}}{i(n+|n|^{1/2}\partial_x\beta(x))}\,dx = \int_{\mathbb{T}} \frac{-i\partial_{xx}\beta(x)\,e^{i(nx+|n|^{1/2}\beta(x))}}{|n|^{3/2}(1+|n|^{1/2}n^{-1}\partial_x\beta(x))^2}\,dx,$$

so that, for $|\partial_x\beta| \le 1/2$,

$$|c_n| \le C\|\beta\|_{H^2}|n|^{-3/2} \qquad \forall n \in \mathbb{Z}\setminus\{0\}.$$

Thus $(\sum |c_n|^2)^{1/2} \le C\|\beta\|_{H^2}$, and by Hölder's inequality the lemma follows. □

8. Controllability

Consider an operator of the form

$$Q := \partial_t + W\partial_x + iL + R,$$

where W is a real-valued function and R is an operator of order 0. In this section we study the following control problem: given a time $T > 0$, a subset $\omega \subset \mathbb{T}$ and an initial data $w_{\mathrm{in}} \in L^2(\mathbb{T})$, find a (possibly) complex-valued function $f \in C^0([0,T];L^2)$ such that the unique solution $w \in C^0([0,T];L^2)$ of

$$Qw = \chi_\omega \operatorname{Re} f, \qquad w(0) = w_{\mathrm{in}}, \tag{8.1}$$

satisfies $w(T) = 0$. We study this control problem by means of an adaptation of the classical HUM method. We need to adapt the standard argument since we want to prove the existence of a real-valued control, while the unknown is complex-valued. In particular, for this reason, one cannot obtain $w(T) = 0$. We prove instead that, for any real-valued function M such that the L^∞-norm of $M - 1$ is small enough, one can find a control such that $w(T,x) = ibM(x)$ for some constant $b \in \mathbb{R}$. We remark that, given f and w_{in}, the existence of a unique solution w to (8.1) is proved in the appendix (Lemma B.3).

We prove not only a control result but also a *contraction estimate*, which is the main technical result of this section. This means that we estimate the difference of two controls

f and f' associated with different functions W, W' or remainders R, R'. This is the key estimate to prove later that the nonlinear scheme converges (using a Cauchy sequence argument). To prove this contraction estimate we introduce an auxiliary control problem which, loosely speaking, interpolates the two control problems. Since the original nonlinear problem is quasi-linear, a loss of derivative appears. This means that to estimate the $C^0([0,T];L^2)$-norm of $f - f'$ we need to have a bound for the $C^0([0,T];H^1)$-norms of f and f'. That is why we prove and use a regularity property of the control: the control is in $C^0([0,T];H^\mu(\mathbb{T}))$ whenever $w_{\rm in} \in H^\mu(\mathbb{T})$. This is proved by adapting an argument used by Dehman–Lebeau [20] and Laurent [32]. Before stating the result, we recall the definition of the adjoint operator Q^*:

$$Q^* = -\mathcal{Q}, \qquad \mathcal{Q} := \partial_t + W\partial_x + iL + \mathcal{R}, \qquad \mathcal{R} := -R^* + \partial_x W. \tag{8.2}$$

Proposition 8.1. *Consider an open domain $\omega \subset \mathbb{T}$. There exist r and six increasing functions $\mathcal{F}_j : \mathbb{R}_+^* \to \mathbb{R}_+^*$ $(0 \le j \le 5)$, satisfying $\lim_{T\to 0} \mathcal{F}_j(T) = 0$, such that for any $T > 0$ and any real-valued function $M \in H^{3/2}(\mathbb{T})$ with $\|M - 1\|_{H^{3/2}} \le \mathcal{F}_0(T)$, the following results hold.*

(i) (Existence) *Consider $R \in C^0([0,T];\mathcal{L}(L^2))$ and a function W satisfying*

$$\int_{\mathbb{T}} W(t,x)\,dx = 0 \quad \text{for any } t \in [0,T].$$

Assume that the norm

$$\|(W,R)\|_{r,T} := \sum_{1\le k\le 3} \|\partial_t^k W\|_{C^0([0,T];H^1)} + \|W\|_{C^0([0,T];H^r)} + \|R\|_{C^0([0,T];\mathcal{L}(L^2))},$$

satisfies

$$\|(W,R)\|_{r,T} \le \mathcal{F}_1(T). \tag{8.3}$$

Then there exists an operator $\Theta_{M,T} : L^2 \to C^0([0,T];L^2)$ such that for any $w_{\rm in} \in L^2$, setting $f := \Theta_{M,T}(w_{\rm in})$, the unique solution $w \in C^0([0,T];L^2)$ of

$$Qw = \chi_\omega \operatorname{Re} f, \qquad w(0) = w_{\rm in}, \tag{8.4}$$

satisfies

$$w(T,x) = ibM(x) \tag{8.5}$$

for some constant $b \in \mathbb{R}$, and

$$\|f\|_{C^0([0,T];L^2)} \le \|w_{\rm in}\|_{L^2}/\mathcal{F}_2(T). \tag{8.6}$$

(ii) (Uniqueness) *For any $w_{\rm in} \in L^2(\mathbb{T})$ and any $T > 0$, $\Theta_{M,T}(w_{\rm in})$ is the unique function $f \in C^0([0,T];L^2(\mathbb{T}))$ satisfying the following two conditions:*

(1) *$Q^* f = 0$ and $\operatorname{Im} \int_{\mathbb{T}} M(x) f(T,x)\,dx = 0$.*
(2) *The solution w of (8.4) satisfies (8.5) for some constant $b \in \mathbb{R}$.*

(iii) (Regularity) *Let* $\mu \in [0, 3/2]$ *and* $w_{\text{in}} \in H^{\mu}(\mathbb{T})$. *If*

$$\|(W, R)\|_{r,T} + \|R\|_{C^0([0,T];\mathcal{L}(H^{\mu}))} \le \mathcal{F}_1(T), \tag{8.7}$$

then $\Theta_{M,T}(w_{\text{in}}) \in C^0([0, T]; H^{\mu}(\mathbb{T}))$ *and*

$$\|\Theta_{M,T}(w_{\text{in}})\|_{C^0([0,T];H^{\mu})} \le \|w_{\text{in}}\|_{H^{\mu}}/\mathcal{F}_3(T). \tag{8.8}$$

(iv) (Stability) *Suppose* (W, R) *is defined for* $t \in [0, T]$ *and satisfies* (8.7) *with* $\mu = 3/2$, *and* (W', R') *is defined for* $t \in [0, T']$ *and satisfies* (8.7) *with* $\mu = 3/2$ *and* T' *instead of* T. *Denote by* $\Theta_{M,T}$ *and* $\Theta'_{M,T'}$ *the operators associated to these two pairs. Consider the time-rescaling operator* $\mathcal{T}$ *defined by*

$$(\mathcal{T}h)(t) := h(\lambda t), \qquad \lambda := T/T', \tag{8.9}$$

and let $\tilde{W} := \mathcal{T}W$, $\tilde{R} := \mathcal{T}R$, *so* $\tilde{R}(t) = R(\lambda t)$. *Then, given any* $w_{\text{in}} \in L^2(\mathbb{T})$,

$$\begin{aligned}&\|\Theta'_{M,T'}(w_{\text{in}}) - \mathcal{T}\Theta_{M,T}(w_{\text{in}})\|_{C^0([0,T'];L^2)}\\ &\quad\le \frac{\|w_{\text{in}}\|_{H^{3/2}}}{\mathcal{F}_4(T)}\big(|1-\lambda| + \|W' - \tilde{W}\|_{C^0([0,T'];H^2)} + \|R' - \tilde{R}\|_{C^0([0,T'];\mathcal{L}(L^2))}\big).\end{aligned} \tag{8.10}$$

(v) (Dependence on M) *Let* $M, M' \in H^{3/2}(\mathbb{T})$ *with* $\|M - 1\|_{L^\infty} + \|M' - 1\|_{L^\infty} \le \mathcal{F}_0(T)$. *If* $\|(W, R)\|_{r,T} \le \mathcal{F}_1(T)$, *then, for all* $w_{\text{in}} \in H^1(\mathbb{T})$,

$$\|(\Theta_{M,T} - \Theta_{M',T})(w_{\text{in}})\|_{C^0([0,T];L^2)} \le \frac{1}{\mathcal{F}_5(T)}\|M - M'\|_{L^\infty}\|w_{\text{in}}\|_{H^1}. \tag{8.11}$$

In this section we often use the notation $A \lesssim B$ to say that $A \le CB$ for some constant C depending only on T. The key result is the following lemma.

Lemma 8.2. *Introduce the space*

$$L^2_M := \Big\{\varphi \in L^2(\mathbb{T}; \mathbb{C})\,;\ \operatorname{Im}\int_{\mathbb{T}} M(x)\varphi(x)\,dx = 0\Big\}.$$

For any $w_{\text{in}} \in L^2(\mathbb{T})$, *there exists a unique* $f_1 \in L^2_M$ *such that*

$$\forall \phi_1 \in L^2_M, \qquad \operatorname{Re}\int_0^T \big(\chi_\omega \operatorname{Re} f(t), \phi(t)\big)\,dt = -\operatorname{Re}(w_{\text{in}}, \phi(0)),$$

where f *and* ϕ *are the unique functions in* $C^0([0, T]; L^2(\mathbb{T}))$ *satisfying*

$$\begin{cases}\mathcal{Q}f = 0,\\ f(T) = f_1,\end{cases} \qquad \begin{cases}\mathcal{Q}\phi = 0,\\ \phi(T) = \phi_1,\end{cases} \tag{8.12}$$

with $\mathcal{Q}$ *given by* (8.2) (*the existence of* f *and* ϕ *follows from Lemma* B.3). *Set*

$$\Theta_{M,T}(w_{\text{in}}) := f.$$

Moreover, (8.6) *holds.*

Proof. The space L^2_M is an $\mathbb{R}$-vector space. Introduce the $\mathbb{R}$-bilinear symmetric map $a(\cdot,\ \cdot)$ defined by

$$\begin{aligned} a(f_1,\phi_1) &:= \operatorname{Re}\int_0^T\int_{\mathbb{T}} \chi_\omega(x)\operatorname{Re}(f(t,x))\,\overline{\phi(t,x)}\,dx\,dt \\ &= \int_0^T\int_{\mathbb{T}} \chi_\omega(x)\operatorname{Re}(f(t,x))\ \operatorname{Re}\phi(t,x)\,dx\,dt. \end{aligned} \tag{8.13}$$

This map is well-defined and continuous. Indeed, it follows from the L^2-energy estimate (see (B.11)) that

$$\begin{aligned} |a(f_1,\phi_1)| &\le \int_0^T\int_{\mathbb{T}} |f|\,|\phi|\,dx\,dt \\ &\le T\|f\|_{C^0([0,T];L^2)}\|\phi\|_{C^0([0,T];L^2)} \le C(T)\|f_1\|_{L^2}\|\phi_1\|_{L^2}. \end{aligned} \tag{8.14}$$

Since $\chi_\omega(x)=1$ for x in an open subset $\omega_1\subset\omega$, one has

$$a(f_1,f_1)\ge \int_0^T\int_{\omega_1}(\operatorname{Re} f)^2\,dx\,dt.$$

If $f_1\in L^2_M$ then $\operatorname{Im}\int_{\mathbb{T}} Mf_1\,dx=0$ and

$$\left|\operatorname{Im}\int_{\mathbb{T}} f_1(x)\,dx\right| = \left|\operatorname{Im}\int_{\mathbb{T}}(1-M(x))f_1(x)\,dx\right| \le \|M-1\|_{L^\infty}\sqrt{2\pi}\,\|f_1\|_{L^2},$$

from which (using $|\operatorname{Re} z|\ge|z|-|\operatorname{Im} z|$) we deduce that

$$|\operatorname{Re}\langle f_1\rangle| \ge |\langle f_1\rangle| - \|M-1\|_{L^\infty}\sqrt{2\pi}\,\|f_1\|_{L^2}.$$

For $\|M-1\|_{L^\infty}$ small enough, one can apply the observability inequality proved in the previous section (see Corollary 7.4 and Remark 7.5) to conclude that

$$C_1(T)\|f_1\|^2_{L^2}\le a(f_1,f_1). \tag{8.15}$$

On the other hand, (8.14) implies that $a(f_1,f_1)\le C(T)\|f_1\|^2_{L^2}$. Hence $a(\cdot,\ \cdot)$ is a real scalar product on L^2_M which induces the norm $N(f_1)=\sqrt{a(f_1,f_1)}$, equivalent to the norm $\|\cdot\|_{L^2(\mathbb{T};\mathbb{C})}$ on L^2_M. Now, Lemma B.3 implies that the mapping $\phi_1\mapsto\phi(0)$ is $\mathbb{R}$-linear and bounded from L^2_M into L^2, and hence $\phi_1\mapsto\Lambda(\phi_1):=-\operatorname{Re}(w_{\text{in}},\phi(0))$ is a bounded $\mathbb{R}$-linear form on L^2_M. Therefore, the Riesz theorem implies that, for any $\mathbb{R}$-linear form Λ on L^2_M, there is a unique $f_1\in L^2_M$ such that $a(f_1,\phi_1)=\Lambda(\phi_1)$ for all $\phi_1\in L^2_M$, together with

$$\|f_1\|_{L^2}\le\|\Lambda\|/C_1(T). \tag{8.16}$$

Moreover (8.6) follows from (8.16) and the bound $\|f\|_{C^0([0,T];L^2)}\lesssim\|f_1\|_{L^2}$ already used. □

Proof of Proposition 8.1. (i) We begin by proving that if $M\in H^{3/2}(\mathbb{T})$ then $H^{3/2}(\mathbb{T})\cap L^2_M$ is dense in $(L^2_M,\|\cdot\|_{L^2})$. To see this, let Π_N be the Fourier truncation operator defined by $\Pi_N h(x)=\sum_{|j|\le N}h_je^{ijx}$ where $h(x)=\sum_{j\in\mathbb{Z}}h_je^{ijx}$. Given

$u \in L^2_M$, define $u_N := M^{-1}\Pi_N(Mu)$. Since Π_N preserves the mean, one has $u_N \in L^2_M$. Moreover, since $u \in L^2$, one finds that $Mu \in L^2(\mathbb{T})$, $\Pi_N(Mu) \in C^\infty(\mathbb{T})$, and hence $M^{-1}\Pi_N(Mu) \in H^{3/2}(\mathbb{T})$ since $M^{-1} \in H^{3/2}(\mathbb{T})$. Since (u_N) converges to u, this proves that $H^{3/2}(\mathbb{T}) \cap L^2_M$ is dense in L^2_M.

Now let f be as given by the previous lemma. It is proved in the appendix that there is a unique solution w in $C^0([0,T]; L^2(\mathbb{T}))$ of (8.4). Our goal is to prove that $w(T)$ satisfies (8.5). To do so we first check that (8.5) will be proved if $\operatorname{Re}(w(T),\phi_1) = 0$ for all ϕ_1 in L^2_M. Indeed,

$$\operatorname{Re}(w(T),\phi_1) = \int (\operatorname{Re} w(T,x)) \operatorname{Re}\phi_1(x)\, dx + \int (\operatorname{Im} w(T,x)) \operatorname{Im}\phi_1(x)\, dx = 0$$

for all $\phi_1 \in L^2_M$. Therefore $\int (\operatorname{Re} w(T,x)) f(x)\, dx = 0$ for any real-valued function f, and $\int (\operatorname{Im} M(x)^{-1} w(T,x)) g(x)\, dx = 0$ for any real-valued function g with $\int g(x)\, dx = 0$. This implies that (8.5) holds.

We now have to prove that $\operatorname{Re}(w(T),\phi_1) = 0$ for any ϕ_1 in L^2_M. By the density argument proved above, it is enough to assume that $\phi_1 \in L^2_M \cap H^{3/2}(\mathbb{T})$. Given such a ϕ_1, let $\phi \in C^0([0,T]; H^{3/2}(\mathbb{T}))$ be such that

$$\mathcal{Q}\phi = 0, \qquad \phi(T) = \phi_1. \tag{8.17}$$

Since $\mathcal{Q} = -Q^*$, multiplying (8.4) by $\overline{\phi}$ and integrating by parts, we find that

$$(w(T),\phi_1) = (w(0),\phi(0)) + \int_0^T (\chi_\omega \operatorname{Re} f, \phi)\, dt + \int_0^T (w, \mathcal{Q}\phi)\, dt. \tag{8.18}$$

Notice that the integration by parts is justified since $\phi \in C^1([0,T]; L^2(\mathbb{T}))$. By definition of ϕ the last term on the right-hand side vanishes, and by definition of f the real part of the sum of the first and second terms vanishes. This proves that $\operatorname{Re}(w(T),\phi_1) = 0$, which concludes the proof of (i).

(ii) Recall that $\mathcal{Q} = -Q^*$ is given by (8.2). Consider $\phi_1 \in L^2_M$ and denote by ϕ the unique function in $C^0([0,T]; L^2(\mathbb{T}))$ satisfying (8.17). As in (8.18), multiplying the equation $Qw = \chi_\omega \operatorname{Re} f$ by ϕ and integrating by parts one obtains (8.18). Since $\phi_1 \in L^2_M$ and $w(T,x) = ibM(x)$ for some constant $b \in \mathbb{R}$, one has $\operatorname{Re}(w(T),\phi_1) = 0$. Therefore, since $\mathcal{Q}\phi = 0$,

$$\operatorname{Re} \int_0^T (\chi_\omega \operatorname{Re} f, \phi)\, dt = -\operatorname{Re}(w_{\text{in}}, \phi(0)).$$

Since $\mathcal{Q}f = 0$ and $f(T) \in L^2_M$ by assumption, and since the function f_1 whose existence is given by Lemma 8.2 is unique, one deduces that $f(T) = f_1$. Hence $f = \Theta_{M,T}(w_{\text{in}})$ by uniqueness of the solution to the Cauchy problem (8.12).

(iii) We prove (8.8). Recall that $\Theta_{M,T}(w_{\text{in}}) = f$ where f is given by

$$\begin{cases} \mathcal{Q}f = 0, \\ f(T) = f_1, \end{cases}$$

for some function $f_1 \in L^2_M$. Then the unique solution $w \in C^0([0,T]; L^2)$ of

$$Qw = \chi_\omega \operatorname{Re} f, \qquad w(0) = w_{\text{in}},$$

satisfies $w(T,x) = ibM(x)$ for some constant $b \in \mathbb{R}$. In view of the energy estimate (B.3), to prove the desired result it is sufficient to show that $\|f_1\|_{H^\mu}$ is controlled by $\|w_{\text{in}}\|_{H^\mu}$. We only prove an *a priori* estimate, assuming that $f_1 \in H^\mu(\mathbb{T})$. To estimate $\|f_1\|_{H^\mu}$, we adapt to our setting an argument used by Dehman–Lebeau [20, Theorem 4.1] and Laurent [32, Lemma 3.1].

First, given any $u \in L^2(\mathbb{T})$, consider the decomposition $u = \Pi(u) + i\lambda(u)$ where $\lambda(u) := (\operatorname{Im} \int_{\mathbb{T}} M(x)u(x)\,dx)/(\int_{\mathbb{T}} M(x)\,dx)$, which is a real number (recall that $M-1$ is small by assumption, so one can divide by the mean of M, which is a positive number). In this way u is the sum of the function $\Pi(u) = u - i\lambda(u)$, which is in L^2_M, and $i\lambda(u)$, which is a purely imaginary constant.

Consider next the mapping

$$S\colon L^2(\mathbb{T}) \to L^2(\mathbb{T}), \qquad S\colon y \mapsto f \mapsto w \mapsto w(0) \in L^2(\mathbb{T}),$$

where f and w are the unique functions in $C^0([0,T]; L^2(\mathbb{T}))$ successively determined by the backward Cauchy problems with data at time T:

$$\begin{cases} \mathcal{Q}f = 0, \\ f(T) = \Pi(y), \end{cases} \qquad \begin{cases} Qw = \chi_\omega \operatorname{Re} f, \\ w(T) = i\lambda(y)M. \end{cases}$$

Notice that S is $\mathbb{R}$-linear. It follows from (i) and (ii) that S is an isomorphism of $L^2(\mathbb{T})$ onto $L^2(\mathbb{T})$ (it is onto by (i); to prove that it is one-to-one, we use the uniqueness property). On the other hand, S is bounded (this follows from the L^2-estimate (B.13) and the fact that $y \mapsto (\Pi(y), \lambda(y))$ is obviously bounded). The open mapping theorem implies that S^{-1} is bounded. As a result, with $\Lambda^\mu = (I - \partial_x^2)^{\mu/2}$,

$$\|f_1\|_{H^\mu} = \|\Lambda^\mu f_1\|_{L^2} \lesssim \|S\Lambda^\mu f_1\|_{L^2}. \tag{8.19}$$

Now we have to conjugate S with Λ^μ. To do so, we want to compare $(\Lambda^\mu f, \Lambda^\mu w)$ with (f', w') defined by

$$\begin{cases} \mathcal{Q}f' = 0, \\ f'(T) = \Pi(\Lambda^\mu f_1), \end{cases} \qquad \begin{cases} Qw' = \chi_\omega \operatorname{Re} f', \\ w'(T) = i\lambda(\Lambda^\mu f_1)M. \end{cases}$$

We have introduced this system because $w'(0) = S(\Lambda^\mu f_1)$.

Claim 8.3. *We have*

$$\|w' - \Lambda^\mu w\|_{C^0([0,T];L^2)} \lesssim \|w_{\text{in}}\|_{H^\mu} + \|f_1\|_{H^{\max(\mu-1,0)}} + a\|f_1\|_{H^\mu}, \tag{8.20}$$

where

$$a := \|W\|_{C^0([0,T];H^3)} + \|R\|_{C^0([0,T];\mathcal{L}(H^\mu)\cap\mathcal{L}(L^2))}.$$

Granting this claim, we conclude the proof of (iii). We use the following consequence of (8.20): at $t = 0$,

$$\|w'(0) - \Lambda^\mu w(0)\|_{L^2} \lesssim \|w_{\text{in}}\|_{H^\mu} + \|f_1\|_{H^{\max(\mu-1,0)}} + a\|f_1\|_{H^\mu}.$$

Now, by definition, $w'(0) = S\Lambda^\mu f_1$ while $\Lambda^\mu w(0) = \Lambda^\mu w_{\text{in}}$. Therefore, by the triangle inequality,

$$\|S\Lambda^\mu f_1\|_{L^2} \lesssim \|w_{\text{in}}\|_{H^\mu} + \|f_1\|_{H^{\max(\mu-1,0)}} + a\|f_1\|_{H^\mu}. \tag{8.21}$$

For $\mu \in [0, 1]$, one has $\|f_1\|_{H^{\max(\mu-1,0)}} = \|f_1\|_{L^2} \lesssim \|w_{\text{in}}\|_{L^2}$, and therefore

$$\|S\Lambda^\mu f_1\|_{L^2} \lesssim \|w_{\text{in}}\|_{H^\mu} + a\|f_1\|_{H^\mu}. \tag{8.22}$$

Plugging this bound into (8.19) yields $\|f_1\|_{H^\mu} \lesssim \|w_{\text{in}}\|_{H^\mu} + a\|f_1\|_{H^\mu}$. Notice that $a \leq \|(W, R)\|_{r,T} + \|R\|_{C^0([0,T];\mathcal{L}(H^\mu))}$ where $\|(W, R)\|_{r,T}$ is defined above (8.3). So the assumption (8.7) implies that, by taking $\mathcal{F}_1(T)$ small enough, we obtain the desired result

$$\|f_1\|_{H^\mu} \lesssim \|w_{\text{in}}\|_{H^\mu}. \tag{8.23}$$

For $\mu \in (1, 3/2]$ we go back to (8.21) and deduce from (8.23) that $\|f_1\|_{H^{\mu-1}} \lesssim \|w_{\text{in}}\|_{H^{\mu-1}}$ because $\mu - 1 \in [0, 1]$. Hence (8.22) holds, and we reach the same conclusion as above. This completes the proof of (iii).

It remains to prove Claim 8.3. To do so, we first estimate $f' - \Lambda^\mu f$ and then deduce an estimate for $w' - \Lambda^\mu w$. Write

$$\mathcal{Q}(f' - \Lambda^\mu f) = [\Lambda^\mu, \mathcal{R}]f + [\Lambda^\mu, W]\partial_x f, \qquad (f' - \Lambda^\mu f)|_{t=T} = \Pi(\Lambda^\mu f_1) - \Lambda^\mu f_1,$$

and use the energy estimate (B.13) to find that

$$\begin{aligned}&\|f' - \Lambda^\mu f\|_{C^0([0,T];L^2)}\\ &\quad\lesssim \|\Pi(\Lambda^\mu f_1) - \Lambda^\mu f_1\|_{L^2} + \|[\Lambda^\mu, \mathcal{R}]f + [\Lambda^\mu, W]\partial_x f\|_{L^1([0,T];L^2)}.\end{aligned} \tag{8.24}$$

Similarly,

$$\begin{aligned}&\|w' - \Lambda^\mu w\|_{C^0([0,T];L^2)} \lesssim \|w'(T) - \Lambda^\mu w(T)\|_{L^2} + \|\mathcal{F}\|_{L^1([0,T];L^2)} \quad \text{where}\\ &\mathcal{F} := \chi_\omega \operatorname{Re}(f' - \Lambda^\mu f) + [\Lambda^\mu, R]w + [\Lambda^\mu, W]\partial_x w - [\Lambda^\mu, \chi_\omega] \operatorname{Re} f.\end{aligned} \tag{8.25}$$

By (8.24) and the obvious embedding $C^0([0, T]; L^2) \subset L^1([0, T]; L^2)$, we deduce that

$$\begin{aligned}\|w' - \Lambda^\mu w\|_{C^0([0,T];L^2)} \lesssim{}& \|w'(T) - \Lambda^\mu w(T)\|_{L^2} + \|\Pi(\Lambda^\mu f_1) - \Lambda^\mu f_1\|_{L^2}\\ &+ \|[\Lambda^\mu, \chi_\omega] \operatorname{Re} f\|_{C^0([0,T];L^2)}\\ &+ \|[\Lambda^\mu, \mathcal{R}]f\|_{C^0([0,T];L^2)} + \|[\Lambda^\mu, R]w\|_{C^0([0,T];L^2)}\\ &+ \|[\Lambda^\mu, W]\partial_x f\|_{C^0([0,T];L^2)} + \|[\Lambda^\mu, W]\partial_x w\|_{C^0([0,T];L^2)}.\end{aligned}$$

To estimate the commutators $[\Lambda^\mu, \chi_\omega]$ and $[\Lambda^\mu, W]$, we use the classical estimate

$$[s > 3/2,\ 0 \leq \mu \leq s] \ \Rightarrow\ \|[\Lambda^\mu, W]u\|_{L^2} \leq K\|W\|_{H^s}\|u\|_{H^{\mu-1}}.$$

On the other hand, to estimate the commutator $[\Lambda^\mu, \mathcal{R}]$ (or $[\Lambda^\mu, R]$) we estimate $\Lambda^\mu \mathcal{R}$ and $\mathcal{R}\Lambda^\mu$ separately. Recalling that $\mathcal{R} = -R^* + \partial_x W$, we conclude that

$$\begin{aligned}\|w' - \Lambda^\mu w\|_{C^0([0,T];L^2)} &\lesssim \|w'(T) - \Lambda^\mu w(T)\|_{L^2} + \|\Pi(\Lambda^\mu f_1) - \Lambda^\mu f_1\|_{L^2} \\ &\quad + \|f\|_{C^0([0,T];H^{\mu-1})} + a\|(f,w)\|_{C^0([0,T];H^\mu)},\end{aligned}$$

where recall that, by definition,

$$a = \|W\|_{C^0([0,T];H^3)} + \|R\|_{C^0([0,T];\mathcal{L}(H^\mu)\cap\mathcal{L}(L^2))}.$$

To complete the proof of (8.20), it remains to prove the five estimates

$$\|\Pi(\Lambda^\mu f_1) - \Lambda^\mu f_1\|_{L^2} \lesssim \|w_{\mathrm{in}}\|_{H^\mu}, \tag{8.26}$$

$$\|w'(T) - \Lambda^\mu w(T)\|_{L^2} \lesssim \|w_{\mathrm{in}}\|_{H^\mu}, \tag{8.27}$$

$$\|f\|_{C^0([0,T];H^{\mu-1})} \lesssim \|f_1\|_{H^{\max(\mu-1,0)}}, \tag{8.28}$$

$$\|f\|_{C^0([0,T];H^\mu)} \lesssim \|f_1\|_{H^\mu}, \tag{8.29}$$

$$\|w\|_{C^0([0,T];H^\mu)} \lesssim \|w_{\mathrm{in}}\|_{H^\mu} + \|f_1\|_{H^\mu}. \tag{8.30}$$

Let us prove (8.26). By definition of Π, one has $\Pi(\Lambda^\mu f_1) - \Lambda^\mu f_1 = -i\lambda(\Lambda^\mu f_1)$ with

$$\lambda(\Lambda^\mu f_1) = \frac{1}{\int_{\mathbb{T}} M\,dx}\operatorname{Im}\Big(\int_{\mathbb{T}} M\Lambda^\mu f_1\,dx\Big) = \frac{1}{\int_{\mathbb{T}} M\,dx}\operatorname{Im}\Big(\int_{\mathbb{T}} (\Lambda^\mu M) f_1\,dx\Big)$$

since Λ^μ is self-adjoint. This implies that

$$|\lambda(\Lambda^\mu f_1)| \le \Big(\int_{\mathbb{T}} M\,dx\Big)^{-1} \|\Lambda^\mu M\|_{L^2}\|f_1\|_{L^2} \le 2\|f_1\|_{L^2}, \tag{8.31}$$

provided $\|M-1\|_{H^\mu} \le \|M-1\|_{H^{3/2}}$ is small enough. Since $\|f_1\|_{L^2} \lesssim \|w_{\mathrm{in}}\|_{L^2}$, this proves (8.26). As regards (8.27), we will estimate $w'(T)$ and $\Lambda^\mu w(T)$ separately. Firstly, since $w'(T) = i\lambda(\Lambda^\mu f_1)M$ and since $\|M\|_{L^2} \lesssim 1$, (8.31) implies that $\|w'(T)\|_{L^2} \lesssim \|w_{\mathrm{in}}\|_{L^2}$. So to prove (8.27), it is enough to show that $\|\Lambda^\mu w(T)\|_{L^2}$ satisfies the same bound. Since $w(T) = ibM$ and $\|M\|_{H^\mu} \le \|M\|_{H^{3/2}} \lesssim 1$, we have $\|\Lambda^\mu w(T)\|_{L^2} \lesssim |b|$. So, we need only estimate $|b|$. In doing so, we use the fact that w solves

$$Qw = \chi_\omega \operatorname{Re} f, \qquad w(0) = w_{\mathrm{in}}, \tag{8.32}$$

to deduce from the L^2-energy estimate (B.10) that

$$\|w\|_{C^0([0,T];L^2)} \lesssim \|w_{\mathrm{in}}\|_{L^2} + \|f\|_{L^1([0,T];L^2)}.$$

Using the bound (8.6) for f yields $\|w(T)\|_{L^2} \le \|w\|_{C^0([0,T];L^2)} \lesssim \|w_{\mathrm{in}}\|_{L^2}$. Now, since $w(T) = ibM$ and one can assume that $\|M\|_{L^2} \ge 1/2$, this gives $|b| \le \|w_{\mathrm{in}}\|_{L^2}$. Remembering that $\|\Lambda^\mu w(T)\|_{L^2} \lesssim |b|$, we have proved that $\|\Lambda^\mu w(T)\|_{L^2} \lesssim \|w_{\mathrm{in}}\|_{L^2}$, which completes the proof of (8.27). The estimates (8.28) and (8.29) follow from (B.12) (in fact we use an estimate analogous to (B.13) with data at time T). Finally, (8.30) follows from (B.12) applied to (8.32), using the bound (8.29) to estimate the source term.

(iv) Given w_{in}, let f_1 and f_1' in L^2_M be as given by Lemma 8.2, so that $f := \Theta_{M,T}(w_{\mathrm{in}})$ and $f' := \Theta'_{M,T'}(w_{\mathrm{in}})$ are determined by the Cauchy problems

$$\begin{cases} \mathcal{Q}f = 0 \text{ on } [0,T], \\ f(T) = f_1, \end{cases} \qquad \begin{cases} \mathcal{Q}'f' = 0 \text{ on } [0,T'], \\ f'(T') = f_1', \end{cases}$$

where

$$\mathcal{Q}' := \partial_t + iL + W'\partial_x + \mathcal{R}', \qquad \mathcal{R}' := -(R')^* + \partial_x W'.$$

Similarly, we denote $Q' = -(\mathcal{Q}')^* = \partial_t + iL + W'\partial_x + R'$. By definition of f, f', the unique solutions $w \in C^0([0,T]; L^2(\mathbb{T}))$ and $w' \in C^0([0,T']; L^2(\mathbb{T}))$ of the two Cauchy problems

$$\begin{cases} Qw = \chi_\omega \operatorname{Re} f \text{ on } [0,T], \\ w(0) = w_{\mathrm{in}}, \end{cases} \qquad \begin{cases} Q'w' = \chi_\omega \operatorname{Re} f' \text{ on } [0,T'], \\ w'(0) = w_{\mathrm{in}}, \end{cases} \tag{8.33}$$

satisfy $w(T) = ibM$ and $w'(T') = ib'M$ for some $b, b' \in \mathbb{R}$. The idea now is to introduce an auxiliary control problem. Let $f'' \in C^0([0,T]; L^2(\mathbb{T}))$ be the unique solution of

$$\mathcal{Q}f'' = 0 \quad \text{on } [0,T], \qquad f''(T) = f_1', \tag{8.34}$$

so that f'' solves the same equation as f and it has the same Cauchy data as f'. Then introduce w'' as the unique solution to

$$Qw'' = \chi_\omega \operatorname{Re} f'' \quad \text{on } [0,T], \qquad w''(T) = ib'M, \tag{8.35}$$

and set $w''_{\mathrm{in}} := w''(0)$. By uniqueness (see (ii)) we deduce that f'' is the control for the operator Q associated to w''_{in}, that is,

$$f'' = \Theta_{M,T}(w''_{\mathrm{in}}).$$

Then, by continuity (see (i)),

$$\|f - f''\|_{C^0([0,T];L^2)} = \|\Theta_{M,T}(w_{\mathrm{in}} - w''_{\mathrm{in}})\|_{C^0([0,T];L^2)} \lesssim \|w_{\mathrm{in}} - w''_{\mathrm{in}}\|_{L^2}.$$

Let $\tilde{f} := \mathcal{T}f$ and $\tilde{f}'' := \mathcal{T}f''$. Then

$$\|\tilde{f} - \tilde{f}''\|_{C^0([0,T'];L^2)} = \|f - f''\|_{C^0([0,T];L^2)} \tag{8.36}$$

because

$$\forall h \in C^0([0,T]; L^2), \qquad \|\mathcal{T}h\|_{C^0([0,T'];L^2)} = \|h\|_{C^0([0,T];L^2)}. \tag{8.37}$$

It remains to estimate $\|f' - \tilde{f}''\|_{C^0([0,T'];L^2)}$ and $\|w_{\mathrm{in}} - w''_{\mathrm{in}}\|_{L^2}$. Since f'' solves (8.34), $\tilde{f}''$ satisfies

$$\tilde{\mathcal{Q}}\tilde{f}'' = 0 \quad \text{on } [0,T'], \qquad \tilde{f}''(T') = f_1',$$

where

$$\tilde{\mathcal{Q}} := \partial_t + i\lambda L + \lambda\tilde{W}\partial_x + \lambda\tilde{\mathcal{R}},$$

and $\tilde{W} := \mathcal{T}W$, $\tilde{\mathcal{R}} := \mathcal{T}\mathcal{R}$ (so $\tilde{\mathcal{R}}(t) := \mathcal{R}(\lambda t)$). Subtracting yields

$$\tilde{\mathcal{Q}}(f' - \tilde{f}'') = F_0, \quad (f' - \tilde{f}'')(T') = 0,$$

where

$$F_0 := (\lambda - 1)(iL + \tilde{W}\partial_x + \tilde{\mathcal{R}})f' + (\tilde{W} - W')\partial_x f' + (\tilde{\mathcal{R}} - \mathcal{R}')f'.$$

In order to apply the L^2-energy bound (B.13), we estimate F_0. Using the regularity property of the control operator $\Theta_{M,T}$ (see (iii)) we have

$$\|Lf'\|_{C^0([0,T'];L^2)} \lesssim \|f'\|_{C^0([0,T'];H^{3/2})} \lesssim \|w_{\text{in}}\|_{H^{3/2}}, \tag{8.38}$$

$$\begin{aligned}\|(\tilde{W} - W')\partial_x f'\|_{C^0([0,T'];L^2)} &\lesssim \|\tilde{W} - W'\|_{C^0([0,T'];H^1)}\|f'\|_{C^0([0,T'];H^1)} \\ &\lesssim \|\tilde{W} - W'\|_{C^0([0,T'];H^1)}\|w_{\text{in}}\|_{H^1}.\end{aligned} \tag{8.39}$$

Similarly

$$\begin{aligned}\|(\tilde{\mathcal{R}} - \mathcal{R}')f'\|_{C^0([0,T'];L^2)} &\lesssim \|\tilde{\mathcal{R}} - \mathcal{R}'\|_{C^0([0,T'];\mathcal{L}(L^2))}\|f'\|_{C^0([0,T'];L^2)} \\ &\lesssim (\|\tilde{R} - R'\|_{C^0([0,T'];\mathcal{L}(L^2))} + \|\tilde{W} - W'\|_{C^0([0,T'];H^2)})\|w_{\text{in}}\|_{L^2}\end{aligned}$$

where $\tilde{R} := \mathcal{T}R$ (so $\tilde{R}(t) = R(\lambda t)$). Using (B.13) we conclude that

$$\begin{aligned}&\|f' - \tilde{f}''\|_{C^0([0,T'];L^2)} \\ &\quad \lesssim \|w_{\text{in}}\|_{H^{3/2}}(|\lambda - 1| + \|\tilde{W} - W'\|_{C^0([0,T'];H^2)} + \|\tilde{R} - R'\|_{C^0([0,T'];\mathcal{L}(L^2))}).\end{aligned} \tag{8.40}$$

It remains to estimate $\|w_{\text{in}} - w''_{\text{in}}\|_{L^2}$. Let $\tilde{w}'' := \mathcal{T}w''$. At $t = 0$ one has $w'(0) - \tilde{w}''(0) = w_{\text{in}} - w''_{\text{in}}$, hence we study the difference $w' - \tilde{w}''$. Since w'' solves (8.35), $\tilde{w}''$ satisfies

$$\tilde{Q}\tilde{w}'' = \lambda\chi_\omega \operatorname{Re} \tilde{f}'' \quad \text{on } [0, T'], \qquad \tilde{w}''(T') = ib'M, \tag{8.41}$$

where

$$\tilde{Q} := \partial_t + i\lambda L + \lambda\tilde{W}\partial_x + \lambda\tilde{R}.$$

Subtracting yields

$$\tilde{Q}(w' - \tilde{w}'') = F, \quad (w' - \tilde{w}'')(T') = 0,$$

where

$$F := \chi_\omega \operatorname{Re}(f' - \lambda\tilde{f}'') + (\lambda - 1)(iL + \tilde{W}\partial_x + \tilde{R})w' + (\tilde{W} - W')\partial_x w' + (\tilde{R} - R')w'.$$

To apply the L^2-energy bound (B.13), we estimate F. First, $f' - \lambda\tilde{f}'' = \lambda(f' - \tilde{f}'') + (1-\lambda)f'$, and we have already estimated both $f' - \tilde{f}''$ (see (8.40)) and f'. For the other terms in F we proceed as above, recalling that $\|w'\|_{C^0([0,T'];L^2)} \lesssim \|w_{\text{in}}\|_{L^2}$. Also, since w' solves the Cauchy problem (8.33), we deduce from (B.12) and the second inequality in (8.38) that $\|w'\|_{C^0([0,T'];H^{3/2})} \lesssim \|w_{\text{in}}\|_{H^{3/2}}$. As a consequence, also $\|F\|_{C^0([0,T'];L^2)}$ is bounded by the right-hand side of (8.40). Then, applying (B.13), we deduce that

$\|w' - \tilde{w}''\|_{C^0([0,T'];L^2)}$ satisfies the same bound. In particular, at time $t = 0$, this yields the desired bound for $w_{\text{in}} - w''_{\text{in}} = (w' - \tilde{w}'')(0)$.

(v) We begin by introducing some notation. As already mentioned, Lemma B.3 yields an operator $E_T : L^2(\mathbb{T}) \to C^0([0,T]; L^2(\mathbb{T}))$ such that $v = E_T(v_1)$ is the unique solution to the Cauchy problem $\mathcal{Q}v = 0$ with $v(T) = v_1$. Moreover

$$\|E_T(v_1)\|_{C^0([0,T];L^2(\mathbb{T}))} \lesssim \|v_1\|_{L^2}. \tag{8.42}$$

Now recall that by definition

$$a_T(f_1, \phi_1) := \operatorname{Re} \int_0^T \int_{\mathbb{T}} \chi_\omega(x) \operatorname{Re}(E_T(f_1)) \overline{E_T(\phi_1)}\, dx\, dt. \tag{8.43}$$

Also introduce the mapping $\Lambda : L^2(\mathbb{T}) \to \mathbb{R}$ defined by $\Lambda(v_1) = -\operatorname{Re}(w_{\text{in}}, E_T(v_1)(0))$ where $E_T(v_1)(0) = E_T(v_1)|_{t=0}$. It follows from Lemma 8.2 that there exist functions $f_1 \in L^2_M$ and $f_1' \in L^2_{M'}$ such that

$$\forall \phi_1 \in L^2_M, \quad a_T(f_1, \phi_1) = \Lambda(\phi_1), \qquad \forall \phi_1 \in L^2_{M'}, \quad a_T(f_1', \phi_1) = \Lambda(\phi_1).$$

Then $\Theta_{M,T}(w_{\text{in}}) - \Theta_{M',T}(w_{\text{in}}) = E_T(f_1 - f_1')$. In view of (8.42), to prove (v) it is sufficient to estimate $f_1 - f_1'$. To do so, we need to compare elements in L^2_M and those in $L^2_{M'}$. Observe that, by definition of L^2_M, if $\varphi \in L^2_{M'}$ then $(M'/M)\varphi \in L^2_M$. Therefore $\varphi_1 := f_1 - (M'/M) f_1' \in L^2_M$ and we can use (8.15) to deduce that

$$\left\| f_1 - \frac{M'}{M} f_1' \right\|_{L^2}^2 \lesssim a_T\left(f_1 - \frac{M'}{M} f_1', f_1 - \frac{M'}{M} f_1' \right) = a_T\left(f_1 - \frac{M'}{M} f_1', \varphi_1 \right). \tag{8.44}$$

Now write the last term as $(I) + (II) + (III)$, where

$$(I) = a_T(f_1, \varphi_1) - a_T\left(f_1', \frac{M}{M'} \varphi_1 \right),$$
$$(II) = a_T\left(f_1', \frac{M}{M'} \varphi_1 \right) - a_T(f_1', \varphi_1),$$
$$(III) = a_T(f_1', \varphi_1) - a_T\left(\frac{M'}{M} f_1', \varphi_1 \right).$$

(Notice that both M/M' and M'/M appear.) Since $(M/M')\varphi_1 \in L^2_{M'}$, we can write $a_T(f_1', (M/M')\varphi_1) = \Lambda((M/M')\varphi_1)$ to deduce that

$$(I) = \Lambda(\varphi_1) - \Lambda\left(\frac{M}{M'} \varphi_1 \right) = \Lambda\left(\frac{M' - M}{M'} \varphi_1 \right),$$

so that $|(I)| \lesssim \|M' - M\|_{L^\infty} \|w_{\text{in}}\|_{L^2} \|\varphi_1\|_{L^2}$. On the other hand, it follows from the easy estimates (8.14) and (8.42) that

$$|(II)| + |(III)| \lesssim \|M - M'\|_{L^\infty} \|f_1'\|_{L^2} \|\varphi_1\|_{L^2} \lesssim \|M - M'\|_{L^\infty} \|w_{\text{in}}\|_{L^2} \|\varphi_1\|_{L^2}.$$

By combining (8.44) with the previous estimates we conclude that

$$\left\| f_1 - \frac{M'}{M} f_1' \right\|_{L^2} \lesssim \|M - M'\|_{L^\infty} \|w_{\text{in}}\|_{L^2}.$$

Now write

$$\|f_1 - f_1'\|_{L^2} \lesssim \|M - M'\|_{L^\infty}\|f_1'\|_{L^2} + \left\| f_1 - \frac{M}{M'} f_1' \right\|_{L^2}$$

and $\|f_1'\|_{L^2} \lesssim \|w_{\rm in}\|_{L^2}$ to complete the proof of (v). □

9. Controllability for the paradifferential equation

From the results of the previous sections we now deduce that the original equation of Section 3 is controllable, together with Sobolev estimates for the control.

Consider a paradifferential operator of the form

$$P = \partial_t + T_V\partial_x + iL^{1/2}(T_c L^{1/2}\,\cdot) + R, \tag{9.1}$$

where R is an operator of order 0. Assume that P satisfies Assumption 3.1, so that as above V and c are real-valued, $c-1$ is small enough and P has the following structural property:

$$Pu \text{ real-valued} \quad\Rightarrow\quad \frac{d}{dt}\int_{\mathbb{T}} \operatorname{Im} u(t,x)\,dx = 0. \tag{9.2}$$

Introduce the norm

$$\begin{aligned}\|(c-1,V,R)\|_{X^{s_0,s}(T)} &:= \|(c-1,\partial_t c,V)\|_{C^0([0,T];H^{s_0})} + \sum_{k=2,3,4}\|\partial_t^k c\|_{C^0([0,T];H^1)} \\ &+ \sum_{k=1,2,3}\|\partial_t^k V\|_{C^0([0,T];H^1)} + \|R\|_{C^0([0,T];\mathcal{L}(H^s))} + \|R\|_{C^0([0,T];\mathcal{L}(H^{s+3/2}))}.\end{aligned} \tag{9.3}$$

We recall that $p := c^{-1/3} + \frac{5}{18i}\frac{\chi(\xi)\partial_\xi \ell(\xi)}{\ell(\xi)}c^{-4/3}\partial_x c$ (see (2.12)).

Proposition 9.1. *Consider an open domain $\omega \subset \mathbb{T}$. There exists s_0 large enough and for any $s \geq s_0$ there exist increasing functions $\mathcal{F}_j\colon \mathbb{R}_+^* \to \mathbb{R}_+^*$ $(1 \leq j \leq 3)$, with $\lim_{T\to 0}\mathcal{F}_j(T) = 0$, such that, for any $T \in (0,1]$, the following holds.*

(i) *If*

$$\|(c-1,V,R)\|_{X^{s_0,s}(T)} \leq \mathcal{F}_1(T), \tag{9.4}$$

then there exists a bounded operator

$$\Theta_{s,T}[(V,c,R)]\colon H^{s+3/2}(\mathbb{T}) \to C^0([0,T];H^{s+3/2}(\mathbb{T}))$$

such that, for any $v_{\rm in} \in H^{s+3/2}(\mathbb{T})$ satisfying

$$\operatorname{Im}\int_{\mathbb{T}} v_{\rm in}(x)\,dx = 0,$$

setting $f := \Theta_{s,T}[(V,c,R)](v_{\rm in})$ one has

$$\|f\|_{C^0([0,T];H^{s+3/2})} \leq \|v_{\rm in}\|_{H^{s+3/2}}/\mathcal{F}_2(T), \tag{9.5}$$

and the unique solution v to $Pv = T_p\chi_\omega \operatorname{Re} f$, $v|_{t=0} = v_{\rm in}$, satisfies

$$v(T) = 0.$$

(ii) *Assume that the triple* (c, V, R) *satisfies* (9.4) *and*

$$\|\partial_t^2 c\|_{C^0([0,T];H^{s_0})} + \|\partial_t V\|_{C^0([0,T];H^{s_0})} + \|\partial_t R\|_{C^0([0,T];\mathcal{L}(H^s))} \le 1. \tag{9.6}$$

Let (c', V', R') *be another triple also satisfying the same (corresponding) bounds* (9.4) *and* (9.6). *Then*

$$\begin{aligned}&\|\Theta_{s,T}[(V,c,R)](v_{\rm in}) - \Theta_{s,T}[(V',c',R')](v_{\rm in})\|_{C^0([0,T];H^s)}\\ &\le \frac{\|v_{\rm in}\|_{H^{s+3/2}}}{\mathcal{F}_3(T)}\{\|(c-c', \partial_t(c-c'), V-V')\|_{C^0([0,T];H^{s_0})} + \|R-R'\|_{C^0([0,T];\mathcal{L}(H^s))}\}.\end{aligned} \tag{9.7}$$

Proof. Let P be given by (9.1), with V, c, R satisfying (9.4). We begin by recalling how the various linear operators have been defined in the previous sections starting from P:

$$\begin{aligned}\widetilde{P} &:= \Lambda_{h,s} P \Lambda_{h,s}^{-1} = \partial_t + T_V\partial_x + iL^{1/2}(T_c L^{1/2}\cdot) + R_1 = \partial_t + V\partial_x + iL^{1/2}(cL^{1/2}\cdot) + R_2,\\ \widetilde{P}_3 &:= \Phi m^{-1}\widetilde{P}\Phi^{-1} = \partial_t + W\partial_x + iL + R_3,\\ \mathcal{P} &:= -(\widetilde{P}_3)^* = \partial_t + W\partial_x + iL + R_4,\end{aligned}$$

where Φ, m, W are given in Proposition 5.1,

$$\begin{aligned}R_1 &:= \Lambda_{h,s} R \Lambda_{h,s}^{-1} + [\Lambda_{h,s}, \partial_t]\Lambda_{h,s}^{-1} + [\Lambda_{h,s}, T_V\partial_x]\Lambda_{h,s}^{-1} + i[\Lambda_{h,s}, L^{1/2}(T_c L^{1/2}\cdot)]\Lambda_{h,s}^{-1},\\ R_2 u &:= R_1 u + T_V\partial_x u - V\partial_x u + i(L^{1/2}T_c L^{1/2}u - L^{1/2}(cL^{1/2}u)),\\ R_4 w &:= -R_3^* w + (\partial_x W)w,\end{aligned} \tag{9.8}$$

and R_3 has a more involved expression, obtained in Appendix C. Moreover $\widetilde{P} = m\Phi^{-1}\widetilde{P}_3\Phi$. As a first step in the proof of Proposition 9.1, we study the control problem for $\widetilde{P}$.

Lemma 9.2. *There exist* s_0 *large enough and increasing functions* $\mathcal{F}_j : \mathbb{R}_+^* \to \mathbb{R}_+^*$ $(j = 1, 2, 3)$, *satisfying* $\lim_{T\to 0}\mathcal{F}_j(T) = 0$, *such that for any* $T > 0$ *the following holds.*

(i) *If*

$$\begin{aligned}&\|(c-1, \partial_t c, V)\|_{C^0([0,T];H^{s_0})} + \sum_{k=2,3,4}\|\partial_t^k c\|_{C^0([0,T];H^1)}\\ &\qquad + \sum_{k=1,2,3}\|\partial_t^k V\|_{C^0([0,T];H^1)} + \|R_2\|_{C^0([0,T];\mathcal{L}(L^2))} \le \mathcal{F}_1(T),\end{aligned} \tag{9.9}$$

then there exists an operator $\widetilde{\Theta}_T : L^2 \to C^0([0,T];L^2)$ *such that for any* $u_{\rm in} \in L^2$, *setting* $f := \widetilde{\Theta}_T(u_{\rm in})$, *one has*

$$\|f\|_{C^0([0,T];L^2)} \le \|u_{\rm in}\|_{L^2}/\mathcal{F}_2(T), \tag{9.10}$$

and the unique solution u *of*

$$\widetilde{P}u = \chi_\omega \operatorname{Re} f, \qquad u(0) = u_{\rm in},$$

satisfies $u(T, x) = ib$ *for some* $b \in \mathbb{R}$ *and all* $x \in \mathbb{T}$. *If, in addition,*

$$\|R_2\|_{C^0([0,T];\mathcal{L}(H^{3/2}))} \le \mathcal{F}_1(T), \tag{9.11}$$

then

$$\|f\|_{C^0([0,T];H^{3/2})} \le \|u_{\rm in}\|_{H^{3/2}}/\mathcal{F}_2(T). \tag{9.12}$$

(ii) *Assume that* (V, c, R_2) *satisfies* (9.9), (9.11) *and*

$$\|\partial_t V\|_{C^0([0,T];H^2)} + \|R_2\|_{C^0([0,T];\mathcal{L}(H^1))} + \|\partial_t R_2\|_{C^0([0,T];\mathcal{L}(L^2))} \le 1, \tag{9.13}$$

and consider another triple (V', c', R_2') *also satisfying the same (corresponding) bounds* (9.9), (9.11) *and* (9.13). *Then*

$$\begin{aligned}\|(\widetilde{\Theta}_T - \widetilde{\Theta}'_T)(u_{\rm in})\|_{C^0([0,T];L^2)} &\le \frac{\|u_{\rm in}\|_{H^{3/2}}}{\mathcal{F}_3(T)}\{\|c-c'\|_{C^0([0,T];H^{r+1})} + \|\partial_t c - \partial_t c'\|_{C^0([0,T];H^1)} \\ &\quad + \|V - V'\|_{C^0([0,T];H^2)} + \|R_2 - R_2'\|_{C^0([0,T];\mathcal{L}(L^2))}\}. \end{aligned} \tag{9.14}$$

Proof. Recall that the cut-off function $\chi_\omega(x)$ is supported on ω and $\chi_\omega = 1$ on the open interval $\omega_1 \subset \omega$. Consider another open interval ω_2 and a cut-off function $\chi_2(x)$ such that

$$\begin{cases} \text{(i)} \ \operatorname{supp}(\chi_2) \subseteq \omega_2; \\ \text{(ii)} \ \operatorname{supp}(h) \subseteq \omega_2 \Rightarrow \operatorname{supp}(\Phi^{-1}h) \subseteq \omega_1 \quad \forall t \in [0,T], \ \forall h \in L^2(\mathbb{T}). \end{cases} \tag{9.15}$$

We want to apply Proposition 8.1 for $Q = \widetilde{P}_3$. The hypothesis (8.3) of Proposition 8.1, i.e. $\|(W, R_3)\|_{r,T} \le \mathcal{F}_1(T)$, follows from (9.9), by using (5.3) and (C.43) with $\sigma = 3/2$. Hence, by Proposition 8.1(i) (applied with T_1 instead of T and χ_2 instead of χ_ω), given $w_{\rm in} \in L^2(\mathbb{T})$, the unique solution w of the Cauchy problem

$$\widetilde{P}_3 w = \chi_2 \operatorname{Re} f_2 \quad \forall t \in [0, T_1], \qquad w(0) = w_{\rm in}, \tag{9.16}$$

satisfies $w(T_1) = ibM$ for some real constant b if we choose $f_2 = \Theta_{M,T_1}(w_{\rm in})$, where Θ_{M,T_1} is the operator given by Proposition 8.1, and the function M will be fixed below in this proof (with $M - 1$ small enough so that the assumption $\|M-1\|_{H^{3/2}} \le \mathcal{F}_0(T)$ in Proposition 8.1 will be satisfied). Also, by (8.6),

$$\|f_2\|_{C^0([0,T_1];L^2)} \le \|w_{\rm in}\|_{L^2}/\mathcal{F}_2(T_1). \tag{9.17}$$

Moreover, if (9.11) also holds, then, using (C.42) with $\sigma = 3/2$, we deduce the bound (8.7) for W, R_3 with $\mu = 3/2$. Therefore, by Proposition 8.1(iii),

$$\|f_2\|_{C^0([0,T_1];H^{3/2})} \le \|w_{\rm in}\|_{H^{3/2}}/\mathcal{F}_3(T_1). \tag{9.18}$$

Now let $u_{\rm in} \in L^2(\mathbb{T})$ and define $w_{\rm in} \in L^2(\mathbb{T})$ by $w_{\rm in} := \Phi|_{t=0} u_{\rm in}$. The previous argument gives a function w satisfying (9.16) and $w(T_1) = ibM$. Set $u := \Phi^{-1} w$. Since $\widetilde{P} u = m\Phi^{-1}\widetilde{P}_3 \Phi u$, it follows from (9.16) that

$$\widetilde{P} u = m\Phi^{-1}(\chi_2 \operatorname{Re} f_2) \quad \forall t \in [0,T], \qquad u(0) = u_{\rm in}, \tag{9.19}$$

and $u(T) = \Phi^{-1}|_{t=T}(ibM)$. Then we set $f := m\Phi^{-1}(\chi_2 f_2)$, so

$$f = \widetilde{\Theta}_T(u_{\text{in}}) := m\Phi^{-1}(\chi_2 \Theta_{M,T_1}(\Phi u_{\text{in}})), \tag{9.20}$$

where $\Phi u_{\text{in}} = \Phi|_{t=0} u_{\text{in}}$. By (9.15)(ii), f is supported in ω_1, and therefore $f = \chi_\omega f$. Then, since $m\Phi^{-1}(\chi_2 \operatorname{Re} f_2) = \operatorname{Re}(m\Phi^{-1}(\chi_2 f_2)) = \operatorname{Re} f$,

$$\widetilde{P}u = \chi_\omega \operatorname{Re} f, \qquad u(0) = u_{\text{in}},$$

and we have to choose M so that $u(T) = ib$. By definition of Φ, recall that $w = \Phi u$ means that

$$w(t,x) = \{1 + \partial_x \tilde{\beta}_1(\psi^{-1}(t), x - p(t))\}^{1/2} u\big(\psi^{-1}(t), x - p(t) + \tilde{\beta}_1(\psi^{-1}(t), x - p(t))\big)$$

for $t \in [0, T_1]$ and $x \in \mathbb{T}$. Since $\psi^{-1}(T_1) = T$, we see that $u(T) = ib$ provided that $w(T_1, x) = ibM(x)$ with

$$M(x) = \{1 + \partial_x \tilde{\beta}_1(T, x - p(T_1))\}^{1/2}, \tag{9.21}$$

and $p(T_1)$ is given in (C.41). Now the estimates (9.10) and (9.12) follow from (9.17), (9.18) and Proposition 5.1. This completes the proof of (i).

(ii) In what follows, we add $'$ to denote objects associated to (V', c', R_2'). Let $f = \widetilde{\Theta}_T(u_{\text{in}})$ be defined by (9.20), and let $f' = \widetilde{\Theta}'_T(u_{\text{in}})$ be the corresponding function obtained by taking (V', c', R_2') instead of (V, c, R_2). We have to estimate the difference $f - f'$. If the constant $\mathcal{F}_1(T)$ in (9.9) is sufficiently small, then ω_2, χ_2 can be chosen so that (9.15) holds for both Φ and Φ'. Hence

$$f - f' = m\Phi^{-1}(\chi_2 \Theta_{M,T_1}(\Phi u_{\text{in}})) - m'\Phi'^{-1}(\chi_2 \Theta'_{M',T_1'}(\Phi' u_{\text{in}})).$$

We split this difference into $A_1 + \cdots + A_6$, where

$$\begin{aligned}
A_1 &:= (m - m')\Phi^{-1}(\chi_2 \Theta_{M,T_1}(\Phi u_{\text{in}})),\\
A_2 &:= m'\Phi^{-1}[\chi_2 \Theta_{M,T_1}(\Phi u_{\text{in}} - \Phi' u_{\text{in}})],\\
A_3 &:= m'\Phi^{-1}[\chi_2(\Theta_{M,T_1} - \Theta_{M',T_1})(\Phi' u_{\text{in}})],\\
A_4 &:= m'(\Psi_1^{-1} - \Psi_1'^{-1})\psi_* \varphi_* [\chi_2 \Theta_{M',T_1}(\Phi' u_{\text{in}})],\\
A_5 &:= m'\Psi_1'^{-1}(\psi_* \varphi_* - \psi'_* \varphi'_* \mathcal{T})[\chi_2 \Theta_{M',T_1}(\Phi' u_{\text{in}})],\\
A_6 &:= m'\Phi'^{-1}[\chi_2\{\mathcal{T}\Theta_{M',T_1}(\Phi' u_{\text{in}}) - \Theta'_{M',T_1'}(\Phi' u_{\text{in}})\}],
\end{aligned}$$

and $\mathcal{T}$ is the time-rescaling operator defined above, $(\mathcal{T}h)(t,x) := h(\lambda t, x)$ with $\lambda := T_1/T_1'$. Let us estimate each A_i.

Estimate for A_1: Apply (C.47). *Estimate for A_2*: By construction (see Appendix C), $\psi^{-1}(0) = 0$, $p(0) = 0$, and therefore $\Phi u_{\text{in}} = \Phi|_{t=0} u_{\text{in}} = \Psi_1^{-1}|_{t=0}(u_{\text{in}})$. Hence the estimate for A_2 follows from (C.44) and (8.6). *Estimate for A_3*: Apply (C.48). *Estimate for A_4*: Apply (C.44) and (8.8) with $\mu = 1$. *Estimate for A_5*: Apply (C.45). To estimate $\partial_t f_2$, use the fact that f_2 solves $\widetilde{P}_3^* f_2 = 0$ (Proposition 8.1(ii)), and similarly for f_2'.

Estimate for A_6: (9.9) and (9.11) imply that W, R_3 and W', R_3' satisfy (8.7) with $\mu = 3/2$, which is the hypothesis of Proposition 8.1(iv). Then (8.10) holds:

$$\|\mathcal{T}\Theta_{M',T_1}(\Phi' u_{\text{in}}) - \Theta'_{M',T_1'}(\Phi' u_{\text{in}})\|_{C^0([0,T_1'];L^2)}$$
$$\lesssim \|\Phi' u_{\text{in}}\|_{H^{3/2}}(|1-\lambda| + \|W' - \mathcal{T}W\|_{C^0([0,T_1'];H^2)} + \|R_3' - \mathcal{T}R_3\|_{C^0([0,T_1'];\mathcal{L}(L^2))}).$$

Now $\|\Phi' u_{\text{in}}\|_{H^{3/2}} \lesssim \|u_{\text{in}}\|_{H^{3/2}}$, and the bounds for the last three differences are given in (C.47), (C.50) (with $\sigma = 2$) and (C.52). Note that assumptions (9.9), (9.11) and (9.13) imply (C.49), (C.51), which imply (C.50) and (C.52). □

Remark 9.3. The function W contains the terms $\partial_t c$ and V (see Appendix C and the bound (5.2)). For this reason we assume that $\partial_t^4 c$ and $\partial_t^3 V$ are bounded in (9.9) in order to get a bound for $\partial_t^3 W$, as required by Proposition 8.1.

Lemma 9.4. *If the* $W^{3/2,\infty}$*-norms of* $c-1$ *and* $c'-1$ *are small enough, then*

$$\|\Lambda_{h,s} - \Lambda'_{h,s}\|_{\mathcal{L}(H^s,L^2)} + \|\Lambda_{h,s}^{-1} - (\Lambda'_{h,s})^{-1}\|_{\mathcal{L}(L^2,H^s)} \lesssim \|c - c'\|_{H^1}.$$

Proof. By definition (4.1) of $\Lambda_{h,s}$ one has

$$\Lambda_{h,s} - \Lambda'_{h,s} = h^s T_{c^{2s/3} - c'^{2s/3}} L^{2s/3}.$$

So the bound for $\Lambda_{h,s} - \Lambda'_{h,s}$ follows from the paradifferential rule (A.10) and the Sobolev embedding $H^1(\mathbb{T}) \subset L^\infty(\mathbb{T})$. To prove the other bound, we use the identity (4.3) to obtain

$$\Lambda_{h,s}^{-1} - (\Lambda'_{h,s})^{-1} = (I + h^s L^{2s/3})^{-1}[(I+B)^{-1} - (I+B')^{-1}].$$

Recall that $\|B\|_{\mathcal{L}(L^2)} \le 1/2$ and $\|B'\|_{\mathcal{L}(L^2)} \le 1/2$, so the identity

$$(I+B)^{-1} - (I+B')^{-1} = (I+B)^{-1}(B'-B)(I+B')^{-1}$$

implies that

$$\|(I+B)^{-1} - (I+B')^{-1}\|_{\mathcal{L}(L^2)} \le 4\|B - B'\|_{\mathcal{L}(L^2)}, \tag{9.22}$$

and the bound follows from the definition of B, B' and (A.10) as above. □

End of the proof of Proposition 9.1. We recall that $\widetilde{\Theta}_T$ is the control operator given by Lemma 9.2, and the operator $\mathcal{K}$ is introduced in (4.15), with $\|(I+\mathcal{K})^{-1}\|_{\mathcal{L}(L^2)} \le 2$ if (c, V, R_2) satisfy (9.9), (9.11) and $\|c-1\|_{C^0([0,T];H^2)}$ is small enough. Set

$$\Theta_{s,T}[(V,c,R)] := \Lambda_{h,s}^{-1}\widetilde{\Theta}_T(I+\mathcal{K})^{-1}\Lambda_{h,s}, \tag{9.23}$$

and let $f := \Theta_{s,T}[(V,c,R)](v_{\text{in}})$. Then it follows from the previous construction (see Section 4, in particular *Proof of Proposition 3.2 given Proposition 4.6*) that the unique solution v to $Pv = T_p\chi_\omega f$, $v|_{t=0} = v_{\text{in}}$, satisfies $v(T) = ib$ for some constant $b \in \mathbb{R}$. Since $\operatorname{Im}\int_{\mathbb{T}} v_{\text{in}}(x)\,dx = 0$ by assumption, from (9.2) we deduce that

$$\operatorname{Im}\int_{\mathbb{T}} v(T,x)\,dx = 0.$$

Therefore $b = 0$ and $v(T) = 0$. Thus it remains to prove (9.5). Following the argument used to prove (4.16), one shows that $\|\mathcal{K}\|_{\mathcal{L}(H^{3/2})} \leq 1/2$, whence $\|(I+\mathcal{K})^{-1}\|_{\mathcal{L}(H^{3/2})} \leq 2$. By combining this estimate with (9.12), we have

$$\begin{aligned}\|f\|_{C^0([0,T];H^{s+3/2})} &\lesssim \|\widetilde{\Theta}_T(I+\mathcal{K})^{-1}\Lambda_{h,s}v_{\text{in}}\|_{C^0([0,T];H^{3/2})} \lesssim \|\Lambda_{h,s}v_{\text{in}}\|_{H^{3/2}} \\ &\lesssim \|v_{\text{in}}\|_{H^{s+3/2}},\end{aligned}$$

which is (9.5). Finally, we observe that

$$\begin{aligned}&\|R_2\|_{C^0([0,T];\mathcal{L}(L^2))} + \|R_2\|_{C^0([0,T];\mathcal{L}(H^{3/2}))} \\ &\quad\lesssim \|(c-1, \partial_t c, V)\|_{C^0([0,T];H^{s_0})} + \|R\|_{C^0([0,T];\mathcal{L}(H^s))} + \|R\|_{C^0([0,T];\mathcal{L}(H^{s+3/2}))}.\end{aligned} \tag{9.24}$$

This bound follows easily from the arguments used in the proofs of Lemmas 4.5 and 9.4. Hence, if (c, V, R) satisfies (9.4), then (c, V, R_2) satisfies (9.9), (9.11). This completes the proof of (i).

(ii) Given $y \in H^{3/2}(\mathbb{T})$, we estimate $\Theta_{s,T}[(V, c, R)](v_{\text{in}}) - \Theta_{s,T}[(V', c', R')](v_{\text{in}})$, which is, by definition,

$$\Lambda_{h,s}^{-1}\widetilde{\Theta}_T(I+\mathcal{K})^{-1}\Lambda_{h,s}v_{\text{in}} - (\Lambda'_{h,s})^{-1}\widetilde{\Theta}'_T(I+\mathcal{K}')^{-1}\Lambda'_{h,s}v_{\text{in}}.$$

We write it as $B_1 + \cdots + B_4$ with

$$\begin{aligned}B_1 &:= \{\Lambda_{h,s}^{-1} - (\Lambda'_{h,s})^{-1}\}\widetilde{\Theta}_T(I+\mathcal{K})^{-1}\Lambda_{h,s}v_{\text{in}}, \\ B_2 &:= (\Lambda'_{h,s})^{-1}(\widetilde{\Theta}_T - \widetilde{\Theta}'_T)(I+\mathcal{K})^{-1}\Lambda_{h,s}v_{\text{in}}, \\ B_3 &:= (\Lambda'_{h,s})^{-1}\widetilde{\Theta}'_T\{(I+\mathcal{K})^{-1} - (I+\mathcal{K}')^{-1}\}\Lambda_{h,s}v_{\text{in}}, \\ B_4 &:= (\Lambda'_{h,s})^{-1}\widetilde{\Theta}'_T(I+\mathcal{K}')^{-1}(\Lambda_{h,s} - \Lambda'_{h,s})v_{\text{in}}.\end{aligned}$$

If (c, V, R) satisfies (9.4), then (c, V, R_2) satisfies (9.9) and (9.11), and $\|\mathcal{K}\|_{\mathcal{L}(L^2)} \leq 1/2$ (see (4.16)). Then, using Lemma 9.4 and (9.10), we bound the $C^0([0,T]; H^s)$-norm of B_1 and B_4 by $\|c-c'\|_{H^1}\|v_{\text{in}}\|_{H^s}$. To estimate B_2, we want to use (9.14), which holds provided that (c, V, R_2) and (c', V', R'_2) satisfy (9.13). One proves that if $\|c-1\|_{C^0([0,T];H^3)}$ is small enough, then

$$\begin{aligned}&\|R_2 - R'_2\|_{C^0([0,T];\mathcal{L}(L^2))} \\ &\quad\lesssim \|(c-c', \partial_t(c-c'), V-V')\|_{C^0([0,T];H^{s_0})} + \|R-R'\|_{C^0([0,T];\mathcal{L}(H^s))},\end{aligned} \tag{9.25}$$

$$\begin{aligned}&\|\partial_t R_2\|_{C^0([0,T];\mathcal{L}(L^2))} \\ &\quad\lesssim \|(c-1, \partial_t c, \partial_t^2 c, V, \partial_t V)\|_{C^0([0,T];H^{s_0})} + \|(R, \partial_t R)\|_{C^0([0,T];\mathcal{L}(H^s))}.\end{aligned} \tag{9.26}$$

These bounds follow from the arguments used in the proofs of Lemmas 4.5 and 9.4. Hence (9.3) and (9.6) imply (9.13), which implies (9.14). We have $\|(I+\mathcal{K})^{-1}\Lambda_{h,s}v_{\text{in}}\|_{H^{3/2}} \leq 2\|\Lambda_{h,s}v_{\text{in}}\|_{H^{3/2}} \lesssim \|v_{\text{in}}\|_{H^{s+3/2}}$. Using (9.25) to estimate the last term in (9.14), we deduce that

$$\begin{aligned}&\|B_2\|_{C^0([0,T];H^s)} \\ &\quad\lesssim \|v_{\text{in}}\|_{H^{s+3/2}}\{\|(c-c', \partial_t(c-c'), V-V')\|_{C^0([0,T];H^{s_0})} + \|R-R'\|_{C^0([0,T];\mathcal{L}(H^s))}\}.\end{aligned}$$

It remains to estimate B_3. We have

$$\|(\mathcal{K}-\mathcal{K}')y\|_{L^2} \lesssim \|y\|_{H^{3/2}}\{\|(c-c', V-V')\|_{C^0([0,T];H^1)} + \|R_2-R_2'\|_{C^0([0,T];\mathcal{L}(L^2))}\}.$$

To see this, recall that $\mathcal{K}$ is defined by solving an evolution equation, and then, as above, use the energy estimates proved in the appendix to bound the difference of two solutions satisfying two equations. Since $\|(I+\mathcal{K})^{-1}\|_{\mathcal{L}(L^2)} \le 2$, $\|(I+\mathcal{K}')^{-1}\|_{\mathcal{L}(H^{3/2})} \le 2$, and

$$(I+\mathcal{K})^{-1} - (I+\mathcal{K}')^{-1} = (I+\mathcal{K})^{-1}(\mathcal{K}'-\mathcal{K})(I+\mathcal{K}')^{-1},$$

we deduce that B_3 satisfies the same bound as B_2. The proof of Proposition 9.1 is complete. □

10. Iterative scheme

In this section we conclude the proof of Theorem 1.1. It is sufficient to prove this result with $(\eta_{\text{final}}, \psi_{\text{final}}) = (0,0)$. Indeed, since the equation is reversible in time, one can exchange initial and final states, and hence it is sufficient to consider the case where the final state vanishes. Also, as explained in the introduction, we seek P_{ext} as the real part of the limit of solutions to approximate control problems with variable coefficients.

Consider the unknown $u = T_p\omega - iT_q\eta$ as introduced in Proposition 2.5. As proved in §2.3 (see also Section 3), this new unknown u solves an equation of the form

$$\partial_t u + T_{V(u)}\partial_x u + iL^{1/2}(T_{c(u)}L^{1/2}u)u + R(u)u = T_{p(u)}P_{\text{ext}}, \tag{10.1}$$

where, with a little abuse of notation, we write $V(u)$, $c(u)$, ... as shorthand for $V(\eta)\psi$ (see (2.3)), $c = (1+(\partial_x\eta)^2)^{-3/4}$, ... where (η,ψ) is expressed in terms of u by means of Lemma 2.8.

Fix $T > 0$. We claim that there is $\varepsilon > 0$ such that, for all initial data whose $H^s(\mathbb{T})$-norm (with s large enough) is smaller than ε, and all source term P_{ext} whose $L^1([0,T]; H^s(\mathbb{T}))$-norm is smaller than ε, the Cauchy problem for (10.1) has a unique solution in $C^0([0,T]; H^s(\mathbb{T}))$. The existence of a solution follows from the analysis given below. The uniqueness is obtained by estimating the difference of two solutions (as in [2]) and we omit its proof.

Recall that $\tilde{H}^\mu(\mathbb{T};\mathbb{C})$ denotes the space of H^μ-functions whose imaginary part has zero mean (see Notation 2.7).

Proposition 10.1. *Let $T > 0$. For all $u_{\text{in}} \in \tilde{H}^\sigma(\mathbb{T};\mathbb{C})$ for some σ large enough such that $\|u_{\text{in}}\|_{H^\sigma}$ is small enough, there exists a real-valued function*

$$P_{\text{ext}} \in C^0([0,T]; H^\sigma(\mathbb{T})) \quad \textit{with} \quad \operatorname{supp} P_{\text{ext}}(t,\cdot) \subset \omega \textit{ for all } t \in [0,T],$$

such that the unique solution $u \in C^0([0,T]; H^\sigma(\mathbb{T}))$ to (10.1) with initial data u_{in} satisfies $u(T) = 0$.

Before proving this proposition, let us explain how to deduce Theorem 1.1 from it. Recall that it is sufficient to consider the case where $(\eta_{\text{final}}, \psi_{\text{final}}) = (0, 0)$. Once P_{ext} is defined by means of Proposition 10.1 applied with $u_{\text{in}} = T_{p_{\text{in}}}\omega_{\text{in}} - iT_{q_{\text{in}}}\eta_{\text{in}}$, we solve the water waves system (2.1) for (η, ψ) with data $(\eta_{\text{in}}, \psi_{\text{in}})$ with this pressure seen as a source term. Then $u = T_p\omega - iT_q\eta$ solves (10.1), so $u(T) = 0$, which in turn implies that $(\eta, \psi)(T) = 0$ in view of Lemma 2.8.

Proof of Proposition 10.1. Set $s = \sigma - 3/2$. Given $u_{\text{in}} \in \tilde{H}^{s+3/2}(\mathbb{T}; \mathbb{C})$ and $T > 0$, introduce the following scheme: define $(u_0, f_0) := (0, 0)$, and then, for $n \geq 0$, (u_{n+1}, f_{n+1}) are defined by induction in this way: f_{n+1} is determined by asking that the unique solution u_{n+1} to the Cauchy problem

$$\begin{aligned} &\partial_t u_{n+1} + T_{V(u_n)}\partial_x u_{n+1} + iL^{1/2}T_{c(u_n)}L^{1/2}u_{n+1} + R(u_n)u_{n+1} = T_{p(u_n)}\chi_\omega \operatorname{Re} f_{n+1}, \\ &u_{n+1}|_{t=0} = u_{\text{in}}, \end{aligned} \tag{10.2}$$

satisfies $u_{n+1}(T) = 0$.

Our goal is to prove that this scheme converges. Then we define P_{ext} as the limit of $\operatorname{Re} f_n$ when $n \to \infty$. With the operator $\Theta_{s,T}$ defined in Proposition 9.1, the scheme corresponds to defining (u_n) and (f_n) as follows:

$$f_{n+1} := \Theta_{s,T}[X_n](u_{\text{in}}) \quad \text{where} \quad X_n := (V(u_n), c(u_n), R(u_n)), \tag{10.3}$$

and u_{n+1} is defined as the unique solution to the Cauchy problem (10.2); by definition of f_{n+1} we then have $u_{n+1}(T) = 0$. Our goal is to prove that, for any $T > 0$, if u_{in} is small enough, then this scheme is well-defined and (u_n, f_n) converges to a solution (u, f) of the desired nonlinear control problem. This will be a consequence of the following result.

Lemma 10.2. *Consider $T > 0$. There exists s_0 large enough and for any $s \geq s_0 + 6$ there exist $\varepsilon_0 > 0$ and positive constants $K_1, \ldots, K_7$ such that, for any $\varepsilon \in (0, \varepsilon_0]$, if*

$$\|u_{\text{in}}\|_{H^{s+3/2}(\mathbb{T})} \leq \varepsilon$$

then, for any $n \geq 0$,

$$\|u_n\|_{C^0([0,T];H^{s+3/2})} \leq K_1\varepsilon, \tag{10.4}$$

$$\|\partial_t^k u_n\|_{C^0([0,T];H^{s_0})} \leq K_2\varepsilon \quad \textit{for } 1 \leq k \leq 4. \tag{10.5}$$

Moreover, for any $n \geq 0$,

$$\|u_{n+1} - u_n\|_{C^0([0,T];H^s)} \leq K_3\varepsilon 2^{-n}, \tag{10.6}$$

$$\|\partial_t(u_{n+1} - u_n)\|_{C^0([0,T];H^{s-3/2})} \leq K_4\varepsilon 2^{-n}; \tag{10.7}$$

and for any $n \geq 1$,

$$\|f_n\|_{C^0([0,T];H^{s+3/2})} \leq K_5\varepsilon, \tag{10.8}$$

$$\|\partial_t^k f_n\|_{C^0([0,T];H^{s_0})} \leq K_6\varepsilon \quad \textit{for } 1 \leq k \leq 3, \tag{10.9}$$

$$\|f_{n+1} - f_n\|_{C^0([0,T];H^s)} \leq K_7\varepsilon^2 2^{-n}. \tag{10.10}$$

Proof. For this proof we denote by C various constants depending only on T, s, s_0 or ω. Also we denote by $\mathcal{F}$ various increasing functions $\mathcal{F}\colon \mathbb{R}_+ \to \mathbb{R}_+$ depending on parameters that are considered fixed.

Step 1: proof of (10.4), (10.5), (10.8) and (10.9). We prove these estimates by induction. They hold for $n = 0$ since $(u_0, f_0) = (0, 0)$. We now assume that they hold at rank n and prove that they hold at rank $n + 1$.

We begin by checking that the fact that the properties (10.4)–(10.5) hold at rank n implies that one can apply Proposition 9.1 to prove that the scheme is well-defined. This means that we have to prove that the smallness assumption (9.4) is satisfied. To do so, we first recall that (see (2.4)) $\|V(u_n)\|_{H^{s_0}} \le \mathcal{F}(\|\eta_n\|_{H^{s_0+1}})\|\psi_n\|_{H^{s_0+1}}$. Then the estimate (2.22) (applied with s replaced by $s_0 + 1$) implies that $\|V(u_n)\|_{H^{s_0}} \lesssim \|u_n\|_{H^{s_0+1}}$. Similarly, the estimates (2.14) and (2.22) yield

$$\|R(u_n)\|_{\mathcal{L}(H^{s+3/2})} \le \mathcal{F}(\|\eta_n\|_{H^{s+3/2}})\|\eta_n\|_{H^{s+3/2}} \le \mathcal{F}(\|u_n\|_{H^{s+3/2}})\|u_n\|_{H^{s+3/2}},$$

and, directly from the definition $c = (1 + (\partial_x\eta)^2)^{-3/4}$, one has

$$\|c(u_n) - 1\|_{H^{s_0}} \le \mathcal{F}(\|\eta_n\|_{H^{s_0+1}})\|\eta_n\|_{H^{s_0+1}} \lesssim \|u_n\|_{H^{s_0+1/2}}.$$

Gathering these estimates and recalling that $s_0 + 1 \le s$, we conclude that

$$\|V(u_n)\|_{H^{s_0}} + \|c(u_n) - 1\|_{H^{s_0}} + \|R(u_n)\|_{\mathcal{L}(H^{s+3/2})} \lesssim \|u_n\|_{H^{s+3/2}}. \tag{10.11}$$

Consequently, the property (10.4) at rank n implies that the part of the smallness condition (9.4) concerning V, c, R is satisfied. Concerning the estimates of the time derivatives $\partial_t^k V$ and $\partial_t^k c$, we use the equations (2.1) and the rule (see [30])

$$\partial_t G(\eta)\psi = G(\eta)\{\partial_t\psi - (B(\eta)\psi)\partial_t\eta\} - \partial_x\big((V(\eta)\psi)\partial_t\eta\big)$$

(where $B(\eta)\psi$ and $V(\eta)\psi$ are given by (2.3)) to express the time derivatives $\partial_t^k V$ and $\partial_t^k c$ in terms of spatial derivatives and of the operators $B(\eta)$, $V(\eta)$ (see [4, Appendix A.3] or [31, 38]). Then, as above, the desired estimates follow from (2.4) and the usual nonlinear estimates in Sobolev spaces.

We now prove (10.4) and (10.8) at rank $n + 1$. By (B.2) we obtain

$$\|u_{n+1}\|_{C^0([0,T];H^{s+3/2})} \le C(\|u_{\text{in}}\|_{H^{s+3/2}} + \|T_{p_n}\chi_\omega \operatorname{Re} f_{n+1}\|_{C^0([0,T];H^{s+3/2})}), \tag{10.12}$$

where the constant C depends on s, T (by (10.11) and (10.4) at rank n, the constant M in Proposition B.1 is bounded by 1 if K_1 is large enough and ε_0 is small enough). Now observe that since T_{p_n} acts on any Sobolev space with operator norm bounded by $M_0^0(p_n) \le \mathcal{F}(\|u_n\|_{H^s}) \le \mathcal{F}(1)$, one has

$$\|T_{p_n}\chi_\omega \operatorname{Re} f_{n+1}\|_{C^0([0,T];H^{s+3/2})} \le C\|f_{n+1}\|_{C^0([0,T];H^{s+3/2})}.$$

Moreover, by (9.5), $\|f_{n+1}\|_{C^0([0,T];H^{s+3/2})} \le K_0\|u_{\text{in}}\|_{H^{s+3/2}}$ for some K_0 depending only on T. We conclude that if we choose K_1 large enough and ε_0 small enough, then (10.4) holds at rank $n + 1$. Also (10.8) at rank $n + 1$ follows by the same argument.

It remains to prove (10.5) and (10.9). Directly from (10.2), expressing $\partial_t u_{n+1}$ in terms of u_n, u_{n+1} and f_{n+1} and using the operator norm estimate (A.10) for paradifferential operators, one deduces (10.5) for $k = 1$ from the bounds (10.4) and (10.8). We next prove (10.9) for $k = 1$. To do so, the key point is to make explicit the equation satisfied by f_{n+1}. We recall from (10.3), (9.23) and (9.20) that

$$f_{n+1} := (\Lambda^n_{h,s})^{-1}\big(m^n(\Phi^n)^{-1}(\chi_2 \tilde{f}_{n+1})\big), \qquad \tilde{f}_{n+1} := \Theta_{M_n,T_1^n}\big(\Phi^n(I+\mathcal{K}_n)^{-1}\Lambda^n_{h,s}u_{\mathrm{in}}\big),$$

where $\Lambda^n_{h,s}, \Phi^n, m^n, M_n, \mathcal{K}_n, T_1^n$ are given by replacing (V, c) with $(V(u_n), c(u_n))$ in the definition of $\Lambda_{h,s}, \Phi, m, M, \mathcal{K}, T_1$. By definition of $\Theta_{M,T}$ (Lemma 8.2) one has

$$\begin{cases} \partial_t \tilde{f}_{n+1} + W(u_n)\partial_x \tilde{f}_{n+1} + iL\tilde{f}_{n+1} + R_4(u_n)\tilde{f}_{n+1} = 0, \\ \tilde{f}_{n+1}|_{t=T_1^n} = \tilde{f}^1_{n+1}, \end{cases} \tag{10.13}$$

where $R_4(u_n)$ is given by (9.8) and the initial data $\tilde{f}^1_{n+1}$ is given by Lemma 8.2. It follows from (8.8) that

$$\|\tilde{f}_{n+1}\|_{C^0([0,T];H^{3/2})} \le K\|\Phi^n(I+\mathcal{K}_n)^{-1}\Lambda^n_{h,s}u_{\mathrm{in}}\|_{H^{3/2}} \le K\|u_{\mathrm{in}}\|_{H^{s+3/2}}.$$

Using (10.13) we thus estimate the $C^0([0,T];L^2)$-norm of $\partial_t \tilde{f}_{n+1}$, from which we estimate $\partial_t f_{n+1}$ in $C^0([0,T];H^s)$. This gives (10.9) for $k = 1$ since $s \ge s_0$. Now we obtain (10.5) for $k = 2, 3, 4$ as well as (10.9) for $k = 2, 3$ by differentiating in time the equations satisfied by u_{n+1} and f_{n+1}.

Step 2: proof of (10.6), (10.7), (10.10). The estimate (10.10) will be deduced from (10.6) and (10.7). To prove (10.6) and (10.7) we proceed by induction. We assume that they hold at rank $n-1$ and prove that they hold at rank n.

The key point is to estimate $\delta_n := u_{n+1} - u_n$. Write

$$\partial_t \delta_n + T_{V(u_n)}\partial_x \delta_n + iL^{1/2}T_{c(u_n)}L^{1/2}\delta_n + R(u_n)\delta_n = G_n \tag{10.14}$$

with

$$\begin{aligned} G_n := {}& (T_{V(u_{n-1})} - T_{V(u_n)})\partial_x u_n + iL^{1/2}(T_{c(u_{n-1})} - T_{c(u_n)})L^{1/2}u_n + (R(u_{n-1}) - R(u_n))u_n \\ & + T_{p(u_n)}\chi_\omega(f_{n+1} - f_n) + (T_{p(u_n)} - T_{p(u_{n-1})})\chi_\omega f_n. \end{aligned} \tag{10.15}$$

As in the previous step, it follows from Proposition B.1 (noticing that $\delta_{n+1}(0) = 0$) that $\|\delta_n\|_{C^0([0,T];H^s)} \le C_0\|G_n\|_{C^0([0,T];H^s)}$ for some C_0 depending on s, T.

Estimate for G_n. We claim that

$$\|G_n\|_{C^0([0,T];H^s)} \le \varepsilon K(T)\|\delta_{n-1}\|_{C^0([0,T];H^s)} + \varepsilon K(T)\|\partial_t \delta_{n-1}\|_{C^0([0,T];H^{s-3/2})}. \tag{10.16}$$

Let us prove this claim. At each $t \in [0,T]$, using (A.10) one has

$$\|(T_{V(u_{n-1})} - T_{V(u_n)})\partial_x u_n\|_{H^s} \lesssim \|V(u_{n-1}) - V(u_n)\|_{L^\infty}\|\partial_x u_n\|_{H^s}.$$

It follows from (10.4) that $\|\partial_x u_n\|_{H^s} \le K_1\varepsilon$. To estimate $V(u_{n-1}) - V(u_n)$ we use the following consequence of [3, Lemma 5.3]: Assume $s > 3/2$ and consider (η_1, η_2) such that $\|\eta_1\|_{H^s} + \|\eta_2\|_{H^s} \le 1$. Then

$$\|G(\eta_1)f_1 - G(\eta_2)f_2\|_{H^{s-3/2}} \le K\|\eta_1 - \eta_2\|_{H^{s-1/2}}\|f_1\|_{H^s} + K\|f_1 - f_2\|_{H^{s-1/2}}.$$

Then, directly from the definition of $V(\eta)\psi$ one deduces that

$$\|V(\eta_1)\psi_1 - V(\eta_2)\psi_2\|_{H^1} \le K\|\eta_1 - \eta_2\|_{H^2}\|\psi_1\|_{H^{5/2}} + K\|\psi_1 - \psi_2\|_{H^2}.$$

Since $H^1(\mathbb{T}) \subset L^\infty(\mathbb{T})$, we then conclude that

$$\|V(u_{n-1}) - V(u_n)\|_{L^\infty} \lesssim \|\eta_n - \eta_{n-1}\|_{H^s} + \|\psi_n - \psi_{n-1}\|_{H^s} \lesssim \|u_n - u_{n-1}\|_{H^s}.$$

The estimate of the H^s-norm of $L^{1/2}(T_{c(u_{n-1})} - T_{c(u_n)})L^{1/2}u_n$ is similar. To estimate $(R(u_{n-1}) - R(u_n))u_n$ recall that $R(\underline{u})u$ is given by Proposition 2.5. This operator is defined by means of the remainder $F(\eta)\psi$ in (2.7) and also in terms of explicit expressions involving symbolic calculus or the paralinearization of products. The only delicate point is to estimate $F(\eta_n)\psi_n - F(\eta_{n-1})\psi_{n-1}$. To do so one uses [2, Lemma 6.8].

It remains to estimate the last two terms on the right-hand side of (10.15). Directly from (A.4) we find that

$$\|(T_{p(u_n)} - T_{p(u_{n-1})})\chi_\omega f_n\|_{H^s} \lesssim M_0^0(p(u_n) - p(u_{n-1}))\|\chi_\omega f_n\|_{H^s}.$$

Now $\|\chi_\omega f_n\|_{H^s} \lesssim \|\chi_\omega\|_{H^s}\|f_n\|_{H^s} \lesssim \varepsilon$ by (10.8), and $M_0^0(p(u_n) - p(u_{n-1}))$ is bounded by $K\|u_n - u_{n-1}\|_{H^s}$. Eventually, to estimate the H^s-norm of $T_{p(u_n)}\chi_\omega(f_{n+1} - f_n)$ we use again (A.4) to bound this expression in terms of $\|f_{n+1} - f_n\|_{H^s}$. We use (9.7) to obtain

$$\begin{aligned}\|f_{n+1} - f_n\|_{C^0([0,T];H^s)} &\lesssim \|u_{\mathrm{in}}\|_{H^{s+3/2}}\{\|(c_n - c_{n-1}, \partial_t(c_n - c_{n-1}), V_n - V_{n-1})\|_{C^0([0,T];H^{s_0})} \\ &\qquad + \|R_n - R_{n-1}\|_{C^0([0,T];\mathcal{L}(H^s))}\} \\ &\lesssim \|u_{\mathrm{in}}\|_{H^{s+3/2}}\{\|u_n - u_{n-1}\|_{H^{s_0}} + \|\partial_t(u_n - u_{n-1})\|_{H^{s_0}}\}, \qquad (10.17)\end{aligned}$$

and then we use (10.6) and (10.7) at rank $n-1$.

Estimate for $u_{n+1} - u_n$. For $\varepsilon_0 K(T)C_0 \le 1/2$, it follows from (10.6) and (10.7) at rank $n-1$ and (10.16) that the desired result (10.6) at rank n holds.

Estimate for $f_{n+1} - f_n$. The estimate (10.10) follows from (10.17) and the assumptions (10.6)–(10.7) at rank $n-1$.

Estimate for $\partial_t(u_{n+1} - u_n)$. By (10.14),

$$\partial_t\delta_n = -T_{V(u_n)}\partial_x\delta_n - iL^{1/2}T_{c(u_n)}L^{1/2}\delta_n - R(u_n)\delta_n + G_n. \qquad (10.18)$$

As above,

$$\|T_{V(u_n)}\|_{\mathcal{L}(H^s, H^{s-1})} + \|L^{1/2}T_{c(u_n)}L^{1/2}\|_{\mathcal{L}(H^s, H^{s-3/2})} + \|R(u_n)\|_{\mathcal{L}(H^s, H^s)} \le C\|u_n\|_{H^s}.$$

Therefore one can use (10.6) and (10.4) to estimate the first three terms on the right-hand side of (10.18). The last term G_n is estimated by means of (10.16) and the induction assumptions. Consequently, $\|\partial_t \delta_n\|_{C^0([0,T];H^{s-3/2})} \leq C\varepsilon^2 2^{-n}$, and for $\varepsilon \leq \varepsilon_0$ with ε_0 small enough, we deduce (10.7). □

We can now conclude the proof of Proposition 10.1. Recall that $s = \sigma - 3/2$. By (10.6) and (10.10), $(u_n)_{n\in\mathbb{N}}$, and $(f_n)_{n\in\mathbb{N}}$ are Cauchy sequences in $C^0([0,T];H^s)$ and therefore converge to some limits u and f in $C^0([0,T];H^s)$. Using the uniform bounds (10.4) and (10.8) and the interpolation inequality in Sobolev spaces, we infer that $(u_n)_{n\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$ converge in $C^0([0,T];H^{s'+3/2})$ for all $s' < s$. Furthermore, u and f belong to $C^0([0,T];H^{s'+3/2}) \cap L^\infty([0,T];H^{s+3/2})$ for all $s' < s$. Passing to the limit in (10.2), we conclude that u and f satisfy (10.1) and $u(T) = 0$. Eventually, using Lemma B.1 (seeing (10.1) as a linear equation of the type (B.1) with unknown u and coefficients in $L^\infty([0,T];H^s)$), we deduce $u \in C^0([0,T];H^{s+3/2})$.

It remains to prove that $f \in C^0([0,T];H^{s+3/2})$. We know that $u_n \to u$ in $C^0([0,T];H^s) \subset C^0([0,T];H^{s_0+6})$. As a consequence, $V(u_n) \to V(u)$, $c(u_n) \to c(u)$, $\partial_t c(u_n) \to \partial_t c(u)$, $p(u_n) \to p(u)$ in $C^0([0,T];H^{s_0})$, and $R(u_n) \to R(u)$ in $C^0([0,T];\mathcal{L}(H^s))$. Now consider $f_\infty := \Theta_{s,T}[V(u), c(u), R(u)](u_{\text{in}})$, and recall the definition (10.3). By (9.7), $\|f_n - f_\infty\|_{C^0([0,T];H^s)} \to 0$ as $n \to \infty$. On the other hand, $f = \lim f_n$ in $C^0([0,T];H^s)$, and therefore $f = f_\infty$. By Proposition 9.1(i), $f_\infty \in C^0([0,T];H^{s+3/2})$, with estimate (9.5).

This concludes the proof of Proposition 10.1 and hence the proof of Theorem 1.1. □

Appendix A. Paradifferential operators

Notation A.1. For $\rho \in \mathbb{N}$, we denote by $W^{\rho,\infty}(\mathbb{T})$ the Sobolev space of L^∞ functions whose derivatives of order ρ are in L^∞. For $\rho \in (0,\infty) \setminus \mathbb{N}$, we denote by $W^{\rho,\infty}(\mathbb{T})$ the space of functions in $W^{[\rho],\infty}(\mathbb{T})$ whose derivatives of order $[\rho]$ are uniformly Hölder continuous with exponent $\rho - [\rho]$.

Definition A.2. Given $\rho \geq 0$ and $m \in \mathbb{R}$, Γ^m_ρ denotes the space of functions $a(x,\xi)$ on $\mathbb{T} \times \mathbb{R}$ which are C^∞ with respect to ξ and such that, for all $\alpha \in \mathbb{N}$ and all ξ, the function $x \mapsto \partial_\xi^\alpha a(x,\xi)$ belongs to $W^{\rho,\infty}(\mathbb{T})$ and

$$\|\partial_\xi^\alpha a(\cdot,\xi)\|_{W^{\rho,\infty}(\mathbb{T})} \leq C_\alpha (1+|\xi|)^{m-|\alpha|}.$$

Definition A.3. For $m \in \mathbb{R}$, $\rho \in [0,1]$ and $a \in \Gamma^m_\rho(\mathbb{R}^d)$, we set

$$M^m_\rho(a) = \sup_{|\alpha|\leq 6+\rho} \sup_{\xi\in\mathbb{R}} \|(1+|\xi|)^{|\alpha|-m} \partial_\xi^\alpha a(\cdot,\xi)\|_{W^{\rho,\infty}(\mathbb{T})}. \tag{A.1}$$

Now consider a C^∞ function χ homogeneous of degree 0 and satisfying, for $0 < \varepsilon_1 < \varepsilon_2$ small enough,

$$\chi(\theta,\eta) = 1 \quad \text{if } |\theta| \leq \varepsilon_1|\eta|, \qquad \chi(\theta,\eta) = 0 \quad \text{if } |\theta| \geq \varepsilon_2|\eta|.$$

Given a symbol a, we define the paradifferential operator T_a by

$$\widehat{T_a u}(\xi) = (2\pi)^{-1} \sum_{\eta \in \mathbb{Z}} \chi(\xi - \eta, \eta) \widehat{a}(\xi - \eta, \eta) \widehat{u}(\eta), \tag{A.2}$$

where $\widehat{a}(\theta, \xi) = \int e^{-ix \cdot \theta} a(x, \xi)\, dx$ is the Fourier transform of a with respect to the first variable.

In addition, we assume that χ satisfies the following symmetry conditions:

$$\chi(\xi_1, \xi_2) = \chi(-\xi_1, -\xi_2) = \chi(-\xi_1, \xi_2). \tag{A.3}$$

It follows from (A.3) that if a and u are real-valued functions, so is $T_a u$.

The main features of symbolic calculus for paradifferential operators are given by the following theorem (see the original article by Bony [13] and the books by Taylor [46] and Métivier [39]).

Definition A.4. Let $m \in \mathbb{R}$. An operator T is said of *order* m if, for any $\mu \in \mathbb{R}$, it is bounded from $H^\mu(\mathbb{T})$ to $H^{\mu - m}(\mathbb{T})$.

Theorem A.5. *Let* $m \in \mathbb{R}$.

(i) *If* $a \in \Gamma^m_0$, *then* T_a *is of order* m. *Moreover, for any* $\mu \in \mathbb{R}$ *there exists* $K > 0$ *such that*

$$\| T_a \|_{\mathcal{L}(H^\mu, H^{\mu - m})} \le K M^m_0(a). \tag{A.4}$$

(ii) *Let* $(m, m') \in \mathbb{R}^2$ *and* $\rho \in (0, \infty)$. *If* $a \in \Gamma^m_\rho$ *and* $b \in \Gamma^{m'}_\rho$, *then* $T_a T_b - T_{a\sharp b}$ *is of order* $m + m' - \rho$, *where*

$$a \sharp b = \sum_{|\alpha| < \rho} \frac{1}{i^{|\alpha|} \alpha!} \partial^\alpha_\xi a \partial^\alpha_x b. \tag{A.5}$$

Furthermore, for any $\mu \in \mathbb{R}$ *there exists* $K > 0$ *such that*

$$\| T_a T_b - T_{a\sharp b} \|_{\mathcal{L}(H^\mu, H^{\mu - m - m' + \rho})} \le K M^m_\rho(a) M^{m'}_\rho(b). \tag{A.6}$$

In particular, if $\rho \in (0, 1]$, $a \in \Gamma^m_\rho$, $b \in \Gamma^{m'}_\rho$ *then*

$$\| T_a T_b - T_{ab} \|_{\mathcal{L}(H^\mu, H^{\mu - m - m' + \rho})} \le K M^m_\rho(a) M^{m'}_\rho(b). \tag{A.7}$$

(iii) *Let* $m \in \mathbb{R}$, $\rho > 0$ *and* $a \in \Gamma^m_\rho(\mathbb{R}^d)$. *Denote by* $(T_a)^*$ *the adjoint operator of* T_a *and by* $\overline{a}$ *the complex-conjugate of* a. *Then* $(T_a)^* - T_{a^*}$ *is of order* $m - \rho$, *where*

$$a^* = \sum_{|\alpha| < \rho} \frac{1}{i^{|\alpha|} \alpha!} \partial^\alpha_\xi \partial^\alpha_x \overline{a}.$$

Moreover, for all μ *there exists a constant* K *such that*

$$\| (T_a)^* - T_{a^*} \|_{\mathcal{L}(H^\mu, H^{\mu - m + \rho})} \le K M^m_\rho(a). \tag{A.8}$$

Remark A.6. These properties are well-known when Sobolev spaces of periodic functions are replaced by Sobolev spaces on the real line. To prove these results for periodic functions, one can use the results of [3] about the general case of uniformly local Sobolev spaces $H^s_{\mathrm{ul}}(\mathbb{R})$. In particular it is proved that

$$\|(T_aT_b - T_{ab})u\|_{H^{\mu-m-m'+\rho}_{\mathrm{ul}}} \le K M^m_\rho(a) M^{m'}_\rho(b)\|u\|_{H^\mu_{\mathrm{ul}}}.$$

Since $\|u\|_{H^s_{\mathrm{ul}}} \lesssim \|u\|_{H^s(\mathbb{T})}$, it follows that

$$\|(T_aT_b - T_{ab})u\|_{H^{\mu-m-m'+\rho}_{\mathrm{ul}}(\mathbb{R})} \le K M^m_\rho(a) M^{m'}_\rho(b)\|u\|_{H^\mu(\mathbb{T})}.$$

Now, if u is a periodic function and a and b are periodic in x, then so is $(T_aT_b - T_{ab})u$, and we deduce that

$$\|(T_aT_b - T_{ab})u\|_{H^{\mu-m-m'+\rho}(\mathbb{T})} \lesssim \|(T_aT_b - T_{ab})u\|_{H^{\mu-m-m'+\rho}_{\mathrm{ul}}(\mathbb{R})}.$$

By combining the previous estimates we obtain (A.7). The other estimates are proved in a similar way.

It follows from (A.7) applied with $\rho = 1$ that if $a \in \Gamma^m_1$ and $b \in \Gamma^{m'}_1$ then

$$\|[T_a, T_b]\|_{\mathcal{L}(H^\mu, H^{\mu-m-m'+1})} \le K M^m_1(a) M^{m'}_1(b). \tag{A.9}$$

If $a = a(x)$ is a function of x only, then T_a is called a *paraproduct*. We often use the following consequence of (A.4): if $a \in L^\infty(\mathbb{T})$ then T_a is an operator of order 0, together with the estimate

$$\forall \sigma \in \mathbb{R}, \qquad \|T_a u\|_{H^\sigma} \lesssim \|a\|_{L^\infty}\|u\|_{H^\sigma}. \tag{A.10}$$

If $a = a(x)$ and $b = b(x)$ then (A.5) simplifies to $a\sharp b = ab$, and hence (A.6) implies that, for any $\rho > 0$,

$$\|T_aT_b - T_{ab}\|_{\mathcal{L}(H^\mu, H^{\mu-m-m'+\rho})} \le K\|a\|_{W^{\rho,\infty}}\|b\|_{W^{\rho,\infty}}, \tag{A.11}$$

provided that $a, b \in W^{\rho,\infty}(\mathbb{T})$.

Theorem A.7. (i) *Given two functions a, b defined on $\mathbb{R}$ we define the remainder*

$$\mathcal{R}(a, u) = au - T_a u - T_u a. \tag{A.12}$$

Let $\alpha \in \mathbb{R}_+$ and $\beta \in \mathbb{R}$ be such that $\alpha + \beta > 0$. Then

$$\|\mathcal{R}(a, u)\|_{H^{\alpha+\beta-1/2}} \le K\|a\|_{H^\alpha}\|u\|_{H^\beta}. \tag{A.13}$$

(ii) *Let $\alpha > 1/2$. For all C^∞ functions F with $F(0) = 0$, if $a \in H^\alpha(\mathbb{T})$ then*

$$\|F(a) - T_{F'(a)}a\|_{H^{2\alpha-1/2}} \le C(\|a\|_{H^\alpha})\|a\|_{H^\alpha}. \tag{A.14}$$

Proposition A.8. *Let $r, \mu \in \mathbb{R}$ be such that $r + \mu > 0$. If $\gamma \in \mathbb{R}$ satisfies*

$$\gamma \leq r \quad \text{and} \quad \gamma < r + \mu - 1/2,$$

then there exists a constant K such that, for all $a \in H^r(\mathbb{T})$ and all $u \in H^\mu(\mathbb{T})$,

$$\|au - T_a u\|_{H^\gamma} \leq K \|a\|_{H^r} \|u\|_{H^\mu}. \tag{A.15}$$

We also recall two well-known nonlinear properties. Firstly, if $u_1, u_2 \in H^s(\mathbb{T}) \cap L^\infty(\mathbb{T})$ and $s \geq 0$ then

$$\|u_1 u_2\|_{H^s} \leq K \|u_1\|_{L^\infty} \|u_2\|_{H^s} + K \|u_2\|_{L^\infty} \|u_1\|_{H^s}, \tag{A.16}$$

and hence, for $s > 1/2$,

$$\|u_1 u_2\|_{H^s} \leq K \|u_1\|_{H^s} \|u_2\|_{H^s}. \tag{A.17}$$

Similarly, for $s > 0$ and $F \in C^\infty(\mathbb{C}^N)$ such that $F(0) = 0$, there exists a nondecreasing function $C: \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|F(U)\|_{H^s} \leq C(\|U\|_{L^\infty}) \|U\|_{H^s} \tag{A.18}$$

for any $U \in (H^s(\mathbb{T}) \cap L^\infty(\mathbb{T}))^N$.

Appendix B. Energy estimates and well-posedness of some linear equations

Recall the linearized equation $\partial_t u + iLu = 0$, where $L := ((g - \partial_x^2)|D_x|)^{1/2}$.

In this section we gather Sobolev energy estimates for linear equations of the form

$$\partial_t \varphi + V \partial_x \varphi + i L^{1/2}(c L^{1/2} \varphi) + R\varphi = F,$$

where $V = V(t, x)$ is a real-valued coefficient, $c = c(t, x)$ is a real-valued coefficient bounded from below by $1/2$, $F = F(t, x)$ is a given complex-valued source term, and R is a time-dependent operator of order 0, which means that $R\varphi$ is defined by $(R\varphi)(t) = R(t)\varphi(t)$ and R belongs to $C^0(\mathbb{R}_+; \mathcal{L}(H^\mu))$ (for some μ) where $\mathcal{L}(H^\mu)$ denotes the set of bounded operators on $H^\mu(\mathbb{T})$. Below we consider various equations of this form where, for instance, R is either multiplication by some function or the commutator between $V\partial_x$ and a Fourier multiplier.

We also consider paradifferential equations of the form

$$\partial_t \varphi + T_V \partial_x \varphi + i\mathcal{L}\varphi + R\varphi = F,$$

where $\mathcal{L} = L^{1/2}(T_c L^{1/2} \cdot)$ and V, c, R are as above.

Proposition B.1. *Let $T > 0$ and $\mu \in [0,\infty)$. Consider $R \in C^0([0,T];\mathcal{L}(H^\mu))$ and real-valued coefficients V, c satisfying*

$$V \in C^0([0,T]; W^{1,\infty}(\mathbb{T})), \qquad c \in C^0([0,T]; W^{3/2,\infty}(\mathbb{T})),$$

with the $L^\infty_{t,x}$-norm of $c-1$ small enough. For any $\varphi_{\mathrm{in}} \in H^\mu(\mathbb{T})$ and any $F \in L^1([0,T]; H^\mu(\mathbb{T}))$, there exists a unique $\varphi \in C^0([0,T]; H^\mu(\mathbb{T}))$ such that

$$\partial_t\varphi + T_V\partial_x\varphi + R\varphi + i\mathcal{L}\varphi = F, \qquad \varphi|_{t=0} = \varphi_{\mathrm{in}}. \tag{B.1}$$

Moreover, for any $t \geq 0$,

$$\|\varphi(t)\|_{H^\mu} \leq e^{Ct}(\|\varphi_{\mathrm{in}}\|_{H^\mu} + \|F\|_{L^1([0,t];H^\mu)}) \tag{B.2}$$

for some constant $C = C(\mu, M)$ depending only on μ and

$$M = \sup_{t\in[0,T]} \{\|\partial_x V(t)\|_{L^\infty} + \|c(t)\|_{W^{3/2,\infty}} + \|R(t)\|_{\mathcal{L}(H^\mu)}\}.$$

Remark B.2. We often use energy estimates for backward Cauchy problems, that is, for Cauchy problems on time intervals $[0,T]$ with data prescribed at time T. Then the energy estimates read

$$\|\varphi(t)\|_{H^\mu} \leq e^{CT}(\|\varphi(T)\|_{H^\mu} + \|F\|_{L^1([0,T];H^\mu)}). \tag{B.3}$$

Proof of Proposition B.1. As already seen in (2.21), $\mathcal{L} = L^{1/2}T_cL^{1/2} = T_\gamma + R'$ where R' is of order 0 and

$$\gamma = c\ell + \frac{1}{i}(\partial_\xi\sqrt{\ell})\sqrt{\ell}\,\partial_x c.$$

Up to replacing in (B.1) the remainder R by $R + iR'$, we prove the existence of the solution as limits of approximate problems of the form

$$\partial_t\varphi + T_V\partial_x J_\varepsilon\varphi + iT_\gamma J_\varepsilon\varphi + R\varphi = F, \qquad \varphi|_{t=0} = J_\varepsilon\varphi_{\mathrm{in}}, \tag{B.4}$$

where J_ε are smoothing operators. Then (B.4) is an ODE in Banach spaces and admits a global in time solution denoted by φ_ε.

Set $\gamma^{(3/2)}(t,x,\xi) = c(t,x)\ell(\xi)$, which is the principal symbol of γ. As in [2], consider the paradifferential operator Λ_μ with symbol $1 + (c(t,x)\ell(\xi))^{2\mu/3}$, and given $\varepsilon \in [0,1]$, define J_ε as the paradifferential operator with symbol $\jmath_\varepsilon = \jmath_\varepsilon(t,x,\xi)$ given by

$$\jmath_\varepsilon = \jmath_\varepsilon^{(0)} + \jmath_\varepsilon^{(-1)} = \exp(-\varepsilon\gamma^{(3/2)}) - \frac{i}{2}(\partial_x\partial_\xi)\exp(-\varepsilon\gamma^{(3/2)}).$$

Recall that the Poisson bracket of two symbols is $\{a,b\} = (\partial_x a)(\partial_\xi b) - (\partial_\xi a)(\partial_x b)$. Then

$$\{\jmath_\varepsilon^{(0)}, \gamma^{(3/2)}\} = 0, \qquad \{\jmath_\varepsilon^{(0)}, (c\ell)^{2\mu/3}\} = 0, \qquad \{\gamma^{(3/2)}, (c\ell)^{2\mu/3}\} = 0, \tag{B.5}$$

and

$$\operatorname{Im} \jmath_\varepsilon^{(-1)} = -\tfrac{1}{2}(\partial_x\partial_\xi)\jmath_\varepsilon^{(0)}.$$

Of course, for any $\varepsilon > 0$, $j_\varepsilon \in C^0([0,T];\Gamma^m_{3/2}(\mathbb{R}^d))$ for all $m \le 0$, so that $T_{j_\varepsilon}u \in C^0([0,T];H^\infty(\mathbb{T}))$ for any $u \in C^0([0,T];H^{-\infty}(\mathbb{T}))$. Also j_ε is uniformly bounded in $C^0([0,T];\Gamma^0_{3/2}(\mathbb{R}^d))$ for all $\varepsilon \in [0,1]$. Hence, using (A.6) with $\rho = 3/2$ or (A.7) with $\rho = 1$, we have the following estimates (uniformly in ε):

$$\begin{aligned}
&\|[J_\varepsilon, T_\gamma]u\|_{H^\mu} \le C\|u\|_{H^\mu}, && \|(J_\varepsilon)^* u - J_\varepsilon u\|_{H^{\mu+3/2}} \le C\|u\|_{H^\mu},\\
&\|[\Lambda_\mu, L^{1/2}(T_c L^{1/2}\,\cdot)]u\|_{L^2} \le C\|u\|_{H^\mu}, && \|[\Lambda_\mu, J_\varepsilon]u\|_{H^{3/2}} \le C\|u\|_{H^\mu},\\
&\|[\Lambda_\mu, T_V\partial_x J_\varepsilon]u\|_{L^2} \le C\|V\|_{W^{1,\infty}}\|u\|_{H^\mu}, && \|[J_\varepsilon, T_V\partial_x]u\|_{H^\mu} \le C\|V\|_{W^{1,\infty}}\|u\|_{H^\mu},
\end{aligned} \tag{B.6}$$

for some constant C depending only on $\|c\|_{W^{3/2,\infty}}$ and uniform in $\varepsilon \in [0,1]$.

Recall that φ_ε is the unique solution to (B.4), and set $\dot\varphi_\varepsilon := \Lambda_\mu\varphi_\varepsilon$. Using the fact that Λ_μ is invertible (for $c-1$ small enough) and the preceding estimates, we deduce that

$$\partial_t\dot\varphi_\varepsilon + T_V\partial_x J_\varepsilon\dot\varphi_\varepsilon + \Lambda_\mu R\Lambda_\mu^{-1}\dot\varphi_\varepsilon + iT_\gamma J_\varepsilon\dot\varphi_\varepsilon = F_\varepsilon, \qquad \dot\varphi_\varepsilon|_{t=0} = \Lambda_\mu J_\varepsilon\varphi_{\mathrm{in}}, \tag{B.7}$$

where

$$\|F_\varepsilon\|_{L^1([0,T];L^2)} \le C(M)\{\|\varphi_\varepsilon\|_{L^1([0,T];H^\mu)} + \|F\|_{L^1([0,T];H^\mu)}\}. \tag{B.8}$$

Write $\frac{d}{dt}\|\dot\varphi_\varepsilon\|^2_{L^2} = 2\,\mathrm{Re}\,\langle\partial_t\dot\varphi_\varepsilon, \dot\varphi_\varepsilon\rangle$, where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $L^2(\mathbb{T})$, and hence

$$\frac{d}{dt}\|\dot\varphi_\varepsilon\|^2_{L^2} = -\langle(P+P^*)\dot\varphi_\varepsilon, \dot\varphi_\varepsilon\rangle \quad \text{with}$$
$$P = T_V\partial_x J_\varepsilon + \Lambda_\mu R\Lambda_\mu^{-1} + iT_\gamma J_\varepsilon.$$

To estimate the operator norm of $P+P^*$, there are two ingredients. Firstly, we replace J_ε^* by $J_\varepsilon + (J_\varepsilon^* - J_\varepsilon)$ and conjugate J_ε with $T_V\partial_x$ and T_γ. This produces remainder terms that are estimated by means of (B.6). The proof is then reduced to the case without J_ε and it suffices to estimate the operator norm of $\tilde P + \tilde P^*$ where $\widetilde P = T_V\partial_x + \Lambda_\mu R\Lambda_\mu^{-1} + iT_\gamma$. Since $\Lambda_\mu R\Lambda_\mu^{-1}$ is bounded from $L^\infty([0,T];L^2(\mathbb{T}))$ into itself with operator norm estimated by M, it remains to estimate $T_V\partial_x + iT_\gamma + (T_V\partial_x + iT_\gamma)^*$, which can be done directly by means of the paradifferential rule (A.8). We conclude that

$$\frac{d}{dt}\|\dot\varphi_\varepsilon\|^2_{L^2} \le C(M)\|\dot\varphi_\varepsilon\|^2_{L^2} + |\langle 2F_\varepsilon, \dot\varphi_\varepsilon\rangle|. \tag{B.9}$$

We thus obtain a uniform estimate for the $L^\infty([0,T];L^2)$-norm of $\dot\varphi_\varepsilon$ (from Gronwall's inequality and (B.8)), which gives a uniform estimate for the $L^\infty([0,T];H^\mu)$-norm of φ_ε. From this uniform estimate and classical arguments (see [39]), one deduces the existence of a solution in $L^\infty([0,T];H^\mu(\mathbb{T}))$. The uniqueness is obtained by considering the equation satisfied by the difference of two solutions and performing an L^2-energy estimate (using similar arguments to those used above). The continuity in time of the solution is proved as in [2, §6.4]. □

Lemma B.3. *Consider real-valued coefficients V, c satisfying*

$$V \in C^0([0,T]; W^{1,\infty}(\mathbb{T})), \qquad c \in C^0([0,T]; W^{3/2,\infty}(\mathbb{T})),$$

with the $L^\infty_{t,x}$-norm of $c-1$ small enough. Let also $R \in C^0([0,T]; \mathcal{L}(L^2))$.

(i) *For any $\varphi_{\rm in} \in L^2(\mathbb{T})$ and any $F \in L^1([0,T]; L^2(\mathbb{T}))$, there exists a unique $\varphi \in C^0([0,T]; L^2(\mathbb{T}))$ such that*

$$\partial_t\varphi + V\partial_x\varphi + R\varphi + iL^{1/2}(cL^{1/2}\varphi) = F, \qquad \varphi|_{t=0} = \varphi_{\rm in}. \tag{B.10}$$

Moreover, for any $t \geq 0$,

$$\|\varphi(t)\|_{L^2} \leq \exp\Big(\int_0^t M(t')\,dt'\Big)(\|\varphi_{\rm in}\|_{L^2} + \|F\|_{L^1([0,t];L^2)}) \tag{B.11}$$

with $M(t') = \|\partial_x V(t')\|_{L^\infty} + \|R(t')\|_{\mathcal{L}(L^2)}$.

(ii) *Let $\mu \in [0,3/2]$. Assume that $V \in C^0([0,T]; H^2(\mathbb{T}))$, $c \in C^0([0,T]; H^3(\mathbb{T}))$ and $R \in C^0([0,T]; \mathcal{L}(H^\mu))$. If $\varphi_{\rm in} \in H^\mu(\mathbb{T})$ and $F \in L^1([0,T]; H^\mu(\mathbb{T}))$, then, for any $t \geq 0$,*

$$\|\varphi(t)\|_{H^\mu} \leq \exp\Big(\int_0^t M(t')\,dt'\Big)(\|\varphi_{\rm in}\|_{H^\mu} + \|F\|_{L^1([0,t];H^\mu)}) \tag{B.12}$$

with $M(t') = \|V(t')\|_{H^2} + \|c(t')\|_{H^3} + \|R(t')\|_{\mathcal{L}(H^\mu)}$.

Remark B.4. Consider a backward Cauchy problem, that is, a Cauchy problem with data prescribed at time T. Then (B.11) implies that

$$\|\varphi(t)\|_{L^2} \leq \exp\Big(\int_0^T M(t')\,dt'\Big)(\|\varphi(T)\|_{L^2} + \|F\|_{L^1([0,T];L^2)}) \tag{B.13}$$

with $M(t') = \|\partial_x V(t')\|_{L^\infty} + \|R(t')\|_{\mathcal{L}(L^2)}$.

Proof of Lemma B.3. (i) The existence of the solution can be deduced from the previous proposition, writing

$$\partial_t\varphi + V\partial_x\varphi + R\varphi + iL^{1/2}(cL^{1/2}\varphi)$$

in the form

$$\partial_t\varphi + T_V\partial_x\varphi + R'\varphi + iL^{1/2}(T_cL^{1/2}\varphi),$$

where

$$R'\varphi = R\varphi + (V\partial_x\varphi - T_V\partial_x\varphi) + iL^{1/2}((c-T_c)L^{1/2}\varphi). \tag{B.14}$$

Indeed, $R' \in C^0([0,T]; \mathcal{L}(L^2))$ in view of (A.13) and (A.15).

In order to see that the energy estimate does not depend on the norm of c, start from $\frac{d}{dt}\|\varphi\|_{L^2}^2 = 2\,\mathrm{Re}\,\langle\partial_t\varphi, \varphi\rangle$. Since $\mathrm{Re}\,\langle iL^{1/2}(cL^{1/2}\varphi), \varphi\rangle = 0$, we obtain

$$\frac{d}{dt}\|\varphi\|_{L^2}^2 = 2\,\mathrm{Re}\,\langle -V\partial_x\varphi - R\varphi - F, \varphi\rangle.$$

Hence, integrating by parts yields

$$\frac{d}{dt}\|\varphi\|_{L^2}^2 = \mathrm{Re}\,\langle((\partial_x V) - 2R)\varphi - 2F, \varphi\rangle, \tag{B.15}$$

and the result easily follows from Gronwall's inequality.

(ii) This follows from (B.2) and the fact that the remainder R' in (B.14) belongs to $C^0([0, T]; \mathcal{L}(H^\mu))$ in view of (A.13) and (A.15). □

Appendix C. Changes of variables

Recall the operator

$$\widetilde{P} := \partial_t + V\partial_x + iL^{1/2}(cL^{1/2}\,\cdot) + R_2, \tag{C.1}$$

where $L := (g - \partial_{xx})^{1/2}G(0)^{1/2}$, the operator R_2 is of order zero, and $c(t, x)$, $V(t, x)$ are real-valued functions. Consider a time-dependent change of the space variable (a diffeomorphism of $\mathbb{T}$) and its inverse,

$$x = y + \tilde{\beta}_1(t, y) \;\Leftrightarrow\; y = x + \beta_1(t, x), \tag{C.2}$$

for $x, y \in \mathbb{T}$, $t \in \mathbb{R}$, with $\|\partial_y\tilde{\beta}_1\|_{L^\infty}, \|\partial_x\beta_1\|_{L^\infty} \le 1/2$. Introduce a self-adjoint variant of the pull-back operators, defined by

$$(\Psi_1 h)(t, y) := (1 + \partial_y\tilde{\beta}_1(t, y))^{1/2} h(t, y + \tilde{\beta}_1(t, y)), \tag{C.3}$$

$$(\Psi_1^{-1} h)(t, x) := (1 + \partial_x\beta_1(t, x))^{1/2} h(t, x + \beta_1(t, x)), \tag{C.4}$$

and note that Ψ_1, Ψ_1^{-1} are self-adjoint with respect to the standard $L^2(\mathbb{T})$ scalar product in space, for any t. We want to compute $\Psi_1 Q_0 \Psi_1^{-1}$ when Q_0 is a Fourier multiplier (the analysis below applies more generally assuming only that Q_0 is a pseudo-differential operator), using the Egorov theorem (see also [9]).

C.1. Change of variable as a flow map

Introduce a parameter $\tau \in [0, 1]$ and consider a diffeomorphism of $\mathbb{T}$ (depending on (τ, t)) and its inverse,

$$x = y + \tilde{\beta}(\tau, t, y) \;\Leftrightarrow\; y = x + \beta(\tau, t, x),$$

for $x, y \in \mathbb{T}$, $\tau \in [0, 1]$, $t \in \mathbb{R}$, where β and $\tilde{\beta}$ are such that $\|\partial_y\tilde{\beta}\|_{L^\infty}, \|\partial_x\beta\|_{L^\infty} \le 1/2$ and

$$\tilde{\beta}|_{\tau=0} = 0, \qquad \beta|_{\tau=0} = 0, \qquad \tilde{\beta}|_{\tau=1} = \tilde{\beta}_1, \qquad \beta|_{\tau=1} = \beta_1.$$

We denote

$$(\Psi(\tau)h)(t, y) := (1 + \partial_y\tilde{\beta}(\tau, t, y))^{1/2} h(t, y + \tilde{\beta}(\tau, t, y)), \tag{C.5}$$

$$(\Psi(\tau)^{-1}h)(t, x) := (1 + \partial_x\beta(\tau, t, x))^{1/2} h(t, x + \beta(\tau, t, x)). \tag{C.6}$$

Then $\Psi_1 = \Psi(1)$. The reason for introducing the parameter τ is that $\Psi(\tau)$ satisfies an equation of the form

$$\partial_\tau \Psi(\tau) = F(\tau)\Psi(\tau), \qquad \Psi(0) = I, \tag{C.7}$$

that is, $\partial_\tau(\Psi(\tau)h) = F(\tau)(\Psi(\tau)h)$, $\Psi(0)h = h$ for all h, where

$$F(\tau) = b_0(\tau, t, y)\partial_y + \tfrac{1}{2}(\partial_y b_0)(\tau, t, y), \qquad b_0(\tau, t, y) := \frac{\partial_\tau \tilde{\beta}(\tau, t, y)}{1 + \partial_y \tilde{\beta}(\tau, t, y)}. \tag{C.8}$$

Assume that Q_0 is a Fourier multiplier with symbol $q_0(\xi)$ of order $m \le 3/2$. We seek a pseudo-differential operator $Q(\tau)$ of order m such that the difference

$$R(\tau) := Q(\tau)\Psi(\tau) - \Psi(\tau)Q_0 \tag{C.9}$$

is an operator of order 0. Conjugating the equation $\partial_\tau \Psi(\tau) = F(\tau)\Psi(\tau)$ with $Q(\tau)$ one obtains

$$\begin{aligned}\partial_\tau(Q(\tau)\Psi(\tau)) &= Q(\tau)F(\tau)\Psi(\tau) + (\partial_\tau Q(\tau))\Psi(\tau)\\ &= F(\tau)Q(\tau)\Psi(\tau) + \big([Q(\tau), F(\tau)]\Psi(\tau) + (\partial_\tau Q(\tau))\Psi(\tau)\big).\end{aligned}$$

On the other hand, $\partial_\tau(\Psi(\tau)Q_0) = F(\tau)\Psi(\tau)Q_0$. By combining both equations we find that R satisfies

$$\partial_\tau R(\tau) = F(\tau)R(\tau) + \mathcal{R}_1(\tau)\Psi(\tau), \qquad \mathcal{R}_1(\tau) := [Q(\tau), F(\tau)] + \partial_\tau Q(\tau). \tag{C.10}$$

The analysis is then in two steps. The main step consists in proving that $Q(\tau)$ can be so chosen that $Q(\tau = 0) = Q_0$ (then $R(0) = 0$) and $\mathcal{R}_1(\tau)$ is of order 0. Then, by using an L^2-energy estimate for the hyperbolic equation $\partial_\tau u = b_0\partial_y u + f$, one deduces an estimate for the operator norm of $R(\tau)$ uniform in τ (and hence the desired estimate for $\tau = 1$). Here we describe in detail only the main step, as the L^2-energy estimate is a standard argument.

C.2. Expansion of the symbol

Let $p(\tau, t, x, \xi)$ be the symbol of $Q(\tau)$. To obtain $\mathcal{R}_1$ of order zero amounts to seeking p such that $\partial_\tau p - \sigma_{[F,Q]}$ has order zero (where $\sigma_{[F,Q]}$ is the symbol of $[F, Q]$), and $p|_{\tau=0} = q_0$. The asymptotic expansion of $\sigma_{[F,Q]}$ is

$$\sigma_{[F,Q]} \sim \sum_{\alpha=1}^{\infty} \frac{1}{i^\alpha\, \alpha!}\{(\partial_\xi^\alpha f)(\partial_x^\alpha p) - (\partial_\xi^\alpha p)(\partial_x^\alpha f)\}, \tag{C.11}$$

where $f(\tau, t, x, \xi) := ib_0(\tau, t, x)\xi + \frac{1}{2}(\partial_x b_0)(\tau, t, x)$ is the symbol of $F(\tau)$ (we rename the space variable to be x). Since $m \le 3/2 \le 2$ by assumption, it is enough to determine the principal and the subprincipal symbols of p. Thus we write $p = p_0 + p_1$, where p_0 has order m and p_1 has order $m - 1$. The equations for p_0, p_1 are

$$\partial_\tau p_0 = b_0\partial_x p_0 - \xi(\partial_x b_0)\partial_\xi p_0, \qquad p_0|_{\tau=0} = q_0, \tag{C.12}$$

$$\partial_\tau p_1 = b_0\partial_x p_1 - \xi(\partial_x b_0)\partial_\xi p_1 + z, \quad p_1|_{\tau=0} = 0, \tag{C.13}$$

where

$$z := \frac{i}{2}(\partial_{xx} b_0)(\partial_\xi p_0 + \xi \partial_{\xi\xi} p_0). \tag{C.14}$$

If p_0, p_1 satisfy (C.12) and (C.13), then it follows from standard symbolic calculus for pseudo-differential operators (similar to (A.6)) that $\mathcal{R}_1(\tau)$, defined in (C.10), is an operator of order 0 satisfying

$$\|\mathcal{R}_1(\tau)\|_{\mathcal{L}(L^2)} + \|\mathcal{R}_1(\tau)\|_{\mathcal{L}(H^{3/2})} \lesssim (M_r^m(p_0) + M_r^{m-1}(p_1))\|b_0(\tau)\|_{W^{r,\infty}} \tag{C.15}$$

with r large enough (here the seminorms M_ρ^m are as defined by (A.1); one has to consider r large enough because we are here considering pseudo-differential operators instead of paradifferential ones).

Equation (C.12) can be solved by the characteristics method: if $x(\tau)$, $\xi(\tau)$ solve

$$\frac{d}{d\tau}x(\tau) = -b_0(\tau, t, x(\tau)), \qquad \frac{d}{d\tau}\xi(\tau) = \xi(\tau)(\partial_x b_0)(\tau, t, x(\tau)), \tag{C.16}$$

then

$$p_0(\tau, t, x(\tau), \xi(\tau)) = p_0(0, t, x(0), \xi(0)) \quad \forall \tau. \tag{C.17}$$

Now, by (C.8), the first equation in (C.16) is

$$0 = \{1 + (\partial_x \tilde{\beta})(\tau, t, x(\tau))\}\, x'(\tau) + (\partial_\tau \tilde{\beta})(\tau, t, x(\tau)) = \frac{d}{d\tau}\{x(\tau) + \tilde{\beta}(\tau, t, x(\tau))\},$$

whence

$$x(\tau) + \tilde{\beta}(\tau, t, x(\tau)) = x(0) + \tilde{\beta}(0, t, x(0)) = x(0). \tag{C.18}$$

Applying the inverse diffeomorphism, we get $x(\tau) = x(0) + \beta(\tau, t, x(0))$. This is the solution $x(\tau)$ of the first equation in (C.16) with initial data $x(0)$. Also, one verifies that

$$\xi(\tau) = \xi(0)\big(1 + (\partial_x \tilde{\beta})(\tau, t, x(\tau))\big) \tag{C.19}$$

satisfies the second equation in (C.16), because $x(\tau)$ satisfies the first equation in (C.16), b_0 is given by (C.8), and

$$\partial_x b_0(\tau, t, x) = \frac{\partial_{\tau x}\tilde{\beta}(\tau, t, x)}{1 + \partial_x \tilde{\beta}(\tau, t, x)} - \frac{\partial_\tau \tilde{\beta}(\tau, t, x)\partial_{xx}\tilde{\beta}(\tau, t, x)}{[1 + \partial_x \tilde{\beta}(\tau, t, x)]^2}.$$

Hence we deduce a formula for the backward flow of (C.16): for any $\tau_1 \in [0, 1]$ and any (x_1, ξ_1), the solution $(x(\tau), \xi(\tau))$ of (C.16) with initial data $(x(0), \xi(0)) = (x_0, \xi_0)$ satisfies $(x(\tau_1), \xi(\tau_1)) = (x_1, \xi_1)$ if the initial data is

$$x_0 = x_1 + \tilde{\beta}(\tau_1, t, x_1), \qquad \xi_0 = \frac{\xi_1}{1 + \partial_x \tilde{\beta}(\tau_1, t, x_1)}. \tag{C.20}$$

As a consequence, using (C.17) and the initial data in (C.12), we get

$$\begin{aligned} p_0(\tau_1, t, x_1, \xi_1) &= p_0(0, t, x_0, \xi_0) = q_0(t, x_0, \xi_0) \\ &= q_0\left(t, x_1 + \tilde{\beta}(\tau_1, t, x_1), \frac{\xi_1}{1 + \partial_x \tilde{\beta}(\tau_1, t, x_1)}\right). \end{aligned}$$

We have a formula for the solution $p_0(\tau, t, x, \xi)$ of (C.12):

$$p_0(\tau, t, x, \xi) = q_0\Big(t, x + \tilde{\beta}(\tau, t, x), \frac{\xi}{1 + \partial_x \tilde{\beta}(\tau, t, x)}\Big). \tag{C.21}$$

Now we study equation (C.13). By the definition of $(x(\tau), \xi(\tau))$,

$$p_1(\tau, t, x(\tau), \xi(\tau)) = \int_0^\tau z(s, t, x(s), \xi(s))\, ds, \tag{C.22}$$

where z is given in (C.14). We examine z in detail. By (C.21), for $k = 1, 2$,

$$\partial_\xi^k p_0(\tau, t, x, \xi) = (\partial_\xi^k q_0)\Big(t, x + \tilde{\beta}(\tau, t, x), \frac{\xi}{1 + \partial_x \tilde{\beta}(\tau, t, x)}\Big) \frac{1}{[1 + \partial_x \tilde{\beta}(\tau, t, x)]^k}$$

for all τ, t, x, ξ. Hence along the curves $(x(s), \xi(s))$, by (C.18) and (C.19),

$$(\partial_\xi^k p_0)(s, t, x(s), \xi(s)) = \frac{(\partial_\xi^k q_0)(t, x_0, \xi_0)}{[1 + \partial_x \tilde{\beta}(s, t, x(s))]^k},$$

where $(x_0, \xi_0) := (x(0), \xi(0))$, and therefore, using (C.19) again, we get

$$(\partial_\xi p_0 + \xi \partial_{\xi\xi} p_0)(s, t, x(s), \xi(s)) = \frac{\partial_\xi q_0(t, x_0, \xi_0) + \xi_0 \partial_{\xi\xi} q_0(t, x_0, \xi_0)}{1 + \partial_x \tilde{\beta}(s, t, x(s))}.$$

Now we note that

$$\frac{(\partial_{xx} b_0)(s, t, x(s), \xi(s))}{1 + (\partial_x \tilde{\beta})(s, t, x(s))} = \frac{d}{ds}\Big\{ \frac{(\partial_{xx} \tilde{\beta})(s, t, x(s))}{[1 + (\partial_x \tilde{\beta})(s, t, x(s))]^2} \Big\},$$

as can be verified by a straightforward calculation, using also (C.16) and the definition (C.8) of b_0. Hence, recalling the definition (C.14) of z, we obtain

$$z(s, t, x(s), \xi(s)) = \frac{i}{2}\{\partial_\xi q_0(t, x_0, \xi_0) + \xi_0 \partial_{\xi\xi} q_0(t, x_0, \xi_0)\} \frac{d}{ds}\Big\{ \frac{(\partial_{xx} \tilde{\beta})(s, t, x(s))}{[1 + (\partial_x \tilde{\beta})(s, t, x(s))]^2} \Big\},$$

and, by (C.22),

$$p_1(\tau, t, x(\tau), \xi(\tau)) = \frac{i}{2}\{\partial_\xi q_0(t, x_0, \xi_0) + \xi_0 \partial_{\xi\xi} q_0(t, x_0, \xi_0)\} \frac{(\partial_{xx} \tilde{\beta})(\tau, t, x(\tau))}{[1 + (\partial_x \tilde{\beta})(\tau, t, x(\tau))]^2}$$

because $\tilde{\beta}|_{\tau=0} = 0$. We use the backward flow as above: given τ_1, x_1, ξ_1, the solution $(x(\tau), \xi(\tau))$ of (C.16) with initial data $(x(0), \xi(0)) = (x_0, \xi_0)$ satisfies $(x(\tau_1), \xi(\tau_1)) = (x_1, \xi_1)$ if the initial data is (C.20). Therefore, replacing (x_0, ξ_0) by (C.20) in the last equality, we get a formula for p_1, with τ, x, ξ instead of τ_1, x_1, ξ_1:

$$\begin{aligned} p_1(\tau, t, x, \xi) = \frac{i}{2}\Big\{ &(\partial_\xi q_0)\Big(t, x + \tilde{\beta}(\tau, t, x), \frac{\xi}{1 + \partial_x \tilde{\beta}(\tau, t, x)}\Big) \\ &+ \frac{\xi}{1 + \partial_x \tilde{\beta}(\tau, t, x)} (\partial_{\xi\xi} q_0)\Big(t, x + \tilde{\beta}(\tau, t, x), \frac{\xi}{1 + \partial_x \tilde{\beta}(\tau, t, x)}\Big)\Big\} \frac{\partial_{xx} \tilde{\beta}(\tau, t, x)}{(1 + \partial_x \tilde{\beta}(\tau, t, x))^2}. \end{aligned} \tag{C.23}$$

C.3. Conjugation of L

We fix $q_0(\xi)$ to be the symbol of L (see (2.11)) with a cut-off around $\xi = 0$, namely

$$q_0(\xi) := (g+\xi^2)^{1/2}\lambda(\xi)^{1/2}\chi(\xi) = (g+\xi^2)^{1/2}|\xi|^{1/2}\tanh^{1/2}(b|\xi|)\chi(\xi),$$

where $\chi(\xi)$ is the cut-off function of Proposition 2.5. Note that $\mathrm{Op}(q_0) = L$ on the periodic functions, as their symbols coincide at any $\xi \in \mathbb{Z}$, and therefore no remainder is produced by replacing L with $\mathrm{Op}(q_0)$. In the previous section we have constructed p_0, p_1, and we have defined $p := p_0 + p_1$ and $Q(\tau) := \mathrm{Op}(p)$. Then $\mathcal{R}_1(\tau)$ defined in (C.10) is an operator of order zero and it satisfies estimate (C.15). Now observe that, in view of (C.10), for any $u_0 \in L^2(\mathbb{T})$, $R(\tau)u_0$ solves a hyperbolic evolution equation. Using the energy estimate (B.11), we deduce that the difference $R(\tau) := Q(\tau)\Psi(\tau) - \Psi(\tau)L$ (see (C.9)) is also of order zero, and it satisfies the same estimate (C.15) as $\mathcal{R}_1(\tau)$. As a consequence, the conjugate of L is

$$\Psi(\tau)L\Psi(\tau)^{-1} = Q(\tau) + \mathcal{R}_2(\tau), \qquad \mathcal{R}_2(\tau) := -R(\tau)\Psi(\tau)^{-1}, \tag{C.24}$$

and $\mathcal{R}_2(\tau)$ satisfies the same estimate (C.15) as $R(\tau)$. By (C.21), $p_0 = q_0(\xi(1+\partial_x\tilde{\beta})^{-1})$. We expand

$$p_0 = (1+\partial_x\tilde{\beta})^{-3/2}q_0 + r, \tag{C.25}$$

where the remainder r satisfies $\|\mathrm{Op}(r)\|_{\mathcal{L}(H^\mu,H^{\mu+1/2})} \lesssim \|\partial_x\tilde{\beta}\|_{H^{\mu+\rho}}$ for all $\mu \geq 0$, for some absolute constant ρ large enough, because

$$g+\xi^2h^2 = h^2(g+\xi^2)\left(1+\frac{g(1-h^2)}{h^2(g+\xi^2)}\right), \qquad h := (1+\partial_x\tilde{\beta})^{-1},$$

and then we use Taylor expansion for the square root of the last factor. The second component p_1 is given by (C.23). By Taylor expansion,

$$\left|q_0'(\xi) - \tfrac{3}{2}|\xi|^{-1/2}\xi\right| \lesssim (1+|\xi|)^{-3/2}, \qquad \left|q_0''(\xi) - \tfrac{3}{4}|\xi|^{-1/2}\right| \lesssim (1+|\xi|)^{-5/2},$$

so that we calculate

$$p_1 = i\tfrac{9}{8}(\partial_{xx}\tilde{\beta})(1+\partial_x\tilde{\beta})^{-5/2}|\xi|^{-1/2}\xi\,\chi(\xi) + r, \tag{C.26}$$

where the remainder r satisfies $\|\mathrm{Op}(r)\|_{\mathcal{L}(H^\mu,H^{\mu+3/2})} \lesssim \|\partial_x\tilde{\beta}\|_{H^{\mu+\rho}}$ for all $\mu \geq 0$, for some ρ large enough. Assume that $\|\partial_\tau\tilde{\beta}\|_{H^\mu} \lesssim \|\tilde{\beta}\|_{H^\mu}$ (this bound holds for the choice of β we make below). By (C.24)–(C.26), we have

$$\Psi(\tau)L\Psi(\tau)^{-1} = (1+\partial_x\tilde{\beta})^{-3/2}L + \tfrac{9}{8}(\partial_{xx}\tilde{\beta})(1+\partial_x\tilde{\beta})^{-5/2}|D_x|^{-1/2}\partial_x + \mathcal{R}_{0,1}, \tag{C.27}$$

where $\mathcal{R}_{0,1}$ is defined to be the difference and it satisfies $\|\mathcal{R}_{0,1}\|_{\mathcal{L}(H^\mu,H^\mu)} \lesssim \|\partial_x\tilde{\beta}\|_{H^{\mu+\rho}}$ for all $\mu \geq 0$, for some ρ large enough. With similar calculations, one proves that for any $r \in \mathbb{R}$,

$$\Psi(\tau)|D_x|^r\Psi(\tau)^{-1} = (1+\partial_x\tilde{\beta})^{-r}|D_x|^r + \mathcal{R}_{0,2}, \tag{C.28}$$

where the difference $\mathcal{R}_{0,2}$ satisfies $\|\mathcal{R}_{0,2}\|_{\mathcal{L}(H^\mu,H^{\mu-r+1})} \lesssim \|\partial_x\tilde{\beta}\|_{H^{\mu+\rho}}$.

C.4. Conjugation of $\widetilde{P}$

We conjugate the operator in (C.1) with $\Psi_1 := \Psi(1) = \Psi(\tau)|_{\tau=1}$. By symbolic calculus,

$$L^{1/2} c L^{1/2} = cL - \tfrac{3}{4}(\partial_x c)\partial_x |D_x|^{-1/2} + \mathcal{R}_{0,3}, \tag{C.29}$$

where the difference $\mathcal{R}_{0,3}$ satisfies $\|\mathcal{R}_{0,3}\|_{\mathcal{L}(H^\mu, H^{\mu+1/2})} \lesssim \|\partial_x c\|_{H^{\mu+\rho}}$ for all $\mu \geq 0$, for some ρ large enough. We recall that $c-1$ is small, and therefore $\partial_x c$ is small. By definition (see (C.5), (C.6)), and recalling that $\tilde{\beta}|_{\tau=1} = \tilde{\beta}_1$, $\beta|_{\tau=1} = \beta_1$, we directly calculate

$$\Psi_1 \partial_t \Psi_1^{-1} = \partial_t + a_1 \partial_x + r_1, \qquad \Psi_1 \partial_x \Psi_1^{-1} = a_2 \partial_x + r_2,$$

where

$$a_1(t,x) := (\partial_t \beta_1)(t, x + \tilde{\beta}_1(t,x)), \qquad a_2(t,x) := (1 + \partial_x \tilde{\beta}_1(t,x))^{-1}, \tag{C.30}$$

and

$$\begin{aligned} r_1(t,x) &:= \tfrac{1}{2}(\partial_{tx}\beta_1)(t, x + \tilde{\beta}_1(t,x))\,(1 + \partial_x \tilde{\beta}_1(t,x)), \\ r_2(t,x) &:= \tfrac{1}{2}(1 + \partial_x \tilde{\beta}_1(t,x))\,(\partial_{xx}\beta_1)(t, x + \tilde{\beta}_1(t,x)). \end{aligned}$$

The conjugate of any multiplication operator $h \mapsto ah$ is the multiplication operator $h \mapsto (\tilde{B}a)h$,

$$\Psi_1 a \Psi_1^{-1} = \tilde{B}a, \qquad (\tilde{B}a)(t,x) := a(t, x + \tilde{\beta}_1(t,x)).$$

Thus

$$\Psi_1 \widetilde{P} \Psi_1^{-1} = \partial_t + a_3 \partial_x + i a_4 L + i a_5 \partial_x |D_x|^{-1/2} + \tilde{R}_3,$$

where

$$\begin{aligned} a_3 &:= a_1 + (\tilde{B}V)a_2, \qquad a_4 := (\tilde{B}c)(1 + \partial_x \tilde{\beta})^{-3/2}, \\ a_5 &:= -\tfrac{3}{4}\big\{-\tfrac{3}{2}(\tilde{B}c)(1 + \partial_x\tilde{\beta})^{-5/2}(\partial_{xx}\tilde{\beta}) + (\tilde{B}(\partial_x c))(1 + \partial_x \tilde{\beta})^{-1/2}\big\}, \\ \tilde{R}_3 &:= r_1 + (\tilde{B}V)r_2 + i(\tilde{B}c)\mathcal{R}_{0,1} + -i\tfrac{3}{4}(\tilde{B}\partial_x c) r_2 (1 + \partial_x \tilde{\beta})^{1/2} |D_x|^{-1/2} \\ &\qquad + \mathcal{R}_{0,2} + i \Psi_1 \mathcal{R}_{0,3} \Psi_1^{-1} + \Psi_1 R_2 \Psi_1^{-1}, \end{aligned} \tag{C.31}$$

$\mathcal{R}_{0,1}$ is defined in (C.27) with $\tau = 1$, $\mathcal{R}_{0,2}$ is defined in (C.28) with $\tau = 1$ and $r = -1/2$, and $\mathcal{R}_{0,3}$ is defined in (C.29). The remainder $\tilde{R}_3$ is of order zero and it is estimated in Lemma C.1. Moreover, as is immediate to verify, $a_5 = -\frac{3}{4}\partial_x a_4$. We choose $\beta_1, \tilde{\beta}_1$ such that the highest order coefficient a_4 is independent of x. This means

$$a_4(t,x) = c(t, x + \tilde{\beta}_1(t,x))\,(1 + \partial_x \tilde{\beta}_1(t,x))^{-3/2} = m(t) \quad \forall x \in \mathbb{T}, \tag{C.32}$$

for some function $m(t)$ independent of x. By applying the inverse diffeomorphism, this is equivalent to

$$c(t,x)\,(1 + \partial_x \beta_1(t,x))^{3/2} = m(t) \quad \forall x \in \mathbb{T}.$$

This implies $1 + \partial_x \beta_1(t,x) = m(t)^{2/3} c(t,x)^{-2/3}$, which, after integration over x, gives

$$m(t) = \left(\frac{1}{2\pi} \int_{\mathbb{T}} c(t,x)^{-2/3}\, dx \right)^{-3/2}. \tag{C.33}$$

Hence m in (C.32) is determined. We fix β_1 as

$$\beta_1(t,x) = \partial_x^{-1}[m(t)^{2/3} c(t,x)^{-2/3} - 1], \tag{C.34}$$

and then we fix $\beta(\tau,t,x) := \tau \beta_1(t,x)$. As a consequence, $\tilde{\beta}(\tau,t,y)$ and $\tilde{\beta}_1$ are also determined. Since $a_4(t,x) = m(t)$ is independent of x, it follows that $a_5 = -\frac{3}{4}\partial_x a_4 = 0$ (as was natural to expect, because the vector field in $\widetilde{P}$ is anti-selfadjoint and the transformation Ψ preserves this structure). We have conjugated $\widetilde{P}$ to

$$\widetilde{P}_1 := \Psi_1 \widetilde{P} \Psi_1^{-1} = \partial_t + i m(t) L + a_3 \partial_x + \tilde{R}_3. \tag{C.35}$$

We underline that the coefficient $m(t)$ is a function of time, independent of space.

Lemma C.1. *There exists a universal constant $\delta_0 \in (0,1)$ such that if*

$$\|c(t) - 1\|_{L^\infty} < \delta_0$$

then $\|\partial_x \beta_1(t)\|_{L^\infty} + \|\partial_x \tilde{\beta}_1(t)\|_{L^\infty} < 1/2$ and

$$\|\partial_x \beta_1(t)\|_{W^{\mu,\infty}} + \|\partial_x \tilde{\beta}_1(t)\|_{W^{\mu,\infty}} \le C_\mu \|c(t) - 1\|_{W^{\mu,\infty}} \quad \forall \mu \ge 0$$

for some positive constant C_μ depending only on μ. As a consequence, $\Psi_1(t)$ and $\Psi_1(t)^{-1}$ are bounded transformations of $H^\mu(\mathbb{T})$ with

$$\|\Psi_1(t)\|_{\mathcal{L}(H^\mu)} + \|\Psi_1(t)^{-1}\|_{\mathcal{L}(H^\mu)} \le C_\mu (1 + \|c(t) - 1\|_{H^\mu}) \quad \forall \mu \ge 0.$$

Moreover $|m(t) - 1| \le C \|c(t) - 1\|_{H^1}$, and

$$\|a_3(t)\|_{H^\mu} \le C_\mu (\|c(t) - 1\|_{H^\mu} + \|\partial_t c(t)\|_{H^{\mu-1}}) \quad \forall \mu \ge 1.$$

The remainder $\tilde{R}_3(t)$ maps $L^2(\mathbb{T})$ into itself with

$$\|\tilde{R}_3(t)\|_{\mathcal{L}(L^2)} \le C \big(\|c(t) - 1\|_{H^r} + \|V(t)\|_{L^\infty} + \|\partial_t c(t)\|_{L^\infty} + \|R_2(t)\|_{\mathcal{L}(L^2)} \big),$$

and, for all $\mu > 1/2$, $\tilde{R}_3(t)$ also maps $H^\mu(\mathbb{T})$ into itself with

$$\|\tilde{R}_3(t)\|_{\mathcal{L}(H^\mu)} \le C_\mu \big(\|c(t) - 1\|_{H^{\mu+r}} + \|V(t)\|_{H^\mu} + \|\partial_t c(t)\|_{H^\mu} + \|R_2(t)\|_{\mathcal{L}(H^\mu)} \big),$$

where $r > 0$ is a universal constant.

Proof. The estimates follow from the explicit formulas above, the usual estimates for the composition of functions (see, e.g., [7, Appendix B]) and Sobolev estimates for pseudo-differential operators (see (C.15)). The estimate of the pseudo-differential remainder term is the reason why r further space derivatives are required on c. The term $\partial_t c$ appears only in a_1 and r_1. The term V appears only in a_3 and $\tilde{R}_3$ where it is explicitly written, and nowhere else. The operator R_2 only appears in $\tilde{R}_3$ in the term $\Psi_1 R_2 \Psi_1^{-1}$. All the other terms depend only on c and its space derivatives. □

C.5. Reparametrization of time

Now we want to replace the coefficient $m(t)$ in (C.35) with a constant coefficient. We consider a diffeomorphism of the time interval

$$\psi : [0, T] \to [0, T_1], \quad \psi(0) = 0, \quad \psi(T) = T_1, \quad \psi'(t) > 0,$$

where $T_1 > 0$ has to be determined. We consider the pull-back ψ_* defined by $(\psi_* h)(t, x) := h(\psi(t), x)$, and similarly for its inverse ψ^{-1}. Then we calculate the conjugate

$$\psi_*^{-1}(\partial_t + im(t)L)\psi_* = \psi'(\psi^{-1}(t))\partial_t + im(\psi^{-1}(t))L.$$

The two time-dependent coefficients are equal if $m(t) = \psi'(t)$ for all $t \in [0, T]$.

We define

$$\psi(t) := \int_0^t m(s)\, ds, \quad T_1 := \int_0^T m(t)\, dt, \quad \rho(t) := m(\psi^{-1}(t)). \tag{C.36}$$

Since $|m - 1|$ is small, the ratio T_1/T is close to 1, and also $\psi'(t)$ is close to 1 for all t. We have the conjugate

$$\psi_*^{-1}\widetilde{P}_1\psi_* = \rho(t)\widetilde{P}_2, \quad \widetilde{P}_2 := \partial_t + iL + a_6\partial_x + \tilde{R}_4, \tag{C.37}$$

where

$$a_6(t, x) := \frac{a_3(\psi^{-1}(t), x)}{\rho(t)}, \quad \tilde{R}_4 := \frac{1}{\rho(t)}\psi_*^{-1}\tilde{R}_3\psi_* \tag{C.38}$$

(and, more explicitly, $(\psi_*^{-1}\tilde{R}_3\psi_*)(t) = \tilde{R}_3(\psi^{-1}(t))$). Now the coefficient of the highest order term L is constant.

C.6. Translation of the space variable

The goal of this section is to eliminate the space average of the coefficient $a_6(t, x)$ in front of ∂_x. Consider a time-dependent change of the space variable which is simply a translation,

$$y = \varphi(t, x) = x + p(t) \;\Leftrightarrow\; x = \varphi^{-1}(t, y) = y - p(t),$$

and its pull-back $(\varphi_* h)(t, x) = h(t, \varphi(t, x)) = h(t, x + p(t))$, and similarly for φ^{-1}. Thus $\varphi_*^{-1}\partial_t\varphi_* = \partial_t + p'(t)\partial_x$, and φ_* commutes with every Fourier multiplier like ∂_x, $|D_x|^r$, L. We calculate the conjugate

$$\widetilde{P}_3 := \varphi_*^{-1}\widetilde{P}_2\varphi_* = \partial_t + iL + a_7\partial_x + \tilde{R}_5,$$

where

$$a_7 := p'(t) + (\varphi_*^{-1}a_6), \quad \tilde{R}_5 := \varphi_*^{-1}\tilde{R}_4\varphi_*. \tag{C.39}$$

Since φ_* and φ_*^{-1} preserve the space average, we fix

$$p(t) := -\frac{1}{2\pi}\int_0^t \int_{\mathbb{T}} a_6(s, x)\, dx\, ds. \tag{C.40}$$

It follows that $\int_{\mathbb{T}} a_7(t,x)\,dx = 0$ for all $t \in [0, T_1]$. Note that φ_* commutes with the multiplication operator $h \mapsto \rho(t)h$, because $\rho(t)$ is independent of x. Moreover, by the change of time variable $s = \psi(t)$, $ds = m(t)dt$ in the integral, we get

$$p(T_1) = -\frac{1}{2\pi}\int_0^{T_1}\int_{\mathbb{T}} a_6(s,x)\,dx\,ds = -\frac{1}{2\pi}\int_0^{T}\int_{\mathbb{T}} a_3(t,x)\,dx\,dt. \tag{C.41}$$

Proof of Proposition 5.1 concluded. The composition $\Phi := \varphi_*^{-1}\psi_*^{-1}\Psi_1$ of the previous three transformations gives $\widetilde{P} = \Phi^{-1}\rho\widetilde{P}_3\Phi$. Also note that $\Phi^{-1}(\rho u) = m\Phi^{-1}u$ for all u. The transformation Ψ_1 is estimated in Lemma C.1. The estimates for ψ_* and φ_* are straightforward. Finally, rename $W := a_7$ and $R_3 := \tilde{R}_5$. □

Notation. In the following proposition we use the shorter notation $\|u\|_{T,X}$ for the $C^0([0,T];X)$ norm of any u, with $X = L^2(\mathbb{T}), L^\infty(\mathbb{T}), H^\mu(\mathbb{T}), \mathcal{L}(L^2(\mathbb{T}))$, etc.

Proposition C.2. *Assume the hypotheses of Proposition 5.1.*

(i) (Regularity) *In addition, suppose that $\mu > 1/2$, $\|c-1\|_{T,H^\mu} \le K < \infty$, and*

$$\mathcal{N}_\mu := \|c-1\|_{T,H^{\mu+r}} + \|V\|_{T,H^\mu} + \|\partial_t c\|_{T,H^\mu} + \|R_2\|_{T,\mathcal{L}(H^\mu)} < \infty.$$

Then R_3 maps $C^0([0,T_1];H^\mu(\mathbb{T}))$ into itself with

$$\|R_3\|_{T_1,\mathcal{L}(H^\mu)} \le C_{\mu,K}\mathcal{N}_\mu \tag{C.42}$$

for some constant $C_{\mu,K}$ depending on μ, K. For $\mu \ge 1$,

$$\|W\|_{T_1,H^\mu} \le C_\mu(\|c-1\|_{T,H^\mu} + \|\partial_t c\|_{T,H^{\mu-1}} + \|V\|_{T,H^\mu}) \tag{C.43}$$

and

$$\|\Phi u\|_{T_1,H^\mu} \le C_\mu\|c\|_{T,H^\mu}\|u\|_{T,H^\mu}, \qquad \|\Phi^{-1}u\|_{T,H^\mu} \le C_\mu\|c\|_{T,H^\mu}\|u\|_{T_1,H^\mu}$$

for all $u = u(t,x)$, for some constant C_μ depending only on μ.

(ii) (Stability) *Consider another triple (c', V', R_2') such that c' also satisfies (5.1), and $\mathcal{N}_0 < \infty$ also for (c', V', R_2'). Let $\Phi', \Psi_1', \varphi_*', \psi_*', T_1', W', R_3'$ be the corresponding objects for the triple (c', V', R_2'). Then for all $u \in L^2(\mathbb{T})$ and $t \in [0,T]$,*

$$\|\Psi_1(t)u - \Psi_1'(t)u\|_{L^2} + \|\Psi_1(t)^{-1}u - \Psi_1'(t)^{-1}u\|_{L^2} \le C\|c(t) - c'(t)\|_{L^2}\|u\|_{H^1}. \tag{C.44}$$

Let $\lambda := T_1/T_1'$, and let $\mathcal{T}$ be the time-rescaling operator $(\mathcal{T}v)(t,x) := v(\lambda t, x)$. Then for all $\mu \ge 0$ and $v = v(t,x)$,

$$\|\psi_*\varphi_* v - \psi_*'\varphi_*'(\mathcal{T}v)\|_{T,H^\mu} \le CT(\|\partial_t v\|_{T_1,H^\mu} + \|v\|_{T_1,H^{\mu+1}})\Delta_0, \tag{C.45}$$

$$\|\varphi_*'^{-1}\psi_*'^{-1}v - \mathcal{T}(\varphi_*^{-1}\psi_*^{-1}v)\|_{T_1',H^\mu} \le CT(\|\partial_t v\|_{T,H^\mu} + \|v\|_{T,H^{\mu+1}})\Delta_0, \tag{C.46}$$

where

$$\Delta_0 := \|c - c'\|_{T,H^1} + \|(\partial_t c - \partial_t c', V - V')\|_{T,L^2}.$$

Also,

$$|1-\lambda| + \|m-m'\|_{C^0([0,T])} \le C\|c-c'\|_{T,L^\infty}, \tag{C.47}$$

and if

$$M(x) := \{1+\partial_x\tilde{\beta}_1(T, x-p(T_1))\}^{1/2}, \qquad M'(x) := \{1+\partial_x\tilde{\beta}_1'(T, x-p'(T_1'))\}^{1/2},$$

then

$$\|M-M'\|_{L^\infty(\mathbb{T})} \le C(\|c-c'\|_{T,H^2} + \|(\partial_t c-\partial_t c', V-V')\|_{T,L^2}). \tag{C.48}$$

For $\mu \ge 1$, *if*

$$\|c-1\|_{T,H^{\mu+1}} + \|\partial_t c\|_{T,H^\mu} + \|\partial_t^2 c\|_{T,H^{\mu-1}} + \|V\|_{T,H^{\mu+1}} + \|\partial_t V\|_{T,H^\mu} \le 1, \tag{C.49}$$

and if (C.49) *also holds for* c', V', *then*

$$\|W'-\mathcal{T}W\|_{T_1',H^\mu} \le C_\mu(\|c-c'\|_{T,H^\mu} + \|\partial_t c-\partial_t c'\|_{T,H^{\mu-1}} + \|V-V'\|_{T,H^\mu}). \tag{C.50}$$

Moreover, if

$$\begin{aligned}\|c-1\|_{T,H^{r+1}} + \|\partial_t c\|_{T,H^{r+1}} + \|\partial_t^2 c\|_{T,L^2} + \|V\|_{T,H^1} + \|\partial_t V\|_{T,L^2} \\ + \|R_2\|_{T,\mathcal{L}(H^1)\cap\mathcal{L}(L^2)} + \|\partial_t R_2\|_{T,\mathcal{L}(L^2)} \le 1,\end{aligned} \tag{C.51}$$

and if (C.51) *also holds for* c', V', R_2', *then*

$$\begin{aligned}\|R_3'-\mathcal{T}R_3\|_{T_1',\mathcal{L}(L^2)} \le C(\|c-c'\|_{T,H^{r+1}} + \|\partial_t c-\partial_t c'\|_{T,H^1} \\ + \|V-V'\|_{T,H^1} + \|R_2-R_2'\|_{T,\mathcal{L}(L^2)}).\end{aligned} \tag{C.52}$$

Proof. To prove (ii) we make repeated use of the triangular inequality and explicit formulas. In particular, to estimate $p(\psi(\lambda t)) - p'(\psi'(t))$, we use explicit formulas similar to (C.41). To estimate $\tilde{R}_5' - \mathcal{T}\tilde{R}_5$ we note that the rescaled operator $\mathcal{T}\tilde{R}_5$ is the composition $\mathcal{T}\tilde{R}_5\mathcal{T}^{-1}$, and then we also use (C.45)–(C.46). Remember that we have renamed $W := a_7$ and $R_3 := \tilde{R}_5$. □

Acknowledgments. We would like to thank the referee for the careful reading of our manuscript.

T.A. was partly supported by the grant "ANAÉ" ANR-13-BS01-0010-03. This research was carried out in the framework of Programme STAR, financially supported by UniNA and Compagnia di San Paolo; it was partially supported by the European Research Council under FP7, and PRIN 2012 "Variational and perturbative aspects of nonlinear differential problems".

References

[1] Alazard, T., Baldi, P.: Gravity capillary standing water waves. Arch. Ration. Mech. Anal. **217**, 741–830 (2015) Zbl 1317.35181 MR 3356988

[2] Alazard, T., Burq, N., Zuily, C.: On the water-wave equations with surface tension. Duke Math. J. **158**, 413–499 (2011) Zbl 1258.35043 MR 2805065

[3] Alazard, T., Burq, N., Zuily, C.: Cauchy theory for the gravity water waves system with non-localized initial data. Ann. Inst. H. Poincaré Anal. Non Linéaire **33**, 337–395 (2016) Zbl 1339.35227 MR 3465379

[4] Alazard, T., Delort, J.: Sobolev estimates for two dimensional gravity water waves. Astérisque **374**, viii + 241 pp. (2015) Zbl 1360.35002 MR 3460636

[5] Alazard, T., Métivier, G.: Paralinearization of the Dirichlet to Neumann operator, and regularity of three-dimensional water waves. Comm. Partial Differential Equations **34**, 1632–1704 (2009) Zbl 1207.35082 MR 2581986

[6] Anantharaman, N., Macià, F.: Semiclassical measures for the Schrödinger equation on the torus. J. Eur. Math. Soc. **16**, 1253–1288 (2014) Zbl 1298.42028 MR 3226742

[7] Baldi, P.: Periodic solutions of fully nonlinear autonomous equations of Benjamin–Ono type. Ann. Inst. H. Poincaré Anal. Non Linéaire **30**, 33–77 (2013) Zbl 1285.35090 MR 3011291

[8] Baldi, P., Berti, M., Montalto, R.: KAM for quasi-linear and fully nonlinear forced perturbations of Airy equation. Math. Ann. **359**, 471–536 (2014) Zbl 1350.37076 MR 3201904

[9] Baldi, P., Berti, M., Montalto, R.: KAM for autonomous quasi-linear perturbations of KdV. Ann. Inst. H. Poincaré Anal. Non Linéaire **33**, 1589–1638 (2016) Zbl 1370.37134 MR 3569244

[10] Ball, J., Slemrod, M.: Nonharmonic Fourier series and the stabilization of distributed semi-linear control systems. Comm. Pure Appl. Math. **37**, 555–587 (1979) Zbl 0394.93041 MR 0528632

[11] Bardos, C., Lebeau, G., Rauch, J.: Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary. SIAM J. Control Optim. **30**, 1024–1065 (1992) Zbl 0786.93009 MR 1178650

[12] Beyer, K., Günther, M.: On the Cauchy problem for a capillary drop. I. Irrotational motion. Math. Methods Appl. Sci. **21**, 1149–1183 (1998) Zbl 0916.35141 MR 1637554

[13] Bony, J.-M.: Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires. Ann. Sci. École Norm. Sup. (4) **14**, 209–246 (1981) Zbl 0495.35024 MR 0631751

[14] Burq, N., Lebeau, G.: Mesures de défaut, application au système de Lamé. Ann. Sci. École Norm. Sup. (4) **34**, 817–870 (2001) Zbl 01750767 MR 1872422

[15] Christianson, H., Hur, V., Staffilani, G.: Strichartz estimates for the water-wave problem with surface tension. Comm. Partial Differential Equations **35**, 2195–2252 (2010) Zbl 1280.35107 MR 2763354

[16] Clamond, D., Fructus, D., Grue, J., Kristiansen, O.: An efficient model for three-dimensional surface wave simulations. Part II: Generation and absorption. J. Comput. Phys. **205**, 686–705 (2005) Zbl 1087.76016 MR 2134998

[17] Coron, J.-M.: Control and Nonlinearity. Math. Surveys Monogr. 136, Amer. Math. Soc., Providence, RI (2007) Zbl 1140.93002 MR 2302744

[18] Craig, W.: An existence theory for water waves and the Boussinesq and Korteweg–deVries scaling limits. Comm. Partial Differential Equations **10**, 787–1003 (1985) Zbl 0577.76030 MR 0795808

[19] Craig, W., Sulem, C.: Numerical simulation of gravity waves. J. Comput. Phys. **108**, 73–83 (1993) Zbl 0778.76072 MR 1239970

[20] Dehman, B., Lebeau, G.: Analysis of the HUM control operator and exact controllability for semilinear waves in uniform time. SIAM J. Control Optim. **48**, 521–550 (2009) Zbl 1194.35268 MR 2486082

[21] Germain, P., Masmoudi, N., Shatah, J.: Global solutions for capillary waves equation in dimension 3. Comm. Pure Appl. Math. **68**, 625–687 (2015) Zbl 1314.35100 MR 3318019

[22] Haraux, A.: Quelques méthodes et résultats récents en théorie de la contrôlabilité exacte. Rapport de recherche No. 1317, INRIA Rocquencourt (1990)

[23] Ifrim, M., Tataru, D.: The lifespan of small data solutions in two dimensional capillary water waves. Arch. Ration. Mech. Anal. **225**, 1279–1346 (2017) Zbl 1375.35347 MR 3667289

[24] Iguchi, T.: A long wave approximation for capillary-gravity waves and an effect of the bottom. Comm. Partial Differential Equations **32**, 37–85 (2007) Zbl 1136.35081 MR 2304142

[25] Ionescu, A., Pusateri, F.: Global analysis of a model for capillary water waves in two dimensions. Comm. Pure Appl. Math. **69**, 2015–2071 (2016) Zbl 1351.35117 MR 3552008

[26] Ionescu, A., Pusateri, F.: Global regularity for 2D water waves with surface tension. arXiv:1408.4428 (2014)

[27] Jaffard, S.: Contrôle interne exact des vibrations d'une plaque rectangulaire. Portugal. Math. **47**, 423–429 (1990) Zbl 0718.49026 MR 1090480

[28] Kahane, J.-P..: Pseudo-périodicité et séries de Fourier lacunaires. Ann. Sci. École Norm. Sup. **79**, 93–150 (1962) Zbl 0105.28601 MR 0154060

[29] Komornik, V., Loreti, P.: Fourier Series in Control Theory. Springer (2005) Zbl 1094.49002 MR 2114325

[30] Lannes, D.: Well-posedness of the water-waves equations. J. Amer. Math. Soc. **18**, 605–654 (2005) Zbl 1069.35056 MR 2138139

[31] Lannes, D.: The Water Waves Problem: Mathematical Analysis and Asymptotics. Math. Surveys Monogr. 188, Amer. Math. Soc. (2013) Zbl 06168816 MR 3060183

[32] Laurent, C.: Global controllability and stabilization for the nonlinear Schrödinger equation on an interval. ESAIM Control Optim. Calc. Var. **16**, 356–379 (2010) Zbl 1217.93032 MR 2654198

[33] Laurent, C., Linares, F., Rosier, L.: Control and stabilization of the Benjamin–Ono equation in $L^2(\mathbb{T})$. Arch. Ration. Mech. Anal. **218**, 1531–1575 (2015) Zbl 1325.35166 MR 3401014

[34] Laurent, C., Rosier, L., Zhang, B.-Y.: Control and stabilization of the Korteweg–de Vries equation on a periodic domain. Comm. Partial Differential Equations **35**, 707–744 (2010) Zbl 1213.93015 MR 2753618

[35] Linares, F., Ortega, J.: On the controllability and stabilization of the linearized Benjamin–Ono equation. ESAIM Control Optim. Calc. Var. **11**, 204–218 (2005) Zbl 1125.93006 MR 2141886

[36] Linares, F., Rosier, L.: Control and stabilization of the Benjamin–Ono equation on a periodic domain. Trans. Amer. Math. Soc. **367**, 4595–4626 (2015) Zbl 1310.93031 MR 3335395

[37] Lissy, P.: On the cost of fast controls for some families of dispersive or parabolic equations in one space dimension. SIAM J. Control Optim. **52**, 2651–2676 (2014) Zbl 1300.93039 MR 3252801

[38] Mésognon-Gireau, B.: The Cauchy problem on large time for the water waves equations with large topography variations. Adv. Differential Equations **22**, 457–504 (2017) Zbl 1372.35231 MR 3646468

[39] Métivier, G.: Para-differential Calculus and Applications to the Cauchy Problem for Nonlinear Systems. Centro di Ricerca Matematica Ennio De Giorgi Ser. 5, Edizioni della Normale, Pisa (2008) Zbl 1156.35002 MR 2418072

[40] Micu, S., Zuazua, E.: An introduction to the controllability of partial differential equations. In: T. Sari (ed.), Contrôle non linéaire et applications, Travaux en Cours 64, Hermann, Paris, 69–150 (2015) Zbl 1231.93042

[41] Miller, L.: Resolvent conditions for the control of unitary groups and their approximations. J. Spectr. Theory **2**, 1–55 (2012) Zbl 1239.93012 MR 2879308

[42] Ming, M., Rousset, F., Tzvetkov, N.: Multi-solitons and related solutions for the water-waves system. SIAM J. Math. Anal. **47**, 897–954 (2015) Zbl 1320.35284 MR 3315224

[43] Nalimov, V. I.: The Cauchy–Poisson problem. Dinamika Sploshn. Sredy **18**, 104–210, 254 (1974) (in Russian) MR 0609882

[44] Rosier, L.: Exact boundary controllability for the Korteweg–de Vries equation on a bounded domain. ESAIM Control Optim. Calc. Var. **2**, 33–55 (1997) Zbl 0873.93008 MR 1440078

[45] Reid, R. M.: Control time for gravity-capillary waves on water. SIAM J. Control Optim. **33**, 1577–1586 (1995) Zbl 0853.76010 MR 1348122

[46] Taylor, M. E.: Pseudodifferential Operators and Nonlinear PDE. Progr. Math. 100, Birkhäuser Boston, Boston, MA (1991) Zbl 0746.35062 MR 1121019

[47] Wehausen, J. V., Laitone, E. V.: Surface waves. In: Handbuch der Physik, Vol. 9, Part 3, Springer, Berlin, 446–778 (1960) Zbl 1339.76009 MR 0119656

[48] Wu, S.: Well-posedness in Sobolev spaces of the full water wave problem in 2-D. Invent. Math. **130**, 39–72 (1997) Zbl 0892.76009 MR 1471885

[49] Yosihara, H.: Gravity waves on the free surface of an incompressible perfect fluid of finite depth. Publ. RIMS Kyoto Univ. **18**, 49–96 (1982) Zbl 0493.76018 MR 0660822

[50] Zakharov, V. E.: Stability of periodic waves of finite amplitude on the surface of a deep fluid. J. Appl. Mech. Tech. Phys. **9**, 190–194 (1968)