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Vectorial nonlinear potential theory

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Abstract. We settle the longstanding problem of establishing pointwise potential estimates for vectorial solutions $u\colon \Omega \to \mathbb{R}^N$ to the non-homogeneous p-Laplacean system

$$-\operatorname{div}(|Du|^{p-2}Du) = \mu \quad \text{in } \Omega \subset \mathbb{R}^n,$$

where μ is an $\mathbb{R}^N$-valued Borel measure with finite total mass. In particular, for solutions $u \in W^{1,p-1}_{\mathrm{loc}}(\mathbb{R}^n)$ with a suitable decay at infinity, the global estimates via Riesz and Wolff potentials,

$$|Du(x_0)|^{p-1} \lesssim \int_{\mathbb{R}^n} \frac{d|\mu|(x)}{|x-x_0|^{n-1}}$$

and

$$|u(x_0)| \lesssim \mathbf{W}^{\mu}_{1,p}(x_0,\infty) = \int_0^\infty \left(\frac{|\mu|(B_\varrho(x_0))}{\varrho^{n-p}}\right)^{1/(p-1)} \frac{d\varrho}{\varrho}$$

respectively, hold at every point x_0 such that the corresponding potentials are finite. The estimates allow sharp descriptions of fine properties of solutions which are the exact analog of the ones in classical linear potential theory. For instance, sharp characterizations of Lebesgue points of u and Du and optimal regularity criteria for solutions are provided exclusively in terms of potentials.

Keywords. Nonlinear potential theory, regularity, degenerate elliptic systems, measure data

Contents

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1. Introduction, results, techniques

One of the main concerns of nonlinear potential theory is to extend the study of the fine properties of classical harmonic functions to solutions to nonlinear, possibly degenerate, elliptic and parabolic equations or systems. Its origins can be traced back to the landmark paper of Havin & Maz'ya [49], where what are nowadays called Wolff potentials were introduced and studied in detail. Almost at the same time, a fundamental contribution of Maz'ya [48] established the sufficiency part of the Wiener criterion for solutions to boundary value problems involving the p-Laplacean equation

$$-\triangle_p u \equiv -\operatorname{div}(|Du|^{p-2}Du) = \mu \quad \text{in } \Omega \subset \mathbb{R}^n. \tag{1.1}$$

Here, as in the rest of the paper, Ω is an open subset and $n \geq 2$. Maz'ya's proof employs techniques and capacitary conditions again involving quantities related to Wolff potentials. The linear case $p = 2$ was originally treated by Wiener [61], while the extension to general linear elliptic equations is an achievement of Littmann, Stampacchia & Weinberger [45]. For a modern description of the mains topics in nonlinear potential theory and the use of nonlinear potentials we refer to the treatise of Heinonen, Kilpeläinen & Martio [27]. A book with special emphasis on the interplay between nonlinear potential theory and regularity theory of nonlinear elliptic equations is the one of Malý & Ziemer [46]. For more classical potential theory we instead refer to Adams & Hedberg [1].

At this point it is worth recalling that, with μ being a Borel measure with finite total mass in $\mathbb{R}^n$, the nonlinear Wolff potential $\mathbf{W}^{\mu}_{\beta,p}$ is defined by

$$\mathbf{W}^{\mu}_{\beta,p}(x,R) := \int_0^R \left(\frac{|\mu|(B_\varrho(x))}{\varrho^{n-\beta p}}\right)^{1/(p-1)} \frac{d\varrho}{\varrho}, \qquad \beta > 0,$$

for $x \in \mathbb{R}^n$ and $0 < R \leq \infty$; see Section 2 below for the general notation used in this paper. Wolff potentials have been first introduced by Havin & Maz'ya [49], while important contributions are in [26]. Wolff potentials play an essential role in studying fine properties of $W^{1,p}$-Sobolev functions and establishing optimal regularity results for solutions to equations of the type (1.1). A landmark result in this direction was achieved by Kilpeläinen & Malý [31, 32], who proved pointwise estimates for solutions to (1.1) in terms of the Wolff potential $\mathbf{W}^{\mu}_{1,p}$, similar to our estimate (1.15) below. See also [33, 58] for different proofs. This eventually implied the long-awaited proof of the necessity part of the Wiener criterion for the p-Laplacean equation, thus completing Maz'ya's original result [48]. The results of Kilpeläinen & Malý actually extend to more general equations of the type

$$-\operatorname{div} a(x, Du) = \mu \tag{1.2}$$

with p-Laplacean structure and measurable dependence on the variable x. The remarkable point here is that pointwise estimates via potentials as (1.15) fully replace those via fundamental solutions valid in the linear case. In this respect, the results in [31, 32] for equations of the type (1.2) are significant and totally nontrivial already in the non-degenerate case $p = 2$, that is, possibly nonlinear equations of the type (1.2) with $a(\cdot)$ having linear growth in the gradient variable.

The question of extending the pointwise estimates via nonlinear potentials to the gradient of solutions to (1.2) has remained a difficult and challenging open problem for nearly 20 years after [31, 32]. The first and complete result for $p = 2$ has been established in [53]. On the other hand, the final, and surprising, answer to this problem in the general case $p > 2 - 1/n$ has been found in [20, 35]; see also [19] for an intermediate result. Indeed, it was proved that, when passing to the gradient level, Riesz potentials reappear as in the linear theory, and they allow one to prove pointwise estimates better than those expected via Wolff potentials (see [39] for a full description of the problems). Riesz potentials are defined as

$$\mathbf{I}^{\mu}_{\beta}(x, R) := \int_0^R \frac{\mu(B_\varrho(x))}{\varrho^{n-\beta}} \frac{d\varrho}{\varrho}, \qquad \beta > 0,$$

for $x \in \mathbb{R}^n$ and $0 < R \leq \infty$, and the estimate proved in [35] is (1.10) below. Extensions can be found in [34, 39]. In particular, the results derived in [39] allow one to reduce the regularity theory of nonlinear equations of the type (1.2) with p-structure to the one of the classical Poisson equation

$$-\triangle u = \mu \tag{1.3}$$

up to the C^1-level. For related Calderón–Zygmund type estimates we refer to [28, 14, 11].

1.1. Results

So far, the validity of the potential estimates proved in [32, 35] for the scalar case and described above has remained an open problem in the case of systems. The aim of this paper is to give a full extension of such results to degenerate systems *in order to start a nonlinear potential theory in the vectorial case*. The prototype and the main model example is precisely the p-Laplacean system

$$-\operatorname{div}(|Du|^{p-2}Du) = \mu. \tag{1.4}$$

Throughout we assume that the vector valued measure $\mu\colon \Omega \to \mathbb{R}^N$ is Borel regular and has finite total mass,

$$|\mu|(\Omega) < \infty.$$

A vector valued measure will be simply referred to as a measure. With no loss of generality, we shall assume that μ is defined on the whole $\mathbb{R}^n$ with $|\mu|(\mathbb{R}^n) < \infty$.

The local $C^{1,\alpha}$-regularity of $W^{1,p}$-solutions in the homogeneous case, for some positive exponent $\alpha \equiv \alpha(n, N, p)$, is a fundamental achievement of Uhlenbeck [59], in turn extending to the vectorial case the analogous result of Ural'tseva [60] valid for the scalar equation (see also [47]). For this reason we shall consider vector valued solutions $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ to the p-Laplacean system with measure data (1.4), considered in an open subset $\Omega \subset \mathbb{R}^n$ for $n \geq 2$. More general systems can be considered: see Remark 1.1 below. We shall anyway confine ourselves to the model case (1.1) in order not to hide the main ideas behind a fog of additional technicalities. To keep the treatment at a reasonable length, when dealing with gradient potential estimates and unless otherwise stated, we shall mainly deal with the range of parameters dictated by

$$p > 2 \qquad \text{(when using Riesz potentials).}$$

The subquadratic case $2-1/n < p < 2$, aimed at proving a vectorial analog of the results in [20], needs a different approach and it will be treated in the forthcoming paper [42] together with additional cases. Needless to say, all our results continue to hold when $p = 2$. This is of course a corollary of the classical linear theory that holds for solutions to (1.3). Instead, when proving Wolff potential estimates, i.e., estimates that do not involve the gradient of solutions, we shall assume the weaker lower bound

$$p > 2 - 1/n \qquad \text{(when using Wolff potentials).} \tag{1.5}$$

We recall that the above inequality also serves to guarantee that the solutions we are considering are Sobolev functions (see (1.6) below).

To fix the functional framework, we clarify the type of solutions we are dealing with, which is on the other hand the usual one adopted in measure data problems. Distributional solutions to measure data problems as the one in (1.4) are not in general energy solutions. Indeed, the so-called fundamental solution

$$G_p(x) = c(n,p) \begin{cases} |x|^{\frac{p-n}{p-1}} - 1 & \text{if } 1 < p \neq n, \\ \log|x| & \text{if } p = n, \end{cases} \tag{1.6}$$

solves, in the sense of distributions, the problem

$$\begin{cases} -\triangle_p u = \delta & \text{in } B_1 \\ u = 0 & \text{on } \partial B_1, \end{cases} \tag{1.7}$$

where δ is the Dirac mass charging the origin and B_1 denotes the unit ball in $\mathbb{R}^n$ (see for instance [12]). Clearly, G_p does not belong $W^{1,p}_{\mathrm{loc}}(B_1)$ for $p \leq n$, but $G_p \in W^{1,q}(B_1)$ for every $q < n(p-1)/(n-p)$ for $p < n$, provided the lower bound in (1.5) holds. Indeed, notice also that if $p < 2 - 1/n$ then $G_p \notin W^{1,1}_{\mathrm{loc}}(B_1)$. For general measure data problems involving the p-Laplacean system (1.1) the notion of solution which is usually used in the literature is the one of SOLA (Solution Obtained as Limits of Approximations):

Definition 1.1 (SOLA). A vector valued map $u \in W^{1,p-1}(\Omega;\mathbb{R}^N)$ for $p > 2-1/n$ is a SOLA to (1.1) if there exists a sequence $\{u_h\} \subset W^{1,p}(\Omega;\mathbb{R}^N)$ of local energy solutions to the systems

$$-\operatorname{div}(|Du_h|^{p-2}Du_h) = \mu_h \tag{1.8}$$

such that $u_h \to u$ locally in $W^{1,p-1}(\Omega;\mathbb{R}^N)$, where $\{\mu_h\} \subset C^\infty(\Omega;\mathbb{R}^N)$ is a sequence of smooth maps that converges to μ weakly in the sense of measures and satisfies

$$\limsup_k |\mu_k|(\overline{B}) \le |\mu|(\overline{B}) \qquad \text{for every ball } B \subset \Omega. \tag{1.9}$$

Observe that the above approximation property immediately implies that u is a distributional solution to (1.1), that is,

$$\int_\Omega |Du|^{p-2}Du : D\varphi\,dx = \int_\Omega \varphi\,d\mu \qquad \forall \varphi \in C_0^\infty(\Omega;\mathbb{R}^N).$$

See Section 2 for notation. Indeed, SOLAs naturally stem from approximation methods built to prove existence theorems. We also observe that if all the components of $\{\mu_k\}$ are nonnegative, then (1.9) is a standard consequence of weak convergence of measures. Here we are dealing with signed measures, and therefore (1.9) must be prescribed to avoid cancellations in the limit. Anyway, usual SOLAs, built by convolution methods, satisfy (1.9). The existence of SOLAs has been proved in [8] for scalar equations and in [15, 16] for the p-Laplacean system (1.1); see also [22] for a special case concerning (1.7) and [43, 44] for more existence theorems. When $p > n$, SOLAs coincide with usual energy solutions. In the scalar case several definitions of solutions are available and all of them coincide for μ nonnegative, as proved in [29]. Instead, in the vectorial case SOLAs are the only ones available, and their uniqueness is still unclear when $p < n$ (see [16]). For the scalar case, we recommend [12] for a thorough discussion of the concept of solution to measure data problems, and [53] for an overview of the results available.

The first result we are going to present is a potential gradient estimate, which is also the most delicate one.

Theorem 1.1 (Riesz potential gradient bound). *Let $u \in W^{1,p-1}(\Omega;\mathbb{R}^N)$ be a SOLA to the system (1.1) with $p > 2$, and let $B_r(x_0) \subset \Omega$ be a ball. If $\mathbf{I}_1^{|\mu|}(x_0,r)$ is finite, then x_0 is a Lebesgue point of Du and the pointwise estimate*

$$|Du(x_0)| \le c[\mathbf{I}_1^{|\mu|}(x_0,r)]^{1/(p-1)} + c \fint_{B_r(x_0)} |Du|\,dx \tag{1.10}$$

holds with a constant c that depends only on n, N, p.

As a matter of fact, Theorem 1.1 follows as a corollary of a more general statement encoding also the oscillation properties of SOLAs.

Theorem 1.2 (Gradient oscillations via potential bounds). *Let $u \in W^{1,p-1}(\Omega;\mathbb{R}^N)$ be a SOLA to* (1.1) *with $p > 2$, and let $B_r(x_0) \subset \Omega$. If*

$$\lim_{\varrho\to 0} \frac{|\mu|(B_\varrho(x_0))}{\varrho^{n-1}} = 0, \tag{1.11}$$

then Du has vanishing mean oscillations at x_0, i.e.,

$$\lim_{\varrho\to 0} \fint_{B_\varrho(x_0)} |Du - (Du)_{B_\varrho(x_0)}|\,dx = 0. \tag{1.12}$$

Moreover, if $\mathbf{I}_1^{|\mu|}(x_0, r)$ is finite, then x_0 is a Lebesgue point of Du, and

$$|Du(x_0) - (Du)_{B_r(x_0)}| \le c[\mathbf{I}_1^{|\mu|}(x_0, r)]^{1/(p-1)} + c \fint_{B_r(x_0)} |Du - (Du)_{B_r(x_0)}|\,dx \tag{1.13}$$

for a constant $c \equiv c(n, N, p)$.

The previous theorem implies the following optimal continuity result.

Theorem 1.3 (Riesz potential continuity criterion). *Let $u \in W^{1,p-1}(\Omega;\mathbb{R}^N)$ be a SOLA to* (1.1) *with $p > 2$, and let $B_r(x_0) \Subset \Omega$. If*

$$\lim_{\varrho\to 0} \sup_{x\in B_r(x_0)} \mathbf{I}_1^{|\mu|}(x, \varrho) = 0, \tag{1.14}$$

then Du is continuous in $B_r(x_0)$.

A few remarks are now in order. In the scalar case $N = 1$ estimate (1.10) has been obtained in [35]; the proof given here is completely different and substantially more difficult. The ultimate effect of Theorems 1.1–1.3 is to reduce the problem of getting sharp gradient estimates for solutions to (1.1) to the study of Riesz potentials, whose properties are well-understood. The gradient regularity theory of the p-Laplacean system can be reduced to the theory of the Poisson equation $-\triangle u = \mu$ and all the results follow verbatim from this case up to the C^1-level thanks to Theorem 1.3. These aspects are briefly discussed in Sections 10.1–10.2. As an example already mentioned in the abstract, we note that if we have a global solution $u \in W^{1,p-1}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^N)$ to (1.1) with suitable decay properties at infinity, then letting $r \to \infty$ in (1.10) yields

$$|Du(x_0)|^{p-1} \le c \int_{\mathbb{R}^n} \frac{d|\mu|(x)}{|x - x_0|^{n-1}}$$

for $c \equiv c(n, N, p)$. This happens for instance when

$$\liminf_{r\to\infty} \fint_{B_r} |Du|\,dx = 0.$$

Notice that this is the case for the fundamental solution to $-\triangle_p u = \delta$, which has $|Du|$ proportional to $|x|^{(1-n)/(p-1)}$. Apart from the exponent $p - 1$ dictated by the scaling of the system, this is exactly the same pointwise estimate that holds for the solution to the Poisson system (1.3) via fundamental solutions. We finally note that the criterion to determine Lebesgue points in Theorem 1.1 is a reproduction of the analogous one from the classical linear potential theory.

We now come to Wolff potential estimates, thereby giving a vectorial version of the results of Kilpeläinen & Malý [31, 32].

Theorem 1.4 (Wolff potential bound). *Let $u \in W^{1,p-1}(\Omega;\mathbb{R}^N)$ be a SOLA to (1.1) with $p > 2-1/n$, and let $B_r(x_0) \subset \Omega$. If $\mathbf{W}^{\mu}_{1,p}(x_0,r)$ is finite, then x_0 is a Lebesgue point of u, and the pointwise estimate*

$$|u(x_0)| \le c\mathbf{W}^{\mu}_{1,p}(x_0,r) + c \fint_{B_r(x_0)} |u|\,dx \tag{1.15}$$

holds with a constant c that depends only on n, N, p.

In the case of a single equation ($N = 1$) a two-sided estimate of the type

$$\mathbf{W}^{\mu}_{1,p}(x,r) \lesssim u(x) \lesssim \mathbf{W}^{\mu}_{1,p}(x,2r) + \inf_{B_r(x)} u$$

actually holds when u, μ are nonnegative and $B_{2r}(x_0) \subset \Omega$ [31, 32]. No analog of that estimate is possible in the vectorial case, which is ultimately related to the lack of maximum principle for systems. Just as the gradient estimates, also Theorem 1.4 follows as a corollary of more general facts:

Theorem 1.5 (Oscillations via Wolff potential bounds). *Let $u \in W^{1,p-1}(\Omega;\mathbb{R}^N)$ be a SOLA to (1.1) with $p > 2-1/n$, and let $B_r(x_0) \subset \Omega$. If*

$$\lim_{\varrho\to 0} \frac{|\mu|(B_\varrho(x_0))}{\varrho^{n-p}} = 0, \tag{1.16}$$

then u has vanishing mean oscillations at x_0, i.e.,

$$\lim_{\varrho\to 0} \fint_{B_\varrho(x_0)} |u-(u)_{B_\varrho(x_0)}|\,dx = 0. \tag{1.17}$$

Furthermore, if $\mathbf{W}^{\mu}_{1,p}(x_0,r)$ is finite, then x_0 is a Lebesgue point of u and

$$|u(x_0)-(u)_{B_r(x_0)}| \le c\mathbf{W}^{\mu}_{1,p}(x_0,r) + c \fint_{B_r(x_0)} |u-(u)_{B_r(x_0)}|\,dx \tag{1.18}$$

with a constant $c \equiv c(n,N,p)$.

We finally have the following optimal continuity criterion.

Theorem 1.6 (Wolff potential continuity criterion). *Let $u \in W^{1,p-1}(\Omega;\mathbb{R}^N)$ be a SOLA to (1.1) with $p > 2-1/n$, and let $B_r(x_0) \Subset \Omega$. Assume that*

$$\lim_{\varrho\to 0} \sup_{x\in B_r(x_0)} \mathbf{W}^{\mu}_{1,p}(x,\varrho) = 0. \tag{1.19}$$

Then u is continuous in $B_r(x_0)$.

Remark 1.1. The methods of this paper open the way to the proof of potential estimates for more general systems than (1.4). The systems in question have quasi-diagonal structure, i.e. they are of the type

$$-\operatorname{div}(g(|Du|)Du) = \mu, \tag{1.20}$$

where $g\colon [0,\infty) \to [0,\infty)$ is a continuous, nondecreasing and $C^1((0,\infty))$-regular function and satisfies the growth and monotonicity assumptions

$$\sqrt{\nu}\, t^{p-2} \le g(t) \le t^{p-2}/\sqrt{\nu}, \qquad \sqrt{\nu}\, t^{p-2} \le t g'(t) \le t^{p-2}/\sqrt{\nu} \quad \forall t \ge 0, \tag{1.21}$$

where $\nu \in (0,1)$ is a fixed ellipticity constant. We still impose $p > 2$ when gradient estimates are considered, otherwise $p > 2 - 1/n$ suffices for Wolff potential estimates. Systems of the type (1.20) with $\mu = 0$ have in fact been considered in the original paper by Uhlenbeck [59]. For a brief summary of the modifications needed in order to treat (1.20), see Section 11.

In contrast to the scalar case $N = 1$, an extension of the potential estimates to general systems of the form

$$-\operatorname{div} a(Du) = \mu, \qquad u\colon \Omega \to \mathbb{R}^N, \tag{1.22}$$

is not possible. Indeed, already in the case of homogeneous systems of the type

$$-\operatorname{div} a(Du) = 0 \tag{1.23}$$

vector valued energy solutions u can be unbounded and develop singularities [54, 57]. This means that potential estimates such as (1.10) and (1.15) do not hold in general, because, for $\mu \equiv 0$, they would imply the local boundedness of Du and u, respectively. The appearance of singularities when considering systems is a genuine feature of vectorial problems and it is unrelated to the presence of the datum μ; see for instance [50] for a survey of examples. When passing to general elliptic systems of the type (1.23), it is still possible to obtain sharp potential estimates, but these cannot hold at every point. Roughly speaking, they hold outside a closed, negligible singular set defined by a so called ε-smallness condition on a suitable excess functional of the gradient. In other words, potential estimates are embedded in the setting of partial regularity [41]. We finally mention that the methods developed here are the starting point to attack the question of parabolic potential estimates in the vectorial case [6]. These are the vectorial versions of the caloric potential estimates obtained in [36, 37, 38].

1.2. Techniques

The proofs of Theorems 1.1–1.6 are very different from those of the analogous results in the scalar case [31, 32, 33, 34, 35, 39, 58] and establish new, basic techniques in nonlinear potential theory. The fundamental difference between the scalar and vectorial cases ultimately lies in the lack of a comparison principle in the latter. To overcome this point, we combine methods from classical potential theory and measure data problems theory with those coming from the partial regularity theory for elliptic systems, as for instance described in [24] and [59]. In turn, such methods find their origins in the regularity techniques developed in the context of geometric measure theory, where the idea of local linearization through so called ε-regularity theorems has been initially introduced by De Giorgi [13] (see [50] for a historical overview and more details). In particular, the main approach here relies on a connection between the classical potential-theoretic methods developed in the last decades and the regularity theory of degenerate systems started in the fundamental paper of Uhlenbeck [59]. In order to achieve this, we develop new tools that, we believe, will be useful to attack other problems in the future.

We here give a brief outline of the proof of estimate (1.10), which in our opinion is the main result of the paper. Assume, for simplicity, that x_0 is a Lebesgue point of Du. We consider a sequence $\{B^j\}$ of nested balls, with radii $\{r_j\}$, shrinking to x_0, and the related gradient averages

$$A_j := |(Du)_{B^j}|.$$

We also consider the excess functional defined by

$$E_j := \fint_{B^j} |Du - (Du)_{B^j}|\, dx, \tag{1.24}$$

which, roughly speaking, provides an integral measure of the oscillations of Du in the ball B^j. The first, preliminary step in the proof is to implement a comparison method in which, at all scales, we argue via alternatives. Either the so-called degenerate alternative holds, that is, A_j can be controlled by E_j, or A_j is much larger than E_j, in which case we analyze the size of A_j and $|\mu|(B^j)$ simultaneously. When the first alternative holds, we are considering conditions of the type (5.7); this is treated in Section 5.1. We can then find a p-harmonic map v, a solution to the homogeneous system $\triangle_p v = 0$ in $B^j/2$, such that u and v are suitably close (see (5.8)). In order to achieve this we need to develop a technical tool, which we call *measure data p-harmonic approximation*. This is the main concern of Section 4. The outcome is Theorem 4.1, which states that if a map u has controlled L^1-norm in the sense of (4.1), and is almost p-harmonic in the sense of (4.3), then there exists a truly p-harmonic map v which is close to u in $W^{1,q}$ as described in (4.4). Lemmas of this type for the p-Laplacean operator are already contained in the literature (see for instance [17, 18, 21]) and they essentially go back to the pioneering paper of De Giorgi in the setting of minimal surfaces [13, 55], where a similar result is proved for harmonic maps. The one presented here differs from them in two essential points. In fact, the typical energy bounds assumed in the known lemmas are not of the type (4.1) but rather

$$\fint_{B_r(x_0)} |Dv|^p\, dx \lesssim 1, \tag{1.25}$$

i.e. they involve the natural energy space $W^{1,p}$ associated to the p-Laplacean operator. Bounds of the type (1.25) do not hold in the context of measure data problems since, as already noticed above, solutions to measure data problems do not belong to $W^{1,p}$. Therefore we are led to assume energy bounds which are compatible to the degree of integrability of the solutions in question; the one considered in (4.1) fits our purposes. The second difference from standard p-harmonic approximation lemmas is that now the measure data problem forces us to consider the closeness of u to being p-harmonic in L^∞-norm instead of the natural $W^{1,p}$-norm (see (4.3)). Proving the lemma under bounds like (4.1) and (4.3) rather than (1.25) requires significant technical efforts and very different means. We rely on two basic ingredients to build a proof by contradiction: energy estimates and convergence. For energy estimates we use truncation arguments that work in the vectorial case due to the quasi-diagonal structure of the p-Laplacean operator. As for the convergence, we adapt to our situation powerful blow-up arguments developed in [15] for which once again the specific structure of the p-Laplacean is essential. Indeed, general systems of the type (1.23) cannot be considered here.

Going back to the proof, we then analyze the case when A_j is much larger than E_j, that is, we consider conditions as in (7.1). Moreover, these are considered together with a simultaneous smallness condition on $|\mu|(B^j)$ as described in (7.2). In this case we find that our solution u is suitably close, in the sense of (7.4), to a solution h of a linearized system

$$-\operatorname{div}(\tilde{A} : Dh) = 0$$

where $\tilde{A}$ is a constant elliptic tensor. This is essentially the content of Section 7. In particular, the key estimate (7.43) provides a crucial reverse type inequality for certain intermediate rescalings of the original solution.

The two alternatives described above are then combined in Section 8. In both cases we see that on each ball B^j the original solution u can be approximated either by a p-harmonic map or by an $\tilde{A}$-harmonic map, that is, a solution to a linear system. Therefore we can prove a decay estimate for the excess quantities E_j defined in (1.24) of the type

$$E_{j+1} \le \varepsilon E_j + c(m, N, p, \varepsilon)[|\mu|(B^j)/r_j^{n-1}]^{1/(p-1)}, \qquad \varepsilon \in (0, 1).$$

This resembles the classical decay properties of harmonic and p-harmonic functions. The last inequality is the real starting point of the proof of (1.10), which now proceeds via an iteration procedure based on two ingredients. The first one is the key Lemma 8.4. It allows us to prove by induction certain bounds on the size of the gradient averages A_j, namely

$$A_j \le \lambda \tag{1.26}$$

for a certain $\lambda > 0$, and for a set of indices j satisfying certain properties in the induction scheme applied. The second and final ingredient is developed in Section 8.4. This time we compare A_j not to E_j, but rather to an initial parameter λ of the form

$$\lambda \approx [\mathbf{I}_1^{|\mu|}(x_0, r_0)]^{1/(p-1)} + \fint_{B_0} |Du - (Du)_{B_0}|\, dx.$$

It is in fact part of the proof to show that λ can really be chosen in this way. At this stage we still consider degenerate and nondegenerate cases (Steps 3 and 4 in Section 8.4), but the use of Lemma 8.4 together with the choice of λ enables reducing ourselves to the nondegenerate case. This allows us to prove bounds of the type (1.26) for every j. Since $A_j \to |Du(x_0)|$ as $j \to \infty$, because x_0 was assumed to be a Lebesgue point of the gradient, we finally conclude the proof of (1.10).

The rest of the paper is structured as follows. In Section 2 we establish some notation, while in Section 3 we restate, in suitable forms, some basic regularity properties of p-harmonic maps. Section 4 is entirely dedicated to the proof of measure data p-harmonic approximation, while in Section 5 we derive a few corollaries of it. Section 6 is dedicated to proving reverse inequalities satisfied by p-harmonic maps; this is linked to some basic properties used in the classical Gehring lemma [23], and it is formulated directly on level sets. Section 7 is the last technical section before the proofs of the potential estimates, and features a series of lemmas aimed at implementing the necessary linearization procedures in order to treat the nondegenerate case. In Section 8 we give the proof of the Riesz potential estimates, in Section 9 we prove the Wolff potential estimates and in the final Section 10 we give a few selected consequences.

2. Notation

Let us recall some basic notation that will be used throughout the paper. In what follows we denote by c a general positive constant, possibly varying from line to line; special occurrences will be denoted by $\bar{c}$ or the like. All such constants will always be *larger than or equal to* 1; moreover, relevant dependencies on parameters will be emphasized using parentheses, e.g., $\bar{c} \equiv \bar{c}(n, p, q)$ means that $\bar{c}$ depends only on n, p, q. Throughout, we use Einstein's convention of summing over repeated indices. With t being a real number, we shall sometimes denote $t_+ := \max\{t, 0\}$. We denote by

$$B_r(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < r\}$$

the open ball with center x_0 and radius $r > 0$; when not important, or clear from the context, we shall omit denoting the center as follows: $B_r \equiv B_r(x_0)$. Unless otherwise stated, different balls in the same context will have the same center; we shall often denote $B_1 \equiv B_1(0)$. With $\gamma > 0$ and B being a given ball, we denote by γB $[B/\gamma]$ the ball with the same center and radius magnified [demagnified] by a factor of γ. With $\mathcal{B} \subset \mathbb{R}^n$ being a measurable subset with positive measure $|\mathcal{B}|$, and with $g \colon \mathcal{B} \to \mathbb{R}^k$, $k \geq 1$, being an integrable map, we shall denote by

$$(g)_{\mathcal{B}} \equiv \fint_{\mathcal{B}} g(x)\, dx := \frac{1}{|\mathcal{B}|} \int_{\mathcal{B}} g(x)\, dx$$

its integral average. Moreover, the oscillation of g on $\mathcal{B}$ is defined as

$$\operatorname*{osc}_{\mathcal{B}} g := \sup_{x,y \in \mathcal{B}} |g(x) - g(y)|.$$

We finally recall a general and elementary property of the so-called excess functionals:

$$\fint_{\mathcal{B}} |g - (g)_{\mathcal{B}}|\, dx \leq 2 \fint_{\mathcal{B}} |g - \xi|\, dx \quad \forall \xi \in \mathbb{R}^k. \tag{2.1}$$

We shall very often deal with vector valued maps and related function spaces. The usual notation indicates the number of components of the map in question. For instance when considering a vector field $g\colon \Omega \to \mathbb{R}^k$ whose components belong to a certain function space $X(\Omega)$, it is customary to write $g \in X(\Omega; \mathbb{R}^k)$. However, we shall often abbreviate $X(\Omega) \equiv X(\Omega; \mathbb{R}^k)$, for instance $L^p(\Omega) \equiv L^p(\Omega; \mathbb{R}^k)$, $W^{1,p}(\Omega) \equiv W^{1,p}(\Omega; \mathbb{R}^k)$ etc. The number of components involved will always be clear from the context.

Let next $\{e^\alpha\}_{\alpha=1}^N$ and $\{e_j\}_{j=1}^n$ stand for Cartesian bases for $\mathbb{R}^N$ and $\mathbb{R}^n$, respectively. We will denote a second-order tensor ζ of size (N, n) as $\zeta = \zeta_j^\alpha e^\alpha \otimes e_j$, where repeated indices are summed. The Frobenius product of second-order tensors ξ and ζ is defined as $\xi : \zeta = \xi_j^\alpha \zeta_j^\alpha$ so that obviously $\xi : \xi = |\xi|^2$. The linear space of all such tensors is isomorphic to matrices in $\mathbb{R}^{N\times n}$, and indeed we will denote the space of these by $\mathbb{R}^{N\times n}$. We let the Greek superscripts indicate components and Latin subscripts indicate partial derivatives. The gradient of a map $u = u^\alpha e^\alpha$ is thus defined as $Du = \partial_{x_j} u^\alpha e^\alpha \otimes e_j$, and the divergence of a tensor $\zeta = \zeta_j^\alpha e^\alpha \otimes e_j$ as $\operatorname{div} \zeta = \partial_{x_j} \zeta_j^\alpha e^\alpha$. Again we sum over repeated indices. Concerning the tensor field

$$A_p(z) := |z|^{p-2} z, \qquad A_p(z) = |z|^{p-2} z_j^\alpha e^\alpha \otimes e_j, \tag{2.2}$$

defined on second-order tensors $z \in \mathbb{R}^{N\times n}$ for $p > 2$, we interpret the differential of A_p as a fourth-order tensor defined as

$$\partial A_p(z) = |z|^{p-2}\left(\delta_{\alpha\beta}\delta_{ij} + (p-2)\frac{z_i^\alpha z_j^\beta}{|z|^2}\right)(e^\alpha \otimes e_i) \otimes (e^\beta \otimes e_j), \tag{2.3}$$

so that by contracting we have

$$\partial A_p(z) : \xi = |z|^{p-2}\left[\xi + (p-2)\frac{(z:\xi)z}{|z|^2}\right]$$

and

$$(\partial A_p(z) : \xi) : \xi = |z|^{p-2}\left[|\xi|^2 + (p-2)\frac{(z:\xi)^2}{|z|^2}\right]$$

whenever ξ, z are second-order tensors, i.e. for every choice of $\xi, z \in \mathbb{R}^{N\times n}$. Notice that we still denote the contraction of a fourth-order tensor and a second-order one by :, exactly as in the above display. Notice also that A_p is of class C^1 and smooth outside the origin. In what follows, it is convenient for us to write

$$L(z) = \left(\delta_{\alpha\beta}\delta_{ij} + (p-2)\frac{z_i^\alpha z_j^\beta}{|z|^2}\right)(e^\alpha \otimes e_i) \otimes (e^\beta \otimes e_j) \tag{2.4}$$

for $z \neq 0$, so that

$$\partial A_p(z) = |z|^{p-2} L(z) \tag{2.5}$$

for all $z \in \mathbb{R}^{nN}$, defined obviously at the origin as $\partial A_p(0) = 0$ for $p > 2$. Notice that L is zero-homogeneous in the sense that

$$L(\lambda z) = L(z) \quad \text{ for all } \lambda \in \mathbb{R} \text{ and } z \in \mathbb{R}^{N\times n} \setminus \{0\}. \tag{2.6}$$

Moreover, L satisfies the ellipticity conditions

$$|\xi|^2 \leq (L(z):\xi):\xi, \qquad |L(z):\xi| \leq (p-1)|\xi| \tag{2.7}$$

for every $z \in \mathbb{R}^{N\times n} \setminus \{0\}$ and $\xi \in \mathbb{R}^{N\times n}$.

3. Harmonic and p-harmonic maps

In this section we collect, and reformulate in a suitable way, a few basic facts concerning p-harmonic and A-harmonic maps. First, the definitions. For $p > 1$, $v \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$ is a *p-harmonic map* in $\Omega \subset \mathbb{R}^n$ provided that $v \in W^{1,p}(\Omega; \mathbb{R}^N)$ and

$$\int_\Omega |Dv|^{p-2} Dv : D\varphi \, dx = 0 \quad \forall \varphi \in C_0^\infty(\Omega; \mathbb{R}^N). \tag{3.1}$$

Analogously, for a given constant coefficient fourth-order tensor A we say that h is an *A-harmonic map* in Ω if $h \in W^{1,2}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$ and

$$\int_\Omega (A : Dh) : D\varphi \, dx = 0 \quad \forall \varphi \in C_0^\infty(\Omega; \mathbb{R}^N).$$

It is well-known [59] that a p-harmonic map is locally $C^{1,\alpha}$-regular for some $\alpha \equiv \alpha(n, N, p) \in (0, 1)$, while an A-harmonic map is smooth (real analytic) provided that A is uniformly elliptic, i.e.,

$$\Lambda^{-1}|\xi|^2 \le (A : \xi) : \xi \le \Lambda|\xi|^2 \quad \forall \xi \in \mathbb{R}^{N\times n} \tag{3.2}$$

with some parameter $\Lambda \in [1, \infty)$. The following theorem reports a classical regularity property of solutions to elliptic systems with constant coefficients that essentially goes back to the work of Campanato. The proof can be found for instance in [24, Chapter 10].

Theorem 3.1. *Suppose that a map v is A-harmonic in $B_r \equiv B_r(x_0)$, with A satisfying (3.2) for some $\Lambda \in [1, \infty)$. Then there exists a positive constant $c_{\mathrm{hol},A} \ge 1$, depending only on n, N, Λ, such that*

$$\operatorname*{osc}_{B_{\delta r}} Dv \le c_{\mathrm{hol},A}\, \delta \fint_{B_r} |Dv - (Dv)_{B_r}| \, dx \quad \forall \delta \in (0, 1/2]. \tag{3.3}$$

The next theorem reports an analogous property of p-harmonic maps. The proof is considerably more delicate and rests on some classical a priori estimates of Uhlenbeck [59] and a priori estimates for solutions to constant coefficient elliptic systems. The following theorem is stated for the general case $p > 1$, although we shall use it only for $p > 2$.

Theorem 3.2. *Let $v \in W^{1,p}(B_r)$ be a p-harmonic map, $p > 1$, in $B_r \equiv B_r(x_0)$. Then there exist constants $c_{\mathrm{hol},p} \ge 1$, $\alpha_{\mathrm{hol}} \in (0, 1)$ and $\sigma_0 \in (0, 1/2]$, depending only on n, N, p, such that*

$$\operatorname*{osc}_{B_{\delta r}} Dv \le c_{\mathrm{hol},p}\, \delta^{\alpha_{\mathrm{hol}}} \fint_{B_r} |Dv - (Dv)_{B_r}| \, dx \quad \forall \delta \in (0, \sigma_0]. \tag{3.4}$$

Proof. We first recall a few a priori regularity estimates "below the natural growth exponent" that hold for p-harmonic maps. The classical local Lipschitz estimate for p-harmonic maps states that for every $\delta < 1$ there exists a constant c, depending only on n, N, p and $1-\delta$, such that

$$\sup_{B_{\delta r}} |Dv| \le c \left(\fint_{B_r} |Dv|^p \, dx \right)^{1/p} \tag{3.5}$$

(see [59]). By a standard interpolation technique (see for instance [24, Chapter 10] or [51]) the previous estimate implies

$$\sup_{B_{\delta r}} |Dv| \le c \fint_{B_r} |Dv| \, dx. \tag{3.6}$$

Here c again depends only on n, N, p and $1-\delta$. The estimate (3.6) then allows one to use the following form of the standard $C^{1,\alpha}$ a priori local estimate for p-harmonic maps:

$$\operatorname*{osc}_{B_{\delta r}} Dv \le \bar{c}\delta^{\alpha_{\rm hol}} \fint_{B_r} |Dv| \, dx. \tag{3.7}$$

This holds whenever $\delta \in (0, 2/3)$ and for a constant $\bar{c} \equiv \bar{c}(n, N, p) \ge 1$ and an exponent $\alpha_{\rm hol} \equiv \alpha_{\rm hol}(n, N, p) \in (0, 1)$. See for instance [14], where the right hand side features the L^p-norm of Dv instead of the L^1-norm, as in the above display. Combining the result in [14] with (3.6) yields (3.7). We then proceed via alternatives, that is, we consider the two cases

$$\begin{cases} \fint_{B_r} |Dv - (Dv)_{B_r}| \, dx \ge \bar{\theta} |(Dv)_{B_r}| \\ \text{or} \\ \fint_{B_r} |Dv - (Dv)_{B_r}| \, dx < \bar{\theta} |(Dv)_{B_r}| \end{cases} \tag{3.8}$$

where, with $\bar{c}$ being the constant appearing in (3.7),

$$\bar{\theta} := \frac{1}{40\sigma_0^n} \quad \text{and} \quad \sigma_0 := \left(\frac{1}{160\bar{c}} \right)^{1/\alpha_{\rm hol}}. \tag{3.9}$$

Notice that $\bar{\theta} \equiv \bar{\theta}(n, N, p) \in (0, 1)$ and $\sigma_0 \le 1/160$. If the first inequality in (3.8) holds then by (3.7) we have

$$\begin{aligned} \operatorname*{osc}_{B_{\delta r}} Dv &\le \bar{c}\delta^{\alpha_{\rm hol}} \fint_{B_r} |Dv - (Dv)_{B_r}| \, dx + \bar{c}\delta^{\alpha_{\rm hol}} |(Dv)_{B_r}| \\ &\le \bar{c}\left(1 + \frac{1}{\bar{\theta}}\right) \delta^{\alpha_{\rm hol}} \fint_{B_r} |Dv - (Dv)_{B_r}| \, dx, \end{aligned}$$

so that (3.4) holds with $\sigma_0 = 1/2$ since $\bar{\theta}$ depends only on n, N, p.

We then consider the case when the second inequality in (3.8) holds, and we may assume that $|(Dv)_{B_r}| > 0$ since otherwise (3.4) follows trivially. The triangle inequality gives

$$\fint_{B_r} |Dv| \, dx \le 2|(Dv)_{B_r}|. \tag{3.10}$$

By (3.8)$_2$ we observe that

$$\fint_{B_{2\sigma_0 r}} |Dv - (Dv)_{B_r}|\,dx \le (2\sigma_0)^{-n} \fint_{B_r} |Dv - (Dv)_{B_r}|\,dx \le (2\sigma_0)^{-n}\bar\theta |(Dv)_{B_r}|,$$

and by (3.9) we conclude that there exists a point $\tilde{x} \in B_{2\sigma_0 r}$ such that

$$|Dv(\tilde{x}) - (Dv)_{B_r}| \le |(Dv)_{B_r}|/20.$$

Then, by (3.7) and (3.10) we notice that

$$\operatorname*{osc}_{B_{2\sigma_0 r}} Dv \le 4\bar{c}\sigma_0^{\alpha_{\rm hol}} |(Dv)_{B_r}| \le |(Dv)_{B_r}|/20,$$

where we have used the definition of σ_0 in (3.9). The information in the last two displays then allows us to conclude that

$$|Dv(x) - (Dv)_{B_r}| \le |(Dv)_{B_r}|/10 \qquad \forall x \in B_{2\sigma_0 r}.$$

On the other hand, notice that (3.6) and (3.10) imply

$$|Dv(x)| \le c \fint_{B_r} |Dv|\,dy \le 2c|(Dv)_{B_r}|$$

for every $x \in B_{2r/3}$ and for $c \equiv c(n, N, p)$. All in all, the last two displays yield

$$|(Dv)_{B_r}|/c \le |Dv(x)| \le c|(Dv)_{B_r}|, \qquad \forall x \in B_{2\sigma_0 r}, \tag{3.11}$$

for a constant $c \equiv c(n, N, p)$. In particular, the gradient Dv never vanishes in $B_{2\sigma_0 r}$ and thus the system (3.1) becomes nondegenerate in $B_{2\sigma_0 r}$. Therefore smoothness of the solution v (i.e., local Hölder regularity of the gradient) can be gained by a standard perturbation argument (but on the other hand this already comes from the standard regularity theory of Uhlenbeck). The main point is the quantification of such smoothness, in turn yielding suitable a priori estimates. To this end, let us first observe that whenever $x, y \in B_{r/2}$, (3.7) and (3.10) give

$$|Dv(x) - Dv(y)| \le c(n, N, p)|(Dv)_{B_r}|\,|x - y|^{\alpha_{\rm hol}}. \tag{3.12}$$

By differentiating (3.1) we see that the partial derivatives $w_i := \partial_{x_i} v$ satisfy the system

$$-\operatorname{div}(B(x) : Dw_i) = 0, \qquad B(x) := \left(\frac{|Dv|}{|(Dv)_{B_r}|}\right)^{p-2} L(Dv(x)),$$

where L is defined in (2.4). This is a linear elliptic system since by (3.11)

$$c^{-1}|\xi|^2 \le (B(x) : \xi) : \xi \le c|\xi|^2 \qquad \forall \xi \in \mathbb{R}^{N\times n} \text{ and } \forall x \in B_{2\sigma_0 r},$$

where the constant $c \ge 1$ depends only on n, N, p. By using the mean value theorem and again (3.11), from (3.12) it then follows that

$$|B(x) - B(x_0)| \le c|x - x_0|^{\alpha_{\rm hol}} \qquad \forall x, x_0 \in B_{2\sigma_0 r},$$

where $c \geq 1$ again depends on n, N, p. We can therefore use a standard perturbation argument, based on the classical trick of freezing the coefficients. This shows that w_i is locally Hölder continuous in $B_{\sigma_0 r}$ with *every* exponent $\alpha < 1$, with a related local a priori estimate. In particular, this applies with the exponent $\alpha_{\rm hol}$ determined in (3.7) and gives

$$\operatorname*{osc}_{B_{\delta r}} w_i \leq c\delta^{\alpha_{\rm hol}} \fint_{B_{2\sigma_0 r}} |w_i - (w_i)_{B_{\sigma_0 r}}|\, dx \leq 2c\sigma_0^{-n}\delta^{\alpha_{\rm hol}} \fint_{B_r} |w_i - (Dv)_{B_r}|\, dx$$

whenever $\delta \in (0, \sigma_0)$ and with $c \equiv c(n, N, p)$; notice that we have used (2.1). From this last inequality, recalling that $\sigma_0 \equiv \sigma_0(n, N, p)$, we conclude that (3.4) follows for $\delta \in (0, \sigma_0]$, as required. □

Theorem 3.3. *Let $v \in W^{1,p}(B_r)$ be a p-harmonic map, $p > 1$, in $B_r \equiv B_r(x_0)$. Then there exists a constant $c \equiv c(n, N, p)$ such that*

$$\operatorname*{osc}_{B_{\delta r}} v \leq c\delta \fint_{B_r} |v - (v)_{B_r}|\, dx \quad \textit{whenever } \delta \in (0, 1/2]. \tag{3.13}$$

Proof. Once again we start by recalling a few basic regularity estimates for p-harmonic maps; this time we deal with the regularity of v rather than Dv. We take a ball $B_r \subset B_r$, not necessarily concentric to B_r. The standard Caccioppoli type estimate for p-harmonic mappings reads

$$\fint_{B_{\sigma' r}} |Dv|^p\, dx \leq \frac{c}{(\sigma' - \sigma)^p} \fint_{B_{\sigma r}} \left|\frac{v - \lambda}{r}\right|^p dx \tag{3.14}$$

for all $\lambda \in \mathbb{R}^N$ and $7/8 \leq \sigma' < \sigma \leq 1$ and some $c \equiv c(n, N, p)$ (see for instance [24, Chapter 6]). Applying the Sobolev–Poincaré inequality gives

$$\left(\fint_{B_{\sigma' r}} \left|\frac{v - (v)_{B_{\sigma' r}}}{r}\right|^\gamma dx\right)^{1/\gamma} \leq \frac{c}{\sigma' - \sigma} \left(\fint_{B_{\sigma r}} \left|\frac{v - \lambda}{r}\right|^p dx\right)^{1/p}$$

for some $\gamma > p$. Since v is p-harmonic, so is $v + (v)_{B_{\sigma' r}}$; applying to this last function the inequality above with $\lambda = (v)_{B_{\sigma' r}}$ yields

$$\left(\fint_{B_{\sigma' r}} |v|^\gamma\, dx\right)^{1/\gamma} \leq \frac{c}{\sigma' - \sigma} \left(\fint_{B_{\sigma r}} |v|^p\, dx\right)^{1/p}.$$

We can at this point use Lemma 3.1 below with $w = |v|$, $U \equiv B_r$ $d\tilde{\mu} \equiv dx$ and $\kappa = 7/8$, and this gives the estimate

$$\left(\fint_{B_{7r/8}} |v|^p\, dx\right)^{1/p} \leq c \fint_{B_r} |v|\, dx,$$

again with $c \equiv c(n, N, p)$. Applying this inequality to $v - (v)_{B_r}$, which is still p-harmonic, yields

$$\left(\fint_{B_{7r/8}} \left|\frac{v - (v)_{B_r}}{r}\right|^p dx\right)^{1/p} \leq c \fint_{B_r} \left|\frac{v - (v)_{B_r}}{r}\right| dx.$$

We then combine (3.5), (3.14) and the above display to get

$$\sup_{B_{r/2}} |Dv| \leq c\left(\fint_{B_{3r/4}} |Dv|^p \, dx\right)^{1/p} \leq c \fint_{B_r} \left|\frac{v-(v)_{B_r}}{r}\right| dx.$$

Then the mean value principle implies

$$\operatorname*{osc}_{B_{\delta r}} v \leq 2\delta r \sup_{B_{\delta r}} |Dv| \leq c\delta \fint_{B_r} |v-(v)_{B_r}| \, dx,$$

thereby finishing the proof of (3.13). □

The following lemma reports a well-known self-improving property of reverse Hölder inequalities; see [27, Lemma 3.38] for the proof.

Lemma 3.1. *Let $\tilde{\mu}$ be a nonnegative Borel measure with finite total mass. Let $0 < q < p < \gamma < \infty$ and $\xi, M \geq 0$, and let $\{\sigma U\}_\sigma$ be a family of open sets with the property $\sigma' U \subset \sigma U \subset 1U = U$ whenever $0 < \sigma' < \sigma \leq 1$. Suppose that $w \in L^p_{\tilde{\mu}}(U)$ is a nonnegative function satisfying*

$$\left(\int_{\sigma' U} w^\gamma \, d\tilde{\mu}\right)^{1/\gamma} \leq \frac{c_0}{(\sigma-\sigma')^\xi}\left(\int_{\sigma U} w^p \, d\tilde{\mu}\right)^{1/p} + M$$

for all $\kappa \leq \sigma' < \sigma \leq 1$, where $\kappa \in (0,1)$. Then there exists a positive constant c depending only on c_0, ξ, γ, p, and q such that

$$\left(\int_{\sigma U} w^\gamma \, d\tilde{\mu}\right)^{1/\gamma} \leq \frac{c}{(1-\sigma)^{\bar{\xi}}}\left[\left(\int_{U} w^q \, d\tilde{\mu}\right)^{1/q} + M\right]$$

for all $\sigma \in (\kappa, 1)$, where

$$\bar{\xi} := \frac{\xi p(\gamma-q)}{q(\gamma-p)}.$$

4. Measure data p-harmonic approximation

In this section we prove a result usually called a compactness or blow-up lemma. We think that it is of independent interest and, due to its generality, could be employed in other settings in the future. The novelty here, making the approach nontrivial, is that this is tailored to measure data problems. This is essentially reflected in the assumed energy bound (4.1), and in the closeness condition (4.3). These are weaker than those usually used to formulate such results in the setting of finite energy problems (i.e. solutions which are uniformly in $W^{1,p}$). Our theorem states that a map u with possibly arbitrarily large $W^{1,p}$-energy, which is almost p-harmonic in the distributional sense, that is, satisfies (4.3), is then actually close to a real $W^{1,p}$-energy function, that is, (4.4) holds. Such a closeness is prescribed in Sobolev spaces natural for solutions to measure data problems.

Theorem 4.1 (Measure data p-harmonic approximation). *Let $u \in W^{1,p}(B_r(x_0))$ with $p > 2 - 1/n$ satisfy*

$$\fint_{B_r(x_0)} |u|\, dx \le Mr, \qquad M \ge 1. \tag{4.1}$$

Let q be such that

$$\max\{1, p-1\} \le q < q_{\mathrm{m}}, \quad \textit{where} \quad q_{\mathrm{m}} := \min\left\{\frac{n(p-1)}{n-1}, p\right\}. \tag{4.2}$$

Let $\varepsilon > 0$. There exists a positive constant $\delta \equiv \delta(n, N, p, q, M, \varepsilon) \in (0, 1]$ such that if

$$\left| \fint_{B_r(x_0)} |Du|^{p-2} Du : D\varphi\, dx \right| \le \frac{\delta}{r} \|\varphi\|_{L^\infty(B_r(x_0))} \tag{4.3}$$

for every test function $\varphi \in W^{1,p}_0(B_r(x_0)) \cap L^\infty(B_r(x_0))$, then there exists a p-harmonic map $v \in W^{1,p}(B_{r/2}(x_0))$ satisfying

$$\left(\fint_{B_{r/2}(x_0)} |Du - Dv|^q\, dx \right)^{1/q} \le \varepsilon \tag{4.4}$$

together with

$$\fint_{B_{r/2}(x_0)} |v|\, dx \le M2^n r \quad \textit{and} \quad \left(\fint_{B_{r/2}(x_0)} |Dv|^q\, dx \right)^{1/q} \le cM, \tag{4.5}$$

for a constant c depending only on n, N, p, q.

Remark 4.1. This section is the only one place where the exponent p may be less than 2. Indeed, our assumption here is $p > 2 - 1/n$. This range is natural as it guarantees that solutions to measure data problems belong to Sobolev spaces. Indeed, $p > 2 - 1/n$ implies that $q_{\mathrm{m}} > 1$. Notice that the lower bound $p > 2 - 1/n$ is exactly what is needed in the proof of gradient potential estimates in [20]. We also remark that if the above theorem is proved for a certain value of $q \ge p - 1$, then it automatically holds for all smaller values. Therefore, with no loss of generality, we can assume that $q > p - 1$.

Proof of Theorem 4.1. The plan is to first establish suitable a priori estimates and then proceed with the proof via contradiction. The proof is in five steps.

Step 1: Scaling and testing. In this and in the next step we first derive the needed a priori integrability estimates for a function u satisfying (4.1) and (4.3) for some $\delta \in (0, 1]$; more precisely, we derive a priori estimates for a certain rescaled function defined in the unit ball B_1. Indeed, we set

$$\bar{u}(x) = \frac{u(x_0 + rx)}{Mr} \quad \text{and} \quad \eta(x) = \frac{\varphi(x_0 + rx)}{r},$$

where $\varphi \in W_0^{1,p}(B_r(x_0)) \cap L^\infty(B_r(x_0))$. Then (4.1) and (4.3) imply that

$$\fint_{B_1} |\bar u|\, dx \le 1, \tag{4.6}$$

$$\left| \fint_{B_1} |D\bar u|^{p-2} D\bar u : D\eta\, dx \right| \le M^{1-p}\delta \|\eta\|_{L^\infty(B_1)}, \tag{4.7}$$

where, as usual, we have denoted $B_\sigma \equiv B_\sigma(0)$ for every $\sigma > 0$. We now want to test (4.7) with suitable test functions φ; these have to be bounded, so we introduce the truncation operator $T_t : \mathbb{R}^N \to \mathbb{R}^N$ defined by

$$T_t(z) := \min\{1, t/|z|\} z, \tag{4.8}$$

so that $DT_t : \mathbb{R}^N \to \mathbb{R}^N$ turns out to be

$$DT_t(z) := \begin{cases} \mathrm{Id} & \text{if } |z| \le t, \\ \dfrac{t}{|z|}\left(\mathrm{Id} - \dfrac{z \otimes z}{|z|^2}\right) & \text{if } |z| > t, \end{cases} \tag{4.9}$$

where Id is the identity operator on $\mathbb{R}^N$ (it is an $N \times N$ tensor) and t is a nonnegative real number. We then choose

$$\eta := \phi^p T_t(\bar u) \quad \text{with } \phi \in C_0^\infty(B_1),\ 0 \le \phi \le 1, \tag{4.10}$$

as a test function in (4.7). We have

$$\begin{aligned} D\eta &= \chi_{\{|\bar u| \le t\}}(\phi^p D\bar u + p\phi^{p-1}\bar u \otimes D\phi) \\ &\quad + \chi_{\{|\bar u|>t\}} \frac{t}{|\bar u|}\left(\phi^p(\mathrm{Id} - P)D\bar u + p\phi^{p-1}\bar u \otimes D\phi\right), \qquad P := \frac{\bar u \otimes \bar u}{|\bar u|^2}, \end{aligned}$$

and P is evaluated when $|\bar u| > 0$. This is the case since $t > 0$. We then plug the identity into (4.7). Notice that

$$D\bar u : [(\mathrm{Id} - P)D\bar u] = |D\bar u|^2 - \frac{D_j \bar u^\alpha \bar u^\alpha D_j \bar u^\beta \bar u^\beta}{|u|^2} = |D\bar u|^2 - \frac{\sum_{j=1}^n \langle D_j u, u\rangle^2}{|u|^2} \ge 0 \tag{4.11}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^N$. Using (4.11), we can then easily establish the following standard energy inequality, after a reabsorption with the aid of Young's inequality:

$$\begin{aligned} \int_{B_1 \cap \{|\bar u| < t\}} |D\bar u|^p \phi^p\, dx &\le c \int_{B_1 \cap \{|\bar u| < t\}} |\bar u|^p |D\phi|^p\, dx + cM^{1-p}\delta t \\ &\quad + ct \int_{B_1 \cap \{|\bar u| \ge t\}} |D\bar u|^{p-1} |D\phi| \phi^{p-1}\, dx \end{aligned} \tag{4.12}$$

with $c \equiv c(n, N, p)$. In what follows, a constant c will only depend on n, N, p, unless otherwise stated. It may vary from line to line.

Step 2: Summability of $\bar{u}$ and $D\bar{u}$. We multiply (4.12) by $(1+t)^{-2-\gamma}$, $\gamma > 0$, and integrate from zero to infinity, to get, after a few elementary estimations,

$$\frac{1}{1+\gamma}\int_{B_1}\frac{|D\bar{u}|^p\phi^p}{(1+|\bar{u}|)^{1+\gamma}}\,dx \le \frac{c}{1+\gamma}\int_{B_1}(1+|\bar{u}|)^{p-1-\gamma}|D\phi|^p\,dx + \frac{c}{\gamma}\delta + \frac{c}{\gamma}\int_{B_1}|D\bar{u}|^{p-1}|D\phi|\phi^{p-1}\,dx. \tag{4.13}$$

Here we have made use of the Cavalieri principle to obtain

$$\int_0^\infty \frac{\nu_j(\{|\bar{u}|<t\})}{(1+t)^{2+\gamma}}\,dt = \frac{1}{1+\gamma}\int_{\mathbb{R}^n}\frac{d\nu_j}{(1+|\bar{u}|)^{1+\gamma}}$$

for $j=1,2$ and measures $d\nu_1 = |D\bar{u}|^p\phi^p\,dx$ and $d\nu_2 = |\bar{u}|^p|D\phi|^p\,dx$. Observe that we have estimated $M^{1-p}\le 1$ as $M\ge 1$ here. Moreover, we have simply estimated

$$\int_0^\infty \frac{t}{(1+t)^{\gamma+2}}\int_{B_1\cap\{|\bar{u}|\ge t\}}|D\bar{u}|^{p-1}|D\phi|\phi^{p-1}\,dx\,dt \le \int_0^\infty \frac{dt}{(1+t)^{\gamma+1}}\int_{B_1}|D\bar{u}|^{p-1}|D\phi|\phi^{p-1}\,dx \le \frac{c}{\gamma}\int_{B_1}|D\bar{u}|^{p-1}|D\phi|\phi^{p-1}\,dx.$$

The last term appearing in (4.13) can be estimated by Young's inequality as

$$\frac{c}{\gamma}\int_{B_1}|D\bar{u}|^{p-1}|D\phi|\phi^{p-1}\,dx \le \frac{1}{2(1+\gamma)}\int_{B_1}\frac{|D\bar{u}|^p\phi^p}{(1+|\bar{u}|)^{1+\gamma}}\,dx + \frac{c(1+\gamma)^{p-1}}{\gamma^p}\int_{B_1}(1+|\bar{u}|)^{(p-1)(1+\gamma)}|D\phi|^p\,dx,$$

leading immediately to

$$\int_{B_1}\frac{|D\bar{u}|^p\phi^p}{(1+|\bar{u}|)^{1+\gamma}}\,dx \le c\left[1+\frac{(1+\gamma)^p}{\gamma^p}\right]\int_{B_1}(1+|\bar{u}|)^{(p-1)(1+\gamma)}|D\phi|^p\,dx + \frac{c(1+\gamma)\delta}{\gamma} \tag{4.14}$$

for every $\gamma > 0$. It is now convenient to denote

$$g := (1+|\bar{u}|)^{p-1-\gamma}, \qquad 0<\gamma<p-1. \tag{4.15}$$

In fact, we are considering values of γ as in the last inequality. The pointwise inequality $|D|\bar{u}||\le|D\bar{u}|$ implies

$$|D(g^{1/p}\phi)|^p \le \frac{c|D\bar{u}|^p}{(1+|\bar{u}|)^{1+\gamma}}\phi^p + cg|D\phi|^p.$$

Using this last inequality together with (4.14) gives

$$\int_{B_1}|D(g^{1/p}\phi)|^p\,dx \le \frac{c}{\gamma^p}\int_{B_1}(1+|\bar{u}|)^{\gamma p}g|D\phi|^p\,dx + \frac{c}{\gamma}, \tag{4.16}$$

therefore Sobolev's inequality in turn yields

$$\left(\int_{B_1} g^{\theta}\phi^{p^*}\,dx\right)^{1/\theta} \le \frac{c}{\gamma^p}\int_{B_1}(1+|\bar u|)^{\gamma p} g|D\phi|^p\,dx + \frac{c}{\gamma} \tag{4.17}$$

with

$$\theta := \frac{p^*}{p} > 1 \quad \text{where} \quad p^* = \begin{cases} \dfrac{np}{n-p}, & p<n,\\ \text{any number larger than } p, & p\ge n,\end{cases} \tag{4.18}$$

that is, p^* is the usual Sobolev conjugate exponent of p. We now consider a number $\tilde\theta$ such that $1<\tilde\theta<\theta$ and apply Hölder's inequality to obtain

$$\int_{B_1}(1+|\bar u|)^{\gamma p} g|D\phi|^p\,dx \le \left(\int_{B_1} g^{\tilde\theta}|D\phi|^{p\tilde\theta}\,dx\right)^{1/\tilde\theta}\left(\int_{B_1}(1+|\bar u|)^{\gamma p\tilde\theta/(\tilde\theta-1)}\,dx\right)^{1-1/\tilde\theta}. \tag{4.19}$$

We then choose $\gamma>0$ such that $\gamma p\tilde\theta/(\tilde\theta-1)\le 1$ so that by (4.6) we have

$$\int_{B_1}(1+|\bar u|)^{\gamma p\tilde\theta/(\tilde\theta-1)}\,dx \le 1+|B_1|. \tag{4.20}$$

Notice that by the definition (4.18) the admissible values of γ are

$$0<\gamma<\min\{1/n, 1/p\}. \tag{4.21}$$

Combining (4.17)–(4.20) yields the following inequality of reverse Hölder type:

$$\left(\int_{B_1} g^{\theta}\phi^{p^*}\,dx\right)^{1/\theta} \le \frac{c}{\gamma^p}\left(\int_{B_1} g^{\tilde\theta}|D\phi|^{p\tilde\theta}\,dx\right)^{1/\tilde\theta} + \frac{c}{\gamma}$$

with a constant c that depends only on n, N, p. Now, with $7/8\le\sigma'<\sigma\le 1$, let us take a cut-off function ϕ as in (4.10) with the additional feature that $\phi\equiv 1$ on $B_{\sigma'}$ and $|D\phi|\le 100/(\sigma-\sigma')$; we obtain

$$\left(\int_{B_{\sigma'}} g^{\theta}\,dx\right)^{1/\theta} \le \frac{c}{\gamma^p(\sigma-\sigma')^p}\left(\int_{B_\sigma} g^{\tilde\theta}\,dx\right)^{1/\tilde\theta} + \frac{c}{\gamma}.$$

Applying Lemma 3.1 with a suitable choice of parameters gives

$$\left(\int_{B_{7/8}} g^{\theta}\,dx\right)^{1/\theta} \le c\left(\int_{B_1} g^{1/(p-1-\gamma)}\,dx\right)^{p-1-\gamma} + \frac{c}{\gamma}.$$

Recalling the definition of g in (4.15) and the bound in (4.6) we deduce the desired integrability a priori estimate for $\bar u$:

$$\int_{B_{7/8}} |\bar u|^{\bar q}\,dx \le c, \quad \bar q < q_0 := \begin{cases} \dfrac{n(p-1)}{n-p}, & p<n,\\ \text{any number larger than } p, & p\ge n,\end{cases} \tag{4.22}$$

with $c \equiv c(n, N, p, \bar{q})$. It remains to prove the gradient integrability. To this end, observe that (4.14) and (4.22) easily imply that

$$\int_{B_{3/4}} \frac{|D\bar{u}|^p \phi^p}{(1+|\bar{u}|)^{1+\gamma}}\, dx \le c(n, N, p, \gamma)$$

for every $\gamma > 0$ (indeed, the inequality holds for small values of γ as prescribed in (4.21), and then trivially also for larger ones). With $q_{\rm m}$ as in (4.2) and $q < q_{\rm m}$, we use Hölder's inequality to estimate

$$\int_{B_{3/4}} |D\bar{u}|^q\, dx \le \left(\int_{B_{3/4}} \frac{|D\bar{u}|^p}{(1+|\bar{u}|)^{1+\gamma}}\, dx\right)^{q/p} \left(\int_{B_{3/4}} (1+|\bar{u}|)^{(1+\gamma)q/(p-q)}\, dx\right)^{1-q/p}.$$

By observing that $q < q_{\rm m}$ leads to the existence of $\gamma > 0$ such that $(1+\gamma)q/(p-q) < q_0$, where q_0 has been defined in (4.22), we deduce the desired integrability result:

$$\int_{B_{3/4}} |D\bar{u}|^q\, dx \le c_{\rm ap} \equiv c_{\rm ap}(n, N, p, q) \quad \text{for every } q < q_{\rm m}. \tag{4.23}$$

Step 3: Contradiction argument. After establishing the needed energy estimates in Steps 1 and 2, we continue the proof by making a counter-assumption. Let q be as in the statement of Theorem 4.1 (with no loss of generality we may assume $q > p-1$) and define, in addition, $q_1 := (q + q_{\rm m})/2$, so that

$$\max\{p-1, 1\} < q < q_1 < q_{\rm m} \tag{4.24}$$

with $q_{\rm m}$ as in (4.2). Then, we assume that there exist $\varepsilon > 0$ and sequences of balls $\{B_{r_j}(x_j)\}$ and almost p-harmonic maps $\{u_j\} \subset W^{1,p}(B_{r_j}(x_j))$ such that

$$-\!\!\!\!\!\!\int_{B_{r_j}(x_j)} |u_j|\, dx \le M r_j, \tag{4.25}$$

$$\left| -\!\!\!\!\!\!\int_{B_{r_j}(x_j)} |Du_j|^{p-2} Du_j : D\varphi\, dx \right| \le \frac{2^{-j}}{r_j} \|\varphi\|_{L^\infty(B_{r_j}(x_j))} \tag{4.26}$$

for all $\varphi \in W^{1,p}_0(B_{r_j}(x_j)) \cap L^\infty(B_{r_j}(x_j))$, while the counter-assumption is that

$$\left(-\!\!\!\!\!\!\int_{B_{r_j/2}(x_j)} |Du_j - Dv|^q\, dx\right)^{1/q} > \varepsilon$$

whenever $v \in W^{1,p}(B_{r_j/2}(x_j))$ is a p-harmonic map in $B_{r_j/2}(x_j)$ satisfying

$$-\!\!\!\!\!\!\int_{B_{r_j/2}(x_j)} |v|\, dx \le M 2^n r_j \quad \text{and} \quad \left(-\!\!\!\!\!\!\int_{B_{r_j/2}(x_j)} |Dv|^q\, dx\right)^{1/q} \le \left(\frac{2^n c_{\rm ap}}{|B_1|}\right)^{1/q} M,$$

where $c_{\mathrm{ap}} \equiv c_{\mathrm{ap}}(n, N, p, q)$ is the constant appearing in (4.23). This also fixes the constant c appearing in the statement, and more precisely the one appearing in the second inequality of (4.5). Let $\bar{u}_j$ be scaled as in the first step, but using x_j and r_j instead, that is,

$$\bar{u}_j(x) := \frac{u(x_j + r_j x)}{M r_j}.$$

We then have

$$\fint_{B_1} |\bar{u}_j|\,dx \le 1 \tag{4.27}$$

by (4.25), and (4.26) implies that

$$\left| \fint_{B_1} |D\bar{u}_j|^{p-2} D\bar{u}_j : D\varphi\,dx \right| \le \frac{2^{-j}}{M^{p-1}} \|\varphi\|_{L^\infty(B_1)} \tag{4.28}$$

for all $\varphi \in W_0^{1,p}(B_1) \cap L^\infty(B_1)$. Moreover,

$$\left(\fint_{B_{1/2}} |D\bar{u}_j - D\bar{v}|^q\,dx \right)^{1/q} > \frac{\varepsilon}{M} \tag{4.29}$$

whenever $\bar{v} \in W^{1,p}(B_{1/2})$ is a p-harmonic map in $B_{1/2}$ such that

$$\fint_{B_{1/2}} |\bar{v}|\,dx \le 2^n \quad \text{and} \quad \int_{B_{1/2}} |D\bar{v}|^q\,dx \le c_{\mathrm{ap}}. \tag{4.30}$$

Recalling (4.27), according to the a priori estimate (4.23) (which we actually apply with the exponents q and q_1 selected in (4.24)), we have

$$\int_{B_{3/4}} |D\bar{u}_j|^q\,dx \le c_{\mathrm{ap}} \quad \text{and} \quad \int_{B_{3/4}} |D\bar{u}_j|^{q_1}\,dx \le \tilde{c}_{\mathrm{ap}} \tag{4.31}$$

uniformly in $j \in \mathbb{N}$. Then by (4.24) we can assume that there exist $\tilde{u} \in W^{1,q}(B_{3/4})$, $b \in L^{q/(p-1)}(B_{3/4})$ and $h \in L^q(B_{3/4})$ such that

$$\begin{cases} \displaystyle\int_{B_{3/4}} |D\tilde{u}|^q\,dx + \sup_j \int_{B_{3/4}} |Du_j|^q\,dx + \sup_j \int_{B_{3/4}} |Du_j|^{q_1}\,dx < \infty, \\ D\bar{u}_j \rightharpoonup D\tilde{u}, \quad |D\bar{u}_j - D\tilde{u}| \rightharpoonup h \quad \text{in } L^q(B_{3/4}), \\ |D\bar{u}_j|^{p-2} D\bar{u}_j \rightharpoonup b \quad \text{in } L^{q/(p-1)}(B_{3/4}), \\ \bar{u}_j \to \tilde{u} \quad \text{strongly in } L^q(B_{3/4}) \text{ and pointwise in } B_{3/4}, \end{cases} \tag{4.32}$$

up to a subsequence. Using lower semicontinuity, by (4.27) and the first bound in (4.31), we get

$$\fint_{B_{1/2}} |\tilde{u}|\,dx \le 2^n \quad \text{and} \quad \int_{B_{1/2}} |D\tilde{u}|^q\,dx \le c_{\mathrm{ap}}. \tag{4.33}$$

In the next step we shall prove that actually the gradients converge strongly,

$$D\bar{u}_j \to D\tilde{u} \quad \text{in } L^q(B_{3/4}). \tag{4.34}$$

This allows us to pass to the limit in (4.28), provided the test function φ is smooth. Specifically, we get

$$\fint_{B_{1/2}} |D\tilde{u}|^{p-2} D\tilde{u} : D\varphi \, dx = 0 \tag{4.35}$$

for all $\varphi \in C_0^\infty(B_{1/2})$. Later on, in the final step, we shall prove that $D\tilde{u} \in L^p(B_{1/2})$ and therefore (4.35) holds whenever $W_0^{1,p}(B_{1/2})$. This means that $\tilde{u}$ is a p-harmonic function as claimed in the statement of the theorem.

Remark 4.2. Notice that in the contradiction assumptions we have considered an arbitrary sequence of balls $B_{r_j}(x_j)$, and this guarantees that in the statement of Theorem 4.1 the number δ is actually independent of the radius r. Alternatively, one can directly prove the statement of Theorem 4.1 in the special case $r = 1$ and then retrieve the general statement by a final scaling, as done for instance in the proof of the p-harmonic approximation lemma valid for the energy range from [17, Section 3]. The two ways are equivalent.

Step 4: Strong convergence of gradients. In this section we use arguments developed in [15] to prove the strong L^q-convergence of the gradients Du_j, that is, (4.34). This will be established by showing that $h = 0$ almost everywhere, where $h \in L^q(B_{3/4})$ has been defined via $(4.32)_2$. Let $\bar{x} \in B_{3/4}$ be a Lebesgue point simultaneously for $\tilde{u}$, $D\tilde{u}$, h, and b in the sense that

$$\lim_{\varrho \to 0} \fint_{B_\varrho(\bar{x})} \left(|\tilde{u}-\tilde{u}(\bar{x})|+|D\tilde{u}-D\tilde{u}(\bar{x})|+|h-h(\bar{x})|+|b-b(\bar{x})|^{1/(p-1)}\right)^q dx = 0 \tag{4.36}$$

together with

$$|\tilde{u}(\bar{x})| + |D\tilde{u}(\bar{x})| + |h(\bar{x})| + |b(\bar{x})| < \infty. \tag{4.37}$$

By classical Lebesgue theory, almost every point in $B_{3/4}$ satisfies these conditions and we aim at showing that

$$h(\bar{x}) = 0. \tag{4.38}$$

This will imply (4.34). Indeed, notice that (4.38) implies that $D\bar{u}_j \to D\tilde{u}$ strongly in $L^1(B_{3/4})$. This, together with the second bound in (4.30) and the standard interpolation inequality

$$\|D\bar{u}_j - D\tilde{u}\|_{L^q(B_{3/4})} \le \|D\bar{u}_j - D\tilde{u}\|_{L^1(B_{3/4})}^{\theta} \|D\bar{u}_j - D\tilde{u}\|_{L^{q_1}(B_{3/4})}^{1-\theta}$$

(for $1/q = \theta + (1-\theta)/q_1$), finally gives (4.34). We also set

$$\ell_\varrho(x) := (\tilde{u})_{B_\varrho(\bar{x})} + \langle D\tilde{u}(\bar{x}), x - \bar{x} \rangle \tag{4.39}$$

as the linearization of $\tilde{u}$ at $\bar{x}$. Note that Poincaré's inequality implies

$$\lim_{\varrho \to 0} \fint_{B_\varrho(\bar{x})} \left| \frac{\tilde{u} - \ell_\varrho}{\varrho} \right|^q dx \le c \lim_{\varrho \to 0} \fint_{B_\varrho(\bar{x})} |D\tilde{u} - D\tilde{u}(\bar{x})|^q \, dx = 0. \tag{4.40}$$

From now on ϱ will be considered to be small enough to guarantee that $B_\varrho(\bar{x}) \subset B_{3/4}$, so that all the information in (4.32) is available. By the weak convergence of $|D\bar{u}_j - D\tilde{u}|$ to h and the fact that $\bar{x}$ is a Lebesgue point of h, we get

$$h(\bar{x}) = \lim_{\varrho\to 0}\lim_{j\to\infty} \fint_{B_{\varrho/2}(\bar{x})} |D\bar{u}_j - D\tilde{u}|\,dx. \tag{4.41}$$

Rewrite

$$\begin{aligned}\fint_{B_{\varrho/2}(\bar{x})} |D\bar{u}_j - D\tilde{u}|\,dx &= \fint_{B_{\varrho/2}(\bar{x})} \chi_{\{|\bar{u}_j-\ell_\varrho|<\varrho\}}|D\bar{u}_j - D\tilde{u}|\,dx \\ &\quad + \fint_{B_{\varrho/2}(\bar{x})} \chi_{\{|\bar{u}_j-\ell_\varrho|\geq\varrho\}}|D\bar{u}_j - D\tilde{u}|\,dx.\end{aligned} \tag{4.42}$$

The second term tends to zero as first $j \to \infty$ and then $\varrho \to 0$, because

$$\begin{aligned}\fint_{B_{\varrho/2}(\bar{x})} \chi_{\{|\bar{u}_j-\ell_\varrho|\geq\varrho\}}|D\bar{u}_j - D\tilde{u}|\,dx &\leq \fint_{B_{\varrho/2}(\bar{x})} \chi_{\{|\bar{u}_j-\tilde{u}|\geq\varrho/2\}}|D\bar{u}_j - D\tilde{u}|\,dx \\ &\quad + \fint_{B_{\varrho/2}(\bar{x})} \chi_{\{|\tilde{u}-\ell_\varrho|\geq\varrho/2\}}|D\bar{u}_j - D\tilde{u}|\,dx \\ &\xrightarrow{j\to\infty} \fint_{B_{\varrho/2}(\bar{x})} \chi_{\{|\tilde{u}-\ell_\varrho|\geq\varrho/2\}}h\,dx \xrightarrow{\varrho\to 0} 0\end{aligned} \tag{4.43}$$

since $\bar{x}$ is a Lebesgue point of h. Indeed, in order to prove the first convergence in (4.43) we appeal to $(4.32)_1$ and observe that

$$\begin{aligned}&\fint_{B_{\varrho/2}(\bar{x})} \chi_{\{|\bar{u}_j-\tilde{u}|\geq\varrho/2\}}|D\bar{u}_j - D\tilde{u}|\,dx \\ &\quad\leq \left(\fint_{B_{\varrho/2}(\bar{x})} |D\bar{u}_j - D\tilde{u}|^q\,dx\right)^{1/q}\left(\frac{|\{x\in B_{3/4} : |\bar{u}_j-\tilde{u}|\geq\varrho/2\}|}{|B_{\varrho/2}(\bar{x})|}\right)^{1-1/q} \xrightarrow{j\to\infty} 0\end{aligned}$$

since $\bar{u}_j$ converges to $\tilde{u}$ strongly in $L^q(B_{3/4})$. In order to prove the last convergence in (4.43) we use (4.36)–(4.37) and (4.40) in the estimate

$$\begin{aligned}\fint_{B_{\varrho/2}(\bar{x})} \chi_{\{|\tilde{u}-\ell_\varrho|\geq\varrho/2\}}h\,dx &\leq \left(\fint_{B_\varrho(\bar{x})} h^q\,dx\right)^{1/q}\left(\fint_{B_\varrho(\bar{x})} \chi_{\{|\tilde{u}-\ell_\varrho|\geq\varrho/2\}}\,dx\right)^{1-1/q} \\ &\leq c\left[\left(\fint_{B_\varrho(\bar{x})} |h-h(\bar{x})|^q\,dx\right)^{1/q} + h(\bar{x})\right]\left(\fint_{B_\varrho(\bar{x})}\left|\frac{\tilde{u}-\ell_\varrho}{\varrho}\right|^q dx\right)^{1-1/q}.\end{aligned}$$

By (4.41)–(4.43), in order to establish (4.38), it remains to prove that

$$\lim_{\varrho\to 0}\lim_{j\to\infty} \fint_{B_{\varrho/2}(\bar{x})} \chi_{\{|\bar{u}_j-\ell_\varrho|<\varrho\}}|D\bar{u}_j - D\tilde{u}|\,dx = 0. \tag{4.44}$$

For this, we start by estimating

$$\fint_{B_{\varrho/2}(\bar{x})} \chi_{\{|\bar{u}_j-\ell_\varrho|<\varrho\}}|D\bar{u}_j - D\tilde{u}|\,dx \le \fint_{B_{\varrho/2}(\bar{x})} \chi_{\{|\bar{u}_j-\ell_\varrho|<\varrho\}}|D\bar{u}_j - D\ell_\varrho|\,dx + 2^n \fint_{B_\varrho(\bar{x})} \chi_{\{|\bar{u}_j-\ell_\varrho|<\varrho\}}|D\tilde{u} - D\ell_\varrho|\,dx.$$

Since by (4.36) we have

$$\lim_{\varrho\to 0}\limsup_{j\to\infty} \fint_{B_\varrho(\bar{x})} \chi_{\{|\bar{u}_j-\ell_\varrho|<\varrho\}}|D\tilde{u} - D\ell_\varrho|\,dx \le \lim_{\varrho\to 0}\fint_{B_\varrho(\bar{x})} |D\tilde{u} - D\tilde{u}(\bar{x})|\,dx = 0,$$

it remains to show that

$$\limsup_{\varrho\to 0}\limsup_{j\to\infty} \fint_{B_{\varrho/2}(\bar{x})} \chi_{\{|\bar{u}_j-\ell_\varrho|<\varrho\}}|D\bar{u}_j - D\ell_\varrho|\,dx = 0. \tag{4.45}$$

For (4.45), consider $\phi \in C_0^\infty(B_\varrho(\bar{x}))$ with $0 \le \phi \le 1$, $\phi \equiv 1$ on $B_{\varrho/2}(\bar{x})$ and $|D\phi| \le 4/\varrho$. Let $\eta := \phi T_\varrho(\bar{u}_j - \ell_\varrho)$, where the truncation operator T_ϱ has been introduced in (4.8). Then, also recalling (4.9), we have

$$\begin{aligned}
&(|D\bar{u}_j|^{p-2}D\bar{u}_j - |D\ell_\varrho|^{p-2}D\ell_\varrho) : D\eta \\
&\quad = \chi_{\{|\bar{u}_j-\ell_\varrho|<\varrho\}}[(|D\bar{u}_j|^{p-2}D\bar{u}_j - |D\ell_\varrho|^{p-2}D\ell_\varrho) : D(\bar{u}_j - \ell_\varrho)]\phi \\
&\qquad + \frac{\varrho\chi_{\{|\bar{u}_j-\ell_\varrho|\ge\varrho\}}}{|\bar{u}_j - \ell_\varrho|}[(|D\bar{u}_j|^{p-2}D\bar{u}_j - |D\ell_\varrho|^{p-2}D\ell_\varrho) : (\mathrm{Id} - P_j)D(\bar{u}_j - \ell_\varrho)]\phi \\
&\qquad + (|D\bar{u}_j|^{p-2}D\bar{u}_j - |D\ell_\varrho|^{p-2}D\ell_\varrho) : [T_\varrho(\bar{u}_j - \ell_\varrho) \otimes D\phi] \\
&\quad =: G^1_{j,\varrho}(x) + G^2_{j,\varrho}(x) + G^3_{j,\varrho}(x).
\end{aligned}$$

Here and in what follows, P_j and P are defined by

$$P_j := \frac{(\bar{u}_j - \ell_\varrho)\otimes(\bar{u}_j - \ell_\varrho)}{|\bar{u}_j - \ell_\varrho|^2}, \qquad P := \frac{(\tilde{u} - \ell_\varrho)\otimes(\tilde{u} - \ell_\varrho)}{|\tilde{u} - \ell_\varrho|^2}$$

when $|\bar{u}_j - \ell_\varrho| \ne 0$ and $|\tilde{u} - \ell_\varrho| \ne 0$, respectively. Observe also that by the definition of ℓ_ϱ in (4.39) it follows that

$$\int_B |D\ell_\varrho|^{p-2}D\ell_\varrho : D\varphi\,dx = 0 \qquad \forall \varphi \in W_0^{1,1}(B;\mathbb{R}^N)$$

whenever $B \subset B_1$ is a ball, since ℓ_ϱ is affine. By (4.28) and the previous display applied to $\varphi \equiv \eta$, we find

$$0 \le \fint_{B_\varrho(\bar{x})} G^1_{j,\varrho}(x)\,dx \le 2^{-j}\varrho^{1-n} - \fint_{B_\varrho(\bar{x})} G^2_{j,\varrho}(x)\,dx - \fint_{B_\varrho(\bar{x})} G^3_{j,\varrho}(x)\,dx. \tag{4.46}$$

Notice that the positivity of the first integral above follows from the monotonicity of the vector field $z \mapsto |z|^{p-2}z$ (see also (4.51) below). We now estimate the various terms on the right hand side, starting from $G^3_{j,\varrho}$. We have

$$\lim_{j\to\infty} \fint_{B_\varrho(\bar{x})} G^3_{j,\varrho}(x)\,dx = \fint_{B_\varrho(\bar{x})} (b - |D\ell_\varrho|^{p-2}D\ell_\varrho) : [T_\varrho(\tilde{u}-\ell_\varrho)\otimes D\phi]\,dx$$

by the weak convergence of $|D\bar{u}_j|^{p-2}D\bar{u}_j$ to b, the equiboundedness of $|D\bar{u}_j|^{p-2}D\bar{u}_j$ in $L^{q/(p-1)}$ and the strong convergence of $\bar{u}_j$ to $\tilde{u}$, as described in (4.32). We further find by Hölder's inequality that

$$\begin{aligned}
&\left|\fint_{B_\varrho(\bar{x})} (b - |D\ell_\varrho|^{p-2}D\ell_\varrho) : [T_\varrho(\tilde{u}-\ell_\varrho)\otimes D\phi]\,dx\right| \\
&\qquad \le c\left(\fint_{B_\varrho(\bar{x})} \left(|b-b(\bar{x})|^{q/(p-1)} + |b(\bar{x})|^{q/(p-1)} + |D\tilde{u}(\bar{x})|^q\right)dx\right)^{(p-1)/q} \\
&\qquad\quad \cdot \left(\fint_{B_\varrho(\bar{x})} \left(\frac{\min\{\varrho, |\tilde{u}-\ell_\varrho|\}}{\varrho}\right)^{q/(q-(p-1))} dx\right)^{1-(p-1)/q}.
\end{aligned}$$

The first integral on the right stays bounded and the second one tends to zero as $\varrho \to 0$ by (4.40); indeed, since $p > q_{\mathrm{m}} > q$ implies $q/(q-(p-1)) > q$, we have

$$\begin{aligned}
\fint_{B_\varrho(\bar{x})} \left(\frac{\min\{\varrho, |\tilde{u}-\ell_\varrho|\}}{\varrho}\right)^{q/(q-(p-1))} dx &\le \fint_{B_\varrho(\bar{x})} \left(\frac{\min\{\varrho, |\tilde{u}-\ell_\varrho|\}}{\varrho}\right)^{q} dx \\
&\le \fint_{B_\varrho(\bar{x})} \left|\frac{\tilde{u}-\ell_\varrho}{\varrho}\right|^q dx \xrightarrow{\varrho\to 0} 0.
\end{aligned}$$

Therefore

$$\lim_{\varrho\to 0}\lim_{j\to\infty}\left|\fint_{B_\varrho(\bar{x})} G^3_{j,\varrho}(x)\,dx\right| = 0. \tag{4.47}$$

We then focus on $G^2_{j,\varrho}(x)$. For this we use $D\bar{u}_j : [(\mathrm{Id}-P_j)D\bar{u}_j] \ge 0$ (see (4.11)), so that

$$\begin{aligned}
&(|D\bar{u}_j|^{p-2}D\bar{u}_j - |D\ell_\varrho|^{p-2}D\ell_\varrho) : (\mathrm{Id}-P_j)D(\bar{u}_j-\ell_\varrho) \\
&\quad \ge -|D\bar{u}_j|^{p-2}D\bar{u}_j : (\mathrm{Id}-P_j)D\ell_\varrho - |D\ell_\varrho|^{p-2}D\ell_\varrho : (\mathrm{Id}-P_j)D(\bar{u}_j-\ell_\varrho).
\end{aligned} \tag{4.48}$$

We have $\chi_{\{|\bar{u}_j-\ell_\varrho|\ge\varrho\}}P_j \to \chi_{\{|\tilde{u}-\ell_\varrho|\ge\varrho\}}P$ almost everywhere and therefore strongly in $L^t(B_{3/4})$ for every $t \ge 1$; the same holds for the (uniformly bounded) functions $\chi_{\{|\bar{u}_j-\ell_\varrho|\ge\varrho\}}|\bar{u}_j-\ell_\varrho|^{-1}$, which converge to $\chi_{\{|\tilde{u}-\ell_\varrho|\ge\varrho\}}|\tilde{u}-\ell_\varrho|^{-1}$. Using these facts and (4.48), by (4.32) we obtain

$$\begin{aligned}
\limsup_{j\to\infty}\left(-\fint_{B_\varrho(\bar{x})} G^2_{j,\varrho}(x)\,dx\right) &\le \fint_{B_\varrho(\bar{x})} b : (\mathrm{Id}-P)D\ell_\varrho \frac{\varrho\chi_{\{|\tilde{u}-\ell_\varrho|\ge\varrho\}}}{|\tilde{u}-\ell_\varrho|}\,dx \\
&\quad + \fint_{B_\varrho(\bar{x})} |D\ell_\varrho|^{p-2}D\ell_\varrho : (\mathrm{Id}-P)D(\tilde{u}-\ell_\varrho)\frac{\varrho\chi_{\{|\tilde{u}-\ell_\varrho|\ge\varrho\}}}{|\tilde{u}-\ell_\varrho|}\,dx.
\end{aligned} \tag{4.49}$$

We now estimate the two terms on the right hand side. Since $q > p-1$ we can find $t > 1$ such that $q(t-1)/(q-p+1) \leq q$; therefore

$$\begin{aligned}\left|\fint_{B_\varrho(\bar{x})} b : (\mathrm{Id}-P)D\ell_\varrho \frac{\varrho\chi_{\{|\tilde{u}-\ell_\varrho|\geq\varrho\}}}{|\tilde{u}-\ell_\varrho|}\,dx\right| &\leq c\fint_{B_\varrho(\bar{x})} |b|\left|\frac{\tilde{u}-\ell_\varrho}{\varrho}\right|^{t-1} dx\\ &\leq c\left(\fint_{B_\varrho(\bar{x})} |b|^{q/(p-1)}\,dx\right)^{(p-1)/q}\left(\fint_{B_\varrho(\bar{x})}\left|\frac{\tilde{u}-\ell_\varrho}{\varrho}\right|^q dx\right)^{(t-1)/q}.\end{aligned}$$

As for the remaining integral in (4.49), we have

$$\begin{aligned}&\left|\fint_{B_\varrho(\bar{x})} |D\ell_\varrho|^{p-2}D\ell_\varrho : (\mathrm{Id}-P)D(\tilde{u}-\ell_\varrho)\frac{\varrho\chi_{\{|\tilde{u}-\ell_\varrho|\geq\varrho\}}}{|\tilde{u}-\ell_\varrho|}\,dx\right|\\ &\qquad\leq c\fint_{B_\varrho(\bar{x})} |D(\tilde{u}-\ell_\varrho)|\left|\frac{\tilde{u}-\ell_\varrho}{\varrho}\right|^{q-1} dx\\ &\qquad\leq c\left(\fint_{B_\varrho(\bar{x})} |D\tilde{u}-D\tilde{u}(\bar{x})|^q\,dx\right)^{1/q}\left(\fint_{B_\varrho(\bar{x})}\left|\frac{\tilde{u}-\ell_\varrho}{\varrho}\right|^q dx\right)^{1-1/q}.\end{aligned}$$

By using (4.40), the last three displays lead to

$$\limsup_{\varrho\to 0}\limsup_{j\to\infty}\left(-\fint_{B_\varrho(\bar{x})} G^2_{j,\varrho}(x)\,dx\right) \leq 0.$$

Recalling also (4.46) and (4.47) we conclude that

$$\limsup_{\varrho\to 0}\limsup_{j\to\infty}\fint_{B_\varrho(\bar{x})} G^1_{j,\varrho}(x)\,dx = 0. \tag{4.50}$$

We proceed with the proof of (4.45); we are going to use the following, well-known inequality (see for instance [51] and the references therein):

$$(|z_2|^{p-2}z_2-|z_1|^{p-2}z_1) : (z_2-z_1) \geq \frac{1}{c}(|z_2|^2+|z_1|^2)^{(p-2)/2}|z_2-z_1|^2 \tag{4.51}$$

with a constant $c \equiv c(n,N,p) \geq 1$, and therefore from (4.50) it follows that

$$\limsup_{\varrho\to 0}\limsup_{j\to\infty}\fint_{B_\varrho(\bar{x})} \chi_{\{|\bar{u}_j-\ell_\varrho|<\varrho\}}(|D\bar{u}_j|+|D\ell_\varrho|)^{p-2}|D\bar{u}_j-D\ell_\varrho|^2\phi\,dx = 0. \tag{4.52}$$

Now, as $p \geq 2$ and recalling that $\phi \equiv 1$ on $B_{\varrho/2}$, it follows that

$$\fint_{B_\varrho(\bar{x})} G^1_{j,\varrho}(x)\,dx \geq \frac{1}{c}\fint_{B_{\varrho/2}(\bar{x})} \chi_{\{|\bar{u}_j-\ell_\varrho|<\varrho\}}|D\bar{u}_j-D\ell_\varrho|^p\,dx \geq 0,$$

proving also (4.45) by Hölder's inequality in the case $p \geq 2$. For $2-1/n < p < 2$ we instead have, again by Hölder's inequality,

$$\begin{aligned}&\fint_{B_{\varrho/2}(\bar{x})} \chi_{\{|\bar{u}_j-\ell_\varrho|<\varrho\}}|D\bar{u}_j - D\ell_\varrho|\,dx\\&\qquad\qquad \leq \left(\fint_{B_{\varrho/2}(\bar{x})} \chi_{\{|\bar{u}_j-\ell_\varrho|<\varrho\}}(|D\bar{u}_j| + |D\ell_\varrho|)^{p-2}|D\bar{u}_j - D\ell_\varrho|^2\,dx\right)^{1/2}\\&\qquad\qquad\quad \cdot\left(\fint_{B_{\varrho/2}(\bar{x})} (|D\bar{u}_j| + |D\ell_\varrho|)^{2-p}\,dx\right)^{1/2},\end{aligned}$$

and therefore, by (4.52), and recalling that $\bar{x}$ is a Lebesgue point of $D\tilde{u}$, we again deduce (4.45), which is thus established in the full range $p > 2 - 1/n$. This shows that $h = 0$ (h has been defined in (4.41)), thereby establishing (4.34).

Step 5: p-harmonicity of the limit map and contradiction. In the final step we prove that the function $\tilde{u}$ obtained in the previous step is p-harmonic. Indeed, $D\tilde{u} \in L^p(B_{1/2})$ would then imply that $\tilde{u}$ is a p-harmonic map by (4.35). Since (4.33) holds, we may take $\bar{v} = \tilde{u}$ in (4.29) as a test function, thereby obtaining a contradiction after taking j large enough using (4.34); notice that bounds in (4.30) are satisfied with $\bar{v} = \tilde{u}$. The final goal is hence to show that $D\tilde{u} \in L^p(B_{1/2})$, and this will finish the proof. In order to achieve this, let us write (4.22) for $\bar{u} \equiv \bar{u}_j$; by lower semicontinuity we then get

$$\int_{B_{7/8}} |\tilde{u}|^{\bar{q}} \leq c, \quad \bar{q} < q_0 := \begin{cases} \dfrac{n(p-1)}{n-p}, & p < n,\\ \text{any number larger than } p, & p \geq n, \end{cases} \tag{4.53}$$

with $c \equiv c(n, N, p, \bar{q})$. Similarly, we consider (4.12) with $\bar{u} \equiv \bar{u}_j$ and $\delta = 2^{-j}$. Letting $j \to \infty$ in the resulting inequality, and using Fatou's lemma to handle the convergence on the left hand side and (4.34) on the right hand side, we arrive at

$$\begin{aligned}\int_{B_{3/4}\cap\{|\tilde{u}|<t\}} |D\tilde{u}|^p\phi^p\,dx &\leq c\int_{B_{3/4}\cap\{|\tilde{u}|<t\}} |\tilde{u}|^p|D\phi|^p\,dx\\&\quad + ct\int_{B_{3/4}\cap\{|\tilde{u}|\geq t\}} |D\tilde{u}|^{p-1}|D\phi|\phi^{p-1}\,dx\end{aligned}$$

for all $t > 0$ and $\phi \in C_0^\infty(B_{3/4})$ with $\phi \geq 0$, and $c \equiv c(n, N, p)$. We then multiply the inequality above by $(1+t)^{-1-\gamma}$, $\gamma > 0$, and integrate over $(0, \infty)$ with respect to t. Manipulations based on Fubini's theorem similar to those in Step 2 lead to

$$\begin{aligned}\frac{1}{\gamma}\int_{B_{3/4}} \frac{|D\tilde{u}|^p\phi^p}{(1+|\tilde{u}|)^\gamma}\,dx &\leq \frac{c}{\gamma}\int_{B_{3/4}} (1+|\tilde{u}|)^{p-\gamma}|D\phi|^p\,dx\\&\quad + c\int_0^\infty \frac{1}{(1+t)^\gamma}\int_{B_{3/4}\cap\{|\tilde{u}|\geq t\}} |D\tilde{u}|^{p-1}|D\phi|\phi^{p-1}\,dx\,dt\end{aligned}$$

with $c \equiv c(p)$. Considering $\gamma \in (0, 1)$, again by Fubini's theorem and Young's inequality we obtain

$$
\begin{aligned}
c\int_0^{\infty}\frac{1}{(1+t)^{\gamma}}&\int_{B_{3/4}\cap\{|\tilde u|\ge t\}}|D\tilde u|^{p-1}|D\phi|\phi^{p-1}\,dx\,dt\\
&\le \frac{c}{1-\gamma}\int_{B_{3/4}}|D\tilde u|^{p-1}(1+|\tilde u|)^{1-\gamma}|D\phi|\phi^{p-1}\,dx\\
&\le \frac{1}{2\gamma}\int_{B_{3/4}}\frac{|D\tilde u|^{p}\phi^{p}}{(1+|\tilde u|)^{\gamma}}\,dx+\frac{c\gamma^{p-1}}{(1-\gamma)^{p}}\int_{B_{3/4}}(1+|\tilde u|)^{p-\gamma}|D\phi|^{p}\,dx.
\end{aligned}
$$

Combining the last two displays, and reabsorbing terms, finally yields

$$
\int_{B_{3/4}}\frac{|D\tilde u|^{p}\phi^{p}}{(1+|\tilde u|)^{\gamma}}\,dx\le\frac{c}{(1-\gamma)^{p}}\int_{B_{3/4}}(1+|\tilde u|)^{p-\gamma}|D\phi|^{p}\,dx \tag{4.54}
$$

with $c\equiv c(p)$. The idea of proving the L^p-integrability of Du is now as follows. Note first that the constant c in the last display is independent of $\gamma\in(0,1)$. We may thus let $\gamma\to 0$ provided the right hand side of (4.54) remains finite. This will be achieved via a finite iteration scheme that resembles Moser's iteration; the only difference is that we stop after a number of iterations. As a matter of fact, in the case $n<p^2$ by (4.53) we have $u\in L^p(B_{7/8})$; therefore letting $\gamma\to 0$ in (4.54) and choosing ϕ properly yields $Du\in L^p(B_{1/2})$ and we are done. We can therefore confine ourselves to $p^2\le n$, where we again observe that by (4.53) the right hand side in (4.54) is finite for $\gamma>(n-p^2)/(n-p)$; notice that this last quantity is smaller than 1. Set $\tilde g=1+|\tilde u|$. Since

$$
|D\tilde g^{(p-\gamma)/p}|^{p}\le(1-\gamma/p)^{p}|D\tilde u|^{p}\tilde g^{-\gamma},
$$

(4.54) implies that

$$
\int_{B_{3/4}}|D(\tilde g^{(p-\gamma)/p}\phi)|^{p}\,dx\le\frac{c}{(1-\gamma)^{p}}\int_{B_{3/4}}\tilde g^{p-\gamma}|D\phi|^{p}\,dx \tag{4.55}
$$

with $c\equiv c(p)$, provided that $\gamma\in(0,1)$. For $\theta=n/(n-p)=p^*/p$ we apply Sobolev's inequality to get

$$
\left(\int_{B_{3/4}}(\tilde g^{1-\gamma/p}\phi)^{\theta p}\,dx\right)^{1/\theta}\le c\int_{B_{3/4}}|D(\tilde g^{1-\gamma/p}\phi)|^{p}\,dx,
$$

which, in conjunction with (4.55), leads to

$$
\left(\int_{B_{3/4}}(\tilde g^{1-\gamma/p}\phi)^{\theta p}\,dx\right)^{1/\theta}\le\frac{c}{(1-\gamma)^{p}}\int_{B_{3/4}}\tilde g^{p-\gamma}|D\phi|^{p}\,dx. \tag{4.56}
$$

We have thus improved the integrability of $\tilde g$ on $\{\phi\equiv 1\}\cap B_{3/4}$. We now iterate this argument: we take a fixed $\gamma_0:=\gamma>(n-p^2)/(n-p)$ with $\gamma<1$, and define

$$
q_j:=\theta^j(p-\gamma_0)\quad\text{and}\quad\gamma_j:=p-q_j
$$

for every integer $j\ge 0$; we have $q_{j+1}:=\theta q_j$. We also notice that $\{\gamma_j\}$ is a decreasing sequence. We then choose a shrinking sequence $\{B^j\}$ of concentric balls centered at the origin such that

$$
B^{j+1}\Subset B^j\quad\forall j\quad\text{and}\quad B_{5/8}\subset\bigcap_j B^j,
$$

and related cut-off functions $\{\phi_j\} \subset C_0^\infty(B^j)$ such that $0 \le \phi_j \le 1$, $\phi_{j+1} \le \phi_j$, and $\phi_j \equiv 1$ on B_{j+1} for every j. Using (4.56) with the obvious choices leads to

$$\left(\int_{B_{3/4}} \tilde{g}^{\theta(p-\gamma_j)}\phi_j^{\theta p}\,dx\right)^{1/\theta} \le c\int_{B_{3/4}} \tilde{g}^{p-\gamma_j}|D\phi_j|^p\,dx$$

whenever $\gamma_j > 0$. Therefore $u \in L^{q_j}(B^j)$ implies $u \in L^{q_{j+1}}(B^{j+1})$ for every positive integer j such that $\gamma_j > 0$. We proceed inductively and the iteration stops at the first step $\bar{j}$ such that $\gamma_{\bar{j}+1} := p - q_{\bar{j}+1} \le 0$, when the inequality in the last display yields $u \in L^{q_{\bar{j}+1}}(B^{\bar{j}+1})$. It then follows in particular that $\tilde{u} \in L^p(B_{5/8})$. With this information at hand we go back to (4.54), letting $\gamma \to 0$, which gives, upon a suitable choice of ϕ, that $D\tilde{u} \in L^p(B_{1/2})$. This finishes the proof of Theorem 4.1 as seen at the beginning of Step 5. □

5. Degenerate p-linearization

In this section we derive two corollaries of Theorem 4.1. The central result is Proposition 5.1 below, where we exploit the consequence of a degeneracy condition (5.7). The condition tells us that the gradient is suitably small provided it is measured in the right scale dictated by the excess functional.

Lemma 5.1. *Let $u \in W^{1,p}(B_r)$ be a weak solution to (1.1) in $B_r \equiv B_r(x_0)$ with $p > 2 - 1/n$, where $\mu \in C^\infty(B_r)$. Then, for every $\bar{q} \in (1, q_{\rm m})$ with $q_{\rm m}$ defined in (4.2), there exists a constant $c_{\rm h} \equiv c_{\rm h}(n, N, p, \bar{q})$ such that*

$$\left(\fint_{B_{r/2}} |Du|^q\,dx\right)^{1/q} \le c_{\rm h}\fint_{B_r} |Du|\,dx + c_{\rm h}\left[\frac{|\mu|(B_r)}{r^{n-1}}\right]^{1/(p-1)} \tag{5.1}$$

whenever $q \in [1, \bar{q}]$.

Proof. It is obviously sufficient to prove (5.1) for $q = \bar{q}$, the remaining cases then following by the Jensen inequality. Let $\varepsilon > 0$ be a free parameter for the moment, and set

$$\lambda := \frac{1}{\varepsilon}\fint_{B_r} |Du|\,dx + \left[\frac{2^n r}{\delta}\,\frac{|\mu|(B_r)}{|B_r|}\right]^{1/(p-1)}, \tag{5.2}$$

where $\delta \equiv \delta(n, N, p, \bar{q}, \varepsilon)$ corresponds to the constant in Theorem 4.1 with parameters ε and $M = 1$; notice that we can assume that $\lambda > 0$ since otherwise the statement becomes trivial. We use a scaling argument with

$$\bar{u} := \frac{u - (u)_{B_r}}{\lambda}, \quad \bar{\mu} := \frac{\mu}{\lambda^{p-1}}, \quad -\Delta_p \bar{u} = \bar{\mu} \quad \text{in } B_r. \tag{5.3}$$

Then for $y \in B_{r/2}(x_0)$ we have

$$\left|\fint_{B_{r/2}(y)} |D\bar{u}|^{p-2}D\bar{u} : D\varphi\,dx\right| \le \frac{2^n\|\varphi\|_{L^\infty(B_{r/2}(y))}}{\lambda^{p-1}}\,\frac{|\mu|(B_r)}{|B_r|} \le \frac{\delta}{r}\|\varphi\|_{L^\infty(B_{r/2}(y))}$$

whenever $\varphi \in W^{1,p}_0(B_{r/2}(y)) \cap L^\infty(B_{r/2}(y))$. Scaling gives

$$\fint_{B_r} |D\bar{u}|\,dx = \frac{1}{\lambda} \fint_{B_r} |Du|\,dx \le \varepsilon. \tag{5.4}$$

Moreover, Poincaré's inequality implies

$$\fint_{B_r} |\bar{u}|\,dx \le \frac{c(n)r}{\lambda} \fint_{B_r} |Du|\,dx$$

for a constant $c(n)$ depending indeed only on n, therefore

$$\fint_{B_{r/2}(y)} |\bar{u}|\,dx \le 2^n \fint_{B_r} |\bar{u}|\,dx \le c(n) r\varepsilon = \frac{r}{2}$$

provided $\varepsilon := 1/[2c(n)]$. This fixes ε and therefore also δ, which is now a function of $n, N, p, \bar{q}$. Theorem 4.1 then implies that there exists a p-harmonic map $\bar{v} \equiv \bar{v}_y$ in $B_{r/4}(y)$ such that

$$\left(\fint_{B_{r/4}(y)} |D\bar{u} - D\bar{v}|^{\bar{q}}\,dx\right)^{1/\bar{q}} \le \varepsilon. \tag{5.5}$$

Furthermore, by the Lipschitz estimate for $D\bar{v}$ (see (3.6)), and by (5.4) and (5.5),

$$\begin{aligned}\sup_{B_{r/8}(y)} |D\bar{v}| &\le c \fint_{B_{r/4}(y)} |D\bar{v}|\,dx\\ &\le c \fint_{B_{r/4}(y)} |D\bar{u} - D\bar{v}|\,dx + c \fint_{B_{r/4}(y)} |D\bar{u}|\,dx \le 2c\varepsilon.\end{aligned}$$

This translates to $D\bar{u}$ via (5.5) and the triangle inequality:

$$\left(\fint_{B_{r/8}(y)} |D\bar{u}|^{\bar{q}}\,dx\right)^{1/\bar{q}} \le \sup_{B_{r/8}(y)} |D\bar{v}| + \left(\fint_{B_{r/8}(y)} |D\bar{u} - D\bar{v}|^{\bar{q}}\,dx\right)^{1/\bar{q}} \le c\varepsilon.$$

A simple covering argument then gives

$$\left(\fint_{B_{r/2}(x_0)} |D\bar{u}|^{\bar{q}}\,dx\right)^{1/\bar{q}} \le c\varepsilon,$$

and (5.1) follows by recalling the definitions in (5.2)–(5.3), i.e. scaling back to u. □

Lemma 5.2. *Let $u \in W^{1,p}(B_r)$ be a weak solution to (1.1) in $B_r \equiv B_r(x_0)$ with $p > 2 - 1/n$, where $\mu \in C^\infty(B_r)$. Let $q \in (1, q_{\mathrm{m}})$ with q_{m} defined in (4.2), and $\varepsilon \in (0,1)$. There exists a positive constant $c_{\mathrm{s}} \equiv c_{\mathrm{s}}(n, N, p, q, \varepsilon)$ and a p-harmonic map v in $B_{r/2}$ such that*

$$\left(\fint_{B_{r/2}} |Du - Dv|^q\,dx\right)^{1/q} \le \frac{\varepsilon}{r} \fint_{B_r} |u - (u)_{B_r}|\,dx + c_{\mathrm{s}} \left[\frac{|\mu|(B_r)}{r^{n-1}}\right]^{1/(p-1)}. \tag{5.6}$$

Proof. We rescale u as in (5.3), this time with

$$\lambda := \frac{1}{r} \fint_{B_r} |u-(u)_{B_r}|\,dx + \left[\frac{r}{\delta}\,\frac{|\mu|(B_r)}{|B_r|}\right]^{1/(p-1)},$$

where $\delta \equiv \delta(n, N, p, q, \varepsilon)$ corresponds to the parameter δ in Theorem 4.1 with $M \equiv 1$, and $\varepsilon \in (0,1)$ is the number appearing on the statement of the lemma; again we can assume that $\lambda > 0$ since otherwise u is constant and we deduce the conclusion with $v = u$. As in the proof of the previous lemma we have

$$\left|\fint_{B_r} |D\bar{u}|^{p-2} D\bar{u} : D\varphi\,dx\right| \le \frac{\|\varphi\|_{L^\infty(B_r}}{\lambda^{p-1}}\,\frac{|\mu|(B_r)}{|B_r|} \le \frac{\delta}{r}\|\varphi\|_{L^\infty(B_r)},$$

while by the very definition of $\bar{u}$ and λ it follows that $\fint_{B_r} |\bar{u}|\,dx \le r$. Therefore Theorem 4.1 applies to $\bar{u}$ and provides a p-harmonic map $\bar{v}$ in $B_{r/2}$ such that

$$\left(\fint_{B_{r/2}} |D\bar{u} - D\bar{v}|^q\,dx\right)^{1/q} \le \varepsilon.$$

Inequality (5.6) now follows by scaling back to u with $v = \lambda\bar{v}$, which is of course still p-harmonic. □

The next proposition is finally the degenerate p-harmonic approximation result the title of this section is alluding to.

Proposition 5.1. *Let $u \in W^{1,p}(B_r)$ be a weak solution to (1.1) in $B_r \equiv B_r(x_0)$ with $p > 2 - 1/n$, where $\mu \in C^\infty(B_r)$. Let $q \in (1, q_{\rm m})$ with $q_{\rm m}$ defined in (4.2), and $\varepsilon, \theta \in (0,1)$. There exists a positive constant $c_{\rm d} \equiv c_{\rm d}(n, N, p, q, \varepsilon, \theta)$ such that if*

$$\fint_{B_r} |Du - (Du)_{B_r}|\,dx \ge \theta |(Du)_{B_r}| \tag{5.7}$$

then there exists a p-harmonic map v in $B_{r/2}$, that is, a solution to

$$-\triangle_p v = 0 \quad \text{in } B_{r/2},$$

satisfying

$$\left(\fint_{B_{r/2}} |Du - Dv|^q\,dx\right)^{1/q} \le \varepsilon \fint_{B_r} |Du-(Du)_{B_r}|\,dx + c_{\rm d}\left[\frac{|\mu|(B_r)}{r^{n-1}}\right]^{1/(p-1)}. \tag{5.8}$$

Proof. We notice that (5.7) implies

$$\fint_{B_r} |Du|\,dx \le \fint_{B_r} |Du-(Du)_{B_r}|\,dx + |(Du)_{B_r}| \le \frac{1+\theta}{\theta}\fint_{B_r} |Du-(Du)_{B_r}|\,dx,$$

and Poincaré's inequality gives

$$\fint_{B_r} |u-(u)_{B_r}|\,dx \le c(n) r \fint_{B_r} |Du|\,dx \le \frac{c(n)(1+\theta)r}{\theta}\fint_{B_r} |Du-(Du)_{B_r}|\,dx.$$

We then proceed as in the proof of Lemma 5.2. Indeed, we scale as in (5.3), this time with

$$\lambda := \frac{c(n)(1+\theta)}{\theta}\fint_{B_r} |Du-(Du)_{B_r}|\,dx + \left[\frac{r}{\delta}\,\frac{|\mu|(B_r)}{|B_r|}\right]^{1/(p-1)},$$

where $\delta \equiv \delta(n, N, p, q, \varepsilon, \theta)$ corresponds to the parameter δ in Theorem 4.1 with

$$M \leftrightarrow 1 \quad \text{and} \quad \varepsilon \leftrightarrow \frac{\varepsilon\theta}{c(n)(1+\theta)}.$$

In the previous line, the last ε appearing on the right hand side is the one that will eventually appear in (5.8) and that comes from the statement of the proposition. Notice that again we may assume that $\lambda > 0$, otherwise Du is constant and we can take $v = u$. Applying Theorem 4.1 then gives the existence of a p-harmonic map $\bar{v}$ in $B_{r/2}$ such that

$$\left(\fint_{B_{r/2}} |D\bar{u} - D\bar{v}|^q \, dx\right)^{1/q} \le \frac{\varepsilon\theta}{c(n)(1+\theta)}.$$

Then (5.8) follows by scaling back to u and letting $v := \lambda v$; notice that this time

$$c_{\mathrm{d}} = \frac{\varepsilon\theta}{c(n)(1+\theta)\delta^{1/(p-1)}}$$

with dangerous dependence on θ now incorporated in δ. □

6. A pre-reverse Hölder inequality on level sets

In this section we prove Lemma 6.1 below; it involves a level set consequence of the reverse Hölder inequality (5.1). The proof is based on a Calderón–Zygmund type exit time argument and on Vitali's covering lemma. The "pre-reverse" terminology refers to the fact that, in the usual energy setting, inequalities like (6.2) below are used to prove reverse Hölder type inequalities. In the present, subenergy setting, we take an inverse path: we start from the reverse Hölder type inequality of Lemma 5.1 and derive level set inequalities as in (6.2) below.

Lemma 6.1. *Let $u \in W^{1,p}(B_r)$ be a weak solution to (1.1) in $B_r \equiv B_r(x_0)$ with $p > 2 - 1/n$, where $\mu \in C^\infty(B_r)$. Then, for every $\bar{q} \in (1, q_{\mathrm{m}})$ with q_{m} defined in (4.2), there exists a constant $c_* \equiv c_*(n, N, p, \bar{q})$ such that if*

$$t > 20^n c_{\mathrm{h}} \fint_{B_r} |Du| \, dx + 20^n c_{\mathrm{h}} \left[\frac{|\mu|(B_r)}{r^{n-1}}\right]^{1/(p-1)}, \tag{6.1}$$

where $c_{\mathrm{h}} \equiv c_{\mathrm{h}}(n, N, p, \bar{q})$ is the constant appearing in (5.1), then

$$\int_{B_{r/2}\cap\{|Du|>t\}} |Du|^\gamma \, dx \le c_* t^{\gamma-1} \int_{B_r\cap\{|Du|>t/c_*\}} |Du| \, dx + c_* t^{\gamma+1-p} r |\mu|(B_r) \tag{6.2}$$

whenever $\gamma \in [1, \bar{q}]$.

Proof. Let t be as in (6.1), and consider the level set $E_t := B_{r/2} \cap \{|Du| > t\}$. Then, by classical Lebesgue theory, for almost every $y \in E_t$ we have

$$\lim_{\varrho\to 0}\left(\fint_{B_\varrho(y)} |Du|^\gamma \, dx\right)^{1/\gamma} = |Du(y)| > t.$$

Moreover, using the reverse Hölder inequality of Lemma 5.1 we get

$$\left(\fint_{B_{r/20}(y)} |Du|^\gamma \, dx\right)^{1/\gamma} \le c_{\rm h} \fint_{B_{r/10}} |Du|\, dx + c_{\rm h}\left[\frac{|\mu|(B_{r/10})}{(r/10)^{n-1}}\right]^{1/(p-1)}$$
$$\le 10^n c_{\rm h} \fint_{B_r} |Du|\, dx + 10^{\frac{n-1}{p-1}} c_{\rm h}\left[\frac{|\mu|(B_r)}{r^{n-1}}\right]^{1/(p-1)}.$$

Estimating the right hand side by means of (6.1) we have

$$\left(\fint_{B_{r/20}(y)} |Du|^\gamma \, dx\right)^{1/\gamma} < t.$$

Therefore for almost every $y \in E_t$ we find an exit time radius $r_y \in (0, r/20)$ such that

$$\left(\fint_{B_{r_y}(y)} |Du|^\gamma \, dx\right)^{1/\gamma} = t, \qquad \max_{\varrho\in[r_y, r/20]}\left(\fint_{B_\varrho(y)} |Du|^\gamma \, dx\right)^{1/\gamma} \le t. \tag{6.3}$$

Applying again the reverse Hölder inequality of Lemma 5.1 then gives

$$t = \left(\fint_{B_{r_y}(y)} |Du|^\gamma \, dx\right)^{1/\gamma} \le c_{\rm h} \fint_{B_{2r_y}(y)} |Du|\, dx + c_{\rm h}\left[\frac{|\mu|(B_{2r_y}(y))}{r_y^{n-1}}\right]^{1/(p-1)}.$$

Therefore at least one of the two inequalities holds:

$$\frac{t}{2} \le c_{\rm h} \fint_{B_{2r_y}(y)} |Du|\, dx, \qquad \frac{t}{2} \le c_{\rm h}\left[\frac{|\mu|(B_{2r_y}(y))}{r_y^{n-1}}\right]^{1/(p-1)}. \tag{6.4}$$

In case the first inequality holds we use

$$\frac{t}{2} \le c_{\rm h} \fint_{B_{2r_y}(y)} |Du|\, dx \le \frac{c_{\rm h}}{|B_{2r_y}(y)|} \int_{B_{2r_y}(y)\cap\{|Du|>t/(4c_{\rm h})\}} |Du|\, dx + \frac{t}{4}$$

to obtain

$$|B_{2r_y}(y)| \le \frac{4c_{\rm h}}{t} \int_{B_{2r_y}(y)\cap\{|Du|>t/(4c_{\rm h})\}} |Du|\, dx.$$

In case the second inequality in (6.4) holds we have

$$|B_{2r_y}(y)| \le (2c_{\rm h}/t)^{p-1} r_y |\mu|(B_{2r_y}(y)).$$

Combining both cases we conclude that

$$|B_{2r_y}(y)| \le \frac{4c_{\rm h}}{t} \int_{B_{2r_y}(y)\cap\{|Du|>t/(4c_{\rm h})\}} |Du|\, dx + \left(\frac{2c_{\rm h}}{t}\right)^{p-1} r_y |\mu|(B_{2r_y}(y)). \tag{6.5}$$

We next appeal to the classical Vitali covering theorem to find a countable family $\{B_{2r_{y_j}}(y_j)\}$ of disjoint balls satisfying

$$E_t \subset \left(\bigcup_j B_{10r_{y_j}}(y_j)\right) \cup \text{negligible set}. \tag{6.6}$$

Applying (6.5) to each ball $B_{2r_{y_j}}(y_j)$, and recalling also (6.3), yields

$$\int_{B_{10r_{y_j}}(y_j)} |Du|^\gamma \, dx \le t^\gamma |B_{10r_{y_j}}(y_j)| \le 5^n t^\gamma |B_{2r_{y_j}}(y_j)|$$
$$\le ct^{\gamma-1} \int_{B_{2r_{y_j}}(y_j)\cap\{|Du|>t/(4c)\}} |Du| \, dx + crt^{\gamma+1-p}|\mu|(B_{2r_{y_j}}(y_j))$$

with $c \equiv c(n, N, p, \bar{q})$. Summing the above inequality over j, recalling (6.6) and the fact that the family $\{B_{2r_{y_j}}(y_j)\}$ is disjoint, we get (6.2), upon renaming the constant c. □

7. Nondegenerate linearization

This section is dedicated to implementing the counterpart of Proposition 5.1, that is, Proposition 7.1 below. This features a linearization lemma that holds for solutions u to the p-Laplacean system $-\triangle_p u = \mu$ in case the problem is nondegenerate on a single scale, that is, when the average $(Du)_{B_r} \neq 0$ is large enough in a certain quantitative way, described in (7.1) below.

Proposition 7.1. *Let $u \in W^{1,p}(B_r)$ be a weak solution to the system $-\triangle_p u = \mu$ in $B_r \equiv B_r(x_0)$ with $p > 2$, where $\mu \in C^\infty(B_r)$, and assume that $(Du)_{B_r} \neq 0$. Then for every $\varepsilon \in (0, 1]$ there exist positive constants $\theta_{\rm nd} \equiv \theta_{\rm nd}(n, N, p, \varepsilon) \in (0, 1)$ and $q \equiv q(n, p) \in (1, n/(n-1))$ such that if both*

$$⨍_{B_r} |Du - (Du)_{B_r}| \, dx \le \theta_{\rm nd} |(Du)_{B_r}| \tag{7.1}$$

and

$$\frac{|\mu|(B_r)}{r^{n-1}} \le \theta_{\rm nd} |(Du)_{B_r}|^{p-2} ⨍_{B_r} |Du - (Du)_{B_r}| \, dx, \tag{7.2}$$

then there exists an A-harmonic map $h \in W^{1,2}(B_{r/4})$ in $B_{r/4}$, that is,

$$-\operatorname{div}(A : Dh) = 0 \quad \text{weakly in } B_{r/4}, \qquad A := L((Du)_{B_r}), \tag{7.3}$$

satisfying

$$\left(⨍_{B_{r/4}} |Du - Dh|^q \, dx \right)^{1/q} \le \varepsilon ⨍_{B_r} |Du - (Du)_{B_r}| \, dx. \tag{7.4}$$

The tensor $L(\cdot)$ defining A in (7.3) has been introduced in (2.4).

The proof requires a series of technical lemmas, which are included in Section 7.1. The proof proper will then be given in Section 7.2.

Remark 7.1. The proof will reveal that $\theta_{\rm nd}$ can be chosen as $\theta_{\rm nd} := (\varepsilon/c)^{1/\gamma}$ with $c \equiv c(n, N, p) \ge 1$ and $\gamma \equiv \gamma(n, N, p) \in (0, 1)$.

7.1. Preparatory lemmas

In this subsection we work out preliminary estimates, mainly of technical nature, to be used in the proof of Proposition 7.1. Let us first clarify the setting. We shall deal with a weak solution $\bar{u} \in W^{1,p}(B_1)$ to the p-Laplacean system

$$-\triangle_p \bar{u} = \bar{\mu} \in C^{\infty}(B_1), \qquad p > 2, \tag{7.5}$$

which is such that

$$|(D\bar{u})_{B_1}| = 1 \quad \text{and} \quad |(\bar{u})_{B_1}| = 0. \tag{7.6}$$

We then define the maps $\ell \equiv (\ell^{\alpha})_{1\leq\alpha\leq N}$ and v as

$$\ell^{\alpha}(x) := \langle (D\bar{u}^{\alpha})_{B_1}, x\rangle \quad \text{and} \quad v = \bar{u} - \ell. \tag{7.7}$$

It follows that

$$|(Dv)_{B_1}| = |(v)_{B_1}| = 0. \tag{7.8}$$

Finally, we shall assume that

$$\fint_{B_1} |Dv|\, dx \leq 1 \quad \text{and} \quad |\bar{\mu}|(B_1) \leq 1. \tag{7.9}$$

For $\tau > 1$, we have the following trivial inclusions:

$$\begin{cases} B_1 \cap \{|v| > \tau\} \subset B_1 \cap \{|\bar{u}| > \tau - 1\}, \\ B_1 \cap \{|\bar{u}| > \tau\} \subset B_1 \cap \{|v| > \tau - 1\}, \\ B_1 \cap \{|Dv| > \tau\} \subset B_1 \cap \{|D\bar{u}| > \tau - 1\}, \\ B_1 \cap \{|D\bar{u}| > \tau\} \subset B_1 \cap \{|Dv| > \tau - 1\}. \end{cases} \tag{7.10}$$

We also notice that, as a consequence of (7.6) and (7.9),

$$\fint_{B_1} |D\bar{u}|\, dx \leq \fint_{B_1} |Dv|\, dx + |(D\bar{u})_{B_1}| \leq 2. \tag{7.11}$$

Lemma 7.1. *Let $\bar{u}, \bar{\mu}, v$ be as in (7.5)–(7.9). Let $\tau > 3$ and $q \geq 1$. Then*

$$\int_{B_{7/8}\cap\{3<|v|\leq\tau\}} (|Dv|^2 + |Dv|^p)\, dx \leq c \int_{B_1} |Dv|^q\, dx + c|\bar{\mu}|(B_1)$$

with a constant $c \equiv c(n, N, p, q, \tau)$.

Proof. We take $m \in \mathbb{R}$ such that $m > \frac{1}{2(\tau-1)}$, so $2 + \frac{1}{m} < 2\tau$. Define

$$\xi(t) := \min\{1, 2\tau/t\} \min\{m(t-2)_+, 1\}.$$

The number m will eventually be sent to infinity. Keeping also (4.9) in mind, we notice that

$$D(\xi(|\bar{u}|)\bar{u}) = \begin{cases} 0, & |\bar{u}| \le 2, \\ m((|\bar{u}|-2)\mathrm{Id} + |\bar{u}|P)D\bar{u}, & 2 < |\bar{u}| < 2 + 1/m, \\ D\bar{u}, & 2 + 1/m < |\bar{u}| < 2\tau, \\ \dfrac{2\tau}{|\bar{u}|}(\mathrm{Id} - P)D\bar{u}, & |\bar{u}| \ge 2\tau, \end{cases} \tag{7.12}$$

where P is, as usual, the projection defined by

$$P := \frac{\bar{u} \otimes \bar{u}}{|\bar{u}|^2}$$

and which is here computed only when $|\bar{u}| > 2$. Recalling (4.11) we have

$$D\bar{u} : D(\xi(|\bar{u}|)\bar{u})\chi_{\{|\bar{u}|\ge 2\tau\}} = \frac{2\tau}{|\bar{u}|} D\bar{u} : [(\mathrm{Id} - P)D\bar{u}]\chi_{\{|\bar{u}|\ge 2\tau\}} \ge 0,$$

and therefore

$$D\bar{u} : D(\xi(|\bar{u}|)\bar{u}) \ge \chi_{\{2+1/m<|\bar{u}|<2\tau\}} |D\bar{u}|^2. \tag{7.13}$$

We then test the weak form of (7.5), that is,

$$\int_{B_1} |D\bar{u}|^{p-2} D\bar{u} : D\varphi \, dx = \int_{B_1} \varphi \, d\bar{\mu} \quad \forall \varphi \in W_0^{1,p}(B_1; \mathbb{R}^N) \tag{7.14}$$

(notice that the enlarged space of test functions, that is, $W_0^{1,p}(B_1; \mathbb{R}^N)$, is allowed since $D\bar{u} \in L^p$ and $\bar{\mu}$ is a smooth map) with

$$\begin{aligned} \varphi &:= \phi \min\{1, 2\tau/|\bar{u}|\} \min\{m(|\bar{u}| - 2)_+, 1\}\bar{u} \\ &= \phi \min\{m(|\bar{u}| - 2)_+, 1\} T_{2\tau}(\bar{u}) = \phi\xi(|\bar{u}|)\bar{u} \end{aligned}$$

where $\phi \in C_0^\infty(B_{15/16})$, $0 \le \phi \le 1$, $\phi = 1$ in $B_{7/8}$ and $|D\phi| \le 2^8$, and where the operator $T_{2\tau}(\cdot)$ has been defined in (4.8). Using (7.12)–(7.13), and eventually letting $m \to \infty$, yields

$$\int_{B_1 \cap \{2<|\bar{u}|<2\tau\}} |D\bar{u}|^p \phi \, dx \le 2\tau \int_{B_1 \cap \{|\bar{u}|>2\}} |D\bar{u}|^{p-1} |D\phi| \, dx + 2\tau |\bar{\mu}|(B_1). \tag{7.15}$$

We proceed by splitting the last integral as

$$\begin{aligned} \int_{B_1 \cap \{|\bar{u}|>2\}} |D\bar{u}|^{p-1} |D\phi| \, dx &\le \|D\phi\|_{L^\infty} \int_{B_{15/16} \cap \{|D\bar{u}|>H\}} |D\bar{u}|^{p-1} \, dx \\ &\quad + H^{p-1} |B_{15/16} \cap \{|\bar{u}| > 2\}| \end{aligned} \tag{7.16}$$

for $H \ge 1$ to be chosen in a moment. By (7.9) and (7.11), we are able to apply Lemma 6.1 with $\gamma = p - 1$, which together with a standard covering argument yields the existence of a constant $H \equiv H(n, N, p) \ge 1$ such that

$$\int_{B_{15/16} \cap \{|D\bar{u}|>H\}} |D\bar{u}|^{p-1} \, dx \le c \int_{B_1 \cap \{|D\bar{u}|>2\}} |D\bar{u}|^q \, dx + c|\bar{\mu}|(B_1) \tag{7.17}$$

with $c \equiv c(n, N, p, q)$. Indeed, we can find a finite covering $\{\mathcal{B}_i\}$ of $B_{15/16}$ with smaller balls $\mathcal{B}_i$, each of radius $1/128$ and touching $B_{15/16}$; the number of balls in the covering depends only on n. Notice that such a choice implies that $2\mathcal{B}_i \subset B_1$ for every i. On each such ball we can estimate

$$20^n c_{\rm h} \fint_{2\mathcal{B}_i} |D\bar{u}|\,dx + 20^n c_{\rm h}\left[\frac{|\bar{\mu}|(2\mathcal{B}_i)}{|2\mathcal{B}_i|^{1-1/n}}\right]^{1/(p-1)}$$
$$\leq c \fint_{B_1} |D\bar{u}|\,dx + c[|\bar{\mu}|(B_1)]^{1/(p-1)} \leq c(n,N,p) \leq H,$$

where we can choose $H \equiv H(n,N,p)$ large enough to have $H/c_* \geq 2$, and $c_*, c_{\rm h} \equiv c_*, c_{\rm h}(n,N,p)$ are the constants detailed in Lemma 6.1 (which we consider here with $\gamma = p-1$). Notice that in the last estimate we have used both (7.9) and (7.11). We therefore use (6.2) with $\gamma = p-1$ for each $\mathcal{B}_i$, getting

$$\begin{aligned}\int_{B_{15/16}\cap\{|D\bar{u}|>H\}} |D\bar{u}|^{p-1}\,dx &\leq \sum_i \int_{\mathcal{B}_i\cap\{|D\bar{u}|>H\}} |D\bar{u}|^{p-1}\,dx \\ &\leq c_* H^{p-2} \sum_i \int_{2\mathcal{B}_i\cap\{|D\bar{u}|>H/c_*\}} |D\bar{u}|\,dx + c_* \sum_i |\bar{\mu}|(2\mathcal{B}_i) \\ &\leq c \int_{B_1\cap\{|D\bar{u}|>H/c_*\}} |D\bar{u}|\,dx + c|\bar{\mu}|(B_1)\end{aligned}$$

with $c \equiv c(n,N,p)$. From this we get (7.17) as $H/c_* \geq 2$. To proceed with the proof, we observe that the inclusion $B_1 \cap \{|\bar{u}| > 2\} \subset B_1 \cap \{|v| > 1\}$ (by (7.10)) together with Poincaré's inequality (recall (7.8)) gives

$$|B_{15/16} \cap \{|\bar{u}| > 2\}| \leq |B_1 \cap \{|v| > 1\}| \leq \int_{B_1} |v|^q\,dx \leq c\int_{B_1} |Dv|^q\,dx. \tag{7.18}$$

Combining (7.16)–(7.18) and using the fact that in $B_1 \cap \{|D\bar{u}| > 2\}$ we also have $|D\bar{u}| \leq 2|Dv|$ (indeed, $2|Dv| \geq |D\bar{u}| + |D\bar{u}| - 2 \geq |D\bar{u}|$), we obtain

$$\int_{B_1\cap\{|\bar{u}|>2\}} |D\bar{u}|^{p-1}|D\phi|\,dx \leq c\int_{B_1} |Dv|^q\,dx + c|\bar{\mu}|(B_1),$$

again with $c \equiv c(n,N,p,q)$. Using this together with (7.15), we have

$$\int_{B_1\cap\{2<|\bar{u}|<2\tau\}} |D\bar{u}|^p \phi\,dx \leq c\int_{B_1} |Dv|^q\,dx + c|\bar{\mu}|(B_1) \tag{7.19}$$

with $c \equiv c(n,N,p,q,\tau)$. By further splitting the supports of the integrals in the sets $\{|D\bar{u}| \geq 1\}$ and $\{|D\bar{u}| < 1\}$, using Poincaré's inequality and the estimate in (7.19) we conclude the proof as follows:

$$\begin{aligned}&\int_{B_{7/8}\cap\{3<|v|\leq\tau\}} (|Dv|^2 + |Dv|^p)\,dx \\ &\quad\leq c|B_{7/8} \cap \{3 < |v| \leq \tau\}| + c\int_{B_{7/8}\cap\{2<|\bar{u}|\leq 2\tau\}} |D\bar{u}|^p\,dx \\ &\quad\leq c\int_{B_1} |v|^q\,dx + c\int_{B_{7/8}\cap\{2<|\bar{u}|\leq 2\tau\}} |D\bar{u}|^p\,dx \leq c\int_{B_1} |Dv|^q\,dx + c|\bar{\mu}|(B_1). \qquad \square\end{aligned}$$

The next lemma is the key technical result of this section. It includes a weighted inequality with weight given by $|v|^{-\gamma}$. Indeed, the inequality captures the integrability properties around the zero set of $|v|$. For this reason, the range of exponents γ considered cannot be too large.

Lemma 7.2. *Let $\bar{u}, \bar{\mu}, v$ be as in (7.5)–(7.9). Let $\tau \geq 3$ and $q \in [1, 3/2]$. Then*

$$\int_{B_{1/2}\cap\{|v|\leq\tau\}} \frac{|Dv|^2+|Dv|^p}{|v|^\gamma}\,dx \leq c\int_{B_1} |Dv|^q\,dx + c|\bar{\mu}|(B_1) \tag{7.20}$$

with a constant $c \equiv c(n, N, p, \tau, \gamma)$ whenever

$$0 \leq \gamma \leq 1/(4p). \tag{7.21}$$

The integrand on the left hand side of (7.20) is meant to be zero when $|v| = 0$.

Proof. The proof is in four steps. In the first four steps we prove (7.20) in the special case $\gamma = 1/(4p)$; then in the last step we prove it in the whole range (7.21).

Step 1: Testing the system. The system in (7.5) can be rewritten as

$$-\operatorname{div}(|D\bar{u}|^{p-2}D\bar{u} - |D\ell|^{p-2}D\ell) = \bar{\mu}, \tag{7.22}$$

which in turn can be linearized as follows:

$$-\operatorname{div}(B(x): Dv) = \bar{\mu} \tag{7.23}$$

with

$$B(x) := \int_0^1 |D\ell + sDv|^{p-2}L(D\ell + sDv)\,ds, \qquad v = \bar{u} - \ell, \tag{7.24}$$

and $L(\cdot)$ has been defined in (2.4) (keep in mind (2.5)). Notice that $B(\cdot)$ satisfies the ellipticity condition

$$c^{-1}|\xi|^2 \leq \frac{(B(x):\xi):\xi}{(1+|Dv(x)|)^{p-2}} \leq c|\xi|^2 \qquad \forall \xi \in \mathbb{R}^{N\times n} \tag{7.25}$$

with $c \equiv c(p)$; this is indeed a consequence (2.7), (7.6) and the general algebraic fact that

$$\frac{1}{c}(|\xi_1|+|\xi_2|)^t \leq \int_0^1 |\xi_1 + s\xi_2|^t\,ds \leq \frac{c}{t+1}(|\xi_1|+|\xi_2|)^t \qquad \forall \xi_1, \xi_2 \in \mathbb{R}^{N\times n} \tag{7.26}$$

for every real $t > -1$, with c being an absolute constant (see for instance [24, Chapter 8]). We also notice that with

$$P = \frac{v\otimes v}{|v|^2}, \qquad |v| \neq 0 \tag{7.27}$$

(we define it to be the null tensor otherwise) we have

$$Q := \sup_{x\in B_1}\left|\frac{(B(x):Dv):(\mathrm{Id}-P)Dv}{(B(x):Dv):Dv}\right| \leq p-1, \tag{7.28}$$

and this fact will be proved in Step 4 below. Now, let $\Phi \in C_0^\infty((-6,6))$ be such that

$\Phi \equiv 1$ on $[-3,3]$ and $|\Phi'| \leq 1$. Let $\phi \in C_0^\infty(B_{3/4})$ with $0 \leq \phi \leq 1$, $\phi = 1$ in $B_{1/2}$ and $|D\phi| \leq 128$. We test the weak formulation of (7.22), that is,

$$\int_{B_1} (B(x) : Dv) : D\varphi \, dx = \int_{B_1} \varphi \, d\bar{\mu} \qquad \forall \varphi \in W_0^{1,p}(B_1; \mathbb{R}^N)$$

(in view of (7.23)) with

$$\varphi := \phi \, \Phi(|v|/\tau) T_t(v) \qquad \text{where } 6\tau \geq t > 0.$$

We recall that the truncation operator $T_t(\cdot)$ has been defined in (4.8). Notice that this is an admissible test function as we indeed have additional regularity of v (as $\bar{\mu}$ is here assumed to be smooth). We get

$$\begin{aligned}
\int_{B_1} \varphi \, d\bar{\mu} &= \int_{B_1 \cap \{|v| \leq t\}} [(B(x) : Dv) : Dv] \Phi(|v|/\tau) \phi \, dx \\
&\quad + t \int_{B_1 \cap \{|v| > t\}} \frac{(B(x) : Dv) : [(\mathrm{Id} - P) Dv]}{|v|} \Phi(|v|/\tau) \phi \, dx \\
&\quad + \frac{1}{\tau} \int_{B_1 \cap \{|v| \leq t\}} [(B(x) : Dv) : (P Dv)] \Phi'(|v|/\tau) |v| \phi \, dx \\
&\quad + \frac{t}{\tau} \int_{B_1 \cap \{|v| > t\}} [(B(x) : Dv) : (P Dv)] \Phi'(|v|/\tau) \phi \, dx \\
&\quad + \int_{B_1 \cap \{|v| \leq t\}} [(B(x) : Dv) : (v \otimes D\phi)] \Phi(|v|/\tau) \, dx \\
&\quad + t \int_{B_1 \cap \{|v| > t\}} \left[(B(x) : Dv) : \left(\frac{v}{|v|} \otimes D\phi \right) \right] \Phi(|v|/\tau) \, dx,
\end{aligned} \tag{7.29}$$

where P has been defined in (7.27). Take $0 < \varepsilon < \tau/2$ and $K \geq 6$ (to be chosen later). Multiplying (7.29) by $t^{-1-\gamma}$ (we take initially $\gamma \in (0,1)$), using (7.25) and integrating on the interval $(\varepsilon, K\tau)$ with respect to t, and making a few simple estimations, we arrive at

$$\begin{aligned}
\frac{(K\tau)^{1-\gamma}}{1-\gamma} |\bar{\mu}|(B_1) &\geq \int_\varepsilon^{K\tau} t^{-1-\gamma} \int_{B_1 \cap \{|v| \leq t\}} [(B : Dv) : Dv] \Phi(|v|/\tau) \phi \, dx \, dt \\
&\quad - Q \int_\varepsilon^{K\tau} t^{-\gamma} \int_{B_1 \cap \{|v| > t\}} \frac{(B : Dv) : Dv}{|v|} \Phi(|v|/\tau) \phi \, dx \, dt \\
&\quad - c \int_0^{K\tau} \frac{t^{-1-\gamma}}{\tau} \int_{B_1 \cap \{3\tau < |v| < 6\tau\} \cap \{|v| \leq t\}} (1 + |Dv|)^{p-2} |Dv|^2 \Phi'(|v|/\tau) |v| \phi \, dx \, dt \\
&\quad - c \int_0^{K\tau} \frac{t^{-\gamma}}{\tau} \int_{B_1 \cap \{3\tau < |v| < 6\tau\} \cap \{|v| > t\}} (1 + |Dv|)^{p-2} |Dv|^2 \Phi'(|v|/\tau) \phi \, dx \, dt \\
&\quad - c \int_0^{K\tau} t^{-1-\gamma} \int_{B_1 \cap \{|v| \leq t\}} (1 + |Dv|)^{p-2} |Dv| \, |D\phi| |v| \Phi(|v|/\tau) \, dx \, dt \\
&\quad - c \int_0^{K\tau} t^{-\gamma} \int_{B_1 \cap \{|v| > t\}} (1 + |Dv|)^{p-2} |Dv| \, |D\phi| \Phi(|v|/\tau) \, dx \, dt \\
&=: I_1(\varepsilon) - Q I_2(\varepsilon) - I_3 - I_4 - I_5 - I_6,
\end{aligned} \tag{7.30}$$

where $c \equiv c(n, N, p)$ and we recall that Q has been defined and estimated in (7.28). We now apply Fubini's theorem to estimate $I_1(\varepsilon), I_2(\varepsilon), I_3, \ldots, I_6$. Notice that the application of Fubini's theorem is legitimate as all the terms displayed in (7.30) are finite, including those that do not involve ε and the corresponding integrands are nonnegative. This is obvious for as $I_1(\varepsilon), I_2(\varepsilon), I_4, I_6$ as $\gamma < 1$, while for I_3 it is sufficient to remark that the corresponding domain of integration forces $t \geq 3\tau$. And for I_5, we have

$$I_5 \leq c \int_0^{K\tau} t^{-\gamma} \int_{B_1} (1+|Dv|)^{p-2}|Dv|\,|D\phi|\Phi(|v|/\tau)\,dx\,dt < \infty.$$

Therefore, Fubini's theorem yields

$$I_5 \leq \frac{c}{\gamma} \int_{B_1} (1+|Dv|)^{p-2}|Dv|\,|v|^{1-\gamma}|D\phi|\Phi(|v|/\tau)\,dx.$$

Furthermore,

$$\begin{aligned} I_3 &\leq \frac{c}{\tau}\int_{3\tau}^{\infty} t^{-1-\gamma}\,dt \int_{B_1\cap\{3\tau<|v|<6\tau\}} (1+|Dv|)^{p-2}|Dv|^2\Phi'(|v|/\tau)|v|\phi\,dx\,dt \\ &\leq \frac{c}{\gamma\tau^{1+\gamma}} \int_{B_1\cap\{3\tau<|v|<6\tau\}} (1+|Dv|)^{p-2}|Dv|^2|v|\phi\,dx. \qquad (7.31) \end{aligned}$$

Notice that we have used $|\Phi'| \leq 1$. Recalling that $\gamma < 1$ and $\Phi(|v|/\tau) = 0$ for $|v| \geq K\tau$, and keeping (7.24) in mind, we also have

$$\begin{aligned} I_2(\varepsilon) &= \frac{1}{1-\gamma}\int_{B_1\cap\{|v|\geq\varepsilon\}} [\max\{|v|,\varepsilon\}^{1-\gamma} - \varepsilon^{1-\gamma}]\frac{(B:Dv):Dv}{|v|}\Phi(|v|/\tau)\phi\,dx \\ &\leq \frac{1}{1-\gamma}\int_{B_1\cap\{|v|\geq\varepsilon\}} \frac{(B:Dv):Dv}{|v|^{\gamma}}\Phi(|v|/\tau)\phi\,dx, \qquad (7.32) \\ I_4 &\leq \frac{c}{(1-\gamma)\tau}\int_{B_1\cap\{3\tau<|v|<6\tau\}} (1+|Dv|)^{p-2}|Dv|^2|v|^{1-\gamma}\Phi'(|v|/\tau)\phi\,dx, \qquad (7.33) \end{aligned}$$

and finally

$$I_6 \leq \frac{c}{1-\gamma}\int_{B_1} (1+|Dv|)^{p-2}|Dv|\,|v|^{1-\gamma}|D\phi|\Phi(|v|/\tau)\,dx.$$

In all the above estimates the constant c depends only on n, N, p. It remains to estimate $I_1(\varepsilon)$. We have

$$I_1(\varepsilon) = \frac{1}{\gamma}\int_{B_1}\left[\frac{1}{\max\{|v|,\varepsilon\}^{\gamma}} - \frac{1}{K^{\gamma}\tau^{\gamma}}\right][(B:Dv):Dv]\Phi(|v|/\tau)\phi\,dx,$$

and therefore

$$I_1(\varepsilon) \geq \frac{1}{\gamma}\int_{B_1\cap\{|v|\geq\varepsilon\}}\left[\frac{1}{|v|^{\gamma}} - \frac{1}{K^{\gamma}\tau^{\gamma}}\right][(B:Dv):Dv]\Phi(|v|/\tau)\phi\,dx. \qquad (7.34)$$

Again, notice that $\Phi(|v|/\tau) \equiv 0$ provided $|v| \geq 6\tau$ and so

$$\frac{\Phi(|v|/\tau)}{K^\gamma \tau^\gamma} \leq \frac{6^\gamma}{K^\gamma} \frac{\Phi(|v|/\tau)}{|v|^\gamma}.$$

Using this in the preceding display yields

$$I_1(\varepsilon) \geq \frac{1}{\gamma} \int_{B_1 \cap \{|v| \geq \varepsilon\}} \left[1 - \frac{6^\gamma}{K^\gamma}\right] \frac{(B : Dv) : Dv}{|v|^\gamma} \Phi(|v|/\tau)\phi \, dx,$$

so that by choosing

$$K \equiv K(\gamma) = 6(2^{1/\gamma}) \tag{7.35}$$

we conclude that

$$I_1(\varepsilon) \geq \frac{1}{2\gamma} \int_{B_1 \cap \{|v| \geq \varepsilon\}} \frac{(B : Dv) : Dv}{|v|^\gamma} \Phi(|v|/\tau)\phi \, dx. \tag{7.36}$$

Step 2: An estimate for $I_1(\varepsilon) - QI_2(\varepsilon) - I_3 - I_4$. Estimates (7.36) and (7.32) imply

$$I_1(\varepsilon) - QI_2(\varepsilon) \geq \int_{B_1 \cap \{\varepsilon \leq |v| < 6\tau\}} \left[\frac{1}{2\gamma} - \frac{Q}{1-\gamma}\right] \frac{(B : Dv) : Dv}{|v|^\gamma} \Phi(|v|/\tau)\phi \, dx.$$

By recalling the upper bound in (7.28), we notice that

$$\gamma = \frac{1}{4p} \left(\leq \frac{1}{4Q}\right), \quad \text{so} \quad \frac{1}{2\gamma} - \frac{Q}{1-\gamma} \geq \frac{1}{2\gamma} - \frac{p-1}{1-\gamma} \geq \frac{1}{4\gamma}, \tag{7.37}$$

which also fixes the choice of $K \equiv K(p)$ in (7.35). Combining the last two displays yields

$$I_1(\varepsilon) - QI_2(\varepsilon) \geq \frac{1}{4\gamma} \int_{B_1 \cap \{\varepsilon \leq |v| < 6\tau\}} \frac{(B : Dv) : Dv}{|v|^\gamma} \Phi(|v|/\tau)\phi \, dx. \tag{7.38}$$

Next, recalling (7.31) and (7.33), we notice that

$$I_3 + I_4 \leq \frac{c}{\gamma \tau^\gamma} \int_{B_1 \cap \{3\tau < |v| < 6\tau\}} (|Dv|^2 + |Dv|^p)\phi \, dx,$$

and therefore appealing to Lemma 7.1 (with 6τ instead of τ) and recalling that ϕ vanishes outside $B_{3/4}$, we deduce

$$I_3 + I_4 \leq \frac{c}{\gamma \tau^\gamma} \left(\int_{B_1} |Dv|^q \, dx + |\bar{\mu}|(B_1) \right).$$

Combining the last four displays yields

$$\begin{aligned} I_1(\varepsilon) - QI_2(\varepsilon) - I_3 - I_4 \geq {} & \frac{1}{4\gamma} \int_{B_1 \cap \{\varepsilon \leq |v| < 6\tau\}} \frac{(B : Dv) : Dv}{|v|^\gamma} \Phi(|v|/\tau)\phi \, dx \\ & - \frac{c}{\gamma} \left(\int_{B_1} |Dv|^q \, dx + |\bar{\mu}|(B_1) \right) \end{aligned}$$

with $c \equiv c(n, N, p)$ (recall that $\tau \geq 3$). Finally, using (7.25), we conclude that

$$
\begin{aligned}
I_1(\varepsilon) - QI_2(\varepsilon) - I_3 - I_4 \geq \frac{1}{c} \int_{B_1 \cap \{\varepsilon \leq |v| \leq \tau\}} \frac{|Dv|^2 + |Dv|^p}{|v|^\gamma} \phi \, dx \\
- c\left(\int_{B_1} |Dv|^q \, dx + |\bar{\mu}|(B_1) \right),
\end{aligned} \tag{7.39}
$$

again with $c \equiv c(n, N, p, \tau)$, since γ has been fixed in (7.37).

Step 3: Estimates for I_5 and I_6. We start by observing that

$$
I_5 + I_6 \leq c\left[\frac{1}{\gamma} + \frac{1}{1-\gamma}\right] \int_{B_1 \cap \{|v| < 6\tau\}} (1 + |Dv|)^{p-2} |Dv| \, |v|^{1-\gamma} |D\phi| \, dx, \tag{7.40}
$$

where $c \equiv c(n, N, p)$. Then, by Lemma 6.1 (which we can apply with $\gamma = p - 1$ and since (7.9) and (7.11) are in force), for $H \equiv H(n, N, p) \geq 3$ large enough we estimate

$$
\begin{aligned}
&\int_{B_1 \cap \{|v| < 6\tau\} \cap \{|Dv| > H\}} (1 + |Dv|)^{p-2} |Dv| \, |v|^{1-\gamma} |D\phi| \, dx \\
&\quad \leq c\tau^{1-\gamma} \int_{B_{3/4} \cap \{|Dv| > H\}} |Dv|^{p-1} \, dx \leq c \int_{B_{3/4} \cap \{|D\bar{u}| > H-1\}} |D\bar{u}|^{p-1} \, dx \\
&\quad \leq c \int_{B_1 \cap \{|D\bar{u}| > 2\}} |D\bar{u}|^q \, dx + c|\bar{\mu}|(B_1) \leq c \int_{B_1} |Dv|^q \, dx + c|\bar{\mu}|(B_1)
\end{aligned} \tag{7.41}
$$

with $c \equiv c(n, N, p, \tau)$; recall that the support of ϕ is contained in $B_{3/4}$. We remark that, in order to derive the last estimate we have applied Lemma 6.1 via a covering argument which is the same as the one detailed after (7.17). On the other hand, by Hölder's inequality we have

$$
\begin{aligned}
&\int_{B_1 \cap \{|v| < 6\tau\} \cap \{|Dv| \leq H\}} (1 + |Dv|)^{p-2} |Dv| \, |v|^{1-\gamma} |D\phi| \, dx \\
&\quad \leq (1 + H)^{p-2} \left(\int_{B_1} |Dv|^q \, dx \right)^{1/q} \left(\int_{B_1 \cap \{|v| < 6\tau\}} |v|^{(1-\gamma)q/(q-1)} \, dx \right)^{1-1/q} \\
&\quad \leq c\tau^{2-\gamma-q} \left(\int_{B_1} |Dv|^q \, dx \right)^{1/q} \left(\int_{B_1} |v|^q \, dx \right)^{1-1/q}.
\end{aligned}
$$

In the last estimate we have used $(1-\gamma)q/(q-1) \geq q$; this holds whenever $q \leq 2 - \gamma$ and $\gamma \leq 1/2$, and this is the case since we take $q \leq 3/2$ and $\gamma \leq \bar{\gamma} \leq 1/4$. Using the Poincaré inequality, by (7.8) we obtain

$$
\int_{B_1 \cap \{|v| < 6\tau\} \cap \{|Dv| \leq H\}} (1 + |Dv|)^{p-2} |Dv| \, |v|^{1-\gamma} |D\phi| \, dx \leq c(\tau) \int_{B_1} |Dv|^q \, dx.
$$

Hence, combining this last estimate with (7.40)–(7.41), we conclude that

$$I_5 + I_6 \leq c \int_{B_1} |Dv|^q \, dx + c|\bar{\mu}|(B_1)$$

with $c \equiv c(n, N, p, \tau)$. Combining this with (7.30) and (7.39) gives

$$\int_{B_1 \cap \{\varepsilon \leq |v| \leq \tau\}} \frac{|Dv|^2 + |Dv|^p}{|v|^\gamma} \phi \, dx \leq c \left(\int_{B_1} |Dv|^q \, dx + K|\bar{\mu}|(B_1) \right)$$

for $c \equiv c(n, N, p, \tau)$. At this point (7.20) follows by letting $\varepsilon \to 0$, and recalling the way K has been chosen as a function of γ in (7.35), and that γ has in turn been fixed in (7.37).

Step 4: Proof of (7.28). For completeness we give the full proof of inequality (7.28). By the definition in (7.24) and (2.4) it follows that

$$(B(x) : Dv) : Dv \geq \int_0^1 |D\ell + s Dv|^{p-2} \, ds \, |Dv|^2.$$

Next, notice that for $j \in \{1, \dots, n\}$, the vector $(|v|^{-2} v^\alpha v^\beta D_j v^\beta)_{1 \leq \alpha \leq N}$ is the projection of $(D_j v^\beta)_{1 \leq \beta \leq N}$ onto the one-dimensional subspace generated by v. Hence $|(\mathrm{Id} - P)Dv| \leq |Dv|$ and using (2.7) we have

$$|(B(x) : Dv) : (\mathrm{Id} - P)Dv| \leq (p-1) \int_0^1 |D\ell + s Dv|^{p-2} \, ds \, |Dv|^2,$$

so that (7.28) follows by combining the last two displays.

Step 5: Validity of (7.20) in the whole range (7.21). Indeed, observe that for $0 \leq \gamma < 1/(4p)$ we have

$$\begin{aligned} \int_{B_{1/2} \cap \{|v| \leq \tau\}} \frac{|Dv|^2 + |Dv|^p}{|v|^\gamma} \, dx &\leq \tau^{1/(4p) - \gamma} \int_{B_{1/2} \cap \{|v| \leq \tau\}} \frac{|Dv|^2 + |Dv|^p}{|v|^{1/(4p)}} \, dx \\ &\leq c \left(\int_{B_1} |Dv|^q \, dx + |\bar{\mu}|(B_1) \right), \end{aligned}$$

and the proof is finally complete. □

Remark 7.2. Actually, the previous lemma will be used with the specific choice $\gamma = 1/(4p)$ that was used in the main computations.

Lemma 7.2 is the key to achieving the next lemma, which features a sort of reverse Hölder inequality: the integral of a higher power of the gradient can be estimated by integrals of smaller powers.

Lemma 7.3. *Let $\bar{u}, \bar{\mu}, v$ be as in (7.5)–(7.9). There exist constants $t \equiv t(n,p)$ and $\delta \equiv \delta(n,p)$ with*

$$1 < t < \min\left\{\frac{n}{n-1}, \frac{p}{p-1}\right\} \quad \textit{and} \quad 0 < \delta < 1 \tag{7.42}$$

and $c \equiv c(n,N,p)$ such that

$$\int_{B_{1/4}} [(1+|D\bar{u}|)^{p-3}|Dv|^2]^t \, dx \leq c\left(\int_{B_1} |Dv| \, dx\right)^{t+\delta-1}\left(\int_{B_1} |Dv| \, dx + |\bar{\mu}|(B_1)\right). \tag{7.43}$$

Proof. Set

$$E := \int_{B_1} |Dv| \, dx$$

and assume, without loss of generality, that $E > 0$. Recall also that we are assuming, by (7.9), that $E \leq |B_1|$. We will now show that

$$\int_{B_{1/4}} [(1+|D\bar{u}|)^{p-3}|Dv|^2]^t \, dx \leq cE^{t+\delta}[1 + E^{-1}|\bar{\mu}|(B_1)] \tag{7.44}$$

with $c \equiv c(n,N,p)$ for a suitable choice of the exponents t and δ as described in the statement; this is (7.43). Indeed, we start by taking t and δ so that

$$t + \delta \leq 1 + \frac{1}{n^2 + 4pn}. \tag{7.45}$$

It follows in particular that $t < n/(n-1)$. Then, consider the function $f(\cdot)$ defined by

$$f(s,t,\delta) := \frac{t(p-1)}{s} + 1^*\left(1 - \frac{t(p-1)}{s}\right) - t - \delta$$

for

$$\begin{cases} t(p-1) < s < q_{\mathrm{m}} = \min\{1^*(p-1), p\}, \\ 1 < t < \min\{1^*, p/(p-1)\}, \quad \delta \in (0,1). \end{cases} \tag{7.46}$$

where $1^* = n/(n-1)$. We notice that $f(\cdot)$ is decreasing in t and increasing in s, and moreover

$$\lim_{\substack{s \to q_{\mathrm{m}} \\ t \to 1}} f(s,t) = \frac{1}{(n-1)\max\{n,p\}} - \delta.$$

We can therefore select $s \equiv s(n,p)$ as in (7.46) and sufficiently close to q_{m}; t as in (7.46) and sufficiently close to 1; and δ sufficiently close to 0 such that

$$f(s,t,\delta) \geq 0 \ \Leftrightarrow \ t + \delta \leq \frac{t(p-1)}{s} + 1^*\left(1 - \frac{t(p-1)}{s}\right). \tag{7.47}$$

Therefore, from now on, $t > 1$ and $\delta \in (0,1)$ are fixed in such a way that (7.45) and (7.47) hold. We split the integral in (7.44) as

$$\begin{aligned}\int_{B_{1/4}}[(1+|D\bar u|)^{p-3}|Dv|^2]^t\,dx &= \int_{B_{1/4}\cap\{|v|<3\}\cap\{|Dv|\le H\}}[(1+|D\bar u|)^{p-3}|Dv|^2]^t\,dx\\ &\quad+\int_{B_{1/4}\cap\{|v|<3\}\cap\{|Dv|>H\}}[(1+|D\bar u|)^{p-3}|Dv|^2]^t\,dx\\ &\quad+\int_{B_{1/4}\cap\{|v|\ge 3\}\cap\{|Dv|\le H\}}[(1+|D\bar u|)^{p-3}|Dv|^2]^t\,dx\\ &\quad+\int_{B_{1/4}\cap\{|v|\ge 3\}\cap\{|Dv|>H\}}[(1+|D\bar u|)^{p-3}|Dv|^2]^t\,dx\\ &=: J_1+J_2+J_3+J_4 \end{aligned} \tag{7.48}$$

for large enough $H\ge 3$ to be fixed in (7.53) below. We are going to use Lemma 7.2 with $\gamma = 1/(4p)$, $\tau = 3$ and $q = 1$. Let $\varepsilon_1, \varepsilon_2 \in (0,1)$ be such that

$$\frac{\gamma(2-\varepsilon_1)}{\varepsilon_1} = \frac{\gamma(p-\varepsilon_2)}{\varepsilon_2} = 1^* = \frac{n}{n-1}; \tag{7.49}$$

this implies

$$\frac{\varepsilon_1}{2} = \frac{\gamma}{\gamma+1^*}, \qquad \frac{\varepsilon_2}{p} = \frac{\gamma}{\gamma+1^*}. \tag{7.50}$$

Then, using $|Dv - D\bar u| = |D\ell| = 1$ and (7.10), we start by estimating

$$J_1 \le c(H)\int_{B_{1/4}\cap\{|v|<3\}\cap\{|Dv|\le H\}}|Dv|^{2-\varepsilon_1}\,dx,$$

$$J_2 \le c\int_{B_{1/4}\cap\{|v|<3\}\cap\{|Dv|>H\}}|Dv|^{p-\varepsilon_2}\,dx,$$

so that Hölder's inequality gives

$$J_1 \le c\left(\int_{B_{1/4}\cap\{|v|<3\}}|v|^{-\gamma}|Dv|^2\,dx\right)^{1-\varepsilon_1/2}\left(\int_{B_1}|v|^{\gamma(2-\varepsilon_1)/\varepsilon_1}\,dx\right)^{\varepsilon_1/2},$$

$$J_2 \le c\left(\int_{B_{1/4}\cap\{|v|<3\}}|v|^{-\gamma}|Dv|^p\,dx\right)^{1-\varepsilon_2/p}\left(\int_{B_1}|v|^{\gamma(p-\varepsilon_2)/\varepsilon_2}\,dx\right)^{\varepsilon_2/p}.$$

Using Lemma 7.2, (7.8), (7.49)–(7.50), and Sobolev–Poincaré's inequality we get

$$\begin{aligned}J_1+J_2 &\le c[E+|\bar\mu|(B_1)]^{1-\gamma/(\gamma+1^*)}E^{1^*\gamma/(\gamma+1^*)}\\ &= c[1+E^{-1}|\bar\mu|(B_1)]^{1-\gamma/(\gamma+1^*)}E^{1+(1^*-1)\gamma/(\gamma+1^*)}.\end{aligned}$$

Now observe that

$$t+\delta \le 1+(1^*-1)\frac{\gamma}{\gamma+1^*} = 1+\frac{1}{n(4p+1)-1}$$

as a consequence of (7.45). Therefore, using $E\le |B_1|$, we get

$$J_1+J_2 \le cE^{t+\delta}[1+E^{-1}|\bar\mu|(B_1)]$$

where the constant c depends on n, N, p. We then move on to estimate J_3, which is a rather simple task. Indeed, we have

$$|B_{1/4}\cap\{|v|\geq 3\}|\leq \frac{1}{3^{1^*}}\int_{B_{1/4}}|v|^{1^*}\,dx\leq c\left(\int_{B_1}|Dv|\,dx\right)^{1^*}=cE^{1^*} \tag{7.51}$$

with $c\equiv c(n,N,p)$, and hence

$$J_3\leq cE^{t+\delta}\leq cE^{t+\delta}[1+E^{-1}|\bar{\mu}|(B_1)]$$

with $c\equiv c(n,N,p,H)$ provided

$$t+\delta\leq 1^*=1+\frac{1}{n-1},$$

which is again ensured by (7.45). Next, to estimate J_4 we also use Hölder's inequality and (7.10) to obtain

$$\begin{aligned} J_4 &\leq c(p)\int_{B_{1/4}\cap\{|Dv|>H\}}|Dv|^{t(p-1)}\,dx \\ &\leq c\left(\int_{B_{1/4}\cap\{|Dv|>H\}}|Dv|^s\,dx\right)^{t(p-1)/s}|B_{1/4}\cap\{|v|>3\}|^{1-t(p-1)/s}. \end{aligned} \tag{7.52}$$

We recall that s has been defined through (7.46)–(7.47). In order to estimate the last integral, we use Lemma 6.1 with $\bar{q}\equiv\gamma=s$ (which is admissible since $s<q_{\rm m}$) and $t\equiv H$ where

$$H:=20^n2c_{\rm h}\geq 20^nc_{\rm h}\fint_{B_1}|Dv|\,dx+20^nc_{\rm h}[|\mu|(B_1)]^{1/(p-1)} \tag{7.53}$$

where, finally, the constant $c_{\rm h}\equiv c_{\rm h}(n,N,p,\bar{q})$ is determined by taking $\bar{q}\equiv s$; notice that in the last estimate, (7.8) has been used. In this way we finally see that H depends only on n, N, p, which fixes the dependence of all the constants above depending on H. Therefore Lemma 6.1 can be applied directly to v and yields

$$\int_{B_{1/4}\cap\{|Dv|>H\}}|Dv|^s\,dx\leq c\int_{B_1\cap\{|D\bar{u}|>H/c_*\}}|Dv|\,dx+c|\bar{\mu}|(B_1)\leq c[E+|\bar{\mu}|(B_1)].$$

Combining the last display with (7.52), and using (7.51) again, we obtain

$$\begin{aligned} J_4 &\leq c[E+|\bar{\mu}|(B_1)]^{t(p-1)/s}E^{1^*[1-t(p-1)/s]} \\ &= c[1+E^{-1}|\bar{\mu}|(B_1)]E^{t(p-1)/s+1^*[1-t(p-1)/s]}\leq cE^{t+\delta}[1+E^{-1}|\bar{\mu}|(B_1)]. \end{aligned}$$

In the last line we have used (7.47) as $E\leq|B_1|$. Using the estimates found for $J_1,\ldots,J_4$ in (7.48) finally yields (7.44), and the proof is complete. □

7.2. Proof of Proposition 7.1

We look for the constants $\theta_{\rm nd}$ and q in the form

$$\theta_{\rm nd}:=(\varepsilon/H)^{q/\delta}\quad\text{and}\quad q:=t, \tag{7.54}$$

where $\varepsilon\in(0,1]$ is as in the statement of Proposition 7.1, while t and δ are provided in (7.42) by Lemma 7.3, and therefore depend only on n and p; finally, $H\geq 1$ will be chosen later depending only on n, N, p. When $(Du)_{B_r} \neq 0$ (this is the relevant case to consider here, otherwise the assertion of Proposition 7.1 becomes trivial), we assume that both

$$\fint_{B_r} |Du - (Du)_{B_r}|\, dx \leq \left(\frac{\varepsilon}{H}\right)^{q/\delta} |(Du)_{B_r}| \tag{7.55}$$

and

$$\frac{|\mu|(B_r)}{r^{n-1}} \leq \left(\frac{\varepsilon}{H}\right)^{q/\delta} |(Du)_{B_r}|^{p-2} \fint_{B_r} |Du - (Du)_{B_r}|\, dx. \tag{7.56}$$

We then construct an A-harmonic map h in $B_{r/4}$ satisfying (7.4), with A defined in (7.3) and $L(\cdot)$ defined in (2.4). To this end, we start by rescaling u and μ in B_1 as

$$\bar{u}(x) := \frac{u(x_0 + rx) - (u)_{B_r}}{r|(Du)_{B_r}|}, \qquad \bar{\mu}(x) = \frac{r\mu(x_0 + rx)}{|(Du)_{B_r}|^{p-1}}.$$

In this way, also recalling (2.6), it follows that

$$|(D\bar{u})_{B_1}| = 1, \quad (\bar{u})_{B_1} = 0, \quad L((D\bar{u})_{B_1}) = L((Du)_{B_r}). \tag{7.57}$$

Moreover, $\bar{u}$ solves the system

$$-\triangle_p \bar{u} = \bar{\mu} \quad \text{in } B_1,$$

and by (7.55)–(7.56) we have

$$E := \fint_{B_1} |D\bar{u} - (D\bar{u})_{B_1}|\, dx \leq \left(\frac{\varepsilon}{H}\right)^{q/\delta} \quad \text{and} \quad |\bar{\mu}|(B_1) \leq \left(\frac{\varepsilon}{H}\right)^{q/\delta} E; \tag{7.58}$$

in particular, $|\bar{\mu}|(B_1) \leq 1$. Next, again for $x \in B_1$, we define the maps $\ell \equiv (\ell^\alpha)_{1\leq\alpha\leq N}$ and v exactly as in (7.7), so that

$$\fint_{B_1} |Dv|\, dx = E \leq \left(\frac{\varepsilon}{H}\right)^{q/\delta} \leq 1$$

together with

$$|D\ell| = 1, \tag{7.59}$$

which is a consequence of the first relation in (7.57). We are thus in the setup of Section 7.1. Then, by Lemma 7.3, there exists a positive constant $c \equiv c(n, N, p)$ such that

$$\begin{aligned} \int_{B_{1/4}} [(1 + |D\bar{u}|)^{p-3}|Dv|^2]^q\, dx &\leq c\left(\int_{B_1} |Dv|\, dx\right)^{t+\delta-1} \left(\int_{B_1} |Dv|\, dx + |\bar{\mu}|(B_1)\right) \\ &\leq cE^{q+\delta} \leq c\left(\frac{\varepsilon}{H}\right)^q E^q, \end{aligned} \tag{7.60}$$

because we have chosen $q = t$ at the beginning of the proof. Observe that we have used (7.58) repeatedly. Now, as in Lemma 7.2, rewrite

$$|D\bar{u}|^{p-2}D\bar{u} - |D\ell|^{p-2}D\ell - |D\ell|^{p-2}A:(D\bar{u}-D\ell)$$
$$= \int_0^1 \big(G(D\ell+\tau(D\bar{u}-D\ell)) - G(D\ell)\big)\,d\tau:(D\bar{u}-D\ell) =: W, \tag{7.61}$$

where $G(z) := \partial A_p(z) = |z|^{p-2}L(z)$, and $L(\cdot)$ and A have been defined in (2.4) and (7.3), respectively. In particular, $G(z)$ is a tensor obtained by linearizing the vector field $|z|^{p-2}z$. Its components are

$$G_{ij}^{\alpha\beta}(z) = |z|^{p-2}\left(\delta_{\alpha\beta}\delta_{ij} + (p-2)\frac{z_i^\alpha z_j^\beta}{|z|^2}\right)$$

(see (2.3)). A direct computation gives the existence of a constant $c \equiv c(n, N)$ such that

$$|\partial G(z)| \le c(p-2)|z|^{p-3} \tag{7.62}$$

for every $z \neq 0$. Therefore, using also (7.26) with $t = p-3 > -1$, we have

$$\int_0^1 |G(D\ell+\tau(D\bar{u}-D\ell)) - G(D\ell)|\,d\tau$$
$$\le c(p-2)\int_0^1\int_0^1 |D\ell+\tau s(D\bar{u}-D\ell))|^{p-3}\,d\tau\,ds\,|D\bar{u}-D\ell|$$
$$\le c(|D\ell|+|D\bar{u}|)^{p-3}|D\bar{u}-D\ell| = c(1+|D\bar{u}|)^{p-3}|Dv|$$

with $c \equiv c(n, N)$. Here we have used $p > 2$ although this information does not appear in a quantitative way, i.e., by (7.62) the constant in the above inequality is stable as $p \to 2$. Thus, by the definitions of W in (7.61) and of v we get

$$|W| \le c(1+|D\bar{u}|)^{p-3}|Dv|\,|D\bar{u}-D\ell| = c(1+|D\bar{u}|)^{p-3}|Dv|^2,$$

again with $c \equiv c(n, N, p)$, and hence (7.60) implies

$$\left(\fint_{B_{1/4}} |W|^q\,dx\right)^{1/q} \le c\left(\frac{\varepsilon}{H}\right)E. \tag{7.63}$$

We now define $\bar{h} \in W^{1,2}(B_{1/4})$ as the solution to

$$\begin{cases} -\operatorname{div}(A:D\bar{h}) = 0 & \text{in } B_{1/4}, \\ \bar{h} = \bar{u} & \text{on } \partial B_{1/4}, \end{cases} \tag{7.64}$$

where we recall that $A = L((D\bar{u})_{B_1})$ thanks to (7.57). By (2.7) the system (7.64) is a strongly elliptic system with constant coefficients, and therefore standard Calderón–Zygmund theory applies. In particular, since $\bar{u}$ is in $W^{1,p}(B_{1/4})$, so is $\bar{h}$. Using the identity $-\triangle_p\bar{u} = \bar{\mu}$ together with (7.59) and (7.61) we get

$$\operatorname{div}[A:(D\bar{h}-D\bar{u})] = \operatorname{div} W + \bar{\mu}.$$

We can now use classical linear Calderón–Zygmund theory for linear elliptic systems to find a constant $c \equiv c(n, N, p, q) \equiv c(n, N, p)$ such that

$$\left(\fint_{B_{1/4}} |D\bar{u}-D\bar{h}|^q\,dx\right)^{1/q} \le c\left(\fint_{B_{1/4}} |W|^q\,dx\right)^{1/q} + c|\bar{\mu}|(B_1). \tag{7.65}$$

Here we are in fact using the smoothness of $\bar{\mu}$ and $1 < q < n/(n-1)$. We continue to estimate using (7.63) and the second inequality in (7.58); this yields

$$\left(\fint_{B_{1/4}} |D\bar{u} - D\bar{h}|^q \, dx\right)^{1/q} \leq c\left(\frac{\varepsilon}{H}\right)E + c\left(\frac{\varepsilon}{H}\right)^{q/\delta} E \leq c\left(\frac{\varepsilon}{H}\right)E.$$

Scaling back to original solutions, setting also

$$h(x) = r|(Du)_{B_r}|\,\bar{h}\left(\frac{x - x_0}{r}\right), \qquad x \in B_{r/4}(x_0),$$

we get

$$\left(\fint_{B_{r/4}} |Du - Dh|^q \, dx\right)^{1/q} \leq c\left(\frac{\varepsilon}{H}\right)E \fint_{B_r} |Du - (Du)_{B_r}| \, dx$$

with $c \equiv c(n, N, p)$. Choosing $H := c$ finishes the proof, since h is an A-harmonic map in $B_{r/4}$. □

8. Riesz potential estimates: Proof of Theorems 1.1–1.3

In this section we give the proof of Theorems 1.1–1.3. We first prove a few lemmas for energy solutions, $u \in W^{1,p}(\Omega)$ and assuming that $\mu \in C^\infty(\Omega)$ (Section 8.2). Then, starting from Section 8.3, we start treating the general case of SOLAs. In the rest of the section we shall always assume that $p > 2$.

8.1. Setting up the parameters

In this section we fix a few quantities and objects that will be used throughout the following pages; these are the main actors in the proofs, and they are independent of whether we are considering an energy solution or a SOLA to (1.1). In particular, we choose some constants in order to apply Propositions 5.1 and 7.1; in this connection, a key point is that while the choice of $\theta_{\rm nd}$ in Proposition 7.1 is determined by the choice of ε, the choice of θ is essentially free in Proposition 5.1, and this allows us to combine the two results. Let $c_{{\rm hol},p} \equiv c_{{\rm hol},p}(n, N, p) \geq 1$, $\alpha_{\rm hol} \equiv \alpha_{\rm hol}(n, N, p)$ and $\sigma_0 \equiv \sigma_0(n, N, p) \in (0, 1/2)$ be the constants from Theorem 3.2; moreover, let $c_{{\rm hol},A} \equiv c_{{\rm hol},A}(n, N, \Lambda) \equiv c_{{\rm hol},A}(n, N, p-1) \geq 1$ be the constant from Theorem 3.1 with $\Lambda = p - 1$. We finally set

$$c_{\rm hol} := \max\{c_{{\rm hol},A}(n, N, p-1), c_{{\rm hol},p}(n, N, p)\} \equiv c_{\rm hol}(n, N, p). \tag{8.1}$$

We observe that the choice of $c_{{\rm hol},A}$ is dictated by the fact that we are going to apply Theorem 3.1 to A-harmonic maps with A being a fourth-order tensor satisfying the ellipticity condition

$$|\xi|^2 \leq (A : \xi) : \xi \leq (p-1)|\xi|^2 \qquad \forall z, \xi \in \mathbb{R}^{N\times n}. \tag{8.2}$$

This is actually the condition satisfied by $L(\cdot)$ in (2.7). We proceed by fixing

$$\sigma := \min\left\{\frac{1}{32}, \frac{\sigma_0}{20}, \left(\frac{1}{8^{n+20}c_{\rm hol}}\right)^{1/\alpha_{\rm hol}}\right\}, \tag{8.3}$$

which then depends only on n, N, p. With σ chosen as in the above display and with $B_r(x_0) \subset \Omega$ being a fixed ball, we set

$$r_j := \sigma^{j+1} r \quad \text{and} \quad B^j := \overline{B_{r_j}(x_0)}, \quad \forall j \in \mathbb{N} \cup \{0\}, \quad r_{-1} := r, \tag{8.4}$$

thereby defining a sequence of *closed* balls shrinking to x_0:

$$\cdots \subset B_{r_{j+1}}(x_0) \equiv B^{j+1} \subset B^j \equiv B_{r_j}(x_0) \subset \cdots \subset B^0 \equiv B_{\sigma r}(x_0).$$

Remark 8.1. We are here using closed balls instead of the usual open ones since this will be useful when passing from $W^{1,p}$ solutions to SOLAs, in Lemma 8.4 below.

We now look at Proposition 7.1 with the choice

$$\varepsilon := \sigma^n / 2^{10}, \tag{8.5}$$

which again fixes ε as a quantity depending only on n, N, p, and determine the corresponding quantity

$$\theta_{\text{nd}} \equiv \theta_{\text{nd}}(n, N, p, \varepsilon) \equiv \theta_{\text{nd}}(n, N, p).$$

We then set

$$\theta := \min\{\theta_{\text{nd}}, \sigma^{4n}\} \tag{8.6}$$

and let

$$c_{\text{d}} \equiv c_{\text{d}}(n, N, p, 1, \varepsilon, \theta) \equiv c_{\text{d}}(n, N, p)$$

be as in Proposition 5.1 corresponding to ε and θ in (8.5) and (8.6), respectively, and with $q = 1$. Set finally

$$H_1 := \frac{2^8}{\theta} \geq \frac{2^8}{\sigma^{4n}}, \tag{8.7}$$

$$H_2 := \frac{2^{20p} c_{\text{d}} H_1}{\sigma^{8n} \theta}. \tag{8.8}$$

Note that since σ depends only on n, N, p, so do the constants $\varepsilon, \theta, H_1, H_2$.

8.2. *Three preliminary lemmas*

In this section we restrict our attention to the system (1.1) when $u \in W^{1,p}(\Omega)$ and $\mu \in C^\infty(\Omega)$. In other words, we are considering standard energy solutions. Anyway, we shall use this result only in a qualitative way, while all the constants will be independent of this fact. We define, for every integer $j \geq 0$,

$$A_j := |(Du)_{B^j}| \quad \text{and} \quad E_j := \fint_{B^j} |Du - (Du)_{B^j}| \, dx, \tag{8.9}$$

together with the composite quantity

$$C_j := A_j + H_1 E_j \equiv |(Du)_{B^j}| + H_1 \fint_{B^j} |Du - (Du)_{B^j}| \, dx. \tag{8.10}$$

We now prove a sequence of preliminary lemmas, which are eventually employed to prove the main theorems. In the following, j will always denote a nonnegative integer. The first lemma analyses the case where it is possible to apply Proposition 5.1.

Lemma 8.1 (Decay in the degenerate case). *Let* $u \in W^{1,p}(\Omega)$ *be an energy solution to the system* (1.1) *with* $\mu \in C^{\infty}(\Omega)$. *Suppose that*

$$E_j \geq \theta A_j \tag{8.11}$$

for some integer $j \geq 0$. *Then*

$$E_{j+1} \leq \frac{E_j}{2^7} + \frac{2c_{\mathrm{d}}}{\sigma^n}\left[\frac{|\mu|(B^j)}{r_j^{n-1}}\right]^{1/(p-1)}. \tag{8.12}$$

Proof. Since (8.11) holds, Proposition 5.1 applied in $B_r \equiv B_{r_j}(x_0)$ implies that there exists a p-harmonic map $v_j \in W^{1,p}(B_{r_j/2}(x_0))$ in $B_{r_j/2}(x_0)$ such that

$$\fint_{\frac{1}{2}B^j} |Du - Dv_j|\,dx \leq \varepsilon E_j + c_{\mathrm{d}}\left[\frac{|\mu|(B^j)}{r_j^{n-1}}\right]^{1/(p-1)}. \tag{8.13}$$

Therefore the triangle inequality implies that

$$\begin{aligned}
&\fint_{\frac{1}{2}B^j} |Dv_j - (Dv_j)_{\frac{1}{2}B^j}|\,dx \leq 2\fint_{\frac{1}{2}B^j} |Dv_j - (Du_j)_{B^j}|\,dx \\
&\quad\leq 2^{n+1}E_j + 2\fint_{\frac{1}{2}B^j} |Du - Dv_j|\,dx \leq 2^{n+2}E_j + 2c_{\mathrm{d}}\left[\frac{|\mu|(B^j)}{r_j^{n-1}}\right]^{1/(p-1)}.
\end{aligned} \tag{8.14}$$

In the first line above we have used (2.1). By appealing to Theorem 3.2, inequality (3.4) applied to $v \equiv v_j$, $B_r \equiv B^j/2$ and $\delta \equiv 2\sigma$, together with (8.14), leads to

$$\begin{aligned}
\operatorname*{osc}_{B^{j+1}} Dv_j &\leq 2c_{\mathrm{hol}}\sigma^{\alpha_{\mathrm{hol}}} \fint_{\frac{1}{2}B^j} |Dv_j - (Dv_j)_{\frac{1}{2}B^j}|\,dx \\
&\leq 2c_{\mathrm{hol}}\sigma^{\alpha_{\mathrm{hol}}}\left(2^{n+2}E_j + 2c_{\mathrm{d}}\left[\frac{|\mu|(B^j)}{r_j^{n-1}}\right]^{1/(p-1)}\right).
\end{aligned}$$

Combining this with (8.13) and again using (2.1) then gives

$$\begin{aligned}
E_{j+1} &\leq 2\fint_{B^{j+1}} |Du - (Dv_j)_{B_{j+1}}|\,dx \\
&\leq 2\fint_{B^{j+1}} |Du - Dv_j|\,dx + 2\fint_{B^{j+1}} |Dv_j - (Dv_j)_{B_{j+1}}|\,dx \\
&\leq 2\fint_{B^{j+1}} |Du - Dv_j|\,dx + 2\operatorname*{osc}_{B^{j+1}} Dv_j \\
&\leq \sigma^{-n}\fint_{\frac{1}{2}B^j} |Du - Dv_j|\,dx + 2\operatorname*{osc}_{B^{j+1}} Dv_j \\
&\leq (2^{n+4}c_{\mathrm{hol}}\sigma^{\alpha_{\mathrm{hol}}} + \sigma^{-n}\varepsilon)E_j + (8c_{\mathrm{hol}}c_{\mathrm{d}}\sigma^{\alpha_{\mathrm{hol}}} + \sigma^{-n}c_{\mathrm{d}})\left[\frac{|\mu|(B^j)}{r_j^{n-1}}\right]^{1/(p-1)} \\
&\leq \frac{E_j}{2^7} + \frac{2c_{\mathrm{d}}}{\sigma^n}\left[\frac{|\mu|(B^j)}{r_j^{n-1}}\right]^{1/(p-1)}.
\end{aligned}$$

In deriving the last estimate we have also used the definitions of the constants σ and ε in (8.3) and (8.5), respectively. This proves (8.12). □

Lemma 8.2 (Decay in the nondegenerate case I). *Let $u \in W^{1,p}(\Omega)$ be an energy solution to the system* (1.1) *with $\mu \in C^\infty(\Omega)$. Suppose that*

$$E_j \le \theta A_j \quad \textit{and} \quad \frac{|\mu|(B^j)}{r_j^{n-1}} \le \theta A_j^{p-2} E_j \tag{8.15}$$

for some integer $j \ge 0$. Then

$$E_{j+1} \le E_j/4. \tag{8.16}$$

Proof. First, if $A_j = 0$, then (8.15) implies that also $E_j = 0$, and hence $E_{j+1} = 0$ as well, because then Du is a constant in B^j. Therefore (8.16) holds trivially. So we can assume that $A_j = |(Du)_{B^j}| > 0$. Then, since by (8.6) we have $\theta \le \theta_{\mathrm{nd}}$, Proposition 7.1 applied in $B_r \equiv B_{r_j}(x_0)$ implies that there exists an A-harmonic map $h_j \in W^{1,p}(B_{r_j/4}(x_0))$ in $B_{r_j/4}(x_0)$ that solves

$$\operatorname{div}(A : Dh_j) = 0 \quad \text{in } B_{r_j/4}(x_0), \qquad A := L((Du)_{B^j}),$$

such that

$$\fint_{\frac14 B^j} |Du - Dh_j|\,dx \le \varepsilon E_j.$$

We recall that $L(\cdot)$ has been defined in (2.4), while ε has been fixed in (8.5). Theorem 3.1 applied to h_j in $B_r \equiv B^j/4$ and with $\delta = 4\sigma$ further gives

$$\operatorname*{osc}_{B^{j+1}} Dh_j \le 4c_{\mathrm{hol}}\sigma^{\alpha_{\mathrm{hol}}} \fint_{\frac14 B^j} |Dh_j - (Dh_j)_{\frac14 B^j}|\,dx.$$

The application of Theorem 3.1 here is allowed by the choice of the constant c_{hol} in (8.1), since the tensor A satisfies (8.2). Combining the last two displays as in the proof of Lemma 8.1, and using the elementary property in (2.1) repeatedly, we obtain

$$\begin{aligned}
E_{j+1} &\le 2 \fint_{B^{j+1}} |Du - (Dh_j)_{B^{j+1}}|\,dx \\
&\le 2 \fint_{B^{j+1}} |Du - Dh_j|\,dx + 2 \fint_{B^{j+1}} |Dh_j - (Dh_j)_{B^{j+1}}|\,dx \\
&\le \sigma^{-n} \fint_{\frac14 B^j} |Du - Dh_j|\,dx + 2 \operatorname*{osc}_{B^{j+1}} Dh_j \\
&\le \sigma^{-n}\varepsilon E_j + 16c_{\mathrm{hol}}\sigma^{\alpha_{\mathrm{hol}}} \fint_{\frac14 B^j} |Dh_j - (Du)_{B^j}|\,dx \\
&\le (\sigma^{-n}\varepsilon + 4^{n+2}c_{\mathrm{hol}}\sigma^{\alpha_{\mathrm{hol}}})E_j + 16c_{\mathrm{hol}}\sigma^{\alpha_{\mathrm{hol}}} \fint_{\frac14 B^j} |Du - Dh_j|\,dx \\
&\le (\sigma^{-n}\varepsilon + 4^{n+2}c_{\mathrm{hol}}\sigma^{\alpha_{\mathrm{hol}}} + 16c_{\mathrm{hol}}\sigma^{\alpha_{\mathrm{hol}}}\varepsilon)E_j.
\end{aligned}$$

Now (8.3) and (8.5) yield

$$\sigma^{-n}\varepsilon + 4^{n+2}c_{\mathrm{hol}}\sigma^{\alpha_{\mathrm{hol}}} + 16c_{\mathrm{hol}}\sigma^{\alpha_{\mathrm{hol}}}\varepsilon \le 1/4,$$

so that (8.16) follows. □

Lemma 8.3 (Decay in the nondegenerate case II). *Let $u \in W^{1,p}(\Omega)$ be an energy solution to the system (1.1) with $\mu \in C^{\infty}(\Omega)$. Suppose that*

$$E_j \leq \theta A_j \quad \text{and} \quad \frac{|\mu|(B^j)}{r_j^{n-1}} > \theta A_j^{p-2} E_j \tag{8.17}$$

for some integer $j \geq 0$. Then

$$E_j \leq \frac{1}{\theta}\left[\frac{|\mu|(B^j)}{r_j^{n-1}}\right]^{1/(p-1)}. \tag{8.18}$$

Proof. By (8.17) and since we are assuming that $p > 2$, we simply estimate

$$E_j = E_j^{(p-2)/(p-1)} E_j^{1/(p-1)} \leq [(\theta A_j)^{p-2} E_j]^{1/(p-1)} \leq \left[\theta^{p-3} \frac{|\mu|(B^j)}{r_j^{n-1}}\right]^{1/(p-1)},$$

and (8.18) follows since $\theta \leq 1$ and again $p > 2$. □

8.3. The key lemma for SOLAs

In this section, after the the first preliminary results obtained for standard energy solutions in Lemmas 8.1–8.3, we treat the transition to SOLAs. We stress that, for a single SOLA u to (1.1), the quantities A_j, C_j and E_j are defined exactly as in (8.9)–(8.10).

Lemma 8.4 (Iterative step). *Let $u \in W^{1,p-1}(\Omega; \mathbb{R}^N)$ be a SOLA to the system (1.1). Suppose that λ is a positive number and that for some integers $k \geq m \geq 0$ we have*

$$C_j < \lambda, \quad C_{j+1} > \lambda/16 \quad \forall j \in \{m, \dots, k\}, \qquad C_m < \lambda/4, \tag{8.19}$$

$$\left[\sum_{j=m}^{k} \frac{|\mu|(B^j)}{r_j^{n-1}}\right]^{1/(p-1)} < \frac{2\lambda}{\sigma^n H_2}. \tag{8.20}$$

Then, inductively,

$$C_{k+1} < \lambda. \tag{8.21}$$

Moreover,

$$\sum_{j=m}^{k+1} E_j \leq \frac{\sigma^{4n}}{32}\lambda \quad \text{and} \quad \sum_{j=m}^{k+1} E_j \leq 2E_m + \frac{H_2 \sigma^n}{64 H_1} \lambda^{2-p} \sum_{j=m}^{k} \frac{|\mu|(B^j)}{r_j^{n-1}}. \tag{8.22}$$

Proof. The proof is in several steps; with $m \leq k$ fixed as in the statement of the lemma, the following arguments hold for any $j \in \{m, \dots, k\}$.

Step 0: Reduction to the case of energy solutions. Here we first show that we can reduce to the case when $u \in W^{1,p}(\Omega)$ is an energy solution and $\mu \in C^{\infty}(\Omega)$. More precisely, under the additional condition $u \in W^{1,p}(\Omega)$, in Steps 1–4 below we will show that if

$$C_j \leq \lambda, \quad C_{j+1} \geq \lambda/16 \quad \forall j \in \{m, \ldots, k\}, \quad C_m \leq \lambda/4, \tag{8.23}$$

$$\left[\sum_{j=m}^{k} \frac{|\mu|(B^j)}{r_j^{n-1}}\right]^{1/(p-1)} \leq \frac{2\lambda}{\sigma^n H_2} \tag{8.24}$$

for a $W^{1,p}$-solution u and $\mu \in C^{\infty}(\Omega)$, then

$$C_{k+1} \leq 11\lambda/12, \tag{8.25}$$

and (8.22) holds as in the statement of the lemma. Taking this for granted, we now show the validity of the lemma for SOLAs. We therefore take a SOLA u as in Definition 1.1 and, as in (8.9)–(8.10), we define the corresponding quantities

$$A_j^h := |(Du_h)_{B^j}|, \quad E_j^h := \fint_{B^j} |Du_h - (Du_h)_{B^j}|\, dx,$$

$$C_j^h := A_j^h + H_1 E_j^h \equiv |(Du_h)_{B^j}| + H_1 \fint_{B^j} |Du_h - (Du_h)_{B^j}|\, dx.$$

Since the sets B^j are the closures of open balls, we can use (1.9) to conclude that

$$\limsup_h |\mu_h|(B^j) \leq |\mu|(B^j) \quad \text{for every } j \in \mathbb{N}. \tag{8.26}$$

On the other hand, the $W^{1,p-1}$-convergence of the gradient also gives

$$A_j^h \to A_j, \quad E_j^h \to E, \quad C_j^h \to C_j, \quad \text{as } h \to \infty, \tag{8.27}$$

for every j. Therefore, by (8.19), there exists an integer $\bar{h}$ such that

$$C_j^h \leq \lambda, \quad C_{j+1}^h \geq \lambda/16 \quad \forall j \in \{m, \ldots, k\}, \quad C_m^h \leq \lambda/4,$$

$$\left[\sum_{j=m}^{k} \frac{|\mu^h|(B^j)}{r_j^{n-1}}\right]^{1/(p-1)} \leq \frac{2\lambda}{\sigma^n H_2}$$

provided $h \geq \bar{h}$. Applying the result for $W^{1,p}$-solutions as described at the beginning of this step we will actually show that

$$C_{k+1}^h \leq 11\lambda/12$$

and

$$\sum_{j=m}^{k+1} E_j^h \leq \frac{\sigma^{4n}}{32}\lambda \quad \text{and} \quad \sum_{j=m}^{k+1} E_j^h \leq 2E_m^h + \frac{H_2 \sigma^n}{64 H_1} \lambda^{2-p} \sum_{j=m}^{k} \frac{|\mu_h|(B^j)}{r_j^{n-1}}$$

whenever $h \geq \bar{h}$. Letting $h \to \infty$ and recalling that the balls B^j are closed yields (8.25) and (8.22). We have therefore reduced the matter to proving that (8.23)–(8.24) imply (8.25) and (8.22) for $W^{1,p}$-solutions.

Step 1: The degenerate case never occurs. Assume first that the basic inequality used in Lemma 8.1 to treat the so-called degenerate case, that is,

$$E_j > \theta A_j, \tag{8.28}$$

holds. We will show that this leads to a contradiction. Indeed, Lemma 8.1 and (8.20), together with the definition of H_2 in (8.8), imply

$$H_1 E_{j+1} \le \frac{H_1 E_j}{2^7} + \frac{4c_{\mathrm{d}} H_1}{\sigma^{2n} H_2}\lambda \le \frac{C_j}{2^7} + \frac{\lambda}{2^7} \le \frac{\lambda}{2^6}.$$

Furthermore,

$$\begin{aligned} |A_{j+1} - A_j| &\le |(Du)_{B^{j+1}} - (Du)_{B^j}| \\ &\le \fint_{B^{j+1}} |Du - (Du)_{B^j}|\, dx \le \frac{E_j}{\sigma^n} \le \frac{C_j}{\sigma^n H_1} \le \frac{\lambda}{2^6} \end{aligned} \tag{8.29}$$

by the definition of H_1 in (8.7). Then (8.28) and $C_j \le \lambda$, which is also an assumption in (8.19), imply

$$A_j \le \frac{E_j}{\theta} \le \frac{C_j}{H_1\theta} \le \frac{\lambda}{H_1\theta} \le \frac{\lambda}{2^6},$$

and therefore

$$\frac{\lambda}{16} \le C_{j+1} \le |A_{j+1} - A_j| + A_j + H_1 E_{j+1} \le \frac{\lambda}{2^6} + \frac{\lambda}{2^6} + \frac{\lambda}{2^6} < \frac{\lambda}{16},$$

a contradiction. Thus we must have

$$E_j \le \theta A_j \tag{8.30}$$

for every $j \in \{m, \dots, k\}$, under the assumptions considered in (8.19)–(8.20).

Step 2: Nondegenerate case I. Assume now that

$$\frac{|\mu|(B^j)}{r_j^{n-1}} \le \theta A_j^{p-2} E_j.$$

Then (8.30) allows us to apply Lemma 8.2, which gives

$$E_{j+1} \le E_j/4 \quad \forall j \in \{m, \dots, k\}. \tag{8.31}$$

Step 3: Nondegenerate case II. Assume finally that

$$\frac{|\mu|(B^j)}{r_j^{n-1}} > \theta A_j^{p-2} E_j. \tag{8.32}$$

This last inequality with (8.30) allows us to apply Lemma 8.3, which together with (8.20) implies

$$E_j \le \frac{1}{\theta}\left[\frac{|\mu|(B^j)}{r_j^{n-1}}\right]^{1/(p-1)} \le \frac{2\lambda}{\theta\sigma^n H_2}.$$

Therefore, since by the definitions of H_1, H_2 in (8.7)–(8.8) we have

$$|A_{j+1} - A_j| \le \fint_{B^{j+1}} |Du - (Du)_{B^j}|\,dx \le \frac{E_j}{\sigma^n} \le \frac{2\lambda}{\theta\sigma^{2n} H_2} \le \frac{\lambda}{2^6},$$

and also by means of (2.1),

$$H_1 E_{j+1} \le 2H_1 \fint_{B^{j+1}} |Du - (Du)_{B^j}|\,dx \le \frac{2H_1 E_j}{\sigma^n} \le \frac{4H_1\lambda}{\theta\sigma^{2n} H_2} \le \frac{\lambda}{2^6}.$$

The last two displays allow us to conclude that

$$C_{j+1} \equiv A_{j+1} + H_1 E_{j+1} \le A_j + |A_{j+1} - A_j| + H_1 E_{j+1} \le A_j + \lambda/2^5.$$

Recalling that in (8.19) we are assuming $C_{j+1} \ge \lambda/16$, we find that

$$A_j \ge \lambda/2^5,$$

which together with (8.32) further gives (keep again property (2.1) in mind)

$$E_{j+1} \le \frac{2E_j}{\sigma^n} \le \frac{2A_j^{2-p}}{\theta\sigma^n}\frac{|\mu|(B^j)}{r_j^{n-1}} \le \frac{2^{1+5(p-2)}}{\theta\sigma^n}\lambda^{2-p}\frac{|\mu|(B^j)}{r_j^{n-1}} \tag{8.33}$$

for every $j \in \{m, \dots, k\}$.

Step 4: Induction step. Estimates (8.31) and (8.33) allow us to conclude that

$$E_{j+1} \le E_j/4 + \frac{2^{1+5(p-2)}}{\theta\sigma^n}\lambda^{2-p}\frac{|\mu|(B^j)}{r_j^{n-1}} \qquad \forall j \in \{m, \dots, k\}.$$

Summing the above inequalities, reabsorbing terms, still adding E_m to both sides and finally using (8.20), leads to

$$\begin{aligned}
\sum_{j=m}^{k+1} E_j &\le \frac{4E_m}{3} + \frac{8}{3}\frac{2^{5(p-2)}}{\theta\sigma^n}\lambda^{2-p}\sum_{j=m}^{k}\frac{|\mu|(B^j)}{r_j^{n-1}} \\
&= \frac{4E_m}{3} + \frac{2}{3H_1}\left(\frac{2^{5p}H_1}{H_2\sigma^n}\right)H_2\sigma^n\lambda^{2-p}\sum_{j=m}^{k}\frac{|\mu|(B^j)}{r_j^{n-1}} \\
&\le \frac{4E_m}{3} + \frac{H_2\sigma^n}{2^{10p}H_1}\lambda^{2-p}\sum_{j=m}^{k}\frac{|\mu|(B^j)}{r_j^{n-1}} \\
&\le \frac{\lambda}{3H_1} + \frac{2^{p-1}}{2^{10p}H_1}\left(\frac{1}{H_2\sigma^n}\right)^{p-2}\lambda \le \frac{5\lambda}{12H_1}.
\end{aligned} \tag{8.34}$$

Notice that we have used (8.19) to estimate $H_1 E_m \leq \lambda/4$, and the very definition of H_1, H_2 in (8.7)–(8.8). Moreover, by the definition of H_1, we conclude that

$$\sigma^{-4n} \sum_{j=m}^{k+1} E_j \leq \frac{H_1}{16} \sum_{j=m}^{k+1} E_j \leq \frac{\lambda}{32}. \tag{8.35}$$

The last inequality and (8.34) prove the two estimates in (8.22). Furthermore, since

$$|A_{k+1} - A_m| \leq \sum_{j=m}^{k} |A_{j+1} - A_j| \leq \sum_{j=m}^{k} \fint_{B^{j+1}} |Du - (Du)_{B^j}|\, dx \leq \sigma^{-n} \sum_{j=m}^{k} E_j \leq \lambda/4, \tag{8.36}$$

we get

$$A_{k+1} \leq A_m + |A_{k+1} - A_m| \leq C_m + |A_{k+1} - A_m| \leq \lambda/4 + \lambda/4 = \lambda/2.$$

Therefore, using (8.34) and (8.35), we conclude that

$$C_{k+1} = A_{k+1} + H_1 E_{k+1} \leq \lambda/2 + 5\lambda/12 \leq 11\lambda/12,$$

proving (8.25) and thereby finishing the proof. □

We conclude this section with another consequence of Lemmas 8.1–8.3.

Lemma 8.5 (Excess decay). *Let $u \in W^{1,p-1}(\Omega; \mathbb{R}^N)$ be a SOLA to the system (1.1). There exist positive constants $c_V \equiv c_V(n, N, p) \geq 1$ and $\alpha_V \equiv \alpha_V(n, N, p) \in (0, 1)$ such that*

$$\begin{aligned} &\fint_{B_{\tau r}(x_0)} |Du - (Du)_{B_{\tau r}(x_0)}|\, dx \\ &\quad \leq c_V \tau^{\alpha_V} \fint_{B_r(x_0)} |Du - (Du)_{B_r(x_0)}|\, dx + c_V \sup_{0<\varrho<r} \left[\frac{|\mu|(B_\varrho(x_0))}{\varrho^{n-1}}\right]^{1/(p-1)} \end{aligned} \tag{8.37}$$

for all $\tau \in (0, 1]$.

Proof. As in Lemma 8.4, we reduce to the case when we are dealing with energy solutions and μ is smooth. Indeed, let $u \in W^{1,p-1}(\Omega)$ be the SOLA mentioned in the statement, and define $\{u_h\}$ and $\{\mu_h\}$ as in Definition 1.1; in particular, $-\triangle_p u_h = \mu_h$ for every h. This means that, taking (8.37) for granted in the case of energy solutions, we have

$$\begin{aligned} &\fint_{B_{\tau r}(x_0)} |Du_h - (Du_h)_{B_{\tau r}(x_0)}|\, dx \\ &\quad \leq c_V \tau^{\alpha_V} \fint_{B_r(x_0)} |Du_h - (Du_h)_{B_r(x_0)}|\, dx + c_V \sup_{0<\varrho<r} \left[\frac{|\mu_h|(B_\varrho(x_0))}{\varrho^{n-1}}\right]^{1/(p-1)} \end{aligned} \tag{8.38}$$

uniformly in h. At this point (8.37) follows by letting $h \to \infty$, and recalling (1.9). Therefore, it remains to prove (8.37) for every energy solution to a system with smooth right hand side. Using Lemmas 8.1–8.3 we conclude that

$$E_{j+1} \leq E_j/4 + \max\{2\sigma^{-n} c_d, \theta^{-1}\} \left[\frac{|\mu|(B^j)}{r_j^{n-1}}\right]^{1/(p-1)}$$

for all $j \in \mathbb{N} \cup \{0\}$. Iterating the above inequality gives, by induction,

$$E_{k+1} \leq \frac{E_0}{4^{k+1}} + c \sum_{j=0}^{k} 4^{j-k} \left[\frac{|\mu|(B^j)}{r_j^{n-1}} \right]^{1/(p-1)}$$

for every $k \in \mathbb{N} \cup \{0\}$ with $c \equiv c(n, N, p)$ (recall the choice of the constants made in Section 8.1 that makes σ, θ and c_d depend only on n, N, p). By using (2.1) and some standard manipulation we then conclude that

$$E_k \leq \frac{\bar{c}}{4^k} \fint_{B_r} |Du - (Du)_{B_r}|\, dx + \bar{c} \sup_{0<\varrho<r} \left[\frac{|\mu|(B_\varrho(x_0))}{\varrho^{n-1}} \right]^{1/(p-1)}, \tag{8.39}$$

again with $\bar{c} \equiv \bar{c}(n, N, p)$ and for every $k \in \mathbb{N} \cup \{0\}$. (8.37) now follows via an interpolation argument that we report here for completeness. We first consider the case $\tau \in (0, \sigma)$. We take the integer $k \geq 1$ such that $\sigma^{k+1} < \tau \leq \sigma^k$; then, by (2.1) and (8.39),

$$\fint_{B_{\tau r}(x_0)} |Du - (Du)_{B_{\tau r}(x_0)}|\, dx \leq \frac{2\sigma^{kn}}{\tau^n} \fint_{B_{\sigma^k r}(x_0)} |Du - (Du)_{B^k}|\, dx \leq \frac{2E_{k-1}}{\sigma^n}$$
$$\leq \frac{8E_0}{4^k \sigma^n} + \frac{2\bar{c}}{\sigma^n} \sup_{0<\varrho<r} \left[\frac{|\mu|(B_\varrho(x_0))}{\varrho^{n-1}} \right]^{1/(p-1)} = \frac{8\sigma^{k\alpha_V} E_0}{\sigma^n} + \frac{2\bar{c}}{\sigma^n} \sup_{0<\varrho<r} \left[\frac{|\mu|(B_\varrho(x_0))}{\varrho^{n-1}} \right]^{1/(p-1)}$$

where $\alpha_V := [\log(1/4)]/\log \sigma$. This defines the exponent α_V mentioned in the statement, with the dependence on n, N, p coming only from that of σ, introduced in (8.3). Recalling the definition of E_0 ((8.4)), and using again (2.1), we conclude that

$$\fint_{B_{\tau r}(x_0)} |Du - (Du)_{B_{\tau r}(x_0)}|\, dx$$
$$\leq \frac{16\sigma^{k\alpha_V}}{\sigma^{2n}} \fint_{B_r(x_0)} |Du - (Du)_{B_r(x_0)}|\, dx + \frac{2\bar{c}}{\sigma^n} \sup_{0<\varrho<r} \left[\frac{|\mu|(B_\varrho(x_0))}{\varrho^{n-1}} \right]^{1/(p-1)}.$$

Now (8.37) follows with $c_V = 16(1 + \bar{c})\sigma^{-2n-1}$ for $\tau \in (0, \sigma r)$; notice that c_V again depends only on n, N, p by the choices made in Section 8.1. Here we have used $\sigma^{k+1} < \tau$. For $\tau \in [\sigma, 1]$ we then trivially have

$$\fint_{B_{\tau r}(x_0)} |Du - (Du)_{B_{\tau r}(x_0)}|\, dx \leq \frac{2}{\sigma^n} \fint_{B_r(x_0)} |Du - (Du)_{B_r(x_0)}|\, dx,$$

and (8.37) follows in every case. □

8.4. Proof of Theorems 1.1 and 1.2

We will now prove Theorem 1.2, from which Theorem 1.1 immediately follows. The proof is in several steps. Here the ball $B_r(x_0)$ is the one fixed in the statements of Theorems 1.1 and 1.2, while we notice that the arguments developed in Sections 8.1 and 8.2 work for any ball $B_r \subset \Omega$.

Step 1: Vanishing mean oscillations at x_0. We will first show that if (1.11) holds, then Du has vanishing mean oscillations at x_0, i.e., (1.12) holds. To this end, fix $\delta \in (0,1)$. By (1.11) we find a positive radius $r_{1,\delta} < r$ such that

$$c_V \sup_{0<\varrho<r_{1,\delta}} \left[\frac{|\mu|(B_\varrho(x_0))}{\varrho^{n-1}}\right]^{1/(p-1)} \leq \frac{\delta}{2},$$

and then τ_δ so small that

$$c_V \tau_\delta^\alpha \fint_{B_{r_{1,\delta}}(x_0)} |Du - (Du)_{B_{r_{1,\delta}}(x_0)}|\,dx \leq \frac{\delta}{2}.$$

Thus, for $r_\delta := \tau_\delta r_{1,\delta}$ we deduce from (8.37) (which we apply with $r \equiv r_{1,\delta}$, see the remarks at the beginning of the proof) that

$$\sup_{0<\varrho<r_\delta} \fint_{B_\varrho(x_0)} |Du - (Du)_{B_\varrho(x_0)}|\,dx \leq \delta,$$

thereby proving that Du has vanishing mean oscillations at x_0 if (1.11) holds. We have thus proved (1.12) and the first assertion of Theorem 1.2.

Step 2: Notation. For a fixed ball $B_r(x_0) \subset \Omega$ we now define the quantity

$$\begin{aligned}\Psi &\equiv \Psi(x_0, r)\\ &:= \frac{64H_1}{\sigma^n} \fint_{B_r(x_0)} |Du - (Du)_{B_r(x_0)}|\,dx + H_2[\mathbf{I}_1^{|\mu|}(x_0,r)]^{1/(p-1)}.\end{aligned} \tag{8.40}$$

Here $\sigma \in (0,1)$, and H_1, H_2 are the large positive constants defined in Section 8.1. Recall also the balls $\{B^j\}$ defined in (8.4) and the quantities A_j, E_j and C_j introduced in (8.9)–(8.10). We shall assume that

$$\mathbf{I}_1^{|\mu|}(x_0, r) < \infty, \tag{8.41}$$

so that Ψ is also finite. At this point, with the definition of H_2 in (8.8) we can control the measure terms as follows:

$$\begin{aligned}\left[\sum_{j=0}^{\infty} \frac{|\mu|(B^j)}{r_j^{n-1}}\right]^{1/(p-1)} &\leq \left[\sum_{j=0}^{\infty} \frac{r_j^{1-n}}{-\log\sigma} \int_{r_j}^{r_{j-1}} |\mu|(B_\varrho)\,\frac{d\varrho}{\varrho}\right]^{1/(p-1)}\\ &\leq \left[\frac{\sigma^{1-n}}{-\log\sigma} \sum_{j=0}^{\infty} \int_{r_j}^{r_{j-1}} \frac{|\mu|(B_\varrho)}{\varrho^{n-1}}\,\frac{d\varrho}{\varrho}\right]^{1/(p-1)}\\ &\leq \sigma^{-n}[\mathbf{I}_1^{|\mu|}(x_0,r)]^{1/(p-1)}\\ &\leq \frac{\Psi}{H_2\sigma^n}.\end{aligned} \tag{8.42}$$

This and (8.41) readily imply

$$\lim_{\varrho\to 0}\left(\frac{|\mu|(B_\varrho(x_0))}{\varrho^{n-1}} + \mathbf{I}_1^{|\mu|}(x_0,\varrho)\right) = 0,$$

and, by the result of Step 1,

$$\lim_{\varrho \to 0} \Psi(x_0, \varrho) = 0. \tag{8.43}$$

The rest of the proof of Theorem 1.2 now splits into two different cases, treated in Steps 3 and 4 below.

Step 3: The nondegenerate case. This is the case in which

$$A_0 > \Psi/16 \equiv \Psi(x_0, r)/16. \tag{8.44}$$

Then we set

$$\lambda := 8A_0. \tag{8.45}$$

Estimating

$$E_0 \leq \frac{2}{\sigma^n} \fint_{B_r(x_0)} |Du - (Du)_{B_r(x_0)}| \, dx \leq \frac{\Psi}{32H_1}, \tag{8.46}$$

by (8.44) we have

$$C_0 = A_0 + H_1 E_0 \leq A_0 + \Psi/16 < 2A_0 = \lambda/4. \tag{8.47}$$

Moreover, by (8.42) and (8.44), and recalling (8.45), it follows that

$$\left[\sum_{j=0}^{\infty} \frac{|\mu|(B^j)}{r_j^{n-1}} \right]^{1/(p-1)} \leq \frac{\Psi}{H_2 \sigma^n} \leq \frac{16A_0}{H_2 \sigma^n} = \frac{2\lambda}{H_2 \sigma^n}. \tag{8.48}$$

Next, notice that

$$A_1 > A_0/2. \tag{8.49}$$

Indeed, recalling (8.44), (8.46) and (8.47) and using again $|A_1 - A_0| \leq \sigma^{-n} E_0$ as in (8.36), we have

$$\begin{aligned} A_1 \geq A_0 - |A_1 - A_0| \geq A_0 - \frac{E_0}{\sigma^n} &\geq A_0 - \frac{\Psi}{32\sigma^n H_1} \\ &\geq A_0 - \Psi/64 \geq 3A_0/4 > A_0/2. \end{aligned}$$

We have used the definition of H_1 in (8.7). We now prove that

$$A_j > A_0/2 \quad \forall j \geq 0. \tag{8.50}$$

To reach a contradiction, by (8.49), assume that there exists a finite (exit time) index $J \geq 2$ such that

$$A_J \leq A_0/2 \quad \text{and} \quad A_j > A_0/2 \quad \forall j \in \{0, \ldots, J-1\}. \tag{8.51}$$

First we are going to prove by induction that

$$C_j < \lambda \quad \forall j \in \{0, \ldots, J-1\}. \tag{8.52}$$

For this we notice that $C_0 < \lambda$ by (8.47). Assuming that the condition

$$C_j < \lambda \quad \forall j \in \{0, \ldots, k\}$$

holds for some $k \leq J-2$, we will prove that $C_{k+1} < \lambda$. Notice that if $A_j > A_0/2 = \lambda/16$ for every $j \in \{0, \ldots, J-1\}$, then, by the very definition of C_j,

$$C_j > \lambda/16 \quad \forall j \in \{0, \ldots, J-1\}. \tag{8.53}$$

In particular, $C_{j+1} > \lambda/16$ for every $j \in \{0, \dots, k\}$. Recalling (8.47), we can therefore apply Lemma 8.4 with $m = 0$, obtaining $C_{k+1} < \lambda$ as desired.

We can now proceed with the proof of (8.50). By (8.47), (8.52) and (8.53) we can apply Lemma 8.4 with $m = 0$ and $k = J - 2$; in particular, we gain the validity of (8.22), whose first inequality reads

$$\sum_{j=0}^{J-1} E_j \le \frac{\sigma^{4n}}{32}\lambda \le \frac{\sigma^{3n} A_0}{32}.$$

Therefore, arguing as for (8.36), we have

$$|A_J - A_0| \le \sum_{j=0}^{J-1} |A_{j+1} - A_j| \le \sigma^{-n} \sum_{j=0}^{J-1} E_j \le A_0/4,$$

and hence

$$A_J \ge A_0 - |A_J - A_0| \ge A_0 - A_0/4 = 3A_0/4,$$

contrary to (8.51). This proves (8.50).

Now, with (8.50) at our disposal, we may repeat the induction argument used above to prove that (8.52) holds for every J, so $C_j < \lambda$ for every $j \in \mathbb{N} \cup \{0\}$. Moreover, again by the very definition of C_j, (8.50) implies that $C_j > \lambda/16$ for every $j \in \mathbb{N} \cup \{0\}$, and this together with (8.47) allows us to apply Lemma 8.4 with $m = 0$ and for every integer k. The second inequality in (8.22), after letting $k \to \infty$, then yields

$$\sum_{j=0}^{\infty} E_j \le 2E_0 + \frac{H_2\sigma^n}{64H_1} A_0^{2-p} \sum_{j=0}^{\infty} \frac{|\mu|(B^j)}{r_j^{n-1}},$$

and, recalling (8.44) and (8.48), in particular estimating $A_0^{2-p} \le 16^{p-2}\Psi^{2-p}$, we have

$$\sum_{j=0}^{\infty} E_j \le 2E_0 + \frac{(H_2\sigma^n)^{2-p} A_0^{2-p}}{64H_1} \Psi^{p-1} \le \frac{\Psi}{32H_1} + \frac{\Psi}{64H_1} \le \frac{\Psi}{H_1}. \tag{8.54}$$

Now observe that if $m < k$ then

$$|(Du)_{B^k} - (Du)_{B^m}| \le \sum_{j=m}^{k-1} |(Du)_{B^{j+1}} - (Du)_{B^j}| \le \sigma^{-n} \sum_{j=m}^{\infty} E_j \le \frac{\Psi}{\sigma^n H_1}, \tag{8.55}$$

which, together with the second-last display, implies that $\{(Du)_{B^j}\}$ is a Cauchy sequence. Thus we may define

$$Du(x_0) := \lim_{j\to\infty} (Du)_{B^j}. \tag{8.56}$$

This is in fact the precise representative of Du at x_0. Indeed, for any positive $\varrho \le \sigma r$ we get the integer $j_\varrho \ge 1$ such that $\sigma^{j_\varrho+1} r < \varrho \le \sigma^{j_\varrho} r$; thus, recalling that the series in (8.55) converges, we have

$$\lim_{\varrho\to 0} |Du(x_0) - (Du)_{B_\varrho(x_0)}| \le \lim_{j_\varrho\to\infty} \left(|Du(x_0) - (Du)_{B^{j_\varrho}}| + \sigma^{-n} E_{j_\varrho}\right) = 0.$$

We have thus proved that

$$\exists \lim_{\varrho \to 0} (Du)_{B_\varrho(x_0)} = Du(x_0). \tag{8.57}$$

Next, letting $k \to \infty$ in (8.55) with $m = 0$ yields

$$|Du(x_0) - (Du)_{B_{\sigma r}(x_0)}| \leq \sigma^{-n} \sum_{j=0}^{\infty} E_j,$$

so that by means of (8.54) we obtain

$$\begin{aligned} |Du(x_0) - (Du)_{B_r(x_0)}| &\leq |Du(x_0) - (Du)_{B_{\sigma r}(x_0)}| + |(Du)_{B_r(x_0)} - (Du)_{B_{\sigma r}(x_0)}| \\ &\leq \sigma^{-n} \sum_{j=0}^{\infty} E_j + \sigma^{-n} \fint_{B_r(x_0)} |Du - (Du)_{B_r(x_0)}|\, dx \\ &\leq \frac{2\Psi}{\sigma^n H_1} \leq \Psi \equiv \Psi(x_0, r). \end{aligned} \tag{8.58}$$

This last estimate is exactly (1.13) (recall the definition of Ψ). In conclusion, if (8.44) holds we have proved that x_0 is a Lebesgue point, that is, (8.57) holds, and estimate (1.13) holds. To conclude the proof, it remains to establish the same facts when (8.44) does not hold. This will be done in Step 4 below.

Step 4: The degenerate case. Here we treat the remaining case when (8.44) does not hold, that is,

$$A_0 = |(Du)_{B_0}| \leq \Psi/16 \equiv \Psi(x_0, r)/16. \tag{8.59}$$

There is no loss of generality in assuming $\Psi > 0$, since otherwise Du is constant in $B_r(x_0)$ and there is nothing to prove. This time we define

$$\lambda := \Psi/2. \tag{8.60}$$

Now, recall that by the definition of Ψ and using (8.46), we have

$$C_0 \equiv A_0 + H_1 E_0 \leq \Psi/16 + \Psi/32 < \Psi/16 + \Psi/16 = \Psi/8 = \lambda/4. \tag{8.61}$$

Next, we consider chains $\mathcal{C}_m^k$ of indices where Lemma 8.4 can be applied; specifically, we set

$$\mathcal{C}_m^k := \{j \in \mathbb{N} : m \leq j \leq k \text{ and the conditions in (8.19) are satisfied}\}. \tag{8.62}$$

We are now able to prove that

$$C_j < \lambda = \Psi/2 \quad \forall j \in \mathbb{N}. \tag{8.63}$$

Indeed, for contradiction define

$$k := \min\{s \in \mathbb{N} \cup \{0\} : C_{s+1} \geq \lambda\}, \tag{8.64}$$

that is, k is the smallest integer such that

$$C_{k+1} \geq \lambda. \tag{8.65}$$

We then define

$$\mathcal{J}_k := \{j \in \mathbb{N} \cup \{0\} : C_j < \lambda/4, \ \ j < k+1\} \quad \text{and} \quad m := \max \mathcal{J}_k.$$

Notice that $\mathcal{J}_k$ is nonempty since $C_0 < \lambda/4$ (see (8.61)). Also, $j \in \{m, \dots, k\}$ implies that $C_j \geq \lambda/4 > \lambda/16$; hence recalling the definition of k in (8.64), we conclude that $\mathcal{C}_m^k$ is a chain of the type considered in (8.62). We can therefore apply Lemma 8.4 to get $C_{k+1} < \lambda$, which contradicts (8.65) and finally implies (8.63).

To proceed with the proof, we take $\varrho \in (0, \sigma r]$ and $k \geq 0$ the largest integer in $\mathbb{N} \cup \{0\}$ such that $r_k = \sigma^{k+1} r \geq \varrho$, so that $(r_k/\varrho)^{-n} \leq (r_k/r_{k+1})^{-n} = \sigma^{-n}$; we obtain

$$\begin{aligned} |(Du)_{B_\varrho(x_0)}| &\leq A_k + |(Du)_{B_\varrho(x_0)} - (Du)_{B^k}| \leq A_k + \sigma^{-n} \fint_{B^k} |Du - (Du)_{B^k}|\, dx \\ &\leq C_k \leq \lambda = \Psi/2. \end{aligned} \tag{8.66}$$

In the last lines we have used the definitions in (8.7)–(8.10), (8.60) and (8.63). If, on the other hand, $\varrho \in (\sigma r, r]$, then in a similar way we have

$$\begin{aligned} |(Du)_{B_\varrho(x_0)}| &\leq A_0 + |(Du)_{B_\varrho(x_0)} - (Du)_{B^0}| \leq A_0 + 2\sigma^{-n} \fint_{B_r(x_0)} |Du - (Du)_{B_r(x_0)}|\, dx \\ &\leq \Psi/16 + \Psi/16 \leq \Psi/2, \end{aligned} \tag{8.67}$$

where this time we have also used (8.59). All in all, we have proved that

$$|(Du)_{B_{\sigma r}}| \leq \Psi(x_0, r)/16, \quad \text{so} \quad \sup_{\varrho < r} |(Du)_{B_\varrho(x_0)}| \leq \Psi(x_0, r)/2. \tag{8.68}$$

We are now ready to finish the proof, and we start by showing that x_0 is a Lebesgue point for Du. For contradiction, assume that

$$\nexists \lim_{\varrho \to 0} (Du)_{B_\varrho(x_0)}. \tag{8.69}$$

This implies that the nondegenerate case (8.44) cannot occur for any $\varrho \in (0, \sigma r)$, since then the limit would exist. Thus

$$|(Du)_{B_\varrho(x_0)}| \leq \Psi(x_0, \varrho/\sigma)/16 \quad \forall \varrho \in (0, \sigma r).$$

But, recalling (8.43), this immediately implies that $\lim_{\varrho \to 0} (Du)_{B_\varrho(x_0)} = 0$, contradicting (8.69). Therefore also in this case (8.57) holds, that is, x_0 is a Lebesgue point of Du. It remains to prove (1.13). Now, as we are assuming that the degenerate case holds for r, that is, (8.59) is valid, it follows that (8.66)–(8.67) hold and give

$$|(Du)_{B_\varrho(x_0)} - (Du)_{B_r(x_0)}| \leq \sup_{\varrho < r} |(Du)_{B_\varrho(x_0)}| + |(Du)_{B_r(x_0)}| \leq \Psi(x_0, r).$$

By letting $\varrho \to 0$ in the above inequality, and using (8.57), we get (1.13), and the proof of Theorems 1.1 and 1.2 is complete.

8.5. Proof of Theorem 1.3

We now assume that (1.14) holds and show that Du is continuous in $B_r(x_0)$. For this we show that for every $\delta > 0$ and $x_1 \in B_r(x_0)$ we can a find positive radius $r_\delta < \operatorname{dist}(B_r(x_0), \partial\Omega)$ such that

$$\operatorname*{osc}_{B_{r_\delta}(x_1)} Du < \delta. \tag{8.70}$$

Let $c \equiv c(n, N, p)$ be the constant from Theorem 1.2. Choose first ϱ_1 so small that

$$\sup_{x \in B_r(x_0)} [\mathbf{I}_1^{|\mu|}(x, \varrho_1)]^{1/(p-1)} \le \frac{\delta}{16c}, \tag{8.71}$$

which is certainly possible by (1.14) (here recall that we are assuming for simplicity that the measure μ is defined on the whole space $\mathbb{R}^n$; see the Introduction). By Theorem 1.2, Du has vanishing mean oscillations at x_1, and hence there is $\varrho_2 \in (0, \min\{\varrho_1, \operatorname{dist}(x_1, \partial B_r(x_0))/4\})$ such that

$$\fint_{B_{\varrho_2}(x_1)} |Du - (Du)_{B_{\varrho_2}(x_1)}| \, dx \le \frac{\delta}{2^{n+5}c}.$$

Set $r_\delta := \varrho_2/2$ and suppose that $x_2 \in B_{r_\delta}(x_1)$. Then

$$\begin{aligned}\fint_{B_{r_\delta}(x_2)} |Du - (Du)_{B_{r_\delta}(x_2)}| \, dx &\le 2 \fint_{B_{r_\delta}(x_2)} |Du - (Du)_{B_{\varrho_2}(x_1)}| \, dx \\ &\le 2^{n+1} \fint_{B_{\varrho_2}(x_1)} |Du - (Du)_{B_{\varrho_2}(x_1)}| \, dx \le \frac{\delta}{16c}\end{aligned}$$

and similarly

$$|(Du)_{B_{r_\delta}(x_2)} - (Du)_{B_{2r_\delta}(x_1)}| \le \fint_{B_{r_\delta}(x_2)} |Du - (Du)_{B_{\varrho_2}(x_1)}| \, dx \le \delta/16.$$

By (8.71) and Theorem 1.2, both x_1 and x_2 are Lebesgue points and

$$|Du(x_1) - (Du)_{B_{2r_\delta}(x_1)}| + |Du(x_2) - (Du)_{B_{r_\delta}(x_2)}| \le \delta/4.$$

Hence $|Du(x_1) - Du(x_2)| < \delta/2$ by the triangle inequality and the above two displays. This immediately implies (8.70), because x_2 was an arbitrary point of $B_{r_\delta}(x_1)$.

9. Wolff potential estimates: Proof of Theorems 1.4–1.6

In this section we shall always assume that $p > 2 - 1/n$, as prescribed in (1.5). We start by proving Theorem 1.5; Theorem 1.4 will then follow as a corollary exactly as estimate (1.10) follows from Theorem 1.2. The proof of Theorem 1.5 is actually considerably simpler than the one for the analogous Theorem 1.2, although some of the ideas are similar. Specifically, as is classical in the case of p-harmonic functions, the low-order regularity analysis does not need to distinguish between degenerate and nondegenerate cases, since Du is not considered. As a consequence we only need the measure data p-harmonic approximation in Theorem 4.1, while the corresponding nondegenerate version of Proposition 7.1 is not needed here. This ultimately allows us to consider the full range $p > 2-1/n$. We notice that the proof reported here is substantially different from the ones originally offered [32, 58], while it partially relies on the different argument introduced in [19].

The proof now goes in two parts. As in the case of the gradient estimates, we shall first carry out the estimates for energy solutions

$$u \in W^{1,p}(\Omega) \quad \text{and} \quad \mu \in C^{\infty}(\Omega). \tag{9.1}$$

Therefore, assuming (9.1), let us now fix a ball $B_r \equiv B_r(x_0) \subset \Omega$, and all the other balls will be concentric. By Lemma 5.2, for every $\varepsilon \in (0,1)$ there exists a p-harmonic map $v \equiv v_r$ in $B_{r/2}$ such that

$$\fint_{B_{r/2}} |Du - Dv|\,dx \le \frac{\varepsilon}{r}\fint_{B_r} |u-(u)_{B_r}|\,dx + c_{\mathrm{s}}\left[\frac{|\mu|(B_r)}{r^{n-1}}\right]^{1/(p-1)}$$

with $c_{\mathrm{s}} \equiv c_{\mathrm{s}}(n,N,p,\varepsilon)$. Set $w = u - v$. By Poincaré's inequality it then follows that

$$\begin{aligned}\fint_{B_{r/2}} |w-(w)_{B_{r/2}}|\,dx &\le cr\fint_{B_{r/2}} |Du - Dv|\,dx\\ &\le \varepsilon c_{\mathrm{P}}\fint_{B_r} |u-(u)_{B_r}|\,dx + c_{\mathrm{P}}c_{\mathrm{s}}\left[\frac{|\mu|(B_r)}{r^{n-p}}\right]^{1/(p-1)}\end{aligned}$$

with $c_{\mathrm{P}} \equiv c_{\mathrm{P}}(n,N)$. Fix $\sigma \in (0,1/4)$ to be determined in a few lines. Theorem 3.3 and the triangle inequality together with the above comparison estimate now imply

$$\begin{aligned}\fint_{B_{\sigma r}} |u-(u)_{B_{\sigma r}}|\,dx &\le \fint_{B_{\sigma r}} |v-(v)_{B_{\sigma r}}|\,dx + \fint_{B_{\sigma r}} |w-(w)_{B_{\sigma r}}|\,dx\\ &\le \operatorname*{osc}_{B_{\sigma r}} v + 2\fint_{B_{\sigma r}} |w-(w)_{B_{r/2}}|\,dx\\ &\le c\sigma\fint_{B_{r/2}} |v-(v)_{B_{r/2}}|\,dx + 2^{1-n}\sigma^{-n}\fint_{B_{r/2}} |w-(w)_{B_{r/2}}|\,dx\\ &\le c\sigma\fint_{B_{r/2}} |u-(u)_{B_{r/2}}|\,dx + c(\sigma+\sigma^{-n})\fint_{B_{r/2}} |w-(w)_{B_{r/2}}|\,dx\\ &\le \bar{c}(\sigma+\varepsilon\sigma^{-n})\fint_{B_r} |u-(u)_{B_r}|\,dx + \bar{c}\sigma^{-n}c_{\mathrm{s}}\left[\frac{|\mu|(B_r)}{r^{n-p}}\right]^{1/(p-1)}.\end{aligned}$$

The constant $\bar{c} \equiv \bar{c}(n,N,p)$ is independent of σ and ε, while c_{s} still depends on ε; notice that we have used property (2.1). Choosing then

$$\sigma := \frac{1}{4\bar{c}} \quad \text{and} \quad \varepsilon := \frac{\sigma^n}{4\bar{c}},$$

thereby also fixing the constant $c_{\mathrm{s}} \equiv c_{\mathrm{s}}(n, N, p)$, we arrive at

$$\fint_{B_{\sigma r}} |u - (u)_{B_{\sigma r}}|\, dx \leq \frac{1}{2} \fint_{B_r} |u - (u)_{B_r}|\, dx + c\left[\frac{|\mu|(B_r)}{r^{n-p}}\right]^{1/(p-1)} \tag{9.2}$$

with $c \equiv c(n, N, p)$. This time we set, for every $j \in \mathbb{N} \cup \{0\}$,

$$\tilde{E}_j := \fint_{B^j} |u - (u)_{B^j}|\, dx \quad \text{and} \quad B^j := \overline{B_{r_j}(x_0)},\ r_j := \sigma^{j+1} r,$$

with $r_{-1} \equiv r$, so that applying (9.2) with $r \equiv r_j$ gives

$$\tilde{E}_{j+1} \leq \frac{1}{2}\tilde{E}_j + c\left[\frac{|\mu|(B^j)}{r_j^{n-p}}\right]^{1/(p-1)} \qquad \forall j \in \mathbb{N} \cup \{0\}. \tag{9.3}$$

By (9.3), and proceeding as for Lemma 8.5, we can find positive constants $\tilde{c}_{\mathrm{V}} \equiv \tilde{c}_{\mathrm{V}}(n, N, p) \geq 1$ and $\tilde{\alpha}_{\mathrm{V}} \equiv \tilde{\alpha}_{\mathrm{V}}(n, N, p) \in (0, 1)$ such that

$$\fint_{B_{\tau r}} |u - (u)_{B_{\tau r}}|\, dx \leq c_{\mathrm{V}} \tau^{\tilde{\alpha}_{\mathrm{V}}} \fint_{B_r} |u - (u)_{B_r}|\, dx + \tilde{c}_{\mathrm{V}} \sup_{0<\varrho<r}\left[\frac{|\mu|(B_\varrho)}{\varrho^{n-p}}\right]^{1/(p-1)} \tag{9.4}$$

for all $\tau \in (0, 1)$; this is an analog of (8.37). The last two displays have been obtained assuming (9.1). At this point we pass from energy solutions as in (9.1) to general SOLAs; the scheme is the same of the one described in Step 1 of the proof of Lemma 8.4 to which we refer for more details. Indeed, take now a SOLA $u \in W^{1,p-1}(\Omega)$ as in the statements of Theorems 1.4–1.6, and take $\{u_h\}$ and $\{\mu_h\}$ as in Definition 1.1. Then we write inequalities (9.3)–(9.4) for u_h and μ_h and finally let $h \to \infty$. Recalling (1.9) (and also (8.26)) then leads to establishing that (9.3)–(9.4) hold for the original SOLA too. From now on this the kind of solution considered. Proceeding as in Section 8.4 and using the analog of the property in the last display, Step 1, it now easily follows that u is VMO at x_0, that is, (1.17) holds, provided (1.16) is assumed. To proceed with the proof, summing up inequalities in (9.3) yields

$$\sum_{j=1}^{k+1} \tilde{E}_j \leq \frac{1}{2}\sum_{j=0}^{k} \tilde{E}_j + c\sum_{j=0}^{k}\left[\frac{|\mu|(B^j)}{r_j^{n-p}}\right]^{1/(p-1)}$$

whenever $k \in \mathbb{N} \cup \{0\}$; reabsorbing terms yields

$$\sum_{j=0}^{k+1} \tilde{E}_j \leq 2\tilde{E}_0 + c\sum_{j=0}^{k}\left[\frac{|\mu|(B^j)}{r_j^{n-p}}\right]^{1/(p-1)}.$$

On the other hand, arguing as in (8.42) we have

$$\sum_{j=0}^{\infty}\left[\frac{|\mu|(B^j)}{r_j^{n-p}}\right]^{1/(p-1)} \leq c\int_0^r \left[\frac{|\mu|(B_\varrho)}{\varrho^{n-p}}\right]^{1/(p-1)} \frac{d\varrho}{\varrho} \equiv c\mathbf{W}^{\mu}_{1,p}(x_0, r),$$

so that, combining the last two displays, we arrive at

$$\sum_{j=0}^{k+1} \tilde{E}_j \le 2\tilde{E}_0 + c\mathbf{W}^{\mu}_{1,p}(x_0, r) \tag{9.5}$$

for $c \equiv c(n, N, p)$. Similarly to (8.29) and (8.55), for every $m, k \in \mathbb{N}$ such that $m < k$ we have

$$\begin{aligned} |(u)_{B^k} - (u)_{B^m}| &\le \sum_{j=m}^{k-1} |(u)_{B^{j+1}} - (u)_{B^j}| \le \sigma^{-n} \sum_{j=m}^{k+1} \tilde{E}_j \le \sigma^{-n} \sum_{j=0}^{k+1} \tilde{E}_j \\ &\le c\tilde{E}_0 + c\mathbf{W}^{\mu}_{1,p}(x_0, r) \le 2c\sigma^{-n} \fint_{B_r(x_0)} |u - (u)_{B_r(x_0)}|\, dx + c\mathbf{W}^{\mu}_{1,p}(x_0, r), \end{aligned} \tag{9.6}$$

where $\sigma \equiv \sigma(n, N, p)$ has been chosen in (9.6). The set of inequalities in the last display allows us to proceed as after (8.55), thereby discovering that $\{(u)_{B^j}\}$ is a Cauchy sequence and eventually

$$\exists \lim_{\varrho \to 0} (u)_{B_\varrho(x_0)} = u(x_0), \tag{9.7}$$

that is, x_0 is a Lebesgue point of u and moreover (1.18) follows as well. This completes the proof of Theorem 1.5 and, as described above, Theorem 1.4 now follows as a corollary. Finally, once the above results have been obtained, Theorem 1.6 follows just as Theorem 1.3, with minor modifications.

10. Selected corollaries

In this final section we report a few corollaries of Theorems 1.1–1.6. These are of two types, and therefore fall into two different subsections. In the first one we list those corollaries aimed at recovering a few well-known results from the literature. The second one involves new results. We recall that the best possible regularity for solutions to (1.1) is of course reached when $\mu \equiv 0$ and does not go beyond the Hölder continuity of the gradient for some exponent > 1. Therefore, in any case, the corollaries will not involve a better degree of regularity for the gradient of solutions.

10.1. Known corollaries

We start with results applying in the so-called energy range, that is, when solutions belong to $W^{1,p}$. In this case we recast some by now classical theorems for the model case (1.1). The first one we mention is concerned with the higher order Calderón–Zygmund theory, and it deals with a result of Iwaniec and DiBenedetto & Manfredi, which when applied to (1.1) reads as follows:

Theorem 10.1 ([14, 28]). *Let $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ be a weak solution to the system (1.1) with $p > 2$. Then*

$$\mu \in L^q_{\mathrm{loc}}(\Omega) \;\Rightarrow\; |Du|^{p-1} \in L^{nq/(n-q)}_{\mathrm{loc}}(\Omega) \quad \textit{whenever} \quad \frac{np}{np - n + p} \le q < n.$$

The above result follows from estimate (1.10) simply by using the mapping properties of Riesz potentials in Lebesgue spaces (see (10.2) below); for more details we refer for instance to [25, 52]. As a matter of fact, a more general statement holds, and leads us to consider a rather large and important family of function spaces, the Lorentz spaces. We recall that, with $\Omega \subset \mathbb{R}^n$ being an open set, the *Lorentz space* $L(q,\gamma) \equiv L(q,\gamma)(\Omega;\mathbb{R}^k)$ with $q \geq 1$ and $0 < \gamma < \infty$ is the set of all measurable maps $f\colon \Omega \to \mathbb{R}^k$ such that

$$\int_0^\infty (\lambda^q |\{x \in \Omega : |f(x)| > \lambda\}|)^{\gamma/q} \frac{d\lambda}{\lambda} < \infty.$$

The Lorentz space $L(q,\infty)$ is by definition the weak-L^q space, also called the *Marcinkiewicz space*, and denoted by $\mathcal{M}^q$. This is defined by prescribing that $f \in \mathcal{M}^q(\Omega)$ iff

$$\sup_{\lambda>0} \lambda^q |\{x \in \Omega : |f(x)| > \lambda\}| < \infty.$$

The local variants of such spaces are defined in the usual fashion (see for instance [52]). We have $L(q,q) \equiv L^q$ for every $q \geq 1$. The spaces $L(q,\gamma)$ decrease in q, while increase in γ; moreover, they interpolate Lebesgue spaces as γ tunes q in the following sense: whenever $0 < \gamma < q < r \leq \infty$ we have, with continuous embeddings,

$$L^r \equiv L(r,r) \subset L(q,\gamma) \subset L(q,q) \subset L(q,r) \subset L(\gamma,\gamma) \equiv L^\gamma.$$

Useful references for Lorentz spaces are for instance [25, 56]. We now have the following extension of Theorem 10.1:

Theorem 10.2 ([4]). *Let $u \in W^{1,p}(\Omega;\mathbb{R}^N)$ be a weak solution to the system (1.1) with $p > 2$. Then*

$$\mu \in L(q,\gamma) \ \Rightarrow\ |Du|^{p-1} \in L(nq/(n-q),\gamma)$$

locally whenever

$$\frac{np}{np-n+p} < q < n \quad \text{and} \quad 0 < \gamma \leq \infty. \tag{10.1}$$

Theorems 10.1–10.2 are a direct consequence of estimate (1.10) and of the basic mapping property of Riesz potentials,

$$\mu \in L(q,\gamma) \ \Rightarrow\ \mathbf{I}_1^{|\mu|} \in L(nq/(n-q),\gamma) \tag{10.2}$$

whenever $1 < q < n$ and $0 < \gamma \leq \infty$. For a discussion of these aspects we refer for instance to [39, 52]. The following result deals with the borderline case $q = n$ in (10.1), yielding gradient continuity and thereby our main result in [40] as a corollary.

Theorem 10.3 ([40]). *Let $u \in W^{1,p}(\Omega;\mathbb{R}^N)$ be a solution to the system (1.1) with $\mu \in L(n,1)$ and $p > 2$. Then Du is continuous in Ω.*

This follows from Theorem 1.3, since assuming $\mu \in L(n,1)$ allows one to verify assumption (1.14) for every ball $B_r(x_0) \Subset \Omega$; see [35, 39] for this fact. We remark that the proof in [40] of Theorem 10.3 is rather long and nontrivial.

We then mention another consequence of Theorem 1.10, this time concerned with measure data. Thus SOLAs must be considered. This is a local version of an important result of Dolzmann, Hungerbühler & Müller.

Theorem 10.4 ([15, 16]). *Let $u \in W^{1,p-1}(\Omega;\mathbb{R}^N)$ be a SOLA to the system (1.1) for $2 < p \leq n$. Then $|Du|^{p-1} \in \mathcal{M}^{n/(n-1)}$ locally in Ω.*

This simply follows from estimate (1.10) and the fact that $\mathbf{I}_1^{|\mu|} \in \mathcal{M}^{n/(n-1)}$ whenever μ is a Borel measure with finite total mass. We remark that in [15, 16] more general systems are considered through a very beautiful proof. Namely, systems featuring measurable coefficients as for instance

$$-\operatorname{div}(a(x)|Du|^{p-2}Du) = \mu$$

are considered, with $a(\cdot)$ bounded away from zero and infinity. For such systems estimate (1.10) *cannot hold*, as already in the scalar case with $\mu = 0$ solutions are not in general Lipschitz continuous.

10.2. New corollaries

We now come to the *new results* implied by Theorems 1.1–1.3. These are theorems that have been proved in the case of scalar equations, but whose validity in the case of systems has remained an open issue. They now follow from Theorem 1.1 straight away.

Theorem 10.5 (Calderón–Zygmund theory below the duality exponent). *Let $u \in W^{1,p-1}(\Omega;\mathbb{R}^N)$ be a SOLA to the system (1.1) for $2 < p < n$. Then the implication*

$$\mu \in L(q,\gamma) \;\Rightarrow\; |Du|^{p-1} \in L(nq/(n-q),\gamma) \tag{10.3}$$

holds locally in Ω for

$$1 < q \leq \frac{np}{np-n+p} \quad \text{and} \quad 0 < \gamma \leq \infty. \tag{10.4}$$

In the case $\gamma = q < np/(np-n+p) = (p^*)'$ the implication in (10.3) recovers, in a local fashion, the classical result in [9] that reads

$$\mu \in L^q_{\mathrm{loc}}(\Omega) \;\Rightarrow\; |Du|^{p-1} \in L^{nq/(n-q)}_{\mathrm{loc}}(\Omega).$$

In the case $1 < q < np/(n-p)$ and $\gamma = \infty$ we are considering the case of Marcinkiewicz spaces and (10.3) reads

$$\mu \in \mathcal{M}^q_{\mathrm{loc}}(\Omega) \;\Rightarrow\; |Du|^{p-1} \in \mathcal{M}^{nq/(n-q)}_{\mathrm{loc}}(\Omega).$$

This last result has originally been proved in the scalar case in [7, 30]. The full range of Lorentz spaces and Morrey–Lorentz spaces together with the hard borderline case $q = np/(np-n+p)$ has been obtained in [52] in the scalar case, via a rather involved proof. It is worth observing that the scalar techniques used in [7, 9, 30, 52] do not apply in the vectorial case. More generally, in [52] a whole set of local estimates for various function spaces has been obtained for the gradient in the range (10.4); several of them also extend to the vectorial case again by means of Theorem 1.1.

We finally have the following continuity result for solutions, and this is a direct consequence of Theorem 1.6. The main assumption (10.5) below forces once again the right hand side to belong to the dual space of $W^{1,p}$, therefore the statement is about energy solutions.

Theorem 10.6. *Let $u \in W^{1,p}(\Omega;\mathbb{R}^N)$ be a weak solution to the system (1.1) with $2 < p < n$. If*

$$\mu \in L(n/p, 1/(p-1)), \tag{10.5}$$

then u is continuous.

This follows by just observing that (10.5) yields (1.19) in every ball of Ω and then Theorem 1.6 applies.

For further results and consequences of Theorems 1.1–1.6 we refer for instance to the very recent paper [2].

11. Final remarks and developments

More general structures can be considered instead of the purely p-Laplacean system (1.1). Here we briefly describe the modifications needed to treat the case (1.20) and outlined in Remark 1.1. The part of the proof in Sections 3–6 still works with minor modifications since it is essentially based on p-monotonicity and on the quasi-diagonal structure of the p-Laplacean system. Both properties are satisfied by (1.20) under assumptions (1.21). The only somewhat delicate modifications are necessary in Section 7, and now we briefly describe them. The definition in (2.2) is modified to $A_g(z) := g(|z|)z$, and therefore, replacing (2.4) and (2.5), we have

$$L(z) = \left(\delta_{\alpha\beta}\delta_{ij} + \frac{g'(|z|)|z|}{g(|z|)}\frac{z_i^\alpha z_j^\beta}{|z|^2}\right)(e^\alpha \otimes e_i) \otimes (e^\beta \otimes e_j)$$

for $z \neq 0$ and

$$\partial A_g(z) = g(|z|)L(z), \tag{11.1}$$

respectively. With this new definition, by using (1.21), we have

$$|\xi|^2 \le (L(z):\xi):\xi, \quad |L(z):\xi| \le \frac{2}{\nu}|\xi| \tag{11.2}$$

for all $z, \xi, |z| \neq 0$. The modifications then start from the proof of Lemma 7.2, where the range of admissible exponents γ now depends on the number ν appearing in (1.21). More precisely, the lemma continues to hold provided (7.21) is replaced by

$$0 \le \gamma \le \nu/8. \tag{11.3}$$

For this, observe that instead of (7.22) we linearize as

$$-\operatorname{div}(g(|D\bar{u}|)D\bar{u} - g(|D\ell|)D\ell) = \bar{\mu}$$

with (7.23) which now holds with

$$B(x) := \int_0^1 g(|D\ell + sDv|)L(D\ell + sDv)\,ds, \qquad v = \bar{u} - \ell, \tag{11.4}$$

according to the identity (11.1), and where $L(\cdot)$ is as in (11). Using (1.21) and the reasoning from Step 4 of the proof of Lemma 7.2, we find that in (7.28) we have the new estimate

$$Q \le 2/\nu. \tag{11.5}$$

Indeed, by using (11.2) this time we have

$$
\begin{aligned}
(B(x):Dv):Dv &\geq \int_0^1 g(|D\ell + sDv|)\,ds\,|Dv|^2,\\
|(B(x):Dv):(\mathrm{Id}-P)Dv| &\leq \frac{2}{\nu}\int_0^1 g(|D\ell + sDv|)\,ds\,|Dv|^2,
\end{aligned}
$$

so that (11.5) follows. Estimate (11.5) can now be used in the estimation of $I_1 - QI_2(\varepsilon)$ from Step 2, therefore the new bound in (7.21) comes from choosing $\gamma \leq 1/(4Q)$ in (7.37). This new choice of γ influences the size of $t \equiv t(\nu)$ and $\delta \equiv \delta(\nu)$ in (7.42), which are now potentially smaller values. This eventually leads to reformulating the condition in (7.45), which becomes

$$
t + \delta < 1 + o(n, 1/\nu).
$$

Finally, the new choice of t, δ as above influences the choice of q and θ_{nd} in the subsequent proof of Proposition 7.1. In particular, the smaller δ affects the choice of θ_{nd} in (7.54). Finally, the smaller $t \equiv q > 1$ implies a larger c in (7.65), since Calderón–Zygmund estimates degenerate as q approaches 1 (the borderline case). The rest of the proof then follows as in Section 8 and all the constants exhibit an additional dependence on ν. The proofs are at this stage very similar to the ones given above.

It appears an interesting issue to consider more general structures as in (1.20), but without the polynomial condition $g(t) \approx t^{p-2}$, in the spirit of the scalar case already considered in [5]. This seems to be a nontrivial task and the issue will be considered in future work.

Finally, we mention the recent paper [3], where a fractional differentiability analog of the potential estimate appears in the scalar case. Namely, for SOLAs of the p-Laplacean equation (1.1) the authors prove that $|Du|^{p-2}Du \in W^{1-\varepsilon,1}_{\mathrm{loc}}(\Omega)$ for every $\varepsilon \in (0,1)$. This is in a sense a singular integral analog of the fractional integral estimate (1.10), as, once again, the exponent p does not appear in the space involved. This result was originally conjectured in [51].

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