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Boundedness of moduli of varieties of general type

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Abstract. We show that the family of semi log canonical pairs with ample log canonical class and with fixed volume is bounded.

Keywords. Moduli, boundedness, general type, minimal model program, abundance

Contents

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1. Introduction

The aim of this paper is to show that the moduli functor of semi log canonical stable pairs is bounded:

Theorem 1.1. *Fix an integer n, a positive rational number d and a set $I \subset [0, 1]$ which satisfies the DCC. Then the set $\mathfrak{F}_{\mathrm{slc}}(n, d, I)$ of all log pairs (X, Δ) such that*

(i) *X is projective of dimension n,*
(ii) *(X, Δ) is semi log canonical,*
(iii) *the coefficients of Δ belong to I,*
(iv) *$K_X + \Delta$ is an ample $\mathbb{Q}$-divisor, and*
(v) *$(K_X + \Delta)^n = d$,*

is bounded. In particular there is a finite set I_0 such that $\mathfrak{F}_{\mathrm{slc}}(n, d, I) = \mathfrak{F}_{\mathrm{slc}}(n, d, I_0)$.

The main new technical result we need to prove 1.1 is to show that abundance behaves well in families:

Theorem 1.2. *Suppose that (X, Δ) is a log pair where the coefficients of Δ belong to $(0, 1] \cap \mathbb{Q}$. Let $\pi : X \to U$ be a projective morphism to a smooth variety U. Suppose that (X, Δ) is log smooth over U. If there is a closed point $0 \in U$ such that the fibre (X_0, Δ_0) has a good minimal model then (X, Δ) has a good minimal model over U and every fibre has a good minimal model.*

Corollary 1.3. *Let (X, Δ) be a log pair where Δ is a $\mathbb{Q}$-divisor and let $X \to U$ be a flat projective morphism to a variety U. Suppose that U is smooth and the support of Δ contains neither a component of any fibre nor a codimension one component of the singular locus of a fibre. Then the subset $U_0 \subset U$ of points $u \in U$ such that the fibre (X_u, Δ_u) is divisorially log terminal and has a good minimal model is constructible.*

Corollary 1.4. *Let $\pi : X \to U$ be a projective morphism to a smooth variety U and let (X, Δ) be log smooth over U. Suppose that the coefficients of Δ belong to $(0, 1] \cap \mathbb{Q}$. If there is a closed point $0 \in U$ such that the fibre (X_0, Δ_0) has a good minimal model then the restriction morphism*

$$\pi_* \mathcal{O}_X(m(K_X + \Delta)) \to H^0(X_u, \mathcal{O}_{X_u}(m(K_{X_u} + \Delta_u)))$$

is surjective for any $m \in \mathbb{N}$ such that $m\Delta$ is integral and for any closed point $u \in U$. In particular if $\psi : X \dashrightarrow Z$ is the ample model of (X, Δ) then $\psi_u : X_u \dashrightarrow Z_u$ is the ample model of (X_u, Δ_u) for every closed point $u \in U$.

The moduli space of stable curves is one of the most intensively studied varieties. The moduli space of stable varieties of general type is the higher dimensional analogue of the moduli space of curves. Unfortunately, constructing this moduli space is more complicated than constructing the moduli space of curves. In particular it does not seem easy to use GIT to construct the moduli space in higher dimensions; for example see [33] for a precise example of how badly behaved the situation can be. Instead Kollár and Shepherd-Barron initiated a program to construct the moduli space in all dimensions in [28]. This program was carried out in large part by Alexeev for surfaces [1, 2].

We recall the definition of the moduli functor. For simplicity, in the definition of the functor, we restrict ourselves to the case with no boundary. We refer to the forthcoming book [20] for a detailed discussion of this subject and to [25] for a more concise survey.

Definition 1.5 (Moduli of slc models, [25, Definition 29]). Let $H(m)$ be an integer valued function. The *moduli functor of semi log canonical models* with Hilbert function H is

$$\mathcal{M}^{\mathrm{slc}}_H(S) = \left\{\begin{array}{l}\text{flat projective morphisms } X \to S \text{ whose fibres are slc models with}\\ \text{ample canonical class and Hilbert function } H(m),\ \omega_X \text{ is flat over } S\\ \text{and all reflexive powers of } \omega_X \text{ commute with base change}\end{array}\right\}.$$

In this paper we focus on the problem of showing that the moduli functor is bounded, so that if we fix the degree, we get a bounded family. The precise statement is given in 1.1. We now describe its proof. We first explain the reduction to 1.2.

For curves, if one fixes the genus g then the moduli space is irreducible. In particular stable curves are always limits of smooth curves. This fails in higher dimensions, so that there are components of the moduli space whose general point corresponds to a non-normal variety, or better, a semi log canonical variety.

Fortunately ([24, paragraphs 23, 24] and [26, 5.13]), one can reduce boundedness of semi log canonical pairs to boundedness of log canonical pairs in a straightforward manner. If (X, Δ) is semi log canonical then let $n\colon Y \to X$ be the normalisation. X has nodal singularities in codimension one, so informally it is obtained from Y by identifying points of the double locus, the closure of the codimension one singular locus. More precisely, we may write

$$K_Y + \Gamma = n^*(K_X + \Delta),$$

where Γ is the sum of the strict transform of Δ plus the double locus and (Y, Γ) is log canonical. If $K_X + \Delta$ is ample then (X, Δ) is determined by (Y, Γ) and the data of the involution $\tau\colon S \to S$ of the normalisation of the double locus. Note that the involution τ fixes the *different*, the divisor Θ defined by adjunction in the following formula:

$$(K_Y + \Gamma)|_S = K_S + \Theta.$$

Conversely, if (Y, Γ) is log canonical and $K_Y + \Gamma$ is ample, and if τ is an involution of the normalisation S of a divisor supported on $\lfloor \Gamma \rfloor$ which fixes the different, then we may construct a semi log canonical pair (X, Δ) whose normalisation is (Y, Γ) and whose double locus is S.

Note that τ fixes the pullback L of the very ample line bundle determined by a multiple of $K_X + \Delta$. The group of all automorphisms of S which fixes L is a linear algebraic group. It follows by standard arguments that if (Y, Γ) is bounded then τ is bounded.

Thus to prove 1.1 it suffices to prove the result when X is normal, that is, (X, Δ) is log canonical (see 7.3). The first problem is that a priori X might have arbitrarily many components. Note that if $X = C$ is a curve of genus g then K_X has degree $2g - 2$, and so X has at most $2g - 2$ components. In higher dimensions the situation is more complicated since K_X is not necessarily Cartier and so d is not necessarily an integer.

Instead we use [15, 1.3.1], which was conjectured by Alexeev [1] and Kollár [22]:

Theorem 1.6. *Fix a positive integer n and a set $I \subset [0, 1]$ which satisfies the DCC. Let $\mathfrak{D}$ be the set of log canonical pairs (X, Δ) such that the dimension of X is n and the coefficients of Δ belong to I. Then the set*

$$\{\mathrm{vol}(X, K_X + \Delta) \mid (X, \Delta) \in \mathfrak{D}\}$$

also satisfies the DCC.

Since there are only finitely many ways to write d as a sum of elements $d_1, \dots, d_k$ taken from a set which satisfies the DCC (see 2.4.1), we are reduced to proving 1.1 when X is normal and irreducible.

Let $\mathfrak{F} \subset \mathfrak{F}_{\mathrm{slc}}(n, d, I)$ be the subset of all log canonical pairs (X, Δ) where X is irreducible. Since the coefficients of Δ belong to a set which satisfies the DCC, [15, 1.3.3] implies that some fixed multiple of $K_X + \Delta$ defines a birational map to projective space. As the degree of $K_X + \Delta$ is bounded by assumption, $\mathfrak{F}$ is log birationally bounded, that is, there is a log pair (Z, B) and a projective morphism $\pi : Z \to U$ such that given any $(X, \Delta) \in \mathfrak{F}$, we may find $u \in U$ such that X is birational to Z_u and the strict transform Φ of Δ plus the exceptional divisors are components of B_u.

To fix ideas, it might help to introduce an example to illustrate some of the ideas that go into the proof that $\mathfrak{F}$ is bounded. We start with $\mathbb{P}^2$ and $k \geq 4$ lines. The subscript $_0$ will indicate we are working with this example. The variety U_0 is the set of all configurations of k lines, $Z_0 = \mathbb{P}^2 \times U_0$ and B_0 is the reduced divisor corresponding to the lines. We take $I_0 = \{1/2, 1\}$.

[15, 1.6] proves that $\mathfrak{F}$ is a bounded family provided we assume in addition that the total log discrepancy of (X, Δ) is bounded away from zero (meaning that the coefficients of Δ are bounded away from one as well as the log discrepancy is bounded away from zero). For applications to moduli this is far too strong; the double locus occurs with coefficient one.

Instead we proceed as follows. By standard arguments we may assume that U is smooth, the morphism π is smooth and its restriction to any strata of B is smooth, that is, (Z, B) is log smooth over U. In the case of lines in $\mathbb{P}^2$, we simply replace U_0 by the open subset of lines in linear general position; the case of lines not in general position is handled by Noetherian induction. We first reduce to the case when $\mathrm{vol}(Z_u, K_{Z_u} + \Phi) = d$. We are looking for a higher model $Y \to Z$ such that if C is the strict transform of B plus the exceptionals and u is a point then $\mathrm{vol}(Y_u, K_{Y_u} + \Gamma) = d$ where Γ is the transform of Δ plus the exceptionals. At this point we use some of the ideas that go into the proof of [14, 1.9].

We describe how to reduce to the case when U is a point. We illustrate the argument for lines in $\mathbb{P}^2$; the argument in the general case is very similar. In this case the elements $(X, \Delta) \in \mathfrak{F}$ are constructed as follows. Start with $\mathbb{P}^2$ and a collection of k lines in general position. Let $S \to \mathbb{P}^2$ be any sequence of smooth blowups and let D be the strict transform of the lines plus the exceptional divisors. Now blow down some -1-curves on S to obtain X. Let Δ be any divisor supported on the pushforward of D whose coefficients are 0, $1/2$ or 1. Note that there are some restrictions on which -1-curves we blow down: we are only allowed to blow down components of D and we are also assuming that (X, Δ) is log smooth.

To proceed further we want to understand how the volume changes for one smooth blowup of a smooth surface $\pi : T \to S$. Working locally, we may assume that $S = \mathbb{A}^2$, D is the sum of the two coordinate lines L_1 and L_2 and π blows up the origin. Let E be the exceptional divisor and let M_1 and M_2 be the strict transform of the two lines. By assumption $\Delta = a_1 L_1 + a_2 L_2$ where $a_i = 0, 1/2$ or 1. If we write

$$K_T + a_1 M_1 + a_2 M_2 + eE = \pi^*(K_S + a_1 L_1 + a_2 L_2),$$

then $e = a_1 + a_2 - 1$.

Globally we have a pair (T, Θ) such that $\pi_* \Theta = \Delta$. If the volume of the pair (T, Θ) is smaller than the volume of the pair (S, Δ) then the coefficient E of Θ is smaller than e.

In particular, since $e \leq 1$, if we increase the coefficient of any -1-curve we blow down $S \to X$ to 1 then the volume is unchanged. So there is no harm in assuming that $S = X$. Note also that if we blow up $T \to S$ a point which does not belong to D then $e \leq 0$ so that the volume is unchanged. Therefore we may also assume that $X \to Z$ only blows up strata of a fibre of B, since blowups away from the strata do not change the volume. Since (Z, B) is log smooth over U, any sequence of blowups of the strata of a particular fibre can be realised in the whole family. By deformation invariance of log plurigenera we may therefore assume that U is a point (see 7.2).

In general $\text{vol}(Z_u, K_{Z_u} + \Phi) \geq \text{vol}(X, K_X + \Delta) = d$. Our goal is to find a higher model $Y \to Z$ where we always have equality. This follows using some results from [14] (see 7.1). We give an example at the end of §1 which illustrates some of the subtleties behind the statement and proof of 7.1.

So we may assume that $\text{vol}(Z_u, K_{Z_u} + \Phi) = d$. Since (X, Δ) is log canonical and $K_X + \Delta$ is ample, we can recover (X, Δ) from (Z_u, Φ) as the log canonical model (see 2.2.2). Conversely, if $u \in U$ is a point such that (Z_u, Φ) has a log canonical model $f : Z_u \dashrightarrow X$, where

$$X = \text{Proj}\, R(Z_u, K_{Z_u} + \Phi), \qquad \Delta = f_* \Phi,$$

the coefficients of $0 \leq \Phi \leq B_u$ belong to I and $\text{vol}(Z_u, K_{Z_u} + \Phi) = d$, then $(X, \Delta) \in \mathfrak{F}$.

It therefore suffices to prove that the set of fibres with a log canonical model is constructible. Note that (X, Δ) has a log canonical model if and only if the log canonical section ring

$$R(X, K_X + \Delta) = \bigoplus_{m \in \mathbb{N}} H^0(X, \mathcal{O}_X(m(K_X + \Delta)))$$

is finitely generated. Conjecturally every fibre has a log canonical model. Once again the problem is the components of Δ with coefficient one. The main result of [7] implies that if there are no such components, that is, (X, Δ) is Kawamata log terminal, then the log canonical section ring is finitely generated.

In general (see 2.9.1), the existence of the log canonical model Z is equivalent to the existence of a good minimal model $f: X \dashrightarrow Y$, that is, a model (Y, Γ) such that $K_Y + \Gamma$ is semi-ample. In this case the log canonical model is simply the model $Y \to Z$ such that $K_Y + \Gamma$ is the pullback of an ample divisor.

In fact we prove in 1.2 a much stronger result: if one fibre (X_0, Δ_0) has a good minimal model then every fibre has a good minimal model. By [17, 1.1] it suffices to prove that every fibre over an open subset has a good minimal model, or equivalently, the generic fibre has a good minimal model.

Let $\eta \in U$ be the generic point. We may assume that U is affine. We prove the existence of a good minimal model for the pair (X_η, Δ_η) in two steps. We first show that (X_η, Δ_η) has a minimal model. For this we run the $(K_X + \Delta)$-MMP with scaling of an ample divisor. We know that if we run the $(K_{X_0} + \Delta_0)$-MMP with scaling of an ample divisor then this MMP terminates with a good minimal model. Using [17, 2.10] and 5.3 we can reduce to the case when the diminished stable base locus of $K_{X_0} + \Delta_0$ does not contain any non-canonical centres. In this case we show (see 3.1) that every step of the $(K_X + \Delta)$-MMP induces a $(K_{X_0} + \Delta_0)$-negative map. This generalises [14, 4.1], which assumes that U is a curve and (X, Δ) is terminal. This MMP $f: X \dashrightarrow Y$ ends with a minimal model for the generic fibre (see 3.2).

To finish off we need to show that the minimal model is a good minimal model. There are two cases. We may write $(X, \Delta = S + B)$, where $S = \lfloor \Delta \rfloor$.

In the first case, if $K_X + (1-\epsilon)S + B$ is not pseudo-effective for any $\epsilon > 0$ then we may run the $(K_X + (1-\epsilon)S + B)$-MMP $Y \dashrightarrow W$ until we reach a Mori fibre space $W \to Z$ (see 5.2). If $\epsilon > 0$ is sufficiently small, this MMP induces a $(K_{X_0} + \Delta_0)$-non-positive map (see 5.1). It follows that this MMP is $(K_X + \Delta)$-non-positive. We know that there is a component D of S whose image dominates the base Z of the Mori fibre space. By induction the generic fibre of the image E of D in Y is a good minimal model. The restriction $E \dashrightarrow F$ of the map $Y \dashrightarrow W$ need not be a birational contraction but we will not lose semi-ampleness. The image of the divisor is pulled back from Z and so $K_X + \Delta$ has a semi-ample model.

In the second case $K_X + (1-\epsilon)S + B$ is pseudo-effective. As $K_X + (1-\epsilon)S + B$ is Kawamata log terminal, it follows by work of B. Berndtsson and M. Păun (see 4.1) that the Kodaira dimension is invariant (see 4.2). As $K_X + (1-\epsilon)S + B$ is pseudo-effective and (X_0, Δ_0) has a good minimal model, it follows that $K_{X_0} + \Delta_0$ is abundant, that is, the Kodaira dimension is the same as the numerical dimension. By deformation invariance of log plurigenera the generic fibre is abundant. As the restriction of $K_Y + \Gamma$ to every component of coefficient one is semi-ample, the restriction of $K_Y + \Gamma$ to the sum of the coefficient one part is semi-ample by 2.5.1, and we are done by 2.6.1.

As promised, here is an example to illustrate some of the subtleties of the argument in the proof of 7.1. We go back to the example of lines in $\mathbb{P}^2$. We start with four lines L_1, L_2, L_3 and L_4 in $\mathbb{P}^2$, all with coefficient one. In this case U_0 is a point since there is no moduli to four lines in linear general position. The volume of the pair in $\mathbb{P}^2$ is then 1. Now suppose that $(X, \Delta) \in \mathfrak{F}$. As already pointed out, $d \leq 1$ and there is no harm in assuming that X is a blowup of $\mathbb{P}^2$, $f: X \to \mathbb{P}^2$. We may even assume that all of the blowups lie over the six points where the four lines intersect. Fix the point p

where the two lines L_1 and L_2 meet and assume that all blowups are over p. Then X is a toric variety and $f\colon X \to \mathbb{P}^2$ is a toric morphism. Let us simplify matters even more and assume that we only alter one coefficient of one exceptional divisor E over p; let us suppose that we do not include E in Θ, that is, we make its coefficient zero. In this case, since every other divisor occurs with coefficient one, we can compute the volume on the weighted blowup of $\mathbb{P}^2$ corresponding to the divisor E, $g\colon S \to \mathbb{P}^2$. The problem is that unless we fix the degree d, there is no constraint on how many times we blow up over p, that is, there is no constraint on the weighted blowup g. Let M_1, M_2, M_3 and M_4 be the strict transforms of the four lines. Then $(S, M_1 + M_2 + M_3 + E)$ is a toric pair, so that $K_S + M_1 + M_2 + M_3 + E \sim 0$. It follows that

$$(K_S + M_1 + M_2 + M_3 + M_4)^2 = (M_4 - E)^2 = M_4^2 + E^2 = 1 + E^2.$$

It is a simple exercise in toric geometry to compute E^2. If we make a weighted blow up of type (a, b) then

$$E^2 = -\frac{1}{ab},$$

so that the volume is

$$\frac{ab-1}{ab}.$$

As expected, the volume satisfies the DCC. If we fix the volume d then there are only finitely many possible values for (a, b). This is the content of 7.1 in this example.

2. Preliminaries

2.1. Notation and conventions

We will follow the terminology from [27]. Let $f\colon X \dashrightarrow Y$ be a proper birational map of normal quasi-projective varieties and let $p\colon W \to X$ and $q\colon W \to Y$ be a common resolution of f. We say that f is a *birational contraction* if every p-exceptional divisor is q-exceptional. If D is an $\mathbb{R}$-Cartier divisor on Y then f^*D is the $\mathbb{R}$-Weil divisor q_*p^*D. Equivalently, if U is the domain of f then f^*D is the $\mathbb{R}$-Weil divisor on X corresponding to the $\mathbb{R}$-Cartier divisor $(f|_U)^*D$ on U.

If D is an $\mathbb{R}$-Cartier divisor on X such that $D' := f_*D$ is $\mathbb{R}$-Cartier then we say that f is *D-non-positive* (resp. *D-negative*) if $p^*D = q^*D' + E$ where $E \geq 0$ and E is q-exceptional (respectively E is q-exceptional and the support of E contains the strict transform of the f-exceptional divisors).

We say a proper morphism $\pi\colon X \to U$ is a *contraction morphism* if $\pi_*\mathcal{O}_X = \mathcal{O}_U$. Recall that for any $\mathbb{R}$-divisor D on X, the sheaf $\pi_*\mathcal{O}_X(D)$ is defined to be $\pi_*\mathcal{O}_X(\lfloor D \rfloor)$.

If X is a normal variety and B is a divisor whose components all have coefficient one then the *strata* of B are the irreducible components of the intersections

$$B_I = \bigcap_{j \in I} B_j = B_{i_1} \cap \cdots \cap B_{i_r}$$

of components of B, where $I = \{i_1, \dots, i_r\}$ is a subset of the indices, including the empty intersection $X = B_\emptyset$. If (X, Δ) is a log pair then the *strata* of (X, Δ) are the strata of the support B of Δ.

If we are given a morphism $X \to U$, then we say that (X, Δ) is *log smooth over U* if (X, Δ) has simple normal crossings and both X and the strata of (X, D) are smooth over U, where D is the support of Δ. If $\pi \colon X \to U$ and $Y \to U$ are projective morphisms, $f \colon X \dashrightarrow Y$ is a birational contraction over U and (X, Δ) is a log canonical pair (respectively divisorially log terminal $\mathbb{Q}$-factorial pair) such that f is $(K_X+\Delta)$-non-positive (respectively $(K_X+\Delta)$-negative) and $K_Y+\Gamma$ is nef over U (respectively and Y is $\mathbb{Q}$-factorial), then we say that $f \colon X \dashrightarrow Y$ is a *weak log canonical model* (respectively a *minimal model*) of $K_X+\Delta$ over U.

We say $K_Y+\Gamma$ is *semi-ample* over U if there exists a contraction morphism $\psi \colon Y \to Z$ over U such that $K_Y+\Gamma \sim_{\mathbb{R}} \psi^* A$ for some $\mathbb{R}$-divisor A on Z which is ample over U. Equivalently, when $K_Y+\Gamma$ is $\mathbb{Q}$-Cartier, $K_Y+\Gamma$ is semi-ample over U if there exists an integer $m > 0$ such that $\mathcal{O}_Y(m(K_Y+\Gamma))$ is generated over U. Note that in this case

$$R(Y/U, K_Y+\Gamma) := \bigoplus_{m \geq 0} \pi_* \mathcal{O}_Y(m(K_Y+\Gamma))$$

is a finitely generated $\mathcal{O}_U$-algebra, and

$$Z = \operatorname{Proj} R(Y/U, K_Y+\Gamma).$$

If $K_Y+\Gamma$ is semi-ample and big over U, then Z is the *log canonical model* of (X, Δ) over U. A weak log canonical model $f \colon X \dashrightarrow Y$ is called a *semi-ample model* if $K_Y+\Gamma$ is semi-ample.

Suppose that $\pi \colon X \to U$ is a projective morphism of normal varieties. Let D be an $\mathbb{R}$-Cartier divisor on X. Let C be a prime divisor. If D is big over U then

$$\sigma_C(X/U, D) = \inf\{\operatorname{mult}_C(D') \mid D' \sim_{\mathbb{R},U} D, D' \geq 0\}.$$

Now let A be any ample $\mathbb{Q}$-divisor over U and suppose that D is pseudo-effective over U. Following [31], let

$$\sigma_C(X/U, D) = \lim_{\epsilon \to 0} \sigma_C(X/U, D+\epsilon A).$$

Then $\sigma_C(X/U, D)$ exists (where we allow ∞ as a limit) and is independent of the choice of A. There are only finitely many prime divisors C such that $\sigma_C(X/U, D) > 0$, this number only depends on the numerical equivalence class of D over U, and if we replace U by an open subset which contains the image of the generic point of C then σ_C is unchanged. However, with no more assumptions there are examples when $\sigma_C(X/U, D) = \infty$ [30]. On the other hand, if $\pi(C)$ has codimension no more than one then $\sigma_C(X/U, D) < \infty$. In this case the $\mathbb{R}$-divisor $N_\sigma(X/U, D) = \sum_C \sigma_C(D)C$ is determined by the numerical equivalence class of D (see [7, 3.3.1] and [31] for more details). Note that if the fibres of π are irreducible and all of the same dimension then $\pi(C)$ automatically has codimension at most one for every prime divisor C on X.

Now suppose that D is only an $\mathbb{R}$-divisor. The *real linear system* associated to D over U is

$$|D/U|_{\mathbb{R}} = \{C \geq 0 \mid C \sim_{\mathbb{R},U} D\}.$$

The *stable base locus* of D over U is the Zariski closed set $\mathbf{B}(X/U, D)$ given by the intersection of the supports of the elements of $|D/U|_{\mathbb{R}}$. If $|D/U|_{\mathbb{R}} = \emptyset$, then we let $\mathbf{B}(X/U, D) = X$.

The *diminished stable base locus* of D over U is

$$\mathbf{B}_{-}(X/U, D) = \bigcup_{A} \mathbf{B}(X/U, D + A),$$

where the union runs over all divisors A which are ample over U.

Suppose that U is a point. Following [31], if D is pseudo-effective we define the *numerical dimension*

$$\kappa_\sigma(X, D) = \max_{H \in \mathrm{Pic}(X)} \left\{ k \in \mathbb{N} \;\middle|\; \limsup_{m \to \infty} \frac{h^0(X, \mathcal{O}_X(mD + H))}{m^k} > 0 \right\}.$$

If D is nef then this is the same as

$$\nu(X, D) = \max\{k \in \mathbb{N} \mid H^{n-k} \cdot D^k > 0\}$$

for any ample divisor H [31]. If D is $\mathbb{Q}$-Cartier then D is called *abundant* if $\kappa_\sigma(X, D) = \kappa(X, D)$, that is, the numerical dimension is equal to the Iitaka dimension. If we drop the condition that X is projective and instead we have a projective morphism $\pi \colon X \to U$, then a $\mathbb{Q}$-Cartier divisor D on X is called *abundant* over U if its restriction to the generic fibre is abundant.

If (X, Δ) is a log pair then a *non-canonical centre* is the centre of a valuation of log discrepancy less than one.

We say a family $\mathfrak{D}$ of log pairs is *bounded* if there is a morphism $Z \to U$ of varieties, where U is smooth, Z is flat over U, and a log pair (Z, Σ), where the support of Σ contains neither a component of a fibre nor a codimension one singular point of any fibre, such that for every $(X, \Delta) \in \mathfrak{D}$ there is a closed point $u \in U$ and an isomorphism of log pairs between (X, Δ) and (Z_u, Σ_u). In particular the coefficients of Δ belong to a finite set.

2.2. *The volume*

Definition 2.2.1. Let X be a normal n-dimensional irreducible projective variety and let D be an $\mathbb{R}$-divisor. The *volume* of D is

$$\mathrm{vol}(X, D) = \limsup_{m \to \infty} \frac{n! h^0(X, \mathcal{O}_X(mD))}{m^n}.$$

Let $V \subset X$ be a normal irreducible subvariety of dimension d. Suppose that D is $\mathbb{R}$-Cartier with support not containing V. The *restricted volume* of D along V is

$$\mathrm{vol}(X|V, D) = \limsup_{m \to \infty} \frac{d!(\dim \mathrm{Im}(H^0(X, \mathcal{O}_X(mD)) \to H^0(V, \mathcal{O}_V(mD|_V))))}{m^d}.$$

Lemma 2.2.2. *Let $f: X \to Z$ be a birational morphism between log canonical pairs (X, Δ) and (Z, B). Suppose that $K_X + \Delta$ is big and that (X, Δ) has a log canonical model $g: X \dashrightarrow Y$. If $f_*\Delta \leq B$ and $\mathrm{vol}(X, K_X + \Delta) = \mathrm{vol}(Z, K_Z + B)$ then the induced birational map $Z \dashrightarrow Y$ is the log canonical model of (Z, B).*

Proof. Let $\pi: W \to X$ be a log resolution of $(X, C + F)$ which also resolves the map g, where C is the strict transform of B and F is the sum of the f-exceptional divisors. We may write

$$K_W + \Theta = \pi^*(K_X + \Delta) + E,$$

where $\Theta, E \geq 0$ have no common components, $\pi_*\Theta = \Delta$ and $\pi_*E = 0$. Then the log canonical model of (W, Θ) is the same as the log canonical model of (X, Δ). Replacing (X, Δ) by (W, Θ) we may assume that $(X, C + F)$ is log smooth and $g: X \to Y$ is a morphism. Replacing (Z, B) by $(X, D = C + F)$ we may assume $Z = X$.

If $A = g_*(K_X + \Delta)$ and $H = g^*A$ then A is ample and $K_X + \Delta - H \geq 0$. Let $L = D - \Delta \geq 0$, let S be a component of L with coefficient a and let

$$v(t) = \mathrm{vol}(X, H + tS).$$

Then $v(t)$ is a non-decreasing function of t and

$$\begin{aligned} v(0) = \mathrm{vol}(X, H) = \mathrm{vol}(X, K_X + \Delta) &= \mathrm{vol}(X, K_X + D) \\ \geq \mathrm{vol}(X, H + L) &\geq \mathrm{vol}(X, H + aS) = v(a). \end{aligned}$$

Thus $v(t)$ is constant over the range $[0, a]$. As [29, 4.25(iii)] implies that

$$\frac{1}{n} \frac{dv}{dt}\bigg|_{t=0} = \mathrm{vol}_{X|S}(H) \geq S \cdot H^{n-1} = g_*S \cdot A^{n-1},$$

we have $g_*S = 0$. But then every component of L is exceptional for g, and g is the log canonical model of (X, D). □

2.3. Deformation invariance

Lemma 2.3.1. *Let $\pi: X \to U$ be a projective morphism to a smooth variety U and let (X, Δ) be a log smooth pair over U. Let A be a relatively ample Cartier divisor such that $\lfloor \Delta \rfloor + A \sim A'$ where $(X, \Delta + A')$ is log smooth over U. If the coefficients of Δ belong to $[0, 1]$ then*

$$f_*\mathcal{O}_X(m(K_X + \Delta) + A) \to H^0(X_u, \mathcal{O}_{X_u}(m(K_{X_u} + \Delta_u) + A_u))$$

is surjective for all positive integers m such that $m\Delta$ is integral and for every $u \in U$.

Proof. We have

$$m(K_X + \Delta) + A \sim m\left(K_X + \Delta - \frac{1}{m}\lfloor \Delta \rfloor + \frac{1}{m}A'\right) = m(K_X + \Delta'),$$

where (X, Δ') is log smooth over U, $\lfloor \Delta' \rfloor = 0$ and Δ' is big over U, so that we may apply [14, 1.8.1]. □

Lemma 2.3.2. *Let $\pi : X \to U$ be a projective morphism to a smooth variety U and let (X, Δ) be a log smooth pair over U. Assume that $K_X + \Delta$ is pseudo-effective over U. If the coefficients of Δ belong to* $[0, 1]$ *then*

$$N_\sigma(X/U, K_X + \Delta)|_{X_u} = N_\sigma(X_u, K_{X_u} + \Delta_u)$$

for every $u \in U$.

Proof. Since this result is local about every point $u \in U$, we may assume that U is affine. Pick a relatively ample Cartier divisor A such that $\lfloor \Delta \rfloor + A \sim A'$ where $(X, \Delta + A')$ is log smooth over U. Fix $u \in U$. Then 2.3.1 implies that

$$f_*\mathcal{O}_X(m(K_X + \Delta) + A) \to H^0(X_u, \mathcal{O}_{X_u}(m(K_{X_u} + \Delta_u) + A_u))$$

is surjective for all positive integers m such that $m\Delta$ is integral. It follows that

$$N_\sigma(X/U, K_X + \Delta)|_{X_u} \le N_\sigma(X_u, K_{X_u} + \Delta_u),$$

and the reverse inequality is clear. □

Lemma 2.3.3. *Let $\pi : X \to U$ be a projective morphism to a smooth variety U and let (X, Δ) be a log smooth pair over U such that the strata of Δ have irreducible fibers over U and $K_X + \Delta$ is pseudo-effective over U. Let $0 \in U$ be a closed point, let*

$$\Theta_0 = \Delta_0 - \Delta_0 \wedge N_\sigma(X_0, K_{X_0} + \Delta_0)$$

and let $0 \le \Theta \le \Delta$ be the unique divisor such that $\Theta_0 = \Theta|_{X_0}$. If the coefficients of Δ belong to $[0, 1]$ *then*

$$\Theta = \Delta - \Delta \wedge N_\sigma(X/U, K_X + \Delta).$$

Proof. Replacing U by an open neighbourhood of $0 \in U$ we may assume that U is affine. Pick a relatively ample Cartier divisor H with the property that for every integral divisor $0 \le S \le \lfloor \Delta \rfloor$ we may find $S + H \sim H'$ such that $(X, \Delta + H')$ is log smooth over U. Given a positive integer m, let

$$\Phi_0 = \Delta_0 - \Delta_0 \wedge N_\sigma\left(X_0, K_{X_0} + \Delta_0 + \frac{1}{m}H_0\right),$$

and let $0 \le \Phi \le \Delta$ be the unique divisor such that $\Phi_0 = \Phi|_{X_0}$. Consider the commutative diagram

$$\begin{array}{ccc} \pi_*\mathcal{O}_X(m(K_X + \Phi) + H) & \longrightarrow & \pi_*\mathcal{O}_X(m(K_X + \Delta) + H) \\ \downarrow & & \downarrow \\ H^0(X_0, \mathcal{O}_{X_0}(m(K_{X_0} + \Phi_0) + H_0)) & \longrightarrow & H^0(X_0, \mathcal{O}_{X_0}(m(K_{X_0} + \Delta_0) + H_0)) \end{array}$$

The top row is an inclusion and the bottom row is an isomorphism by assumption. The left vertical map is surjective by 2.3.1. Nakayama's Lemma implies that the top row is an isomorphism in a neighbourhood of X_0. It follows that

$$\Phi \ge \Delta - \Delta \wedge N_\sigma\left(X/U, K_X + \Delta + \frac{1}{m}H\right).$$

Taking the limit as m goes to infinity we get

$$\Theta \geq \Delta - \Delta \wedge N_\sigma(X/U, K_X + \Delta),$$

and the reverse inequality follows by 2.3.2. □

Lemma 2.3.4. *Let $\pi : X \to U$ be a projective morphism to a smooth variety U and let (X, D) be log smooth over U, where the coefficients of D are all one. Let $0 \in U$ be a closed point. Then the restriction morphism*

$$\pi_* \mathcal{O}_X(K_X + D) \to H^0(X_0, \mathcal{O}_{X_0}(K_{X_0} + D_0))$$

is surjective.

Proof. Since the result is local, we may assume that U is affine. Cutting by hyperplanes we may assume that U is a curve. Thus we want to show that the restriction map

$$H^0(X, \mathcal{O}_X(K_X + X_0 + D)) \to H^0(X_0, \mathcal{O}_{X_0}(K_{X_0} + D_0))$$

is surjective. This is equivalent to showing that multiplication by a local parameter

$$H^1(X, \mathcal{O}_X(K_X + D)) \to H^1(X, \mathcal{O}_X(K_X + D + X_0))$$

is injective.

By assumption the image of every stratum of D is the whole of U, and $0 = (K_X + D) - (K_X + D)$ is semi-ample. Therefore a generalisation of Kollár's injectivity theorem (see [21], [9, 6.3] and [4, 5.4]) implies that the last displayed map is indeed injective. □

2.4. DCC sets

Lemma 2.4.1. *Let $I \subset \mathbb{R}$ be a set of positive real numbers which satisfies the DCC. Fix a constant d. Then the set*

$$T = \left\{ (d_1, \dots, d_k) \,\middle|\, k \in \mathbb{N},\ d_i \in I,\ \sum d_i = d \right\}$$

is finite.

Proof. As I satisfies the DCC, there is a real number $\delta > 0$ such that if $i \in I$ then $i \geq \delta$. Thus

$$k \leq d/\delta.$$

It is enough to show that any infinite sequence $t_1, t_2, \dots$ of elements of T has a constant subsequence. Possibly passing to a subsequence we may assume that the number of entries k of each vector $t_i = (d_{i\,1}, \dots, d_{i\,k})$ is constant. Since I satisfies the DCC, possibly passing to a subsequence we may assume that the entries are non-decreasing. Since the sum is constant, it is clear that the entries are constant, so that $t_1, t_2, \dots$ is a constant sequence. □

Lemma 2.4.2. *Let J be a finite set of real numbers at most one. Then the set*

$$I = \left\{ a \in (0, 1] \,\middle|\, a = 1 + \sum_{i \leq k} a_i - k,\ a_1, \dots, a_k \in J \right\}$$

is finite.

Proof. If $a_k = 1$ then
$$\sum_{i \leq k} a_i - k = \sum_{i \leq k-1} a_i - (k-1).$$
Thus there is no harm in assuming that $1 \notin J$. If $a_k < 0$ then
$$1 + \sum_{i \leq k} a_i - k < 0.$$
Thus we may assume that $J \subset [0, 1)$.

Note that
$$1 + \sum_{i \leq k} a_i - k > 0 \quad \text{if and only if} \quad \sum_{i \leq k} (1 - a_i) < 1.$$
Since J is finite, we may find $\delta > 0$ such that if $a \in J$ then $1 - a \geq \delta$. This bounds k and the result is clear. □

2.5. *Semi log canonical varieties*

We will need the definition of certain singularities of semi-normal pairs [23, 7.2.1]. Let X be a semi-normal variety which satisfies Serre's condition S_2. We say that X is *demi-normal* if X has nodal singularities in codimension one [26, 5.1]. Let Δ be an $\mathbb{R}$-divisor on X such that no component of Δ is contained in the singular locus of X and $K_X + \Delta$ is $\mathbb{R}$-Cartier. Let $n \colon Y \to X$ be the normalisation of X and write
$$K_Y + \Gamma = n^*(K_X + \Delta),$$
where Γ is the sum of the strict transform of Δ and the double locus. We say that (X, Δ) is *semi log canonical* if (Y, Γ) is log canonical. See [26] for more details about semi log canonical singularities.

Theorem 2.5.1. *Let (X, Δ) be a semi log canonical pair and let $n \colon Y \to X$ be the normalisation. By adjunction we may write*
$$K_Y + \Gamma = n^*(K_X + \Delta),$$
where (Y, Γ) is log canonical. If X is projective and Δ is a $\mathbb{Q}$-divisor then $K_X + \Delta$ is semi-ample if and only if $K_Y + \Gamma$ is semi-ample.

Proof. See [11] or [16, 1.4]. □

Suppose that (X, Δ) is log canonical and $\pi \colon X \to U$ is a morphism of quasi-projective varieties. Suppose that U is smooth, the fibres of π all have the same dimension and the support of Δ does not contain any fibre.

If (X_0, Δ_0) is the fibre over a closed point $0 \in U$ and X_0 is integral and normal then note that
$$(K_X + \Delta)|_{X_0} = K_{X_0} + \Delta_0.$$

2.6. Base point free theorem

Recall the following generalisation of Kawamata's theorem:

Theorem 2.6.1. *Let $(X, \Delta = S + B)$ be a divisorially log terminal pair, where $S = \lfloor \Delta \rfloor$ and B is a $\mathbb{Q}$-divisor. Let H be a $\mathbb{Q}$-Cartier divisor on X and let $X \to U$ be a proper surjective morphism of varieties. If there is a constant a_0 such that*

(1) *$H|_S$ is semi-ample over U,*
(2) *$aH - (K_X + \Delta)$ is nef and abundant over U for all $a > a_0$,*

then H is semi-ample over U.

Proof. See [18], [12, 3.2], [3], [8], [9], [10], [17, 4.1] and [11]. □

2.7. Minimal models

Lemma 2.7.1. *Let (X, Δ) be a log canonical pair where X is a projective variety, and let $f : X \dashrightarrow Y$ be a weak log canonical model. Suppose that the rational map ϕ associated to the linear system $|r(K_X + \Delta)|$ is birational. Then:*

(1) *Every component of $N_\sigma(X, K_X + \Delta)$ is f-exceptional.*
(2) *If P is a prime divisor such that P is not a component of the base locus of $|r(K_X+\Delta)|$ and the restriction of ϕ to P is birational then P is not f-exceptional.*

Proof. Let $p : W \to X$ and $q : W \to Y$ resolve f. As f is a weak log canonical model of (X, Δ), we may write

$$p^*(K_X + \Delta) = q^*(K_Y + \Gamma) + E,$$

where $E \geq 0$ is q-exceptional. As $q^*(K_Y + \Gamma)$ is nef, it follows that

$$N_\sigma(X, K_X + \Delta) = p_*E.$$

In particular (1) holds.

If Q is the strict transform of P and ψ is the birational map associated to the linear system $|rp^*(K_X + \Delta)|$ then the restriction of ψ to Q is birational. On the other hand,

$$|rp^*(K_X + \Delta)| = |rq^*(K_Y + \Gamma)| + rE.$$

Therefore ψ is the birational map associated to the linear system $|rq^*(K_Y + \Gamma)|$. In particular Q is not q-exceptional, so that P is not f-exceptional. □

Lemma 2.7.2. *Let (X, Δ) be a divisorially log terminal pair where X is $\mathbb{Q}$-factorial and projective. Assume that $K_X + \Delta$ is pseudo-effective. Suppose that we run the $(K_X + \Delta)$-MMP $f : X \dashrightarrow Y$ with scaling of an ample divisor A, so that $(Y, \Gamma + tB)$ is nef, where $\Gamma = f_*\Delta$ and $B = f_*A$.*

(1) *If F is f-exceptional then F is a component of $N_\sigma(X, K_X + \Delta)$.*
(2) *If $t > 0$ is sufficiently small then every component of $N_\sigma(X, K_X+\Delta)$ is f-exceptional.*
(3) *If (X, Δ) has a minimal model and $K_X + \Delta$ is $\mathbb{Q}$-Cartier then $N_\sigma(X, K_X + \Delta)$ is a $\mathbb{Q}$-divisor.*

Proof. Let $p\colon W \to X$ and $q\colon W \to Y$ resolve f. As f is a minimal model of $(X, tA+\Delta)$ for some $t \geq 0$, we may write

$$p^*(K_X + tA + \Delta) = q^*(K_Y + tB + \Gamma) + E,$$

where $E = E_t \geq 0$ is q-exceptional. As $q^*(K_Y + tB + \Gamma)$ is nef, it follows that

$$N_\sigma(X, K_X + tA + \Delta) = p_*E.$$

As A is ample, (1) holds. If t is sufficiently small then

$$N_\sigma(X, K_X + tA + \Delta) \quad \text{and} \quad N_\sigma(X, K_X + \Delta)$$

have the same support and so (2) holds.

If (X, Δ) has a minimal model then we may assume that $t = 0$ and so $N_\sigma(X, K_X+\Delta) = p_*E_0$ is a $\mathbb{Q}$-divisor. □

Lemma 2.7.3. *Let (X, Δ) be a divisorially log terminal pair where X is $\mathbb{Q}$-factorial and projective. Assume that $K_X + \Delta$ is pseudo-effective. If $f\colon X \dashrightarrow Y$ is a birational contraction such that Y is $\mathbb{Q}$-factorial, $K_Y+\Gamma = f_*(K_X+\Delta)$ is nef and f only contracts components of $N_\sigma(X, K_X + \Delta)$ then f is a minimal model of (X, Δ).*

Proof. Let $p\colon W \to X$ and $q\colon W \to Y$ resolve f. We may write

$$p^*(K_X + \Delta) + E = q^*(K_Y + \Gamma) + F,$$

where $E, F \geq 0$ have no common components and both E and F are q-exceptional.

As $K_Y + \Gamma$ is nef, the supports of F and of $N_\sigma(W, q^*(K_Y + \Gamma) + F)$ coincide. On the other hand, every component of E is a component of $N_\sigma(W, p^*(K_X + \Delta) + E)$. Thus $E = 0$ and any divisor contracted by f is a component of F. □

2.8. Blowing up log pairs

Lemma 2.8.1. *Let (X, Δ) be a log smooth pair. If $\lfloor \Delta \rfloor = 0$ then there is a sequence $\pi : Y \to X$ of smooth blowups of the strata of (X, Δ) such that if we write*

$$K_Y + \Gamma = \pi^*(K_X + \Delta) + E,$$

where $\Gamma, E \geq 0$ have no common components, $\pi_*\Gamma = \Delta$ and $\pi_*E = 0$, then no two components of Γ intersect.

Proof. This is standard: see for example [13, 6.5]. □

Lemma 2.8.2. *Let (X, Δ) be a sub log canonical pair (so that some of the coefficients of Δ might be negative). We may find a finite set $I \subset (0, 1]$ such that if $\pi : Y \to X$ is any birational morphism and we write*

$$K_Y + \Gamma = \pi^*(K_X + \Delta)$$

then those coefficients of Γ which are positive belong to I.

Proof. Replacing (X, Δ) by a log resolution we may assume that (X, Δ) is log smooth. Let J be the set of coefficients of Δ and let I be the set given by 2.4.2.

Supppose that $\pi : Y \to X$ is a birational morphism. We may write

$$K_Y + \Gamma = \pi^*(K_X + \Delta).$$

We claim that those coefficients of Γ which are positive belong to I. Possibly blowing up more we may assume that π is a sequence of smooth blowups. If $Z \subset X$ is smooth of codimension k and $a_1, \dots, a_k$ are the coefficients of the components of Δ containing Z then the coefficient of the exceptional divisor is

$$a = 1 + \sum_{i \leq k} a_i - k.$$

If $a > 0$ then $a \in I$ and we are done by induction on the number of blowups. □

Lemma 2.8.3. *Let (X, Δ) be a log smooth pair where the coefficients of Δ belong to $(0, 1]$. Suppose that there is a projective morphism $\psi: X \to U$, where U is an affine variety. If (X, Δ) has a weak log canonical model then there is a sequence $\pi : Y \to X$ of smooth blowups of the strata of Δ such that if we write*

$$K_Y + \Gamma = \pi^*(K_X + \Delta) + E$$

where $\Gamma, E \geq 0$ have no common components, $\pi_*\Gamma = \Delta$ and $\pi_* E = 0$, and if we write

$$\Gamma' = \Gamma - \Gamma \wedge N_\sigma(Y, K_Y + \Gamma),$$

then $\mathbf{B}_-(Y, K_Y + \Gamma')$ contains no strata of Γ'. If Δ is a $\mathbb{Q}$-divisor then Γ' is a $\mathbb{Q}$-divisor.

Proof. Let $f: X \dashrightarrow W$ be a weak log canonical model of (X, Δ). Let $\Phi = f_*\Delta$. Let I be the finite set whose existence is guaranteed by 2.8.2 applied to (W, Φ).

Suppose that $\pi : Y \to X$ is a sequence of smooth blowups of the strata of Δ. We may write

$$K_Y + \Gamma = \pi^*(K_X + \Delta) + E,$$

where $\Gamma, E \geq 0$ have no common components, $\pi_*\Gamma = \Delta$ and $\pi_* E = 0$.

Note that if $g: Y \dashrightarrow W$ is the induced birational map then g is a weak log canonical model of (Y, Γ). In particular if we write

$$K_Y + \Gamma = g^*(K_W + \Phi) + E_1$$

then $E_1 = N_\sigma(Y, K_Y + \Gamma)$. Thus if we write

$$K_Y + \Gamma_0 = g^*(K_W + \Phi) + E_0,$$

where $\Gamma_0, E_0 \geq 0$ have no common components, $g_*\Gamma_0 = \Phi$ and $g_* E_0 = 0$, then

$$\Gamma_0 = \Gamma' = \Gamma - \Gamma \wedge N_\sigma(Y, K_Y + \Gamma).$$

Let $p\colon V \to Y$ and $q\colon V \to W$ resolve g, so that the strict transform of Φ and the exceptional locus of q have global normal crossings. We may write

$$K_V + \Psi = q^*(K_W + \Phi) + F,$$

where $\Psi, F \geq 0$ have no common components, $q_*\Psi = \Phi$ and $q_*F = 0$. Note that the coefficients of Ψ belong to I.

As $q^*(K_W + \Phi)$ is nef, Ψ has no components in common with $N_\sigma(V, K_V + \Psi) = F$. As

$$K_Y + p_*\Psi = g^*(K_W + \Phi) + p_*F, \qquad K_Y + \Gamma_0 = g^*(K_W + \Phi) + E_0,$$

we have

$$\Gamma_0 + p_*F = p_*\Psi + E_0.$$

As Γ_0 and E_0, and also $p_*\Psi$ and p_*F, have no common components, it follows that $\Gamma' = p_*\Psi$, so that the coefficients of Γ' belong to I.

Suppose that Z is a stratum of (X, Δ) which is contained in $N_\sigma(X, K_X + \Delta)$. Let $\pi\colon Y \to X$ blow up Z and let E be the exceptional divisor. The coefficient of E in Γ is no more than the minimum coefficient of any component of Δ containing Z. Either the coefficient of E in Γ' is zero or E is a component of $\Gamma - \Gamma'$, so that, either way, the coefficient of E in Γ' is strictly less than the coefficient of any component of Δ containing Z. Since I is a finite set and (X, Δ) has only finitely many strata, it is clear that after finitely many blowups no stratum of (Y, Γ') is contained in $N_\sigma(Y, K_Y + \Gamma')$. □

Lemma 2.8.4. *Let (X, Δ) be a log pair and let $\pi\colon X \to U$ be a morphism of quasi-projective varieties. Suppose that U is smooth, π is flat and the support of Δ contains neither a component of a fibre nor a codimension one singular point of any fibre. Then the subset $U_0 \subset U$ of points $u \in U$ such that the fibre (X_u, Δ_u) is divisorially log terminal is constructible. Further, if U_0 is dense in U then we may find a smooth dense open subset U_1 of U, contained in U_0, such that the restriction of (X, Δ) to U_1 is divisorially log terminal.*

Proof. Let V be a smooth open subset of the closure of U_0. We may assume that V is irreducible. Replacing U by V we may assume that U_0 is dense in V.

Let $f\colon Y \to X$ be a log resolution. We may write

$$K_Y + \Gamma = f^*(K_X + \Delta) + E,$$

where $\Gamma, E \geq 0$ have no common components. Passing to an open subset of U we may assume that (Y, Γ) is log smooth over U. As Γ_u is a boundary for a dense set of points $u \in U_0$, it follows that Γ is a boundary.

Suppose that F is an exceptional divisor of log discrepancy zero with respect to (X, Δ), that is, of coefficient one in Γ. Let $Z = f(F)$ be the centre of F in X. Note that F_u has log discrepancy zero with respect to (X_u, Δ_u) for any $u \in U_0$. As (X_u, Δ_u) is divisorially log terminal, it follows that (X_u, Δ_u) is log smooth in a neighbourhood of the generic point of Z_u. But then (X, Δ) is log smooth in a neighbourhood of the generic point of Z and so (X, Δ) is divisorially log terminal. This is the second statement.

As (X, Δ) is log smooth in a neighbourhood of the generic point of Z, we may find an open subset $U_2 \subset U_0$ such that if $u \in U_2$ then (X_u, Δ_u) is log smooth in a neighbourhood of the generic point of Z_u. Possibly shrinking U_2 we may also assume that the non-Kawamata log terminal locus of (X_u, Δ_u) is the restriction of the non-Kawamata log terminal locus of (X, Δ). It follows that if $u \in U_2$ then (X_u, Δ_u) is divisorially log terminal. □

2.9. Good minimal models

Lemma 2.9.1. *Let (X, Δ) be a divisorially log terminal pair, where X is projective and $\mathbb{Q}$-factorial. If (X, Δ) has a weak log canonical model then the following are equivalent:*

(1) *every weak log canonical model of (X, Δ) is semi-ample,*
(2) *(X, Δ) has a semi-ample model, and*
(3) *(X, Δ) has a good minimal model.*

Proof. (1) implies (2) is clear.

We show that (2) implies (3). Suppose that $g\colon X \dashrightarrow Z$ is a semi-ample model of (X, Δ). Let $p\colon W \to X$ be a log resolution of (X, Δ) which also resolves g, so that the induced rational map is a morphism $q\colon W \to Z$. We may write

$$K_W + \Phi = p^*(K_X + \Delta) + E,$$

where $\Phi, E \geq 0$ have no common components, $p_*\Phi = \Delta$ and $p_*E = 0$. [17, 2.10] implies that (X, Δ) has a good minimal model if and only if (W, Φ) has a good minimal model.

Replacing (X, Δ) by (W, Φ) we may assume that g is a morphism. We run the $(K_X + \Delta)$-MMP $f\colon X \dashrightarrow Y$ with scaling of an ample divisor over Z. Note that running the $(K_X + \Delta)$-MMP over Z is the same as running the absolute $(K_X + \Delta + H)$-MMP, where H is the pullback of a sufficiently ample divisor from Z. Note also that $N_\sigma(X, K_X + \Delta)$ and $N_\sigma(X, K_X + \Delta + H)$ have the same components. By (2) of 2.7.2 we may run the $(K_X + \Delta)$-MMP with scaling over Z until f contracts every component of $N_\sigma(X, K_X + \Delta)$. If $\Gamma = f_*\Delta$ and $h\colon Y \to Z$ is the induced birational morphism then h only contracts the divisor on which $K_Y + \Gamma$ is trivial. As $h_*(K_Y + \Gamma) = g_*(K_X + \Delta)$ is semi-ample, it follows that

$$K_Y + \Gamma = h^*h_*(K_Y + \Gamma)$$

is semi-ample and so f is a good minimal model. Thus (2) implies (3).

Suppose that $f\colon X \dashrightarrow Y$ is a minimal model and $g\colon X \dashrightarrow Z$ is a weak log canonical model. Let $p\colon W \to Y$ and $q\colon W \to Z$ be common resolutions of Y and Z over X, with induced morphism $r\colon W \to X$. Then we may write

$$p^*(K_Y + \Gamma) + E_1 = r^*(K_X + \Delta) = q^*(K_Z + \Phi) + E_2,$$

where $\Gamma = f_*\Delta$, $\Phi = g_*\Delta$, $E_1 \geq 0$ is p-exceptional and $E_2 \geq 0$ is q-exceptional. As f is a minimal model and g is a weak log canonical model, every f-exceptional divisor is g-exceptional. Thus

$$p^*(K_Y + \Gamma) + E = q^*(K_Z + \Phi),$$

where $E = E_1 - E_2$ is q-exceptional. Negativity of contraction applied to q implies that $E \geq 0$, so that $E \geq 0$ is p-exceptional. Negativity of contraction applied to p implies that $E = 0$. But then $K_Y + \Gamma$ is semi-ample if and only if $K_Z + \Phi$ is semi-ample. Thus (3) implies (1). □

Lemma 2.9.2. *Let (X, Δ) be a divisorially log terminal pair, where X is projective. Let A be an ample dvisor. Let $\pi\colon V \to X$ be a divisorially log terminal modification of X such that π is small and if we write*

$$K_V + \Sigma = \pi^*(K_X + \Delta),$$

then (V, Σ) is divisorially log terminal and V is $\mathbb{Q}$-factorial. If (V, Σ) has a good minimal model then there is a constant $\epsilon > 0$ with the following properties:

(1) *If $g_t\colon X \dashrightarrow Z_t$ is the log canonical model of $(X, \Delta + tA)$ then Z_t is independent of $t \in (0, \epsilon)$ and there is a morphism $Z_t \to Z_0$.*
(2) *If $h\colon X \dashrightarrow Y$ is a weak log canonical model of $(X, \Delta + tA)$ for some $t \in [0, \epsilon)$ then h is a semi-ample model of (X, Δ).*

Proof. Note that as A is ample, $(X, \Delta + tA)$ has a log canonical model Z_t for $t > 0$ by [7, 1.1]. Note also that since π is small, V and X have the same log canonical models and weak log canonical models. At the expense of dropping the hypothesis that A is ample, replacing X by V we may assume that X is $\mathbb{Q}$-factorial.

Suppose that we run the $(K_X + \Delta)$-MMP $f_t\colon X \dashrightarrow W_t$ with scaling of A. Then [6, 1.9.iii] implies that this MMP terminates with a minimal model, so that we may find $\epsilon > 0$ such that $f = f_0 = f_t\colon X \dashrightarrow W = W_t$ is independent of $t \in [0, \epsilon)$. Let $\Phi = f_*\Delta$ and $B = f_*A$. If $C \subset W$ is a curve such that $(K_W + \Phi + sB) \cdot C = 0$ for some $s \in (0, \epsilon)$, then

$$(K_W + \Phi + tB) \cdot C = 0 \qquad \text{for all } t \in [0, \epsilon),$$

since $K_W + \Phi + \lambda B$ is nef for all $\lambda \in (0, \epsilon)$. The induced contraction morphism $W \to Z_t$ to the ample model contracts those curves C such that $(K_W + \Phi + tB) \cdot C = 0$ so that $Z = Z_t$ is independent of $t \in (0, \epsilon)$ and there is a contraction morphism $Z_t \to Z_0$. This is (1).

Let $h\colon X \dashrightarrow Y$ be a weak log canonical model of $(X, \Delta + tA)$. Then h is a semi-ample model of $(X, \Delta + tA)$ and there is an induced morphism $\psi\colon Y \to Z$.

Possibly replacing ϵ with a smaller number, we see that 2.7.1 implies that h contracts every component of $N_\sigma(X, K_X + \Delta)$, independently of the choice of weak log canonical model. Note that if P is a prime divisor which is not a component of $N_\sigma(X, K_X + \Delta)$ then the restriction to P of the birational map associated to some multiple of $K_X + \Delta + tA$ is birational. In particular 2.7.1 implies that h does not contract P. Thus h contracts the components of $N_\sigma(X, K_X + \Delta)$ and no other divisors. Since Z is a log canonical model of $(X, \Delta + tA)$, the morphism $X \dashrightarrow Z$ also contracts the components of $N_\sigma(X, K_X + \Delta)$ and no other divisors. It follows that ψ is a small morphism.

If $\Gamma = h_*\Delta$, $B = h_*A$, $\Psi = \psi_*\Gamma$ and $C = \psi_*B$ then

$$K_Y + \Gamma + sB = \psi^*(K_Z + \Psi + sC)$$

for any s. By assumption $K_Z + \Psi + sC$ is ample for $s \in (0, \epsilon)$ and so $K_Y + \Gamma + sB$ is nef for $s \in (0, \epsilon)$. Thus $K_Y + \Gamma$ is nef and so h is a semi-ample model of (X, Δ) by 2.9.1. □

Lemma 2.9.3. *Let k be any field of characteristic zero and let (X, Δ) be a log pair over k, where X is a projective variety. Let $(\bar{X}, \bar{\Delta})$ be the corresponding pair over the algebraic closure $\bar{k}$ of k. Assume that $(\bar{X}, \bar{\Delta})$ is divisorially log terminal and $\bar{X}$ is $\mathbb{Q}$-factorial. Then (X, Δ) has a good minimal model if and only if $(\bar{X}, \bar{\Delta})$ has a good minimal model.*

Proof. If W is a scheme over k then $\bar{W}$ denotes the corresponding scheme over $\bar{k}$. If $f: X \dashrightarrow Y$ is a good minimal model of (X, Δ) then $\bar{f}: \bar{X} \dashrightarrow \bar{Y}$ is a semi-ample model of $(\bar{X}, \bar{\Delta})$ and so $(\bar{X}, \bar{\Delta})$ has a good minimal model by 2.9.1.

Conversely, suppose that $(\bar{X}, \bar{\Delta})$ has a good minimal model. Pick an ample divisor A on X. We run the $(K_X + \Delta)$-MMP $f: X \dashrightarrow Y$ with scaling of A. Then f is a weak log canonical model of $(X, \Delta + tA)$ and so $\bar{f}: \bar{X} \dashrightarrow \bar{Y}$ is a weak log canonical model of $(\bar{X}, \bar{\Delta} + t\bar{A})$. 2.9.2 implies that we may find $\epsilon > 0$ such that $\bar{f}$ is a semi-ample model of $(\bar{X}, \bar{\Delta})$ for $t \in [0, \epsilon)$. If $\Gamma = f_*\Delta$ then $K_{\bar{Y}} + \bar{\Gamma}$ is semi-ample so that $K_Y + \Gamma$ is semi-ample. But then f is a good minimal model of (X, Δ). □

3. The MMP in families I

Lemma 3.1. *Let (X, Δ) be a divisorially log terminal pair and let $\pi: X \to U$ be a projective contraction morphism, where U is smooth, affine and of dimension k, and X is $\mathbb{Q}$-factorial. Let $0 \in U$ be a closed point such that*

(1) *there are k divisors $D_1, \ldots, D_k$ containing 0 such that if $H_i = \pi^* D_i$ and $H = H_1 + \cdots + H_k$ then $(X, H + \Delta)$ is divisorially log terminal,*
(2) *X_0 is integral, $\dim X_0 = \dim X - \dim U$ and $\dim V_0 = \dim V - \dim U$ for all non-canonical centres V of (X, Δ), and*
(3) *$\mathbf{B}_-(X_0, K_{X_0} + \Delta_0)$ contains no non-canonical centres of (X_0, Δ_0).*

Let $f: X \dashrightarrow Y$ be a step of the $(K_X + \Delta)$-MMP. If f is birational and V is a non-canonical centre of (X, Δ) then f is an isomorphism in a neighbourhood of the generic point of V and f_0 is an isomorphism in a neighbourhood of the generic point of V_0. In particular the induced maps $\phi: V \dashrightarrow W$ and $\phi_0: V_0 \dashrightarrow W_0$ are birational, where $W = f(V)$. Let $\Gamma = f_*\Delta$. Further,

(1) *if G_i is the pullback of D_i to Y and $G = G_1 + \cdots + G_k$ then $(Y, G + \Gamma)$ is divisorially log terminal,*
(2) *Y_0 is integral, $\dim Y_0 = \dim Y - \dim U$ and $\dim W_0 = \dim W - \dim U$ for all non-canonical centres W of (Y, Γ), and*
(3) *$\mathbf{B}_-(Y_0, K_{Y_0} + \Gamma_0)$ contains no non-canonical centres of (Y_0, Γ_0).*

If V is a non-Kawamata log terminal centre or $V = X$, then $\phi: V \dashrightarrow W$ and $\phi_0: V_0 \dashrightarrow W_0$ are birational contractions. On the other hand, if f is a Mori fibre space then f_0 is not birational.

Proof. Suppose that f is birational.

As f is a step of the $(K_X+\Delta)$-MMP and H is pulled back from U, it follows that f is also a step of the $(K_X + H + \Delta)$-MMP, and so $(Y, G + \Gamma)$ is divisorially log terminal. As every component of Y_0 is a non-Kawamata log terminal centre of (Y, G) and X_0 is irreducible, it follows that Y_0 is irreducible.

Let V be a non-canonical centre of (X, Δ). Then V is a non-canonical centre of $(X, H + \Delta)$. Let $g\colon X \to Z$ be the contraction of the extremal ray associated to f (so that $f = g$ unless f is a flip). Every component of V_0 is a non-canonical centre of (X_0, Δ_0) [7, 1.4.5], and so no component of V_0 is contained in $\mathbf{B}_-(X_0, K_{X_0} + \Delta_0)$ by hypothesis. On the other hand, note that the locus where g is not an isomorphism is the locus of curves C such that $(K_X + H + \Delta)\cdot C < 0$. Thus the locus where g_0 is not an isomorphism is equal to the locus of curves $C_0 \subset X_0$ such that $(K_{X_0} + \Delta_0)\cdot C_0 < 0$. As every such curve C_0 is contained in $\mathbf{B}_-(X_0, K_{X_0}+\Delta_0)$, it follows that the locus where g_0 (respectively g) is not an isomorphism intersects V_0 (respectively V) in a proper closed subset. In particular both $\phi\colon V \dashrightarrow W$ and $\phi_0\colon V_0 \dashrightarrow W_0$ are birational.

Now suppose that V is a non-Kawamata log terminal centre or $V = X$. If V is a non-Kawamata log terminal centre then V is a non-canonical centre and so $\phi\colon V \dashrightarrow W$ and $\phi_0\colon V_0 \dashrightarrow W_0$ are both birational. We can define divisors Σ_0 and Θ_0 on V_0 and W_0 by adjunction:

$$(K_{X_0} + \Delta_0)|_{V_0} = K_{V_0} + \Sigma_0 \quad \text{and} \quad (K_{Y_0} + \Gamma_0)|_{W_0} = K_{W_0} + \Theta_0.$$

If P is a divisor on W_0 and f is not an isomorphism at the generic point of the centre N of P on V_0 then

$$a(P; V_0, \Sigma_0) < a(P; W_0, \Theta_0) \leq 1.$$

Thus N is a non-canonical centre of (X, Δ). Therefore N is birational to P, so that N is a divisor on V_0. Thus $\phi_0\colon V_0 \dashrightarrow W_0$ is a birational contraction. In particular $f_0\colon X_0 \dashrightarrow Y_0$ is a birational contraction and so (1)–(3) clearly hold. As $\phi_0\colon V_0 \dashrightarrow W_0$ is a birational contraction, it follows that $\phi\colon V \dashrightarrow W$ is a birational contraction in a neighbourhood of V_0.

Suppose that f is a Mori fibre space. As the dimension of the fibres of $f\colon X \to Y$ is upper-semicontinuous, f_0 is not birational. □

Lemma 3.2. *Let (X, Δ) be a divisorially log terminal pair and let $\pi\colon X \to U$ be a projective morphism, where U is smooth and affine and X is $\mathbb{Q}$-factorial. Let $\eta \in U$ be the generic point and let* $0 \in U$ *be a closed point. Suppose that either* (1)–(3) *below hold where*

(1) *there are k divisors $D_1, \dots, D_k$ containing* 0 *such that if $H_i = \pi^* D_i$ and $H = H_1 + \cdots + H_k$ then $(X, H + \Delta)$ is divisorially log terminal,*
(2) *X_0 is integral,* $\dim X_0 = \dim X - \dim U$ *and* $\dim V_0 = \dim V - \dim U$ *for all non-canonical centres V of (X, Δ),*
(3) $\mathbf{B}_-(X_0, K_{X_0} + \Delta_0)$ *contains no non-canonical centres of* (X_0, Δ_0)*,*

or (X, Δ) *is log smooth over* U *and* (3) *holds. If* (X_0, Δ_0) *has a good minimal model then we may run the* $(K_X+\Delta)$*-MMP* $f: X \dashrightarrow Y$ *until* $f_\eta: X_\eta \dashrightarrow Y_\eta$ *is an* (X_η, Δ_η)*-minimal model and* $f_0: X_0 \dashrightarrow Y_0$ *is a semi-ample model of* (X_0, Δ_0)*. If* D *is a component of* $\lfloor \Delta \rfloor$*,* E *is the image of* D *and* $\phi: D \dashrightarrow E$ *is the restriction of* f *to* D *then the induced map* $\phi_0: D_0 \dashrightarrow E_0$ *is a semi-ample model of* (D_0, Σ_0)*, where* Σ_0 *is defined by adjunction*

$$(K_{X_0} + \Delta_0)|_{D_0} = K_{D_0} + \Sigma_0.$$

Further, $\mathbf{B}_-(X, K_X + \Delta)$ *contains no non-canonical centres of* (X_0, Δ_0)*.*

Proof. Suppose that (X, Δ) is log smooth over U. If $D_1, \dots, D_k$ are k general divisors containing 0 then $(X, H+\Delta)$ is log smooth, so that (1) and (2) hold. Thus we may assume (1)–(3) hold.

We run the $(K_X + \Delta)$-MMP $f: X \dashrightarrow Y$ with scaling of an ample divisor A. Let $\Gamma = f_*\Delta$ and $B = f_*A$. By construction $K_Y + tB + \Gamma$ is nef for some $t > 0$. Since $\pi: X \to U$ satisfies the hypotheses of 3.1, $f_0: X_0 \dashrightarrow Y_0$ is a weak log canonical model of $(X_0, tA_0 + \Delta_0)$.

If $K_X + \Delta$ is not pseudo-effective then this MMP ends with a Mori fibre space for some $t > 0$ and so Y_0 is covered by curves on which $K_{Y_0} + tB_0 + \Gamma_0$ is negative by 3.1. This contradicts the fact that $K_{X_0} + tA_0 + \Delta_0$ is big. Thus $K_X + \Delta$ is pseudo-effective and given any $\epsilon > 0$ we may run the MMP until $t < \epsilon$.

Since $K_{X_0} + \Delta_0$ has a good minimal model, 2.9.2 implies that there is a constant $\epsilon > 0$ such that if $t \in (0, \epsilon)$ then any more steps of this MMP are isomorphisms in a neighbourhood of Y_0. It follows that $K_{Y_\eta} + tB_\eta + \Gamma_\eta$ is nef for all $t \in (0, \epsilon)$, so that $K_{Y_\eta} + \Gamma_\eta$ is nef. As we are running a MMP, Y is $\mathbb{Q}$-factorial and so Y_η is $\mathbb{Q}$-factorial. Thus $f_\eta: X_\eta \dashrightarrow Y_\eta$ is a minimal model of (X_η, Δ_η).

Suppose that D is a component of $\lfloor \Delta \rfloor$. Then 3.1 implies that the induced map $\phi_0: D_0 \dashrightarrow E_0$ is a birational contraction, so that ϕ_0 is a semi-ample model of (D_0, Σ_0).

As

$$(K_Y + \Gamma)|_{Y_0} = K_{Y_0} + \Gamma_0$$

is nef, it follows that $\mathbf{B}_-(Y, K_Y + \Gamma)$ does not intersect Y_0. Let G be an ample $\mathbb{Q}$-divisor on Y. Then the stable base locus of $K_Y + \Gamma + tG$ does not intersect Y_0 for any $t > 0$. If $x \in X_0$ is a point where f is an isomorphism then x is not a point of the stable base locus of $K_X + \Delta + f^*(tG)$. As $t > 0$ is arbitrary, it follows that $\mathbf{B}_-(X, K_X + \Delta)|_{X_0}$ is contained in the locus where $f: X \dashrightarrow Y$ is not an isomorphism. By 3.1, f is an isomorphism in a neighbourhood of any non-canonical centre. It follows that $\mathbf{B}_-(X, K_X + \Delta)$ contains no non-canonical centres of (X_0, Δ_0). □

4. Invariance of plurigenera

We will need the following result of B. Berndtsson and M. Păun.

Theorem 4.1. *Let* $f: X \to \mathbb{D}$ *be a projective contraction morphism to the unit disk* $\mathbb{D}$ *and let* (X, Δ) *be a log pair. If*

(1) (X, Δ) *is log smooth over* $\mathbb{D}$ *and* $\lfloor \Delta \rfloor = 0$*,*
(2) *the components of* Δ *do not intersect,*

(3) $K_X + \Delta$ *is pseudo-effective, and*
(4) $\mathbf{B}_-(X, K_X + \Delta)$ *does not contain any components of* Δ_0,

then

$$H^0(X, \mathcal{O}_X(m(K_X + \Delta))) \to H^0(X_0, \mathcal{O}_{X_0}(m(K_{X_0} + \Delta_0)))$$

is surjective for any integer m *such that* $m\Delta$ *is integral.*

Proof. Note that the case $\Delta = 0$ is proven in [32]. Therefore we may assume that $\Delta \neq 0$. We check that the hypotheses of [5, Theorem 0.2] are satisfied and we will use the notation introduced there.

We take $\alpha = 0$ and $p = m$ so that if $L = \mathcal{O}_X(m\Delta)$ then

$$p([\Delta] + \alpha) = m[\Delta] \in c_1(L)$$

is automatic. $K_X + \Delta$ is pseudo-effective by assumption. As we are assuming (4), we have $\nu_{\min}(\{K_X + \Delta\}, X_0) = 0$ and $\rho^j_{\min,\infty} = 0$. In particular $J = J'$ and $\Xi = 0$. As we are assuming that the components of Δ do not intersect, the transversality hypothesis is automatically satisfied.

If $u \in H^0(X_0, \mathcal{O}_{X_0}(m(K_{X_0} + \Delta_0)))$ is a non-zero section then we choose $h_0 = e^{-\varphi_0}$ such that $\varphi_0 \leq 0 = \varphi_\Xi$ and

$$\Theta_{h_0}(K_{X_0} + \Delta_0) \geq 0.$$

Since u has no poles and $\lfloor \Delta \rfloor = 0$, we have

$$\int_{X_0} e^{\varphi_0 - \frac{1}{m}\varphi_{m\Delta}} < \infty.$$

Now [5, Theorem 0.2] implies that we can extend u to $U \in H^0(X, \mathcal{O}_X(m(K_X+\Delta)))$. $\square$

Theorem 4.2. *Let* $\pi : X \to U$ *be a projective contraction morphism to a smooth variety* U *and let* (X, Δ) *be a log smooth pair over* U *such that* $\lfloor \Delta \rfloor = 0$. *Then*

$$h^0(X_u, \mathcal{O}_{X_u}(m(K_{X_u} + \Delta_u)))$$

is independent of the point $u \in U$, *for all positive integers* m. *In particular* $\kappa(X_u, K_{X_u} + \Delta_u)$ *is independent of* $u \in U$, *and*

$$f_*\mathcal{O}_X(m(K_X + \Delta)) \to H^0(X_u, \mathcal{O}_{X_u}(m(K_{X_u} + \Delta_u)))$$

is surjective for all positive integers $m > 0$ *and all* $u \in U$.

Proof. Fix a positive integer m. We may assume that U is affine. We may also assume that the strata of Δ have irreducible fibers over U (cf. the proof of [14, 4.2]).

Replacing Δ by $\Delta_m = \lfloor m\Delta \rfloor / m$ we may assume that $m\Delta$ is integral.

By 2.8.1 there is a composition of smooth blowups of the strata of Δ such that if we write

$$K_Y + \Gamma = \pi^*(K_X + \Delta) + E,$$

where $\Gamma, E \geq 0$ have no common components, $\pi_*\Gamma = \Delta$ and $\pi_* E = 0$, then no two components of Γ intersect. Then (Y, Γ) is log smooth over U, $m\Gamma$ is integral and $\lfloor \Gamma \rfloor = 0$.

As

$$h^0(Y_u, \mathcal{O}_{Y_u}(m(K_{Y_u} + \Gamma_u))) = h^0(X_u, \mathcal{O}_{X_u}(m(K_{X_u} + \Delta_u))),$$

replacing (X, Δ) by (Y, Γ) we may assume that no two components of Δ intersect.

We may assume that

$$h^0(X_u, \mathcal{O}_{X_u}(m(K_{X_u} + \Delta_u))) \neq 0$$

for some $u \in U$. Let F be the fixed divisor of the linear system $|m(K_{X_u} + \Delta_u)|$ and let

$$\Theta_u = \Delta_u - \Delta_u \wedge F/m.$$

There is a unique divisor $0 \leq \Theta \leq \Delta$ such that $\Theta|_{X_u} = \Theta_u$. Note that $m\Theta$ is integral,

$$f_*\mathcal{O}_X(m(K_X + \Theta)) \subset f_*\mathcal{O}_X(m(K_X + \Delta))$$

and

$$H^0(X_u, \mathcal{O}_{X_u}(m(K_{X_u} + \Theta_u))) = H^0(X_u, \mathcal{O}_{X_u}(m(K_{X_u} + \Delta_u))).$$

Replacing (X, Δ) by (X, Θ) we may assume that no component of Δ_u is in the base locus of $|m(K_{X_u} + \Delta_u)|$. In particular $\mathbf{B}_-(X_u, K_{X_u} + \Delta_u)$ does not contain any components of Δ_u. Let A be an ample divisor on X. We may assume that $(X, \Delta + A)$ is log smooth over U. Since $K_{X_u} + \Delta_u + tA_u$ is big and $(X_u, \Delta_u + tA_u)$ is Kawamata log terminal for any $0 < t < 1$, it follows that $(X_u, \Delta_u + tA_u)$ has a good minimal model. 3.2 implies that $\mathbf{B}_-(X, K_X + \Delta + tA)$ does not contain any components of Δ_u for any $0 < t < 1$. Since

$$\mathbf{B}_-(X, K_X + \Delta) = \bigcap_{t>0} \mathbf{B}_-(X, K_X + \Delta + tA),$$

it follows that $\mathbf{B}_-(X, K_X + \Delta)$ contains no components of Δ_u and we may apply 4.1. □

Using 4.2 we can give another proof of [15, 1.8]:

Corollary 4.3. *Let $\pi : X \to U$ be a projective contraction morphism to a smooth variety U. If (X, Δ) is a log smooth pair over U and the coefficients of Δ are all at most one then* $\mathrm{vol}(X_u, K_{X_u} + \Delta_u)$ *is independent of $u \in U$.*

Proof. If $\epsilon \in (0, 1]$ is rational then we have $\lfloor (1-\epsilon)\Delta \rfloor = 0$ and so 4.2 implies that $h^0(X_u, \mathcal{O}_{X_u}(m(K_{X_u} + (1-\epsilon)\Delta_u)))$ is independent of the point $u \in U$, for all sufficiently divisible integers $m > 0$. In particular $\mathrm{vol}(X_u, K_{X_u} + (1-\epsilon)\Delta_u)$ is independent of $u \in U$. By continuity, $\mathrm{vol}(X_u, K_{X_u} + \Delta_u)$ is independent of $u \in U$. □

5. The MMP in families II

Lemma 5.1. *Let (X, Δ) be a log canonical pair and let (X, Φ) be a divisorially log terminal pair, where X is $\mathbb{Q}$-factorial of dimension n. Let*

$$\Delta(t) = (1 - t)\Delta + t\Phi.$$

Suppose that $X \to U$ is projective, U is smooth and affine, and the fibres of π all have the same dimension. Let $f : X \dashrightarrow Y$ be a step of the $(K_X + \Delta(t))$-MMP over U and let $\Gamma = f_*\Delta$. Suppose $0 \in U$ is a closed point such that X_0 is reduced, no component of X_0 is contained in the support of Δ, $K_{X_0} + \Delta_0$ is nef and (X_0, Δ_0) is log canonical. Let r be a positive integer such that $r(K_{X_0} + \Delta_0)$ is Cartier. If

$$0 < t \leq \frac{1}{1 + 2nr}$$

then f is $(K_X+\Delta)$-trivial in a neighbourhood of X_0. In particular (Y_0, Γ_0) is log canonical, $K_{Y_0}+\Gamma_0$ is nef, $r(K_{Y_0}+\Gamma_0)$ is Cartier and (Y, Γ) is log canonical in a neighbourhood of Y_0.

Proof. Let R be the extremal ray corresponding to f.

If f is an isomorphism in a neighbourhood of X_0 there is nothing to prove, and if $(K_X + \Delta) \cdot R = 0$, the result follows by [27, 3.17].

Otherwise, as $K_{X_0} + \Delta_0$ is nef, $(K_X + \Delta) \cdot R > 0$ and so $(K_X + \Phi) \cdot R < 0$. Now [19] (see also [7, 3.8.1]) implies that R is spanned by a rational curve C contained in X_0 such that

$$-(K_X + \Phi) \cdot C \leq 2n.$$

As $r(K_{X_0} + \Delta_0)$ is Cartier,

$$(K_X + \Delta) \cdot C = (K_{X_0} + \Delta_0) \cdot C \geq 1/r.$$

Thus

$$\begin{aligned} 0 > (K_X + \Delta(t)) \cdot C &= (1 - t)(K_X + \Delta) \cdot C + t(K_X + \Phi) \cdot C \\ &\geq (1 - t)/r - 2nt = 1/r - t(1 + 2nr)/r \geq 0, \end{aligned}$$

a contradiction. □

Lemma 5.2. *Let $(X, \Delta = S + B)$ be a divisorially log terminal pair, where $S \leq \lfloor \Delta \rfloor$ and X is $\mathbb{Q}$-factorial. Let $\pi : X \to U$ be a projective morphism, where U is smooth and affine, and the fibres of π all have the same dimension. Let $0 \in U$ be a closed point such that X_0 is integral, let n be the dimension of X and let r be a positive integer such that $r(K_{X_0} + \Delta_0)$ is Cartier. Suppose that X_0 is not contained in the support of Δ. Fix*

$$\epsilon < \frac{1}{2nr + 1}.$$

If (X_0, Δ_0) is log canonical, and $K_{X_0} + \Delta_0$ is nef but $K_X + (1 - \epsilon)S + B$ is not pseudo-effective, then we may run the $(K_X+(1-\epsilon)S+B)$-MMP $f : X \dashrightarrow Y$ over U, the steps of which are all $(K_X + \Delta)$-trivial in a neighbourhood of X_0, until we arrive at a Mori fibre space $\psi : Y \to Z$ such that the strict transform of S dominates Z and $K_Y + \Gamma \sim_{\mathbb{Q}} \psi^*L$ for some divisor L on Z.

Proof. We run the $(K_X + (1-\epsilon)S + B)$-MMP $f\colon X \dashrightarrow Y$ with scaling of an ample divisor over U. Then 5.1 implies that every step of this MMP is $(K_X + \Delta)$-trivial in a neighbourhood of X_0. As $K_X + (1-\epsilon)S + B$ is not pseudo-effective, this MMP ends with a Mori fibre space $\psi\colon Y \to Z$. As every step of this MMP is $(K_X + \Delta)$-trivial in a neighbourhood of X_0, it follows that the strict transform of S dominates Z. □

Lemma 5.3. *Let (X, Δ) be a divisorially log terminal pair, where X is $\mathbb{Q}$-factorial and projective and Δ is a $\mathbb{Q}$-divisor. If Φ is a $\mathbb{Q}$-divisor such that*

$$0 \le \Delta - \Phi \le N_\sigma(X, K_X + \Delta),$$

then (X, Φ) has a good minimal model if and only if (X, Δ) has a good minimal model.

Proof. Suppose that $f\colon X \dashrightarrow Y$ is a minimal model of (X, Δ). Let $\Gamma = f_*\Delta$. Then (2) of 2.7.2 implies that f contracts every component of $N_\sigma(X, K_X + \Delta)$, so that

$$f_*(K_X + \Delta) = K_Y + \Gamma = f_*(K_X + \Phi).$$

Let $p\colon W \to X$ and $q\colon W \to Y$ resolve f. If we write

$$p^*(K_X + \Delta) = q^*(K_Y + \Gamma) + E,$$

then $E \ge 0$ is q-exceptional and $p_*E = N_\sigma(X, K_X + \Delta)$. It follows that if we write

$$p^*(K_X + \Phi) = q^*(K_Y + \Gamma) + F,$$

then

$$F = E - p^*(\Delta - \Phi) \ge E - p^*(N_\sigma(X, K_X + \Delta)) = E - p^*p_*E.$$

As $E - p^*p_*E$ is p-exceptional, $p_*F \ge 0$ by the negativity lemma and so f is a weak log canonical model of (X, Φ). If f is a good minimal model of (X, Δ) then f is a semi-ample model of (X, Φ) and so (X, Φ) has a good minimal model by 2.9.1.

Now suppose that (X, Φ) has a good minimal model. We may run the $(K_X + \Phi)$-MMP until we get a minimal model $f\colon X \dashrightarrow Y$ of (X, Φ). Let $Y \to Z$ be the ample model of $K_X + \Phi$.

If $t > 0$ is sufficiently small then f is also a run of the $(K_X + \Delta_t)$-MMP, where

$$\Delta_t = \Phi + t(\Delta - \Phi).$$

Let n be the dimension of X and let r be a positive integer such that $r(K_X + \Phi)$ is Cartier. If

$$0 < t < \frac{1}{1 + 2nr}$$

and we continue to run the $(K_X + \Delta_t)$-MMP with scaling of an ample divisor then 5.1 (with U taken to be a point) implies that every step of this MMP is $(K_X + \Phi)$-trivial, so that every step is over Z. After finitely many steps, 2.7.2 implies that we obtain a model $g\colon X \dashrightarrow W$ which contracts the components of $N_\sigma(X, K_X + \Delta_t)$. As the supports of

$N_\sigma(X, K_X + \Delta)$ and $N_\sigma(X, K_X + \Delta_t)$ are the same and the support of $\Delta - \Phi$ is contained in $N_\sigma(X, K_X + \Delta)$, it follows that

$$g_*(K_X + \Delta) = g_*(K_X + \Phi).$$

Thus $g_*(K_X + \Delta)$ is semi-ample. On the other hand, g only contracts divisors in $N_\sigma(X, K_X + \Delta)$, so that 2.7.3 implies that g is a minimal model of (X, Δ). Thus $g\colon X \dashrightarrow W$ is a good minimal model of (X, Δ). □

6. Abundance in families

Lemma 6.1. *Suppose that (X, Δ) is a log pair where the coefficients of Δ belong to $(0, 1] \cap \mathbb{Q}$. Let $\pi\colon X \to U$ be a projective morphism to a smooth affine variety U. Suppose that (X, Δ) is log smooth over U. If there is a closed point $0 \in U$ such that the fibre (X_0, Δ_0) has a good minimal model then the generic fibre (X_η, Δ_η) has a good minimal model.*

Proof. By 2.9.3 it is enough to prove that the geometric generic fibre has a good minimal model. Replacing U by a finite cover we may therefore assume that π is a contraction morphism and the strata of Δ have irreducible fibres over U.

Let $f_0\colon Y_0 \to X_0$ be the birational morphism given by 2.8.3. As (X, Δ) is log smooth over U, the strata of Δ have irreducible fibres over U and f_0 blows up strata of Δ_0, we may extend f_0 to a birational morphism $f\colon Y \to X$ which is a composition of smooth blowups of strata of Δ. We may write

$$K_Y + \Gamma = f^*(K_X + \Delta) + E,$$

where $\Gamma, E \geq 0$ have no common components, $f_*\Gamma = \Delta$ and $f_*E = 0$. Then (Y, Γ) is log smooth and the fibres of the components of Γ are irreducible. [17, 2.10] implies that (Y_0, Γ_0) has a good minimal model, as (X_0, Δ_0) has a good minimal model; similarly [17, 2.10] also implies that if (Y_η, Γ_η) has a good minimal model then so does (X_η, Δ_η).

Replacing (X, Δ) by (Y, Γ) we may assume that if

$$\Theta_0 = \Delta_0 - \Delta_0 \wedge N_\sigma(X_0, K_{X_0} + \Delta_0)$$

then $\mathbf{B}_-(X_0, K_{X_0} + \Theta_0)$ contains no strata of Θ_0. There is a unique divisor $0 \leq \Theta \leq \Delta$ such that $\Theta|_{X_0} = \Theta_0$. Then 2.3.3 implies that

$$\Theta = \Delta - \Delta \wedge N_\sigma(X, K_X + \Delta),$$

so that

$$\Delta - \Theta \leq N_\sigma(X, K_X + \Delta).$$

Hence by 5.3 and 2.9.3 it suffices to prove that (X_η, Θ_η) has a good minimal model. Replacing (X, Δ) by (X, Θ) we may assume that $\mathbf{B}_-(X_0, K_{X_0} + \Delta_0)$ contains no strata of Δ_0. Then 3.2 implies that we can run the $(K_X + \Delta)$-MMP $f\colon X \dashrightarrow Y$ over U to obtain a minimal model of the generic fibre. Let $\Gamma = f_*\Delta$.

Pick a component D of $\lfloor\Delta\rfloor$. Let $\phi\colon D \dashrightarrow E$ be the restriction of f to D. Then 3.2 implies that ϕ_0 is a semi-ample model of $(D_0, (\Delta_0 - D_0)|_{D_0})$, and 2.9.1 implies that $(D_0, (\Delta_0 - D_0)|_{D_0})$ has a good minimal model. By induction on the dimension, $(D_\eta, (\Delta_\eta - D_\eta)|_{D_\eta})$ has a good minimal model. But then $\phi_\eta\colon D_\eta \dashrightarrow E_\eta$ is a semi-ample model of $(D_\eta, (\Delta_\eta - D_\eta)|_{D_\eta})$.

Let $S = \lfloor\Delta\rfloor$ and $B = \{\Delta\} = \Delta - S$. Let $T = f_*S$ and $C = f_*B$. Suppose that $K_{Y_0} + (1-\epsilon)T_0 + C_0$ is not pseudo-effective for any $\epsilon > 0$. Then $K_{X_0} + (1-\epsilon)S_0 + B_0$ is not pseudo-effective for any $\epsilon > 0$. It follows easily that $K_X + (1-\epsilon)S + B$ is not pseudo-effective for any $\epsilon > 0$. But then $K_Y + (1-\epsilon)T + C$ is not either. 5.2 implies that we may run the $(K_Y + (1-\epsilon)T + C)$-MMP until we get to a Mori fibre space $g\colon Y \dashrightarrow W$, $\psi\colon W \to V$ over U. By assumption $g_*(K_Y + \Gamma) \sim_{\mathbb{Q}} \psi^*L$ for some divisor L.

Pick a component D of S whose image F in W dominates V. Let E be the image of D in Y. As we already observed, $\phi_\eta\colon D_\eta \dashrightarrow E_\eta$ is a semi-ample model of $(D_\eta, (\Delta_\eta - D_\eta)|_{D_\eta})$. As the birational map $g_0\colon Y_0 \dashrightarrow W_0$ is $(K_{Y_0} + \Gamma_0)$-trivial, the birational map $g_\eta\colon Y_\eta \dashrightarrow W_\eta$ is also $(K_{Y_\eta} + \Gamma_\eta)$-trivial. Then L_η is semi-ample as $(\psi^*L)|_{F_\eta}$ is semi-ample. The composition $X_\eta \dashrightarrow W_\eta$ is a semi-ample model of (X_η, Δ_η) and so (X_η, Δ_η) has a good minimal model by 2.9.1.

Otherwise, $K_{Y_0} + (1-\epsilon)T_0 + C_0$ is pseudo-effective for some $\epsilon > 0$. If $Y_0 \to Z_0$ is the log canonical model of (Y_0, Γ_0) then T_0 does not dominate Z_0 and so if ϵ is sufficiently small then $K_{X_0} + (1-\epsilon)S_0 + B_0$ has the same Kodaira dimension as $K_{X_0} + \Delta_0$. We have

$$\begin{aligned}\kappa(X_\eta, K_{X_\eta} + \Delta_\eta) &\geq \kappa(X_\eta, K_{X_\eta} + (1-\epsilon)S_\eta + B_\eta) = \kappa(X_0, K_{X_0} + (1-\epsilon)S_0 + B_0)\\ &= \kappa(X_0, K_{X_0} + \Delta_0) = \kappa_\sigma(X_0, K_{X_0} + \Delta_0)\\ &= \nu(Y_0, K_{Y_0} + \Gamma_0) = \nu(Y_\eta, K_{Y_\eta} + \Gamma_\eta).\end{aligned}$$

The first inequality holds as $S_\eta \geq 0$, the second equality holds by 4.2 (note that $(X_0, (1-\epsilon)S_0 + B_0)$ is Kawamata log terminal as (X_0, Δ_0) is divisorially log terminal) and the last equality holds as intersection numbers are deformation invariant.

We have already seen that if E is a component of T then $(K_Y + \Gamma)|_{E_\eta}$ is semi-ample. 2.5.1 implies that $(K_Y + \Gamma)|_{T_\eta}$ is semi-ample. Let $H = K_{Y_\eta} + \Gamma_\eta$. Then $H|_{T_\eta}$ is semi-ample and $aH - (K_{Y_\eta} + \Gamma_\eta)$ is nef and abundant for all $a > 1$. Thus $f_\eta\colon X_\eta \dashrightarrow Y_\eta$ is a good minimal model by 2.6.1. □

Lemma 6.2. *Suppose that (X, Δ) is a log pair where the coefficients of Δ belong to $(0, 1] \cap \mathbb{Q}$. Let $\pi\colon X \to U$ be a projective morphism to a smooth affine variety U. Suppose that (X, Δ) is log smooth over U. If (X, Δ) has a good minimal model then every fibre (X_u, Δ_u) has a good minimal model.*

Proof. Replacing U by a finite cover we may assume that π is a contraction morphism and the strata of Δ have irreducible fibres over U.

Let $f\colon Y \to X$ be the birational morphism given by 2.8.3. We may write

$$K_Y + \Gamma = f^*(K_X + \Delta) + E,$$

where $\Gamma, E \geq 0$ have no common components, $f_*\Gamma = \Delta$ and $f_*E = 0$. Then (Y, Γ) is log smooth. [17, 2.10] implies that (Y, Γ) has a good minimal model, as (X, Δ) does;

similarly [17, 2.10] also implies that if (Y_u, Γ_u) has a good minimal model then so does (X_u, Δ_u).

Replacing (X, Δ) by (Y, Γ) we may assume that if

$$\Theta = \Delta - \Delta \wedge N_\sigma(X, K_X + \Delta)$$

then $\mathbf{B}_-(X, K_X + \Theta)$ contains no strata of Θ. As

$$\Delta - \Theta \leq N_\sigma(X, K_X + \Delta),$$

5.3 implies that (X, Θ) has a good minimal model. 2.3.3 implies that

$$\Theta_u = \Delta_u - \Delta_u \wedge N_\sigma(X_u, K_{X_u} + \Delta_u),$$

so that $\mathbf{B}_-(X_u, K_{X_u} + \Theta_u)$ contains no strata of Θ_u. Hence

$$\Delta_u - \Theta_u \leq N_\sigma(X_u, K_{X_u} + \Delta_u).$$

Hence by 5.3 it suffices to prove that (X_u, Θ_u) has a good minimal model. Replacing (X, Δ) by (X, Θ) we may assume that $\mathbf{B}_-(X_u, K_{X_u} + \Delta_u)$ contains no strata of Δ_u.

Let A be an ample divisor over U. Then [17, 2.7] implies that the $(K_X + \Delta)$-MMP with scaling of A terminates $\pi : X \dashrightarrow Y$ with a good minimal model for (X, Δ) over U. Since $\mathbf{B}_-(X_u, K_{X_u} + \Delta_u)$ contains no strata of Δ_u, 3.1 implies that $\pi_u : X_u \dashrightarrow Y_u$ is a semi-ample model of (X_u, Δ_u). Finally, 2.9.1 implies that (X_u, Δ_u) has a good minimal model. □

Proof of 1.2. By 6.1 the generic fibre (X_η, Δ_η) has a good minimal model. Hence we may find a good minimal model of $\pi^{-1}(U_0)$ over an open subset U_0 of U. As (X, Δ) is log smooth over U, every stratum of $S = \lfloor \Delta \rfloor$ intersects $\pi^{-1}(U_0)$. Therefore we may apply [17, 1.1] to conclude that (X, Δ) has a good minimal model over U. Finally, 6.2 implies that every fibre has a good minimal model. □

Proof of 1.3. By 2.8.4 we may assume that (X, Δ) is divisorially log terminal and every fibre (X_u, Δ_u) is divisorially log terminal.

It suffices to prove that if U_0 is dense then it contains an open subset. Let $\pi : Y \to X$ be a log resolution. We may write

$$K_Y + \Gamma = \pi^*(K_X + \Delta) + E,$$

where $\Gamma, E \geq 0$ have no common components. Passing to an open subset we may assume that (Y, Γ) is log smooth over U, so that

$$K_{Y_u} + \Gamma_u = \pi^*(K_{X_u} + \Delta_u) + E_u$$

for all $u \in U$. Now [17, 2.10] implies that if (Y, Γ) has a good minimal model over U then so does (X, Δ). Similarly [17, 2.10] implies that if (X_u, Δ_u) has a good minimal model then so does (Y_u, Γ_u).

Replacing (X, Δ) by (Y, Γ) we may assume that (X, Δ) is log smooth over U. Then 1.2 implies that $U_0 = U$. □

Lemma 6.3. *Let $\pi: X \to U$ be a projective morphism to a smooth variety U and let (X, Δ) be log smooth over U. Suppose that the coefficients of Δ belong to $(0, 1] \cap \mathbb{Q}$. If there is a closed point $0 \in U$ such that the fibre (X_0, Δ_0) has a good minimal model then the restriction morphism*

$$\pi_* \mathcal{O}_X(m(K_X + \Delta)) \to H^0(X_0, \mathcal{O}_{X_0}(m(K_{X_0} + \Delta_0)))$$

is surjective for any $m \in \mathbb{N}$ such that $m\Delta$ is integral.

Proof. 2.3.4 implies that we may assume that $m \geq 2$. Replacing U by a finite cover we may assume that π is a contraction morphism and the strata of Δ have irreducible fibres over U. Since the result is local we may assume that U is affine and so we want to show that the restriction map

$$H^0(X, \mathcal{O}_X(m(K_X + \Delta))) \to H^0(X_0, \mathcal{O}_{X_0}(m(K_{X_0} + \Delta_0)))$$

is surjective. Cutting by hyperplanes we may assume that U is a curve. Let $f_0: Y_0 \to X_0$ be the birational morphism given by 2.8.3. As (X, Δ) is log smooth over U and the strata of Δ have irreducible fibres over U, and as f_0 blows up strata of Δ_0, we may extend f_0 to a birational morphism $f: Y \to X$ which is a composition of smooth blowups of strata of Δ. We may write

$$K_Y + \Gamma = f^*(K_X + \Delta) + E,$$

where $\Gamma, E \geq 0$ have no common components, $f_*\Gamma = \Delta$ and $f_*E = 0$. Then (Y, Γ) is log smooth and the fibres of the components of Γ are irreducible. Note that $m\Gamma$ is integral and the natural maps induce isomorphisms

$$H^0(X, \mathcal{O}_X(m(K_X + \Delta))) \simeq H^0(Y, \mathcal{O}_Y(m(K_Y + \Gamma)))$$

and

$$H^0(X_0, \mathcal{O}_{X_0}(m(K_{X_0} + \Delta_0))) \simeq H^0(Y_0, \mathcal{O}_{Y_0}(m(K_{Y_0} + \Gamma_0)))$$

Replacing (X, Δ) by (Y, Γ) we may assume that if

$$\Theta_0 = \Delta_0 - \Delta_0 \wedge N_\sigma(X_0, K_{X_0} + \Delta_0)$$

then $\mathbf{B}_-(X_0, K_{X_0} + \Theta_0)$ contains no strata of Θ_0. There is a unique divisor $0 \leq \Theta \leq \Delta$ such that $\Theta|_{X_0} = \Theta_0$. Now 1.2 implies that $K_X + \Delta$ is pseudo-effective and so 2.3.3 implies that

$$\Theta = \Delta - \Delta \wedge N_\sigma(X, K_X + \Delta).$$

As (X_0, Δ_0) has a good minimal model, 5.3 implies that (X_0, Θ_0) has a good minimal model. Therefore 1.2 implies that (X, Θ) has a good minimal model over U and so [17, 2.9] implies that any run of the $(K_X + \Theta)$-MMP over U with scaling of an ample divisor always terminates. 3.2 implies that we may run the $(K_X + \Theta)$-MMP $f: X \dashrightarrow Y$ over U until we get a semi-ample model of the generic fibre; 3.1 implies that f is an isomorphism in a neighbourhood of the generic point of every non-Kawamata log terminal centre of $(X, X_0 + \Theta)$. Since any MMP over U terminates, we may continue this MMP until we get to a good minimal model over U, without changing the fibre over 0.

Let $V \subset X \times Y$ be the graph. Then $V \to X$ is an isomorphism in a neighbourhood of the generic point of each non-Kawamata log terminal centre of $(X, X_0 + \Theta)$. We may find a log resolution $W \to V$ of the strict transform of Θ and the exceptional divisor of $V \to Y$ which is an isomorphism in a neighbourhood of the generic point of each non-Kawamata log terminal centre of $(X, X_0 + \Theta)$. If $p \colon W \to X$ and $q \colon W \to Y$ are the induced morphisms then we may write

$$K_W + \Phi + W_0 = p^*(K_X + X_0 + \Theta) + E,$$

where W_0 is the strict transform of X_0, Φ is the strict transform of $\lfloor \Theta \rfloor$ and $\lceil E \rceil \geq 0$ as p is an isomorphism in a neighbourhood of the generic point of each non-Kawamata log terminal centre of $(X, X_0 + \Theta)$.

We may also write

$$p^*((m-1)(K_X + \Theta)) = q^* f_*((m-1)(K_X + \Theta)) + F.$$

Possibly shrinking U, we may assume X_0 is $\mathbb{Q}$-linearly equivalent to zero. If we set

$$A = p^*(m(K_X + \Theta)) + E - F, \qquad L = \lceil A \rceil \quad \text{and} \quad C = \{-A\}$$

then

$$\begin{aligned} L - W_0 &= p^*(m(K_X + \Theta)) + E - F + C - W_0 \\ &= p^*(K_X + \Theta) + E + p^*((m-1)(K_X + \Theta)) - F + C - W_0 \\ &\sim_{\mathbb{Q}} K_W + \Phi + C + q^* f_*((m-1)(K_X + \Theta)). \end{aligned}$$

$(W, \Phi + C)$ is log canonical, as $(W, \Phi + C)$ is log smooth and $\Phi + C$ is a boundary. Since all non-Kawamata log terminal centres of $(W, \Phi + C)$ dominate U, a generalisation of Kollár's injectivity theorem (see [21], [9, 6.3] and [4, 5.4]) implies that multiplication by a local parameter

$$H^1(W, \mathcal{O}_W(L - W_0)) \to H^1(W, \mathcal{O}_W(L))$$

is an injective morphism and so the restriction morphism

$$H^0(W, \mathcal{O}_W(L)) \to H^0(W_0, \mathcal{O}_{W_0}(L|_{W_0}))$$

is surjective. Note that the support of $L - \lfloor q^* f_*(m(K_X + \Theta)) \rfloor$ does not contain W_0, and

$$\begin{aligned} L - \lfloor q^* f_*(m(K_X + \Theta)) \rfloor &= \lceil A \rceil - \lfloor q^* f_*(m(K_X + \Theta)) \rfloor \\ &\geq \lceil A - q^* f_*(m(K_X + \Theta)) \rceil = \left\lceil E + \frac{1}{m-1} F \right\rceil \geq 0. \end{aligned}$$

We also have

$$|L| \subset |mp^*(K_X + \Delta) + \lceil E - F \rceil| \subset |mp^*(K_X + \Delta) + \lceil E \rceil| = |m(K_X + \Delta)|.$$

Let $q_0 \colon W_0 \to Y_0$ be the restriction of q to W_0. We have

$$\begin{aligned} |m(K_{X_0} + \Delta_0)| &= |m(K_{X_0} + \Theta_0)| = |m(K_{Y_0} + f_{0*}\Theta_0)| = |q_0^* m(K_{Y_0} + f_{0*}\Theta_0)| \\ &\subset |L_{|W_0}| = |L|_{W_0} \subset |m(K_X + \Delta)|_{X_0}. \end{aligned}$$ □

Proof of 1.4. Immediate from 6.3 and 1.2. □

7. Boundedness of moduli

Lemma 7.1. *Let w be a positive real number and let $I \subset [0, 1]$ be a set which satisfies the DCC. Fix a log smooth pair (Z, B), where Z is a projective variety. Let $\mathfrak{F}$ be the set of all log smooth pairs (X, Δ) such that $\operatorname{vol}(X, K_X + \Delta) = w$, the coefficients of Δ belong to I and there is a sequence of smooth blowups $f : X \to Z$ of the strata of B such that $f_*\Delta \leq B$. Then there is a sequence of blowups $Y \to Z$ of the strata of B such that if $(X, \Delta) \in \mathfrak{F}$ then*

$$\operatorname{vol}(Y, K_Y + \Gamma) = w,$$

where Γ is the sum of the strict transform of Δ and the exceptional divisors of the induced birational map $Y \dashrightarrow X$.

Proof. Let $n = \dim Z$. We may suppose that $1 \in I$. Let $\mathfrak{G}$ be the set of log smooth pairs (Y, Γ) such that Y is projective of dimension n and the coefficients of Γ belong to I.

As [15, 1.3.1] implies that

$$V = \{\operatorname{vol}(Y, K_Y + \Gamma) \mid (Y, \Gamma) \in \mathfrak{G}\}$$

satisfies the DCC, we may find $\delta > 0$ such that

$$\operatorname{vol}(Y, K_Y + \Gamma) < w + \delta \quad \text{implies} \quad \operatorname{vol}(Y, K_Y + \Gamma) \leq w.$$

As the set

$$\left\{ \frac{r-1}{r} i \;\middle|\; r \in \mathbb{N},\ i \in I \right\}$$

satisfies the DCC, by [15, 1.5] we may find $r \in \mathbb{N}$ such that $K_Y + \frac{r-1}{r}\Gamma$ is big whenever $(Y, \Gamma) \in \mathfrak{G}$ and $K_Y + \Gamma$ is big.

Pick $\epsilon > 0$ such that

$$(1 - \epsilon)^n > \frac{w}{w + \delta}$$

and set

$$a = 1 - \epsilon/r.$$

If $(Y, \Gamma) \in \mathfrak{G}$ then

$$K_Y + a\Gamma = (1 - \epsilon)(K_Y + \Gamma) + \epsilon\left(K_Y + \frac{r-1}{r}\Gamma\right),$$

so that

$$\operatorname{vol}(Y, K_Y + a\Gamma) \geq \operatorname{vol}(Y, (1 - \epsilon)(K_Y + \Gamma)) = (1 - \epsilon)^n \operatorname{vol}(Y, K_Y + \Gamma).$$

As (Z, aB) is Kawamata log terminal, 2.8.1 implies we may pick a birational morphism $g : Y \to Z$ such that if we write

$$K_Y + \Psi_0 = g^*(K_Z + aB) + E_0,$$

where $\Psi_0, E_0 \geq 0$ have no common components, $g_*\Psi_0 = aB$ and $g_*E_0 = 0$, then no two components of Ψ_0 intersect. In particular (Y, Ψ_0) is terminal.

Pick $(X, \Delta) \in \mathfrak{F}$ and let Γ be the strict transform of Δ plus the exceptional divisors of the induced birational map $Y \dashrightarrow X$. Let $\Phi = g_*(a\Gamma)$. As $\Phi \leq aB$, if we write

$$K_Y + \Psi = g^*(K_Z + \Phi) + E,$$

where $\Psi, E \geq 0$ have no common components, $g_*\Psi = \Phi$ and $g_*E = 0$, then $\Psi \leq \Psi_0$. In particular (Y, Ψ) is terminal.

Let $\Xi = \Psi \wedge a\Gamma$ and let $\Sigma \leq \Delta$ be the strict transform of Ξ on X. We have

$$\begin{aligned}\operatorname{vol}(Y, K_Y + a\Gamma) &= \operatorname{vol}(Y, K_Y + \Xi) = \operatorname{vol}(X, K_X + \Sigma)\\ &\leq \operatorname{vol}(X, K_X + \Delta) = w,\end{aligned}$$

where we used [14, 5.3.2] for the first line and we used the fact that (Y, Ξ) is terminal, as (Y, Ψ) is terminal, to get from the first line to the second.

It follows that

$$w \leq \operatorname{vol}(Y, K_Y + \Gamma) \leq \frac{1}{(1-\epsilon)^n} \operatorname{vol}(Y, K_Y + a\Gamma) < w + \delta,$$

by our choice of ϵ, so that $\operatorname{vol}(Y, K_Y + \Gamma) = w$, by our choice of δ. □

Lemma 7.2. *Let n be a positive integer, let w be a positive real number and let $I \subset [0, 1]$ be a set which satisfies the DCC. Let $\mathfrak{F}$ be a set of log canonical pairs (X, Δ) such that X is projective of dimension n, the coefficients of Δ belong to I and* $\operatorname{vol}(X, K_X + \Delta) = w$. *Then there is a projective morphism $Z \to U$ and a log smooth pair (Z, B) over U such that if $(X, \Delta) \in \mathfrak{F}$ then there is a point $u \in U$ and a birational map $f_u : X \dashrightarrow Z_u$ such that*

$$\operatorname{vol}(Z_u, K_{Z_u} + \Phi) = w,$$

where $\Phi \leq B_u$ is the sum of the strict transform of Δ and the exceptional divisors of f_u^{-1}.

Proof. We may assume that $1 \in I$. We may also assume that $\mathfrak{F}$ consists of all log canonical pairs (X, Δ) such that X is projective of dimension n, the coefficients of Δ belong to I and $\operatorname{vol}(X, K_X + \Delta) = w$.

By [15, 1.3] there is a constant r such that if $(X, \Delta) \in \mathfrak{F}$ then $\phi_{r(K_X+\Delta)}$ is birational. (2.3.4) and (3.1) of [14] imply that the set $\mathfrak{F}$ is log birationally bounded.

Therefore we may find a projective morphism $\pi : Z \to U$ and a log pair (Z, B) such that if $(X, \Delta) \in \mathfrak{F}$ then there is a point $u \in U$ and a birational map $f : X \dashrightarrow Z_u$ such that the support of the strict transform of Δ plus the f^{-1}-exceptional divisors is contained in the support of B_u. By standard arguments (see for example [14, proof of 1.9]), we may assume that (Z, B) is log smooth over U and the intersection of the strata of B with the fibres is irreducible.

Let 0 be a closed point of U. Let $\mathfrak{F}_0 \subset \mathfrak{F}$ be the set of log smooth pairs (X_0, Δ_0) such that there is a sequence of smooth blowups $f : X_0 \to Z_0$ of the strata of B_0 with $f_*\Delta_0 \leq B_0$. By 7.1 there is a sequence of blowups $g : Y_0 \to Z_0$ of the strata of B_0 such that if $(X_0, \Delta_0) \in \mathfrak{F}_0$ and Γ_0 is the strict transform of Δ_0 plus the exceptional divisors then

$$\operatorname{vol}(Y_0, K_{Y_0} + \Gamma_0) = w.$$

Let $g\colon Y \to Z$ be the sequence of blow ups of the strata of B induced by g_0. Replacing (Z, B) by (Y, C), where C is the sum of the strict transform of B and the exceptional divisors of g, we may assume that if $(X, \Delta) \in \mathfrak{F}_0$ then

$$\operatorname{vol}(Z_0, K_{Z_0} + \Psi_0) = w,$$

where $\Psi_0 = f_*\Delta \leq B_0$. Note that if we replace Z by a higher model, $\mathfrak{F}_0$ becomes smaller.

Suppose that $(X, \Delta) \in \mathfrak{F}$. By a standard argument (see [14, proof of 1.9]), we may assume that (X, Δ) is log smooth and $f\colon X \to Z_u$ blows up the strata of B_u. Let $h\colon W \to Z$ blow up the corresponding strata of B, so that $W_u = X$ and $h_u = f$. Let Θ be the divisor on W such that $\Theta_u = \Delta$ and let $f_0\colon W_0 \to Z_0$ be the induced birational morphism. Then

$$\operatorname{vol}(W_0, K_{W_0} + \Theta_0) = \operatorname{vol}(X, K_X + \Delta) = w,$$

by deformation invariance of the volume (see 4.3), so that $(W_0, \Theta_0) \in \mathfrak{F}_0$.

But then

$$\operatorname{vol}(Z_0, K_{Z_0} + \Phi_0) = w,$$

where $\Phi_0 = f_{0*}\Theta_0$. Let $\Phi = h_*\Theta$. Then Φ_u is the strict transform of Δ plus the exceptional divisors and

$$\operatorname{vol}(Z_u, K_{Z_u} + \Phi_u) = w,$$

by deformation invariance of the volume. □

Proposition 7.3. *Fix an integer n, a constant d and a set $I \subset [0, 1]$ which satisfies the DCC. Then the set $\mathfrak{F}_{\mathrm{lc}}(n, d, I)$ of all (X, Δ) such that*

(1) *X is a union of projective varieties of dimension n,*
(2) *(X, Δ) is log canonical,*
(3) *the coefficients of Δ belong to I,*
(4) *$K_X + \Delta$ is an ample $\mathbb{Q}$-divisor, and*
(5) *$(K_X + \Delta)^n = d$,*

is bounded. In particular there is a finite set I_0 such that $\mathfrak{F}_{\mathrm{lc}}(n, d, I) = \mathfrak{F}_{\mathrm{lc}}(n, d, I_0)$.

Proof. If

$$X = \coprod_{i=1}^{k} X_i$$

and (X_i, Δ_i) is the corresponding log canonical pair then $K_{X_i} + \Delta_i$ is ample and if $d_i = (K_{X_i} + \Delta_i)^n$ then $d = \sum d_i$. Now 2.4.1 and 1.6 imply that there are only finitely many tuples $(d_1, \dots, d_k)$.

Thus it is enough to show that the set $\mathfrak{F}$ of irreducible pairs (X, Δ) satisfying (1)–(5) is bounded.

By 7.2 there is a projective morphism $Z \to U$ and a log smooth pair (Z, B) over U such that if $(X, \Delta) \in \mathfrak{F}$ then there is a closed point $u \in U$ and a birational map $f_u\colon Z_u \dashrightarrow X$ such that

$$\operatorname{vol}(Z_u, K_{Z_u} + \Phi) = d,$$

where $\Phi \leq B_u$ is the sum of the strict transform of Δ and the f_u-exceptional divisors. 2.2.2 implies that f_u is the log canonical model of (Z_u, Φ).

On the other hand, 1.3 implies that if we replace U by a finite disjoint union of locally closed subsets then we may assume that every fibre of π has a log canonical model. When we replace (Z, B) by the log canonical model over U, the fibres of π are the elements of $\mathfrak{F}$. □

Lemma 7.4. *Let $\mathfrak{F}$ be a family of log canonical pairs (X, Δ), where X is projective, the coefficients of Δ belong to a finite set I and $K_X + \Delta$ is ample. Let*

$$\mathfrak{T} = \{(X, \Delta, \tau) \mid (X, \Delta) \in \mathfrak{F},\ \tau\colon S \to S\},$$

where S is the normalisation of a divisor supported on $\lfloor \Delta \rfloor$ and τ is an involution which fixes the different of $(K_X + \Delta)|_S$. If $\mathfrak{F}$ is a bounded family then so is $\mathfrak{T}$.

Proof. By assumption there is a projective morphism $\pi\colon Z \to U$ and a log pair (Z, Σ) such that if $(X, \Delta) \in \mathfrak{F}$ then there is a point $u \in U$ and an isomorphism (Z_u, Θ) with (X, Δ), where Θ is a divisor supported on Σ_u. As I is finite, possibly replacing U by a disjoint union of locally closed subsets we may assume that $\Theta = \Sigma_u$.

Let U_1 be the set u of points of U such that (Z_u, Σ_u) is isomorphic to some element (X, Δ) of $\mathfrak{F}$. Replacing U by the closure of U_1 we may assume that U_1 is dense in U. In particular we may assume that $K_Z + \Sigma$ is ample over U. As the set of points where (Z, Σ) is log canonical is constructible, replacing U by a disjoint union of finitely many locally closed subsets we may assume that (Z, Σ) is log canonical; we may also assume that Σ meets each fibre Z_u in a divisor and that (Z_u, Σ_u) is log canonical.

Possibly replacing U by finitely many disjoint copies we may assume that there is a divisor C' on Z such that if $(X, \Delta, \tau) \in \mathfrak{T}$ then S corresponds to C'_u. Possibly replacing U by a disjoint union of locally closed subsets we may assume that if C is the normalisation of C' then S is isomorphic to C_u. Possibly replacing U by a disjoint union of locally closed subsets for the last time, we may assume that if we write

$$(K_Z + \Sigma)|_C = K_C + \Phi \quad \text{and} \quad (K_X + \Delta)|_S = K_S + \Theta,$$

then Θ corresponds to Φ_u.

Recall that the scheme $\mathrm{Isom}_U(C, C)$, which represents the functor which assigns to a scheme T over U the set of all isomorphisms $C_T \to C_T$ over T, is a countable union of quasi-projective schemes over U. Pick m such that $-m(K_Z + \Sigma)$ is Cartier. Since $-m(K_Z + \Sigma)$ is ample over U, the subscheme of $\mathrm{Isom}_U(C, C)$ fixing the line bundle $\mathcal{O}_C(-m(K_Z + \Sigma))$ is a closed subscheme which is quasi-projective over U. The set of involutions fixing the different is then a closed subscheme.

It follows that $\mathfrak{T}$ is a bounded family. □

Proof of 1.1. Let $\mathfrak{T}$ be the set of triples (X, Δ, τ) where $(X, \Delta) \in \mathfrak{F}_{\mathrm{lc}}(n, d, I)$ and $\tau\colon S \to S$ is an involution of the normalisation of a divisor supported on $\lfloor \Delta \rfloor$, which fixes the different of $(K_X + \Delta)|_S$.

By [26, 5.13], it is enough to prove that $\mathfrak{T}$ is bounded. 7.3 implies that $\mathfrak{F}_{\mathrm{lc}}(n, d, I)$ is bounded and so we may apply 7.4. □

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