J. Eur. Math. Soc. 20, 1139–1159

DOI 10.4171/JEMS/784

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On a Turán problem in weakly quasirandom 3-uniform hypergraphs

Received May 6, 2015 and in revised form December 5, 2017

Abstract. Extremal problems for 3-uniform hypergraphs are known to be very difficult and despite considerable effort the progress has been slow. We suggest a more systematic study of extremal problems in the context of quasirandom hypergraphs. We say that a 3-uniform hypergraph $H = (V, E)$ is *weakly* (d, η)*-quasirandom* if for any subset $U \subseteq V$ the number of hyperedges of H contained in U is in the interval $d\binom{|U|}{3} \pm \eta|V|^3$. We show that for any $\varepsilon > 0$ there exists $\eta > 0$ such that every sufficiently large weakly $(1/4+\varepsilon, \eta)$-quasirandom hypergraph contains four vertices spanning at least three hyperedges. This was conjectured by Erdős and Sós and it is known that the density $1/4$ is best possible.

Recently, a computer assisted proof of this result based on the flag algebra method was given by Glebov, Král', and Volec. In contrast to their work our proof is based on the regularity method for hypergraphs and requires no heavy computations. In addition we obtain an ordered version of this result. Our method of proof allows us to study extremal problems of this type in a more systematic way and we discuss a few extensions and open problems.

Keywords. Quasirandom hypergraphs, extremal graph theory, Turán's problem

1. Introduction

1.1. Extremal problems for graphs and hypergraphs

Given a fixed graph F, a typical problem in extremal graph theory asks for the maximum number of edges that a (large) graph G on n vertices containing no copy of F can have. More formally, for a fixed graph F let the *extremal number* $\mathrm{ex}(n, F)$ be the number $|E|$ of edges of an F-free graph $G = (V, E)$ on $|V| = n$ vertices with the maximum number of edges. It is well known and not hard to observe that the sequence $\mathrm{ex}(n, F)/\binom{n}{2}$ is decreasing. Consequently, one may define the *Turán density*

$$\pi(F) = \lim_{n\to\infty} \frac{\mathrm{ex}(n, F)}{\binom{n}{2}},$$

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Mathematics Subject Classification (2010): Primary 05C35; Secondary 05C65, 05C80

which describes the maximum density of large F-free graphs. The systematic study of these extremal parameters was initiated by Turán [34], who determined $\mathrm{ex}(n, K_k)$ for complete graphs K_k. Recalling that the chromatic number $\chi(F)$ of a graph F is the minimum number of colours one can assign to the vertices of F in such a way that any two vertices connected by an edge receive distinct colours, it follows from a result of Erdős and Stone [12] that

$$\pi(F) = 1 - \frac{1}{\chi(F) - 1}$$

(see also [10], where the result in this form appeared first). In particular, the value of $\pi(F)$ can be calculated in finite time. It also follows that the set $\{\pi(F) \colon F \text{ is a graph}\}$ of all Turán densities of graphs is given by

$$\{0, 1/2, 2/3, \dots, (k-1)/k, \dots\}.$$

Already in his original work [34] Turán asked for hypergraph extensions of these extremal problems. We restrict ourselves to 3-*uniform hypergraphs* $H = (V, E)$, where $V = V(H)$ is a finite set of *vertices*, and the set of *hyperedges* $E = E(H) \subseteq V^{(3)}$ is a family of 3-element subsets of V. Despite considerable effort, even for 3-uniform hypergraphs F no similar characterisation (as in the graph case) is known. Determining the value of $\pi(F)$ is a well known and hard problem even for “simple” hypergraphs like the complete 3-uniform hypergraph $K_4^{(3)}$ on four vertices and $K_4^{(3)-}$, the hypergraph with four vertices and three hyperedges. Currently the best known bounds for these Turán densities are

$$5/9 \le \pi(K_4^{(3)}) \le 0.5616 \quad \text{and} \quad 2/7 \le \pi(K_4^{(3)-}) \le 0.2871,$$

where the lower bounds are given by what is believed to be optimal constructions due to Turán (see, e.g., [7]) and Frankl and Füredi [13]. The stated upper bounds are due to Razborov [23] and Baber and Talbot [1] and their proofs are based on the *flag algebra method* introduced by Razborov [22]. For a thorough discussion of Turán type results and problems for hypergraphs we refer to the recent survey of Keevash [17].

1.2. Quasirandom graphs and hypergraphs

We consider a variant of Turán type questions in connection with quasirandom hypergraphs. Roughly speaking, a quasirandom hypergraph “resembles” a random hypergraph of the same edge density, by sharing some of the key properties with it, namely those that hold true for the random hypergraph with probability close to 1.

The investigation of quasirandom graphs was initiated with the observation that several such properties of randomly generated graphs are equivalent in a deterministic sense. This phenomenon turned out to be useful and had a number of applications in combinatorics. The systematic study of quasirandom graphs was initiated by Thomason [32, 33] and by Chung, Graham, and Wilson [4]. A pivotal feature of random graphs is the uniform edge distribution on “large” sets of vertices, and a quantitative version of this property is used to define quasirandom graphs.

More precisely, a graph $G = (V, E)$ is *quasirandom with density* $d > 0$ if for every subset of vertices $U \subseteq V$ the number $e(U)$ of edges contained in U satisfies

$$e(U) = d\binom{|U|}{2} + o(|V|^2), \tag{1.1}$$

where $o(|V|^2)/|V|^2 \to 0$ as $|V(G)|$ tends to infinity. Strictly speaking, we consider here a sequence of graphs $G_n = (V_n, E_n)$ where the number of vertices $|V_n|$ tends to infinity, but for the sake of a simpler presentation we will suppress the sequence in our discussion. The main result in [4] asserts that (1.1) is deterministically equivalent to several other important properties of random graphs. In particular, it implies that for any fixed graph F with v_F vertices and e_F edges the number $N_F(G)$ of labelled copies of F in a quasirandom graph $G = (V, E)$ of density d satisfies

$$N_F(G) = d^{e_F}|V|^{v_F} + o(|V|^{v_F}). \tag{1.2}$$

In other words, the number of copies of F is close to the expected value in a random graph with edge density d.

The analogous statement for hypergraphs fails to be true and uniform edge distribution on vertex sets is not sufficient to enforce a property similar to (1.2) for all fixed 3-uniform hypergraphs F (see, e.g., Example 1.3 below). A stronger notion of quasirandomness for which such an embedding result actually is true, was considered in connection with the regularity lemma for hypergraphs (cf. Theorem 3.2 below). The central notion for the work presented here, however, is the straightforward extension of (1.1) to 3-uniform hypergraphs, which was for example studied in [5, 18].

Definition 1.1. A 3-uniform hypergraph $H = (V, E)$ is *weakly* (d, η)*-quasirandom* if for every subset $U \subseteq V$ of vertices the number $e(U)$ of hyperedges contained in U satisfies

$$\left| e(U) - d\binom{|U|}{3} \right| \leq \eta n^3. \tag{1.3}$$

For future reference we note that a simple application of the sieve formula shows that the condition (1.3) implies

$$\big| e(X, Y, Z) - d\,|X|\,|Y|\,|Z| \big| \leq 7\eta n^3 \tag{1.4}$$

for all $X, Y, Z \subseteq V$, where $e(X, Y, Z)$ denotes the number of triples $(x, y, z) \in X \times Y \times Z$ for which $\{x, y, z\}$ is a hyperedge of H. We shall denote by $\mathscr{Q}^{(3)}_{\therefore}(d, \eta)$ the class of all 3-uniform weakly (d, η)-quasirandom hypergraphs, where the three dots $\therefore$ represent the possible choices for the three sets X, Y, and Z in (1.4). In fact, we will consider other classes of quasirandom 3-uniform hypergraphs, which we will symbolise by $\mathscr{Q}^{(3)}_{\overset{\bullet}{\bullet\!-\!\bullet}}$ and $\mathscr{Q}^{(3)}_{\wedge}$ and which we will investigate in connection with Turán type questions in [27] and [26] (see also Definition 5.2).

1.3. Extremal problems for weakly quasirandom hypergraphs

Since in contrast to graphs, weakly quasirandom hypergraphs H may not contain every fixed hypergraph F, it seems interesting to determine the maximum density d for which a weakly quasirandom F-free hypergraph of density d exists. This leads to the following notion of Turán density for weakly quasirandom hypergraphs.

Definition 1.2. Given a 3-uniform hypergraph F we set

$$\begin{aligned}\pi_{\therefore}(F) = \sup\big\{d \in [0,1]\colon &\text{ for every } \eta > 0 \text{ and } n \in \mathbb{N} \text{ there exists an } F\text{-free,}\\ &\text{3-uniform hypergraph } H \in \mathscr{Q}_{\therefore}^{(3)}(d,\eta) \text{ with } |V(H)| \geq n\big\}.\end{aligned}$$

Erdős and Sós [11] (see also [8]) were the first to raise questions concerning $\pi_{\therefore}(F)$. In particular, they suggested studying the cases when $F = K_4^{(3)-}$ or F is a complete 3-uniform hypergraph $K_k^{(3)}$. The following probabilistic construction, which can be traced back to the work of Erdős and Hajnal [9], yields $\pi_{\therefore}(K_4^{(3)-}) \geq 1/4$.

Example 1.3. Consider a random tournament T_n on the vertex set $[n] = \{1, \dots, n\}$, i.e., an orientation of all edges of the complete graph on the first n positive integers such that each of the two directions (i, j) or (j, i) of every pair of vertices $\{i, j\}$ is chosen independently with probability $1/2$. Given such a tournament T_n we define a 3-uniform hypergraph $H(T_n)$ on the same vertex set, by including a triple $\{i, j, k\}$ in $E(H(T_n))$ if these three vertices span a cyclically oriented cycle of length three, i.e., $\{i, j, k\} \in E(H(T_n))$ if either (i, j), (j, k), and (k, i) are all in $E(T_n)$, or (i, k), (k, j), and (j, i) are all in $E(T_n)$. It is easy to check that for every $\eta > 0$, with probability tending to 1 as $n \to \infty$ the hypergraph $H(T_n)$ is weakly $(1/4, \eta)$-quasirandom. Moreover, no hypergraph H obtained from a tournament in this way contains three hyperedges on four vertices, i.e., every such H is $K_4^{(3)-}$-free, and this establishes $\pi_{\therefore}(K_4^{(3)-}) \geq 1/4$.

Recently, Glebov, Kráľ', and Volec [15] showed that the construction in Example 1.3 is optimal and proved

$$\pi_{\therefore}(K_4^{(3)-}) = 1/4.$$

The proof in [15] is computer assisted and based on the flag algebra method. We present a computer free and very different proof of the same result. Moreover, our proof yields a strengthening of the result which for ordered vertex sets guarantees the appearance of the $K_4^{(3)-}$ in such a way that the apex vertex, that is, the vertex incident to three hyperedges of the $K_4^{(3)-}$, is either the first or the last. Our method of proof seems to indicate an approach to several other problems of this type, and we shall discuss this in more detail in the concluding remarks in Section 5.

Theorem 1.4. *For every $\varepsilon > 0$ there exists an $\eta > 0$ and an integer n_0 such that for every $n \geq n_0$ every 3-uniform weakly $(1/4 + \varepsilon, \eta)$-quasirandom hypergraph H with vertex set $V(H) = [n]$ contains a $K_4^{(3)-}$ whose apex is either its smallest or its largest vertex.*

Strictly speaking, the authors of [11] and [15] considered a notion slightly different from the weak quasirandomness as defined in Definition 1.1. In their formulation they only required for an n-vertex hypergraph a lower bound of the form $e(U) \geq d\binom{|U|}{3} - \eta n^3$ for every set U of vertices. However, a fairly standard application of the so-called weak regularity lemma for hypergraphs (a straightforward extension of Szemerédi's regularity lemma for graphs [31]) implies that such a hypergraph contains a weakly (d', η')-quasirandom hypergraph on cn vertices for some $d' \geq d$, $c = c(d, \eta) > 0$ and η' with $\eta' \to 0$ as $\eta \to 0$, and thus for the statement of Theorem 1.4 both assumption are equivalent (see, e.g., [25, Proposition 2.5]).

Organisation

A central tool in the proof of Theorem 1.4 is the regularity method for 3-uniform hypergraphs and we will introduce the relevant notation and results in Section 3. Roughly speaking, the regularity lemma (Theorem 3.2) allows us to decompose any given large hypergraph into quasirandom blocks. In fact, the blocks will enjoy stronger quasirandom properties (compared to Definition 1.3), which in "appropriate situations" allow the embedding of any fixed hypergraph (see Theorem 3.4). The main work in the proof of Theorem 1.4 is to ensure such "appropriate situations" for embedding $K_4^{(3)-}$ after the application of the regularity lemma. These arguments will require several ideas from Ramsey theory and extremal graph theory. In particular, in the proof of Theorem 1.4 we will establish a mean square degree condition in multipartite graphs that yields the existence of triangles (Theorem 2.1), which might be of independent interest. The proof of Theorem 1.4 will be given in Section 4. We close with a discussion of a few related results and open problems in Section 5.

2. Forcing triangles in multipartite graphs

In this section we shall prove a purely graph-theoretic statement that will later be used in the proof of $\pi_{\therefore}(K_4^{(3)-}) = 1/4$. Essentially what we have to do then is to find a triangle in the link of a vertex of some weakly quasirandom 3-uniform hypergraph H, and after regularization this will become a problem of finding a triangle in some auxiliary multipartite graph. The vertices of this auxiliary graph will actually not correspond to the vertices of H but rather to some bipartite graphs on $V(H)$, but this subtlety can be ignored until we reach Section 4.

The idea to study multipartite versions of, e.g., Mantel's theorem, or more generally of the Erdős–Stone theorem, seems to go back at least to a suggestion by Bollobás (see [2, discussion after the proof of Theorem VI.2.15]). To the best of our knowledge, the first systematic investigations of this kind have been carried out by Bondy et al. [3]. In the case of triangles they showed the following: Let d_m denote the infimal real number with the property that any m-partite graph G contains a triangle as soon as every edge density between two vertex classes of G is greater than d_m. Then d_m tends to $1/2$ as $m \to \infty$, and moreover $d_{\aleph_0} = 1/2$ for infinite-partite graphs with countably many vertex classes. In the other direction Bondy et al. showed that $d_4 > 1/2$. The situation was further clarified by Pfender [21] who proved that actually $d_m = 1/2$ for all $m \geq 12$; determining the smallest m with $d_m = 1/2$ is an interesting open problem.

The theorem that follows is of a similar flavour. We use the following notation. For an m-partite graph $G = (V, E)$ with vertex classes $V_1 \charfusion[\mathbin]{\cup}{\cdot} \cdots \charfusion[\mathbin]{\cup}{\cdot} V_m = V$, for every vertex $x \in V$ and $j \in [m]$ we denote by $d_j(x)$ the size of the neighbourhood of x in V_j.

Theorem 2.1. *For every $\varepsilon > 0$ there exists an integer m such that if an m-partite graph G with nonempty vertex classes $V_1, \ldots, V_m$ satisfies*

$$\sum_{x \in V_i} d_j(x)^2 \geq (1/4 + \varepsilon)|V_i|\,|V_j|^2$$

whenever $1 \leq i < j \leq m$, then G contains a triangle.

Proof. For convenience we work with the hierarchy

$$m^{-1} \ll m_*^{-1} \ll \delta \ll \varepsilon \ll 1,$$

and commence by defining a colouring of the pairs of indices from $[m]$ with integers from the interval $[1, (2\delta)^{-1}]$.

Let any i and j with $1 \leq i < j \leq m$ be given. For each $r \in \mathbb{N}$ we set

$$Q_{ij}(r) = \{x \in V_i : d_j(x) \geq (1/2 + r\delta)|V_j|\}.$$

We contend that $|Q_{ij}(1)| \geq \delta|V_i|$. To see this, we split the right hand side of the inequality in the statement into two parts according to whether x belongs to $Q = Q_{ij}(1)$ or not. This gives

$$(1/4 + \varepsilon)|V_i|\,|V_j|^2 \leq (1/2 + \delta)^2 |V_i - Q|\,|V_j|^2 + |Q|\,|V_j|^2.$$

Dividing by $|V_j|^2$ and using the trivial estimate $|V_i - Q| \leq |V_i|$ we deduce

$$(1/4 + \varepsilon)|V_i| \leq (1/2 + \delta)^2 |V_i| + |Q|;$$

since $\delta \ll \varepsilon$, the desired conclusion follows.

Clearly, the larger r we take, the smaller the set $Q_{ij}(r)$ becomes, and if $r > (2\delta)^{-1}$ then $Q_{ij}(r) = \emptyset$ holds vacuously. Thus there exists a largest positive value of r, denoted by $r(i, j)$, for which $|Q_{ij}(r)| \geq \delta|V_i|$. This concludes the definition of our colouring

$$r \colon [m]^{(2)} \to \{1, \ldots, \lfloor (2\delta)^{-1} \rfloor\}.$$

By Ramsey's theorem, i.e., since we may assume the validity of the partition relation

$$m \longrightarrow (m_*)^2_{\lfloor (2\delta)^{-1} \rfloor},$$

it is allowed to assume that after some relabelling there is a colour r_* such that $r(i, j) = r_*$ whenever $1 \leq i < j \leq m_*$. Of course, we should now find a triangle in G with vertices from $V_1 \cup \cdots \cup V_{m_*}$. It will turn out that there actually is such a triangle with a vertex in V_1.

Next we will single out some vertex from V_1 that will later be shown to appear in some triangle of G. To this end, we recall that $|Q_{1i}(r_*)| \geq \delta|V_1|$ for all $i \in \{2, \ldots, m_*\}$. It follows that the subsets $Q_{12}(r_*), \ldots, Q_{1m_*}(r_*)$ of V_1 cannot be disjoint provided we have chosen m_* large enough. This means that some vertex $x \in V_1$ appears in at least two of them. For notational simplicity we assume $x \in Q_{12}(r_*)$ as well as $x \in Q_{13}(r_*)$ and endeavor to construct a triangle with vertices from $\{x\} \cup V_2 \cup V_3$.

Let A_2 denote the set of neighbours of x in V_2, set $B_2 = V_2 - A_2$, and define A_3 as well as B_3 analogously. The choice of x implies $|A_2| = d_2(x) \geq (1/2 + r_*\delta)|V_2|$ and $A_3(x) \geq (1/2 + r_*\delta)|V_3|$. Any edge between A_2 and A_3 gives rise to a triangle of the desired kind, so for the sake of contradiction we will henceforth assume that no such edge exists.

Then $d_3(y) \leq |B_3| \leq (1/2 - r_*\delta)|V_3|$ for all $y \in A_2$. For $y \in B_2$ we either have $d_3(y) < (1/2+(r_*+1)\delta)|V_3|$, or $y \in C = Q_{23}(r_*+1)$. But the maximality of $r_* = r(2, 3)$ implies $|C| < \delta|V_2|$ and for $y \in C$ we still have $d_3(y) \leq |V_3|$. So dividing the right hand side of the assumption

$$(1/4 + \varepsilon)|V_2|\,|V_3|^2 \leq \sum_{y \in V_2} d_3(y)^2$$

into three parts depending on whether y appears in A_2, $B_2 - C$ or C we derive

$$(1/4+\varepsilon)|V_2|\,|V_3|^2 \leq |A_2|(1/2-r_*\delta)^2|V_3|^2 + |B_2|(1/2+(r_*+1)\delta)^2|V_3|^2 + \delta|V_2|\,|V_3|^2.$$

Since $|A_2| \geq (1/2 + r_*\delta)|V_2| > |V_2|/2$ and $|A_2| + |B_2| = |V_2|$, this implies

$$1/4 + \varepsilon \leq (1/2 + r_*\delta)(1/2 - r_*\delta)^2 + (1/2 - r_*\delta)(1/2 + (r_* + 1)\delta)^2 + \delta.$$

Now $1/2 + r_*\delta \leq 1$ and for each $x \in [0, 1]$ we have $(x + \delta)^2 \leq x^2 + 3\delta$ by $\delta \leq 1$, so

$$1/4 + \varepsilon \leq (1/2 + r_*\delta)(1/2 - r_*\delta)^2 + (1/2 - r_*\delta)(1/2 + r_*\delta)^2 + 4\delta.$$

Here, the sum of the first two terms gives $1/4 - (r_*\delta)^2$ and hence at most $1/4$, so that altogether we get $\varepsilon \leq 4\delta$, contrary to $\delta \ll \varepsilon$. Thereby Theorem 2.1 is proved. □

The authors of the articles cited at the beginning of this section actually studied the more general question of finding larger cliques, or even arbitrary graphs, in dense multipartite graphs, obtaining results comparable to those indicated above. Similarly, the proof of Theorem 2.1 generalises in a straightforward way from triangles to arbitrary cliques K_k; we omit the details.

Theorem 2.2. *For every* $\varepsilon > 0$ *and* $k \geq 3$ *there exists an integer* m *such that if an* m*-partite graph* G *with nonempty vertex classes* $V_1, \ldots, V_m$ *satisfies*

$$\sum_{x \in V_i} d_j(x)^2 \geq \left(\left(\frac{k-2}{k-1}\right)^2 + \varepsilon\right)|V_i|\,|V_j|^2$$

whenever $1 \leq i < j \leq m$, *then* G *contains a clique* K_k. □

In fact, the proof guarantees $\Omega(n^k)$ copies of K_k and as a result we may replace K_k in Theorem 2.2 by an arbitrary graph F with chromatic number $\chi(F) = k$.

3. Hypergraph regularity method

A key tool in the proof of Theorem 1.4 is the regularity lemma for 3-uniform hypergraphs. We follow the approach from [30, 29] combined with the results from [16] and [20]. First, we introduce the necessary notation.

For two disjoint sets X and Y we denote by $K(X, Y)$ the complete bipartite graph with that vertex partition. We say a bipartite graph $P = (X \dot\cup Y, E)$ is *(δ_2, d_2)-regular* if for all subsets $X' \subseteq X$ and $Y' \subseteq Y$ we have

$$\big|e(X', Y') - d_2|X'|\,|Y'|\big| \le \delta_2|X|\,|Y|,$$

where $e(X', Y')$ denotes the number of edges of P with one vertex in X' and the other in Y'. Moreover, for $k \ge 2$ we say a k-partite graph $P = (X_1 \dot\cup \dots \dot\cup X_k, E)$ is (δ_2, d_2)-regular if all its $\binom{k}{2}$ naturally induced bipartite subgraphs $P[X_i, X_j]$ are (δ_2, d_2)-regular. For a tripartite graph $P = (X \dot\cup Y \dot\cup Z, E)$ we denote by $\mathcal{K}_3(P)$ the triples of vertices spanning a triangle in P, i.e.,

$$\mathcal{K}_3(P) = \big\{\{x, y, z\} \subseteq X \cup Y \cup Z \colon xy, xz, yz \in E\big\}.$$

If the tripartite graph P is (δ_2, d_2)-regular, then the so-called *triangle counting lemma* implies that

$$|\mathcal{K}_3(P)| \le d_2^3|X|\,|Y|\,|Z| + 3\delta_2|X|\,|Y|\,|Z|. \tag{3.1}$$

We say a 3-uniform hypergraph $H = (V, E_H)$ is regular with respect to a tripartite graph P if it matches approximately the same proportion of triangles for every subgraph $Q \subseteq P$. This is made precise in the following definition.

Definition 3.1. A 3-uniform hypergraph $H = (V, E_H)$ is *(δ_3, d_3)-regular with respect to a tripartite graph* $P = (X \dot\cup Y \dot\cup Z, E_P)$ with $V \supseteq X \cup Y \cup Z$ if for every tripartite subgraph $Q \subseteq P$ we have

$$\big||E_H \cap \mathcal{K}_3(Q)| - d_3|\mathcal{K}_3(Q)|\big| \le \delta_3|\mathcal{K}_3(P)|.$$

Moreover, we simply say H *is δ_3-regular with respect to* P if it is (δ_3, d_3)-regular for some $d_3 \ge 0$. We also define the *relative density* of H with respect to P by

$$d(H \mid P) = \frac{|E_H \cap \mathcal{K}_3(P)|}{|\mathcal{K}_3(P)|},$$

where we use the convention $d(H \mid P) = 0$ if $\mathcal{K}_3(P) = \emptyset$.

The regularity lemma for 3-uniform hypergraphs, introduced by Frankl and Rödl [14], provides for every hypergraph H a partition of its vertex set and a partition of the edge sets of the complete bipartite graphs induced by the vertex partition such that for appropriate constants δ_3, δ_2 and d_2,

(1) the bipartite graphs given by the partitions are (δ_2, d_2)-regular, and
(2) H is δ_3-regular for "most" tripartite graphs given by the partition.

Here we use a refined version from [30, Theorem 2.3].

Theorem 3.2 (Regularity Lemma). *For all* $\delta_3 > 0$, $\delta_2 \colon \mathbb{N} \to (0,1]$, *and* $t_0 \in \mathbb{N}$ *there exists an integer* T_0 *such that for every* $n \geq t_0$ *and every* n*-vertex* 3*-uniform hypergraph* $H = (V, E_H)$ *the following holds. There are integers* t *and* ℓ *with* $t_0 \leq t \leq T_0$, $\ell \leq T_0$ *and there exists a vertex partition* $V_0 \dot\cup V_1 \dot\cup \dots \dot\cup V_t = V$ *and for all* $1 \leq i < j \leq t$ *there exists a partition*

$$\mathcal{P}^{ij} = \{P^{ij}_\alpha = (V_i \dot\cup V_j, E^{ij}_\alpha) \colon 1 \leq \alpha \leq \ell\}$$

of the edge set of the complete bipartite graph $K(V_i, V_j)$ *satisfying the following properties:*

(i) $|V_0| \leq \delta_3 n$ *and* $|V_1| = \dots = |V_t|$,

(ii) *for all* $1 \leq i < j \leq t$ *and* $\alpha \in [\ell]$ *the bipartite graph* P^{ij}_α *is* $(\delta_2(\ell), 1/\ell)$*-regular, and*

(iii) H *is* δ_3*-regular with respect to* $P^{ijk}_{\alpha\beta\gamma}$ *for all but at most* $\delta_3 t^3 \ell^3$ *tripartite graphs*

$$P^{ijk}_{\alpha\beta\gamma} = P^{ij}_\alpha \dot\cup P^{ik}_\beta \dot\cup P^{jk}_\gamma = (V_i \dot\cup V_j \dot\cup V_k, E^{ij}_\alpha \dot\cup E^{ik}_\beta \dot\cup E^{jk}_\gamma) \tag{3.2}$$

with $1 \leq i < j < k \leq t$ *and* $\alpha, \beta, \gamma \in [\ell]$.

Owing to their special rôle, we shall refer to the tripartite graphs considered in (3.2) as *triads*. In the formulation of the regularity lemma in [30] a more refined version of hypergraph regularity was used. However, owing to the results of [16] and [20, Corollaries 2.1 and 2.3], for our purposes here the version from Definition 3.1 suffices.

Similarly to other proofs based on the regularity method, it will be convenient to "clean" the regular partition provided by Theorem 3.2. In particular, we shall disregard hyperedges of H that "belong" to *irregular* or *sparse* triads of the regular partition. Since by property (iii) *globally* H is not regular for up to at most $\delta_3 t^3 \ell^3$ triads, a simple averaging argument shows that for $m = m(\delta_3)$ (with $m \to \infty$ as $\delta_3 \to 0$) there exist vertex classes $V_{i_1}, \dots, V_{i_m}$ such that for all fixed $1 \leq a < b < c \leq m$ *locally* H is δ_3-regular for all but at most $\sqrt{\delta_3}\,\ell^3$ triads $P^{i_a i_b i_c}_{\alpha\beta\gamma}$ with $\alpha, \beta, \gamma \in [\ell]$. After removal of the hyperedges belonging to irregular or sparse triads, these considerations lead to the following immediate consequence of Theorem 3.2.

Corollary 3.3. *For all* $d_3, \delta_3 > 0$ *and* $m \in \mathbb{N}$, *and every function* $\delta_2 \colon \mathbb{N} \to (0,1]$, *there exist integers* T_0 *and* n_0 *such that for every* $n \geq n_0$ *and every* n*-vertex* 3*-uniform hypergraph* $H = (V, E)$ *the following holds. There exists a subhypergraph* $\hat{H} = (\hat{V}, \hat{E}) \subseteq H$, *a positive integer* $\ell \leq T_0$, *a vertex partition* $V_1 \dot\cup \dots \dot\cup V_m = \hat{V}$, *and for all* $1 \leq i < j \leq m$ *there exists a partition* $\mathcal{P}^{ij} = \{P^{ij}_\alpha = (V_i \dot\cup V_j, E^{ij}_\alpha) \colon 1 \leq \alpha \leq \ell\}$ *of the edge set of* $K(V_i, V_j)$ *satisfying the following properties:*

(i) $|V_1| = \dots = |V_m| \geq (1 - \delta_3) n / T_0$,

(ii) *for every* $1 \leq i < j \leq m$ *and* $\alpha \in [\ell]$ *the bipartite graph* P^{ij}_α *is* $(\delta_2(\ell), 1/\ell)$*-regular,*

(iii) $\hat{H}$ *is* δ_3*-regular with respect to* $P^{ijk}_{\alpha\beta\gamma}$ *for* all *tripartite graphs* $P^{ijk}_{\alpha\beta\gamma}$ *with* $1 \leq i < j < k \leq m$ *and* $\alpha, \beta, \gamma \in [\ell]$, *where either* $d(\hat{H} \mid P) = 0$ *or* $d(\hat{H} \mid P) \geq d_3$, *and*

(iv) *for every* $1 \leq i < j < k \leq m$ *we have*

$$e_{\hat{H}}(V_i, V_j, V_k) \geq e_H(V_i, V_j, V_k) - (d_3 + \delta_3)|V_i||V_j||V_k|.$$

Moreover, if the vertex set V *is* $[n]$ *then we can ensure* $\max(V_i) < \min(V_{i+1})$ *for every* $i = 1, \dots, m-1$.

Proof. For the proof of Corollary 3.3 (including the "moreover" part) we shall apply the regularity lemma (Theorem 3.2) with δ_3' sufficiently small such that

$$\delta_3' < \delta_3^2 \quad \text{and} \quad \big(1 - 11\sqrt{\delta_3'}\big)\binom{m}{3} > \binom{m}{3} - 1 \tag{3.3}$$

and with the integer $t_0 = \max(m, \lceil 1/\delta_3' \rceil, 21)$ and the given function $\delta_2 \colon \mathbb{N} \to (0, 1]$.

We recall that the hypergraph regularity lemma is proved by iterated refinements starting with an arbitrary initial partition. Hence, given the hypergraph $H = (V, E)$ with $V = [n]$ we may split V initially into t_0 equal sized intervals, $I_1 \cup\!\!\!\cdot \cdots \cup\!\!\!\cdot\, I_{t_0} = [n]$, and then the vertex partition $V_1 \cup\!\!\!\cdot \cdots \cup\!\!\!\cdot\, V_t$ provided by the regularity lemma, Theorem 3.2, will refine this initial partition of intervals.

We consider an auxiliary 3-uniform hypergraph $R = ([t], E_R)$ on the vertex set $[t]$, where a hyperedge $\{i, j, k\}$ signifies the following two properties:

(a) at most $\sqrt{\delta_3'}\,\ell^3$ triads $P^{ijk}_{\alpha\beta\gamma}$ are not δ_3'-regular, and
(b) V_i, V_j, and V_k are contained in three different initial intervals I_s.

Property (iii) of Theorem 3.2 and $t \geq t_0 \geq 21$ imply that at most

$$\frac{\delta_3' t^3 \ell^3}{\sqrt{\delta_3'}\,\ell^3} = \sqrt{\delta_3'}\, t^3 < 7\sqrt{\delta_3'}\binom{t}{3}$$

triples $\{i, j, k\}$ fail to satisfy (a). Moreover, at most

$$t_0 \cdot \binom{t/t_0}{2} t < \frac{4}{t_0}\binom{t}{3}$$

triples are excluded because of (b). Consequently, owing to the choice of $t_0 \geq 1/\delta_3'$ we infer that the auxiliary hypergraph R has density at least $1 - 11\sqrt{\delta_3'}$. The choice of δ_3' in (3.3) entails that R contains a clique on m vertices, say $i_1, \ldots, i_m$. Again appealing to (b) of the construction of R we may assume that there are $1 \leq j_1 < \cdots < j_m \leq t_0$ such that $V_{i_k} \subseteq I_{j_k}$ for every $k \in [m]$, and consequently these vertex sets satisfy the "moreover" part of Corollary 3.3.

In order to construct the desired hypergraph $\hat{H}$ we remove hyperedges of H that are contained in triads with density less than d_3, i.e., hyperedges $e \in E_H \cap \mathcal{K}_3(P^{ijk}_{\alpha\beta\gamma})$ when $d(H \mid P^{ijk}_{\alpha\beta\gamma}) < d_3$. Moreover, we remove hyperedges of H that are contained in triads $P^{ijk}_{\alpha\beta\gamma}$ for which H is not δ_3-regular, and let H_0 be the hypergraph that remains after these deletions. Finally, $\hat{H}$ defined to be the subhypergraph of H_0 induced on $V_{i_1} \cup\!\!\!\cdot \cdots \cup\!\!\!\cdot\, V_{i_m}$ has the desired properties. □

We shall use a so-called *counting/embedding lemma*, which allows us to embed hypergraphs of fixed isomorphism type into appropriate, sufficiently regular and dense triads of the regular partition provided by the regularity lemma. The following statement is a direct consequence of [20, Corollary 2.3].

Theorem 3.4 (Embedding Lemma). *For every* 3*-uniform hypergraph* $F = (V_F, E_F)$ *with vertex set* $V_F = [f]$ *and every* $d_3 > 0$ *there exists* $\delta_3 > 0$ *and functions* $\delta_2 \colon \mathbb{N} \to (0, 1]$ *and* $N \colon \mathbb{N} \to \mathbb{N}$ *such that the following holds for every* $\ell \in \mathbb{N}$*. Let* $P = (\dot\bigcup_{i \in [f]} V_i, E_P)$ *be a* $(\delta_2(\ell), 1/\ell)$*-regular,* f*-partite graph with* $|V_1| = \cdots = |V_f| \geq N(\ell)$ *and let* H *be an* f*-partite,* 3*-uniform hypergraph such that for every edge* $ijk \in E_F$*:*

(a) H *is* δ_3*-regular with respect to to the tripartite graph* $P[V_i \dot\cup V_j \dot\cup V_k]$*, and*
(b) $d(H \mid P[V_i \dot\cup V_j \dot\cup V_k]) \geq d_3$*.*

Then H *contains a copy of* F*, where for every* $i \in [f] = V_F$ *the image of* i *is contained in* V_i*.*

In an application of Theorem 3.4 the tripartite graphs $P[V_i \dot\cup V_j \dot\cup V_k]$ in (a) will be given by triads $P^{ijk}_{\alpha\beta\gamma}$ from the partition given by the regularity lemma.

We shall consider weakly quasirandom hypergraphs H of density μ bounded away from 0. In particular, this assumption implies that in any regular partition provided by Theorem 3.2, the density of H induced on any three vertex classes V_i, V_j, and V_k will be close to μ. For fixed i, j and k this only implies that $d(H \mid P^{ijk}_{\alpha\beta\gamma}) \sim \mu$ on the average taken over all ℓ^3 choices of $\alpha, \beta, \gamma \in [\ell]$. This, however, gives only little information on the density of H with respect to a particular $P^{ijk}_{\alpha\beta\gamma}$. Consequently, for the proof of Theorem 1.4 further analysis is required to arrive at a situation ready for an application of Theorem 3.4. This will be the focus of Section 4.

4. Embedding $K_4^{(3)-}$

In this section we deduce Theorem 1.4. The proof will be based on the regularity lemma for hypergraphs in the form of Corollary 3.3 and the embedding lemma (Theorem 3.4). Below we reduce the proof of Theorem 1.4 to a lemma (see Lemma 4.1 below) which locates in a sufficiently regular partition of a weakly quasirandom hypergraph with density $> 1/4$ a collection of triads that are ready for an application of the embedding lemma for $K_4^{(3)-}$.

Proof of Theorem 1.4. Given $\varepsilon > 0$ we have to find appropriate $\eta > 0$ and $n_0 \in \mathbb{N}$. For this purpose we start by choosing some auxiliary constants obeying the hierarchy

$$\delta_3 \ll d_3, m^{-1} \ll \varepsilon.$$

For these choices of δ_3, d_3 and $F = K_4^{(3)-}$ we appeal to Theorem 3.4 to obtain $\delta_2, N \colon \mathbb{N} \to \mathbb{N}$. Without loss of generality we may assume that for all $\ell \in \mathbb{N}$,

$$\delta_2(\ell) \ll \ell^{-1}, \varepsilon.$$

Applying Corollary 3.3 to d_3, δ_3, m, and δ_2 we get two integers n_0' and T_0. Now we claim that any

$$\eta \ll T_0^{-1} \quad \text{and} \quad n_0 \gg n_0', T_0 \cdot N(T_0)$$

are as desired.

To justify this, we let any weakly $(1/4+\varepsilon, \eta)$-quasirandom hypergraph H on $n \geq n_0$ vertices be given. Since $n \geq n_0'$ holds as well, we may apply Corollary 3.3, thus getting a subhypergraph $\hat{H} \subseteq H$ with vertex partition $\hat{V} = V_1 \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} V_m$ and edge partitions $\mathcal{P}^{ij} = \{P_\alpha^{ij} : \alpha \in [\ell]\}$ of $K(V_i, V_j)$ for $1 \leq i < j \leq m$.

In view of the embedding lemma (Theorem 3.4) it remains to locate four vertex classes $V_{i_1}, \dots, V_{i_4}$ with $i_1 < i_2 < i_3 < i_4$, $\max(V_{i_a}) < \min(V_{i_{a+1}})$ for $a = 1, 2, 3$, and six bipartite graphs $P^{ab} \in \mathcal{P}^{i_a i_b}$ for $1 \leq a < b \leq 4$ from the regular partition, such that at least three of the $\binom{4}{3}$ triads

$$P^{abc} = P^{ab} \mathbin{\dot\cup} P^{ac} \mathbin{\dot\cup} P^{bc}$$

with $1 \leq a < b < c \leq 4$ are *dense* and *regular*, i.e., $d(H \mid P^{abc}) \geq d_3$ and H is δ_3-regular with respect to P^{abc}. For the "moreover" part, we also have to make sure that we embed the apex vertex of $K_4^{(3)-}$ either into V_{i_1} or into V_{i_4}. This will be rendered by Lemma 4.1 (stated below).

In fact due to property (iv) of Corollary 3.3 and the weak quasirandomness of H given by the assumption of Theorem 1.4 (see (1.4)) we have

$$\begin{aligned} e_{\hat{H}}(V_i, V_j, V_k) &\geq (1/4+\varepsilon)|V_i|\,|V_j|\,|V_k| - (d_3+\delta_3)|V_i|\,|V_j|\,|V_k| - 7\eta n^3 \\ &\geq (1/4+\varepsilon/2)|V_i|\,|V_j|\,|V_k|, \end{aligned} \tag{4.1}$$

where the last step exploits $d_3, \delta_3 \ll \varepsilon$ and $\eta \ll T_0^{-1}$.

Moreover, since every triad $P_{\alpha\beta\gamma}^{ijk}$ is $(\delta_2(\ell), 1/\ell)$-regular (as a tripartite graph), the triangle counting lemma for graphs (see (3.1)) asserts that it spans at most

$$(1/\ell^3 + 3\delta_2(\ell))|V_i|\,|V_j|\,|V_k|$$

triangles. By our choice of the function δ_2 we deduce from (4.1) that for every fixed $1 \leq i < j < k \leq m$, at least

$$\frac{1/4+\varepsilon/2}{1+3\delta_2(\ell)\ell^3}\ell^3 > (1/4+\varepsilon/4)\ell^3 \tag{4.2}$$

triads $P_{\alpha\beta\gamma}^{ijk}$ satisfy $d(H \mid P_{\alpha\beta\gamma}^{ijk}) \geq d_3$.

For fixed $1 \leq i < j < k \leq m$ we consider an auxiliary tripartite 3-uniform hypergraph $\mathcal{A}^{ijk}$ with vertices corresponding to bipartite graphs from the regular partition and hyperedges representing dense triads. More precisely, we set $V(\mathcal{A}^{ijk}) = \mathcal{P}^{ij} \mathbin{\dot\cup} \mathcal{P}^{ik} \mathbin{\dot\cup} \mathcal{P}^{jk}$ and we include the triple $P_\alpha^{ij} P_\beta^{ik} P_\gamma^{jk}$ in $E(\mathcal{A}^{ijk})$ if $d(\hat{H} \mid P_{\alpha\beta\gamma}^{ijk}) \geq d_3$. This way (4.2) translates into the assertion that $\mathcal{A}^{ijk}$ contains at least $(1/4+\varepsilon/4)\ell^3$ hyperedges.

In Lemma 4.1 below we analyse the $\binom{m}{2}$-partite, 3-uniform hypergraph $\mathcal{A}$ given by the union of all $\mathcal{A}^{ijk}$ with $1 \leq i < j < k \leq m$. Note that only $\binom{m}{3}$ of the $\binom{\binom{m}{2}}{3}$ naturally induced tripartite subhypergraphs of $\mathcal{A}$ span any hyperedges. Lemma 4.1 asserts that such a hypergraph $\mathcal{A}$ contains three hyperedges on six vertices, which translates back into four vertex classes $V_{i_1}, \dots, V_{i_4}$ and six bipartite graphs $P^{ab} \in \mathcal{P}^{i_a i_b}$ for $1 \leq a < b \leq 4$ from the regular partition of $\hat{H}$, such that at least three of the four triads P^{abc} with $1 \leq a < b < c \leq 4$ satisfy $d(\hat{H} \mid P^{abc}) \geq d_3$. Since $\hat{H}$ was δ_3-regular for any triad, this shows that the assumptions of the embedding lemma, Theorem 3.4, are met for $F = K_4^{(3)-}$, and therefore $\hat{H} \subseteq H$ contains a copy of $K_4^{(3)-}$. We also note that the "moreover" part of Lemma 4.1 together with the "moreover" part of Corollary 3.3 implies that there exists indeed a copy of $K_4^{(3)-}$ in $H = ([n], E)$ with the apex vertex either at the front or at the end. This concludes the reduction of Theorem 1.4 to Lemma 4.1 (where ε corresponds to $\varepsilon/4$ in the reduction above). □

Lemma 4.1. *For every $\varepsilon > 0$ there exists an integer m such that the following holds. If $\mathcal{A}$ is an $\binom{m}{2}$-partite 3-uniform hypergraph with*

(i) *nonempty vertex classes $\mathcal{P}^{ij}$ for $1 \leq i < j \leq m$ such that*
(ii) *for each triple $1 \leq i < j < k \leq m$ the restriction $\mathcal{A}^{ijk}$ of $\mathcal{A}$ to $\mathcal{P}^{ij} \cup \mathcal{P}^{ik} \cup \mathcal{P}^{jk}$ contains at least $(1/4 + \varepsilon)|\mathcal{P}^{ij}|\,|\mathcal{P}^{ik}|\,|\mathcal{P}^{jk}|$ triples,*

then there are four distinct indices $i_1, i_2, i_3, i_4 \in [m]$ together with six vertices P^{ab} in $\mathcal{P}^{i_a i_b}$ for $1 \leq a < b \leq 4$ such that $P^{12}P^{14}P^{24}$, $P^{13}P^{14}P^{34}$, and $P^{23}P^{24}P^{34}$ are triples of $\mathcal{A}$. Moreover, there exists such a configuration with

$$i_4 = \max(i_1, i_2, i_3, i_4) \quad \text{or} \quad i_4 = \min(i_1, i_2, i_3, i_4).$$

Proof. Suppose

$$m^{-1} \ll m_*^{-1} \ll \varepsilon,$$

and let a 3-uniform hypergraph $\mathcal{A}$ as in the statement be given. Notice that each of the three vertices P^{12}, P^{13}, and P^{23} appears only once in the conclusion, so we may eliminate them from consideration by "projecting" the nonempty tripartite parts of $\mathcal{A}$ onto appropriate bipartite graphs. That is, for any three distinct indices $i, j, k \in [m]$ we define a bipartite graph Q^i_{jk} with bipartition $(\mathcal{P}^{ij}, \mathcal{P}^{ik})$ by putting an edge between $P^{ij} \in \mathcal{P}^{ij}$ and $P^{ik} \in \mathcal{P}^{ik}$ if and only if for some $P^{jk} \in \mathcal{P}^{jk}$ the triple $P^{ij}P^{ik}P^{jk}$ belongs to $E(\mathcal{A}^{ijk})$.

In the next step, we colour the 3-subsets of $[m]$ with two colours, *red* and *green*, with the intention of applying Ramsey's theorem afterwards. So let any three indices $1 \leq i < j < k \leq m$ be given. Each triple $P^{ij}P^{ik}P^{jk}$ from $E(\mathcal{A}^{ijk})$, with $P^{ij} \in \mathcal{P}^{ij}$, $P^{ik} \in \mathcal{P}^{ik}$, and $P^{jk} \in \mathcal{P}^{jk}$, gives rise to a unique pair $(P^{ij}P^{ik}, P^{ik}P^{jk})$ of edges $P^{ij}P^{ik} \in E(Q^i_{jk})$ and $P^{ik}P^{jk} \in E(Q^k_{ij})$, and hence our assumption on the density of $\mathcal{A}^{ijk}$ yields

$$\sum_{P^{ik} \in \mathcal{P}^{ik}} d_{Q^i_{jk}}(P^{ik}) d_{Q^k_{ij}}(P^{ik}) \geq (1/4 + \varepsilon)|\mathcal{P}^{ij}|\,|\mathcal{P}^{ik}|\,|\mathcal{P}^{jk}|.$$

Thus the Cauchy–Schwarz inequality shows that at least one of the two estimates

$$\sum_{P^{ik} \in \mathcal{P}^{ik}} d_{Q^i_{jk}}(P^{ik})^2 \geq (1/4 + \varepsilon)|\mathcal{P}^{ij}|^2|\mathcal{P}^{ik}| \tag{$*$}$$

or

$$\sum_{P^{ik} \in \mathcal{P}^{ik}} d_{Q^k_{ij}}(P^{ik})^2 \geq (1/4 + \varepsilon)|\mathcal{P}^{jk}|^2|\mathcal{P}^{ik}| \tag{$**$}$$

holds. Hence no clash of colours arises if we colour $\{i, j, k\}$ red when $(*)$ fails and green when $(**)$ fails. If both $(*)$ and $(**)$ are valid, the colour of $\{i, j, k\}$ is irrelevant and we make an arbitrary choice. In other words, if $\{i, j, k\}$ ends up being red, then necessarily $(**)$ holds, whilst if the triple is green, then this indicates the validity of $(*)$.

By Ramsey's theorem, or more precisely as we may assume the partition relation

$$m \longrightarrow (m_*)^3_2,$$

there is a set $X \subseteq [m]$ of size m_* such that all triples from X have the same colour. By symmetry we can assume that this colour is red, and relabelling the indices if necessary we may further suppose that $X = [m_*]$. We contend that a configuration of the desired kind can be found with $1 \le i_1 < i_2 < i_3 < m_*$ and $i_4 = m_*$.

To show this, we define an $(m_* - 1)$-partite graph G with vertex classes $W_i = \mathcal{P}^{im_*}$ for $1 \le i < m_*$ by demanding that the restriction of G to $W_i \cup W_j$ be isomorphic to $Q^{m_*}_{ij}$ whenever $1 \le i < j < m_*$. Notice that for such i and j the triple $\{i, j, m_*\}$ is red, whence $(**)$ implies

$$\sum_{P \in W_i} d_{W_j}(P)^2 \ge (1/4 + \varepsilon)|W_i|\,|W_j|^2.$$

As we could have chosen m_* so large that the conclusion of Theorem 2.1 applies to $m_* - 1$ and ε here in place of m and ε there, we may assume that G contains a triangle, say with vertices $P^{14} \in W_{i_1}$, $P^{24} \in W_{i_2}$, and $P^{34} \in W_{i_3}$, where $i_1 < i_2 < i_3$. Now, for example, $P^{14}P^{24}$ being an edge of G and hence of $Q^{m_*}_{i_1i_2}$ means that there is some vertex $P^{12} \in \mathcal{P}^{i_1i_2}$ such that the triple $P^{12}P^{14}P^{24}$ appears in $\mathcal{A}^{i_1i_2m_*}$. For the same reason, the desired vertices P^{13} and P^{23} exist. Thereby Lemma 4.1, and hence Theorem 1.4, is proved. □

5. Concluding remarks

5.1. *Turán densities of cliques in weakly quasirandom hypergraphs*

Our main result, Theorem 1.4, asserts that the weakly quasirandom Turán density of $K_4^{(3)-}$ is $1/4$, but many open questions remain. It would be very interesting to determine $\pi_{\therefore}(K_4^{(3)})$ or more generally $\pi_{\therefore}(K_k^{(3)})$ for arbitrary $k \ge 4$. We recall a random construction from [28] which shows that

$$\pi_{\therefore}(K_k^{(3)}) \ge \frac{k-3}{k-2}. \tag{5.1}$$

This lower bound is established by considering a random $(k-2)$-colouring φ of the pairs $[n]^{(2)}$, where the colour of each pair is chosen uniformly and independently among all $k-2$ colours. Given such a colouring φ we let H_φ be the 3-uniform hypergraph with vertex set $[n]$ containing only those hyperedges $\{x, y, z\}$ with $1 \le x < y < z \le n$ that satisfy $\varphi(x, y) \ne \varphi(x, z)$. One can check that for any fixed $\eta > 0$, with high probability the hypergraph H_φ is $\left(\frac{k-3}{k-2}, \eta\right)$-quasirandom for sufficiently large n. On the other hand, for any k vertices $1 \leq x_1 \leq \cdots \leq x_k \leq n$, two of the $k-1$ pairs $\{x_1, x_i\}$ with $i = 2, \ldots, k$ containing x_1 must have the same colour in φ. Consequently, $x_1, \ldots, x_k$ cannot span a clique and (5.1) follows. We believe this construction is optimal for $k = 4$ and put forward the following conjecture.

Conjecture 5.1. We have $\pi_{\therefore}(K_4^{(3)}) = 1/2$.

In [27] we establish a weaker version of Conjecture 5.1. This version is based on the following strengthened form of the assumed quasirandom condition.

Definition 5.2. A 3-uniform hypergraph $H = (V, E)$ is $(d, \eta, \overset{\bullet}{\bullet\!-\!\bullet})$*-quasirandom* if for every subset $U \subseteq V$ of vertices and every set $X \subseteq V^{(2)}$ of pairs in V the number $e(U, X)$ of ordered pairs $(u, \{x, x'\})$ satisfying $\{u, x, x'\} \in E$, $u \in U$, and $\{x, x'\} \in X$ satisfies

$$\big|e(U, X) - d|U|\,|X|\big| \leq \eta n^3$$

and we denote by $\mathscr{Q}^{(3)}_{\overset{\bullet}{\bullet\!-\!\bullet}}(d, \eta)$ the class of $(d, \eta, \overset{\bullet}{\bullet\!-\!\bullet})$-quasirandom 3-uniform hypergraphs.

With this definition at hand we define for a 3-uniform hypergraph F the corresponding quasirandom Turán density

$$\begin{aligned}\pi_{\overset{\bullet}{\bullet\!-\!\bullet}}(F) = \sup\big\{d \in [0,1] \colon &\text{for every } \eta > 0 \text{ and } n \in \mathbb{N} \text{ there exists an } F\text{-free},\\ &\text{3-uniform hypergraph } H \in \mathscr{Q}_{\overset{\bullet}{\bullet\!-\!\bullet}}(d, \eta) \text{ with } |V(H)| \geq n\big\}.\end{aligned}$$

One can check that for every $k \geq 3$, with high probability the hypergraph H_φ defined by a random $(k-2)$-colouring φ above is indeed quasirandom in the sense of Definition 5.2, i.e., it is $\big(\frac{k-3}{k-2}, \eta, \overset{\bullet}{\bullet\!-\!\bullet}\big)$-quasirandom for any fixed $\eta > 0$ for sufficiently large n. Consequently, we also have

$$\pi_{\overset{\bullet}{\bullet\!-\!\bullet}}(K_k^{(3)}) \geq \frac{k-3}{k-2}. \tag{5.2}$$

In [27] we establish a matching upper bound for $k = 4$ by a proof based on the regularity method for hypergraphs.

Theorem 5.3. *We have* $\pi_{\overset{\bullet}{\bullet\!-\!\bullet}}(K_4^{(3)}) = 1/2$.

Also for $k > 4$ it might be possible that the lower bound given in (5.1) (and (5.2)) is best possible and we are not aware of any better constructions. However, we remark that for $k = 6$ there is another construction attaining the same bound. For that we consider a random two-colouring of $[n]^{(2)}$ and let H consist of all triples $\{x, y, z\}$ for which the three pairs $\{x, y\}$, $\{x, z\}$, and $\{y, x\}$ are not all of the same colour. Again it is easy to check that with high probability the hypergraph H is $(3/4, \eta)$-quasirandom for every fixed $\eta > 0$, while the simplest instance of Ramsey's theorem, the so called "three in a party of six theorem", shows that H is $K_6^{(3)}$-free. It would be intriguing if both of these constructions were best possible.

5.2. *Hypergraph with vanishing weakly quasirandom Turán density*

For the classical Turán density $\pi(\cdot)$ Erdős [6] characterised all hypergraphs F with $\pi(F) = 0$. Restricting the discussion to 3-uniform hypergraphs, he showed that $\pi(F) = 0$ if and only if F is tripartite, i.e., $V(F)$ can be partitioned into three classes such that every hyperedge of F contains precisely one vertex from each class. Since large, complete, and balanced tripartite 3-uniform hypergraphs have density approaching $2/9$, Erdős deduced that if $\pi(F) > 0$, then $\pi(F) \geq 2/9$.

We establish a similar characterisation of $\{F : \pi_{\therefore}(F) = 0\}$. Clearly, this set contains all tripartite hypergraphs, and the additional quasirandomness assumption considered here increases this set. In fact, it follows from [18] that in addition to all tripartite hypergraphs it contains all linear 3-uniform hypergraphs F, where we say a hypergraph F is *linear* if any pair of hyperedges shares at most one vertex. In [24] we obtain the following characterisation of hypergraphs with vanishing weakly quasirandom Turán density.

Theorem 5.4. *For each* 3*-uniform hypergraph F, the following are equivalent:*

(a) $\pi_{\therefore}(F) = 0$.

(b) *There is an enumeration of the vertices of F as $v_1, \dots, v_f$ together with a colouring of the pairs of vertices of F using red, blue and green colours such that if for $i < j < k$ the triple $\{v_i, v_j, v_k\}$ is a hyperedge of F, then $\{v_i, v_j\}$ is red, $\{v_i, v_k\}$ is blue, and $\{v_j, v_k\}$ is green.*

Theorem 5.4 has the following consequence, which shows that $\pi_{\therefore}$ "jumps" from 0 to at least $1/27$.

Corollary 5.5. *If a* 3*-uniform hypergraph F satisfies $\pi_{\therefore}(F) > 0$, then $\pi_{\therefore}(F) \geq 1/27$.*

For the proof of Corollary 5.5 we will display a weakly quasirandom hypergraph H of density $1/27$ which only contains subhypergraphs satisfying condition (b) of Theorem 5.4 (in fact, it will be universal for all such hypergraphs). Consequently, if $\pi_{\therefore}(F) > 0$, then by Theorem 5.4 the hypergraph F fails to satisfy (b), whence it is not contained in H, and therefore $\pi_{\therefore}(F) \geq 1/27$.

The hypergraph H will given by the following random construction: We consider a random three-colouring $\psi : [n]^{(2)} \to \{\text{red, blue, green}\}$. For a given ψ we define the 3-uniform hypergraph $H = H_\psi$ on the vertex set $[n]$, where we include the triple $\{i, j, k\}$ with $1 \leq i < j < k \leq n$ in $E(H_\psi)$ if $\psi(i, j)$ is red, $\psi(i, k)$ is blue, and $\psi(j, k)$ is green. It follows that for any $\eta > 0$, with high probability the random hypergraph H_ψ is weakly $(1/27, \eta)$-quasirandom for sufficiently large n. Moreover, it follows from the construction that every subhypergraph of H_ψ satisfies condition (b) of Theorem 5.4, and hence Corollary 5.5 follows from Theorem 5.4.

We also note that if F satisfies $\pi_{\therefore}(F) = 0$, then by definition of $\pi_{\therefore}$ the hypergraph F is contained in any weakly quasirandom hypergraph of positive density, and in particular $F \subseteq H_\psi$. Hence, ψ restricted to the vertices of this copy of F shows that F satisfies (b) of Theorem 5.4, which establishes the implication (a)$\Rightarrow$(b). The converse is the main part of Theorem 5.4; it is based on the regularity method for hypergraphs and is the main result of [24].

5.3. Two extensions of Theorem 1.4

We may suggest two extensions of the main result. Theorem 1.4 concerns the weakly quasirandom Turán density for the hypergraphs $K_4^{(3)-}$. This hypergraph consists of one apex vertex a whose link graph, i.e., the set of pairs that together with a form a hyperedge in $K_4^{(3)-}$, is a triangle. It would be of interest to study the case, when the triangle is replaced by a larger clique. We discuss partial results on this problem in Section 5.3.1.

For another extension of Theorem 1.4 we consider $K_4^{(3)-}$ as a 3-uniform hypergraph with three hyperedges on four vertices and, similarly, for $r \geq 3$ we may consider r-uniform hypergraphs with three edges on $(r+1)$-vertices. In fact, we have established the quasirandom Turán density for these hypergraphs, if the r-uniform hyperedges of H are quasirandomly distributed with respect to the $(r-2)$-tuples of the vertex set (see Section 5.3.2).

5.3.1. Extending graph cliques to hypergraphs. We consider the following star-like 3-uniform hypergraphs S_k. For $k \geq 3$ the hypergraph S_k has vertex set $\{a, b_1, \dots, b_k\}$ and for all $\binom{k}{2}$ pairs $1 \leq i < j \leq k$ the triple $\{a, b_i, b_j\}$ is a hyperedge of S_k. We refer to the vertex a which is contained in every hyperedge of S_k as the *apex vertex*. Clearly, for $k = 3$ we have $S_3 = K_4^{(3)-}$ and by Theorem 1.4 we have $\pi_{\therefore}(S_3) = 1/4$. From this point of view the natural question is to determine $\pi_{\therefore}(S_k)$ for $k > 3$. We have partial results in this direction. Let us begin with S_4.

Theorem 5.6. *We have* $1/3 \leq \pi_{\therefore}(S_4) \leq 4/9$.

The upper bound can be proved along the lines of Theorem 1.4 by using Theorem 2.2 for K_4 instead of Theorem 2.1.

The lower bound is given by the following construction. Again we consider a random three-colouring $\psi\colon [n]^{(2)} \to \{\text{red}, \text{blue}, \text{green}\}$. Given ψ we define a 3-uniform hypergraph $H = H_\psi$ on the vertex set $[n]$ containing those hyperedges $\{x, y, z\}$ with $x < y < z$ where the colour pattern of the three pairs $\{x, y\}$, $\{x, z\}$, and $\{y, z\}$ satisfies

(i) $\psi(x, y) = \psi(y, z) \neq \psi(x, z)$, or
(ii) the ordered colour pattern $(\psi(x, y), \psi(x, z), \psi(y, z))$ is one of the three *rainbow* patterns (red, blue, green), (green, red, blue), or (blue, green, red).

Note that there are six patterns of the first kind and so in total for the hyperedges of H we allow nine of the 27 possible combinations. Standard probabilistic tail estimates show that, for any $\eta > 0$, with high probability H is weakly $(1/3, \eta)$-quasirandom provided n is sufficiently large.

It remains to show that H contains no copy of S_4. Supposing the contrary, let $a \in [n]$ be the apex vertex of a copy of S_4 in H and consider its monochromatic neighbourhoods with respect to ψ, i.e., set

$$N^{<}_{\text{red}}(a) = \{x < a \colon \psi(x, a) = \text{red}\} \quad \text{and} \quad N^{>}_{\text{red}}(a) = \{x > a \colon \psi(a, x) = \text{red}\},$$

and let $N^{<}_{\text{green}}(a)$, $N^{>}_{\text{green}}(a)$, $N^{<}_{\text{blue}}(a)$, and $N^{>}_{\text{blue}}(a)$ be defined similarly for the other two colours. By definition these six sets partition the set $V_a = [n] \setminus \{a\}$. We consider the *link* *graph* L_a of a with vertex set V_a where $\{u, v\}$ forms an edge if $\{a, u, v\}$ is a hyperedge of H. Note that due to the colour patterns allowed in (i) and (ii) the six neighbourhood sets are independent sets in L_a. Moreover, one can check that the three sets

$$N^{<}_{\text{red}}(a) \cup N^{>}_{\text{blue}}(a), \quad N^{<}_{\text{blue}}(a) \cup N^{>}_{\text{green}}(a), \quad \text{and} \quad N^{<}_{\text{green}}(a) \cup N^{>}_{\text{red}}(a)$$

are also independent sets and partition V_a. In other words, the link graph L_a is 3-colourable, and hence it cannot contain a copy of K_4. In particular, the vertex a cannot be the apex vertex of a copy of S_4 in H.

For general k we can prove

$$\frac{k^2 - 5k + 7}{(k-1)^2} \le \pi_{\therefore}(S_k) \le \left(\frac{k-2}{k-1}\right)^2. \tag{5.3}$$

The upper bound follows like the proof for $k = 4$ along the lines of Theorem 1.4 with the generalisation of Theorem 2.1 for the clique K_k (see Theorem 2.2).

For the lower bound we consider a random $(k-1)$-colouring

$$\psi\colon [n]^{(2)} \to \{0, \dots, k-2\}.$$

As before, the colour pattern we see on the pairs of three vertices $x < y < z$ determines whether this triple forms a hyperedge of H. In the general case we allow the following patterns:

(i) $\psi(x, y) = \psi(y, z) \ne \psi(x, z)$, or
(ii) the ordered colour pattern $(\psi(x, y), \psi(x, z), \psi(y, z))$ is rainbow (i.e., all three colours are different), but not of the form $(i, j, i+1)$ for $i = 0, \dots, k-2$ and $j \notin \{i, i+1\}$, where addition is taken modulo $k-1$.

This way, of all different $(k-1)^3$ patterns we allow $(k-1)(k-2)$ patterns by part (i) of the definition and $(k-1)(k-2)(k-3) - (k-1)(k-3) = (k-1)(k-3)^2$ patterns in (ii). Hence, with high probability the hypergraph H is weakly (d, η)-quasirandom for any fixed $\eta > 0$ and

$$d = \frac{(k-1)(k-2) + (k-1)(k-3)^2}{(k-1)^3} = \frac{k^2 - 5k + 7}{(k-1)^2}.$$

Moreover, as above one can show that the link graph L_a of every vertex $a \in [n]$ is $(k-1)$-colourable, and hence it contains no K_k. In fact, with similar notation as above it can be checked that the sets

$$N^{<}_{i}(a) \cup N^{>}_{i+1}(a)$$

for $i = 0, \dots, k-2$ form a partition of $[n] \setminus \{a\}$ into independent sets in L_a. This establishes the lower bound of (5.3).

5.3.2. Three r-tuples on $r+1$ vertices. In their concluding remarks in [15], Glebov, Král', and Volec suggested an analogue of Theorem 1.4 in the context of r-uniform hypergraphs. Instead of looking at $K_4^{(3)-}$ they propose to look at the r-uniform hypergraph $F^{(r)}$ on $r+1$ vertices with three edges, so that an r-uniform hypergraph H contains $F^{(r)}$ if and only if the link of some $(r-2)$-set of vertices contains a triangle. This is perfectly suited for the natural generalization of Example 1.3 to this context.

To keep the discussion simple, we stick to the case $r = 4$. Then one may start from a random directed 3-uniform hypergraph $T_n^{(3)}$ with vertex set $[n]$ in which for any 3-element subset of $[n]$ one of its two *cyclic orientations* has been chosen at random with probabilities $1/2$, all of these choices being mutually independent. Then, we consider a 4-element set to be a hyperedge of the corresponding 4-uniform hypergraph $H(T_n^{(3)})$ if and only if each of its 2-element subsets is traversed by the two triples containing it in opposite directions. So $\{w, x, y, z\} \in E(H(T_n^{(3)}))$ happens for example in case $\overrightarrow{xyz}, \overrightarrow{xwy}, \overrightarrow{xzw}, \overrightarrow{ywz} \in E(T_n^{(3)})$. It is not hard to show that such a hypergraph $H(T_n^{(3)})$ is $F^{(4)}$-free. Also, it is easily checked that this hypergraph has density $1/8$ and is *weakly quasirandom* (i.e., it has uniform hyperedge distribution with respect to sets of vertices). This means that in analogy with (1.4), for any $\eta > 0$, if n is sufficiently large, then with high probability all sets U_1, U_2, U_3, and U_4 of vertices satisfy

$$e(U_1, U_2, U_3, U_4) = \tfrac{1}{8}|U_1|\,|U_2|\,|U_3|\,|U_4| \pm \eta n^4,$$

where $e(U_1, U_2, U_3, U_4)$ contains all 4-tuples $(u_1, u_2, u_3, u_4) \in |U_1| \times |U_2| \times |U_3| \times |U_4|$ such that $\{u_1, u_2, u_3, u_4\}$ is a hyperedge of $H(T_n^{(3)})$. This prompted the authors of [15] to conjecture that any weakly quasirandom 4-uniform hypergraph H with density $> 1/8$ contains a copy of $F^{(4)}$.

An interesting hypergraph described by Leader and Tan [19] in a different context shows however that this is not the case, and that at least twice as much density is needed. Their construction starts from a random (graph) tournament T_n on n vertices as in Example 1.3. Depending on T_n, they define a directed 3-uniform hypergraph $D_n^{(3)}$ by assigning the cyclic orientation to any 3-element set $\{x, y, z\}$ in such a way that it coincides with the direction of the three arcs spanned by $\{x, y, z\}$ in T_n either once or three times. So, e.g. if $\overrightarrow{xy}, \overrightarrow{yz}, \overrightarrow{zx} \in E(T_n)$, then $\overrightarrow{xyz} \in E(D_n^{(3)})$, while if $\overrightarrow{xy}, \overrightarrow{xz}, \overrightarrow{yz} \in E(T_2)$, then $\overrightarrow{xzy} \in E(D_n^{(3)})$. Now the 4-uniform hypergraph $H(D_n^{(3)})$ defined as in the previous paragraph but starting from $D_n^{(3)}$ rather than the random orientation $T_n^{(3)}$ is easily shown to have density about $1/4$. Moreover, it is weakly quasirandom and contains no copy of $F^{(4)}$.

In the light of this example, we propose a modification of the original question: it may be observed that the intended extremal example $H(T_n^{(3)})$ has stronger quasirandomness properties than $H(D_n^{(3)})$ does. Notably, it behaves quasirandomly with respect to pairs, which means that for any six graphs $G_{12}, \dots, G_{34}$ on $[n]$, about $1/8$ of the quadruples (x_1, x_2, x_3, x_4) with $\{x_1, x_2\} \in E(G_{12}), \dots, \{x_3, x_4\} \in E(G_{34})$ satisfy $\{x_1, x_2, x_3, x_4\} \in E(H)$. One may also show directly that the hypergraph $H(D_n^{(3)})$ lacks this property.

This may suggest that any 4-uniform hypergraph with density $> 1/8$ that is quasirandom with respect to pairs in this sense does indeed contain a copy of $F^{(4)}$. More generally, we show in [25] that an r-uniform hypergraph of density $> 2^{1-r}$ that is quasirandom with respect to $(r-2)$-tuples has to contain $F^{(r)}$. The proof presented in [25] relies on the regularity method for r-uniform hypergraphs and is considerably more intricate than the argument presented here.

We note that the case $r = 2$ of this result might be viewed as the density version of Mantel's theorem. Thus the "three edge theorem" in [25] provides a common generalisation of Mantel's theorem and Theorem 1.4 to the context of r-uniform hypergraphs.

Acknowledgments. The second author thanks Yoshi Kohayakawa and Endre Szemerédi for early discussions on the problem.

The second author was supported by NSF grants DMS 1301698 and 1102086.

The third author was supported through the Heisenberg-Programme of the DFG.

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