J. Eur. Math. Soc. 20, 747–796
DOI 10.4171/JEMS/776

Tivadar Danka · Vilmos Totik

Christoffel functions with power type weights

Received April 17, 2015

Abstract. Precise asymptotics for Christoffel functions are established for power type weights on unions of Jordan curves and arcs. The asymptotics involve the equilibrium measure of the support of the measure. The result at the endpoints of arc components is obtained from the corresponding asymptotics for internal points with respect to a different power weight. On curve components the asymptotic formula is proved via a sharp form of Hilbert's lemniscate theorem while taking polynomial inverse images. The situation is completely different on arc components, where the local asymptotics is obtained via a discretization of the equilibrium measure with respect to the zeros of an associated Bessel function. The proofs are potential-theoretical, and fast decreasing polynomials play an essential role in them.

Keywords. Christoffel functions, asymptotics, power type weights, Jordan curves and arcs, Bessel functions, fast decreasing polynomials, equilibrium measures, Green's functions

---

Contents

V. Totik: Bolyai Institute, MTA-SZTE Analysis and Stochastics Research Group, University of Szeged, Aradi v. tere 1, Szeged, 6720, Hungary, and Department of Mathematics and Statistics, University of South Florida, 4202 E. Fowler Ave, CMC342, Tampa, FL 33620-5700, USA; e-mail: totik@mail.usf.edu

T. Danka: Potential Analysis Research Group, ERC Advanced Grant No. 267055, Bolyai Institute, University of Szeged, Aradi v. tere 1, Szeged, 6720, Hungary; e-mail: tivadar.danka@math.u-szeged.hu

Mathematics Subject Classification (2010): 26C05, 31A99, 41A10, 42C05

1. Introduction

Christoffel functions have been the subject of many papers (see e.g. [12], [13], [18], and the extended reference lists there). They are intimately connected with orthogonal polynomials, reproducing kernels, spectral properties of Jacobi matrices, convergence of orthogonal expansion and even random matrices (see [5], [13] and [18] for their various connections and applications). The possible applications are growing, for example recently a new domain recovery technique has been devised that uses the asymptotic behavior of Christoffel functions [6]; and in the last years several important methods for proving universality in random matrix theory were based on them [2], [8], [9], [10]. The aim of the present paper is to complete, to a certain extent, the investigations concerning their asymptotic behavior on Jordan curves and arcs.

Let μ be a finite Borel measure on the plane such that its support is compact and consists of infinitely many points. The *Christoffel functions* associated with μ are defined as

$$\lambda_n(\mu, z_0) = \inf_{P_n(z_0)=1} \int |P_n|^2 \, d\mu, \tag{1.1}$$

where the infimum is taken over all polynomials of degree at most n that take the value 1 at z. If $p_k(z) = p_k(\mu, z)$ denote orthonormal polynomials with respect to μ, i.e.

$$\int p_n \overline{p_m} \, d\mu = \delta_{n,m},$$

then λ_n can be expressed as

$$\lambda_n^{-1}(\mu, z) = \sum_{k=0}^{n} |p_k(z)|^2.$$

In other words, $\lambda^{-1}(\mu, z)$ is the diagonal of the reproducing kernel

$$K_n(z, w) = \sum_{k=0}^{n} p_k(z)\overline{p_k(w)},$$

which makes it an essential tool in many problems. It is easy to see that, with this reproducing kernel, the infimum in (1.1) is attained (only) for

$$P_n(z) = \frac{K_n(z, z_0)}{K_n(z_0, z_0)}$$

(see e.g. [20, Theorem 3.1.3]).

The earliest asymptotics for Christoffel functions for measures on the unit circle or on $[-1, 1]$ go back to Szegő [21, Th. I′, p. 461]. He established their behavior outside the support of the measure, and for some special cases he also found that behavior at points of $(-1, 1)$. The first result for a Jordan arc (a circular arc) was given in [4]. By now the asymptotic behavior of Christoffel functions for measures defined on unions of Jordan curves and arcs Γ is well understood: under certain assumptions we have, for points $z \in \Gamma$ that are different from the endpoints of the arc components of Γ,

$$\lim_{n\to\infty} n\lambda_n(\mu, z_0) = \frac{w(z_0)}{\omega_\Gamma(z_0)}, \tag{1.2}$$

where w is the density of μ with respect to the arc measure s_Γ on Γ, and ω_Γ is the density of the equilibrium measure (see below) with respect to s_Γ. For the most general results see [22] and [24].

What is left is to decide the asymptotic behavior at the endpoints of the arc components. It turns out that this problem is closely related to the asymptotic behavior away from the endpoints, but for measures of the form $d\mu(x) = |z - z_0|^\alpha ds_\Gamma(z)$, $\alpha > -1$, and the aim of this paper is to find these asymptotic behaviors. When μ is of the latter form, we shall show (for the exact formulation see the next section)

$$\lim_{n\to\infty} n^{1+\alpha}\lambda_n(\mu, z_0) = \frac{1}{(\pi\omega_\Gamma(z_0))^{\alpha+1}} 2^{\alpha+1}\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{\alpha+3}{2}\right) \tag{1.3}$$

when z_0 is not the endpoint of an arc component of Γ, while at an endpoint

$$\lim_{n\to\infty} n^{2\alpha+2}\lambda_n(\mu, z_0) = \frac{\Gamma(\alpha+1)\Gamma(\alpha+2)}{(\pi M(\Gamma, z_0))^{2\alpha+2}},$$

where $M(\Gamma, z_0)$ is the limit of $\sqrt{|z - z_0|}\,\omega_\Gamma(z)$ as $z \to z_0$ along Γ.

This paper uses some basic notions and results from potential theory. See [1], [3], [16] or [19] for all the concepts we use and for the basic theory. In particular, ν_Γ will denote the equilibrium measure of the compact set Γ.

Since the asymptotics reflect the support of the measure, in all such questions a global condition is needed, stating that the measure is not too small on any part of Γ (for example, if μ is zero on any arc of Γ, then (1.3) does not hold any more). This global condition is the regularity condition from [19]: we say that μ, with support Γ, belongs to **Reg** if

$$\sup_{P_n}\left(\frac{\|P_n\|_\Gamma}{\|P_n\|_{L^2(\mu)}}\right)^{1/n} \to 1$$

as $n \to \infty$, where the supremum is taken over all polynomials of degree at most n, and where $\|P_n\|_\Gamma$ denotes the supremum norm on Γ. The condition says that in the n-th root sense the $L^\infty(\mu)$ and $L^2(\mu)$ norms are almost the same. The assumption $\mu \in$ **Reg** is very weak—see [19] for several reformulations as well as conditions on the measure μ that imply $\mu \in$ **Reg**. For example, if Γ consists of rectifiable Jordan curves and arcs with arc measure s_Γ, then any measure $d\mu(z) = w(z)ds_\Gamma(z)$ with $w(z) > 0$ s_Γ-almost everywhere is regular in this sense.

Actually, it is not even needed that the support Γ of the measure μ be a system of Jordan curves or arcs: the main theorem below holds for any Γ that is a finite union of continua (connected compact sets). However, it is needed that z_0 lies on a smooth arc J of the outer boundary of Γ; the *outer boundary* of Γ is the boundary of the unbounded connected component of $\overline{\mathbb{C}} \setminus \Gamma$. It is known that the equilibrium measure ν_Γ lives on the outer boundary, and if J is a smooth (say C^1-smooth) arc on the outer boundary, then on J the equilibrium measure is absolutely continuous with respect to the arc measure s_J on J: $d\nu_\Gamma(z) = \omega_\Gamma(z)ds_J(z)$. We call this ω_Γ the *equilibrium density* of Γ.

The following theorem describes the asymptotics of the Christoffel function at points that are different from the endpoints of the arc components/parts of Γ (see Figure 1 for illustration).

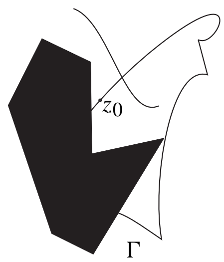

Fig. 1. A typical situation where Theorem 1.1 can be applied.

Theorem 1.1. *Let the support Γ of a measure $\mu \in$ **Reg** consist of finitely many continua, and let z_0 lie on the outer boundary of Γ. Assume that the intersection of Γ with a neighborhood of z_0 is a C^2-smooth arc J which contains z_0 in its (one-dimensional) interior. Assume also that in this neighborhood $d\mu(z) = w(z)|z - z_0|^\alpha ds_J(z)$, where w is a strictly positive continuous function and $\alpha > -1$. Then*

$$\lim_{n\to\infty} n^{1+\alpha}\lambda_n(\mu, z_0) = \frac{w(z_0)}{(\pi\omega_\Gamma(z_0))^{\alpha+1}} 2^{\alpha+1}\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{\alpha+3}{2}\right). \tag{1.4}$$

The second main theorem of this work is about the behavior of the Christoffel function at an endpoint (see Figure 2). If z_0 is an endpoint of a smooth arc J on the outer boundary

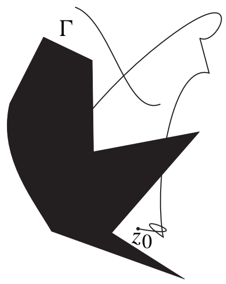

Fig. 2. A typical situation where Theorem 1.2 can be applied.

of Γ, then at z_0 the equilibrium density has a $1/\sqrt{|z-z_0|}$ behavior (see the proof of Theorem 1.2), and we set

$$M(\Gamma, z_0) := \lim_{z\to z_0,\, z\in\Gamma} \sqrt{|z-z_0|}\,\omega_\Gamma(z). \tag{1.5}$$

Theorem 1.2. *Let Γ and μ be as in Theorem 1.1, but now assume that the intersection of Γ with a neighborhood of z_0 is a C^2-smooth Jordan arc J with one endpoint at z_0. Then*

$$\lim_{n\to\infty} n^{2\alpha+2}\lambda_n(\mu, z_0) = \frac{w(z_0)}{(\pi M(\Gamma, z_0))^{2\alpha+2}}\Gamma(\alpha+1)\Gamma(\alpha+2). \tag{1.6}$$

These results can be used, in particular, if the measure is supported on a finite union of intervals on the real line, in which case the quantities $\omega_\Gamma(x)$ and $M(\Gamma, x)$ have a rather explicit form. Let $\Gamma = \bigcup_{j=0}^{k_0}[a_{2j}, a_{2j+1}]$ with disjoint $[a_{2j}, a_{2j+1}]$. Then the equilibrium density of Γ is (see e.g. [23, (40), (41)] or [19, Lemma 4.4.1])

$$\omega_\Gamma(x) = \frac{\prod_{j=0}^{k_0-1}|x-\lambda_j|}{\pi\sqrt{\prod_{j=0}^{2k_0+1}|x-a_j|}}, \qquad x \in \operatorname{Int}(\Gamma), \tag{1.7}$$

where λ_j are the solutions of the system of equations

$$\int_{a_{2k+1}}^{a_{2k+2}} \frac{\prod_{j=0}^{k_0-1}(t-\lambda_j)}{\sqrt{\prod_{j=0}^{2k_0+1}|t-a_j|}}\,dt = 0, \qquad k = 0, \dots, k_0-1. \tag{1.8}$$

It can be easily shown that these λ_j's are uniquely determined and there is one λ_j on every contiguous interval (a_{2j+1}, a_{2j+2}). Now if a is one of the endpoints of the intervals of Γ, say $a = a_{j_0}$, then

$$M(\Gamma, a) = \frac{\prod_{j=0}^{k_0-1}|a-\lambda_j|}{\pi\sqrt{\prod_{j=1,\, j\neq j_0}^{2k_0}|a-a_j|}}. \tag{1.9}$$

This whole work is dedicated to proving Theorems 1.1 and 1.2. Actually, the latter will be a relatively easy consequence of the former, so the main emphasis will be on proving

Theorem 1.1. The main line of reasoning is the following. We start from some known facts for simple measures like $|x|^\alpha dx$ on the real line, and get some elementary results for a model case on the unit circle via a transformation. Then we deduce from these simple cases that Theorem 1.1 is true for *lemniscate sets*, i.e. level sets of polynomials. This part will use the polynomial mapping in question to transform the already known result to the given lemniscate. Then we prove the theorem for finite unions of Jordan curves. Recall that a *Jordan curve* is a homeomorphic image of a circle, while a *Jordan arc* is a homeomorphic image of a segment. From the point of view of finding the asymptotics of Christoffel functions there is a big difference between arcs and curves: Jordan curves have interior and can be exhausted by lemniscates, so the polynomial inverse image method of [23] is applicable for them, while for Jordan arcs that method cannot be applied. Still, the pure Jordan curve case is used when we go over to a Γ which may have arc components, namely it is used in the lower estimate. The upper estimate is the most difficult part of the proof; there Bessel functions enter the picture, and a discretization technique is developed where the discretization of the equilibrium measure of Γ is done using the zeros of appropriate Bessel functions combined with another discretization based on uniform distribution. Once the cases of Jordan curves and arcs have been settled, the proof of Theorem 1.1 will easily follow by approximating a general Γ by a family of Jordan curves and arcs.

2. Tools

In what follows, $\|\cdot\|_K$ denotes the supremum norm on a set K, and s_Γ the arc measure on Γ (when Γ consists of smooth Jordan arcs or curves).

We shall rely on some basic notions and facts from logarithmic potential theory. See the books [1], [3], [16] or [17] for detailed discussion.

We shall often use the trivial fact that if μ, ν are Borel measures, then $\mu \le \nu$ implies $\lambda_n(\mu, x) \le \lambda_n(\nu, x)$ for all x. It is also trivial that $\lambda_n(\mu, z) \le \mu(\mathbb{C})$ (just use the identically 1 polynomial as a test function in the definition of $\lambda_n(\mu, z)$).

Another frequently used fact is the following: if $\{n_k\}$ is a subsequence of the natural numbers such that $n_{k+1}/n_k \to 1$ as $k \to \infty$, then for any $\kappa > 0$,

$$\liminf_{n\to\infty} n^\kappa \lambda_n(\mu, x) = \liminf_{k\to\infty} n_k^\kappa \lambda_{n_k}(\mu, x), \tag{2.1}$$

$$\limsup_{n\to\infty} n^\kappa \lambda_n(\mu, x) = \limsup_{k\to\infty} n_k^\kappa \lambda_{n_k}(\mu, x). \tag{2.2}$$

In fact, since $\lambda_n(\mu, x)$ is a decreasing function of n, for $n_k \le n \le n_{k+1}$ we have

$$(n/n_{k+1})^\kappa n_{k+1}^\kappa \lambda_{n_{k+1}}(\mu, x) \le n^\kappa \lambda_n(\mu, x) \le (n/n_k)^\kappa n_k^\kappa \lambda_{n_k}(\mu, x),$$

and both claims follow because n/n_k and n/n_{k+1} tend to 1 as n (or n_k) tends to infinity.

2.1. Fast decreasing polynomials

The following lemmas on the existence of fast decreasing polynomials will be a constant tool in the proofs.

Lemma 2.1. *Let K be a compact subset of $\mathbb{C}$, Ω the unbounded component of $\mathbb{C} \setminus K$ and let $z_0 \in \partial\Omega$. Suppose that there is a disk in Ω that contains z_0 on its boundary. Then, for every $\gamma > 1$, there are constants $c_\gamma, C_\gamma > 0$ and for every $n \in \mathbb{N}$ polynomials $S_{n,z_0,K}$ of degree at most n such that $S_{n,z_0,K}(z_0) = 1$, $|S_{n,z_0,K}(z)| \leq 1$ for all $z \in K$, and*

$$|S_{n,z_0,K}(z)| \leq C_\gamma e^{-nc_\gamma |z-z_0|^\gamma}, \quad z \in K. \tag{2.3}$$

For details, see [22, Theorem 4.1]. This result will often be used in the following form.

Corollary 2.2. *Under the assumptions of Lemma 2.1, for every $0 < \tau < 1$ there exist constants $c_\tau, C_\tau, \tau_0 > 0$ and for every $n \in \mathbb{N}$ a polynomial $S_{n,z_0,K}$ of degree $o(n)$ such that $S_{n,z_0,K}(z_0) = 1$, $|S_{n,z_0,K}(z)| \leq 1$ for all $z \in K$, and*

$$|S_{n,z_0,K}(z)| \leq C_\tau e^{-c_\tau n^{\tau_0}}, \quad |z - z_0| \geq n^{-\tau}. \tag{2.4}$$

Proof. Let $\varepsilon > 0$ be sufficiently small and select $\gamma > 1$ so that $1 - \varepsilon - \tau\gamma > 0$. Lemma 2.1 tells us that there is a polynomial P_n with $\deg(P_n) \leq n^{1-\varepsilon}$ such that

$$|P_n(z)| \leq C_\gamma e^{-c_\gamma n^{1-(\varepsilon+\tau\gamma)}}, \quad |z - z_0| \geq n^{-\tau},$$

and this proves the claim with $S_{n,z_0,K} = P_n$. □

There is a version of Lemma 2.1 where the decrease is not exponentially small, but starts much earlier than in Lemma 2.1.

Lemma 2.3 ([25, Lemma 4]). *Let K be as in Lemma 2.1. Then, for every $\beta < 1$, there are constants $c_\beta, C_\beta > 0$, and for every $n = 1, 2, \ldots$ polynomials P_n of degree at most n such that $P_n(z_0) = 1$, $|P_n(z)| \leq 1$ for $z \in K$, and*

$$|P_n(z)| \leq C_\beta e^{-c_\beta (n|z-z_0|)^\beta}, \quad z \in K. \tag{2.5}$$

It will be convenient to use these results when $n > 1$ is not necessarily an integer (formally one has to take the integer part of n, but the estimates will hold with possibly smaller constants in the exponents).

2.2. *Polynomial inequalities*

We shall also need some inequalities for polynomials that are used several times in the rest of the paper.

We start with a Bernstein-type inequality.

Lemma 2.4 ([22, Corollary 7.4]). *Let J be a C^2 closed Jordan arc and J_1 a closed subarc of J having no common endpoint with J. Then, for every $D > 0$, there is a constant $C_D > 0$ such that*

$$|P_n'(z)| \leq C_D n \|P_n\|_J, \quad \mathrm{dist}(z, J_1) \leq D/n,$$

for any polynomials P_n of degree $n = 1, 2, \ldots$.

Next, we continue with a Markov-type inequality.

Lemma 2.5. *Let K be a continuum. If Q_n is a polynomial of degree at most $n = 1, 2, \dots,$ then*

$$\|Q_n'\|_K \le \frac{e}{2\,\mathrm{cap}(K)} n^2 \|Q_n\|_K, \tag{2.6}$$

where cap(K) *denotes the logarithmic capacity of K. In particular, if K has diameter* 1, *then*

$$\|Q_n'\|_K \le 2en^2 \|Q_n\|_K. \tag{2.7}$$

For (2.6) see [15, Theorem 1], and for the last statement note that if K has diameter 1, then its capacity is at least $1/4$ [16, Theorem 5.3.2(a)].

Next, we prove a Remez-type inequality.

Lemma 2.6. *Let Γ be a C^1 Jordan curve or arc, and assume that for every $n = 1, 2, \dots,$ J_n is a subarc of Γ, and J_n^* is a subset of J_n such that*

$$s_\Gamma(J_n \setminus J_n^*) = o(n^{-2}) s_\Gamma(J_n),$$

where s_Γ denotes the arc-length measure on Γ. Then, for any sequence $\{Q_n\}$ of polynomials of degree at most $n = 1, 2, \dots,$ we have

$$\|Q_n\|_{J_n} = (1 + o(1)) \|Q_n\|_{J_n^*}. \tag{2.8}$$

Proof. It is clear from the C^1 property that $s_\Gamma(J_n) \sim \mathrm{diam}(J_n)$ uniformly in J_n (meaning that the ratio of the two sides lies between two positive constants).

Make a linear transformation $z \to Cz$ such that, after this transformation, the arc $\tilde{J}_n$ that we obtain from J_n has diameter 1. Under this transformation J_n^* goes into a subset $\tilde{J}_n^*$ of $\tilde{J}_n$ for which

$$s_{\tilde{J}_n}(\tilde{J}_n \setminus \tilde{J}_n^*) = o(n^{-2}) s_{\tilde{J}_n}(\tilde{J}_n), \tag{2.9}$$

and Q_n changes into a polynomial $\tilde{Q}_n$ of degree at most n. (2.8) is clearly equivalent to its $\tilde{}$-version.

Let $M = \|\tilde{Q}_n\|_{\tilde{J}_n}$. By Lemma 2.5, the absolute value of $\tilde{Q}_n'$ is bounded on $\tilde{J}_n$ by $2en^2 M$, hence if $z, w \in \tilde{J}_n$, then

$$|\tilde{Q}_n(z) - \tilde{Q}_n(w)| \le 2en^2 M s_{\tilde{J}_n}(\overline{zw}), \tag{2.10}$$

where $\overline{zw}$ is the arc of $\tilde{J}_n$ between z and w. By the assumption (2.9) for every $z \in \tilde{J}_n$ there is a $w \in \tilde{J}_n^*$ with

$$s_{\tilde{J}_n}(\overline{zw}) = o(n^{-2}) s_{\tilde{J}_n}(\tilde{J}_n) = o(n^{-2})$$

because $s_{\tilde{J}_n}(\tilde{J}_n) \sim \mathrm{diam}(\tilde{J}_n) = 1$. Choose here $z \in \tilde{J}_n$ such that $|\tilde{Q}_n(z)| = M$. Since $|\tilde{Q}_n(w)| \le \|\tilde{Q}_n\|_{\tilde{J}_n^*}$, from (2.10) we get

$$M = |\tilde{Q}_n(z)| \le \|\tilde{Q}_n\|_{\tilde{J}_n^*} + o(1) M,$$

and the claim follows. □

We shall frequently use the following so called Nikolskii-type inequalities for power type weights. Here, we say that a Jordan arc is C^{1+}-*smooth* if it is $C^{1+\theta}$-smooth for some $\theta > 0$.

Lemma 2.7. *Let J be a C^{1+}-smooth Jordan arc and let $J^* \subset J$ be a subarc of J which has no common endpoint with J. Let $z_0 \in J$ be a fixed point, and for $\alpha > -1$ define the measure ν_α on J by $d\nu_\alpha(u) = |u - z_0|^\alpha ds_J(u)$. Then there is a constant C depending only on α, J and J^* such that for any polynomials P_n of degree at most $n = 1, 2, \dots$ we have*

$$\|P_n\|_{J^*} \le Cn^{(1+\alpha)/2}\|P_n\|_{L^2(\nu_\alpha)} \quad \text{if } \alpha \ge 0, \tag{2.11}$$

$$\|P_n\|_{J^*} \le Cn^{1/2}\|P_n\|_{L^2(\nu_\alpha)} \quad \text{if } -1 < \alpha < 0. \tag{2.12}$$

The same is true if $d\nu_\alpha(u) = w(u)|u - z_0|^\alpha ds_J(u)$ with some strictly positive and continuous w.

Proof. In view of [26, Lemmas 3.8 and Corollary 3.9] (and the fact that ν_α is a doubling weight in the sense of [26]), uniformly in $z \in J^*$ we have for large n the relation

$$\lambda_n(\nu_\alpha, z) \sim \nu_\alpha(l_{1/n}(z)),$$

where $l_{1/n}(z)$ is the arc of J consisting of those points of J that lie at distance $\le 1/n$ from z. Then

$$\nu_\alpha(l_{1/n}(z)) \ge c/n^{1+\alpha} \quad \text{if } \alpha \ge 0,$$

$$\nu_\alpha(l_{1/n}(z)) \ge c/n \quad \text{if } -1 < \alpha < 0,$$

with some positive constant c which depends only on α, J and J^*. Therefore we have, for all $z \in J^*$,

$$\lambda_n(\nu_\alpha, z) \ge c/n^{1+\alpha} \quad \text{if } \alpha \ge 0, \tag{2.13}$$

$$\lambda_n(\nu_\alpha, z) \ge c/n \quad \text{if } -1 < \alpha < 0. \tag{2.14}$$

For example, (2.13) means that if $\alpha \ge 0$ and $|P_n(z)| = 1$ for some $z \in J^*$, then necessarily

$$\frac{n^{1+\alpha}}{c}\int_J |P_n|^2\, d\nu_\alpha \ge 1,$$

which is equivalent to saying that for any P_n and $z \in J^*$,

$$\frac{n^{1+\alpha}}{c}\int_J |P_n|^2\, d\nu_\alpha \ge |P_n(z)|^2,$$

and this is (2.11). In a similar manner, (2.12) follows from (2.14).

It is clear that this proof does not change if ν_α is as in the last sentence of the lemma. □

Lemma 2.8. *If $\alpha > -1$, then there is a constant C_α such that for any polynomial P_n of degree at most n,*

$$\|P_n\|_{[-1,1]} \leq C_\alpha n^{(1+\alpha^*)/2}\left(\int_{-1}^{1} |P_n(x)|^2|x|^\alpha\, dx\right)^{1/2} \quad \text{with } \alpha^* = \max(1,\alpha). \tag{2.15}$$

Proof. We follow the preceding proof, but now $J = J^* = [-1,1]$. Let $z_0 = 0$, $\Delta_n(z) = 1/n^2$ if $z \in [-1,-1+1/n^2]$ or $z \in [1-1/n^2, 1]$, and $\Delta_n(z) = \sqrt{1-z^2}/n$ if $z \in [-1+1/n^2, 1-1/n^2]$. If now $l_{1/n}(z) = [z-\Delta_n(z), z+\Delta_n(z)] \cap [-1,1]$, then [26, Lemmas 3.8 and Corollary 3.9] state that for $d\nu_\alpha(x) = |x-z_0|^\alpha dx = |x|^\alpha dx$ on $[-1,1]$ we have

$$\lambda_n(\nu_\alpha, z) \sim \nu_\alpha(l_{1/n}(z)).$$

Then

$$\nu_\alpha(l_{1/n}(z)) \geq c\min(1/n^2, 1/n^{1+\alpha}) \quad \text{if } \alpha \geq 0,$$
$$\nu_\alpha(l_{1/n}(z)) \geq c/n^2 \quad \text{if } -1 < \alpha < 0,$$

with some positive constant c. Hence,

$$\lambda_n(\nu_\alpha, z) \geq c/n^2 \quad \text{if } -1 < \alpha \leq 1,$$
$$\lambda_n(\nu_\alpha, z) \geq c/n^{1+\alpha} \quad \text{if } \alpha \geq 1,$$

from which (2.15) follows exactly as before. □

The Nikolskii inequalities can be combined with the following estimate to get an upper bound for the extremal polynomials that produce $\lambda_n(\mu, z)$.

Lemma 2.9. *Under the assumptions of Theorem 1.1 we have*

$$\lambda_n(\mu, z_0) \leq Cn^{-(\alpha+1)}$$

with some constant C that is independent of n.

Proof. Just use the polynomials P_n from Lemma 2.3 with $\beta = 1/2$ and $K = \Gamma$. Let $\delta > 0$ be so small that in the δ-neighborhood of z_0 we have $d\mu(z) = w(z)|z-z_0|^\alpha ds_\Gamma(z)$. Outside this δ-neighborhood, $|S_{n,z_0,\Gamma}|$ is smaller than $C_\beta \exp(-c_\beta(n\delta)^{1/2})$, so

$$\int |S_{n,z_0,\Gamma}|^2\, d\mu \leq C\int e^{-2c_\beta(n|t|)^{1/2}}|t|^\alpha\, dt + Ce^{-2c_\beta(n\delta)^{1/2}} \leq Cn^{-\alpha-1},$$

which proves the claim. □

We close this section with the classical Bernstein–Walsh lemma [27, p. 77].

Lemma 2.10. *Let $K \subset \mathbb{C}$ be a compact subset of positive logarithmic capacity, let Ω be the unbounded component of $\overline{\mathbb{C}} \setminus K$, and g_Ω the Green's function of this unbounded component with pole at infinity. Then, for polynomials P_n of degree at most $n = 1, 2, \ldots$, and for any $z \in \mathbb{C}$,*

$$|P_n(z)| \leq e^{ng_\Omega(z)}\|P_n\|_K.$$

3. The model cases

3.1. Measures on the real line

Our first goal is to establish asymptotics for the Christoffel function at 0 with respect to the measure $d\mu(x) = |x|^\alpha dx$, $x \in [-1, 1]$. We do this by transforming some previously known results.

In what follows, for simpler notation, if $d\mu(x) = w(x)dx$, then we write $\lambda_n(w(x), z)$ for $\lambda_n(\mu, z)$.

Proposition 3.1. *For $\alpha > -1$ we have*

$$\lim_{n\to\infty} n^{2\alpha+2}\lambda_n\big(|x|^\alpha\big|_{[0,1]}, 0\big) = \Gamma(\alpha+1)\Gamma(\alpha+2). \tag{3.1}$$

Proof. It follows from [10, (1.10)] or [9, Theorem 4.1] that

$$\lim_{n\to\infty} n^{2\alpha+2}\lambda_n\big((1-x)^\alpha\big|_{[-1,1]}, 1\big) = 2^{\alpha+1}\Gamma(\alpha+1)\Gamma(\alpha+2), \tag{3.2}$$

from which the claim is an immediate consequence if we apply the linear transformation $x \to (1-x)/2$. □

Proposition 3.2. *For $\alpha > -1$ we have*

$$\lim_{n\to\infty} n^{\alpha+1}\lambda_n\big(|x|^\alpha\big|_{[-1,1]}, 0\big) = L_\alpha, \tag{3.3}$$

where

$$L_\alpha := 2^{\alpha+1}\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{\alpha+3}{2}\right). \tag{3.4}$$

Proof. In this proof, whenever we write P_n, R_n etc. for polynomials, it is understood that the degree is at most n.

We use the fact that (for continuous f)

$$\int_0^1 f(x)|x|^\alpha\, dx = \int_{-1}^1 f(x^2)|x|^{2\alpha+1}\, dx. \tag{3.5}$$

Assume first that P_{2n} is extremal for $\lambda_{2n}\big(|x|^\alpha\big|_{[-1,1]}, 0\big)$, i.e. $P_{2n}(0) = 1$ and

$$\int_{-1}^1 |P_{2n}(x)|^2|x|^\alpha\, dx = \lambda_{2n}\big(|x|^\alpha\big|_{[-1,1]}, 0\big).$$

Define

$$R_{2n}(x) = \frac{P_{2n}(x) + P_{2n}(-x)}{2}.$$

Then $R_{2n}(0) = 1$, and R_{2n} is a polynomial in x^2, hence $R_{2n}(x) = R_n^*(x^2)$ with some polynomial R_n^* for which $R_n^*(0) = 1$ and $\deg(R_n^*) \le n$. Now we have

$$\int_{-1}^1 |R_{2n}(x)|^2|x|^\alpha\, dx = \int_{-1}^1 |R_n^*(x^2)|^2|x|^\alpha\, dx = \int_0^1 |R_n^*(x)|^2|x|^{(\alpha-1)/2}\, dx$$
$$\ge \lambda_n\big(|x|^{(\alpha-1)/2}\big|_{[0,1]}, 0\big).$$

By the Cauchy–Schwarz inequality and the symmetry of the measure $|x|^\alpha\,dx$,

$$\begin{aligned}\int_{-1}^{1}|R_{2n}(x)|^2|x|^\alpha\,dx &\le \frac{1}{4}\int_{-1}^{1}(|P_{2n}(x)|^2+2|P_{2n}(x)|\,|P_{2n}(-x)|+|P_{2n}(-x)|^2)|x|^\alpha\,dx\\ &\le \frac{1}{2}\int_{-1}^{1}|P_{2n}(x)|^2|x|^\alpha\,dx\\ &\quad+\frac{1}{2}\left(\int_{-1}^{1}|P_{2n}(x)|^2|x|^\alpha\,dx\right)^{1/2}\left(\int_{-1}^{1}|P_{2n}(-x)|^2|x|^\alpha\,dx\right)^{1/2}\\ &=\int_{-1}^{1}|P_{2n}(x)|^2|x|^\alpha\,dx=\lambda_{2n}\big(|x|^\alpha\big|_{[-1,1]},0\big).\end{aligned}$$

Combining these two estimates, we obtain

$$\lambda_n\big(|x|^{(\alpha-1)/2}\big|_{[0,1]},0\big)\le\lambda_{2n}\big(|x|^\alpha\big|_{[-1,1]},0\big).$$

On the other hand, if now P_n is extremal for $\lambda_n\big(|x|^{(\alpha-1)/2}\big|_{[0,1]},0\big)$, then

$$\begin{aligned}\lambda_n\big(|x|^{(\alpha-1)/2}\big|_{[0,1]},0\big)=\int_0^1|P_n(x)|^2|x|^{(\alpha-1)/2}\,dx&=\int_{-1}^{1}|P_n(x^2)|^2|x|^\alpha\,dx\\ &\ge\lambda_{2n}\big(|x|^\alpha\big|_{[-1,1]},0\big),\end{aligned}$$

therefore we actually have

$$\lambda_n\big(|x|^{(\alpha-1)/2}\big|_{[0,1]},0\big)=\lambda_{2n}\big(|x|^\alpha\big|_{[-1,1]},0\big),\tag{3.6}$$

from which the claim follows via Proposition 3.1 (see also (2.1) and (2.2) with $n_k=2k$).

Note also that this proves that if $P_n(x)$ is the n-th degree extremal polynomial for the measure $|x|^{(\alpha-1)/2}\big|_{[0,1]}dx$, then $P_n(x^2)$ is the $2n$-th degree extremal polynomial for the measure $|x|^\alpha\big|_{[-1,1]}dx$. □

3.2. *Measures on the unit circle*

Let $\mu_{\mathbb{T}}$ be the measure on the unit circle $\mathbb{T}$ defined by $d\mu_{\mathbb{T}}(e^{it})=w_{\mathbb{T}}(e^{it})dt$, where

$$w_{\mathbb{T}}(e^{it})=\frac{|e^{2it}+1|^\alpha}{2^\alpha}\,\frac{|e^{2it}-1|}{2},\qquad t\in[-\pi,\pi).\tag{3.7}$$

We shall prove

$$\lim_{n\to\infty}n^{\alpha+1}\lambda_n(\mu_{\mathbb{T}},e^{i\pi/2})=2^{\alpha+1}L_\alpha\tag{3.8}$$

where L_α is from (3.4), by transforming the measure $\mu_{\mathbb{T}}$ into a measure $\mu_{[-1,1]}$ supported on $[-1,1]$ and comparing the Christoffel functions for them. With the transformation $e^{it}\to\cos t$, we have

$$\int_{-\pi}^{\pi}f(\cos t)w_{\mathbb{T}}(e^{it})\,dt=2\int_{-1}^{1}f(x)w_{[-1,1]}(x)\,dx,\quad\text{where}\quad w_{[-1,1]}(x)=|x|^\alpha.$$

Set $d\mu_{[-1,1]}(x)=w_{[-1,1]}(x)dx$.

Let P_n be the extremal polynomial for $\lambda_n(\mu_{[-1,1]}, 0)$ and define

$$S_n(e^{it}) = P_n(\cos t)\left(\frac{1+e^{i(t-\pi/2)}}{2}\right)^{\lfloor \eta n\rfloor} e^{in(t-\pi/2)},$$

where $\eta > 0$ is arbitrary. This S_n is a polynomial of degree $2n+\lfloor \eta n\rfloor$ with $S_n(e^{i\pi/2}) = 1$. For any fixed $0 < \delta < 1$,

$$\begin{aligned}\int_{\pi/2-\delta}^{\pi/2+\delta} |S_n(e^{it})|^2 w_{\mathbb{T}}(e^{it}) dt &\le \int_{\pi/2-\delta}^{\pi/2+\delta} |P_n(\cos t)|^2 w_{\mathbb{T}}(e^{it})\, dt \\ &\le \int_{-1}^{1} |P_n(x)|^2 w_{[-1,1]}(x)\, dx = \lambda_n(\mu_{[-1,1]}, 0). \qquad (3.9)\end{aligned}$$

To estimate the integral over $[-\pi, \pi/2-\delta] \cup [\pi/2+\delta, \pi]$, notice that

$$\max_{t\in[-\pi,\pi]\setminus[\pi/2-\delta,\pi/2+\delta]} \left|\frac{1+e^{i(t-\pi/2)}}{2}\right|^{\lfloor \eta n\rfloor} = O(q^n) \qquad (3.10)$$

for some $q < 1$. From Lemma 2.8 we obtain

$$\|P_n\|_{[-1,1]} \le Cn^{1+|\alpha|/2} \|P_n\|_{L^2(\mu_{[-1,1]})} \le Cn^{1+|\alpha|/2},$$

and so

$$\left(\int_{-\pi}^{\pi/2-\delta} + \int_{\pi/2+\delta}^{\pi}\right) |S_n(e^{it})|^2 w_{\mathbb{T}}(e^{it})\, dt = O(n^{1+|\alpha|/2} q^n) = o(n^{-\alpha-1}).$$

Therefore, using this S_n as a test polynomial for $\lambda_{\deg(S_n)}(\mu_{\mathbb{T}}, e^{i\pi/2})$ we conclude that

$$\lambda_{\deg(S_n)}(\mu_{\mathbb{T}}, e^{i\pi/2}) \le \lambda_n(\mu_{[-1,1]}, 0) + o(n^{-\alpha-1}),$$

and so

$$\begin{aligned}&\limsup_{n\to\infty} (2n + \lfloor \eta n\rfloor)^{\alpha+1} \lambda_{2n+\lfloor \eta n\rfloor}(\mu_{\mathbb{T}}, e^{i\pi/2}) \\ &\qquad \le \limsup_{n\to\infty} (2 + \lfloor \eta n\rfloor/n)^{\alpha+1} n^{\alpha+1} \lambda_n(\mu_{[-1,1]}, 0) = (2+\eta)^{\alpha+1} L_\alpha,\end{aligned}$$

where we have used Proposition 3.2 for the measure $\mu_{[-1,1]}$.

Since $\eta > 0$ was arbitrary, we obtain

$$\limsup_{n\to\infty} n^{\alpha+1} \lambda_n(\mu_{\mathbb{T}}, e^{i\pi/2}) \le 2^{\alpha+1} L_\alpha \qquad (3.11)$$

(see also (2.2)).

Now to prove the matching lower estimate, let $S_{2n}(e^{it})$ be the extremal polynomial for $\lambda_{2n}(\mu_{\mathbb{T}}, e^{i\pi/2})$. Define

$$P_n^*(e^{it}) = S_{2n}(e^{it}) \left(\frac{1+e^{i(t-\pi/2)}}{2}\right)^{2\lfloor \eta n\rfloor} e^{-(n+\lfloor \eta n\rfloor)i(t-\pi/2)}$$

and $P_n(\cos t) = P_n^*(e^{it}) + P_n^*(e^{-it})$. Note that $P_n(\cos t)$ is a polynomial in $\cos t$ with $\deg(P_n) \le n + \lfloor \eta n \rfloor$ and $P_n(0) = 1$. Hence

$$\lambda_{\deg(P_n)}(\mu_{[-1,1]}, 0) \le \int_{-1}^{1} |P_n(x)|^2 w_{[-1,1]}(x)\, dx = \frac{1}{2}\int_{-\pi}^{\pi} |P_n(\cos t)|^2 w_{\mathbb{T}}(e^{it})\, dt. \tag{3.12}$$

First, we claim that for every fixed $0 < \delta < 1$,

$$|P_n(\cos t)|^2 = \begin{cases} |P_n^*(e^{it})|^2 + O(q^n), & t \in [\pi/2 - \delta, \pi/2 + \delta], \\ |P_n^*(e^{-it})|^2 + O(q^n), & t \in [-\pi/2 - \delta, -\pi/2 + \delta], \\ O(q^n), & \text{otherwise}, \end{cases} \tag{3.13}$$

for some $q < 1$. Indeed,

$$|P_n(\cos t)|^2 = |P_n^*(e^{it}) + P_n^*(e^{-it})|^2 \le |P_n^*(e^{it})|^2 + 2|P_n^*(e^{it})|\,|P_n^*(e^{-it})| + |P_n^*(e^{-it})|^2.$$

If we apply Lemma 2.7 to two subarcs (say of length $5\pi/4$) of $\mathbb{T}$ that contain the upper, resp. the lower half of the unit circle, then we obtain

$$\|P_n^*\|_{\mathbb{T}} \le \|S_{2n}\|_{\mathbb{T}} \le Cn^{(1+|\alpha|)/2}\|S_{2n}\|_{L^2(\mu_{\mathbb{T}})} \le Cn^{(1+|\alpha|)/2}.$$

Therefore (use (3.10))

$$|P_n^*(e^{it})| \le Cq^n n^{(1+|\alpha|)/2}, \quad t \in [-\pi, \pi] \setminus [\pi/2 - \delta, \pi/2 + \delta].$$

These imply (3.13).

Now we have

$$\int_{-\pi}^{\pi} |P_n(\cos t)|^2 w_{\mathbb{T}}(e^{it})\, dt = \left(\int_{\pi/2-\delta}^{\pi/2+\delta} + \int_{-\pi/2-\delta}^{-\pi/2+\delta}\right)|P_n(\cos t)|^2 w_{\mathbb{T}}(e^{it})\, dt$$
$$+ \left(\int_{-\pi}^{-\pi/2-\delta} + \int_{-\pi/2+\delta}^{\pi/2-\delta} + \int_{\pi/2+\delta}^{\pi}\right)|P_n(\cos t)|^2 w_{\mathbb{T}}(e^{it})\, dt.$$

(3.13) tells us that the last three terms are $O(q^n)$. For the other two terms we have, again by (3.13),

$$\int_{\pi/2-\delta}^{\pi/2+\delta} |P_n(\cos t)|^2 w_{\mathbb{T}}(e^{it})\, dt = \int_{\pi/2-\delta}^{\pi/2+\delta} |P_n^*(e^{it})|^2 w_{\mathbb{T}}(e^{it})\, dt + O(q^n)$$
$$\le \int_{\pi/2-\delta}^{\pi/2+\delta} |S_{2n}(e^{it})|^2 w_{\mathbb{T}}(e^{it})\, dt + O(q^n) \le \lambda_{2n}(\mu_{\mathbb{T}}, e^{i\pi/2}) + O(q^n),$$

and similarly

$$\int_{-\pi/2-\delta}^{-\pi/2+\delta} |P_n(\cos t)|^2 w_{\mathbb{T}}(e^{it})\, dt \le \lambda_{2n}(\mu_{\mathbb{T}}, e^{i\pi/2}) + O(q^n).$$

Combining these estimates with (3.12), we conclude that

$$\lambda_{\deg(P_n)}(\mu_{[-1,1]}, 0) \le \lambda_{2n}(\mu_{\mathbb{T}}, e^{i\pi/2}) + O(q^n),$$

therefore

$$\liminf_{n\to\infty} \deg(P_n)^{\alpha+1}\lambda_{\deg(P_n)}(\mu_{[-1,1]},0)$$
$$\le \liminf_{n\to\infty}(n+\lfloor \eta n\rfloor)^{\alpha+1}\big(\lambda_{2n}(\mu_{\mathbb{T}},e^{i\pi/2})+O(q^n)\big)$$
$$\le \liminf_{n\to\infty}(1+\lfloor \eta n\rfloor/n)^{\alpha+1}\frac{1}{2^{\alpha+1}}(2n)^{\alpha+1}\lambda_{2n}(\mu_{\mathbb{T}},e^{i\pi/2}).$$

From this, in view of Proposition 3.2 and (2.1), it follows that

$$(1+\eta)^{-(\alpha+1)}2^{\alpha+1}L_\alpha \le \liminf_{n\to\infty}\lambda_n(\mu_{\mathbb{T}},e^{i\pi/2}),$$

and upon letting $\eta \to 0$ we obtain

$$2^{\alpha+1}L_\alpha \le \liminf_{n\to\infty}\lambda_n(\mu_{\mathbb{T}},e^{i\pi/2}). \tag{3.14}$$

This and (3.11) yield (3.8).

Finally, let

$$d\mu_\alpha(e^{it}) = |e^{it}-i|^\alpha\,dt.$$

Let us write $|e^{it}-i|^\alpha$ in the form

$$|e^{it}-i|^\alpha = w(e^{it})w_{\mathbb{T}}(e^{it}).$$

Then w is continuous in a neighborhood of $e^{i\pi/2}$ and it has value 1 at $e^{i\pi/2}$. Let $\tau > 0$ be arbitrary, and choose $0<\delta<1$ in such a way that

$$\frac{1}{1+\tau} \le w(e^{it}) \le 1+\tau, \qquad t\in[\pi/2-\delta,\pi/2+\delta].$$

If we now carry out the preceding arguments with δ and with μ_α replacing $\mu_{\mathbb{T}}$ everywhere, then we find that in (3.11) the limsup is at most $(1+\tau)2^{\alpha+1}L_\alpha$, while in (3.14) the liminf is at least $(1+\tau)^{-1}2^{\alpha+1}L_\alpha$. Since $\tau>0$ is arbitrary, this shows that

$$\lim_{n\to\infty} n^{\alpha+1}\lambda_n(\mu_\alpha,e^{i\pi/2}) = 2^{\alpha+1}L_\alpha. \tag{3.15}$$

This result will serve as our model case in the proof of Theorem 1.1.

4. Lemniscates

In this section, we prove Theorem 1.1 for lemniscates.

Let $\sigma = \{z\in\mathbb{C} : |T_N(z)| = 1\}$ be the level line of a polynomial T_N, and assume that σ has no self-intersections. Let $\deg(T_N) = N$.

The normal derivative of the Green's function with pole at infinity of the outer domain to σ at a point $z\in\sigma$ is $|T_N'(z)|/N$ [22, (2.2)], and since this normal derivative is 2π-times the equilibrium density of σ ([14, II.(4.1)] or [17, Theorem IV.2.3 and (I.4.8)]), the equilibrium density on σ has the form

$$\omega_\sigma(z) = \frac{|T_N'(z)|}{2\pi N}. \tag{4.1}$$

If $z \in \sigma$, then there are n points $z_1, \dots, z_n \in \sigma$ with $T_N(z) = T_n(z_k)$, and for them [22, (2.12)]

$$\int_\sigma \Big(\sum_{i=1}^N f(z_i)\Big)|T_N'(z)|\,ds_\sigma(z) = N\int_\sigma f(z)|T_N'(z)|\,ds_\sigma(z). \tag{4.2}$$

Furthermore, if $g : \mathbb{T} \to \mathbb{C}$ is arbitrary, then [22, (2.14)]

$$\int_\sigma g(T_N(z))|T_N'(z)|\,ds_\sigma(z) = N\int_0^{2\pi} g(e^{it})\,dt. \tag{4.3}$$

Let $z_0 \in \sigma$, and define

$$d\mu_\sigma(z) = |z - z_0|^\alpha ds_\sigma(z), \quad \alpha > -1, \tag{4.4}$$

where s_σ denotes the arc-length measure on σ. Without loss of generality we may assume that $T_N(z_0) = e^{i\pi/2}$. Our plan is to compare the Christoffel functions for the measure μ_σ with those for the measure μ_α supported on the unit circle and defined via

$$d\mu_\alpha(e^{it}) = |e^{it} - e^{i\pi/2}|^\alpha ds_{\mathbb{T}}(e^{it}), \tag{4.5}$$

and for which the asymptotics of the Christoffel functions was calculated in (3.15).

We shall prove that

$$\lim_{n\to\infty} n^{\alpha+1}\lambda_n(\mu_\sigma, z_0) = \frac{L_\alpha}{(\pi\omega_\sigma(z_0))^{\alpha+1}} \tag{4.6}$$

where L_α is taken from (3.4).

4.1. The upper estimate

Let $\eta > 0$ be an arbitrarily small number, and select a $\delta > 0$ such that for every z with $|z - z_0| < \delta$, we have

$$\begin{aligned} \frac{1}{1+\eta}|T_N'(z_0)| \le |T_N'(z)| \le (1+\eta)|T_N'(z_0)|, \\ \frac{1}{1+\eta}|T_N'(z_0)|\,|z-z_0| \le |T_N(z) - T_N(z_0)| \le (1+\eta)|T_N'(z_0)|\,|z-z_0| \end{aligned} \tag{4.7}$$

(note that $T_N'(z_0) \neq 0$ because σ has no self-intersections). Let Q_n be the extremal polynomial for $\lambda_n(\mu_\alpha, e^{i\pi/2})$, where μ_α is from (4.5). Define

$$R_n(z) = Q_n(T_N(z))S_{n,z_0,L}(z),$$

where $S_{n,z_0,L}$ is the fast decreasing polynomial given by Corollary 2.2 for the lemniscate set L enclosed by σ (and for any fixed $0 < \tau < 1$ in Corollary 2.2). Note that R_n is a polynomial of degree $nN + o(n)$ with $R_n(z_0) = 1$. Since $S_{n,z_0,L}$ is fast decreasing, we have

$$\sup_{z\in L\setminus\{t:\,|t-z_0|<\delta\}} |S_{n,z_0,L}(z)| = O(q^{n^{\tau_0}})$$

for some $q < 1$ and $\tau_0 > 0$. The Nikolskii-type inequality of Lemma 2.7, when applied to two subarcs of $\mathbb{T}$ which contain the upper resp. lower part of the unit circle, yields

$$\|Q_n\|_{\mathbb{T}} \le Cn^{(1+|\alpha|)/2}\|Q_n\|_{L^2(\mu_\alpha)} \le Cn^{(1+|\alpha|)/2}.$$

Therefore,

$$\sup_{z\in L\setminus\{t:\,|t-z_0|<\delta\}} |R_n(z)| = O(q^{n^{\tau_0/2}}).$$

It follows that

$$\int_{|z-z_0|\ge\delta} |R_n(z)|^2|z-z_0|^\alpha\, ds_\sigma(z) = O(q^{n^{\tau_0/2}}). \tag{4.8}$$

Using (4.7), we have

$$\begin{aligned}
\int_{|z-z_0|<\delta} |R_n(z)|^2|z-z_0|^\alpha\, ds_\sigma(z) &\le \int_{|z-z_0|<\delta} |Q_n(T_N(z))|^2|z-z_0|^\alpha\, ds_\sigma(z)\\
&\le \frac{(1+\eta)^{|\alpha|+1}}{|T_N'(z_0)|^{\alpha+1}}\int_{|z-z_0|<\delta} |Q_n(T_N(z))|^2|T_N(z)-T_N(z_0)|^\alpha|T_N'(z)|\, ds_\sigma(z)\\
&\le \frac{(1+\eta)^{|\alpha|+1}}{|T_N'(z_0)|^{\alpha+1}}\int_0^{2\pi} |Q_n(e^{it})|^2|e^{it}-e^{i\pi/2}|^\alpha\, dt = (1+\eta)^{|\alpha|+1}\frac{\lambda_n(\mu_\alpha, e^{i\pi/2})}{|T_N'(z_0)|^{\alpha+1}}.
\end{aligned}$$

This and (4.8) imply

$$\lambda_{\deg(R_n)}(\mu_\sigma, z_0) \le (1+\eta)^{|\alpha|+1}\frac{\lambda_n(\mu_\alpha, e^{i\pi/2})}{|T_N'(e^{i\pi/2})|^{\alpha+1}} + O(q^{n^{\tau_0/2}}),$$

from which

$$\begin{aligned}
&\limsup_{n\to\infty} \deg(R_n)^{\alpha+1}\lambda_{\deg(R_n)}(\mu_\sigma, z_0)\\
&\le \limsup_{n\to\infty} (nN+o(n))^{\alpha+1}(1+\eta)^{|\alpha|+1}\frac{\lambda_n(\mu_\alpha, e^{i\pi/2})}{|T_N'(z_0)|^{\alpha+1}} = (1+\eta)^{|\alpha|+1}\frac{N^{\alpha+1}}{|T_N'(z_0)|^{\alpha+1}}2^{\alpha+1}L_\alpha,
\end{aligned}$$

where we have used (3.15). Since $\eta > 0$ is arbitrary, from (4.1) (using also (2.2)) we obtain

$$\limsup_{n\to\infty} n^{\alpha+1}\lambda_n(\mu_\sigma, z_0) \le \frac{N^{\alpha+1}}{|T_N'(z_0)|^{\alpha+1}}2^{\alpha+1}L_\alpha = \frac{L_\alpha}{(\pi\omega_\sigma(z_0))^{\alpha+1}}. \tag{4.9}$$

4.2. *The lower estimate*

Let P_n be the extremal polynomial for $\lambda_n(\mu_\sigma, z_0)$, and let $S_{n,z_0,L}$ be the fast decreasing polynomial given by Corollary 2.2 for the closed lemniscate domain L enclosed by σ (with some fixed $\tau < 1$). As before, from Lemma 2.7 we obtain

$$\|P_n\|_\sigma = O(n^{(1+|\alpha|)/2}). \tag{4.10}$$

Define $R_n(z) = P_n(z)S_{n,z_0,L}(z)$; it is a polynomial of degree $n + o(n)$ and $R_n(z_0) = 1$. Similarly to the previous section, we have

$$\sup_{z\in L\setminus\{t:\,|t-z_0|<\delta\}} |R_n(z)| = O(q^{n^{\tau_0/2}}) \tag{4.11}$$

for some $q < 1$ and $\tau_0 > 0$. Since the expression $\sum_{k=1}^{N} R_n(z_k)$, where $\{z_1, \dots, z_N\} = T_N^{-1}(T_N(z))$, is symmetric in the variables z_k, it is a polynomial in their elementary symmetric polynomials. For more details on this idea, see [23]. Therefore, there is a polynomial Q_n of degree at most $\deg(R_n)/N = (n + o(n))/N$ such that

$$Q_n(T_N(z)) = \sum_{k=1}^{N} R_n(z_k), \qquad z \in \sigma.$$

We claim that for every $z \in \sigma$,

$$|Q_n(T_N(z))|^2 \le \sum_{k=1}^{N} |R_n(z_k)|^2 + O(q^{n^{\tau_0/2}}). \tag{4.12}$$

Indeed, since σ has no self-intersection, $|z_k - z_l|$ cannot be arbitrarily small for distinct k and l. As a consequence, for every z, at most one z_j belongs to the set $\{z : |z - z_0| < \delta\}$ if δ is sufficiently small, and hence, in the sum

$$|Q_n(T_N(z))|^2 \le \sum_{k=1}^{N}\sum_{l=1}^{N} |R_n(z_k)|\,|R_n(z_l)|,$$

every term with $k \ne l$ is $O(q^{n^{\tau_0/2}})$ (use (4.10) and (4.11)).

Now let $\delta > 0$ be so small that for every z with $|z - z_0| < \delta$ the inequalities in (4.7) hold. Then (4.2) and (4.12) give (note that $T_N(z) = T_N(z_k)$ for all k)

$$\begin{aligned}
&\int_\sigma |Q_n(T_N(z))|^2 |T_N'(z)|\,|T_N(z) - T_N(z_0)|^\alpha \, ds_\sigma(z) \\
&\quad\le O(q^{n^{\tau_0/2}}) + \int_\sigma \Big(\sum_{k=1}^{N} |R_n(z_k)|^2\Big)|T_N'(z)|\,|T_N(z) - T_N(z_0)|^\alpha \, ds_\sigma(z) \\
&\quad= O(q^{n^{\tau_0/2}}) + \int_\sigma \Big(\sum_{k=1}^{N} |R_n(z_k)|^2 |T_N(z_k) - T_N(z_0)|^\alpha\Big)|T_N'(z)|\, ds_\sigma(z) \\
&\quad= O(q^{n^{\tau_0/2}}) + N\int_\sigma |R_n(z)|^2 |T_N(z) - T_N(z_0)|^\alpha |T_N'(z)|\, ds_\sigma(z) \\
&\quad\le O(q^{n^{\tau_0/2}}) + (1+\eta)^{|\alpha|+1}|T_N'(z_0)|^{\alpha+1} N \int_{|z-z_0|<\delta} |P_n(z)|^2 |z - z_0|^\alpha \, ds_\sigma \\
&\quad\le O(q^{n^{\tau_0/2}}) + (1+\eta)^{|\alpha|+1}|T_N'(z_0)|^{\alpha+1} N \lambda_n(\mu_\sigma, z_0).
\end{aligned}$$

Since $Q_n(T_N(z_0)) = 1 + o(1)$, from (4.3) we get

$$\int_\sigma |Q_n(T_N(z))|^2 |T_N'(z)|\, |T_N(z) - T_N(z_0)|^\alpha \, ds_\sigma(z)$$
$$= N \int_0^{2\pi} |Q_n(e^{it})|^2 |e^{it} - e^{i\pi/2}|^\alpha \, dt \geq (1 + o(1)) N \lambda_{\deg(Q_n)}(\mu_\alpha, e^{i\pi/2}).$$

Hence,

$$(1 + o(1))\lambda_{\deg(Q_n)}(\mu_\alpha, e^{i\pi/2}) \leq O(q^{n^{\tau_0/2}}) + (1 + \eta)^{|\alpha|+1} |T_N'(z_0)|^{\alpha+1} \lambda_n(\mu_\sigma, z_0).$$

Using $\deg(Q_n) \leq (n + o(n))/N$, we conclude that

$$\liminf_{n\to\infty} \deg(Q_n)^{\alpha+1} \lambda_{\deg(Q_n)}(\mu_\alpha, e^{i\pi/2})$$
$$\leq (1 + \eta)^{|\alpha|+1} |T_N'(z_0)|^{\alpha+1} \liminf_{n\to\infty} \left(\frac{n + o(n)}{N}\right)^{\alpha+1} \lambda_n(\mu_\sigma, z_0)$$
$$\leq (1 + \eta)^{|\alpha|+1} \frac{|T_N'(z_0)|^{\alpha+1}}{N^{\alpha+1}} \liminf_{n\to\infty} n^{\alpha+1} \lambda_n(\mu_\sigma, z_0).$$

Since $\eta > 0$ is arbitrary, we obtain again, from (3.15) and (4.1),

$$\frac{L_\alpha}{(\pi \omega_\sigma(z_0))^{\alpha+1}} \leq \liminf_{n\to\infty} n^{\alpha+1} \lambda_n(\mu_\sigma, z_0),$$

which, along with (4.9), proves (4.6).

5. Smooth Jordan curves

In this section, we verify Theorem 1.1 for a finite union Γ of smooth Jordan curves and for a measure

$$d\mu(z) = w(z)|z - z_0|^\alpha ds_\Gamma(z), \tag{5.1}$$

where s_Γ is the arc measure on Γ. Recall that a Jordan curve is a homeomorphic image of a circle, while a Jordan arc is a homeomorphic image of a segment. From the point of view of our technique there is a big difference between arcs and curves, and in the present section we shall only work with Jordan curves.

Let Γ be a finite system of C^2 Jordan curves exterior to each other and let μ be a measure on Γ given in (5.1) with some $z_0 \in \Gamma$, where w is a continuous and strictly positive function. Our goal is to prove that

$$\lim_{n\to\infty} n^{\alpha+1} \lambda_n(\mu, z_0) = \frac{w(z_0)}{(\pi \omega_\Gamma(z_0))^{\alpha+1}} L_\alpha \tag{5.2}$$

with L_α from (3.4). We shall deduce this from the result for lemniscates proved in the preceding section.

We will approximate Γ by lemniscates using the following theorem, which was proven in [11].

Proposition 5.1. *Let Γ consist of finitely many Jordan curves exterior to each other, let $P \in \Gamma$, and assume that in a neighborhood of P the curve Γ is C^2-smooth. Then, for every $\varepsilon > 0$, there is a lemniscate $\sigma = \sigma_P$ consisting of Jordan curves such that σ touches Γ at P, σ contains Γ in its interior except for the point P, every component of σ contains in its interior precisely one component of Γ, and*

$$\omega_\Gamma(P) \le \omega_\sigma(P) + \varepsilon. \tag{5.3}$$

Also, for every $\varepsilon > 0$, there exists another lemniscate $\sigma = \sigma_P$ consisting of Jordan curves such that σ touches Γ at P, σ lies strictly inside Γ except for the point P, σ has exactly one component inside every component of Γ, and

$$\omega_\sigma(P) \le \omega_\Gamma(P) + \varepsilon. \tag{5.4}$$

Of course, the phrase "σ lies inside Γ" means that the components of σ lie inside (i.e. in the interior of) the corresponding components of Γ (see Figure 3).

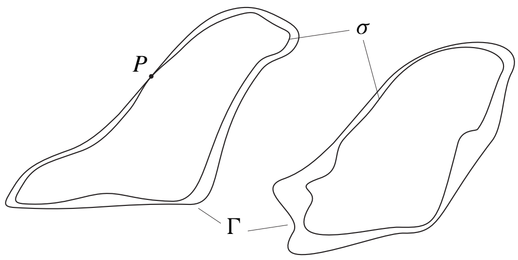

Fig. 3. The Γ and the lemniscate σ as in the second half of Proposition 5.1.

Note that in (5.3) the inequality $\omega_\sigma(P) \le \omega_\Gamma(P)$ is automatic since Γ lies inside σ. In a similar way, in (5.4) the inequality $\omega_\Gamma(P) \le \omega_\sigma(P)$ holds.

Actually, in [11] the conditions (5.3) and (5.4) were formulated in terms of the normal derivatives of the Green's function of the outer domains to Γ and σ, but, in view of the fact that the latter function is just 2π-times the equilibrium density (see [14, II.(4.1)] or [17, Theorem IV.2.3] and [17, (I.4.8)]), the two formulations are equivalent.

5.1. The lower estimate

Let P_n be the extremal polynomial for $\lambda_n(\mu, z_0)$, and for some $\tau > 0$ let $S_{\tau n, z_0, K}$ be the fast decreasing polynomial given by Lemma 2.1 with some $\gamma > 1$ to be chosen below, where K is the set enclosed by Γ. Let $\sigma = \sigma_{z_0}$ be a lemniscate inside Γ given by the second part of Proposition 5.1, and suppose that $\sigma = \{z : |T_N(z)| = 1\}$, where T_N is a polynomial of degree N and $T_N(z_0) = e^{i\pi/2}$. Define $R_n = P_n S_{\tau n, z_0, K}$. Note that R_n is a polynomial of degree at most $(1+\tau)n$ and $R_n(z_0) = 1$. These will be the test polynomials in estimating the Christoffel function for the measure

$$d\mu_\sigma(z) := |z - z_0|^\alpha ds_\sigma(z)$$

on σ, but first we need two nontrivial facts for these polynomials.

Lemma 5.2. *Let $1/2 < \beta < 1$ be fixed. For $z \in \Gamma$ such that $|z - z_0| \le 2n^{-\beta}$, let $z^* \in \sigma$ be the point such that $s_\sigma([z_0, z^*]) = s_\Gamma([z_0, z])$ (actually, there are two such points; we choose as z^* the one closer to z). Then the mapping $q(z) = z^*$ is one-to-one, $|q(z) - z| \le C|z - z_0|^2$, $ds_\Gamma(z) = ds_\sigma(z^*)$, $|q'(z_0)| = 1$, and with the notation $I_n := \{z^* \in \sigma : |z^* - z_0| \le n^{-\beta}\}$, we have*

$$\left|\int_{z^*\in I_n} |R_n(z^*)|^2|z-z_0|^\alpha\, ds_\sigma(z^*) - \int_{z^*\in I_n} |R_n(z)|^2|z-z_0|^\alpha\, ds_\Gamma(z)\right| = o(n^{-(1+\alpha)}). \quad (5.5)$$

On the left-hand side $z = q^{-1}(z^*)$, so the integrand is a function of z^*.

Proof. First of all we mention that $|q'(z_0)| = 1$, i.e. for every $\varepsilon > 0$, if $|z - z_0|$ is small enough, then

$$1 - \varepsilon \le \frac{|q(z) - z_0|}{|z - z_0|} \le 1 + \varepsilon,$$

which is clear since $q(z) = z + O(|z - z_0|^2)$.

We proceed to prove (5.5):

$$\begin{aligned}
&\left|\int_{z^*\in I_n} |R_n(z^*)|^2|z - z_0|^\alpha\, ds_\sigma(z^*) - \int_{z^*\in I_n} |R_n(z)|^2|z - z_0|^\alpha\, ds_\Gamma(z)\right| \\
&\qquad \le \left|\int_{z^*\in I_n} \big(|R_n(z^*)|^2 - |R_n(z)|^2\big)|z - z_0|^\alpha\, ds_\Gamma(z)\right| \\
&\qquad \le \int_{z^*\in I_n} \big||R_n(z^*)|^2 - |R_n(z)|^2\big||z - z_0|^\alpha\, ds_\Gamma(z) =: A.
\end{aligned}$$

Using the Hölder and Minkowski inequalities we get

$$\begin{aligned}
A \le &\left(\int_{z^*\in I_n} |R_n(z^*) - R_n(z)|^2|z - z_0|^\alpha\, ds_\Gamma(z)\right)^{1/2} \\
&\times \left\{\left(\int_{z^*\in I_n} |R_n(z^*)|^2|z - z_0|^\alpha\, ds_\Gamma(z)\right)^{1/2} + \left(\int_{z^*\in I_n} |R_n(z)|^2|z - z_0|^\alpha\, ds_\Gamma(z)\right)^{1/2}\right\}.
\end{aligned} \quad (5.6)$$

We estimate these integrals term by term.

P_n is extremal for $\lambda_n(\mu, z_0) = O(n^{-(\alpha+1)})$ (see Lemma 2.9), therefore (use also $|R_n(z)| \le |P_n(z)|$)

$$\left(\int_{z^*\in I_n} |R_n(z)|^2|z - z_0|^\alpha\, ds_\Gamma(z)\right)^{1/2} \le Cn^{-(\alpha+1)/2}. \quad (5.7)$$

This takes care of the third term in (5.6).

The estimates for the other two terms differ in the cases $\alpha \ge 0$ and $\alpha < 0$.

Assume first that $\alpha \ge 0$. From Lemma 2.7 we get, for any closed subarc $J_1 \subset J$,

$$\|R_n\|_{J_1} \le Cn^{(\alpha+1)/2}\|R_n\|_{L^2(\mu)} \le C,$$

where we have used Lemma 2.9 and $|R_n(z)| \le |P_n(z)|$. Choose J_1 so that it contains z_0 in its interior. Next, note that if $z^* \in I_n$, then $|z^* - z| \le Cn^{-2\beta}$, so $\operatorname{dist}(z^*, z) \le C/n$.

Therefore, an application of Lemma 2.4 yields, for such z,

$$\frac{|R_n(q(z)) - R_n(z)|}{|q(z) - z|} \le Cn\|R_n\|_{J_1},$$

and so

$$|R_n(q(z)) - R_n(z)| \le Cn|q(z) - z| \le Cn^{1-2\beta}. \tag{5.8}$$

Since also $s_\sigma(I_n) \le Cn^{-\beta}$, we have (recall that $z^* = q(z)$)

$$\begin{aligned}\left(\int_{z^*\in I_n} |R_n(z^*) - R_n(z)|^2 |z - z_0|^\alpha \, ds_\Gamma(z)\right)^{1/2} &\le C(n^{-\beta} n^{2-4\beta} n^{-\alpha\beta})^{1/2} \\ &= Cn^{1-(5+\alpha)\beta/2}.\end{aligned}$$

This is the required estimate for the first term in (5.6).

Finally, for the middle term in (5.6), we have

$$\begin{aligned}&\left(\int_{z^*\in I_n} |R_n(z^*)|^2 |z - z_0|^\alpha \, ds_\Gamma(z)\right)^{1/2} \\ &\quad = \left(\int_{z^*\in I_n} \big||R_n(z^*)|^2 - |R_n(z)|^2 + |R_n(z)|^2\big| |z - z_0|^\alpha \, ds_\Gamma(z)\right)^{1/2} \\ &\quad \le \left(\int_{z^*\in I_n} \big||R_n(z^*)|^2 - |R_n(z)|^2\big| |z - z_0|^\alpha \, ds_\Gamma(z)\right)^{1/2} \\ &\qquad + \left(\int_{z^*\in I_n} |R_n(z)|^2 |z - z_0|^\alpha \, ds_\Gamma(z)\right)^{1/2} \\ &\quad \le A^{1/2} + Cn^{-(\alpha+1)/2},\end{aligned}$$

where A is the left-hand side in (5.6), and where we have also used (5.7).

Combining these we get

$$\begin{aligned}A &\le Cn^{1-(5+\alpha)\beta/2}(A^{1/2} + Cn^{-(\alpha+1)/2}) \le CA^{1/2} n^{1-(5+\alpha)\beta/2} + Cn^{1/2-\alpha/2-(5+\alpha)\beta/2} \\ &\le C \max\{A^{1/2} n^{1-(5+\alpha)\beta/2}, n^{1/2-\alpha/2-(5+\alpha)\beta/2}\}.\end{aligned}$$

Therefore $A \le Cn^{2-(5+\alpha)\beta}$ or $A \le Cn^{1/2-\alpha/2-(5+\alpha)\beta/2}$. If $\beta < 1$ is sufficiently close to 1, then both imply $A = o(n^{-(\alpha+1)})$.

Now assume that $\alpha < 0$. From Lemma 2.7 we get, for any closed subarc $J_1 \subset J$,

$$\|R_n\|_{J_1} \le \|P_n\|_{J_1} \le Cn^{1/2}\|P_n\|_{L^2(\mu)} \le Cn^{-\alpha/2},$$

and we may assume that here J_1 contains a neighborhood of z_0. Therefore, in this case (5.8) takes the form

$$|R_n(z^*) - R_n(z)| \le Cn^{1-\alpha/2-2\beta}.$$

Since

$$\int_{z^*\in I_n} |z - z_0|^\alpha \, ds_\Gamma(z) \le Cn^{-\alpha\beta-\beta},$$

we obtain

$$\left(\int_{z^*\in I_n} |R_n(z^*) - R_n(z)|^2 |z - z_0|^\alpha \, ds_\Gamma(z)\right)^{1/2} \le Cn^{1-\alpha/2-2\beta-(\alpha+1)\beta/2},$$

which is the required estimate for the first term in (5.6). Finally, for the middle term in (5.6) we get, as before,

$$\left(\int_{z^*\in I_n} |R_n(z^*)|^2 |z - z_0|^\alpha \, ds_\Gamma(z)\right)^{1/2} \le A^{1/2} + Cn^{-(\alpha+1)/2}.$$

As previously, we deduce from these that

$$A \le Cn^{1-\alpha/2-2\beta-(\alpha+1)\beta/2}(A^{1/2} + n^{-(\alpha+1)/2}),$$

which implies

$$A \le C \max\{n^{2-\alpha-4\beta-(\alpha+1)\beta}, n^{1/2-\alpha-2\beta-(\alpha+1)\beta/2}\}.$$

If β is sufficiently close to 1, then this yields again $A = o(n^{-(\alpha+1)})$, as needed. □

In what follows we keep the notation from the preceding proof. In the following lemma let $\Delta_\delta(z_0) = \{z : |z - z_0| \le \delta\}$ be the disk about z_0 of radius δ.

Note that up to this point the $\gamma > 1$ in Lemma 2.1 was arbitrary. Now we specify how close it should be to 1.

Lemma 5.3. *If* $0 < \beta < 1$ *is fixed and* $\gamma > 1$ *is chosen so that* $\beta\gamma < 1$*, then*

$$\|R_n\|_{K\setminus\Delta_{n^{-\beta}/2}}(z_0) = o(n^{-1-\alpha}). \tag{5.9}$$

Recall that here K is the set enclosed by Γ.

Proof. Let us fix a $\delta > 0$ such that $\Gamma \cap \Delta_\delta(z_0)$ lies in the interior of the arc J from Theorem 1.1. From $\mu \in$ **Reg** and the trivial estimate $\|P_n\|_{L^2(\mu)} = O(1)$ we see that no matter how small $\varepsilon > 0$ is given, for sufficiently large n we have $\|P_n\|_\Gamma \le (1+\varepsilon)^n$. On the other hand, in view of Lemma 2.1 we have, for $z \notin \Delta_\delta(z_0)$, $z \in K$,

$$|S_{\tau n, z_0, K}(z)| \le C_\gamma e^{-c_\gamma \tau n \delta^2},$$

so certainly

$$\|R_n\|_{K\setminus\Delta_\delta}(z_0) = o(n^{-1-\alpha}). \tag{5.10}$$

Consider now $K \cap \Delta_\delta(z_0)$. Its boundary consists of the arc $\Gamma \cap \Delta_\delta(z_0)$, which is part of J, and of an arc on the boundary of $\Delta_\delta(z_0)$, where we already know the bound (5.10). On the other hand, on $\Gamma \cap \Delta_\delta(z_0)$ we have, by Lemma 2.7,

$$|P_n(z)| \le Cn^{(1+|\alpha|)/2}\|P_n\|_{L^2(\mu)} \le Cn^{(1+|\alpha|)/2}.$$

Therefore, by the maximum principle, we obtain the same bound (for large n) on the whole set $K \cap \Delta_\delta(z_0)$. As a consequence, for $z \in K \setminus \Delta_{n^{-\beta}/2}$,

$$|R_n(z)| \le Cn^{(1+|\alpha|)/2} e^{-c_\gamma \tau n (n^{-\beta}/2)^\gamma} = o(n^{-1-\alpha})$$

if we choose $\gamma > 1$ in Lemma 2.1 so that $\beta\gamma < 1$. These prove (5.9). □

After these preliminaries we return to the proof of Theorem 1.1, more precisely to the lower estimate of $\lambda_n(\mu, z_0)$.

Let $\eta > 0$ be arbitrary, and let n be so large that

$$\frac{1}{1+\eta} w(z_0) \le w(z) \le (1+\eta) w(z_0), \qquad \frac{1}{1+\eta}|z - z_0| \le |q(z) - z_0| \le (1+\eta)|z - z_0|$$

for all $z^* \in I_n$, where I_n is the set from Lemma 5.2. Then from Lemma 5.2 we obtain (recall that $z^* = q(z)$)

$$\begin{aligned}
\int_{z^* \in I_n} |R_n(z^*)|^2 |z^* - z_0|^\alpha \, ds_\sigma(z^*) &\le (1+\eta)^{|\alpha|} \int_{z^* \in I_n} |R_n(z^*)|^2 |z - z_0|^\alpha \, ds_\Gamma(z) \\
&\le (1+\eta)^{|\alpha|} \int_{z^* \in I_n} |R_n(z)|^2 |z - z_0|^\alpha \, ds_\Gamma(z) + o(n^{-(\alpha+1)}) \\
&\le \frac{(1+\eta)^{|\alpha|+1}}{w(z_0)} \int_{z^* \in I_n} |R_n(z)|^2 w(z) |z - z_0|^\alpha \, ds_\Gamma(z) + o(n^{-(\alpha+1)}) \\
&\le \frac{(1+\eta)^{|\alpha|+1}}{w(z_0)} \lambda_n(\mu, z_0) + o(n^{-(\alpha+1)}).
\end{aligned}$$

On the other hand, if, for some $z \in \sigma$, we have $z^* \notin I_n$ then necessarily $|z - z_0| \ge n^{-\beta}/2$, so from Lemma 5.3 we obtain

$$\int_{z^* \in \sigma \setminus I_n} |R_n(z^*)|^2 |z^* - z_0|^\alpha \, ds_\sigma(z^*) = o(n^{-(1+\alpha)}).$$

Combining these, we find that

$$\begin{aligned}
\lambda_{\deg(R_n)}(\mu_\sigma, z_0) &\le \int_{z \in \sigma} |R_n(z^*)|^2 |z^* - z_0|^\alpha \, ds_\sigma(z^*) \\
&\le \frac{(1+\eta)^{|\alpha|+1}}{w(z_0)} \lambda_n(\mu, z_0) + o(n^{-(\alpha+1)}).
\end{aligned}$$

Since $\deg(R_n) \le (1+\tau)n$, we conclude from (4.6) (see also (2.1)) that

$$\begin{aligned}
\frac{L_\alpha}{(\pi \omega_\sigma(z_0))^{\alpha+1}} &= \liminf_{n\to\infty} \deg(R_n)^{\alpha+1} \lambda_{\deg(R_n)}(\mu_\sigma, z_0) \\
&\le \liminf_{n\to\infty} (1+\tau)^{\alpha+1} \frac{(1+\eta)^{|\alpha|+1}}{w(z_0)} n^{\alpha+1} \lambda_n(\mu, z_0).
\end{aligned}$$

But here $\tau, \eta > 0$ are arbitrary, so we get

$$\liminf_{n\to\infty} n^{\alpha+1} \lambda_n(\mu, z_0) \ge \frac{w(z_0)}{(\pi \omega_\sigma(z_0))^{\alpha+1}} L_\alpha.$$

As $\omega_\sigma(z_0) \le \omega_\Gamma(z_0) + \varepsilon$ (see (5.4)), for $\varepsilon \to 0$ we finally arrive at the lower estimate

$$\liminf_{n\to\infty} n^{\alpha+1} \lambda_n(\mu, z_0) \ge \frac{w(z_0)}{(\pi \omega_\Gamma(z_0))^{\alpha+1}} L_\alpha. \tag{5.11}$$

5.2. The upper estimate

Let now σ be the lemniscate given by the first part of Proposition 5.1, and let P_n be the polynomial extremal for $\lambda_n(\mu_\sigma, z_0)$. Define, with some $\tau > 0$,

$$R_n(z) = P_n(z) S_{\tau n, z_0, L}(z),$$

where $S_{\tau n, z_0, L}$ is the fast decreasing polynomial given by Lemma 2.1 for the lemniscate set L enclosed by σ (with some $\gamma > 1$). Let $\eta > 0$ be arbitrary, $1/2 < \beta < 1$ as before, and suppose that n is so large that

$$\frac{1}{1+\eta} w(z_0) \le w(z) \le (1+\eta) w(z_0),$$

$$\frac{1}{\eta+1} \le |q'(z)| \le (1+\eta),$$

$$\frac{1}{1+\eta} |z - z_0| \le |q(z) - z_0| \le (1+\eta)|z - z_0|$$

for all $|z - z_0| \le n^{-\beta}$. Using Lemma 5.2 (more precisely, its version when σ encloses Γ) we have (recall again that $z^* = q(z)$)

$$\begin{aligned}\int_{z^* \in I_n} |R_n(z)|^2 w(z) |z - z_0|^\alpha \, ds_\Gamma(z) &\le (1+\eta) w(z_0) \int_{z^* \in I_n} |R_n(z)|^2 |z - z_0|^\alpha \, ds_\Gamma(z) \\ &\le (1+\eta) w(z_0) \int_{z^* \in I_n} |R_n(z^*)|^2 |z - z_0|^\alpha \, ds_\sigma(z^*) + o(n^{-(\alpha+1)}) \\ &\le (1+\eta)^{|\alpha|+1} w(z_0) \int_{z^* \in I_n} |R_n(z^*)|^2 |z^* - z_0|^\alpha \, ds_\sigma(z^*) + o(n^{-(\alpha+1)}) \\ &\le (1+\eta)^{|\alpha|+1} w(z_0) \lambda_n(\mu_\sigma, z_0) + o(n^{-(\alpha+1)}).\end{aligned}$$

On the other hand, Lemma 5.3 (but now applied for the system of curves σ rather than for Γ) implies, as before,

$$\int_{\Gamma \setminus \Delta_{n^{-\beta}/2}(z_0)} |R_n(z)|^2 |z - z_0|^\alpha \, d\mu(z) = o(n^{-(1+\alpha)}).$$

Therefore,

$$\lambda_{\deg(R_n)}(\mu, z_0) \le (1+\eta)^{|\alpha|+1} w(z_0) \lambda_n(\mu_\sigma, z_0) + o(n^{-(\alpha+1)}),$$

which, similarly to the lower estimate, upon using (4.6) and letting τ, η tend to zero, implies (see also (2.2))

$$\limsup_{n\to\infty} n^{\alpha+1} \lambda_n(\mu, z_0) \le \frac{w(z_0)}{(\pi \omega_\sigma(z_0))^{\alpha+1}} L_\alpha.$$

Here, in view of (5.3), $\omega_\Gamma(z_0) \le \omega_\sigma(z_0) + \varepsilon$, hence for $\varepsilon \to 0$ we conclude that

$$\limsup_{n\to\infty} n^{\alpha+1} \lambda_n(\mu, z_0) \le \frac{w(z_0)}{(\pi \omega_\Gamma(z_0))^{\alpha+1}} L_\alpha.$$

This and (5.11) prove (5.2). □

6. Piecewise smooth Jordan curves

The proof in the preceding section can be carried out without any difficulty if Γ consists of piecewise C^2-smooth Jordan curves, provided that Γ is C^2-smooth in a neighborhood of z_0. Indeed, in that case we can still talk about ω_Γ which is continuous where Γ is C^2-smooth [24, Proposition 2.2], and in the above proof the C^2-smoothness was used only in a neighborhood of z_0. Therefore, we have

Proposition 6.1. *Let Γ consist of finitely many disjoint, piecewise C^2-smooth Jordan curves. Let $z_0 \in \Gamma$, and supppose Γ is C^2-smooth in a neighborhood of z_0. Then the measure μ given by* (5.1) *satisfies* (5.2).

7. Arc components

In this section, we prove Theorem 1.1 when Γ is a union of C^2-smooth Jordan curves and arcs, and μ is the measure (5.1) considered before. To be more specific, our aim is to verify

Proposition 7.1. *Let Γ consist of finitely many disjoint C^2-smooth Jordan curves or arcs exterior to each other, and let $z_0 \in \Gamma$. Assume that in a neighborhood of $z_0 \in \Gamma$ the piece of Γ lying in that neighborhood is C^2-smooth, and z_0 is not an endpoint of an arc component of Γ. Then the measure* (5.1), *where w is continuous and positive and $\alpha > -1$, satisfies* (1.4).

We shall need some facts about Bessel functions, and a discretization of the equilibrium measure ν_Γ that uses the zeros of an appropriate Bessel function.

7.1. Bessel functions and some local asymptotics

We shall need the Bessel function of the first kind of order $\beta > 0$:

$$J_\beta(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n+\beta}}{n!\Gamma(n+\beta+1)},$$

as well as the functions [10]

$$\mathbb{J}_\beta(u, v) = \frac{J_\beta(\sqrt{u})\sqrt{v}\, J'_\beta(\sqrt{v}) - J_\beta(\sqrt{v})\sqrt{u}\, J'_\beta(\sqrt{u})}{2(u-v)},$$

$$\mathbb{J}^*_\beta(z) = \frac{J_\beta(z)}{z^\beta}, \qquad \mathbb{J}^*_\beta(u, v) = \frac{\mathbb{J}_\beta(u, v)}{u^{\beta/2} v^{\beta/2}}.$$

The latter are analytic, and we have

$$\mathbb{J}^*_\beta(u, 0) = \frac{1}{2^{2\beta+1} u} \sum_{n=1}^{\infty} \frac{(-1)^n (\sqrt{u}/2)^{2n}}{n!\Gamma(n+\beta+1)} \left(\frac{\beta}{\Gamma(\beta+1)} - \frac{2n+\beta}{\Gamma(\beta+1)} \right) = \frac{\mathbb{J}^*_{\beta+1}(\sqrt{u})}{2^{\beta+1}\Gamma(\beta+1)}.$$

Let $d\nu_0(x)$ be the measure $x^\beta\,dx$ with support $[0,2]$, and $K_n^{(0)}(x,t)$ its n-th reproducing kernel. It is known (see [9, (1.2)] or [20, (4.5.8), p. 72]) that

$$\frac{K_n^{(0)}\big(\frac{x^2}{2n^2},0\big)}{K_n^{(0)}(0,0)}=(1+o(1))\frac{\mathbb{J}_\beta^*(x^2,0)}{\mathbb{J}_\beta^*(0,0)},$$

uniformly for $|x|\le A$ with any fixed A. We have already mentioned (see e.g. [20, Theorem 3.1.3]) that the polynomial $K_n^{(0)}(t,0)/K_n^{(0)}(0,0)$ is the extremal polynomial of degree n for $\lambda_n(\nu_0,0)$, so the preceding relation gives an asymptotic formula for this extremal polynomial on intervals $[0,A/n^2]$. If now $d\nu_1(x)=(2x)^\beta\,dx$ but with support $[0,1]$, and $K_n^{(1)}$ is the associated reproducing kernel, then $K_n^{(1)}(t,0)/K_n^{(1)}(0,0)$ is the extremal polynomial of degree n for $\lambda_n(\nu_1,0)$, and it is clear that this is just a scaled version of the extremal polynomial for ν_0:

$$\frac{K_n^{(1)}(t,0)}{K_n^{(1)}(0,0)}=\frac{K_n^{(0)}(2t,0)}{K_n^{(0)}(0,0)}.$$

Therefore,

$$\frac{K_n^{(1)}\big(\frac{x^2}{4n^2},0\big)}{K_n^{(1)}(0,0)}=(1+o(1))\frac{\mathbb{J}_\beta^*(x^2,0)}{\mathbb{J}_\beta^*(0,0)}.$$

Then the same is true for the measure $2^{-\beta}\,d\nu_1(x)=x^\beta\,dx$ with support $[0,1]$ (multiplying the measure by a constant does not change the extremal polynomial for the Christoffel functions). Next, consider the measure $d\nu_2(x)=|x|^\alpha\,dx$ with support $[-1,1]$. The extremal polynomial for $\lambda_{2n}(\nu_2,0)$ is obtained from the extremal polynomial for $\lambda_n(\nu_1,0)$ with $\beta=(\alpha-1)/2$ by the substitution $t\to t^2$ (see Section 3.1, in particular the last paragraph), i.e.

$$\frac{K_{2n}^{(2)}(t,0)}{K_{2n}^{(2)}(0,0)}=\frac{K_n^{(1)}(t^2,0)}{K_n^{(1)}(0,0)}.$$

Hence, for even n,

$$\frac{K_n^{(2)}(t,0)}{K_n^{(2)}(0,0)}=(1+o(1))\mathcal{J}_{(\alpha+1)/2}(nt),\qquad |t|\le A/n,$$

where

$$\mathcal{J}_{(\alpha+1)/2}(z):=\frac{\mathbb{J}_{(\alpha-1)/2}^*(z^2,0)}{\mathbb{J}_{(\alpha-1)/2}^*(0,0)}=\frac{\mathbb{J}_{(\alpha+1)/2}^*(z)}{\mathbb{J}_{(\alpha+1)/2}^*(0)}.\tag{7.1}$$

Fix a positive number A. According to what we have just seen, for every even n,

$$\begin{aligned}\int_{-A/n}^{A/n}\mathcal{J}_{(\alpha+1)/2}(nt)^2|t|^\alpha\,dt&\le(1+o(1))\int_{-A/n}^{A/n}\left(\frac{K_n^{(2)}(t,0)}{K_n^{(2)}(0,0)}\right)^2|t|^\alpha\,dt\\&\le(1+o(1))\lambda_n(\nu_2,0),\end{aligned}$$

and so for any (even) n,

$$\int_{-A}^{A}\mathcal{J}_{(\alpha+1)/2}(x)^2|x|^\alpha\,dx=n^{\alpha+1}\int_{-A/n}^{A/n}\mathcal{J}_{(\alpha+1)/2}(nt)^2|t|^\alpha\,dt\le(1+o(1))n^{\alpha+1}\lambda_n(\nu_2,0).$$

Now if we let here $n \to \infty$ and use the limit (3.3) for the right-hand side, we obtain

$$\int_{-A}^{A} \mathcal{J}_{(\alpha+1)/2}(x)^2 |x|^\alpha \, dx \le L_\alpha,$$

where L_α is from (3.4). Finally, since here A is arbitrary, we conclude that

$$\int_{-\infty}^{\infty} \mathcal{J}_{(\alpha+1)/2}(x)^2 |x|^\alpha \, dx \le L_\alpha. \tag{7.2}$$

7.2. The upper estimate in Theorem 1.1 for one arc

The aim of this section is to construct polynomials that satisfy the upper estimate for the Christoffel functions in Theorem 1.1 (which is the same as in Proposition 7.1) when Γ consists of a single C^2-smooth arc, and $z_0 \in \Gamma$ is not an endpoint of that arc. In the next subsection we shall indicate what to do when Γ has other components as well.

Let ν_Γ be the equilibrium measure of Γ, and s_Γ the arc measure on Γ. Since Γ is assumed to be C^2-smooth, we have $d\nu_\Gamma(t) = \omega_\Gamma(t) ds_\Gamma(t)$ with an ω_Γ that is continuous and positive away from the endpoints of Γ [24, Proposition 2.2].

We may assume that $z_0 = 0$ and the real line is tangent to Γ at the origin. By assumption, the measure μ we are dealing with, is, in a neighborhood of the origin, of the form $d\mu(z) = w(z)|z|^\alpha \, ds_\Gamma(z)$ with some positive and continuous function $w(z)$.

Since Γ is assumed to be C^2-smooth, in a neighborhood of the origin we have the parametrization $\gamma(t) = \gamma_1(t) + i\gamma_2(t)$, $\gamma_1(t) \equiv t$, where γ_2 is a twice continuously differentiable function such that $\gamma_2(0) = \gamma_2'(0) = 0$. In particular, as $t \to 0$ we have $\gamma_2(t) = O(t^2)$, $\gamma_2'(t) = O(|t|)$. We shall also take an orientation of Γ, and we shall write $z \prec w$ if $z \in \Gamma$ precedes $w \in \Gamma$ in that orientation. We may assume that this orientation is such that around the origin we have $z \prec w \Leftrightarrow \Re z < \Re w$.

It is known that when dealing with $|z|^\alpha$ weights on the real line, Bessel functions of the first kind enter the picture [7], [9], [10]. For a given large n we shall construct the necessary polynomials from two sources: from points on Γ that follow the pattern of the zeros of the Bessel function $\mathcal{J}_{(\alpha+1)/2}$, and from points that are obtained from discretizing the equilibrium measure ν_Γ. The first type will be used close to the origin (at distance $\le 1/n^\tau$ with some appropriate τ), while the latter type will be on the rest of Γ. So first we shall discuss two different divisions of Γ.

7.2.1. Division based on the zeros of Bessel functions. Let $\beta = (\alpha+1)/2$; it is a positive number because $\alpha > -1$. It is known that J_β, and hence also $\mathcal{J}_\beta$ from (7.1), has infinitely many positive zeros which are all simple and tend to infinity; let them be $j_{\beta,1} < j_{\beta,2} < \cdots$. We have the asymptotic formula [28, 15.53]

$$j_{\beta,k} = (k + \beta/2 - 1/4)\pi + o(1), \qquad k \to \infty. \tag{7.3}$$

The negative zeros of $\mathcal{J}_\beta$ are $-j_{\beta,k}$, and we have the product formula [28, 15.41, (3)]

$$J_\beta(z) = \frac{(z/2)^\beta}{\Gamma(\beta+1)} \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{j_{\beta,k}^2}\right).$$

Therefore,

$$\mathcal{J}_\beta(z) = \prod_{k=1}^{\infty}\left(1 - \frac{z^2}{j_{\beta,k}^2}\right). \tag{7.4}$$

Let $a_0 = 0$, and for $k > 0$ let $a_k \in \Gamma$ be the unique point on Γ such that $0 \prec a_k$, and

$$\nu_\Gamma(\overline{0a_k}) = \frac{j_{\beta,k}}{\pi n}, \tag{7.5}$$

where $\overline{0a_k}$ denotes the arc of Γ between 0 and a_k. For negative k let similarly a_k be the unique number for which $a_k \prec 0$ and

$$\nu_\Gamma(\overline{a_k 0}) = \frac{j_{\beta,|k|}}{\pi n}. \tag{7.6}$$

The reader should be aware that these a_k and the whole division depend on n, so a more precise notation would be $a_{k,n}$ for a_k, but we shall suppress the additional parameter n.

This definition makes sense only for finitely many k, say for $-k_0 < k < k_1$, and in view of (7.3) we have $k_0 + k_1 = n + O(1)$, i.e. there are about n such a_k on Γ. The arcs $\overline{a_k a_{k+1}}$ are subarcs of Γ that follow each other according to $\prec$, they satisfy

$$\nu_\Gamma(\overline{a_{k-1}a_k}) = \frac{j_{\beta,k} - j_{\beta,k-1}}{\pi n}, \quad k > 0,$$

$$\nu_\Gamma(\overline{a_{k-1}a_k}) = \frac{j_{\beta,k+1} - j_{\beta,k}}{\pi n}, \quad k < 0,$$

and their union is almost the entire Γ: there can be two additional arcs around the two endpoints with equilibrium measure $< (j_{\beta,k_0} - j_{\beta,k_0-1})/\pi n$ resp. $< (j_{\beta,k_1} - j_{\beta,k_1-1})/\pi n$.

7.2.2. Division based solely on the equilibrium measure. In this subdivision of Γ we follow the procedure in [24, Section 2]. Let $\overline{b_0 b_1} \subset \Gamma$ be the unique arc (at least for large n it is unique) with $0 \in \overline{b_0 b_1}$, $\nu_\Gamma(\overline{b_0 b_1}) = 1/n$, and if ξ_0 is the center of mass of ν_Γ on $\overline{b_0 b_1}$, then $\Re\xi_0 = 0$. For $k > 1$ let $b_k \in \Gamma$ be the point on Γ (if any) with $0 \prec b_k$ and $\nu_\Gamma(\overline{b_1 b_k}) = (k-1)/n$, and similarly for negative k let $b_k \prec 0$ be the point on Γ with $\nu_\Gamma(\overline{b_k b_0}) = |k|/n$. This definition makes sense only for finitely many k, say for $-l_0 < k < l_1$. Thus, the arcs $\overline{b_k b_{k+1}}$, $-l_0 < k < l_1 - 1$, continuously fill Γ_0 (in the orientation of Γ_0) and they all have equal, $1/n$ weight with respect to the equilibrium measure ν_Γ. It may happen that, with this selection, around the endpoints of Γ there still remain two "little" arcs, say $\overline{b_{-l_0} b_{-l_0+1}}$ and $\overline{b_{l_1-1} b_{l_1}}$ of ν_Γ-measure $< 1/n$. We include also these two small arcs into our subdivision of Γ, so in this case we divide Γ into $n+1$ arcs $\overline{b_k b_{k+1}}$, $k = -l_0, \dots, l_1 - 1$.

Let ξ_k be the center of mass of the equilibrium measure ν_Γ on the arc $\overline{b_k b_{k+1}}$:

$$\xi_k = \frac{1}{\nu_\Gamma(\overline{b_k b_{k+1}})} \int_{\overline{b_k b_{k+1}}} u \, d\nu_\Gamma(u). \tag{7.7}$$

Since the length of $\overline{b_k b_{k+1}}$ is at most C/n (note that ω_Γ has a positive lower bound), and Γ is C^2-smooth, it follows that ξ_k lies close to the arc $\overline{b_k b_{k+1}}$:

$$\operatorname{dist}(\xi_k, \overline{b_k b_{k+1}}) \le C/n^2. \tag{7.8}$$

For the polynomials

$$B_n(z) = \prod_{k\neq 0}(z - \xi_k) \tag{7.9}$$

it was proven in [24, Propositions 2.4, 2.5] (see also [24, Section 2.2]) that $B_n(z)/B_n(0)$ are uniformly bounded on Γ:

$$|B_n(z)/B_n(0)| \leq C_0, \qquad z \in \Gamma. \tag{7.10}$$

7.2.3. Construction of the polynomials $\mathcal{C}_n$. Choose a $0 < \tau < 1$ close to 1 (we shall see later how close it has to be), and for an n define $N = N_n = [n^{3(1-\tau)}]$. We set

$$\mathcal{C}_n(z) := \prod_{k=-N_n,\, k\neq 0}^{N_n}\left(1 - \frac{z}{a_k}\right)\prod_{|k|>N_n}\left(1 - \frac{z}{\xi_k}\right). \tag{7.11}$$

Note that the precise range of k in the second factor is $-l_0 \leq k < -N_n$ and $N_n < k \leq l_1 - 1$. Since the number of all ξ_k is $n+1$, this polynomial has degree n, and it takes the value 1 at the origin. This will be the main factor in the test polynomial that will give the appropriate upper bound for $\lambda_n(\mu, 0)$; the other factor will be the fast decreasing polynomial from Corollary 2.2.

We estimate on Γ the two factors

$$\mathcal{A}_n(z) := \prod_{k=-N_n,\ k\neq 0}^{N_n}\left(1 - \frac{z}{a_k}\right) \quad \text{and} \quad \mathcal{B}_n(z) := \prod_{|k|>N_n}\left(1 - \frac{z}{\xi_k}\right)$$

separately. The estimates will be distinctly different for $|z| \leq n^{-\tau}$ and for $|z| > n^{-\tau}$.

7.2.4. Bounds for $\mathcal{A}_n(z)$ for $|z| \leq n^{-\tau}$. In what follows, we shall use N instead of N_n ($= [n^{3(1-\tau)}]$). Consider first

$$\mathcal{A}_n^*(x) := \prod_{k=1}^{N}\left(1 - \frac{(n\pi\omega_\Gamma(0)x)^2}{j_{\beta,k}^2}\right)$$

(recall that $j_{\beta,k}$ are the zeros of the Bessel function J_β with $\beta = (\alpha+1)/2$). In view of (7.4) we can write, for real $|x| \leq n^{-\tau}$,

$$\frac{\mathcal{J}_\beta(n\pi\omega_\Gamma(0)x)}{\mathcal{A}_n^*(x)} = \prod_{k>N}\left(1 - \frac{(n\pi\omega_\Gamma(0)x)^2}{j_{\beta,k}^2}\right).$$

Taking into account (7.3), we get here

$$\frac{n\pi\omega_\Gamma(0)x}{j_{\beta,k}} = O\left(\frac{nn^{-\tau}}{k}\right),$$

hence the product on the right is

$$\exp\left\{O\left(\sum_{k>N}\left(\frac{nn^{-\tau}}{k}\right)^2\right)\right\} = \exp\left(O\left(\frac{n^{2(1-\tau)}}{N}\right)\right) = \exp\left(O\left(\frac{1}{n^{1-\tau}}\right)\right) = 1 + o(1).$$

Thus, our first estimate is

$$\mathcal{A}_n^*(x) = (1 + o(1))\mathcal{J}_\beta(n\pi\omega_\Gamma(0)x), \qquad |x| \le n^{-\tau}. \tag{7.12}$$

Next, we go to a $z \in \Gamma$ with $|z| \le n^{-\tau}$. Let x be the real part of z. Then (recall that Γ is C^2-smooth and the real line is tangent to Γ)

$$z = x + O(x^2) = x + O(n^{-2\tau}).$$

We shall need that the a_k's with $|k| \le N$ are close to $j_{\beta,k}/n\pi\omega_\Gamma(0)$. To prove that, consider the parametrization $\gamma(t) = t + i\gamma_2(t)$ of Γ discussed at the beginning of this section. Then $a_k = \gamma(\Re a_k) = \Re a_k + O((\Re a_k)^2)$. By the definition of the points a_k we have, for $1 \le k \le N$,

$$\frac{j_{\beta,k}}{\pi n} = \nu_\Gamma(\overline{0a_k}) = \int_0^{\Re a_k} \omega_\Gamma(\gamma(t))|\gamma'(t)|\,dt. \tag{7.13}$$

Now since around the origin ω_Γ is C^1-smooth [24, Proposition 2.2], on the right we have

$$\omega_\Gamma(\gamma(t)) = \omega_\Gamma(0) + O(|\gamma(t)|) = \omega_\Gamma(0) + O(|t|),$$

while

$$|\gamma'(t)| = \sqrt{1 + \gamma_2'(t)^2} = \sqrt{1 + O(t^2)} = 1 + O(t^2),$$

hence

$$\frac{j_{\beta,k}}{\pi n} = \int_0^{\Re a_k} (\omega_\Gamma(0) + O(|t|))\,dt = \omega_\Gamma(0)\Re a_k + O((\Re a_k)^2),$$

which implies

$$\Re a_k = \frac{j_{\beta,k}}{\pi n\omega_\Gamma(0)} + O((j_{\beta,k}/n)^2). \tag{7.14}$$

Therefore, since here $j_{\beta,k} \le Ck$ (see (7.3)),

$$a_k - \frac{j_{\beta,k}}{n\pi\omega_\Gamma(0)} = (a_k - \Re a_k) + \Re a_k - \frac{j_{\beta,k}}{n\pi\omega_\Gamma(0)} = O((k/n)^2). \tag{7.15}$$

Let

$$\rho = (\alpha + 9)(1 - \tau), \tag{7.16}$$

and suppose that

$$\left| x - \frac{j_{\beta,k}}{n\pi\omega_\Gamma(0)} \right| \ge \frac{1}{n^{1+\rho}} \qquad \text{for all } -N \le k \le N. \tag{7.17}$$

Then in the product

$$\frac{\mathcal{A}_n(z)}{\mathcal{A}_n^*(x)} = \prod_{k=-N,\,k\ne 0}^{N} \frac{1 - z/a_k}{1 - n\pi\omega_\Gamma(0)x/j_{\beta,k}} = \prod_{k=-N,\,k\ne 0}^{N} \frac{j_{\beta,k} - zj_{\beta,k}/a_k}{j_{\beta,k} - n\pi\omega_\Gamma(0)x}$$

all denominators are $\ge c/n^\rho$. As for the numerators, we have (recall (7.15) and $|a_k| \ge ck/n$)

$$|j_{\beta,k}/a_k - n\pi\omega_\Gamma(0)| = O(k),$$

and hence, because of $z = x + O(x^2)$,

$$|zj_{\beta,k}/a_k - n\pi\omega_\Gamma(0)x| = O(|z|k + nx^2) = O(Nn^{-\tau} + nn^{-2\tau}) \\ = O(n^{3-4\tau} + n^{1-2\tau}) = O(n^{3-4\tau}).$$

Therefore, for the individual factors in $\mathcal{A}_n(z)/\mathcal{A}_n^*(z)$ we have

$$\frac{j_{\beta,k} - zj_{\beta,k}/a_k}{j_{\beta,k} - n\pi\omega_\Gamma(0)x} = 1 + O(n^{3-4\tau}n^\rho),$$

from which we conclude that

$$\frac{\mathcal{A}_n(z)}{\mathcal{A}_n^*(x)} = (1 + O(n^{3-4\tau}n^\rho))^{2N} = \exp(O(n^{3-4\tau}n^\rho N)) \\ = \exp(O(n^{6-7\tau+\rho})) = \exp(O(n^{(15+\alpha)(1-\tau)-\tau})) = 1 + o(1)$$

provided

$$(15+\alpha)(1-\tau) < \tau. \tag{7.18}$$

Let

$$\Gamma_n = \{z \in \Gamma : |z| \le n^{-\tau} \text{ and (7.17) is true with } x = \Re z\}. \tag{7.19}$$

So far we have proved (see (7.12) and the preceding estimates)

$$\mathcal{A}_n(z) = (1 + o(1))\mathcal{J}_\beta(n\pi\omega_\Gamma(0)x), \qquad z \in \Gamma_n. \tag{7.20}$$

As Γ_n is a subset of the arc $\Gamma \cap \Delta_{n^{-\tau}}(0)$ of s_Γ-measure at most $O(Nn^{-1-\rho}) = O(n^{2-3\tau-\rho})$, its relative measure compared to the s_Γ-measure of $\Gamma \cap \Delta_{n^{-\tau}}(0)$ is at most

$$O(n^{2-3\tau-\rho+\tau}) = O(n^{2-2\tau-\rho}) = o(N^{-2})$$

because

$$2 - 2\tau - \rho = -(\alpha+7)(1-\tau) < -6(1-\tau).$$

Since $\mathcal{A}_n$ has degree $2N$, from the Remez-type inequality in Lemma 2.6 we conclude that

$$\sup\{|\mathcal{A}_n(z)| : z \in \Gamma \cap \Delta_{n^{-\tau}}(0)\} \le (1 + o(1)) \sup\{|\mathcal{A}_n(z)| : z \in \Gamma_n\}.$$

But $\mathcal{J}_\beta(t)$ is bounded on the whole real line [28, Section 7.21], therefore by (7.20) there is a constant C_1 such that

$$|\mathcal{A}_n(z)| \le C_1 \qquad \text{for all } z \in \Gamma,\ |z| \le n^{-\tau}. \tag{7.21}$$

7.2.5. Bounds for $\mathcal{B}_n(z)$ for $|z| \le n^{-\tau}$. Consider now, for $z \in \Gamma$, $|z| \le n^{-\tau}$, the expression

$$\mathcal{B}_n(z) = \prod_{|k|>N} \frac{\xi_k - z}{\xi_k}.$$

Recall that the smallest and largest indices here (they are k_{-l_0} and k_{l_1}) refer to ξ_k that were selected for the two additional intervals around the endpoints of Γ, hence for them we have

$$\frac{\xi_k - z}{\xi_k} = 1 + O(|z|) = 1 + o(1), \qquad k = -l_0, l_1 - 1.$$

The other indices refer to points ξ_k which were the centers of mass of the arcs $\overline{b_k b_{k+1}}$ which have ν_Γ-measure $1/n$. We are going to compare $\log|z - \xi_k|$ with the average of $\log|z - t|$ over $\overline{b_k b_{k+1}}$ with respect to ν_Γ:

$$\log|z - \xi_k| - n\int_{\overline{b_k b_{k+1}}} \log|z - t|\, d\nu_\Gamma(t) = -n\int_{\overline{b_k b_{k+1}}} \log\left|\frac{z - t}{z - \xi_k}\right| d\nu_\Gamma(t).$$

Here

$$\frac{z - t}{z - \xi_k} = 1 + \frac{\xi_k - t}{z - \xi_k},$$

and for $t \in \overline{b_k b_{k+1}}$ we have $|\xi_k - t| \le C/n$. Since $|z|$ is small (at most $n^{-\tau}$) and $|\xi_k|$ is comparatively large ($\ge N/n = n^{2(1-\tau)-\tau}$), the second term on the right is small in absolute value, hence

$$\log\left|\frac{z - t}{z - \xi_k}\right| = \Re \log\left(1 + \frac{\xi_k - t}{z - \xi_k}\right) = \Re\frac{\xi_k - t}{z - \xi_k} + O\left(\left|\frac{\xi_k - t}{z - \xi_k}\right|^2\right).$$

Therefore,

$$\left|n\int_{\overline{b_k b_{k+1}}} \log\left|\frac{z - t}{z - \xi_k}\right| d\nu_\Gamma(t)\right| = n\int_{\overline{b_k b_{k+1}}} O\left(\left|\frac{\xi_k - t}{z - \xi_k}\right|^2\right) d\nu_\Gamma(t)$$
$$= O\left(\frac{(1/n)^2}{(k/n)^2}\right) = O\left(\frac{1}{k^2}\right),$$

because the integral

$$\int_{\overline{b_k b_{k+1}}} \Re\frac{\xi_k - t}{z - \xi_k}\, d\nu_\Gamma(t) = \Re\frac{1}{z - \xi_k}\int_{\overline{b_k b_{k+1}}} (\xi_k - t)\, d\nu_\Gamma(t)$$

vanishes by the choice of ξ_k. Hence, if

$$H_n = \bigcup_{-l_0 < k < -N,\; N < k < l_1 - 2} \overline{b_k b_{k+1}},$$

then

$$\log\prod_{|k|>N} |\xi_k - z| - n\int_{H_n} \log|z - t|\, d\nu_\Gamma(t) = o(1) + O\Big(\sum_{|k|>N} k^{-2}\Big)$$
$$= o(1) + O(N^{-1}) = o(1).$$

If we set here $z = 0$, we get

$$\log\prod_{|k|>N} |\xi_k| - n\int_{H_n} \log|t|\, d\nu_\Gamma(t) = o(1).$$

Therefore,

$$\log|\mathcal{B}_n(z)| - n\int_{H_n} \log\frac{|z-t|}{|t|}\,d\nu_\Gamma(t) = o(1). \tag{7.22}$$

As the whole integral

$$\int_\Gamma \log\frac{|z-t|}{|t|}\,d\nu_\Gamma(t)$$

is the value of the logarithmic potential of the equilibrium measure ν_Γ at two points of Γ, and since this logarithmic potential is constant on Γ by Frostman's theorem [16, Theorem 3.3.4], we deduce that this whole integral is 0, and so (7.22) is equivalent to

$$\log|\mathcal{B}_n(z)| + n\int_{\Gamma\setminus H_n} \log\frac{|z-t|}{|t|}\,d\nu_\Gamma(t) = o(1). \tag{7.23}$$

The set $\Gamma\setminus H_n$ consists of the two small additional arcs $\overline{b_{-l_0}b_{-l_0+1}}$, $\overline{b_{l_1-1}b_{l_1}}$ and of the "big" arc $\overline{b_{-N}b_{N+1}}$. The integral, more precisely, n-times the integral, on the left over the two small arcs is $o(1)$ (recall that $|z|$ is small, while on those arcs $|t|$ stays away from 0), and now we estimate the integral over the "big" arc, i.e. we consider

$$n\int_{\overline{b_{-N}b_{N+1}}} \log\frac{|z-t|}{|t|}\,d\nu_\Gamma(t) = n\int_{\Re b_{-N}}^{\Re b_{N+1}} \log\frac{|z-\gamma(t)|}{|\gamma(t)|}\omega_\Gamma(t)|\gamma'(t)|\,dt. \tag{7.24}$$

By the definition of the points b_k we have $b_1 = (1/2+o(1))/n$ and

$$\frac{N}{n} = \nu_\Gamma(\overline{b_1b_{N+1}}) = \int_{\Re b_1}^{\Re b_{N+1}} \omega_\Gamma(\gamma(t))|\gamma'(t)|\,dt,$$

and the same reasoning as between (7.13) and (7.14) yields

$$\Re b_{N+1} = \frac{N+1/2}{n\omega_\Gamma(0)} + O((N/n)^2).$$

Similarly,

$$\Re b_{-N} = \frac{-N+1/2}{n\omega_\Gamma(0)} + O((N/n)^2).$$

If $z=\gamma(\zeta)=\zeta+i\gamma_2(\zeta)$, then in the integrand in (7.24) we have

$$\omega_\Gamma(\gamma(t)) = \omega_\Gamma(0)+O(|t|), \quad |\gamma'(t)| = 1+O(t^2),$$
$$\log|\gamma(t)| = \log(|t|+O(t^2)) = \log|t| + O(|t|),$$

and (with $\gamma(t)=t+i\gamma_2(t)$)

$$\log|\gamma(\zeta)-\gamma(t)| = \log\sqrt{(\zeta-t)^2+(\gamma_2(\zeta)-\gamma_2(t))^2},$$

where

$$\gamma_2(\zeta)-\gamma_2(t) = \gamma_2'(\zeta)(\zeta-t)+O((\zeta-t)^2) = O(|\zeta|\,|\zeta-t|)+O((\zeta-t)^2).$$

Therefore, since $|\zeta|\le n^{-\tau}$ and $|\zeta-t|\le CN/n$, we have

$$\log|\gamma(\zeta)-\gamma(t)| = \log|\zeta-t| + O(n^{-2\tau}) + O((N/n)^2).$$

By substituting all these into (7.24) we find that with

$$M_1 = (-N + 1/2)/n\omega_\Gamma(0), \qquad M_2 = (N + 1/2)/n\omega_\Gamma(0),$$

the expression in (7.24) is equal to

$$n\int_{M_1+O((N/n)^2)}^{M_2+O((N/n)^2))} \log\left|\frac{\zeta - t}{t}\right|\omega_\Gamma(0)\,dt = n\int_{M_1}^{M_2} \log\left|\frac{\zeta - t}{t}\right|\omega_\Gamma(0)\,dt + O((N/n)^2)$$

plus an error term which is at most

$$\begin{aligned} &nO((N/n)^2) + nO(N/n)O(n^{-2\tau}) + nO((N/n)^3) \\ &\quad = O(n^{6(1-\tau)-1}) + O(n^{3(1-\tau)-2\tau}) + O(n^{9(1-\tau)-2}) = o(1) \end{aligned}$$

if (7.18) is satisfied.

From what we have done so far, it follows, say for $0 \le \zeta = \Re z \le n^{-\tau}$, that with $M = N/n\omega_\Gamma(0)$,

$$\log|\mathcal{B}_n(z)| = o(1) - n\omega_\Gamma(0)\int_{-M}^{M} (\log|\zeta - t| - \log|t|)\,dt.$$

But

$$\begin{aligned} \int_{-M}^{M} (\log|\zeta - t| - \log|t|)\,dt &= \int_{M-\zeta}^{M} \log\frac{u+\zeta}{u}\,du = \int_{M-\zeta}^{M} O(\zeta/u)\,du \\ &= O(\zeta^2/M) = O(\zeta^2 n/N), \end{aligned}$$

hence

$$\begin{aligned} \log|\mathcal{B}_n(z)| = O(n\zeta^2(n/N)) + o(1) &= O(n^{2-2\tau-3(1-\tau)}) + o(1) \\ &= O(n^{-(1-\tau)}) + o(1) = o(1) \end{aligned}$$

for all $z \in \Gamma$, $|z| \le n^{-\tau}$, provided τ satisfies (7.18). Thus, when $|z| \le n^{-\tau}$,

$$|\mathcal{B}_n(z)| = 1 + o(1). \tag{7.25}$$

All the reasonings so far used the assumption (7.18), which can be satisfied by choosing $\tau < 1$ sufficiently close to 1.

7.2.6. The square integral of $\mathcal{C}_n$ for $|z| \le n^{-\tau}$. Using (7.20), (7.21) and (7.25) we can now estimate the square integral of $|\mathcal{C}_n(z)|$ against the measure μ over the arc $\Gamma \cap \Delta_{n^{-\tau}}(0)$. Indeed, let $\Re\Gamma_n$ be the projection of Γ_n (see (7.19)) onto the real line. Then $\Re\Gamma_n$ is an interval $[-\alpha_n, \beta_n]$ minus all the intervals

$$I_k = \left(\frac{j_{\beta,k}}{n\pi\omega_\Gamma(0)} - \frac{1}{n^{1+\rho}}, \frac{j_{\beta,k}}{n\pi\omega_\Gamma(0)} + \frac{1}{n^{1+\rho}}\right).$$

Here $\alpha_n, \beta_n \sim n^{-\tau}$, and the $|k|$ in the latter intervals is at most $2n^{1-\tau}$ (see (7.3)). Therefore (use also

$$d\mu(z) = w(z)|z|^\alpha \, ds_\Gamma(z) = (1+o(1))w(0)|x|^\alpha \, dx$$

and $|\gamma'(t)| = 1 + o(1)$ for $t = O(n^{-\tau})$),

$$\int_{\Gamma\cap\Delta_{n^{-\tau}}(0)} |\mathcal{C}_n(z)|^2 \, d\mu(z) = (1+o(1)) \int_{\Re\Gamma_n} \mathcal{J}_\beta(n\pi\omega_\Gamma(0)x)^2 w(0)|x|^\alpha \, dx$$
$$+ C \int_{\bigcup_k I_k} |x|^\alpha \, dx.$$

In view of (7.2) the first integral is at most

$$\frac{(1+o(1))w(0)}{(n\pi\omega_\Gamma(0))^{\alpha+1}} L_\alpha$$

with the L_α defined in (3.4). The second integral is at most

$$C \sum_{k=1}^{2n^{1-\tau}} \frac{1}{n^{1+\rho}} \left(\frac{k}{n}\right)^\alpha = O(n^{(1-\tau)(\alpha+1)-\alpha-1-\rho}) = o(n^{-\alpha-1})$$

because of (7.16).

Combining these we can see that

$$\limsup_{n\to\infty} n^{\alpha+1} \int_{\Gamma\cap\Delta_{n^{-\tau}}(0)} |\mathcal{C}_n(z)|^2 \, d\mu(z) \le \frac{w(0)L_\alpha}{(\pi\omega_\Gamma(0))^{\alpha+1}}. \tag{7.26}$$

7.2.7. The estimate of $\mathcal{C}_n(z)$ for $|z| > n^{-\tau}$. Now let $z \in \Gamma$, $|z| > n^{-\tau}$, say $0 \prec z$. In view of (7.3) and of the definition of a_k and b_k,

$$\nu_\Gamma(\overline{0a_k}) = k/n + O(n^{-1}), \qquad \nu_\Gamma(\overline{0b_k}) = k/n + O(n^{-1}), \qquad k > 0.$$

A similar relation holds for negative k. These imply

$$a_k - b_k = O(n^{-1}), \tag{7.27}$$

and so there is an integer T_0 (independent of n) such that

$$b_{k-T_0} \prec a_k \prec b_{k+T_0} \qquad \text{for } k > T_0,$$

and similarly

$$b_{-k-T_0} \prec a_{-k} \prec b_{-k+T_0} \qquad \text{for } k > T_0.$$

Since Γ is C^2-smooth, this implies the existence of a $\delta > 0$ and a T (actually, $T = T_0 + 1$ will suffice) such that if $|z| \le \delta$ (and if z also satisfies the previous condition that $z \in \Gamma$, $0 \prec z$) then

(i) $z \preceq a_k$, $T < k \le N$ imply

$$|z - a_k| < |z - \xi_{k+T}|, \qquad |a_k| > |\xi_{k-T}|,$$

(ii) $a_k \prec z$, $T < k \le N$ imply

$$|z - a_k| < |z - \xi_{k-T}|, \qquad |a_k| > |\xi_{k-T}|,$$

(iii) $a_k \prec z$, $-N \le k < -T$ imply

$$|z - a_k| < |z - \xi_{k-T}|, \qquad |a_k| > |\xi_{k+T}|.$$

For this particular $z \in \Gamma$, $0 \prec z$, $\delta > |z| > n^{-\tau}$ we shall compare the value $|\mathcal{C}_n(z)|$ with the value of a modified polynomial $|\tilde{\mathcal{C}}_n(z)|$, which we obtain as follows. Remove all factors $|1 - z/a_k|$ from $|\mathcal{C}_n(z)|$ with $|k| \le T$, and then

- **(i′)** for $z \preceq a_k$, $T < k \le N$ replace the factor $|1 - z/a_k| = |a_k - z|/|a_k|$ in $|\mathcal{C}_n(z)|$ by $|z - \xi_{k+T}|/|\xi_{k-T}|$,
- **(ii′)** for $a_k \prec z$, $T < k \le N$ replace the factor $|a_k - z|/|a_k|$ in $|\mathcal{C}_n(z)|$ by $|z - \xi_{k-T}|/|\xi_{k-T}|$,
- **(iii′)** for $a_k \prec z$, $-N \le k < -T$ replace the factor $|a_k - z|/|a_k|$ in $|\mathcal{C}_n(z)|$ by $|z - \xi_{k-T}|/|\xi_{k+T}|$.

Removing a factor $|1 - z/a_k|$ from $|\mathcal{C}_n(z)|$ decreases the absolute value of the polynomial by at most a factor $1/C_2 n$ with some C_2 because each a_k, $k \ne 0$, is $\ge c/n$ in absolute value. On the other hand, the replacements in **(i′)–(iii′)** increase the absolute value of the polynomial at z because of **(i)–(iii)**. Hence,

$$|\mathcal{C}_n(z)| \le C_3 n^{2T} |\tilde{\mathcal{C}}_n(z)|.$$

But $|\tilde{\mathcal{C}}_n(z)|$ has the form

$$|\tilde{\mathcal{C}}_n(z)| = \frac{\prod^* |z - \xi_k|}{\prod^{**} |\xi_k|},$$

where all $|z - \xi_k|$, $-l_0 \le k < l_1$, appear in $\prod^*$ except at most $5T$ of them (at most $2T$ around z, at most $2T$ around 0, and at most T around a_N), and where some $|z - \xi_k|$ may appear twice, but at most T of them (all around a_N). Therefore, if z also satisfies $|z - \xi_k| \ge n^{-4}$ for all $-l_0 \le k \le l_1 - 1$, then

$$\prod\nolimits^* |z - \xi_k| \le \Big(\prod_{k=-l_0,\, k \ne 0}^{l_1 - 1} |z - \xi_k| \Big) \operatorname{diam}(\Gamma)^T (n^4)^{5T}.$$

A similar reasoning gives that in $\prod^{**}$ all $|\xi_k|$ appear except perhaps $2T$ of them, and none of the ξ_k is repeated twice, therefore,

$$\prod\nolimits^{**} |\xi_k| \ge \Big(\prod_{k=-l_0,\, k \ne 0}^{l_1 - 1} |\xi_k| \Big) \frac{1}{\operatorname{diam}(\Gamma)^{2T}}.$$

Hence,

$$|\mathcal{C}_n(z)| \le C_3 n^{2T} |\tilde{\mathcal{C}}_n(z)| \le C_4 n^{22T} \prod_{k=-l_0,\, k \ne 0}^{l_1 - 1} \frac{|z - \xi_k|}{|\xi_k|}.$$

But the product on the right is $|B_n(z)/B_n(0)|$ with B_n from (7.9), for which the bound (7.10) is true. Hence, we conclude that

$$|\mathcal{C}_n(z)| \le C_5 n^{22T} \tag{7.28}$$

under the condition that $|z - \xi_k| \ge n^{-4}$ for all k.

This reasoning was made for $|z| \le \delta$ and $0 \prec z$. The case $|z| \le \delta$, $z \prec 0$ is completely similar. On the other hand, if $z \in \Gamma$, $|z| > \delta$, then we use, for all $-N \le k \le N$, $k \ne 0$,

$$|z - a_k| = |z - \xi_k + O(n^{-1})| = |z - \xi_k|(1 + O(n^{-1}))|$$

because all a_k, ξ_k with $|k| \le N$ lie at distance $\le CN/n = O(n^{3(1-\tau)-1}) = o(1)$ from the origin. Thus, if we replace every $|z - a_k|$ in $\mathcal{C}_n(z)$, $|k| \le N$, $k \ne 0$, by $|z - \xi_k|$, then the value of the polynomial can decrease by at most a factor $(1 + O(n^{-1}))^n = O(1)$. We also want to replace each $|a_k|$ by $|\xi_k|$:

$$\prod_{k=1}^{N} |a_k| \ge \prod_{k=1}^{T} |a_k| \prod_{k=T+1}^{N} |\xi_{k-T}| \ge cn^{-T} \prod_{k=1}^{N} |\xi_k|$$

because $|a_k| \ge |\xi_{k-T}|$ for $k > T$ and $|a_k| \ge c/n$ for all $k \ne 0$. A similar estimate holds for negative values, from which we get

$$|\mathcal{C}_n(z)| \le Cn^{2T} \prod_{k \ne 0} \frac{|z - \xi_k|}{|\xi_k|} \le CC_0 n^{2T},$$

since the last product is just $|B_n(z)/B_n(0)|$, for which we can use (7.10).

Therefore, for $|z| > \delta$ we can again claim the bound (7.28).

All in all, we have proven (7.28) on Γ with the exception of those $z \in \Gamma$ for which there is a ξ_k such that $|z - \xi_k| < n^{-4}$. This exceptional set has arc measure at most $Cn \cdot n^{-4} = Cn^{-3}$, so an application of Lemma 2.6 shows that the bound

$$|\mathcal{C}_n(z)| \le C_5 n^{22T} \tag{7.29}$$

holds throughout Γ.

7.2.8. Completion of the upper estimate for a single arc. Let

$$P_n(z) = \mathcal{C}_n(z) S_{n,0,\Gamma}(z),$$

where $\mathcal{C}_n(z)$ is as in (7.11) and $S_{n,0,\Gamma}(z)$ is the fast decreasing polynomial from Corollary 2.2 for $K = \Gamma$ and for the point 0. This P_n has degree $(1 + o(1))n$, its value is 1 at the origin, and $|P_n(z)| \le |\mathcal{C}_n(z)|$ on Γ. On $\Gamma \cap \Delta_{n^{-\tau}}(0)$ we just use $|P_n(z)| \le |\mathcal{C}_n(z)|$, while for $|z| > n^{-\tau}$ we see from (7.29) and (2.4) that

$$|P_n(z)| \le 2C_5 n^{22T} C_\tau e^{-c_\tau n^{\tau_0}} = o(n^{-\alpha-1}).$$

As a consequence,

$$\limsup_{n\to\infty} n^{\alpha+1} \int_\Gamma |P_n(z)|^2 \, d\mu(z) \le \limsup_{n\to\infty} n^{\alpha+1} \int_{\Gamma \cap \Delta_{n^{-\tau}}(0)} |\mathcal{C}_n(z)|^2 \, d\mu(z).$$

Since the integral on the left is an upper bound for $\lambda_{\deg(P_n)}(\mu, 0)$, from (7.26) (using also (2.2)) we obtain

$$\limsup_{n\to\infty} n^{\alpha+1} \lambda_n(\mu, 0) \le \frac{w(0) L_\alpha}{(\pi \omega_\Gamma(0))^{\alpha+1}}. \tag{7.30}$$

This proves one half of Proposition 7.1 for a single arc.

7.3. *The upper estimate for several components*

In this section, we sketch what to do with the preceding reasoning when Γ may have several components which can be C^2 Jordan curves or arcs. Let $\Gamma_0, \ldots, \Gamma_{k_0}$ be the different components of Γ, and assume that $z_0 = 0$ belongs to Γ_0. Assume that Γ_0 is a Jordan arc; actually this is the only case we shall use below, i.e. when z_0 belongs to an arc component of Γ, and the other components are Jordan curves. On Γ_0 we introduce the points a_k as before; there is no need for them on the other components of Γ (they played a role above only in a small neighborhood of 0).

On the other hand, on the whole Γ we introduce the analogues of the points ξ_k by repeating the process in [24, Section 2]. The outline is as follows. Let $\theta_j = \nu_\Gamma(\Gamma_j)$, consider the integers $n_j = [\theta_j n]$, and divide each Γ_j, $j > 0$, into n_j arcs I_k^j each having equal weight θ_j/n_j with respect to ν_Γ, i.e. $\nu_\Gamma(I_k^j) = \theta_j/n_j$. On Γ_0 introduce the points b_k as before, and the arcs $I_k^0 = \overline{b_k b_{k+1}}$. Let ξ_k^j be the center of mass of the arc I_k^j with respect to ν_Γ, and consider the polynomial

$$R_n(z) = \prod_{j,k} (z - \xi_k^j) \tag{7.31}$$

of degree at most $n + O(1)$. Now the polynomial

$$B_n(z) = R_n(z)/(z - \xi_0^0) \tag{7.32}$$

will have similar properties to the B_n before, namely (7.10) is true (see [24, Section 2], in particular Propositions 2.4 and 2.5).

The rest of the argument in the preceding subsections does not change: the components of Γ_l, $l \geq 1$, are far from $z_0 = 0$, the corresponding estimates in the above proof on them are the same as the estimate in the preceding subsections for $|z| > \delta$.

7.4. *The lower estimate in Theorem 1.1 on Jordan arcs*

In this section, the assumption is the same as before, namely that Γ consists of finitely many C^2-smooth Jordan arcs and curves, z_0 belongs to an arc component of Γ, and μ is given by (5.1). Our aim is to prove the necessary lower bound for $\lambda_n(\mu, z_0)$.

In this proof we shall closely follow the proof of [24, Theorem 3.1].

Let Ω be the unbounded component of $\overline{\mathbb{C}} \setminus \Gamma$, and denote by g_Ω the Green's function of Ω with respect to the pole at infinity (see e.g. [16, Sec. 4.4]).

Assume to the contrary that there are infinitely many n and for each n a polynomial Q_n of degree at most n such that $Q_n(z_0) = 1$ and

$$n^{1+\alpha} \int |Q_n|^2 \, d\mu < (1 - \delta) \frac{w(z_0) L_\alpha}{(\pi \omega_\Gamma(z_0))^{\alpha+1}} \tag{7.33}$$

with some $\delta > 0$, where L_α was defined in (3.4). The strategy will be to show that this implies the following: *there exists another system* Γ^* *of piecewise* C^2*-smooth Jordan* *curves and an extension of* w *to* Γ^* *such that* $\Gamma \subseteq \Gamma^*$, *in a neighborhood* Δ_0 *of* z_0 *we have* $\Gamma \cap \Delta_0 = \Gamma^* \cap \Delta_0$, *and for the measure*

$$d\mu^*(z) = w(z)|z - z_0|^\alpha \, ds_{\Gamma^*}(z) \tag{7.34}$$

with support Γ^*,

$$\liminf_{n\to\infty} n^{1+\alpha}\lambda_n(\mu^*, z_0) < \frac{w(z_0)L_\alpha}{(\pi\omega_{\Gamma^*}(z_0))^{\alpha+1}}. \tag{7.35}$$

Since this contradicts Proposition 6.1, (7.33) cannot be true.

Let $\Gamma_0, \dots, \Gamma_{k_0}$ be the connected components of Γ, Γ_0 being the one that contains z_0. We shall only consider the case when Γ_0 is a Jordan arc; when Γ_0 is a Jordan curve, the argument is similar (see [24, Section 3]).

Let $\mathbf{n}_\pm$ be the two normals to Γ_0 at z_0, and let $A_\pm = \partial g_\Omega(z_0)/\partial\mathbf{n}_\pm$ be the corresponding normal derivatives of the Green's function of Ω with pole at infinity. Assume, for example, that $A_+ \ge A_-$. Note that $A_- > 0$ [24, Section 3].

Let $\varepsilon > 0$ be an arbitrarily small number. For each Γ_j that is a Jordan arc, connect the two endpoints of Γ_j by another C^2-smooth Jordan arc Γ_j' that lies close to Γ_j so that we obtain a system Γ' of $k_0 + 1$ Jordan curves with boundary $(\bigcup_j \Gamma_j) \cup (\bigcup_j \Gamma_j')$. Assume also that Γ_0' is selected so that $\mathbf{n}_+$ is the outer normal to Γ' at z_0. This can be done in such a way that (with Ω' being the unbounded component of $\overline{\mathbb{C}} \setminus \Gamma'$)

$$\frac{\partial g_{\Omega'}(z_0)}{\partial\mathbf{n}_+} > \frac{1}{1+\varepsilon}\frac{\partial g_\Omega(z_0)}{\partial\mathbf{n}_+} \tag{7.36}$$

(see [24, Section 3]).

Select a small disk Δ_0 about z_0 for which $\Gamma' \cap \Delta_0 = \Gamma \cap \Delta_0$, and, as in [24, Section 3], choose a lemniscate $\sigma = \{z : |T_N(z)| = 1\}$ (with some polynomial T_N of degree equal to some integer N) such that Γ' lies in the interior of σ (i.e. in the union of the bounded components of $\mathbb{C} \setminus \sigma$) except for the point z_0, where σ and Γ' touch each other, and (with Ω_σ being the unbounded component of $\overline{\mathbb{C}} \setminus \sigma$)

$$\frac{\partial g_{\Omega_\sigma}(z_0)}{\partial\mathbf{n}_+} > \frac{1}{1+\varepsilon}\frac{\partial g_\Omega(z_0)}{\partial\mathbf{n}_+}. \tag{7.37}$$

For the Green's function associated with the outer domain Ω_σ of σ we have [24, (3.6)]

$$\frac{\partial g_{\Omega_\sigma}(z_0)}{\partial\mathbf{n}_+} = \frac{|T_N'(z_0)|}{N}. \tag{7.38}$$

For a small a let σ_a be the lemniscate $\sigma_a := \{z : |T_N(z)| = e^{-a}\}$. According to [24, Section 3], if $\Delta \subset \Delta_0$ is a fixed small neighborhood of z_0, then for sufficiently small a this σ_a contains $\Gamma' \setminus \Delta$ in its interior, while in Δ the two curves Γ_0 and σ_a intersect in two points U, V (see Figure 4). The points U and V are connected by the arc $\overline{UV}_{\Gamma_0}$ on Γ_0 and also by the arc $\overline{UV}_{\sigma_a}$ on σ_a (there are actually two such arcs on σ_a; we take the one lying in Δ). For each Γ_j which is a Jordan arc, connect the two endpoints of Γ_j by a new C^2 Jordan arc Γ_j^* going inside Γ' so that on Γ_j^* we have

$$g_\Omega(z) \le a^2, \qquad z \in \Gamma_j^*. \tag{7.39}$$

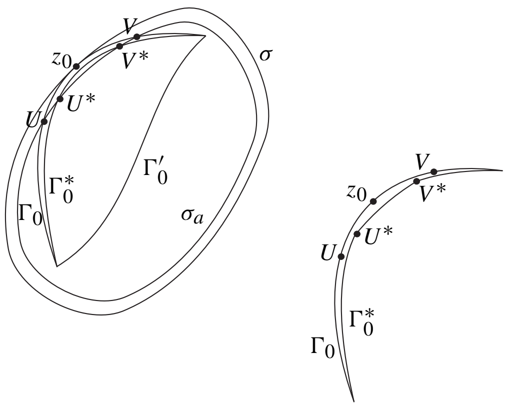

Fig. 4

In addition, Γ_0^* can be selected so that in Δ it intersects σ_a in two points U^*, V^*. Then $\overline{U^*V^*}_{\sigma_a}$ is a subarc of $\overline{UV}_{\sigma_a}$. Let now Γ^* be the union of Γ, of the Γ_j^*'s with $j > 0$, of $\Gamma_0^* \setminus \overline{U^*V^*}_{\Gamma_0^*}$ and of $\overline{U^*V^*}_{\sigma_a}$. Then Γ^* is the union of $k_0 + 1$ piecewise smooth Jordan curves.

Now let

$$m = [(1+\varepsilon)^7 A_- n / N A_+] \tag{7.40}$$

and consider the polynomial

$$P_{n+mN}(z) = Q_n(z) T_N(z)^m \tag{7.41}$$

on Γ^* with the Q_n from (7.33), and let μ^* be the measure in (7.34) on Γ^*. For the polynomials P_{n+Nm} it was shown in [24, (3.18)–(3.20)] that

$$|P_{n+mN}(z)| \le C_1 n^{1/2} e^{na^2 - ma} \qquad \text{on } \Gamma^* \setminus (\overline{UV}_{\Gamma_0} \cup \overline{U^*V^*}_{\sigma_a}), \tag{7.42}$$

$$|P_{n+mN}(z)| \le |Q_n(z)| \qquad \text{on } \overline{UV}_{\Gamma_0}, \tag{7.43}$$

$$|P_{n+mN}(z)| \le C_1 n^{1/2} \exp\big(n(1+\varepsilon)^4 a A_- / |T_N'(z_0)| - ma\big) \qquad \text{on } \overline{U^*V^*}_{\sigma_a}, \tag{7.44}$$

where C_1 is a fixed constant. Here, by the choice of m in (7.40), and by (7.37) and (7.38), the last exponent is at most

$$n\left(\frac{(1+\varepsilon)^5 a A_-}{A_+ N} - \frac{(1+\varepsilon)^6 a A_-}{N A_+}\right) = -\varepsilon n \frac{(1+\varepsilon)^5 a A_-}{N A_+}.$$

Fix a so small that $a^2 - aA_-/NA_+ < 0$. Then the inequality $|T_N(z)| \le 1$ for $z \in \Gamma^*$ and the estimates (7.42)–(7.44) yield

$$\lambda_{n+mN}(\mu^*, z_0) \le \int |P_{n+mN}|^2 \, d\mu^* \le \int |Q_n|^2 \, d\mu + O(n^{-\alpha-2}).$$

Hence, by (7.33), for infinitely many n,

$$(n+mN)^{\alpha+1}\lambda_{n+mN}(\mu^*, z_0) \le \left(\frac{n+mN}{n}\right)^{\alpha+1}(1-\delta)\frac{w(z_0)L_\alpha}{(\pi\omega_\Gamma(z_0))^{\alpha+1}} + o(1). \qquad (7.45)$$

Since [24, (3.22)–(3.23)]

$$\omega_\Gamma(z_0) = \frac{1}{2\pi}\left(\frac{\partial g_\Omega}{\partial \mathbf{n}_+} + \frac{\partial g_\Omega}{\partial \mathbf{n}_-}\right) = \frac{1}{2\pi}(A_+ + A_-), \qquad (7.46)$$

$$\omega_{\Gamma^*}(z_0) = \frac{1}{2\pi}\frac{\partial g_{\Omega^*}(z_0)}{\partial \mathbf{n}_+} \le \frac{1}{2\pi}\frac{\partial g_\Omega(z_0)}{\partial \mathbf{n}_+} = \frac{1}{2\pi}A_+, \qquad (7.47)$$

we have

$$\begin{aligned}
&\left(\frac{n+mN}{n}\right)^{\alpha+1}(1-\delta)\frac{w(z_0)L_\alpha}{(\pi\omega_\Gamma(z_0))^{\alpha+1}}\\
&\quad\le \left(1+(1+\varepsilon)^7\frac{A_-}{A_+}\right)^{\alpha+1}(1-\delta)\frac{w(z_0)L_\alpha}{(\pi\omega_{\Gamma^*}(z_0))^{\alpha+1}}\left(\frac{A_+}{A_+ + A_-}\right)^{\alpha+1}\\
&\quad\le \left(1-\frac{\delta}{2}\right)\frac{w(z_0)L_\alpha}{(\pi\omega_{\Gamma^*}(z_0))^{\alpha+1}}
\end{aligned}$$

if ε is sufficiently small. Therefore, (7.45) implies

$$\liminf_{n\to\infty}(n+mN)^{\alpha+1}\lambda_{n+mN}(\mu^*, z_0) \le \left(1-\frac{\delta}{2}\right)\frac{w(z_0)L_\alpha}{(\pi\omega_{\Gamma^*}(z_0))^{\alpha+1}},$$

which is impossible according to Proposition 6.1. This contradiction shows that (7.33) is impossible, and so

$$\liminf_{n\to\infty} n\lambda_n(\mu, z_0) \ge \frac{w(z_0)L_\alpha}{(\pi\omega_\Gamma(z_0))^{\alpha+1}}. \qquad (7.48)$$

(7.30) and (7.48) prove Proposition 7.1.

8. Proof of Theorem 1.1

Let Γ be as in the theorem, and let $\Gamma = \bigcup_{k=0}^{k_0}\Gamma^k$ be the decomposition of Γ into connected components. Let Ω be the unbounded component of $\overline{\mathbb{C}}\setminus\Gamma$. We may assume that $z_0 \in \Gamma_0$. By assumption, z_0 lies on a C^2-smooth arc J of $\partial\Omega$, and there is an open set O such that $J = \Gamma\cap O$. Let $\Delta_\delta(z_0)$ be a small disk about z_0 that lies in O together with its closure. Now there are two possibilities for J:

Type I: only one side of J belongs to Ω.
Type II: both sides of J belong to Ω.

Type I occurs when $\Gamma^0\setminus\Delta_\delta(z_0)$ is connected, and Type II occurs otherwise.

Let $g_\Omega(z)$ be the Green's function for the domain Ω with pole at infinity, which we assume to be defined to be 0 outside Ω. The proof of Theorem 1.1 is based on the following propositions.

Proposition 8.1. *If J is of Type I, then there is a sequence $\{\Gamma_m\}$ of sets consisting of disjoint C^2-smooth Jordan curves Γ_m^k, $k = 0, 1, \ldots, k_0$, such that with some positive sequence $\{\varepsilon_m\}$ tending to 0 we have*

- **(i)** $z_0 \in \Gamma_m^0$ *and* $\Gamma \cap \overline{\Delta_\delta(z_0)} = \Gamma_m \cap \overline{\Delta_\delta(z_0)}$,
- **(ii)** $\frac{1}{1+\varepsilon_m}\omega_\Gamma(z_0) \le \omega_{\Gamma_m}(z_0) \le (1+\varepsilon_m)\omega_\Gamma(z_0)$,
- **(iii)** $\max_{x\in\Gamma_m} g_\Omega(z) \le \varepsilon_m$ *and* $\max_{x\in\Gamma} g_{\Omega_m}(z) \le \varepsilon_m$,
- **(iv)** *the Hausdorff distance of the outer boundaries of Γ and Γ_m tends to 0 as $m \to \infty$.*

Property **(i)** means that in the δ-neighborhood of z_0 the sets Γ_m and Γ coincide.

Proposition 8.2. *If J is of Type II, then there is a sequence $\{\Gamma_m\}$ of sets consisting of $\Gamma_m^0 := J \cap \overline{\Delta_\delta(z_0)}$ and of disjoint C^2 Jordan curves Γ_m^k, $k = 1, \ldots, k_0 + 2$, lying in the component of Γ_m^0 such that* **(i)**–**(iv)** *above hold.*

Pending the proofs of these propositions we now complete the proof of Theorem 1.1. It follows from **(i)** and **(iv)** that there is a compact set K that contains Γ and all Γ_m such that z_0 lies on the outer boundaries of K, and in a neighborhood of z_0 the outer boundaries of K and Γ are the same. In particular, there is a circle in the unbounded component of $\overline{\mathbb{C}} \setminus K$ that contains z_0 on its boundary, so we can apply Lemma 2.1 to K and z_0.

Fix an m and consider the set Γ_m either from Proposition 8.1 if J is of Type I or from Proposition 8.2 if J is of Type II. We define the measure

$$\mu_m(z) = w(z)|z - z_0|^\alpha \, ds_{\Gamma_m}(z),$$

where w is a continuous and positive extension of the original w (existing on J) from $J \cap \overline{\Delta_\delta(z_0)}$ to Γ_m. It follows from the Erdős–Turán criterion [19, Theorem 4.1.1] that μ_m is in the **Reg** class.

For a positive integer n let P_n be the extremal polynomial of degree n for $\lambda_n(\mu, z_0)$. Consider the polynomial $S_{4n\varepsilon_m/c_2\delta^2, z_0, K}(z)$ from Lemma 2.1 with $\gamma = 2$ (here c_2 is the constant from Lemma 2.1), and form the product $Q_n(z) = P_n(z) S_{4n\varepsilon_m/c_2\delta^2, z_0, K}(z)$. This is a polynomial of degree at most $n(1 + 4\varepsilon_m/c_2\delta^2)$ which takes the value 1 at z_0. On $\Gamma_m \cap \overline{\Delta_\delta(z_0)} = \Gamma \cap \overline{\Delta_\delta(z_0)}$ we have

$$\int_{\Gamma_m \cap \overline{\Delta_\delta(z_0)}} |Q_n(z)|^2 \, d\mu(z) \le \int_{\Gamma \cap \overline{\Delta_\delta(z_0)}} |P_n(z)|^2 \, d\mu(z) \le \lambda_n(\mu, z_0). \tag{8.1}$$

Since the $L^2(\mu)$-norms of $\{P_n\}$ are bounded, it follows from $\mu \in$ **Reg** that there is an n_m such that if $n \ge n_m$ then

$$\|P_n\|_\Gamma \le e^{\varepsilon_m n}.$$

Then, by the Bernstein–Walsh lemma (Lemma 2.10) and by property **(iii)**, for all $z \in \Gamma_m$ we have

$$|P_n(z)| \le \|P_n\|_\Gamma e^{n g_\Omega(z)} \le e^{2n\varepsilon_m}.$$

Therefore, (2.3) and $\Gamma_m \subseteq K$ imply that for $z \in \Gamma_m \setminus \overline{\Delta_\delta(z_0)}$,

$$|Q_n(z)| \leq \exp(2n\varepsilon_m - [4n\varepsilon_m/c_2\delta^2]c_2\delta^2) < e^{-n\varepsilon_m}$$

if n is sufficiently large. As a consequence, the integral of Q_n over $\Gamma_m \setminus \overline{\Delta_\delta(z_0)}$ is exponentially small in n, which, combined with (8.1), yields

$$\lambda_{n(1+4\varepsilon_m/c_2\delta^2)}(\mu_m, z_0) \leq \lambda_n(\mu, z_0) + o(n^{-(1+\alpha)}).$$

Now we multiply both sides by $n(1+4\varepsilon_m/c_2\delta^2)^{1+\alpha}$ and let n tend to infinity. Since Theorem 1.1 has already been proven for Γ_m and for the measure μ_m (see Proposition 7.1), we conclude (using also (2.1)) that

$$\liminf_{n\to\infty} n^{\alpha+1}\lambda_n(\mu, z_0) \geq \frac{1}{1+4\varepsilon_m/c_2\delta^2}\,\frac{w(z_0)}{(\pi\omega_{\Gamma_m}(z_0))^{\alpha+1}}L_\alpha$$

(with the L_α from (3.4)), and an application of property **(ii)** yields

$$\liminf_{n\to\infty} n^{\alpha+1}\lambda_n(\mu, z_0) \geq \frac{1}{(1+\varepsilon_m)^{|\alpha|+1}(1+4\varepsilon_m/c_2\delta^2)}\,\frac{w(z_0)}{(\pi\omega_\Gamma(z_0))^{\alpha+1}}L_\alpha.$$

If we reverse the roles of Γ and Γ_m in this argument, we similarly conclude that

$$\limsup_{n\to\infty} n^{\alpha+1}\lambda_n(\mu, z_0) \leq (1+\varepsilon_m)^{|\alpha|+1}(1+4\varepsilon_m/c_2\delta^2)\frac{w(z_0)}{(\pi\omega_\Gamma(z_0))^2}L_\alpha.$$

Finally, in these last two relations we can let $m \to \infty$, and as $\varepsilon_m \to 0$, the limit in Theorem 1.1 follows.

Thus, it remains to prove Propositions 8.1 and 8.2.

8.1. Proof of Proposition 8.1

Both in this proof and in the next one we shall use the fact that if $\Omega_1 \subset \Omega_2$ (say both with a smooth boundary), and $z \in \Omega_1$, then $g_{\Omega_1}(z) \leq g_{\Omega_2}(z)$. As a consequence, if z is a common point on their boundaries, then the normal derivative of g_{Ω_1} (the normal pointing to the interior of Ω_1) is not larger than the same normal derivative of g_{Ω_2} (because both Green's functions vanish on the common boundary). Since, modulo a factor of $1/2\pi$, the normal derivatives yield the equilibrium densities (see (8.2) and (8.4) below), it also follows that if $\Gamma_1 \subset \Gamma_2$, then on (an arc of) Γ_1 the equilibrium density ω_{Γ_2} is at most as large as ω_{Γ_1} (see also [17, Theorem IV.1.6(e)], according to which the equilibrium measure for Γ_1 is the balayage onto Γ_1 of the equilibrium measure of Γ_2).

Choose, for each m and $1 \leq k \leq k_0$, C^2-smooth Jordan curves Γ_m^k so that they lie in Ω and are at distance $< 1/m$ from Γ^k. For $k = 0$ the choice is somewhat different: let Γ_m^0 be a C^2 Jordan curve that lies in $\overline{\Omega}$, its distance from Γ^0 is smaller than $1/m$, $J \cap \overline{\Delta_\delta(z_0)} \subset \Gamma_m^0$, and $\Gamma_m^0 \setminus J$ lies in Ω (see Figure 5). We can select these so that the outer domains Ω_m of Γ_m are increasing with m. From this construction it is clear that **(i)** and **(iv)** are true. Now $\overline{\mathbb{C}} \setminus \Omega_m$ (the *polynomial convex hull* of Γ_m) is a shrinking

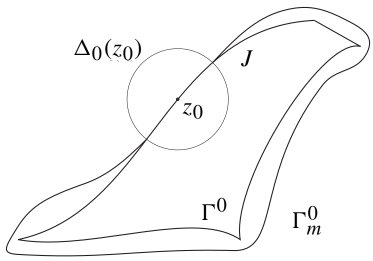

Fig. 5. The arc J and the selection of Γ^0_m.

sequence of compact sets with intersection $\overline{\mathbb{C}} \setminus \Omega$. Therefore, if cap denotes logarithmic capacity, then $\mathrm{cap}(\overline{\mathbb{C}} \setminus \Omega_m) \to \mathrm{cap}(\overline{\mathbb{C}} \setminus \Omega)$ [16, Theorem 5.1.3]. Since $\{g_\Omega(z) - g_{\Omega_m}(z)\}$ is a decreasing sequence of positive harmonic functions (more precisely, the subsequence starting from $g_\Omega(z) - g_{\Omega_l}(z)$ is harmonic in Ω_l) for which [16, Theorem 5.2.1]

$$g_\Omega(\infty) - g_{\Omega_m}(\infty) = \log \frac{1}{\mathrm{cap}(\overline{\mathbb{C}} \setminus \Omega)} - \frac{1}{\mathrm{cap}(\overline{\mathbb{C}} \setminus \Omega_m)} \to 0,$$

we deduce from Harnack's theorem [16, Theorem 1.3.9] that $g_\Omega(z) - g_{\Omega_m}(z) \to 0$ locally uniformly on compact subsets of Ω. This, and the fact that this sequence is defined in $\Omega \cap \Delta_\delta(z_0)$ and has boundary values identically 0 on $\partial\Omega \cap \Delta_\delta(z_0)$, then imply (see e.g. [11, Lemma 7.1]) the following: if $\mathbf{n}$ denotes the normal at z_0 in the direction of Ω then

$$\frac{\partial g_{\Omega_m}(z_0)}{\partial \mathbf{n}} \to \frac{\partial g_\Omega(z_0)}{\partial \mathbf{n}} \quad \text{as } m \to \infty.$$

But in the Type I situation we have (see [14, II.(4.1)] combined with [16, Theorem 4.3.14] or [17, Theorem IV.2.3] and [17, (I.4.8)])

$$\omega_\Gamma(z_0) = \frac{1}{2\pi} \frac{\partial g_\Omega(z_0)}{\partial \mathbf{n}}, \tag{8.2}$$

and a similar formula is true for ω_{Γ_m}, hence

$$\omega_{\Gamma_m}(z_0) \to \omega_\Gamma(z_0) \quad \text{as } m \to \infty.$$

This takes care of **(ii)**.

Finally, we use the following statement from [22, Theorem 7.1]:

Lemma 8.3. *Let S be a continuum. Then the Green's function $g_{\overline{\mathbb{C}}\setminus S}(z, \infty)$ is uniformly $1/2$-Hölder continuous on S, i.e. if $z_0 \in \Omega$, then*

$$g_{\overline{\mathbb{C}}\setminus S}(z_0, \infty) \le C \mathrm{dist}(z_0, S)^{1/2}. \tag{8.3}$$

Furthermore, here C can be chosen to depend only on the diameter of S.

If we apply this with $S = \Gamma^k$, $k = 0, \ldots, k_0$, and use $g_{\Omega_m}(z) \le g_{\Omega_m^k}(z)$ for each k (where, of course, Ω_m^k is the unbounded component of $\overline{\mathbb{C}} \setminus \Gamma_m^k$), then we can deduce the first inequality in **(iii)**. In this case (i.e. when J is of Type I), the second inequality in **(iii)** is trivial, since, by construction, g_{Ω_m} is identically 0 on Γ. □

8.2. Proof of Proposition 8.2

For an m let $J_{1,m}$ resp. $J_{2,m}$ be the two open subarcs of J of diameter $1/m$ that lie outside $\Delta_\delta(z_0)$, but which have one endpoint in $\overline{\Delta_\delta(z_0)}$ (see Figure 6) (for large m these exist).

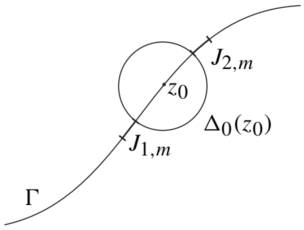

Fig. 6. The arcs $J_{1,m}$ and $J_{2,m}$.

Remove now $J_{1,m}$ and $J_{2,m}$ from Γ. Since we are in the Type II situation, after this removal the unbounded component of the complement of $\Gamma^0 \setminus (J_{1,m} \cup J_{2,m})$ is $\Omega \cup J_{1,m} \cup J_{2,m}$, and $\Gamma^0 \setminus (J_{1,m} \cup J_{2,m})$ splits into three connected components, one of them being $J \cap \overline{\Delta_\delta(z_0)}$; let $\Gamma^{0,1}$, $\Gamma^{0,2}$ be the other two. As $m \to \infty$ we have $\mathrm{cap}(\overline{\mathbb{C}} \setminus (\Omega \cup J_{1,m} \cup J_{2,m})) \to \mathrm{cap}(\overline{\mathbb{C}} \setminus \Omega)$, and since now the domains $\Omega \cup J_{1,m} \cup J_{2,m}$ are shrinking, we infer from Harnack's theorem as before that $g_\Omega(z) - g_{\Omega_m}(z) \to 0$ locally uniformly on compact subsets of Ω. This implies again that if $\mathbf{n}_\pm$ are the two normals to Γ at z_0 (note that now both point to the interior of Ω), then

$$\frac{\partial g_{\Omega \cup J_{1,m} \cup J_{2,m}}(z_0)}{\partial \mathbf{n}_\pm} \to \frac{\partial g_\Omega(z_0)}{\partial \mathbf{n}}$$

as $m \to \infty$. Since now (see [14, II.(4.1)] or [17, Theorem IV.2.3] and [17, (I.4.8)])

$$\omega_\Gamma(z_0) = \frac{1}{2\pi}\left(\frac{\partial g_\Omega(z_0)}{\partial \mathbf{n}_+} + \frac{\partial g_\Omega(z_0)}{\partial \mathbf{n}_-}\right), \tag{8.4}$$

we conclude again that

$$0 \le \omega_{\Gamma \setminus (J_{1,m} \cup J_{2,m})}(z_0) - \omega_\Gamma(z_0) < \varepsilon_m \tag{8.5}$$

with some $\varepsilon_m > 0$ that tends to 0 as $m \to \infty$. By selecting a somewhat larger ε_m we may also assume

$$g_{\Omega \cup J_{1,m} \cup J_{2,m}}(z) < \varepsilon_m, \qquad z \in J_{1,m} \cup J_{2,m} \tag{8.6}$$

(apply Lemma 8.3 to $S = \Gamma \cap \overline{\Delta_\delta(z_0)}$ and use $g_{\Omega \cup J_{1,m} \cup J_{2,m}}(z) \le g_{\overline{\mathbb{C}} \setminus (\Gamma \cap \overline{\Delta_\delta(z_0)})}(z)$).

For the continua $\Gamma^{0,1}, \Gamma^{0,2}, \Gamma_1, \dots, \Gamma_{k_0}$ and for a small $0 < \theta < 1/m$ select C^2-smooth Jordan curves $\gamma^{0,1}, \gamma^{0,2}, \gamma_1, \dots, \gamma_{k_0}$ that lie in $\Omega \cup J_{1,m} \cup J_{2,m}$ and are at distance $< \theta$ from the corresponding continuum. Let $\Gamma_{m,\theta}$ be the union of $J \cap \overline{\Delta_\delta(z_0)}$ and of these Jordan curves. Then $\Gamma_{m,\theta}$ consists (for small θ) of $k_0 + 2$ Jordan curves and one Jordan arc (namely $J \cap \overline{\Delta_\delta(z_0)}$), all of them C^2-smooth. According to the proof of Proposition 8.1 we have

$$\omega_{\Gamma_{m,\theta}}(z_0) \to \omega_{\Gamma \setminus (J_{1,m} \cup J_{2,m})}(z_0)$$

as $\theta \to 0$, therefore, for sufficiently small θ, we have (see (8.5))

$$-\varepsilon_m < \omega_{\Gamma_{m,\theta}}(z_0) - \omega_\Gamma(z_0) < \varepsilon_m.$$

Thus, if θ is sufficiently small, we have properties **(i)**, **(ii)** and **(iv)** in the proposition for $\Gamma_m = \Gamma_{m,\theta}$. The first inequality in **(iii)** follows exactly as at the end of the proof of Proposition 8.1. Finally, the second inequality in **(iii)** follows from (8.6) because

$$g_{\Omega_{m,\theta}}(z) \le g_{\Omega \cup J_{1,m} \cup J_{2,m}}(z)$$

(where $\Omega_{m,\theta}$ is the unbounded component of $\overline{\mathbb{C}} \setminus \Gamma_{m,\theta}$) and $g_{\Omega_{m,\theta}}(z) = 0$ if $z \in \Gamma$ unless $z \in J_{1,m} \cup J_{2,m}$.

These show that for sufficiently small θ we can select Γ_m in Proposition 8.2 to be $\Gamma_{m,\theta}$. □

9. Proof of Theorem 1.2

Let Γ be as in Theorem 1.2, and let $\Gamma = \bigcup_{k=0}^{k_0} \Gamma_k$ be its decomposition into connected components, Γ_0 being the one that contains z_0. We may assume that $z_0 = 0$. Set

$$\tilde{\Gamma} = \{z : z^2 \in \Gamma\}, \qquad \tilde{\Gamma}_k = \{z : z^2 \in \Gamma_k\}.$$

Every $\tilde{\Gamma}_k$ is the union $\Gamma_k^+ \cup \tilde{\Gamma}_k^-$ of two disjoint continua, where $\tilde{\Gamma}_k^- = -\tilde{\Gamma}_k^+$. Set $\tilde{\Gamma}^\pm = \bigcup_k \tilde{\Gamma}_k^\pm$. All the $\tilde{\Gamma}_k^\pm$ are disjoint, except when $k = 0$: then 0 is a common point of $\Gamma_0^\pm$, but except for that point, $\tilde{\Gamma}_0^+$ and $\tilde{\Gamma}_0^-$ are again disjoint. In general, we shall use the notation $\tilde{H}$ for the set of points z such that $z^2 \in H$, and if H is a continuum, then we represent $\tilde{H}$ as the union $\tilde{H}^+ \cup \tilde{H}^-$ of two continua, where $\tilde{H}^- = -\tilde{H}^+$, and $\tilde{H}^-$ and $\tilde{H}^+$ are disjoint except perhaps for the point 0 if 0 belongs to H.

Now $\tilde{\Gamma}_0^+ \cup \tilde{\Gamma}_0^-$ is connected, and if J is the C^2-smooth arc of Γ with one endpoint at $z_0 = 0$, then a direct calculation shows that $\tilde{J}$ is a C^2-smooth arc that lies on the outer boundary of $\tilde{\Gamma}$, and $\tilde{J}$ contains 0 in its (one-dimensional) interior. Thus, $\tilde{\Gamma}$ and $z_0 = 0$ satisfy the assumptions in Theorem 1.1.

For a measure μ defined on Γ let $\tilde{\mu}$ be the measure $d\tilde{\mu}(z) = \frac{1}{2}\, d\mu(z^2)$, i.e. if, say, $E \subset \tilde{\Gamma}^+$ is a Borel set and $E^2 = \{z^2 : z \in E\}$, then $\tilde{\mu}(E) = \frac{1}{2}\mu(E^2)$, and similarly for $E \subset \Gamma^-$. So $\tilde{\mu}$ is an even measure, which has the same total mass as μ has.

Let ν_Γ be the equilibrium measure of Γ. We claim that $\nu_{\tilde{\Gamma}} = \tilde{\nu}_\Gamma$. Indeed, for any $z \in \tilde{\Gamma}$,

$$\begin{aligned}\int \log|z-t|\,d\tilde{\nu}_\Gamma(t) &= \int_{\tilde{\Gamma}^+} (\log|z-t| + \log|z+t|)\,d\tilde{\nu}_\Gamma(t) = \frac{1}{2}\int_\Gamma \log|z^2-t^2|\,d\nu_\Gamma(t^2)\\ &= \frac{1}{2}\int \log|z^2-u|\,d\nu_\Gamma(u) = \text{const}\end{aligned}$$

because the equilibrium potential of ν_Γ is constant on Γ by Frostman's theorem [16, Theorem 3.3.4], and $z^2 \in \Gamma$. Since the equilibrium measure $\nu_{\tilde{\Gamma}}$ is characterized (among all probability measures on $\tilde{\Gamma}$) by the fact that its logarithmic potential is constant on the given set, we conclude that $\tilde{\nu}_\Gamma$ is indeed the equilibrium measure of $\tilde{\Gamma}$ (here we use the fact that all the sets we are considering are unions of finitely many continua, hence the equilibrium potentials for them are continuous everywhere).

Let $\gamma(t)$ be a parametrization of $\tilde{J}^+$ with $\gamma(0) = 0$. Then $\gamma(t)^2$ is a parametrization of J, and the two corresponding arc measures are $|\gamma'(t)|dt$ and $|(\gamma(t)^2)'|dt = 2|\gamma(t)|\,|\gamma'(t)|dt$, resp. Therefore, since the $\nu_{\tilde{\Gamma}}$-measure of an arc $\{\gamma(t) : t_1 \le t \le t_2\}$ is half the ν_Γ-measure of the arc $\{\gamma(t)^2 : t_1 \le t \le t_2\}$, we have

$$\int_{t_1}^{t_2} \omega_{\tilde{\Gamma}}(\gamma(t))|\gamma'(t)|\,dt = \frac{1}{2}\int_{t_1}^{t_2} \omega_\Gamma(\gamma(t)^2)2|\gamma(t)|\,|\gamma'(t)|\,dt,$$

so

$$\omega_{\tilde{\Gamma}}(\gamma(t)) = \omega_\Gamma(\gamma(t)^2)|\gamma(t)|, \qquad t \in \tilde{J}^+$$

(recall that on both sides, ω is the equilibrium density with respect to the corresponding arc measure). A similar formula holds on $\tilde{J}^-$. But $\omega_{\tilde{\Gamma}}(z)$ is continuous and positive at 0 [24, Proposition 2.2], so the preceding formula shows that $\omega_\Gamma(z)$ behaves around 0 as $\omega_{\tilde{\Gamma}}(0)/\sqrt{|z|}$, and we have (see (1.5) for the definition of $M(\Gamma, 0)$)

$$M(\Gamma, 0) = \lim_{z\to 0} \sqrt{|z|}\,\omega_\Gamma(z) = \omega_{\tilde{\Gamma}}(0). \tag{9.1}$$

Now the same argument as in the proof of Proposition 3.2 (see in particular (3.6)) shows that

$$\lambda_{2n}(\tilde{\mu}, 0) = \lambda_n(\mu, 0). \tag{9.2}$$

Since μ was assumed to be of the form $w(z)|z|^\alpha ds_J(z)$ on J, as before we have

$$\int_{t_1}^{t_2} d\tilde{\mu}(t) = \frac{1}{2}\int_{t_1}^{t_2} w(\gamma(t)^2)|\gamma(t)^2|^\alpha 2|\gamma(t)|\,|\gamma'(t)|\,dt,$$

and since $|\gamma'(t)|dt$ is the arc measure on $\tilde{J}^+$, we conclude that on $\tilde{J}^+$ we have $d\tilde{\mu}(z) = w(z^2)|z|^{2\alpha+1}ds_{\tilde{J}}(z)$, and the same representation holds on $\tilde{J}^-$. Therefore, Theorem 1.1 can be applied to the set $\tilde{\Gamma}$, the measure $\tilde{\mu}$ and the point $z_0 = 0$; the only change is that now α has to be replaced by $2\alpha + 1$ when dealing with $\tilde{\mu}$. Now from (9.2) we obtain

$$\lim_{n\to\infty} (2n)^{2\alpha+2}\lambda_{2n}(\tilde{\mu}, 0) = \lim_{n\to\infty} (2n)^{2\alpha+2}\lambda_n(\mu, 0),$$

and since, according to Theorem 1.1, the limit on the left is

$$2^{2\alpha+2}\Gamma\left(\frac{2\alpha+2}{2}\right)\Gamma\left(\frac{2\alpha+4}{2}\right)\frac{w(0)}{(\pi\omega_{\tilde{\Gamma}}(0))^{2\alpha+2}},$$

we obtain

$$\lim_{n\to\infty} n^{2\alpha+2}\lambda_n(\mu, 0) = \Gamma(\alpha+1)\Gamma(\alpha+2)\frac{w(0)}{(\pi\omega_{\tilde{\Gamma}}(0))^{2\alpha+2}},$$

which, in view of (9.1), is the same as (1.6) in Theorem 1.2. □

Acknowledgments. T. Danka was supported by ERC Advanced Grant No. 267055. V. Totik was supported by NSF DMS-1265375.

References

[1] Armitage, D. H., Gardiner, S. J.: Classical Potential Theory. Springer, Berlin (2001) Zbl 0972.31001 MR 1801253

[2] Avila, A., Last, Y., Simon, B.: Bulk universality and clock spacing of zeros for ergodic Jacobi matrices with absolutely continuous spectrum. Anal. PDE **3**, 81–108 (2010) Zbl 1225.26031 MR 2663412

[3] Garnett, J. B., Marshall, D. E.: Harmonic Measure. New Math. Monogr. 2, Cambridge Univ. Press, Cambridge (2008) Zbl 1139.31001 MR 2450237

[4] Golinskii, L.: The Christoffel function for orthogonal polynomials on a circular arc. J. Approx. Theory **101**, 165–174 (1999) Zbl 0939.33005 MR 1726450

[5] Grenander, U., Szegő, G.: Toeplitz Forms and Their Applications. Univ. of California Press, Berkeley and Los Angeles (1958) Zbl 0080.09501 MR 0094840

[6] Gustafsson, B., Putinar, M., Saff, E., Stylianopoulos, N.: Bergman polynomials on an archipelago: Estimates, zeros and shape reconstruction. Adv. Math. **222**, 1405–1460 (2009) Zbl 1194.42030 MR 2554940

[7] Kuijlaars, A. B., Vanlessen, M.: Universality for eigenvalue correlations from the modified Jacobi unitary ensemble. Int. Math. Res. Notices **2002**, 1575–1600 Zbl 1122.30303 MR 1912278

[8] Lubinsky, D. S.: A new approach to universality involving orthogonal polynomials. Ann. of Math. **170**, 915–939 (2009) Zbl 1176.42022 MR 2552113

[9] Lubinsky, D. S.: A new approach to universality limits at the edge of the spectrum. In: Integrable Systems and Random Matrices, Contemp. Math. 458, Amer. Math. Soc., Providence, RI, 281–290 (2008) Zbl 1147.15306 MR 2411912

[10] Lubinsky, D. S.: Universality limits at the hard edge of the spectrum for measures with compact support. Int. Math. Res. Notices **2008**, art. ID rnn 099, 39 pp. Zbl 1172.28006 MR 2439541

[11] Nagy, B., Totik, V.: Sharpening of Hilbert's lemniscate theorem, J. Anal. Math. **96**, 191–223 (2005) Zbl 1093.30003 MR 2177185

[12] Nevai, P.: Orthogonal Polynomials. Mem. Amer Math. Soc. **18**, no. 213, v + 185 pp. (1979) Zbl 0405.33009 MR 0519926

[13] Nevai, P.: Géza Freud, orthogonal polynomials and Christoffel functions. A case study. J. Approx. Theory **48**, 3–167 (1986) Zbl 0606.42020 MR 0862231

[14] Nevanlinna, R.: Analytic Functions. Grundlehren Math. Wiss. 162, Springer, Berlin (1970) Zbl 0199.12501 MR 0279280

[15] Pommerenke, Ch.: On the derivative of a polynomial, Michigan Math. J. **6**. 373–375 (1959) Zbl 0123.26602 MR 0109208

[16] Ransford, T.: Potential Theory in the Complex Plane. Cambridge Univ. Press, Cambridge (1995) Zbl 0828.31001 MR 1334766

[17] Saff, E. B., Totik, V.: Logarithmic Potentials with External Fields. Grundlehren Math. Wiss. 316, Springer, Berlin (1997) Zbl 0881.31001 MR 1485778

[18] Simon, B.: The Christoffel–Darboux kernel. In: Perspectives in PDE, Harmonic Analysis and Applications, in honor of V. G. Maz'ya's 70th birthday, Proc. Sympos. Pure Math. 79, Amer. Math. Soc., Providence, RI, 295–335 (2008) Zbl 1159.42020 MR 2500498

[19] Stahl, H., Totik, V.: General Orthogonal Polynomials. Encyclopedia Math. Appl. 43, Cambridge Univ. Press, Cambridge (1992) Zbl 0791.33009 MR 1163828

[20] Szegő, G.: Orthogonal Polynomials. Amer. Math. Soc. Colloq. Publ. 23, Amer. Math. Soc., Providence, RI (1975) Zbl 0305.42011 MR 0372517

[21] Szegő, G.: Collected Papers. R. Askey (ed.), Birkhaüser, Boston (1982) Zbl 0491.01016 MR 0674482(Vol. 1) MR 0674483(Vol. 2) MR 0674484(Vol. 3)

[22] Totik, V.: Christoffel functions on curves and domains, Trans. Amer. Math. Soc. **362**, 2053–2087 (2010) Zbl 1189.26028 MR 2574887

[23] Totik, V.: The polynomial inverse image method. In: Approximation Theory XIII: San Antonio 2010, Springer Proc. Math. 13, M. Neamtu and L. Schumaker (eds.), Springer, 345–365 (2012) Zbl 1268.41008 MR 3052033

[24] Totik, V.: Asymptotics of Christoffel functions on arcs and curves. Adv. Math. **252**, 114–149 (2014) Zbl 1293.30017 MR 3144225

[25] Totik, V.: Szegő's problem on curves. Amer. J. Math. **135**, 1507–1524 (2013) Zbl 1286.42040 MR 3145002

[26] Varga, T.: Christoffel functions for doubling measures on quasismooth curves and arcs. Acta Math. Hungar. **141**, 161–184 (2013) Zbl 1324.42039 MR 3102977

[27] Walsh, J. L.: Interpolation and Approximation by Rational Functions in the Complex Domain. 3rd ed., Amer. Math. Soc. Colloq. Publ. 20, Amer. Math. Soc., Providence, RI (1960) Zbl 0106.28104 MR 0218587

[28] Watson, G. N.: A Treatise on the Theory of Bessel Functions. Cambridge Univ. Press, Cambridge (1995) (reprint of the second (1944) edition) Zbl 0849.33001 MR 1349110