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Affinizations and R-matrices for quiver Hecke algebras

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Abstract. We introduce the notion of affinizations and R-matrices for arbitrary quiver Hecke algebras. It is shown that they enjoy similar properties to those for symmetric quiver Hecke algebras. We next define a duality datum $\mathcal{D}$ and construct a tensor functor $\mathfrak{F}^{\mathcal{D}}\colon \mathrm{Mod}_{\mathrm{gr}}(R^{\mathcal{D}}) \to \mathrm{Mod}_{\mathrm{gr}}(R)$ between graded module categories of quiver Hecke algebras R and $R^{\mathcal{D}}$ arising from $\mathcal{D}$. The functor $\mathfrak{F}^{\mathcal{D}}$ sends finite-dimensional modules to finite-dimensional modules, and is exact when $R^{\mathcal{D}}$ is of finite type. It is proved that affinizations of real simple modules and their R-matrices give a duality datum. Moreover, the corresponding duality functor sends every simple module to a simple module or zero when $R^{\mathcal{D}}$ is of finite type. We give several examples of the functors $\mathfrak{F}^{\mathcal{D}}$ from the graded module category of the quiver Hecke algebra of type D_ℓ, C_ℓ, $B_{\ell-1}$, $A_{\ell-1}$ to that of type A_ℓ, A_ℓ, B_ℓ, B_ℓ, respectively.

Keywords. Quiver Hecke algebra, affinization, R-matrix, duality functor

Introduction

Quiver Hecke algebras (or *Khovanov–Lauda–Rouquier algebras*), introduced by Khovanov–Lauda [12, 13] and Rouquier [17] independently, are $\mathbb{Z}$-graded algebras which provide a categorification for the negative half of a quantum group. These algebras are a vast generalization of affine Hecke algebras of type A in the direction of categorification [1, 17], and they have special graded quotients, called *cyclotomic quiver Hecke algebras*, which categorify irreducible integrable highest weight modules [4]. When the quiver Hecke algebras are *symmetric*, we can study them more deeply.

- First of all, it is known that the upper global basis corresponds to the set of isomorphism classes of simple modules over symmetric quiver Hecke algebras [18, 19].

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- The KLR-type quantum affine Schur–Weyl duality functor was constructed in [5] using symmetric quiver Hecke algebras and R-matrices of quantum affine algebras. This functor has been studied in various types [6, 7, 9].

The notion of R-*matrices* for symmetric quiver Hecke algebras was introduced in [5]. The R-matrices are special homomorphisms defined by using intertwiners and affinizations. It turned out that the R-matrices have very good properties with respect to *real* simple modules [10]. They also have an important role as a main tool in studying a monoidal categorification of quantum cluster algebras [8].

Let us explain the construction of R-matrices in [5] briefly. We assume that the quiver Hecke algebra R is symmetric. Let M be an R-module and M_z its affinization. The R-module M_z is isomorphic to $\mathbf{k}[z] \otimes_{\mathbf{k}} M$ as a $\mathbf{k}$-vector space. The actions of $e(\nu)$ and τ_i on M_z are the same as those on M, but the action of x_i on M_z is equal to the action of x_i on M with the action of z added (see (1.8)). For R-modules M and N, we next consider the homomorphism $R_{M_z,N_{z'}} \in \mathrm{H\scriptstyle{OM}}_R(M_z \circ N_{z'}, N_{z'} \circ M_z)$ given by using intertwiners (see (1.7)). Here $\mathrm{H\scriptstyle{OM}}$ denotes the non-graded homomorphism space (see (1.5)). We set

$$R^{\mathrm{norm}}_{M_z,N_{z'}} := (z'-z)^{-s} R_{M_z,N_{z'}}, \qquad \mathbf{r}_{M,N} := R^{\mathrm{norm}}_{M_z,N_{z'}}|_{z=z'=0},$$

where s is the order of the zero of $R_{M_z,N_{z'}}$. Then the morphisms $R^{\mathrm{norm}}_{M_z,N_{z'}}$ and $\mathbf{r}_{M,N}$ are non-zero, commute with the spectral parameters z, z', and satisfy the braid relations. Here, in defining M_z and $\mathbf{r}_{M,N}$, we crucially use the fact that R is symmetric.

In this paper, we introduce and investigate the notion of affinizations and R-matrices for *arbitrary* quiver Hecke algebras, and construct a new duality functor between finitely generated graded module categories of quiver Hecke algebras. The affinizations defined in this paper generalize the affinizations M_z for symmetric quiver Hecke algebras. The root modules given in [2] are examples of affinizations.

We then define a tensor functor $\mathfrak{F}^{\mathcal{D}} : \mathrm{Mod}_{\mathrm{gr}}(R^{\mathcal{D}}) \to \mathrm{Mod}_{\mathrm{gr}}(R)$ between the graded module categories of the quiver Hecke algebras R and $R^{\mathcal{D}}$, which arises from a *duality datum* $\mathcal{D}$ consisting of certain R-modules and their homomorphisms. This is inspired by the KLR-type quantum affine Schur–Weyl duality functor of [5]. The functor $\mathfrak{F}^{\mathcal{D}}$ sends finite-dimensional modules to finite-dimensional modules. It is exact when $R^{\mathcal{D}}$ is of finite type. We show that affinizations of real simple modules and their R-matrices give a duality datum. The corresponding duality functor sends every simple module to a simple module or zero when $R^{\mathcal{D}}$ is of finite type.

Here is a brief description of our work. Let $R(\beta)$ be an arbitrary quiver Hecke algebra. We define an affinization $(\mathsf{M}, z_{\mathsf{M}})$ of a simple $R(\beta)$-module $\bar{M}$ to be an $R(\beta)$-module M with a homogeneous endomorphism $z_{\mathsf{M}} \in \mathrm{End}_R(\mathsf{M})$ and an isomorphism $\mathsf{M}/z_{\mathsf{M}}\mathsf{M} \simeq \bar{M}$ satisfying the conditions in Definition 2.2.

We then study the endomorphism rings of affinizations and the homomorphism spaces between convolution products of simple modules and their affinizations. For a non-zero R-module N, let s be the largest integer such that $R_{\mathsf{M},N}(\mathsf{M} \circ N) \subset z_{\mathsf{M}}^s N \circ \mathsf{M}$. We set

$$R^{\mathrm{norm}}_{\mathsf{M},N} = z_{\mathsf{M}}^{-s} R_{\mathsf{M},N} \colon \mathsf{M} \circ N \to N \circ \mathsf{M},$$

and denote by $\mathbf{r}_{\bar{M},N}\colon \bar{M}\circ N \to N\circ\bar{M}$ the homomorphism induced by $R^{\mathrm{norm}}_{\mathsf{M},N}$. By the definition $\mathbf{r}_{\bar{M},N}$ never vanishes. The R-matrix $\mathbf{r}_{\bar{M},N}$ has similar properties to R-matrices for symmetric quiver Hecke algebras (Proposition 2.10). Proposition 2.12 tells us that if $(\mathsf{M}, z_{\mathsf{M}})$ and $(\mathsf{N}, z_{\mathsf{N}})$ are affinizations of simple modules $\bar{M}$ and $\bar{N}$ and one of $\bar{M}$ and $\bar{N}$ is real (see (2.7)), then

(i) $\mathrm{HOM}_{R[z_{\mathsf{M}},z_{\mathsf{N}}]}(\mathsf{M}\circ\mathsf{N}, \mathsf{M}\circ\mathsf{N}) = \mathbf{k}[z_{\mathsf{M}}, z_{\mathsf{N}}]\,\mathrm{id}_{\mathsf{M}\circ\mathsf{N}}$,
(ii) $\mathrm{HOM}_{R[z_{\mathsf{M}},z_{\mathsf{N}}]}(\mathsf{M}\circ\mathsf{N}, \mathsf{N}\circ\mathsf{M})$ is a free $\mathbf{k}[z_{\mathsf{M}}, z_{\mathsf{N}}]$-module of rank one.

Here, HOM denotes the space of non-graded homomorphisms (see (1.5)). We define $R^{\mathrm{norm}}_{\mathsf{M},\mathsf{N}}$ as a generator of the $\mathbf{k}[z_{\mathsf{M}}, z_{\mathsf{N}}]$-module $\mathrm{HOM}_{R[z_{\mathsf{M}},z_{\mathsf{N}}]}(\mathsf{M}\circ\mathsf{N}, \mathsf{N}\circ\mathsf{M})$. Then $R^{\mathrm{norm}}_{\mathsf{M},\mathsf{N}}$ commutes with z_{M} and z_{N} by construction, and we prove that $R^{\mathrm{norm}}_{\mathsf{M},\mathsf{N}}|_{z_{\mathsf{M}}=z_{\mathsf{N}}=0} \in \mathrm{HOM}(\bar{M}\circ\bar{N}, \bar{N}\circ\bar{M})$ does not vanish and coincides with $\mathbf{r}_{\bar{M},\bar{N}}$ up to a constant multiple (Theorem 2.13).

We next define the duality datum $\mathcal{D} = \{\beta_j, M_j, \mathsf{z}_j, \mathsf{r}_j, \mathsf{R}_{j,k}\}_{j,k\in J}$ axiomatically. Here, J is a finite index set, and

$$M_j \in \mathrm{Mod}_{\mathrm{gr}}(R(\beta_j)), \qquad \mathsf{z}_j \in \mathrm{END}_{R(\beta_j)}(M_j),$$
$$\mathsf{r}_j \in \mathrm{END}_{R(2\beta_j)}(M_j\circ M_j), \qquad \mathsf{R}_{j,k} \in \mathrm{HOM}_{R(\beta_j+\beta_k)}(M_j\circ M_k, M_k\circ M_j),$$

satisfying certain conditions given in Definition 4.1. We construct a generalized Cartan matrix $\mathsf{A}^{\mathcal{D}}$ and polynomial parameters $\mathcal{Q}^{\mathcal{D}}_{i,j}(u,v)$ from the duality datum $\mathcal{D}$ and consider the quiver Hecke algebra $R^{\mathcal{D}}$ corresponding to $\mathsf{A}^{\mathcal{D}}$ and $\mathcal{Q}^{\mathcal{D}}_{i,j}(u,v)$. For $\gamma \in \mathsf{Q}^{\mathcal{D}}_{+}$ with $m = \mathrm{ht}(\gamma)$, we define

$$\Delta^{\mathcal{D}}(\gamma) := \bigoplus_{\mu\in J^{\gamma}} \Delta^{\mathcal{D}}_{\mu},$$

where

$$\Delta^{\mathcal{D}}_{\mu} := M_{\mu_1}\circ\cdots\circ M_{\mu_m} \quad \text{for } \mu = (\mu_1,\ldots,\mu_m)\in J^{\gamma}.$$

It turns out that $\Delta^{\mathcal{D}}(\gamma)$ has an $(R, R^{\mathcal{D}})$-bimodule structure (Theorem 4.2), and we obtain the duality functor $\mathfrak{F}^{\mathcal{D}}\colon \mathrm{Mod}_{\mathrm{gr}}(R^{\mathcal{D}}) \to \mathrm{Mod}_{\mathrm{gr}}(R)$ by tensoring $\Delta^{\mathcal{D}}(\gamma)$. Theorem 4.3 tells us that $\mathfrak{F}^{\mathcal{D}}$ is a tensor functor and sends finite-dimensional modules to finite-dimensional modules. Moreover, it is exact when $\mathsf{A}^{\mathcal{D}}$ is of finite type. Affinizations of real simple modules and their R-matrices provide a duality functor which enjoys extra good properties (Theorem 4.4).

Several examples of duality functors $\mathfrak{F}^{\mathcal{D}}$ are given in Sections 5 and 6. In Example 5.2, we construct a duality functor $\mathfrak{F}^{\mathcal{D}}$ from the graded module category of a quiver Hecke algebra of type D_ℓ to that of type A_ℓ. The other examples are in non-symmetric cases. We discuss a duality functor from type C_ℓ to type A_ℓ in Example 6.2, and ones from types $B_{\ell-1}$ and $A_{\ell-1}$ to type B_ℓ in Examples 6.3 and 6.4.

1. Preliminaries

1.1. Quantum groups

Let I be an index set.

Definition 1.1. A *Cartan datum* is a quintuple $(\mathsf{A}, \mathsf{P}, \Pi, \Pi^\vee, (\cdot,\cdot))$ consisting of

(a) a free abelian group P, called the *weight lattice*,
(b) $\Pi = \{\alpha_i \mid i \in I\} \subset \mathsf{P}$, called the set of *simple roots*,
(c) $\Pi^\vee = \{h_i \mid i \in I\} \subset \mathsf{P}^\vee := \mathrm{Hom}(\mathsf{P}, \mathbb{Z})$, called the set of *simple coroots*,
(d) a $\mathbb{Q}$-valued symmetric bilinear form $(\cdot,\cdot)$ on P,

which satisfy

(1) $(\alpha_i, \alpha_i) \in 2\mathbb{Z}_{>0}$ for any $i \in I$,
(2) $\langle h_i, \lambda\rangle = 2(\alpha_i, \lambda)/(\alpha_i, \alpha_i)$ for any $i \in I$ and $\lambda \in \mathsf{P}$,
(3) $\mathsf{A} := (\langle h_i, \alpha_j\rangle)_{i,j\in I}$ is a *generalized Cartan matrix*, i.e., $\langle h_i, \alpha_i\rangle = 2$ for any $i \in I$ and $\langle h_i, \alpha_j\rangle \in \mathbb{Z}_{\le 0}$ if $i \neq j$,
(4) Π is a linearly independent set,
(5) for each $i \in I$, there exists $\Lambda_i \in \mathsf{P}$ such that $\langle h_j, \Lambda_i\rangle = \delta_{ij}$ for any $j \in I$.

Let us write $\mathsf{Q} = \bigoplus_{i\in I} \mathbb{Z}\alpha_i$ and $\mathsf{Q}_+ = \sum_{i\in I} \mathbb{Z}_{\ge 0}\alpha_i$. For $\beta = \sum_{i\in I} k_i\alpha_i \in \mathsf{Q}_+$, set $\mathrm{ht}(\beta) = \sum_{i\in I} k_i$. The *Weyl group* W associated with the Cartan datum is the subgroup of $\mathrm{Aut}(\mathsf{P})$ generated by the reflections $\{r_i\}_{i\in I}$ defined by

$$r_i(\lambda) = \lambda - \langle h_i, \lambda\rangle\alpha_i \quad \text{for } \lambda \in \mathsf{P}.$$

Let $\mathfrak{g}$ be the Kac–Moody algebra associated with a Cartan datum $(\mathsf{A}, \mathsf{P}, \Pi, \Pi^\vee, (\cdot,\cdot))$ and Φ_+ the set of positive roots of $\mathfrak{g}$. We denote by $U_q(\mathfrak{g})$ the corresponding quantum group, which is an associative algebra over $\mathbb{Q}(q)$ generated by e_i, f_i $(i \in I)$ and q^h $(h \in \mathsf{P}^\vee)$ with certain defining relations (see [3, Chap. 3] for details). Set $\mathbf{A} = \mathbb{Z}[q, q^{-1}]$. We denote by $U_{\mathbf{A}}^-(\mathfrak{g})$ the subalgebra of $U_q(\mathfrak{g})$ generated by $f_i^{(n)} := f_i^n/[n]_i!$ for $i \in I$ and $n \in \mathbb{Z}_{\ge 0}$, where $q_i = q^{(\alpha_i,\alpha_i)/2}$ and

$$[n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \qquad [n]_i! = \prod_{k=1}^{n} [k]_i.$$

1.2. Quiver Hecke algebras

Let $\mathbf{k}$ be a field. For $i, j \in I$, we take polynomials $\mathcal{Q}_{i,j}(u,v) \in \mathbf{k}[u,v]$ such that

(i) $\mathcal{Q}_{i,j}(u,v) = \mathcal{Q}_{j,i}(v,u)$,
(ii)

$$\mathcal{Q}_{i,j}(u,v) = \begin{cases} \sum_{2(\alpha_i,\alpha_j)+p(\alpha_i,\alpha_i)+q(\alpha_j,\alpha_j)=0} t_{i,j;p,q} u^p v^q & \text{if } i \neq j, \\ 0 & \text{if } i = j, \end{cases} \tag{1.1}$$

where $t_{i,j;-a_{ij},0} \in \mathbf{k}^\times$. We set

$$\overline{\mathcal{Q}}_{i,j}(u,v,w) = \frac{\mathcal{Q}_{i,j}(u,v) - \mathcal{Q}_{i,j}(w,v)}{u - w} \in \mathbf{k}[u,v,w]. \tag{1.2}$$

For $\beta \in \mathsf{Q}_+$ with $\mathrm{ht}(\beta) = n$, set

$$I^\beta := \Big\{ \nu = (\nu_1, \ldots, \nu_n) \in I^n \;\Big|\; \sum_{k=1}^{n} \alpha_{\nu_k} = \beta \Big\}.$$

The symmetric group $\mathfrak{S}_n = \langle s_k \mid k = 1, \ldots, n-1\rangle$ on n letters, where s_k is the transposition of k and $k+1$, acts on I^β by place permutations.

Definition 1.2. For $\beta \in \mathsf{Q}_+$, the *quiver Hecke algebra* $R(\beta)$ associated with A and $(\mathcal{Q}_{i,j}(u,v))_{i,j\in I}$ is the $\mathbf{k}$-algebra generated by

$$\{e(\nu) \mid \nu \in I^\beta\}, \quad \{x_k \mid 1 \le k \le n\}, \quad \{\tau_l \mid 1 \le l \le n-1\}$$

satisfying the following defining relations:

$$e(\nu)e(\nu') = \delta_{\nu,\nu'}e(\nu), \quad \sum_{\nu\in I^\beta} e(\nu) = 1, \quad x_k e(\nu) = e(\nu)x_k, \quad x_k x_l = x_l x_k,$$

$$\tau_l e(\nu) = e(s_l(\nu))\tau_l, \quad \tau_k\tau_l = \tau_l\tau_k \quad \text{if } |k-l|>1,$$

$$\tau_k^2 e(\nu) = \mathcal{Q}_{\nu_k,\nu_{k+1}}(x_k, x_{k+1})e(\nu),$$

$$(\tau_k x_l - x_{s_k(l)}\tau_k)e(\nu) = \begin{cases} -e(\nu) & \text{if } l = k \text{ and } \nu_k = \nu_{k+1}, \\ e(\nu) & \text{if } l = k+1 \text{ and } \nu_k = \nu_{k+1}, \\ 0 & \text{otherwise,} \end{cases} \tag{1.3}$$

$$(\tau_{k+1}\tau_k\tau_{k+1} - \tau_k\tau_{k+1}\tau_k)e(\nu) = \begin{cases} \overline{\mathcal{Q}}_{\nu_k,\nu_{k+1}}(x_k, x_{k+1}, x_{k+2})e(\nu) & \text{if } \nu_k = \nu_{k+2}, \\ 0 & \text{otherwise.} \end{cases}$$

The algebra $R(\beta)$ has the $\mathbb{Z}$-graded algebra structure given by

$$\deg(e(\nu)) = 0, \quad \deg(x_k e(\nu)) = (\alpha_{\nu_k}, \alpha_{\nu_k}), \quad \deg(\tau_l e(\nu)) = -(\alpha_{\nu_l}, \alpha_{\nu_{l+1}}). \tag{1.4}$$

For $\beta \in \mathsf{Q}_+$, let us denote by $\mathrm{Mod}(R(\beta))$ the category of $R(\beta)$-modules and by $R(\beta)$-mod the category of finite-dimensional $R(\beta)$-modules.

We denote by $\mathrm{Mod}_{\mathrm{gr}}(R(\beta))$ the category of graded $R(\beta)$-modules and by $R(\beta)$-gmod the category of finite-dimensional graded $R(\beta)$-modules. We denote by $\mathrm{Mod}_{\mathrm{fg}}(R(\beta))$ the full subcategory of $\mathrm{Mod}_{\mathrm{gr}}(R(\beta))$ consisting of finitely generated graded $R(\beta)$-modules. Their morphisms are homogeneous of degree zero. Hence, $\mathrm{Mod}(R(\beta))$, $R(\beta)$-mod, $\mathrm{Mod}_{\mathrm{gr}}(R(\beta))$, $R(\beta)$-gmod and $\mathrm{Mod}_{\mathrm{fg}}(R(\beta))$ are abelian categories. We set $\mathrm{Mod}_{\mathrm{gr}}(R) := \bigoplus_{\beta\in\mathsf{Q}^+} \mathrm{Mod}_{\mathrm{gr}}(R(\beta))$, R-mod $:= \bigoplus_{\beta\in\mathsf{Q}^+} R(\beta)$-mod, etc. The objects of $\mathrm{Mod}_{\mathrm{gr}}(R)$ are sometimes simply called R-modules.

We denote by $R(\beta)$-proj the full subcategory of $\mathrm{Mod}_{\mathrm{gr}}(R(\beta))$ consisting of finitely generated projective graded $R(\beta)$-modules.

Let us denote by q the *grading shift functor*, i.e., $(qM)_k = M_{k-1}$ for a graded module $M = \bigoplus_{k\in\mathbb{Z}} M_k$.

For $\nu \in I^\beta$ and $\nu' \in I^{\beta'}$, let $e(\nu, \nu')$ be the idempotent corresponding to the concatenation $\nu * \nu'$ of ν and ν', and set

$$e(\beta, \beta') := \sum_{\nu\in I^\beta,\ \nu'\in I^{\beta'}} e(\nu, \nu').$$

For an $R(\beta)$-module M and an $R(\beta')$-module N, we define an $R(\beta+\beta')$-module $M \circ N$ by

$$M \circ N := R(\beta+\beta')e(\beta,\beta') \underset{R(\beta)\otimes R(\beta')}{\otimes} (M \otimes N).$$

We denote by $M \diamond N$ the head of $M \circ N$.

For a graded $R(\beta)$-module M, the *q-character* of M is defined by

$$\mathrm{ch}_q(M) := \sum_{\nu \in I^\beta} \dim_q(e(\nu)M)\nu.$$

Here, $\dim_q V := \sum_{k\in\mathbb{Z}} \dim(V_k)q^k$ for a graded vector space $V = \bigoplus_{k\in\mathbb{Z}} V_k$. It is well-defined whenever $\dim V_k < \infty$ for all $k \in \mathbb{Z}$.

For $i \in I$, let $L(\alpha_i)$ be the simple graded $R(\alpha_i)$-module such that $\mathrm{ch}_q(L(\alpha_i)) = (i)$. For simplicity, we write $L(i)$ for $L(\alpha_i)$ if no confusion can arise.

For graded $R(\beta)$-modules M and N, let $\mathrm{Hom}_{R(\beta)}(M, N)$ be the space of morphisms in $\mathrm{Mod}_{\mathrm{gr}}(R(\beta))$, i.e., the $\mathbf{k}$-vector space of homogeneous homomorphisms of degree 0, and set

$$\begin{aligned} \mathrm{H{\small OM}}_{R(\beta)}(M, N) &= \bigoplus_{k\in\mathbb{Z}} \mathrm{H{\small OM}}_{R(\beta)}(M, N)_k, \\ \mathrm{H{\small OM}}_{R(\beta)}(M, N)_k &:= \mathrm{Hom}_{R(\beta)}(q^k M, N). \end{aligned} \tag{1.5}$$

We write $\mathrm{E{\small ND}}_{R(\beta)}(M)$ for $\mathrm{H{\small OM}}_{R(\beta)}(M, M)$. When $f \in \mathrm{Hom}_{R(\beta)}(q^k M, N)$, we denote

$$\deg(f) := k.$$

For simplicity, we write $\mathrm{H{\small OM}}_R(M, N)$ for $\mathrm{H{\small OM}}_{R(\beta)}(M, N)$ if no confusion can arise.

We write $[R\text{-proj}]$ and $[R\text{-gmod}]$ for the (split) Grothendieck group of R-proj and the Grothendieck group of R-gmod. Then the $\mathbb{Z}$-grading gives a $\mathbb{Z}[q, q^{-1}]$-module structure on $[R\text{-proj}]$ and $[R\text{-gmod}]$, and convolution gives an algebra structure.

Theorem 1.3 ([12, 13, 17]). *There exist algebra isomorphisms*

$$[R\text{-proj}] \simeq U_{\mathbf{A}}^-(\mathfrak{g}), \qquad [R\text{-gmod}] \simeq A_q(\mathfrak{g}^+).$$

Here, $A_q(\mathfrak{g}^+) := \{a \in U_q^-(\mathfrak{g}) \mid (a, U_{\mathbf{A}}^-(\mathfrak{g})) \subset \mathbf{A}\}$, where $(\cdot,\cdot)$ is the non-degenerate symmetric bilinear form on $U_q^-(\mathfrak{g})$ defined in [11]. Note that $A_q(\mathfrak{g}^+)$ is an $\mathbf{A}$-subalgebra of $U_{\mathbf{A}}^-(\mathfrak{g})$ (cf. [8] where $A_q(\mathfrak{g}^+)$ is denoted by $A_q(\mathfrak{n})_{\mathbb{Z}[q^{\pm 1}]}$).

Definition 1.4. Let $\mathbf{c}$ be a $\mathbb{Z}$-valued skew-symmetric bilinear form on Q. If we redefine $\deg(\tau_l e(\nu))$ to be $-(\alpha_{\nu_l}, \alpha_{\nu_{l+1}}) - \mathbf{c}(\alpha_{\nu_l}, \alpha_{\nu_{l+1}})$, then this gives a well-defined $\mathbb{Z}$-graded algebra structure on $R(\beta)$. We denote by $R_{\mathbf{c}}(\beta)$ the $\mathbb{Z}$-graded algebra thus defined.

The usual grading (1.4) is a special case of such a $\mathbb{Z}$-grading.

We define $R_{\mathbf{c}}(\beta)$-gmod, $R_{\mathbf{c}}$-gmod, etc., similarly.

Let us denote by $\mathrm{Mod}_{\mathrm{gr}}(R_{\mathbf{c}}(\beta))[q^{1/2}]$ the category of $\left(\frac{1}{2}\mathbb{Z}\right)$-graded modules over $R_{\mathbf{c}}(\beta)$. For $\nu \in I^\beta$ we set

$$H(\nu) = \frac{1}{2} \sum_{1\le a<b\le \mathrm{ht}(\beta)} \mathbf{c}(\alpha_{\nu_a}, \alpha_{\nu_b}).$$

Lemma 1.5. *For* $\beta \in \mathsf{Q}_+$ *and* $M \in \mathrm{Mod}_{\mathrm{gr}}(R(\beta))[q^{1/2}]$*, set*

$$(K_{\mathbf{c}}(M))_n = \bigoplus_{\nu \in I^\beta} e(\nu) M_{n-H(\nu)}.$$

Then

(i) $K_{\mathbf{c}}$ *is an equivalence of categories from* $\mathrm{Mod}_{\mathrm{gr}}(R(\beta))[q^{1/2}]$ *to* $\mathrm{Mod}_{\mathrm{gr}}(R_{\mathbf{c}}(\beta))[q^{1/2}]$*,*
(ii) *for* $M \in \mathrm{Mod}_{\mathrm{gr}}(R(\beta))[q^{1/2}]$ *and* $N \in \mathrm{Mod}_{\mathrm{gr}}(R(\gamma))[q^{1/2}]$*, we have*

$$K_{\mathbf{c}}(M \circ N) \simeq q^{\frac{1}{2}\mathbf{c}(\beta,\gamma)} K_{\mathbf{c}}(M) \circ K_{\mathbf{c}}(N).$$

Proof. (i) We have

$$\begin{aligned}\tau_k e(\nu)(K_{\mathbf{c}}(M))_n &= \tau_k e(\nu) M_{n-H(\nu)}\\ &\subset e(s_k\nu) M_{n-H(\nu)-(\alpha_{\nu_k},\alpha_{\nu_{k+1}})}\\ &= e(s_k\nu)(K_{\mathbf{c}}(M))_{n-H(\nu)-(\alpha_{\nu_k},\alpha_{\nu_{k+1}})+H(s_k\nu)}.\end{aligned}$$

Then (i) follows from

$$H(s_k\nu) - H(\nu) = \tfrac{1}{2}\big(\mathbf{c}(\alpha_{\nu_{k+1}},\alpha_{\nu_k}) - \mathbf{c}(\alpha_{\nu_k},\alpha_{\nu_{k+1}})\big) = -\mathbf{c}(\alpha_{\nu_k},\alpha_{\nu_{k+1}}).$$

(ii) For $\nu \in I^\beta$ and $\mu \in I^\gamma$, we have

$$\begin{aligned}e(\nu)K_{\mathbf{c}}(M)_a \otimes e(\mu)K_{\mathbf{c}}(N)_b &= e(\nu)M_{a-H(\nu)} \otimes e(\mu)N_{b-H(\mu)}\\ &\subset e(\nu*\mu)(M\circ N)_{a+b-H(\nu)-H(\mu)}\\ &= e(\nu*\mu)K_{\mathbf{c}}(M\circ N)_{a+b-H(\nu)-H(\mu)+H(\nu*\mu)}.\end{aligned}$$

Since

$$H(\nu*\mu) - H(\nu) - H(\mu) = \tfrac{1}{2}\mathbf{c}(\beta,\gamma),$$

we have

$$K_{\mathbf{c}}(M)_a \otimes K_{\mathbf{c}}(N)_b \subset K_{\mathbf{c}}(M\circ N)_{a+b+\frac{1}{2}\mathbf{c}(\beta,\gamma)}.$$

This yields a map

$$K_{\mathbf{c}}(M)_a \otimes K_{\mathbf{c}}(N)_b \to (q^{-\frac{1}{2}\mathbf{c}(\beta,\gamma)} K_{\mathbf{c}}(M\circ N))_{a+b},$$

which induces an isomorphism

$$K_{\mathbf{c}}(M)\circ K_{\mathbf{c}}(N) \xrightarrow{\sim} q^{-\frac{1}{2}\mathbf{c}(\beta,\gamma)} K_{\mathbf{c}}(M\circ N).$$ □

We define the algebra $U_{\mathbf{A}}^-(\mathfrak{g})_{\mathbf{c}}$ as $\mathbb{Z}[q^{\pm 1/2}] \otimes_{\mathbb{Z}[q^{\pm 1}]} U_{\mathbf{A}}^-(\mathfrak{g})$ with a new multiplication $\circ_{\mathbf{c}}$ given by

$$a \circ_{\mathbf{c}} b = q^{-\frac{1}{2}\mathbf{c}(\alpha,\beta)} ab$$

for $a \in \mathbb{Z}[q^{\pm 1/2}] \otimes_{\mathbb{Z}[q^{\pm 1}]} U_{\mathbf{A}}^-(\mathfrak{g})_\alpha$ and $b \in \mathbb{Z}[q^{\pm 1/2}] \otimes_{\mathbb{Z}[q^{\pm 1}]} U_{\mathbf{A}}^-(\mathfrak{g})_\beta$.

We define $A_q(\mathfrak{g}^+)_{\mathbf{c}}$ similarly.

Corollary 1.6. *There is a* $\mathbb{Z}[q^{\pm 1/2}]$*-algebra isomorphism*

$$\xi_{\mathbf{c}} \colon A_q(\mathfrak{g}^+)_{\mathbf{c}} \xrightarrow{\sim} \mathbb{Z}[q^{\pm 1/2}] \mathop{\otimes}_{\mathbb{Z}[q^{\pm 1}]} [R_{\mathbf{c}}\text{-gmod}].$$

1.3. Remark on parity

Under hypothesis (1) of Definition 1.1, the category $\mathrm{Mod}_{\mathrm{gr}}(R(\beta))$ is divided into two parts according to the parity of degrees for any $\beta \in \mathsf{Q}_+$.

Lemma 1.7. *Let $\beta \in \mathsf{Q}_+$. Then there exists a map $S\colon I^\beta \to \mathbb{Z}/2\mathbb{Z}$ such that*

$$S(s_k\nu) = S(\nu) + (\alpha_{\nu_k}, \alpha_{\nu_{k+1}})$$

for any $\nu \in I^\beta$ and any integer k with $1 \le k < \mathrm{ht}(\beta)$.

Proof. Let $n = \mathrm{ht}(\beta)$. Choose a total order $\prec$ on I and set

$$S(\nu) := \sum_{1 \le a < b \le n,\ \nu_a \prec \nu_b} (\alpha_{\nu_a}, \alpha_{\nu_b}).$$

Then we have

$$\begin{aligned} S(s_k\nu) &= S(\nu) + \big(\delta(\nu_{k+1} \prec \nu_k) - \delta(\nu_k \prec \nu_{k+1})\big)(\alpha_{\nu_k}, \alpha_{\nu_{k+1}}) \\ &\equiv S(\nu) + \big(1 - \delta(\nu_k = \nu_{k+1})\big)(\alpha_{\nu_k}, \alpha_{\nu_{k+1}}) \equiv S(\nu) + (\alpha_{\nu_k}, \alpha_{\nu_{k+1}}) \bmod 2. \end{aligned}$$

Here, for a statement P, we set $\delta(P)$ to be 1 if P is true and 0 if P is false. □

Proposition 1.8. *Let $\beta \in \mathsf{Q}_+$ and $S\colon I^\beta \to \mathbb{Z}/2\mathbb{Z}$ be as in Lemma 1.7. Let $\mathrm{Mod}_{\mathrm{gr}}(R(\beta))^S$ be the full subcategory of $\mathrm{Mod}_{\mathrm{gr}}(R(\beta))$ consisting of graded $R(\beta)$-modules M such that $e(\nu)M_k = 0$ for any $\nu \in I^\beta$ and $k \equiv S(\nu) + 1 \bmod 2$. Then*

$$\mathrm{Mod}_{\mathrm{gr}}(R(\beta)) \simeq \mathrm{Mod}_{\mathrm{gr}}(R(\beta))^S \oplus q\,\mathrm{Mod}_{\mathrm{gr}}(R(\beta))^S.$$

Proof. For any graded $R(\beta)$-module M and $\varepsilon = 0, 1$ set

$$M^\varepsilon := \bigoplus_{\substack{\nu \in I^\beta,\ k \in \mathbb{Z}, \\ k \equiv S(\nu) + \varepsilon \bmod 2}} e(\nu) M_k.$$

Then we can see easily that the M^ε are $R(\beta)$-submodules of M and $M = M^0 \oplus M^1$. Moreover, $M^\varepsilon \in q^\varepsilon\,\mathrm{Mod}_{\mathrm{gr}}(R(\beta))^S$. □

Note that $q^2\,\mathrm{Mod}_{\mathrm{gr}}(R(\beta))^S = \mathrm{Mod}_{\mathrm{gr}}(R(\beta))^S$ and

$$\mathrm{HOM}_{R(\beta)}(M, N)_k = 0 \quad \text{if } k \text{ is odd and } M, N \in \mathrm{Mod}_{\mathrm{gr}}(R(\beta))^S. \tag{1.6}$$

1.4. R-*matrices*

Let $\beta \in \mathsf{Q}_+$ and $m = \mathrm{ht}(\beta)$. For $k = 1, \ldots, m-1$ and $\nu \in I^\beta$, the *intertwiner* $\varphi_k \in R(\beta)$ is defined by

$$\varphi_k e(\nu) := \begin{cases} (\tau_k x_k - x_k \tau_k) e(\nu) & \text{if } \nu_k = \nu_{k+1}, \\ \tau_k e(\nu) & \text{otherwise.} \end{cases}$$

Lemma 1.9 ([5, Lem. 1.5]).

(i) $\varphi_k^2 e(\nu) = (\mathcal{Q}_{\nu_k,\nu_{k+1}}(x_k, x_{k+1}) + \delta_{\nu_k,\nu_{k+1}})e(\nu)$.
(ii) $\{\varphi_k\}_{1\le k\le m-1}$ *satisfies the braid relation.*
(iii) *For a reduced expression* $w = s_{i_1}\cdots s_{i_t} \in \mathfrak{S}_m$, *let* $\varphi_w = \varphi_{i_1}\cdots\varphi_{i_t}$. *Then* φ_w *does not depend on the choice of reduced expressions of* w.
(iv) *For* $w \in \mathfrak{S}_m$ *and* $1 \le k \le m$, *we have* $\varphi_w x_k = x_{w(k)}\varphi_w$.
(v) *For* $w \in \mathfrak{S}_m$ *and* $1 \le k < m$, *if* $w(k+1) = w(k)+1$, *then* $\varphi_w \tau_k = \tau_{w(k)}\varphi_w$.
(vi) $\varphi_{w^{-1}}\varphi_w e(\nu) = \prod_{a<b,\, w(a)>w(b)}(\mathcal{Q}_{\nu_a,\nu_b}(x_a, x_b) + \delta_{\nu_a,\nu_b})e(\nu)$.

For $m, n \in \mathbb{Z}_{\ge 0}$, let $w[m,n]$ be the element of $\mathfrak{S}_{m+n}$ defined by

$$w[m,n](k) = \begin{cases} k+n & \text{if } 1 \le k \le m, \\ k-m & \text{if } m < k \le m+n. \end{cases}$$

Let M be an $R(\beta)$-module with $\mathrm{ht}(\beta) = m$ and N an $R(\beta')$-module with $\mathrm{ht}(\beta') = n$. The $R(\beta)\otimes R(\beta')$-linear map $M\otimes N \to N\circ M$ given by $u\otimes v \mapsto \varphi_{w[n,m]}(v\otimes u)$ can be extended to an $R(\beta+\beta')$-module homomorphism

$$R_{M,N} : M\circ N \to N\circ M. \tag{1.7}$$

For $\beta = \sum_{k=1}^m \alpha_{i_k}$, we set $\mathrm{supp}(\beta) := \{i_k \mid 1 \le k \le m\}$.

Definition 1.10. The quiver Hecke algebra $R(\beta)$ is said to be *symmetric* if $\mathcal{Q}_{i,j}(u,v)$ is a polynomial in $u-v$ for all $i, j \in \mathrm{supp}(\beta)$.

Suppose that $R(\beta)$ is symmetric. Let z be an indeterminate. For an $R(\beta)$-module M, we define an $R(\beta)$-module structure on $M_z := \mathbf{k}[z]\otimes_{\mathbf{k}} M$ by

$$\begin{aligned} &e(\nu)(a\otimes u) = a\otimes e(\nu)u, \quad x_j(a\otimes u) = (za)\otimes u + a\otimes x_j u, \\ &\tau_k(a\otimes u) = a\otimes(\tau_k u), \end{aligned} \tag{1.8}$$

for $\nu \in I^n$, $a \in \mathbf{k}[z]$ and $u \in M$. We call M_z the *affinization* of M. For a non-zero $R(\beta)$-module M and a non-zero $R(\beta')$-module N, let s be the order of the zero of $R_{M_z,N_{z'}} : M_z\circ N_{z'} \to N_{z'}\circ M_z$, and

$$R^{\mathrm{norm}}_{M_z,N_{z'}} := (z'-z)^{-s} R_{M_z,N_{z'}}.$$

We define $\mathbf{r}_{M,N} : M\circ N \to N\circ M$ by

$$\mathbf{r}_{M,N} := R^{\mathrm{norm}}_{M_z,N_{z'}}|_{z=z'=0}.$$

We set $R(\beta)[z_1,\ldots,z_k] := \mathbf{k}[z_1,\ldots,z_k]\otimes_{\mathbf{k}} R(\beta)$. For simplicity, we write $R[z_1,\ldots,z_k]$ for $R(\beta)[z_1,\ldots,z_k]$ if there is no risk of confusion.

Theorem 1.11 ([5, Section 1]). *Suppose that* $R(\beta)$ *and* $R(\beta')$ *are symmetric. Let* M *be a non-zero* $R(\beta)$*-module and* N *a non-zero* $R(\beta')$*-module. Then:*

(i) $R^{\mathrm{norm}}_{M_z,N_{z'}}$ *and* $\mathbf{r}_{M,N}$ *are non-zero.*
(ii) $R^{\mathrm{norm}}_{M_z,N_{z'}}$ *and* $\mathbf{r}_{M,N}$ *satisfy the braid relations.*

(iii) *Set*

$$A = \sum_{\mu \in I^\beta,\, \nu \in I^{\beta'}} \Big(\prod_{1 \le a \le m,\, 1 \le b \le n,\, \mu_a \neq \nu_b} \mathcal{Q}_{\mu_a, \nu_b}(x_a \boxtimes e(\beta'), e(\beta) \boxtimes x_b) \Big) e(\mu) \boxtimes e(\nu).$$

Then A is in the center of $R(\beta) \otimes R(\beta')$, and

$$R_{N_{z'}, M_z} R_{M_z, N_{z'}}(u \otimes v) = A(u \otimes v) \quad \text{for } u \in M_z \text{ and } v \in N_{z'}.$$

(iv) *If M and N are simple modules, then*

$$\mathrm{E{\scriptstyle ND}}_{R(\beta+\beta')[z,z']}(M_z \circ N_{z'}) \simeq \mathbf{k}[z, z'],$$
$$\mathrm{H{\scriptstyle OM}}_{R(\beta+\beta')[z,z']}(M_z \circ N_{z'}, N_{z'} \circ M_z) \simeq \mathbf{k}[z, z'] R^{\mathrm{norm}}_{M_z, N_{z'}}.$$

2. Affinization

2.1. Definition of affinization

Definition 2.1. For any $i \in I$ and $\beta \in \mathsf{Q}_+$ with $\mathrm{ht}(\beta) = m$, we set

$$\mathfrak{p}_{i,\beta} = \sum_{\nu \in I^\beta} \Big(\prod_{a \in [1,m],\, \nu_a = i} x_a \Big) e(\nu),$$

where $[1, m] = \{1, \dots, m\}$.

Note that $\mathfrak{p}_{i,\beta}$ belongs to the center of $R(\beta)$. If there is no danger of confusion, we simply write $\mathfrak{p}_i$ for $\mathfrak{p}_{i,\beta}$.

Definition 2.2. Let $\beta \in \mathsf{Q}_+$ and $\bar{M}$ a simple $R(\beta)$-module. An *affinization* $\mathsf{M} := (\mathsf{M}, z_\mathsf{M})$ of $\bar{M}$ is an $R(\beta)$-module M with an injective homogeneous endomorphism z_M of M of degree $d_\mathsf{M} \in \mathbb{Z}_{>0}$ and an isomorphism $\mathsf{M}/z_\mathsf{M}\mathsf{M} \xrightarrow{\sim} \bar{M}$ satisfying the following conditions:

(a) M is a finitely generated free module over the polynomial ring $\mathbf{k}[z_\mathsf{M}]$,
(b) $\mathfrak{p}_i \mathsf{M} \neq 0$ for any $i \in I$.

If moreover

(c) the exact sequence $0 \to z_\mathsf{M}\mathsf{M}/z_\mathsf{M}^2\mathsf{M} \to \mathsf{M}/z_\mathsf{M}^2\mathsf{M} \to \mathsf{M}/z_\mathsf{M}\mathsf{M} \to 0$ of $R(\beta)$-modules does not split,

then the affinization M is *strong*. We say that the affinization is *even* if d_M is even.

Let us denote by $\pi_\mathsf{M} \colon \mathsf{M} \twoheadrightarrow \bar{M}$ the composition $\mathsf{M} \twoheadrightarrow \mathsf{M}/z_\mathsf{M}\mathsf{M} \xrightarrow{\sim} \bar{M}$.

Remark 2.3. (i) Condition (a) is equivalent to

(a′) The degree of M is bounded from below, that is, $\mathsf{M}_n = 0$ for $n \ll 0$.

Moreover, under these equivalent conditions, we have

$$\mathrm{ch}_q(\mathsf{M}) = (1 - q^{d_\mathsf{M}})^{-1} \mathrm{ch}_q(\bar{M}).$$

Note that every finitely generated R-module M satisfies (a′).

(ii) The non-splitting condition (c) is equivalent to saying that $z_{\mathsf{M}}\mathsf{M}/z_{\mathsf{M}}^2\mathsf{M}$ is a unique proper $R(\beta)$-submodule of $\mathsf{M}/z_{\mathsf{M}}^2\mathsf{M}$.

(iii) If $R(\beta)$ is a symmetric quiver Hecke algebra, then $\bar{M}_z$ is a strong affinization of any simple $R(\beta)$-module $\bar{M}$ for $\beta \neq 0$.

Example 2.4. (i) For $i \in I$, $\mathsf{M} := L(i)_z \circ L(i)$ is not an affinization of $\bar{M} := L(i) \circ L(i)$ In fact, conditions (a) and (c) in Definition 2.2 hold but (b) does not.

(ii) Let $(\mathsf{M}, z_{\mathsf{M}})$ be an affinization of $\bar{M}$. Assume that $d_{\mathsf{M}} = ab$ for $a, b \in \mathbb{Z}_{>0}$ and let z be an indeterminate of homogeneous degree b. Let $\mathbf{k}[z_{\mathsf{M}}] \to \mathbf{k}[z]$ be the algebra homomorphism given by $z_{\mathsf{M}} \mapsto z^a$. Then $(\mathbf{k}[z] \otimes_{\mathbf{k}[z_{\mathsf{M}}]} \mathsf{M},\ z)$ is an affinization of $\bar{M}$. If $a > 1$ then it is never a strong affinization, because

$$(\mathbf{k}[z] \underset{\mathbf{k}[z_{\mathsf{M}}]}{\otimes} \mathsf{M})/(z^a\mathbf{k}[z] \underset{\mathbf{k}[z_{\mathsf{M}}]}{\otimes} \mathsf{M}) \simeq (\mathbf{k}[z]/\mathbf{k}[z]z^a) \underset{\mathbf{k}}{\otimes} \bar{M}$$

is a semisimple $R(\beta)$-module.

As seen in the proposition below, every affinization is essentially even.

Proposition 2.5. *Let $(\mathsf{M}, z_{\mathsf{M}})$ be an affinization of a simple module $\bar{M}$. Assume that the homogeneous degree d_{M} of z_{M} is odd. Then there exists an $R(\beta)$-submodule M' of M such that*

(i) *$z_{\mathsf{M}}^2\mathsf{M}' \subset \mathsf{M}'$, and $(\mathsf{M}',\ z_{\mathsf{M}}^2)$ is an affinization of $\bar{M}$,*
(ii) *$\mathsf{M} \simeq \mathbf{k}[z_{\mathsf{M}}] \otimes_{\mathbf{k}[z_{\mathsf{M}}^2]} \mathsf{M}'$ as $R(\beta)[z_{\mathsf{M}}]$-modules.*

Proof. Let $\mathrm{Mod}_{\mathrm{gr}}(R(\beta)) \simeq \mathrm{Mod}_{\mathrm{gr}}(R(\beta))^S \oplus q\,\mathrm{Mod}_{\mathrm{gr}}(R(\beta))^S$ be the decomposition in Proposition 1.8. We may assume that $\bar{M}$ belongs to $\mathrm{Mod}_{\mathrm{gr}}(R(\beta))^S$. Let $\mathsf{M} = \mathsf{M}' \oplus \mathsf{M}''$ with $\mathsf{M}' \in \mathrm{Mod}_{\mathrm{gr}}(R(\beta))^S$ and $\mathsf{M}'' \in q\,\mathrm{Mod}_{\mathrm{gr}}(R(\beta))^S$. Then $z_{\mathsf{M}}\mathsf{M}' \subset \mathsf{M}''$ and $z_{\mathsf{M}}\mathsf{M}'' \subset \mathsf{M}'$ by (1.6). Hence,

$$\mathsf{M}/z_{\mathsf{M}}\mathsf{M} = (\mathsf{M}'/z_{\mathsf{M}}\mathsf{M}'') \oplus (\mathsf{M}''/z_{\mathsf{M}}\mathsf{M}'),$$

which implies that $\mathsf{M}'/z_{\mathsf{M}}\mathsf{M}'' \simeq \bar{M}$ and $\mathsf{M}'' = z_{\mathsf{M}}\mathsf{M}'$, giving the desired result. □

2.2. *Strong affinization*

Note that Lemmas 2.6 and 2.7 below hold without assumption (b) in Definition 2.2.

Lemma 2.6. *Assume that*

$$\left.\begin{array}{l}\beta \in \mathsf{Q}_+ \text{ and } (\mathsf{M}, z_{\mathsf{M}}) \text{ is a strong affinization of a simple } R(\beta)\text{-module } \bar{M},\\ z_{\mathsf{M}} \text{ has homogeneous degree } d_{\mathsf{M}} \in \mathbb{Z}_{>0}, \text{ and } \pi_{\mathsf{M}}\colon \mathsf{M} \to \bar{M} \text{ is a canoni-}\\ \text{cal projection.}\end{array}\right\} \qquad (2.1)_{\mathrm{strong}}$$

Then:

(i) *The head of the R-module M is isomorphic to $\bar{M}$, or equivalently $z_{\mathsf{M}}\mathsf{M}$ is a unique maximal $R(\beta)$-module.*
(ii) *Let $s := \min\{m \in \mathbb{Z} \mid \mathsf{M}_m \neq 0\}$ and $u \in \mathsf{M}_s \setminus \{0\}$. Then $\mathsf{M} = R(\beta)u$.*
(iii) $\mathrm{END}_{R(\beta)}(\mathsf{M}) \simeq \mathbf{k}[z_{\mathsf{M}}]\,\mathrm{id}_{\mathsf{M}}$.

Proof. (i) Let S be a simple module and $\varphi\colon \mathsf{M} \to S$ be an epimorphism. By homogeneous-degree considerations, we may assume that $\varphi(z_{\mathsf{M}}^k\mathsf{M}) = 0$ for $k \gg 0$. Take $k \geq 0$ such that $\varphi(z_{\mathsf{M}}^k\mathsf{M}) = S$ and $\varphi(z_{\mathsf{M}}^{k+1}\mathsf{M}) = 0$. Since $z_{\mathsf{M}}^k\mathsf{M}/z_{\mathsf{M}}^{k+1}\mathsf{M} \simeq \mathsf{M}/z_{\mathsf{M}}\mathsf{M}$ is simple, φ induces an isomorphism $z_{\mathsf{M}}^k\mathsf{M}/z_{\mathsf{M}}^{k+1}\mathsf{M} \xrightarrow{\sim} S$. It is enough to show that $k = 0$. If $k > 0$ then we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & z_{\mathsf{M}}^k\mathsf{M}/z_{\mathsf{M}}^{k+1}\mathsf{M} & \longrightarrow & z_{\mathsf{M}}^{k-1}\mathsf{M}/z_{\mathsf{M}}^{k+1}\mathsf{M} & \longrightarrow & z_{\mathsf{M}}^{k-1}\mathsf{M}/z_{\mathsf{M}}^{k}\mathsf{M} & \longrightarrow & 0 \\ & & & \overset{\sim}{\searrow} & \downarrow{\scriptstyle\varphi} & & & & \\ & & & & S & & & & \end{array} \tag{2.2}$$

Hence the first row of the above diagram is a split exact sequence, which contradicts Definition 2.2(c).

(ii) Since $u \notin z_{\mathsf{M}}\mathsf{M}$, (i) implies that $\mathsf{M} = R(\beta)u$.

(iii) Let $f \in \mathrm{E{\scriptstyle ND}}_{R(\beta)}(\mathsf{M})$ be a homogeneous endomorphism of degree ℓ. Assume that $f(\mathsf{M}) \subset z_{\mathsf{M}}^k\mathsf{M}$ for $k \in \mathbb{Z}_{\geq 0}$. We shall show $f \in \mathbf{k}[z_{\mathsf{M}}]\,\mathrm{id}_{\mathsf{M}}$ by descending induction on k. If $d_{\mathsf{M}}k > \ell$, then f has to be 0 since $f(u) \notin z_{\mathsf{M}}^k\mathsf{M}$ if $f(u) \neq 0$. Here u is as in (ii). Suppose that $d_{\mathsf{M}}k \leq \ell$. As $\bar{M}$ is the head of M, the composition $\mathsf{M} \xrightarrow{z_{\mathsf{M}}^{-k}f} \mathsf{M} \xrightarrow{\pi_{\mathsf{M}}} \bar{M}$ decomposes as $\mathsf{M} \xrightarrow{\pi_{\mathsf{M}}} \bar{M} \to \bar{M}$. Hence the composition must be equal to $c\pi_{\mathsf{M}}$ for some $c \in \mathbf{k}$, which yields

$$(z_{\mathsf{M}}^{-k}f - c\,\mathrm{id}_{\mathsf{M}})(\mathsf{M}) \subset z_{\mathsf{M}}\mathsf{M}.$$

Therefore, $(f - cz_{\mathsf{M}}^k)(\mathsf{M}) \subset z_{\mathsf{M}}^{k+1}\mathsf{M}$, and the induction proceeds. □

2.3. Normalized R*-matrices*

Lemma 2.7. *Assume that*

$$\left.\begin{array}{l} \beta \in \mathsf{Q}^+ \text{ and } (\mathsf{M}, z_{\mathsf{M}}) \text{ is an affinization of a simple } R(\beta)\text{-module } \bar{M}, \\ z_{\mathsf{M}} \text{ has homogeneous degree } d_{\mathsf{M}} \in \mathbb{Z}_{>0}, \text{ and } \pi_{\mathsf{M}}\colon \mathsf{M} \to \bar{M} \text{ is a} \\ \text{canonical projection.} \end{array}\right\} \tag{2.3$_{\mathrm{weak}}$}$$

Then

(i) $\mathrm{E{\scriptstyle ND}}_{R(\beta)[z_{\mathsf{M}}]}(\mathsf{M}) \simeq \mathbf{k}[z_{\mathsf{M}}]\,\mathrm{id}_{\mathsf{M}}$,

(ii) *for any $i \in I$, there exist $c_i \in \mathbf{k}^\times$ and $d_i \in \mathbb{Z}_{\geq 0}$ such that $\mathfrak{p}_i|_{\mathsf{M}} = c_i z_{\mathsf{M}}^{d_i}$.*

Proof. (i) The proof is similar to that of Lemma 2.6(iii). Let $f \in \mathrm{E{\scriptstyle ND}}_{R(\beta)[z_{\mathsf{M}}]}(\mathsf{M})$ be a homogeneous endomorphism of degree ℓ. Suppose that $f(\mathsf{M}) \subset z_{\mathsf{M}}^k\mathsf{M}$ for $k \in \mathbb{Z}_{\geq 0}$. We shall show $f \in \mathbf{k}[z_{\mathsf{M}}]\,\mathrm{id}_{\mathsf{M}}$ by descending induction on k.

We have $f = 0$ if $d_{\mathsf{M}}k > \ell$ by degree considerations. If $d_{\mathsf{M}}k \leq \ell$, then the endomorphism $z_{\mathsf{M}}^{-k}f$ induces an endomorphism of $\bar{M}$. Hence it must be equal to $c\,\mathrm{id}_{\bar{M}}$ for some $c \in \mathbf{k}$. Then $(f - cz_{\mathsf{M}}^k)(\mathsf{M}) \subset z_{\mathsf{M}}^{k+1}\mathsf{M}$, and the induction proceeds.

(ii) The assertion follows from (i) immediately. □

Lemma 2.8. *Let β, M and $\bar{M}$ be as in (2.3)$_{\mathrm{weak}}$. Assume further that $\beta \neq 0$. Then M is a finitely generated $R(\beta)$-module.*

Proof. Since $\beta \neq 0$, there exists $i \in I$ such that $\mathfrak{p}_{i,\beta}$ has a positive degree. Then there exists $m > 0$ such that $z_{\mathsf{M}}^m \in \mathbf{k}(\mathfrak{p}_{i,\beta}|_{\mathsf{M}}) \subset \mathrm{END}_R(\mathsf{M})$. Since M is finitely generated over $\mathbf{k}[z_{\mathsf{M}}^m]$, we obtain the desired result. □

Lemma 2.9. *Let β, M and $\bar{M}$ be as in (2.3)$_{\mathrm{weak}}$. Let $\gamma \in \mathsf{Q}_+$ and $N \in R(\gamma)$-gmod. Then:*

(i) *The homomorphisms*

$$R_{\mathsf{M}[z_{\mathsf{M}}^{-1}],N}\colon \mathsf{M}[z_{\mathsf{M}}^{-1}]\circ N \to N\circ \mathsf{M}[z_{\mathsf{M}}^{-1}] \quad \textit{and} \quad R_{N,\mathsf{M}[z_{\mathsf{M}}^{-1}]}\colon N\circ \mathsf{M}[z_{\mathsf{M}}^{-1}] \to \mathsf{M}[z_{\mathsf{M}}^{-1}]\circ N$$

are isomorphisms. Here, $\mathsf{M}[z_{\mathsf{M}}^{-1}] = \mathbf{k}[z_{\mathsf{M}}, z_{\mathsf{M}}^{-1}] \otimes_{\mathbf{k}[z_{\mathsf{M}}]} \mathsf{M}$.

(ii) *If N is a simple module, there exist $c \in \mathbf{k}^\times$ and $d \in \mathbb{Z}_{\geq 0}$ such that $R_{N,\mathsf{M}} \circ R_{\mathsf{M},N} = c(z_{\mathsf{M}}^d \circ N)$ and $R_{\mathsf{M},N} \circ R_{N,\mathsf{M}} = c(N \circ z_{\mathsf{M}}^d)$.*

Proof. (i) is an immediate consequence of (ii). Let us show (ii). Set $m = \mathrm{ht}(\beta)$ and $n = \mathrm{ht}(\gamma)$. Then $(R_{N,\mathsf{M}} \circ R_{\mathsf{M},N})|_{\mathsf{M}\otimes N}$ is given by

$$\sum_{\nu \in I^{\beta+\gamma}} \Big(\prod_{1 \leq a \leq m < b \leq m+n,\ \nu_a \neq \nu_b} \mathcal{Q}_{\nu_a,\nu_b}(x_a, x_b) \Big) e(\nu).$$

Since any element in the center of $R(\gamma)$ with positive degree acts by zero on N, it is equal to

$$\sum_{\nu \in I^{\beta+\gamma}} \Big(\prod_{1 \leq a \leq m < b \leq m+n,\ \nu_a \neq \nu_b} \mathcal{Q}_{\nu_a,\nu_b}(x_a, 0) \Big) e(\nu).$$

Consequently, it is a product of $\mathfrak{p}_{i,\beta}|_{\mathsf{M}}$'s up to a constant multiple. Hence Lemma 2.7(ii) implies the desired result. □

Let M and $\bar{M}$ be as in (2.3)$_{\mathrm{weak}}$, and let $N \in R$-gmod be a non-zero module. Let s be the largest integer such that $R_{\mathsf{M},N}(\mathsf{M}\circ N) \subset z_{\mathsf{M}}^s N \circ \mathsf{M}$. Then we set

$$R^{\mathrm{norm}}_{\mathsf{M},N} = z_{\mathsf{M}}^{-s} R_{\mathsf{M},N}\colon \mathsf{M}\circ N \to N \circ \mathsf{M}.$$

We denote by

$$\mathbf{r}_{\bar{M},N}\colon \bar{M}\circ N \to N \circ \bar{M}$$

the homomorphism induced by $R^{\mathrm{norm}}_{\mathsf{M},N}$. By the definition, $\mathbf{r}_{\bar{M},N}$ never vanishes. We set $R^{\mathrm{norm}}_{\mathsf{M},N} = 0$ and $\mathbf{r}_{\bar{M},N} = 0$ when $N = 0$.

We define $R^{\mathrm{norm}}_{N,\mathsf{M}}$ and $\mathbf{r}_{N,\bar{M}}$ similarly.

The arguments in [8, 10] still work under these assumptions, and we obtain similar results. We list some of them without repeating the proofs. A simple module S is called *real* if $S \circ S$ is simple.

Proposition 2.10 ([10, Th. 3.2, Prop. 3.8], [8, Prop. 3.2.9, Th. 4.1.1]). *Assume that*

$$\left.\begin{array}{l}\text{(a) } M \text{ and } N \text{ are simple } R\text{-modules,}\\ \text{(b) one of them is real simple and also admits an affinization.}\end{array}\right\} \qquad (2.4)$$

Then:

(i) *$M \circ N$ has a simple head and a simple socle. Moreover,* $\mathrm{Im}(\mathbf{r}_{M,N})$ *is equal to the head of $M \circ N$ and the socle of $N \circ M$.*
(ii) *We have*

$$\mathrm{H}\text{OM}_R(M \circ N, M \circ N) = \mathbf{k}\,\mathrm{id}_{M \circ N}, \qquad \mathrm{H}\text{OM}_R(M \circ N, N \circ M) = \mathbf{k}\,\mathbf{r}_{M,N}.$$

(iii) *$M \diamond N$ appears only once in a Jordan–Hölder series of $M \circ N$ in R*-mod.

Proposition 2.11. *Let M and $\bar{M}$ be as in* (2.3)$_{\text{weak}}$*, and let N be a simple R-module. Assume that $\bar{M}$ is real. Then*

(i)

$$\mathrm{H}\text{OM}_{R[z_{\mathsf{M}}]}(\mathsf{M} \circ N, \mathsf{M} \circ N) = \mathbf{k}[z_{\mathsf{M}}]\,\mathrm{id}_{\mathsf{M} \circ N}, \tag{2.5}$$
$$\mathrm{H}\text{OM}_{R[z_{\mathsf{M}}]}(N \circ \mathsf{M}, N \circ \mathsf{M}) = \mathbf{k}[z_{\mathsf{M}}]\,\mathrm{id}_{N \circ \mathsf{M}}, \tag{2.6}$$

(ii) $\mathrm{H}\text{OM}_{R[z_{\mathsf{M}}]}(\mathsf{M} \circ N, N \circ \mathsf{M})$ *and* $\mathrm{H}\text{OM}_{R[z_{\mathsf{M}}]}(N \circ \mathsf{M}, \mathsf{M} \circ N)$ *are free* $\mathbf{k}[z_{\mathsf{M}}]$*-modules of rank one.*

Proof. (i) Let us first show (2.5). The idea of the proof is similar to that of Lemma 2.6(iii).

Let $f \in \mathrm{H}\text{OM}_{R[z_{\mathsf{M}}]}(\mathsf{M} \circ N, \mathsf{M} \circ N)$ be of homogeneous degree ℓ. We know that $f(\mathsf{M} \circ N) \subset z_{\mathsf{M}}^s \mathsf{M} \circ N$ for some $s \in \mathbb{Z}_{\geq 0}$. We shall show $f \in \mathbf{k}[z_{\mathsf{M}}]\,\mathrm{id}_{\mathsf{M} \circ N}$ by descending induction on s. If $s \gg 0$, then f is zero by degree considerations. Now, we consider $z_{\mathsf{M}}^{-s} f$. As $z_{\mathsf{M}}^{-s} f$ induces an endomorphism of $\bar{M} \circ N$, by Proposition 2.10(ii) it is equal to $c\,\mathrm{id}_{\bar{M} \circ N}$ for some $c \in \mathbf{k}$. Hence

$$(f - c z_{\mathsf{M}}^s)(\mathsf{M} \circ N) \subset z_{\mathsf{M}}^{s+1} \mathsf{M} \circ N.$$

Thus, the induction hypothesis implies that $f - c z_{\mathsf{M}}^s \in \mathbf{k}[z_{\mathsf{M}}]\,\mathrm{id}_{\bar{M} \circ N}$. The proof of (2.6) is similar.

(ii) By Lemma 2.9, we have an $R[z_{\mathsf{M}}]$-linear monomorphism $N \circ \mathsf{M} \rightarrowtail \mathsf{M} \circ N$. This yields

$$\mathrm{H}\text{OM}_{R[z_{\mathsf{M}}]}(\mathsf{M} \circ N, N \circ \mathsf{M}) \rightarrowtail \mathrm{H}\text{OM}_{R[z_{\mathsf{M}}]}(\mathsf{M} \circ N, \mathsf{M} \circ N) \simeq \mathbf{k}[z_{\mathsf{M}}].$$

As $\mathrm{H}\text{OM}_{R[z_{\mathsf{M}}]}(\mathsf{M} \circ N, N \circ \mathsf{M})$ is non-zero, $\mathrm{H}\text{OM}_{R[z_{\mathsf{M}}]}(\mathsf{M} \circ N, N \circ \mathsf{M})$ is a free $\mathbf{k}[z_{\mathsf{M}}]$-module of rank one. □

Proposition 2.12. *Assume that*

(a) $(\mathsf{M}, z_{\mathsf{M}})$ *and* $(\mathsf{N}, z_{\mathsf{N}})$ *are affinizations of simple modules $\bar{M}$ and $\bar{N}$, respectively,*
(b) *one of $\bar{M}$ and $\bar{N}$ is real.* (2.7)

Then

(i) $\mathrm{H}\text{OM}_{R[z_{\mathsf{M}}, z_{\mathsf{N}}]}(\mathsf{M} \circ \mathsf{N}, \mathsf{M} \circ \mathsf{N}) = \mathbf{k}[z_{\mathsf{M}}, z_{\mathsf{N}}]\,\mathrm{id}_{\mathsf{M} \circ \mathsf{N}}$,
(ii) $\mathrm{H}\text{OM}_{R[z_{\mathsf{M}}, z_{\mathsf{N}}]}(\mathsf{M} \circ \mathsf{N}, \mathsf{N} \circ \mathsf{M})$ *is a free* $\mathbf{k}[z_{\mathsf{M}}, z_{\mathsf{N}}]$*-module of rank one.*

Proof. (i) Assume that $\bar{M}$ is real simple. The other case can be proved similarly.

Let f be a homogeneous element of $\mathrm{HOM}_{R[z_\mathsf{M},z_\mathsf{N}]}(\mathsf{M}\circ\mathsf{N},\mathsf{M}\circ\mathsf{N})$ of degree ℓ. Assuming that $\mathrm{Im}(f)\subset z_\mathsf{N}^k(\mathsf{M}\circ\mathsf{N})$, we shall show $f\in\mathbf{k}[z_\mathsf{M},z_\mathsf{N}]\,\mathrm{id}_{\mathsf{M}\circ\mathsf{N}}$ by descending induction on k. If $k\gg 0$, then f is zero by homogeneous-degree considerations. We now consider $z_\mathsf{N}^{-k}f$. The $R[z_\mathsf{M},z_\mathsf{N}]$-linear homomorphism $z_\mathsf{N}^{-k}f\colon\mathsf{M}\circ\mathsf{N}\to\mathsf{M}\circ\mathsf{N}$ induces an $R[z_\mathsf{M}]$-linear homomorphism $\mathsf{M}\circ\bar{N}\to\mathsf{M}\circ\bar{N}$. By Proposition 2.11, the latter is equal to $\varphi(z_\mathsf{M})\,\mathrm{id}_{\mathsf{M}\circ\bar{N}}$ for some $\varphi(z_\mathsf{M})\in\mathbf{k}[z_\mathsf{M}]$. Hence

$$\mathrm{Im}(f-z_\mathsf{N}^k\varphi(z_\mathsf{M})\,\mathrm{id}_{\mathsf{M}\circ\mathsf{N}})\subset z_\mathsf{N}^{k+1}\mathsf{M}\circ\mathsf{N},$$

which implies

$$f-z_\mathsf{N}^k\varphi(z_\mathsf{M})\,\mathrm{id}_{\mathsf{M}\circ\mathsf{N}}\in\mathbf{k}[z_\mathsf{M},z_\mathsf{N}]\,\mathrm{id}_{\mathsf{M}\circ\mathsf{N}}$$

by the induction hypothesis.

(ii) Since $R_{\mathsf{M},\mathsf{N}}R_{\mathsf{N},\mathsf{M}}|_{z_\mathsf{N}=0}$ is non-zero by Lemma 2.9(ii), assertion (i) tells us that $R_{\mathsf{M},\mathsf{N}}R_{\mathsf{N},\mathsf{M}}\in\mathbf{k}[z_\mathsf{M},z_\mathsf{N}]\,\mathrm{id}_{\mathsf{N}\circ\mathsf{M}}$ is non-zero. The injectivity of $R_{\mathsf{M},\mathsf{N}}R_{\mathsf{N},\mathsf{M}}$ implies that $R_{\mathsf{N},\mathsf{M}}\colon\mathsf{N}\circ\mathsf{M}\rightarrowtail\mathsf{M}\circ\mathsf{N}$ is injective. Thus the composition with $R_{\mathsf{N},\mathsf{M}}$ induces an injective homomorphism

$$\mathrm{HOM}_{R[z_\mathsf{M},z_\mathsf{N}]}(\mathsf{M}\circ\mathsf{N},\mathsf{N}\circ\mathsf{M})\rightarrowtail\mathrm{HOM}_{R[z_\mathsf{M},z_\mathsf{N}]}(\mathsf{M}\circ\mathsf{N},\mathsf{M}\circ\mathsf{N})\simeq\mathbf{k}[z_\mathsf{M},z_\mathsf{N}].$$

We now consider the non-zero $\mathbf{k}[z_\mathsf{M},z_\mathsf{N}]$-module $L:=\mathrm{HOM}_{R[z_\mathsf{M},z_\mathsf{N}]}(\mathsf{M}\circ\mathsf{N},\mathsf{N}\circ\mathsf{M})$. Let $a,b\in\mathbf{k}[z_\mathsf{M},z_\mathsf{N}]$ be non-zero relatively prime elements. If $f\in aL\cap bL$, then $f(\mathsf{M}\circ\mathsf{N})\subset a(\mathsf{N}\circ\mathsf{M})\cap b(\mathsf{N}\circ\mathsf{M})$. Since a and b are relatively prime and $\mathsf{N}\circ\mathsf{M}$ is a free $\mathbf{k}[z_\mathsf{M},z_\mathsf{N}]$-module,

$$f(\mathsf{M}\circ\mathsf{N})\subset ab(\mathsf{N}\circ\mathsf{M}),$$

which implies that $(ab)^{-1}f:\mathsf{M}\circ\mathsf{N}\to\mathsf{N}\circ\mathsf{M}$ is well-defined, i.e., $f\in abL$. Therefore, we conclude that L satisfies the condition:

$$\text{if } a,b\in\mathbf{k}[z_\mathsf{M},z_\mathsf{N}]\setminus\{0\} \text{ are prime to each other, then } aL\cap bL=abL,$$

which implies that L is a free $\mathbf{k}[z_\mathsf{M},z_\mathsf{N}]$-module of rank one. □

We define $R^{\mathrm{norm}}_{\mathsf{M},\mathsf{N}}$ as a generator of the $\mathbf{k}[z_\mathsf{M},z_\mathsf{N}]$-module $\mathrm{HOM}_{R[z_\mathsf{M},z_\mathsf{N}]}(\mathsf{M}\circ\mathsf{N},\mathsf{N}\circ\mathsf{M})$. It is uniquely determined up to a constant multiple. We call it a *normalized* R-*matrix*.

Theorem 2.13. *Assume* (2.7). *Then* $R^{\mathrm{norm}}_{\mathsf{M},\mathsf{N}}|_{z_\mathsf{M}=z_\mathsf{N}=0}\colon\bar{M}\circ\bar{N}\to\bar{N}\circ\bar{M}$ *does not vanish and is equal to* $\mathbf{r}_{\bar{M},\bar{N}}$ *up to a constant multiple.*

Proof. Since any simple R-module is absolutely simple, we may assume that the base field $\mathbf{k}$ is algebraically closed without loss of generality.

By Proposition 2.10(ii), we have

$$\mathrm{HOM}_R(\bar{M}\circ\bar{N},\bar{N}\circ\bar{M})=\mathbf{k}\,\mathbf{r}_{\bar{M},\bar{N}}\tag{2.8}$$

for a non-zero $\mathbf{r}_{\bar{M},\bar{N}}\in\mathrm{HOM}_R(\bar{M}\circ\bar{N},\bar{N}\circ\bar{M})$. Let ℓ be the homogeneous degree of $\mathbf{r}_{\bar{M},\bar{N}}$.

For $a \in \mathbb{Z}$, let $\mathbf{k}[z_{\mathsf{M}}, z_{\mathsf{N}}]_a$ be the homogeneous part of $\mathbf{k}[z_{\mathsf{M}}, z_{\mathsf{N}}]$ of degree a and set $\mathbf{k}[z_{\mathsf{M}}, z_{\mathsf{N}}]_{\ge a} = \bigoplus_{k\ge a} \mathbf{k}[z_{\mathsf{M}}, z_{\mathsf{N}}]_k$. Let $c \in \mathbb{Z}$ be the largest integer such that

$$R^{\mathrm{norm}}_{\mathsf{M},\mathsf{N}}(\mathsf{M}\circ\mathsf{N}) \subset \mathbf{k}[z_{\mathsf{M}}, z_{\mathsf{N}}]_{\ge c}(\mathsf{N}\circ\mathsf{M}).$$

Then $R^{\mathrm{norm}}_{\mathsf{M},\mathsf{N}}$ induces a non-zero map

$$\varphi\colon \bar{M}\circ\bar{N} \to \big(\mathbf{k}[z_{\mathsf{M}}, z_{\mathsf{N}}]_{\ge c}(\mathsf{N}\circ\mathsf{M})\big)/\big(\mathbf{k}[z_{\mathsf{M}}, z_{\mathsf{N}}]_{\ge c+1}(\mathsf{N}\circ\mathsf{M})\big).$$

Since $\mathsf{N}\circ\mathsf{M}$ is a free $\mathbf{k}[z_{\mathsf{M}}, z_{\mathsf{N}}]$-module, we have

$$\big(\mathbf{k}[z_{\mathsf{M}}, z_{\mathsf{N}}]_{\ge c}(\mathsf{N}\circ\mathsf{M})\big)/\big(\mathbf{k}[z_{\mathsf{M}}, z_{\mathsf{N}}]_{\ge c+1}(\mathsf{N}\circ\mathsf{M})\big) \simeq \mathbf{k}[z_{\mathsf{M}}, z_{\mathsf{N}}]_c \otimes (\bar{N}\circ\bar{M}).$$

By (2.8), there exists a non-zero $f(z_{\mathsf{M}}, z_{\mathsf{N}}) \in \mathbf{k}[z_{\mathsf{M}}, z_{\mathsf{N}}]_c$ such that

$$\varphi(u) = f(z_{\mathsf{M}}, z_{\mathsf{N}})\mathbf{r}_{\bar{M},\bar{N}}(u) \quad \text{for any } u \in \bar{M}\circ\bar{N}.$$

Hence, the homogeneous degree of $R^{\mathrm{norm}}_{\mathsf{M},\mathsf{N}}$ is $c+\ell$, and

$$R^{\mathrm{norm}}_{\mathsf{M},\mathsf{N}}(\mathsf{M}\circ\mathsf{N}) \subset f(z_{\mathsf{M}}, z_{\mathsf{N}})(\mathsf{N}\circ\mathsf{M}) + \mathbf{k}[z_{\mathsf{M}}, z_{\mathsf{N}}]_{\ge c+1}(\mathsf{N}\circ\mathsf{M}).$$

Let us show that $f(z_{\mathsf{M}}, z_{\mathsf{N}})$ is a constant function (i.e., $c = 0$). Assuming that $c > 0$, take a prime divisor $a(z_{\mathsf{M}}, z_{\mathsf{N}})$ of $f(z_{\mathsf{M}}, z_{\mathsf{N}})$. Let $(x, y) \in \mathbf{k}^2$ be such that $a(x, y) = 0$. Let d_{M} and d_{N} be the homogeneous degrees of z_{M} and z_{N}, respectively, and let z be an indeterminate of homogeneous degree one. Then $a(xz^{d_{\mathsf{M}}}, yz^{d_{\mathsf{N}}}) = f(xz^{d_{\mathsf{M}}}, yz^{d_{\mathsf{N}}}) = 0$. Let $\mathbf{k}[z_{\mathsf{M}}, z_{\mathsf{N}}] \to \mathbf{k}[z]$ be the map obtained by the substitution $z_{\mathsf{M}} = xz^{d_{\mathsf{M}}}$ and $z_{\mathsf{N}} = yz^{d_{\mathsf{N}}}$. Set

$$K = \mathbf{k}[z] \underset{\mathbf{k}[z_{\mathsf{M}}, z_{\mathsf{N}}]}{\otimes} (\mathsf{M}\circ\mathsf{N}), \qquad K' = \mathbf{k}[z] \underset{\mathbf{k}[z_{\mathsf{M}}, z_{\mathsf{N}}]}{\otimes} (\mathsf{N}\circ\mathsf{M}), \qquad R' = \mathbf{k}[z] \underset{\mathbf{k}[z_{\mathsf{M}}, z_{\mathsf{N}}]}{\otimes} R^{\mathrm{norm}}_{\mathsf{M},\mathsf{N}}.$$

Then we obtain the map

$$R'\colon K \to z^{c+1}K'.$$

Note that $K/zK \simeq \bar{M}\circ\bar{N}$ and $K'/zK' \simeq \bar{N}\circ\bar{M}$. We shall show $R'(K) \subset z^k K'$ for any $k \ge c+1$ by induction on k. Assume that $k \ge c+1$ and $R'(K) \subset z^k K'$. Then the morphism $\bar{M}\circ\bar{N} \to \bar{N}\circ\bar{M}$ induced by $z^{-k}R'$ is equal to $b\,\mathbf{r}_{\bar{M},\bar{N}}$ for some $b \in \mathbf{k}$. If $b \ne 0$, then the homogeneous degree of R' is $k+\ell > c+\ell$, which is a contradiction. Thus $b = 0$ and $R'(K) \subset z^{k+1}K'$. Hence the induction proceeds, and we conclude that

$$R^{\mathrm{norm}}_{\mathsf{M},\mathsf{N}}|_{z_{\mathsf{M}}=xz^{d_{\mathsf{M}}},\, z_{\mathsf{N}}=yz^{d_{\mathsf{N}}}} = 0$$

for any $(x, y) \in \mathbf{k}^2$ such that $a(x, y) = 0$, which implies $R^{\mathrm{norm}}_{\mathsf{M},\mathsf{N}}$ is divisible by $a(z_{\mathsf{M}}, z_{\mathsf{N}})$. This is a contradiction.

Therefore f is a constant function, and $R^{\mathrm{norm}}_{\mathsf{M},\mathsf{N}}$ induces $\mathbf{r}_{\bar{M},\bar{N}}$ (up to a constant multiple) after the specialization $z_{\mathsf{M}} = z_{\mathsf{N}} = 0$. □

Corollary 2.14. *Assume* (2.7). *If $\bar{M}$ is real, then $R^{\mathrm{norm}}_{\mathsf{M},\mathsf{N}}|_{z_{\mathsf{N}}=0} = R^{\mathrm{norm}}_{\mathsf{M},\bar{N}}$ (up to a constant multiple).*

Lemma 2.15. *Assume* (2.7). *Then there exists a homogeneous element* $f(z_{\mathsf{M}}, z_{\mathsf{N}})$ *such that*

(i) $R^{\mathrm{norm}}_{\mathsf{N},\mathsf{M}} \circ R^{\mathrm{norm}}_{\mathsf{M},\mathsf{N}} = f(z_{\mathsf{M}}, z_{\mathsf{N}})\,\mathrm{id}_{\mathsf{M}\circ\mathsf{N}}$ *and* $R^{\mathrm{norm}}_{\mathsf{M},\mathsf{N}} \circ R^{\mathrm{norm}}_{\mathsf{N},\mathsf{M}} = f(z_{\mathsf{M}}, z_{\mathsf{N}})\,\mathrm{id}_{\mathsf{N}\circ\mathsf{M}}$,
(ii) $f(z_{\mathsf{M}}, 0)$ *and* $f(0, z_{\mathsf{N}})$ *are non-zero.*

Proof. This follows from Proposition 2.12, Corollary 2.14 and Lemma 2.9. ◻

Lemma 2.16. *Let* $(\mathsf{M}, z_{\mathsf{M}})$ *and* $(\mathsf{N}, z_{\mathsf{N}})$ *be affinizations of simple modules* $\bar{M}$ *and* $\bar{N}$, *respectively. Assume that either* $\bar{M}$ *or* $\bar{N}$ *is real, and* $\bar{M} \circ \bar{N} \simeq \bar{N} \circ \bar{M}$. *Let d be a common divisor of the homogeneous degrees* d_{M} *of* z_{M} *and* d_{N} *of* z_{N}. *Let z be an indeterminate of homogeneous degree d and let* $\mathbf{k}[z_{\mathsf{M}}, z_{\mathsf{N}}] \to \mathbf{k}[z]$ *be the algebra homomorphism given by* $z_{\mathsf{M}} \mapsto z^{d_{\mathsf{M}}/d}$ *and* $z_{\mathsf{N}} \mapsto z^{d_{\mathsf{N}}/d}$. *Then* $\mathsf{M} \circ_z \mathsf{N} := \mathbf{k}[z] \otimes_{\mathbf{k}[z_{\mathsf{M}}, z_{\mathsf{N}}]} (\mathsf{M} \circ \mathsf{N})$ *is an affinization of* $\bar{M} \circ \bar{N}$.

Proof. By the assumptions, $\bar{M} \circ \bar{N}$ is simple. Condition (a) in Definition 2.2 is obvious. Condition (b) follows from $\mathfrak{p}_i|_{\mathsf{M}\circ_z\mathsf{N}} = (\mathfrak{p}_i|_{\mathsf{M}}) \circ_z (\mathfrak{p}_i|_{\mathsf{N}})$. ◻

Proposition 2.17. *Let* M *and* $\bar{M}$ *be as in* $(2.3)_{\mathrm{weak}}$. *Assume that* $\bar{M}$ *is real. Normalize* $R^{\mathrm{norm}}_{\mathsf{M},\mathsf{M}}$ *so that it induces* $\mathrm{id}_{\bar{M}\circ\bar{M}}$ *after the specialization* $z_{\mathsf{M}} \circ \mathsf{M} = \mathsf{M} \circ z_{\mathsf{M}} = 0$. *Then*

(i) $(R^{\mathrm{norm}}_{\mathsf{M},\mathsf{M}} - \mathrm{id}_{\mathsf{M}\circ\mathsf{M}})(\mathsf{M} \circ \mathsf{M}) \subset (z_{\mathsf{M}} \circ \mathsf{M} - \mathsf{M} \circ z_{\mathsf{M}})(\mathsf{M} \circ \mathsf{M})$,
(ii) $R^{\mathrm{norm}}_{\mathsf{M},\mathsf{M}} \circ R^{\mathrm{norm}}_{\mathsf{M},\mathsf{M}} = \mathrm{id}_{\mathsf{M},\mathsf{M}}$.

Proof. (i) To avoid confusion, let $(\mathsf{N}, z_{\mathsf{N}})$ be a copy of $(\mathsf{M}, z_{\mathsf{M}})$ and regard $R^{\mathrm{norm}}_{\mathsf{M},\mathsf{M}}$ as a homomorphism $\mathsf{M} \circ \mathsf{N} \to \mathsf{N} \circ \mathsf{M}$. We denote by $\iota \colon \mathsf{M} \circ \mathsf{N} \xrightarrow{\sim} \mathsf{N} \circ \mathsf{M}$ the identity. We regard $\mathsf{M} \circ \mathsf{N}$ and $\mathsf{N} \circ \mathsf{M}$ as $R[z_{\mathsf{M}}, z_{\mathsf{N}}]$-modules. Then $R^{\mathrm{norm}}_{\mathsf{M},\mathsf{N}}$ commutes with z_{M} and z_{N}, but ι does not. Precisely, we have $\iota \circ z_{\mathsf{M}} = z_{\mathsf{N}} \circ \iota$ and $\iota \circ z_{\mathsf{N}} = z_{\mathsf{M}} \circ \iota$.

Let z be another indeterminate with the same homogeneous degree d_{M}, and let

$$\mathbf{k}[z_{\mathsf{M}}, z_{\mathsf{N}}] \to \mathbf{k}[z]$$

be the algebra homomorphism given by $z_{\mathsf{M}} \mapsto z$ and $z_{\mathsf{N}} \mapsto z$. Then Lemma 2.16 implies that $K := \mathbf{k}[z] \otimes_{\mathbf{k}[z_{\mathsf{M}}, z_{\mathsf{N}}]} (\mathsf{M} \circ \mathsf{N})$ is an affinization of $\bar{M} \circ \bar{M}$. The homomorphisms $R^{\mathrm{norm}}_{\mathsf{M},\mathsf{N}}$ and ι induce $R[z]$-linear endomorphisms R' and ι' of K. By Lemma 2.7(i), R' and ι' are powers of z up to a constant multiple. Since they are $\mathrm{id}_{\bar{M}\circ\bar{M}}$ after the specialization $z = 0$, we conclude that $R' = \iota'$, which completes the proof.

(ii) This follows from Lemma 2.15 immediately. ◻

Example 2.18. Let $i \in I$. Let $P(i^n)$ be a projective cover of the simple module $L(i)^{\circ n}$. Then $P(i^n)$ is an $R(n\alpha_i)$-module generated by an element u of degree 0 with the defining relation $\tau_k u = 0$ $(1 \le k < n)$. Let $e_k(x_1, \ldots, x_n)$ be the elementary symmetric function of degree k. The center of $R(n\alpha_i)$ is equal to $\mathbf{k}[e_k(x_1, \ldots, x_n) \mid k = 1, \ldots, n] = \mathbf{k}[x_1, \ldots, x_n]^{\mathfrak{S}_n}$. Then we have

$$L(i)^{\circ n} \simeq P(i^n) / \Big(\sum_{k=1}^{n} R(n\alpha_i) e_k(x_1, \ldots, x_n) u \Big).$$

Set

$$\mathrm{K}(i^n) := P(i^n)/\Big(\sum_{k=1}^{n-1} R(n\alpha_i)e_k(x_1, \dots, x_n)u\Big),$$

and define $\mathrm{z}_{\mathrm{K}(i^n)} \in \mathrm{END}_{R(n\alpha_i)}(\mathrm{K}(i^n))$ by

$$\mathrm{z}_{\mathrm{K}(i^n)}u = e_n(x_1, \dots, x_n)u.$$

Then $(\mathrm{K}(i^n), \mathrm{z}_{\mathrm{K}(i^n)})$ is a strong affinization of $L(i)^{\circ n}$. Note that $\mathfrak{p}_i|_{\mathrm{K}(i^n)} = \mathrm{z}_{\mathrm{K}(i^n)}$. The homogeneous degree of $\mathrm{z}_{\mathrm{K}(i^n)}$ is $n(\alpha_i, \alpha_i)$.

3. Root modules

In this section, we shall review the results of McNamara [15] and Brundan–Kleshchev–McNamara [2]. Throughout this section, we assume that the Cartan matrix A is of finite type. Fix a reduced expression $w_0 = r_{i_1} \dots r_{i_N}$ of the longest element $w_0 \in \mathsf{W}$. This expression gives a convex total order $\prec$ on the set Φ_+ of positive roots: $\alpha_{i_1} \prec r_{i_1}\alpha_{i_2} \prec \cdots \prec r_{i_1} \cdots r_{i_{N-1}}\alpha_{i_N}$. For each positive root $\beta \in \Phi_+$, McNamara defined a simple $R(\beta)$-module $L(\beta)$, which he called the *cuspidal module* [14, 15].

Lemma 3.1 ([15, Lem. 3.4]). *For any $\beta \in \Phi_+$, $L(\beta)$ is a real simple module.*

Lemma 3.2 ([2, Lem. 3.2]). *For $n \ge 0$, there exist unique (up to isomorphism) $R(\beta)$-modules $\Delta_n(\beta)$ with $\Delta_0(\beta) = 0$ such that there are short exact sequences*

$$0 \to q_\beta^{2(n-1)}L(\beta) \xrightarrow{i_n} \Delta_n(\beta) \xrightarrow{p_n} \Delta_{n-1}(\beta) \to 0,$$
$$0 \to q_\beta^2\Delta_{n-1}(\beta) \xrightarrow{j_n} \Delta_n(\beta) \xrightarrow{q_n} L(\beta) \to 0 \quad \text{for } n \ge 1,$$

where $q_\beta = q^{(\beta,\beta)/2}$. Moreover,

(i) $[\Delta_n(\beta)] = \frac{1-q_\beta^{2n}}{1-q_\beta^2}[L(\beta)]$,

(ii) *$\Delta_n(\beta)$ is a cyclic module with simple head isomorphic to $L(\beta)$ and socle isomorphic to $q_\beta^{2(n-1)}L(\beta)$,*

(iii) *for $n \ge 1$,*

$$\mathrm{Ext}^k_{R(\beta)}(\Delta_n(\beta), L(\beta)) \simeq \begin{cases} q_\beta^{-2n}\mathbf{k} & \text{if } k = 1, \\ 0 & \text{if } k \ge 2. \end{cases}$$

Define the *root module*

$$\Delta(\beta) := \varprojlim_n \Delta_n(\beta).$$

Theorem 3.3 ([2, Th. 3.3]). *There is a short exact sequence*

$$0 \to q_\beta^2\Delta(\beta) \xrightarrow{z_\beta} \Delta(\beta) \to L(\beta) \to 0.$$

Moreover,

(i) $\Delta(\beta)$ *is a cyclic module with* $[\Delta(\beta)] = [L(\beta)]/(1 - q_\beta^2)$,
(ii) $L(\beta)$ *is the head of* $\Delta(\beta)$,
(iii) $\mathrm{END}_{R(\beta)}(\Delta(\beta)) \simeq \mathbf{k}[z_\beta]$.

Corollary 3.4 ([2, Cor. 3.5]). *Any finitely generated graded* $R(\beta)$*-module with all simple subquotients isomorphic to* $L(\beta)$ *(up to a grading shift) is a finite direct sum of grade-shifted copies of the indecomposable modules* $\Delta_n(\beta)$ $(n \geq 1)$ *and* $\Delta(\beta)$.

Proposition 3.5. *For any* $\beta \in \Phi_+$, $(\Delta(\beta), z_\beta)$ *is a strong affinization of* $L(\beta)$.

Proof. We can easily check that conditions (a) and (c) in Definition 2.2 are satisfied.

We shall show (b) by induction on $\mathrm{ht}(\beta)$. If β is a simple root, then (b) is obvious. Assume that $\mathrm{ht}(\beta) > 1$. Then, by [2, Lemma 4.9, Theorem 4.10], there exist $\alpha, \gamma \in \Phi_+$ such that $\alpha + \gamma = \beta$ and there exists an exact sequence

$$0 \to q^{-(\alpha,\gamma)}\Delta(\gamma) \circ \Delta(\alpha) \xrightarrow{\varphi} \Delta(\alpha) \circ \Delta(\gamma) \to [1+p]\Delta(\beta) \to 0.$$

Here p is some non-negative integer and $[1+p]$ is the q-integer with respect to the short root. Moreover φ is given by

$$\varphi(u \otimes v) = \tau_{w[m,n]}(v \otimes u) \tag{3.1}$$

for any $u \in \Delta(\gamma)$ and $v \in \Delta(\alpha)$. Here $m = \mathrm{ht}(\alpha)$ and $n = \mathrm{ht}(\gamma)$.

By the induction hypothesis, $(\Delta(\alpha), z_\alpha)$ and $(\Delta(\gamma), z_\gamma)$ are affinizations. By (3.1), φ commutes with z_α and z_γ. Then $\varphi = a(z_\alpha, z_\gamma) R^{\mathrm{norm}}_{\Delta(\gamma),\Delta(\alpha)}$ for some $a(z_\alpha, z_\gamma) \in \mathbf{k}[z_\alpha, z_\gamma]$ by Proposition 2.12.

Note that $\mathfrak{p}_i|_{\Delta(\alpha)\circ\Delta(\gamma)} = (\mathfrak{p}_i|_{\Delta(\alpha)}) \circ (\mathfrak{p}_i|_{\Delta(\gamma)})$, and $\mathfrak{p}_i|_{\Delta(\alpha)} = c_1 z_\alpha^{s_1}$ and $\mathfrak{p}_i|_{\Delta(\gamma)} = c_2 z_\gamma^{s_2}$ for $c_1, c_2 \in \mathbf{k}^\times$ and $s_1, s_2 \in \mathbb{Z}_{\geq 0}$. Hence, if (b) fails, then $(z_\alpha z_\gamma)^s|_{\Delta(\beta)} = 0$ for some $s > 0$. Consequently,

$$(z_\alpha z_\gamma)^s \Delta(\alpha) \circ \Delta(\gamma) \subset \mathrm{Im}(\varphi) \subset \mathrm{Im}(R^{\mathrm{norm}}_{\Delta(\gamma),\Delta(\alpha)}).$$

Take $f(z_\alpha, z_\gamma) \in \mathbf{k}[z_\alpha, z_\gamma]$ such that $R^{\mathrm{norm}}_{\Delta(\gamma),\Delta(\alpha)} R^{\mathrm{norm}}_{\Delta(\alpha),\Delta(\gamma)} = f(z_\alpha, z_\gamma)\,\mathrm{id}_{\Delta(\alpha)\circ\Delta(\gamma)}$. Then

$$(z_\alpha z_\beta)^s\, \mathrm{Im}(R^{\mathrm{norm}}_{\Delta(\alpha),\Delta(\gamma)}) \subset f(z_\alpha, z_\gamma)\Delta(\gamma) \circ \Delta(\alpha).$$

By Lemma 2.15, we have $f(z_\alpha, 0) \neq 0$ and $f(0, z_\gamma) \neq 0$, which implies

$$\mathrm{Im}(R^{\mathrm{norm}}_{\Delta(\alpha),\Delta(\gamma)}) \subset f(z_\alpha, z_\gamma)\Delta(\gamma) \circ \Delta(\alpha).$$

Therefore $f(z_\alpha, z_\gamma)^{-1} R^{\mathrm{norm}}_{\Delta(\alpha),\Delta(\gamma)}$ is well-defined, which implies that f is an invertible element of $\mathbf{k}$. Hence $R^{\mathrm{norm}}_{\Delta(\alpha),\Delta(\gamma)}$ is an isomorphism. Then $L(\alpha) \circ L(\gamma)$ is simple, which is a contradiction. ◻

Note that $[L(\beta)] \in [R\text{-gmod}] \simeq A_q(\mathfrak{g}^+)$ coincides with the dual PBW vector $E^*(\beta) \in A_q(\mathfrak{g}^+)$. It is known that $\{E^*(m_1, \ldots, m_N)\}_{(m_1,\ldots,m_N)\in\mathbb{Z}^N_{\geq 0}}$ is a basis of $A_q(\mathfrak{g}^+)$, which is called the *dual PBW basis*. Here, we set

$$E^*(m_1, \ldots, m_N) := \left(q_{\beta_1}^{m_1(m_1-1)/2} E^*(\beta_1)^{m_1}\right) \cdots \left(q_{\beta_N}^{m_N(m_N-1)/2} E^*(\beta_N)^{m_N}\right)$$

with $\beta_{N-k+1} := r_{i_1} \cdots r_{i_{k-1}} \alpha_{i_k}$ and $q_\beta = q^{(\beta,\beta)/2}$ ($k = 1, \dots, N$). On the other hand, $E(\beta) = E^*(\beta)/(E^*(\beta), E^*(\beta))$ is called the *PBW vector* and

$$\{E(\beta_1)^{(m_1)} \cdots E(\beta_N)^{(m_N)}\}_{(m_1,\dots,m_N)\in\mathbb{Z}_{\geq 0}^N}$$

is a basis of $U_{\mathbf{A}}^-(\mathfrak{g})$ called the *PBW basis*. Here $E(\beta)^{(m)} = E(\beta)^m/[m]_i!$ with $i \in I$ such that $(\beta, \beta) = (\alpha_i, \alpha_i)$. Note that the PBW basis and the dual PBW basis are dual to each other.

4. The duality functor

4.1. Duality data

Let R be the quiver Hecke algebra associated with a generalized Cartan matrix A and polynomials $\mathcal{Q}_{i,j}(u, v)$.

Definition 4.1. Let J be a finite index set. We say that $\mathcal{D} = \{\beta_j, M_j, \mathsf{z}_j, \mathsf{r}_j, \mathsf{R}_{j,k}\}_{j,k\in J}$ is a *duality datum* if $\beta_j \in \mathsf{Q}_+ \setminus \{0\}$, $M_j \in \mathrm{Mod}_{\mathrm{gr}}(R(\beta_j))$ and homogeneous homomorphisms

$$\begin{aligned} &\mathsf{z}_j \in \mathrm{E{\scriptstyle ND}}_{R(\beta_j)}(M_j), \qquad \mathsf{r}_j \in \mathrm{E{\scriptstyle ND}}_{R(2\beta_j)}(M_j \circ M_j), \\ &\mathsf{R}_{j,k} \in \mathrm{H{\scriptstyle OM}}_{R(\beta_j+\beta_k)}(M_j \circ M_k, M_k \circ M_j) \quad \text{for } j, k \in J \end{aligned} \tag{4.1}$$

satisfy the following conditions:

($\mathcal{F}$-1) For $j \in J$, $\deg \mathsf{z}_j \in 2\mathbb{Z}_{>0}$. In addition, M_j is a finitely generated free module over the polynomial ring $\mathbf{k}[\mathsf{z}_j]$.

($\mathcal{F}$-2) For $j \in J$, we have $\mathsf{r}_j \in \mathrm{E{\scriptstyle ND}}_{R(2\beta_j)}(M_j \circ M_j)_{-\deg \mathsf{z}_j}$ and

$$\mathsf{R}_{j,j} = (\mathsf{z}_j \circ M_j - M_j \circ \mathsf{z}_j)\, \mathsf{r}_j + \mathrm{id}_{M_j \circ M_j}.$$

($\mathcal{F}$-3) For $k, l \in J$,

(a) $(\mathsf{z}_l \circ M_k)\, \mathsf{R}_{k,l} = \mathsf{R}_{k,l}\, (M_k \circ \mathsf{z}_l)$ in $\mathrm{H{\scriptstyle OM}}_{R(\beta_k+\beta_l)}(M_k \circ M_l, M_l \circ M_k)$,

(b) $(M_l \circ \mathsf{z}_k)\, \mathsf{R}_{k,l} = \mathsf{R}_{k,l}\, (\mathsf{z}_k \circ M_l)$ in $\mathrm{H{\scriptstyle OM}}_{R(\beta_k+\beta_l)}(M_k \circ M_l, M_l \circ M_k)$.

($\mathcal{F}$-4) There exist polynomials $\mathcal{Q}^{\mathcal{D}}_{k,l}(u, v) \in \mathbf{k}[u, v]$ ($k, l \in J$) such that

(a) $\mathcal{Q}^{\mathcal{D}}_{k,k}(u, v) = 0$, and $\mathcal{Q}^{\mathcal{D}}_{k,l}(u, v)$ ($k \neq l$) is of the form

$$\sum_{\deg \mathsf{R}_{k,l}+\deg \mathsf{R}_{l,k}-p \deg \mathsf{z}_k - q \deg \mathsf{z}_l = 0} t_{k,l;p,q}\, u^p v^q,$$

where $t_{k,l;(\deg \mathsf{R}_{k,l}+\deg \mathsf{R}_{l,k})/\deg \mathsf{z}_k,\, 0} \in \mathbf{k}^\times$,

(b) $\mathcal{Q}^{\mathcal{D}}_{k,l}(u, v) = \mathcal{Q}^{\mathcal{D}}_{l,k}(v, u)$,

(c) $\mathsf{R}_{l,k}\mathsf{R}_{k,l} = \begin{cases} 1 & \text{if } k = l, \\ \mathcal{Q}^{\mathcal{D}}_{k,l}(\mathsf{z}_k \circ M_l, M_k \circ \mathsf{z}_l) & \text{if } k \neq l. \end{cases}$

($\mathcal{F}$-5) For any $j, k, l \in J$,

$$(\mathsf{R}_{k,l} \circ M_j)(M_k \circ \mathsf{R}_{j,l})(\mathsf{R}_{j,k} \circ M_l) = (M_l \circ \mathsf{R}_{j,k})(\mathsf{R}_{j,l} \circ M_k)(M_j \circ \mathsf{R}_{k,l})$$

in $\mathrm{H{\scriptstyle OM}}_{R(\beta_j+\beta_k+\beta_l)}(M_j \circ M_k \circ M_l, M_l \circ M_k \circ M_j)$.

For simplicity, we write briefly $\{M_j, \mathsf{z}_j, \mathsf{R}_{j,k}\}_{j,k\in J}$ for $\{\beta_j, M_j, \mathsf{z}_j, \mathsf{r}_j, \mathsf{R}_{j,k}\}_{j,k\in J}$ if there is no risk of confusion.

We now construct a Cartan datum corresponding to the duality datum $\mathcal{D}$ as follows. Let $\{\alpha_j^{\mathcal{D}}\}_{j\in J}$ be the simple roots. Then we define a weight lattice $\mathsf{P}^{\mathcal{D}}$ by $\mathsf{P}^{\mathcal{D}} = \mathsf{Q}^{\mathcal{D}} := \bigoplus_{j\in J} \mathbb{Z}\alpha_j^{\mathcal{D}}$, and define a symmetric bilinear form on $\mathsf{P}^{\mathcal{D}}$ by

$$(\alpha_j^{\mathcal{D}}, \alpha_k^{\mathcal{D}}) = \begin{cases} \deg \mathsf{z}_j & \text{if } j = k, \\ -(\deg \mathcal{Q}_{j,k}^{\mathcal{D}}(\mathsf{z}_j, \mathsf{z}_k))/2 = -(\deg \mathsf{R}_{j,k} + \deg \mathsf{R}_{k,j})/2 & \text{otherwise.} \end{cases} \tag{4.2}$$

Define $h_j^{\mathcal{D}}$ by (2) of Definition 1.1. Then the corresponding generalized Cartan matrix $\mathsf{A}^{\mathcal{D}} := (a_{jk}^{\mathcal{D}})_{j,k\in J}$ is given by

$$a_{jk}^{\mathcal{D}} = \frac{2(\alpha_j^{\mathcal{D}}, \alpha_k^{\mathcal{D}})}{(\alpha_j^{\mathcal{D}}, \alpha_j^{\mathcal{D}})}.$$

Since $\mathcal{Q}_{j,k}^{\mathcal{D}}(\mathsf{z}_j, 0) \in \mathbf{k}^\times \mathsf{z}_j^{-a_{jk}^{\mathcal{D}}}$ for $j \neq k$, $-a_{jk}^{\mathcal{D}}$ is a non-negative integer. Therefore, $\mathsf{A}^{\mathcal{D}}$ is a generalized Cartan matrix. We then define $R^{\mathcal{D}}$ as the quiver Hecke algebra corresponding to the datum $\{\mathcal{Q}_{j,k}^{\mathcal{D}}\}_{j,k\in J}$.

We now have two different quiver Hecke algebras R and $R^{\mathcal{D}}$. To distinguish them, we write

$$x_k^{\mathcal{D}}\ (1 \le k \le \mathrm{ht}(\gamma)) \quad \text{and} \quad \tau_l^{\mathcal{D}}\ (1 \le l \le \mathrm{ht}(\gamma) - 1)$$

for the generators x_k $(1 \le k \le \mathrm{ht}(\gamma))$ and τ_l $(1 \le j \le \mathrm{ht}(\gamma) - 1)$ of $R^{\mathcal{D}}(\gamma)$ $(\gamma \in \mathsf{Q}_+^{\mathcal{D}})$.

The $\mathbb{Z}$-grading on $R^{\mathcal{D}}(\gamma)$ is given as follows:

$$\deg(e(\mu)) = 0, \qquad \deg(e(\mu)x_k^{\mathcal{D}}) = \deg \mathsf{z}_{\mu_k},$$
$$\deg(e(\mu)\tau_l^{\mathcal{D}}) = \begin{cases} -\deg \mathsf{z}_{\mu_l} & \text{if } \mu_l = \mu_{l+1}, \\ \deg \mathsf{R}_{\mu_l,\mu_{l+1}} & \text{if } \mu_l \neq \mu_{l+1}, \end{cases}$$

which is well-defined (see Definition 1.4).

Let $\gamma \in \mathsf{Q}_+^{\mathcal{D}}$ with $m = \mathrm{ht}(\gamma)$, and define

$$\Delta^{\mathcal{D}}(\gamma) := \bigoplus_{\mu\in J^\gamma} \Delta_\mu^{\mathcal{D}},$$

where

$$\Delta_\mu^{\mathcal{D}} := M_{\mu_1} \circ \cdots \circ M_{\mu_m} \quad \text{for } \mu = (\mu_1, \dots, \mu_m) \in J^\gamma.$$

Let $\phi\colon \mathsf{Q}^{\mathcal{D}} \to \mathsf{Q}$ be the linear map defined by $\phi(\alpha_j^{\mathcal{D}}) = \beta_j$ for $j \in J$. Then it is clear that $\Delta^{\mathcal{D}}(\gamma)$ is a left $R(\phi(\gamma))$-module.

We define a right $R^{\mathcal{D}}(\gamma)$-module structure on $\Delta^{\mathcal{D}}(\gamma)$ as follows:

(a) $e(\mu)$ is the projection to the component $\Delta_\mu^{\mathcal{D}}$,
(b) the action of $x_k^{\mathcal{D}}$ on $\Delta_\mu^{\mathcal{D}}$ is given by $M_{\mu_1} \circ \cdots \circ M_{\mu_{k-1}} \circ \mathsf{z}_{\mu_k} \circ M_{\mu_{k+1}} \circ \cdots \circ M_{\mu_m}$,

(c) if $\mu_k \neq \mu_{k+1}$, the action of $\tau_k^{\mathcal{D}}$ on $\Delta_\mu^{\mathcal{D}}$ is given by

$$M_{\mu_1} \circ \cdots \circ M_{\mu_{k-1}} \circ \mathsf{R}_{\mu_k,\, \mu_{k+1}} \circ M_{\mu_{k+2}} \circ \cdots \circ M_{\mu_m},$$

(d) if $\mu_k = \mu_{k+1}$, the action of $\tau_k^{\mathcal{D}}$ on $\Delta_\mu^{\mathcal{D}}$ is given by

$$M_{\mu_1} \circ \cdots \circ M_{\mu_{k-1}} \circ \mathsf{r}_{\mu_k} \circ M_{\mu_{k+2}} \circ \cdots \circ M_{\mu_m}.$$

Theorem 4.2. *The right $R^{\mathcal{D}}(\gamma)$-module structure on $\Delta^{\mathcal{D}}(\gamma)$ is well-defined.*

Proof. Since the proof is easy and similar to the arguments in [5], we omit it. □

By construction, the right $R^{\mathcal{D}}(\gamma)$-module action commutes with the left $R(\phi(\gamma))$-module action, which means that

$$\Delta^{\mathcal{D}}(\gamma) \text{ has an } (R(\phi(\gamma)),\, R^{\mathcal{D}}(\gamma))\text{-bimodule structure.}$$

We now define a functor

$$\mathfrak{F}_\gamma^{\mathcal{D}} \colon \operatorname{Mod}_{\mathrm{gr}}(R^{\mathcal{D}}(\gamma)) \to \operatorname{Mod}_{\mathrm{gr}}(R(\phi(\gamma)))$$

by

$$\mathfrak{F}_\gamma^{\mathcal{D}}(M) := \Delta^{\mathcal{D}}(\gamma) \otimes_{R^{\mathcal{D}}(\gamma)} M.$$

Set

$$\mathfrak{F}^{\mathcal{D}} = \bigoplus_{\gamma \in \mathsf{Q}_+^{\mathcal{D}}} \mathfrak{F}_\gamma^{\mathcal{D}}.$$

For $j \in J$, we write $L^{\mathcal{D}}(j)$ for the simple $R^{\mathcal{D}}(\alpha_j^{\mathcal{D}})$-module $R^{\mathcal{D}}(\alpha_j^{\mathcal{D}})/R^{\mathcal{D}}(\alpha_j^{\mathcal{D}})x_1^{\mathcal{D}}$.

Theorem 4.3. *Let $\mathcal{D} = \{\beta_j, M_j, \mathsf{z}_j, \mathsf{r}_j, \mathsf{R}_{j,k}\}_{j,k \in J}$ be a duality datum. Then:*

(i) *The functor $\mathfrak{F}^{\mathcal{D}} : \operatorname{Mod}_{\mathrm{gr}}(R^{\mathcal{D}}) \to \operatorname{Mod}_{\mathrm{gr}}(R)$ is a tensor functor.*
(ii) *For $j \in J$,*

$$\mathfrak{F}^{\mathcal{D}}(R^{\mathcal{D}}(\alpha_j^{\mathcal{D}})) \simeq \mathsf{M}_j \quad \textit{and} \quad \mathfrak{F}^{\mathcal{D}}(L^{\mathcal{D}}(j)) \simeq \mathsf{M}_j/\mathsf{z}_j\mathsf{M}_j.$$

(iii) *If $\mathsf{A}^{\mathcal{D}}$ is of finite type, then the functor $\mathfrak{F}^{\mathcal{D}}$ is exact.*
(iv) *If a graded $R^{\mathcal{D}}(\gamma)$-module L is finite-dimensional, then so is $\mathfrak{F}^{\mathcal{D}}(L)$. Thus, we have the induced functor $\mathfrak{F}^{\mathcal{D}} \colon R^{\mathcal{D}}$-gmod $\to R$-gmod.*

Proof. Since the proof is easy and similar to one in [5], we omit it. □

4.2. Construction of duality data from affinizations

Let J be a finite index set. Let $\{\beta_j, \mathsf{M}_j, \mathsf{z}_j\}_{j \in J}$ be a datum such that

(a) $\beta_j \in \mathsf{Q}_+ \setminus \{0\}$,
(b) $(\mathsf{M}_j, \mathsf{z}_j)$ is an even affinization of a real simple $R(\beta_j)$-module $\bar{M}_j := \mathsf{M}_j/\mathsf{z}_j\mathsf{M}_j$.

Then we take $\mathsf{R}_{j,k}$ as follows:

(c) $\mathsf{R}_{j,k} = R^{\mathrm{norm}}_{\mathsf{M}_j,\mathsf{M}_k}$. Furthermore, we normalize $\mathsf{R}_{j,j}$ so that $\mathsf{R}_{j,k}|_{\mathsf{z}_j=\mathsf{z}_k=0} = \mathrm{id}_{\bar{M}_j\circ\bar{M}_j}$ when $j = k$.

Then Proposition 2.17 implies that

$$\mathsf{r}_j := (\mathsf{z}_j \circ \mathsf{M}_j - \mathsf{M}_j \circ \mathsf{z}_j)^{-1}(\mathsf{R}_{j,j} - \mathrm{id}_{\mathsf{M}_j\circ\mathsf{M}_j}) \tag{4.3}$$

is a well-defined endomorphism of $\mathsf{M}_j \circ \mathsf{M}_j$.

Note that for any $\{\beta_j, \mathsf{M}_j, \mathsf{z}_j\}_{j\in J}$ satisfying (a) and (b), we can always choose $\mathsf{R}_{j,k}$'s. Moreover, $\mathsf{R}_{j,j}$ is unique and $\mathsf{R}_{j,k}$ $(j \neq k)$ is unique up to constant multiple.

Theorem 4.4. *Under the above assumptions* (a)–(c)*, we have the following.*

(i) *The datum* $\mathcal{D} = \{\beta_j, \mathsf{M}_j, \mathsf{z}_j, \mathsf{r}_j, \mathsf{R}_{j,k}\}_{j,k\in J}$ *is a duality datum.*

(ii) *Assume that* $\mathsf{A}^{\mathcal{D}}$ *is of finite type. Then:*

  (a) $\mathfrak{F}^{\mathcal{D}}(M)$ *is either a simple module or vanishes for any simple* $R^{\mathcal{D}}$*-module* M*. Moreover, if* M *is a real simple module and* $\mathfrak{F}^{\mathcal{D}}(M)$ *is non-zero, then* $\mathfrak{F}^{\mathcal{D}}(M)$ *is real.*

  (b) *Let* $(\mathsf{N}, z_{\mathsf{N}})$ *be an affinization of a simple* $R^{\mathcal{D}}$*-module* $\bar{N}$*. If* $\mathfrak{F}^{\mathcal{D}}(\bar{N})$ *is simple, then* $(\mathfrak{F}^{\mathcal{D}}(\mathsf{N}), \mathfrak{F}^{\mathcal{D}}(z_{\mathsf{N}}))$ *is an affinization of* $\mathfrak{F}^{\mathcal{D}}(\bar{N})$*.*

  (c) *Let* M *and* N *be simple* $R^{\mathcal{D}}$*-modules, and assume that one of them is real and also admits an affinization. Then* $\mathfrak{F}^{\mathcal{D}}(M \diamond N)$ *is either zero or isomorphic to* $\mathfrak{F}^{\mathcal{D}}(M) \diamond \mathfrak{F}^{\mathcal{D}}(N)$*.*

Proof. (i) Let us prove that $\mathcal{D}$ is a duality datum. Since axioms ($\mathcal{F}$-1)–($\mathcal{F}$-4) are obvious, we only give the proof of the braid relation ($\mathcal{F}$-5):

$$\mathsf{R}_{jk} \circ \mathsf{R}_{ik} \circ \mathsf{R}_{ij} = \mathsf{R}_{ij} \circ \mathsf{R}_{ik} \circ \mathsf{R}_{jk} \tag{4.4}$$

as a morphism $\mathsf{M}_i \circ \mathsf{M}_j \circ \mathsf{M}_k \to \mathsf{M}_k \circ \mathsf{M}_j \circ \mathsf{M}_i$ for $i, j, k \in J$. By the definition, we have $R_{\mathsf{M}_i,\mathsf{M}_j} = a(\mathsf{z}_i, \mathsf{z}_j)\mathsf{R}_{i,j}$ for a non-zero polynomial $a(\mathsf{z}_i, \mathsf{z}_j)$. The R-matrices $R_{\mathsf{M}_i,\mathsf{M}_j}$ satisfy the braid relation

$$R_{\mathsf{M}_j,\mathsf{M}_k} \circ R_{\mathsf{M}_i,\mathsf{M}_k} \circ R_{\mathsf{M}_i,\mathsf{M}_j} = R_{\mathsf{M}_i,\mathsf{M}_j} \circ R_{\mathsf{M}_i,\mathsf{M}_k} \circ R_{\mathsf{M}_j,\mathsf{M}_k}.$$

The calculation

$$\begin{aligned} R_{\mathsf{M}_j,\mathsf{M}_k} \circ R_{\mathsf{M}_i,\mathsf{M}_k} \circ R_{\mathsf{M}_i,\mathsf{M}_j} &= a(\mathsf{z}_j, \mathsf{z}_k)\mathsf{R}_{j,k} \circ a(\mathsf{z}_i, \mathsf{z}_k)\mathsf{R}_{i,k} \circ a(\mathsf{z}_i, \mathsf{z}_j)\mathsf{R}_{i,j} \\ &= a(\mathsf{z}_j, \mathsf{z}_k)a(\mathsf{z}_i, \mathsf{z}_k)a(\mathsf{z}_i, \mathsf{z}_j)\mathsf{R}_{j,k} \circ \mathsf{R}_{i,k} \circ \mathsf{R}_{i,j} \end{aligned}$$

and a similar calculation for $R_{\mathsf{M}_i,\mathsf{M}_j} \circ R_{\mathsf{M}_i,\mathsf{M}_k} \circ R_{\mathsf{M}_j,\mathsf{M}_k}$ show that

$$a(\mathsf{z}_j, \mathsf{z}_k)a(\mathsf{z}_i, \mathsf{z}_k)a(\mathsf{z}_i, \mathsf{z}_j)(\mathsf{R}_{jk} \circ \mathsf{R}_{ik} \circ \mathsf{R}_{ij} - \mathsf{R}_{ij} \circ \mathsf{R}_{ik} \circ \mathsf{R}_{jk}) = 0.$$

Hence we obtain (4.4).

(ii)(a) Let us prove that $\mathfrak{F}^{\mathcal{D}}(M)$ is a simple module or zero for every simple $R^{\mathcal{D}}(\gamma)$-module M by induction on $\mathrm{ht}(\gamma)$. Assume $M \simeq N \diamond L^{\mathcal{D}}(j)$ for some $j \in J$ and a simple

$R^{\mathcal{D}}(\gamma - \alpha_j^{\mathcal{D}})$-module N. By the induction hypothesis, $\mathfrak{F}^{\mathcal{D}}(N)$ is a simple module or zero. Let $r\colon N \circ L^{\mathcal{D}}(j) \to L^{\mathcal{D}}(j) \circ N$ be a non-zero homomorphism of $R^{\mathcal{D}}(\gamma)$-modules. Then $\mathrm{Im}(r)$ is isomorphic to $N \diamond L^{\mathcal{D}}(j)$. Since $\mathfrak{F}^{\mathcal{D}}$ is exact, $\mathfrak{F}^{\mathcal{D}}(\mathrm{Im}(r)) \simeq \mathrm{Im}(\mathfrak{F}^{\mathcal{D}}(r)) \simeq \mathfrak{F}^{\mathcal{D}}(M)$. If $\mathfrak{F}^{\mathcal{D}}(N) \simeq 0$, then $\mathfrak{F}^{\mathcal{D}}(M) \simeq 0$. Assume that $\mathfrak{F}^{\mathcal{D}}(N)$ is a simple module. Then $\mathrm{Im}(\mathfrak{F}^{\mathcal{D}}(r))$ is isomorphic to $\mathfrak{F}^{\mathcal{D}}(N) \diamond \mathfrak{F}^{\mathcal{D}}(L^{\mathcal{D}}(j))$ or 0 according as $\mathfrak{F}^{\mathcal{D}}(r)$ is non-zero or zero by Proposition 2.10.

If M is real simple and $\mathfrak{F}^{\mathcal{D}}(M)$ is simple, then $\mathfrak{F}^{\mathcal{D}}(M) \circ \mathfrak{F}^{\mathcal{D}}(M) \simeq \mathfrak{F}^{\mathcal{D}}(M \circ M)$ is simple and hence $\mathfrak{F}^{\mathcal{D}}(M)$ is real.

Thus we obtain (ii)(a).

(ii)(b) Let $\bar{N}$ be a simple $R^{\mathcal{D}}(\gamma)$-module and set $m = \mathrm{ht}(\gamma)$. We write $\mathsf{N}_{\mathfrak{F}} = \mathfrak{F}^{\mathcal{D}}(\mathsf{N})$ and $z_{\mathfrak{F}} = \mathfrak{F}^{\mathcal{D}}(z_{\mathsf{N}})$. Applying the functor $\mathfrak{F}^{\mathcal{D}}$ to the exact sequence

$$0 \to \mathsf{N} \xrightarrow{z_{\mathsf{N}}} \mathsf{N} \to \bar{N} \to 0,$$

we obtain the exact sequence

$$0 \to \mathsf{N}_{\mathfrak{F}} \xrightarrow{z_{\mathfrak{F}}} \mathsf{N}_{\mathfrak{F}} \to \mathfrak{F}^{\mathcal{D}}(\bar{N}) \to 0.$$

Thus, we have an injective homogeneous endomorphism $z_{\mathfrak{F}}$ of $\mathsf{N}_{\mathfrak{F}}$ and $\mathsf{N}_{\mathfrak{F}}/z_{\mathfrak{F}}\mathsf{N}_{\mathfrak{F}} \simeq \mathfrak{F}^{\mathcal{D}}(\bar{N})$. Since M_j is a finitely generated $R(\beta_j)$-module for any j by Lemma 2.8, $\mathsf{N}_{\mathfrak{F}}$ is a finitely generated graded R-module and $\mathfrak{F}^{\mathcal{D}}(\bar{N})$ is a finite-dimensional R-module. Hence, condition (a) of Definition 2.2 holds (see Remark 2.3(i)).

Let us show (b) of Definition 2.2. Let $i \in I$. By Lemma 2.7(ii), for any $j \in J$, there exist $d_j \in \mathbb{Z}_{\ge 0}$ and $c_j \in \mathbf{k}^\times$ such that $\mathfrak{p}_i|_{\mathsf{M}_j} = c_j \mathsf{z}_j^{d_j}$. Since $\mathfrak{p}_i|_{\mathsf{M}_{\mu_1}\circ\cdots\circ\mathsf{M}_{\mu_m}} = (\mathfrak{p}_i|_{\mathsf{M}_{\mu_1}}) \circ \cdots \circ (\mathfrak{p}_i|_{\mathsf{M}_{\mu_m}}) = \prod_{k=1}^m c_{\mu_k} (x_k^{\mathcal{D}})^{d_{\mu_k}}$, we obtain

$$\begin{aligned}
\mathfrak{p}_i|_{\mathsf{N}_{\mathfrak{F}}} &= \sum_{\mu \in J^\gamma} (\mathfrak{p}_i|_{\mathsf{M}_{\mu_1}\circ\cdots\circ\mathsf{M}_{\mu_m}}) \underset{R^{\mathcal{D}}(\gamma)}{\otimes} \mathsf{N} \\
&= \sum_{\mu \in J^\gamma} (\mathsf{M}_{\mu_1} \circ \cdots \circ \mathsf{M}_{\mu_m}) \underset{R^{\mathcal{D}}(\gamma)}{\otimes} \big(e(\mu) c (x_1^{\mathcal{D}})^{d_{\mu_1}} \cdots (x_m^{\mathcal{D}})^{d_{\mu_m}} \big)\big|_{\mathsf{N}} \\
&= \sum_{\mu \in J^\gamma} (\mathsf{M}_{\mu_1} \circ \cdots \circ \mathsf{M}_{\mu_m}) \underset{R^{\mathcal{D}}(\gamma)}{\otimes} \Big(c e(\mu) \prod_{j \in J} \Big(\prod_{k \in [1,m],\ \mu_k = j} (x_k^{\mathcal{D}})^{d_j} \Big)\Big)\Big|_{\mathsf{N}} \\
&= \Delta^{\mathcal{D}}(\gamma) \underset{R^{\mathcal{D}}(\gamma)}{\otimes} \Big(c \prod_{j \in J} \mathfrak{p}_j^{d_j} \Big)\Big|_{\mathsf{N}}
\end{aligned}$$

with $c = \prod_{k=1}^m c_{\mu_k}$ which does not depend on $\mu \in J^\gamma$. Therefore, condition (b) of Definition 2.2 holds.

(ii)(c) immediately follows from (a) and the epimorphism

$$\mathfrak{F}^{\mathcal{D}}(M) \circ \mathfrak{F}^{\mathcal{D}}(N) \twoheadrightarrow \mathfrak{F}^{\mathcal{D}}(M \diamond N)$$

because $M \diamond N$ is simple. □

5. Examples

Let $\mathfrak{g}$ be a Kac–Moody Lie algebra associated with a Cartan matrix A of finite type. Suppose that

$$\left.\begin{array}{l}\text{(a) } \{\beta_j\}_{j\in J} \text{ is a family of elements of } \Phi_+ \text{ which is linearly independent in } \mathsf{Q},\\ \text{(b) } \beta_j - \beta_k \notin \Phi \text{ for any } j,k \in J, \text{ where } \Phi \text{ is the set of roots of } \mathfrak{g}.\end{array}\right\} \tag{5.1}$$

Let $\overline{\mathfrak{g}}$ be the Lie subalgebra of $\mathfrak{g}$ generated by the root vectors of weight β_j and $-\beta_j$ (cf. [16, Th. 1.1]). Then $\overline{\mathfrak{g}}$ is a Kac–Moody Lie algebra associated to

$$\overline{\mathsf{A}} := (\overline{a}_{j,k})_{j,k\in J} \quad \text{with} \quad \overline{a}_{j,k} := 2(\beta_j,\beta_k)/(\beta_j,\beta_j). \tag{5.2}$$

We have an injective algebra homomorphism

$$U^-(\overline{\mathfrak{g}}) \rightarrowtail U^-(\mathfrak{g}). \tag{5.3}$$

Choosing a convex order of the set Φ_+ of positive roots, let $(\Delta(\beta_j), \mathsf{z}_j)$ be the affinization of $L(\beta_j)$ given in Proposition 3.5. Then we have the duality datum

$$\mathcal{D} := \{\Delta(\beta_j), \mathsf{z}_j, \mathsf{R}_{k,l}\}_{j,k,l\in J}.$$

Let $\mathfrak{g}^{\mathcal{D}}$ be the Kac–Moody Lie algebra associated with $\mathsf{A}^{\mathcal{D}}$. Suppose that $\mathsf{A}^{\mathcal{D}}$ is of finite type. Then the functor $\mathfrak{F}^{\mathcal{D}}$ is exact, and gives a $\mathbb{Z}[q^{\pm1}]$-algebra homomorphism

$$[R^{\mathcal{D}}\text{-gmod}] \to [R\text{-gmod}]$$

which gives a $\mathbb{Z}[q^{\pm1/2}]$-algebra homomorphism (see Corollary 1.6)

$$A_q((\mathfrak{g}^{\mathcal{D}})^+)_{\mathbf{c}} \to \mathbb{Z}[q^{\pm1/2}] \underset{\mathbb{Z}[q^{\pm1}]}{\otimes} A_q(\mathfrak{g}^+). \tag{5.4}$$

sending f_j to the dual PBW generator $E^*(\beta_j)$ corresponding to $[\Delta(\beta_j)]$. Here $\mathbf{c}$ is the bilinear form on $\mathsf{Q}^{\mathcal{D}}$ given by $\mathbf{c}(\alpha_j^{\mathcal{D}}, \alpha_k^{\mathcal{D}}) = \frac{1}{2}(\deg \mathsf{R}_{k,j} - \deg \mathsf{R}_{j,k})$.

By applying the exact functor $\mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Q}[q^{\pm1/2}]} \bullet$ to (5.4), we obtain a $\mathbb{Q}(q^{1/2})$-algebra homomorphism

$$U_q^-(\mathfrak{g}^{\mathcal{D}})_{\mathbf{c}} \to \mathbb{Q}(q^{1/2}) \underset{\mathbb{Q}(q)}{\otimes} U_q^-(\mathfrak{g}). \tag{5.5}$$

Set $c_\beta := (E^*(\beta), E^*(\beta))^{-1}$. Then $E(\beta) = c_\beta E^*(\beta)$ is the PBW vector corresponding to $\beta \in \Phi_+$. Let ψ be the algebra automorphism of $U_q^-(\mathfrak{g}^{\mathcal{D}})_{\mathbf{c}}$ sending f_j to $c_{\beta_j} f_j$. Then the composition

$$U_q^-(\mathfrak{g}^{\mathcal{D}})_{\mathbf{c}} \underset{\psi}{\xrightarrow{\sim}} U_q^-(\mathfrak{g}^{\mathcal{D}})_{\mathbf{c}} \to \mathbb{Q}(q^{1/2}) \underset{\mathbb{Q}(q)}{\otimes} U_q^-(\mathfrak{g})$$

sends f_j to $E(\beta_j)$. Since $\deg z_{\mathsf{M}} = (\beta_j, \beta_j)$ by Theorem 3.3, the above homomorphism sends the divided power $f_j^{(m)}$ to the divided power $E(\beta_j)^{(m)}$. Moreover, the $f_j^{(m)}$'s generate the $\mathbf{A}$-algebra $U_{\mathbf{A}}^-(\mathfrak{g}^{\mathcal{D}})_{\mathbf{c}}$, and the $E(\beta_j)^{(m)}$'s are contained in $U_{\mathbf{A}}^-(\mathfrak{g})$. Hence we obtain an algebra homomorphism

$$U_{\mathbf{A}}^-(\mathfrak{g}^{\mathcal{D}})_{\mathbf{c}} \to \mathbb{Q}[q^{\pm 1/2}] \underset{\mathbb{Q}[q^{\pm 1}]}{\otimes} U_{\mathbf{A}}^-(\mathfrak{g}). \tag{5.6}$$

Taking the classical limit $q^{1/2} = 1$, we obtain the induced algebra homomorphism

$$U^-(\mathfrak{g}^{\mathcal{D}}) \to U^-(\mathfrak{g}) \tag{5.7}$$

sending f_j to the root vector corresponding to $-\beta_j$ for $j \in J$.

Proposition 5.1. *If* $\mathsf{A}^{\mathcal{D}} = \overline{\mathsf{A}}$, *then the morphism* $[R^{\mathcal{D}}\text{-gmod}] \to [R\text{-gmod}]$ *induced by* $\mathfrak{F}^{\mathcal{D}}$ *is injective. In particular* $\mathfrak{F}^{\mathcal{D}}$ *sends simple* $R^{\mathcal{D}}$*-modules to simple* R*-modules.*

In such a case, the functor $\mathfrak{F}^{\mathcal{D}}$ categorifies the homomorphism (5.3).

Proof of Proposition 5.1. By assumption, we have $U^-(\mathfrak{g}^{\mathcal{D}}) \simeq U^-(\overline{\mathfrak{g}})$. Hence the map (5.7) is injective, which implies that (5.6) is injective. Hence (5.5) and (5.4) are injective. □

Let us give several examples of such duality data.

Example 5.2. Let $I = \{1, \dots, \ell\}$ and A a Cartan matrix of type A_ℓ. Hence $(\alpha_i, \alpha_j) = 2\delta(i = j) - \delta(|i - j| = 1)$ for $i, j \in I$. Let R be the quiver Hecke algebra associated with A and with the parameter $\mathcal{Q}_{i,j}(u, v)$ defined as follows: for $i, j \in I$ with $i < j$,

$$\mathcal{Q}_{i,j}(u, v) = \begin{cases} u - v & \text{if } j = i + 1, \\ 1 & \text{otherwise.} \end{cases}$$

Let $J = \{1, \dots, \ell\}$ and $\beta_1 := \alpha_1 + \alpha_2$, $\beta_j := \alpha_j$ for $j \in J \setminus \{1\}$. Note that the β_j's do not satisfy condition (5.1)(b). We set

$$\Delta(\beta_1) := L(1, 2)_{\mathsf{z}_1}, \qquad \Delta(\beta_j) := L(j)_{\mathsf{z}_j} \quad (j \in J \setminus \{1\}),$$

where $L(1, 2) := \mathbf{k}v$ is the 1-dimensional $R(\beta_1)$-module with the actions

$$e(\nu)v = \delta_{\nu,(1,2)}v, \qquad x_1 v = x_2 v = \tau_1 v = 0 \qquad \text{for } \nu \in I^{\alpha_1+\alpha_2}.$$

Note that $\deg \mathsf{z}_j = 2$ for $j \in J$ and the $\Delta(\beta_j)$'s are root modules. We set $\mathsf{R}_{j,k} = R^{\mathrm{norm}}_{\Delta(\beta_j),\Delta(\beta_k)}$. By direct computations, the R-matrix $R_{\Delta(\beta_j),\Delta(\beta_k)}$ $(j \neq k)$ is given as follows: for $u \otimes v \in \Delta(\beta_j) \otimes \Delta(\beta_k)$,

$$R_{\Delta(\beta_j),\Delta(\beta_k)}(u \otimes v) = \begin{cases} (\tau_2\tau_1(\mathsf{z}_2 - \mathsf{z}_1) + \tau_1)(v \otimes u) & \text{if } j = 1 \text{ and } k = 2, \\ \tau_2\tau_1(v \otimes u) & \text{if } j = 1 \text{ and } k > 2, \\ \tau_1\tau_2(\mathsf{z}_1 - \mathsf{z}_2)(v \otimes u) & \text{if } j = 2 \text{ and } k = 1, \\ \tau_1\tau_2(v \otimes u) & \text{if } j > 2 \text{ and } k = 1, \\ \tau_1(v \otimes u) & \text{otherwise,} \end{cases}$$

which yields

$$\mathsf{R}_{j,k} = \begin{cases} (\mathsf{z}_1 - \mathsf{z}_2)^{-1} R_{\Delta(\beta_j),\Delta(\beta_k)} & \text{if } j = 2 \text{ and } k = 1, \\ R_{\Delta(\beta_j),\Delta(\beta_k)} & \text{otherwise,} \end{cases}$$

and

$$\deg \mathsf{R}_{j,k} = \begin{cases} 1 & \text{if } |j-k| = 1 \text{ and } (j,k) \neq (2,1), \\ 1 & (j,k) = (1,3), (3,1), \\ -1 & \text{if } (j,k) = (2,1), \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we have

$$\mathsf{A}^{\mathcal{D}} = \begin{pmatrix} 2 & 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix},$$

which is of type D_ℓ, i.e., the quiver Hecke algebra $R^{\mathcal{D}}$ is of type D_ℓ. Note that $\deg(e(1,2)\tau_1^{\mathcal{D}}) = 1$ and $\deg(e(2,1)\tau_1^{\mathcal{D}}) = -1$ (see Definition 1.4). By Theorem 4.4, we have the functor $\mathfrak{F}^{\mathcal{D}}$ between quiver Hecke algebras of type D_ℓ and A_ℓ such that

$$\mathfrak{F}^{\mathcal{D}}(L^{\mathcal{D}}(j)) \simeq L(\beta_j) \qquad \text{for } j \in J.$$

Let us consider the $R^{\mathcal{D}}$-module $L^{\mathcal{D}}(1,3) := L^{\mathcal{D}}(1) \diamond L^{\mathcal{D}}(3)$ and the one-dimensional R-module $L(1,2,3) := L(1,2) \diamond L(3)$. Applying the functor $\mathfrak{F}^{\mathcal{D}}$ to the exact sequence

$$0 \to L^{\mathcal{D}}(1,3) \to L^{\mathcal{D}}(3) \circ L^{\mathcal{D}}(1) \to L^{\mathcal{D}}(1) \circ L^{\mathcal{D}}(3) \to L^{\mathcal{D}}(1,3) \to 0,$$

we obtain

$$0 \to \mathfrak{F}^{\mathcal{D}}(L^{\mathcal{D}}(1,3)) \to L(3) \circ L(1,2) \to L(1,2) \circ L(3) \to \mathfrak{F}^{\mathcal{D}}(L^{\mathcal{D}}(1,3)) \to 0.$$

Since $\mathfrak{F}^{\mathcal{D}}$ sends every simple module to a simple module or zero by Theorem 4.4, and $L(3) \circ L(1,2)$ is not isomorphic to $L(1,2) \circ L(3)$, we have

$$\mathfrak{F}^{\mathcal{D}}(L^{\mathcal{D}}(1,3)) \simeq L(1,2,3).$$

Set $L^{\mathcal{D}}(1,3,2) \simeq L^{\mathcal{D}}(1,3) \diamond L^{\mathcal{D}}(2)$, which is one-dimensional. It is isomorphic to the image of the composition of

$$\begin{aligned} L^{\mathcal{D}}(1) \circ L^{\mathcal{D}}(3) \circ L^{\mathcal{D}}(2) &\to L^{\mathcal{D}}(1) \circ L^{\mathcal{D}}(2) \circ L^{\mathcal{D}}(3) \\ &\to L^{\mathcal{D}}(2) \circ L^{\mathcal{D}}(1) \circ L^{\mathcal{D}}(3) \to L^{\mathcal{D}}(2) \circ L^{\mathcal{D}}(1,3). \end{aligned}$$

By applying $\mathfrak{F}^{\mathcal{D}}$, we obtain a diagram

$$\begin{array}{ccccc} L(1,2)\circ L(3)\circ L(2) & \xrightarrow{f_1} & L(1,2)\circ L(2)\circ L(3) & \xrightarrow{f_2} & L(2)\circ L(1,2)\circ L(3) \\ & & & & \downarrow f_3 \\ & & & & L(2)\circ L(1,2,3) \end{array} \tag{5.8}$$

Hence $\mathfrak{F}^{\mathcal{D}}(L^{\mathcal{D}}(1,3,2))$ is isomorphic to the image of $f_3f_2f_1$. Let $u_{1,2}$, u_2 and u_3 be the generators of $L(1,2)$, $L(2)$ and $L(3)$, respectively. Then

$$f_1(u_{1,2}\otimes u_3\otimes u_2)=\tau_3(u_{1,2}\otimes u_2\otimes u_3),\qquad f_2(u_{1,2}\otimes u_2\otimes u_3)=\tau_1(u_2\otimes u_{1,2}\otimes u_3).$$

Therefore,

$$f_2f_1(u_{1,2}\otimes u_3\otimes u_2)=\tau_3\tau_1(u_2\otimes u_{1,2}\otimes u_3)=\tau_1\tau_3(u_2\otimes u_{1,2}\otimes u_3),$$

which is killed by f_3. Thus $f_3f_2f_1=0$, and hence

$$\mathfrak{F}^{\mathcal{D}}(L^{\mathcal{D}}(1,3,2))\simeq 0.$$

Consequently, $\mathfrak{F}^{\mathcal{D}}$ can send simple modules to zero in this example.

6. Further examples for non-symmetric types

Let $\beta\in \mathsf{Q}_+$ and let $(\mathsf{M}, z_{\mathsf{M}})$ be an affinization of a real simple $R(\beta)$-module $\bar{M}$. We set $J=\{0\}$, $\beta_0=\beta$, $\mathsf{M}_0=\mathsf{M}$. Then

$$\mathcal{D}=\{\mathsf{M}_0, z_{\mathsf{M}}, R^{\mathrm{norm}}_{\mathsf{M},\mathsf{M}}\}$$

is a duality datum. The corresponding simple root $\alpha_0^{\mathcal{D}}$ satisfies $(\alpha_0^{\mathcal{D}},\alpha_0^{\mathcal{D}})=\deg z_{\mathsf{M}}$. Let $(\mathsf{K}(0^n), \mathsf{z}_{\mathsf{K}(0^n)})$ be the affinization of the simple $R^{\mathcal{D}}(n\alpha_0^{\mathcal{D}})$-module $L(\alpha_0^{\mathcal{D}})^{\circ n}$ given in Example 2.18.

Now $\mathsf{M}^{\circ n}:=\mathsf{M}\circ\cdots\circ\mathsf{M}$ (n times) has a structure of $(R(n\beta), R^{\mathcal{D}}(n\alpha_0^{\mathcal{D}}))$-bimodule. We set

$$\mathsf{C}_n(\mathsf{M})=\mathsf{M}^{\circ n}\underset{R^{\mathcal{D}}(n\alpha_0^{\mathcal{D}})}{\otimes}\mathsf{K}(0^n)\simeq\mathfrak{F}^{\mathcal{D}}(\mathsf{K}(0^n)).$$

Then $\mathsf{z}_{\mathsf{K}(0^n)}\in \mathrm{E{\scriptsize ND}}(\mathsf{K}(0^n))$ induces an endomorphism $\mathsf{z}_{\mathsf{C}_n(\mathsf{M})}\in \mathrm{E{\scriptsize ND}}(\mathsf{C}_n(\mathsf{M}))_{n\deg z_{\mathsf{M}}}$. By Theorem 4.4(ii)(b), we obtain the following lemma.

Lemma 6.1. $(\mathsf{C}_n(\mathsf{M}), \mathsf{z}_{\mathsf{C}_n(\mathsf{M})})$ *is an affinization of the real simple module* $\bar{M}^{\circ n}$.

For example,

$$\mathsf{C}_2(\mathsf{M})=\frac{\mathsf{M}\circ\mathsf{M}}{(z_{\mathsf{M}}\circ\mathsf{M}+\mathsf{M}\circ z_{\mathsf{M}})(\mathsf{M}\circ\mathsf{M})+\mathsf{r}(\mathsf{M}\circ\mathsf{M})}, \tag{6.1}$$

where r is the endomorphism given in (4.3), and $\mathsf{z}_{\mathsf{C}_2(\mathsf{M})}$ is the endomorphism induced by $z_{\mathsf{M}}\circ z_{\mathsf{M}}$.

Let $M \in R(\beta)$-gmod and $N \in R(\beta')$-gmod be real simple modules. Suppose that $R(\beta)$ and $R(\beta')$ are symmetric, and

$$R^{\mathrm{norm}}_{N_{\mathsf{z}'},M_{\mathsf{t}}} R^{\mathrm{norm}}_{M_{\mathsf{t}},N_{\mathsf{z}'}} = c(\mathsf{t}-\mathsf{z}')^p \in \mathrm{End}_{R(\beta+\beta')}(M_{\mathsf{t}} \circ N_{\mathsf{z}'})$$

for some $c \in \mathbf{k}^\times$ and $p \in \mathbb{Z}_{\geq 0}$. Set

$$R_1 := (R^{\mathrm{norm}}_{M_{\mathsf{t}_1},N_{\mathsf{z}'}} \circ M_{\mathsf{t}_2})(M_{\mathsf{t}_1} \circ R^{\mathrm{norm}}_{M_{\mathsf{t}_2},N_{\mathsf{z}'}}) \in \mathrm{HOM}_{R(2\beta+\beta')}(M_{\mathsf{t}_1} \circ M_{\mathsf{t}_2} \circ N_{\mathsf{z}'}, N_{\mathsf{z}'} \circ M_{\mathsf{t}_1} \circ M_{\mathsf{t}_2}),$$
$$R_2 := (M_{\mathsf{t}_1} \circ R^{\mathrm{norm}}_{N_{\mathsf{z}'},M_{\mathsf{t}_2}})(R^{\mathrm{norm}}_{N_{\mathsf{z}'},M_{\mathsf{t}_1}} \circ M_{\mathsf{t}_2}) \in \mathrm{HOM}_{R(2\beta+\beta')}(N_{\mathsf{z}'} \circ M_{\mathsf{t}_1} \circ M_{\mathsf{t}_2}, M_{\mathsf{t}_1} \circ M_{\mathsf{t}_2} \circ N_{\mathsf{z}'}).$$

Setting $\mathsf{t}_1 + \mathsf{t}_2 = 0$ and $\mathsf{t}_1\mathsf{t}_2 = \widehat{\mathsf{z}} := \mathsf{z}_{\mathrm{C}_2(M)}$, we regard R_1, R_2 as homomorphisms in $\mathrm{HOM}_{R(2\beta+\beta')}(\mathrm{C}_2(M_{\mathsf{z}}) \circ N_{\mathsf{z}'}, N_{\mathsf{z}'} \circ \mathrm{C}_2(M_{\mathsf{z}}))$, $\mathrm{HOM}_{R(2\beta+\beta')}(N_{\mathsf{z}'} \circ \mathrm{C}_2(M_{\mathsf{z}}), \mathrm{C}_2(M_{\mathsf{z}}) \circ N_{\mathsf{z}'})$ respectively. Then we have

$$\begin{aligned} R_2R_1 &= c^2(\mathsf{t}_1-\mathsf{z}')^p(\mathsf{t}_2-\mathsf{z}')^p = c^2(\mathsf{t}_1\mathsf{t}_2 - (\mathsf{t}_1+\mathsf{t}_2)\mathsf{z}' + {\mathsf{z}'}^2)^p \\ &= c^2(\widehat{\mathsf{z}} + {\mathsf{z}'}^2)^p \end{aligned} \tag{6.2}$$

in $\mathrm{End}_{R(2\beta+\beta')}(\mathrm{C}_2(M_{\mathsf{z}}) \circ N_{\mathsf{z}'})$.

Using (6.2), one can construct functors $\mathfrak{F}^{\mathcal{D}}$ between symmetric and non-symmetric quiver Hecke algebras. In particular, a functor from type C_ℓ (resp. $C^{(1)}_\ell$, $A^{(2)}_{2\ell-1}$) to type A_ℓ (resp. $A_{\ell+1}$, $D_{\ell+1}$) can be constructed. We give such constructions in the following examples.

Example 6.2. We take I, A, and $\mathcal{Q}_{i,j}(u,v)$ given in Example 5.2. In particular, $\mathfrak{g}$ is of type A_ℓ.

Let $J = \{1, \dots, \ell\}$ and

$$\beta_1 = 2\alpha_1, \qquad \beta_j = \alpha_j \quad \text{for } j \in J \setminus \{1\}.$$

Let us denote

$$\mathsf{M}_1 = \mathrm{K}(1^2), \qquad \mathsf{M}_j = L(j)_{\mathsf{z}_j} \quad (j \in J \setminus \{1\}),$$

and $\mathsf{z}_1 := \mathsf{z}_{K(1^2)}$. Then $\deg \mathsf{z}_1 = 4$ and $\deg \mathsf{z}_j = 2$ for $j \neq 1$. Note that

$$R^{\mathrm{norm}}_{L(j)_{\mathsf{z}},L(k)_{\mathsf{w}}} = R_{L(j)_{\mathsf{z}},L(k)_{\mathsf{w}}}.$$

We set $\mathsf{R}_{j,k} := R_{\mathsf{M}_j,\mathsf{M}_k}$. It follows from (6.2) that, for $j, k \in J$ with $j < k$,

$$\mathsf{R}_{k,j}\mathsf{R}_{j,k} = \begin{cases} \mathsf{z}_1 + \mathsf{z}_2^2 & \text{if } (j,k) = (1,2), \\ \mathsf{z}_j - \mathsf{z}_k & \text{if } k = j+1 \text{ and } (j,k) \neq (1,2), \\ 1 & \text{otherwise.} \end{cases} \tag{6.3}$$

We now set

$$\mathcal{D} = \{\mathsf{M}_j, \mathsf{z}_j, \mathsf{R}_{j,k}\}_{j,k \in J}.$$

Then $\mathcal{D}$ is a dual datum, and (6.3) implies that

$$\mathsf{A}^{\mathcal{D}} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -2 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix},$$

which is of type C_ℓ. Therefore, we have the quiver Hecke algebra $R^{\mathcal{D}}$ of type C_ℓ and the functor $\mathfrak{F}^{\mathcal{D}}$ from the category of modules over quiver Hecke algebras of type C_ℓ to that of type A_ℓ with

$$\mathfrak{F}^{\mathcal{D}}(L^{\mathcal{D}}(j)) \simeq \begin{cases} L(1) \circ L(1) & \text{if } j = 1, \\ L(j) & \text{otherwise.} \end{cases}$$

In the following examples for type B_ℓ, we construct affinizations directly.

Example 6.3. Let $I = \{1, \dots, \ell\}$ and A a Cartan matrix of type B_ℓ:

$$\mathsf{A} = \begin{pmatrix} 2 & -2 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix},$$

and $(\alpha_i, \alpha_j) = 2\delta(i = j = 1) + 4\delta(i = j \neq 1) - 2\delta(|i - j| = 1)$ for $i, j \in I$.

Let R be the quiver Hecke algebra associated with A and with the parameter $\mathcal{Q}_{i,j}(u, v)$ defined as follows: for $i, j \in I$ such that $i < j$,

$$\mathcal{Q}_{i,j}(u, v) = \begin{cases} u^2 - v & \text{if } (i, j) = (1, 2), \\ u - v & \text{if } j = i + 1 \text{ and } (i, j) \neq (1, 2), \\ 1 & \text{otherwise.} \end{cases}$$

Let $J = \{1, \dots, \ell - 1\}$ and

$$\beta_1 = \alpha_1 + \alpha_2, \qquad \beta_j = \alpha_{j+1} \quad (j \in J \setminus \{1\}).$$

Note that $(\beta_1, \beta_1) = 2$ and $(\beta_j, \beta_j) = 4$ for $j \neq 1$. We set

$$\Delta(\beta_1) = L(1, 2)_{\mathsf{z}_1}, \qquad \Delta(\beta_j) = L(j + 1)_{\mathsf{z}_j} \quad (j \neq 1),$$

where the $R(\beta_1)$-module $L(1, 2)_{\mathsf{z}_1} := \mathbf{k}[\mathsf{z}_1]v$ is defined by

$$e(\nu)v = \delta_{\nu,(1,2)}v, \qquad x_j v = \mathsf{z}_1^{(\alpha_j,\alpha_j)/2} v, \qquad \tau_1 v = 0.$$

Note that $\Delta(\beta_j)$'s are root modules and $\deg \mathsf{z}_j$ is 2 or 4 according to whether $j = 1$ or not. For $j, k \in J$ with $j \neq k$, we define

$$\mathsf{R}_{j,k} := R_{\Delta(\beta_j),\Delta(\beta_k)} \in \mathrm{Hom}_{R(\beta_j+\beta_k)}(\Delta(\beta_j) \circ \Delta(\beta_k), \Delta(\beta_k) \circ \Delta(\beta_j)),$$

that is,

$$\mathsf{R}_{j,k}(p \otimes q) = \begin{cases} \tau_2\tau_1(q \otimes p) & \text{if } j = 1, \\ \tau_1\tau_2(q \otimes p) & \text{if } k = 1, \\ \tau_1(q \otimes p) & \text{otherwise} \end{cases}$$

for $p \otimes q \in \Delta(\beta_j) \otimes_{\mathbf{k}} \Delta(\beta_k)$. For $j, k \in J$ with $j < k$, we have

$$\mathsf{R}_{k,j}\mathsf{R}_{j,k} = \begin{cases} \mathsf{z}_j^2 - \mathsf{z}_k & \text{if } (j,k) = (1,2), \\ \mathsf{z}_j - \mathsf{z}_k & \text{if } k = j+1 \text{ and } (j,k) \neq (1,2), \\ 1 & \text{otherwise.} \end{cases}$$

Then we have the duality datum $\mathcal{D} = \{\Delta(\beta_j), \mathsf{z}_j, \mathsf{R}_{j,k}\}_{j,k \in J}$ and

$$\mathsf{A}^{\mathcal{D}} = \begin{pmatrix} 2 & -2 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix},$$

which is of type $B_{\ell-1}$. Therefore, we have the functor $\mathfrak{F}^{\mathcal{D}}$ from the category of modules over a quiver Hecke algebra of type $B_{\ell-1}$ to that of type B_ℓ such that

$$\mathfrak{F}^{\mathcal{D}}(L^{\mathcal{D}}(j)) \simeq \begin{cases} \underline{L(1,2)} & \text{if } j = 1, \\ L(j+1) & \text{otherwise,} \end{cases}$$

where $\underline{L(1,2)} = L(1,2)_{\mathsf{z}_1}/\mathsf{z}_1 L(1,2)_{\mathsf{z}_1}$.

It is easy to check that $\{\beta_1, \ldots, \beta_{\ell-1}\}$ satisfies (5.1) and $\mathsf{A}^{\mathcal{D}}$ is equal to the matrix $\overline{\mathsf{A}}$ defined by (5.2). Thus, Proposition 5.1 implies that the functor $\mathfrak{F}^{\mathcal{D}}$ categorifies the injective homomorphism $U^-(\mathfrak{g}^{\mathcal{D}}) \simeq U^-(\overline{\mathfrak{g}}) \to U^-(\mathfrak{g})$ and $\mathfrak{F}^{\mathcal{D}}$ sends simple modules to simple modules. By Theorem 4.4, for a simple $R^{\mathcal{D}}$-module N,

$$\mathfrak{F}^{\mathcal{D}}(N \diamond L^{\mathcal{D}}(j)) \simeq \begin{cases} \mathfrak{F}^{\mathcal{D}}(N) \diamond \underline{L(1,2)} & \text{if } j = 1, \\ \mathfrak{F}^{\mathcal{D}}(N) \diamond L(j+1) & \text{otherwise.} \end{cases}$$

Example 6.4. We use the same notations I, A and $\mathcal{Q}_{i,j}(u,v)$ as in Example 6.3.

Let $J = \{1, \ldots, \ell-1\}$ and

$$\beta_1 = 2\alpha_1 + \alpha_2, \qquad \beta_j = \alpha_{j+1} \quad (j \in J \setminus \{1\}).$$

Note that $(\beta_j, \beta_j) = 4$ for all $j \in J$. We define an $R(\beta_1)$-module structure on $L(1,1,2)_{\mathsf{z}_1} := \mathbf{k}[\mathsf{z}_1] \otimes_{\mathbf{k}} (\mathbf{k}u \oplus \mathbf{k}v)$ by

$$e(\nu)(a \otimes u) = \delta_{\nu,(1,1,2)} a \otimes u, \qquad e(\nu)(a \otimes v) = \delta_{\nu,(1,1,2)} a \otimes v,$$

$$x_j(a \otimes u) = \begin{cases} -\mathsf{z}_1 a \otimes v & \text{if } j = 1, \\ \mathsf{z}_1 a \otimes v & \text{if } j = 2, \\ \mathsf{z}_1 a \otimes u & \text{otherwise}, \end{cases} \qquad x_j(a \otimes v) = \begin{cases} -a \otimes u & \text{if } j = 1, \\ a \otimes u & \text{if } j = 2, \\ \mathsf{z}_1 a \otimes v & \text{otherwise}, \end{cases}$$

$$\tau_k(a \otimes u) = \begin{cases} a \otimes v & \text{if } k = 1, \\ 0 & \text{if } k \neq 1, \end{cases} \qquad \tau_k(a \otimes v) = 0 \quad \text{for any } k.$$

We set

$$\Delta(\beta_1) = L(1,1,2)_{\mathsf{z}_1}, \qquad \Delta(\beta_j) = L(j+1)_{\mathsf{z}_j} \quad (j \neq 1).$$

Note that $\Delta(\beta_j)$'s are root modules and $\deg \mathsf{z}_j = 4$ for $j \in J$. For $j, k \in J$ with $j \neq k$ and $p \otimes q \in \Delta(\beta_j) \otimes_{\mathbf{k}} \Delta(\beta_k)$, we define

$$\mathsf{R}_{j,k} := R_{\Delta(\beta_j),\, \Delta(\beta_k)} \in \mathrm{Hom}_{R(\beta_j+\beta_k)}(\Delta(\beta_j) \circ \Delta(\beta_k), \Delta(\beta_k) \circ \Delta(\beta_j)).$$

Then

$$\mathsf{R}_{k,j}\mathsf{R}_{j,k} = \begin{cases} \mathsf{z}_j - \mathsf{z}_k & \text{if } k = j+1, \\ 1 & \text{otherwise}, \end{cases}$$

for $j, k \in J$ with $j < k$.

Thus, we have the duality datum $\mathcal{D} = \{\Delta(\beta_j), \mathsf{z}_j, \mathsf{R}_{j,k}\}_{j,k \in J}$ and

$$\mathsf{A}^{\mathcal{D}} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}$$

is of type $A_{\ell-1}$. Therefore, we have the quiver Hecke algebra $R^{\mathcal{D}}$ of type $A_{\ell-1}$ and the functor $\mathfrak{F}^{\mathcal{D}}$ between quiver Hecke algebras of type $A_{\ell-1}$ and B_ℓ. Moreover,

$$\mathfrak{F}^{\mathcal{D}}(L^{\mathcal{D}}(j)) \simeq \begin{cases} L(1,1,2) & \text{if } j = 1, \\ L(j+1) & \text{otherwise}, \end{cases}$$

where $L(1,1,2) = L(1,1,2)_{\mathsf{z}_1} / \mathsf{z}_1 L(1,1,2)_{\mathsf{z}_1}$.

One can easily show that $\{\beta_1, \ldots, \beta_{\ell-1}\}$ satisfies (5.1) and $\mathsf{A}^{\mathcal{D}}$ is equal to the matrix $\overline{\mathsf{A}}$ defined by (5.2). Thus, Proposition 5.1 implies that the functor $\mathfrak{F}^{\mathcal{D}}$ categorifies the injective homomorphism $U^-(\overline{\mathfrak{g}}) \to U^-(\mathfrak{g})$ and $\mathfrak{F}^{\mathcal{D}}$ preserves simple modules. We have

$$\mathfrak{F}^{\mathcal{D}}(N \diamond L^{\mathcal{D}}(j)) \simeq \begin{cases} \mathfrak{F}^{\mathcal{D}}(N) \diamond L(1,1,2) & \text{if } j = 1, \\ \mathfrak{F}^{\mathcal{D}}(N) \diamond L(j+1) & \text{otherwise}, \end{cases}$$

for every simple $R^{\mathcal{D}}$-module N by Theorem 4.4.

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References

[1] Brundan, J., Kleshchev, A.: Blocks of cyclotomic Hecke algebras and Khovanov–Lauda algebras. Invent. Math. **178**, 451–484 (2009) Zbl 1201.20004 MR 2551762

[2] Brundan, J., Kleshchev, A., McNamara, P. J.: Homological properties of finite-type Khovanov–Lauda–Rouquier algebras. Duke Math. J. **163**, 1353–1404 (2014) Zbl 1314.16005 MR 3205728

[3] Hong, J., Kang, S.-J.: Introduction to Quantum Groups and Crystal Bases. Grad. Stud. Math. 42, Amer. Math. Soc., Providence, RI (2002) Zbl 1134.17007 MR 1881971

[4] Kang, S.-J., Kashiwara, M.: Categorification of highest weight modules via Khovanov–Lauda–Rouquier algebras. Invent. Math. **190**, 699–742 (2012) Zbl 1280.17017 MR 2995184

[5] Kang, S.-J., Kashiwara, M., Kim, M.: Symmetric quiver Hecke algebras and R-matrices of quantum affine algebras. Invent. Math. **211**, 591–685 (2018) Zbl 06838398 MR 3748315

[6] Kang, S.-J., Kashiwara, M., Kim, M.: Symmetric quiver Hecke algebras and R-matrices of quantum affine algebras II. Duke Math. J. **164**, 1549–1602 (2015) Zbl 1323.81046 MR 3352041

[7] Kang, S.-J., Kashiwara, M., Kim, M., Oh, S.-j.: Symmetric quiver Hecke algebras and R-matrices of quantum affine algebras III. Proc. London Math. Soc. (3) **111**, 420–444 (2015) Zbl 1322.81056 MR 3384517

[8] Kang, S.-J., Kashiwara, M., Kim, M., Oh, S.-j.: Monoidal categorification of cluster algebras. J. Amer. Math. Soc. **31**, 349–426 (2018) Zbl 06836098 MR 3758148

[9] Kang, S.-J., Kashiwara, M., Kim, M., Oh, S.-j.: Symmetric quiver Hecke algebras and R-matrices of quantum affine algebras IV. Selecta Math. (N.S.) **22**, 1987–2015 (2016) Zbl 1354.81030 MR 3573951

[10] Kang, S.-J., Kashiwara, M., Kim, M., Oh, S.-j.: Simplicity of heads and socles of tensor products. Compos. Math. **151**, 377–396 (2015) Zbl 1366.17014 MR 3314831

[11] Kashiwara, M.: On crystal bases of the q-analogue of universal enveloping algebras. Duke Math. J. **63**, 465–516 (1991) Zbl 0739.17005 MR 1115118

[12] Khovanov, M., Lauda, A.: A diagrammatic approach to categorification of quantum groups I. Represent. Theory **13**, 309–347 (2009) Zbl 1188.81117 MR 2525917

[13] Khovanov, M., Lauda, A.: A diagrammatic approach to categorification of quantum groups II. Trans. Amer. Math. Soc. **363**, 2685–2700 (2011) Zbl 1214.81113 MR 2763732

[14] Kleshchev, A., Ram, A.: Representations of Khovanov–Lauda–Rouquier algebras and combinatorics of Lyndon words. Math. Ann. **349**, 943–975 (2011) Zbl 1267.20010 MR 2777040

[15] McNamara, P. J.: Finite dimensional representations of Khovanov–Lauda–Rouquier algebras I: Finite type. J. Reine Angew. Math. **707**, 103–124 (2015) Zbl 1378.17018 MR 3403455

[16] Naito, S.: On regular subalgebras of a symmetrizable Kac–Moody algebra. Proc. Japan Acad. Ser. A Math. Sci. **67**, 117–121 (1991) Zbl 0752.17025 MR 1114951

[17] Rouquier, R.: 2-Kac–Moody algebras. arXiv:0812.5023v1 (2008)

[18] Rouquier, R.: Quiver Hecke algebras and 2-Lie algebras. Algebra Colloq. **19**, 359–410 (2012) Zbl 1247.20002 MR 2908731

[19] Varagnolo, M., Vasserot, E.: Canonical bases and KLR algebras. J. Reine Angew. Math. **659**, 67–100 (2011) Zbl 1229.17019 MR 2837011