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Nicolas Lerner · Toan Nguyen · Benjamin Texier

The onset of instability in first-order systems

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Abstract. We study the Cauchy problem for first-order quasi-linear systems of partial differential equations. When the spectrum of the initial principal symbol is not included in the real line, i.e., hyperbolicity is violated at initial time, the Cauchy problem is strongly unstable in the sense of Hadamard. This phenomenon, which extends the linear Lax–Mizohata theorem, was explained by G. Métivier [Contemp. Math. 368, 2005]. In the present paper, we are interested in the transition from hyperbolicity to non-hyperbolicity, that is, the limiting case where hyperbolicity holds at initial time, but is violated at positive times: under that hypothesis, we generalize a recent work by N. Lerner, Y. Morimoto and C.-J. Xu [Amer. J. Math. 132 (2010)] on complex scalar systems, as we prove that even a weak defect of hyperbolicity implies a strong Hadamard instability. Our examples include Burgers systems, Van der Waals gas dynamics, and Klein–Gordon–Zakharov systems. Our analysis relies on an approximation result for pseudo-differential flows, proved by B. Texier [Indiana Univ. Math. J. 65 (2016)].

Keywords. Cauchy problem, hyperbolicity, ellipticity

Contents

N. Lerner: Institut de Mathématiques de Jussieu-Paris Rive Gauche, UMR CNRS 7586,
Université Pierre-et-Marie-Curie, 4, Place Jussieu, 75252 Paris Cedex 05, France;
e-mail: nicolas.lerner@imj-prg.fr

T. Nguyen: Department of Mathematics, Pennsylvania State University,
University Park, State College, PA 16802, USA; e-mail: nguyen@math.psu.edu

B. Texier: Institut de Mathématiques de Jussieu-Paris Rive Gauche UMR CNRS 7586,
Université Paris-Diderot, 5, rue Thomas Mann, 75205 Paris Cedex 13, France;
e-mail: benjamin.texier@imj-prg.fr

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1. Introduction

We study well-posedness issues in Sobolev spaces for the Cauchy problem for first-order, quasi-linear systems of partial differential equations:

$$\partial_t u + \sum_{1\le j\le d} A_j(t,x,u)\partial_{x_j} u = F(t,x,u), \tag{1.1}$$

where $t \ge 0$, $x \in \mathbb{R}^d$, $u(t,x) \in \mathbb{R}^N$, the maps A_j are smooth from $\mathbb{R}_+ \times \mathbb{R}^d_x \times \mathbb{R}^N_u$ to the space of $N \times N$ real matrices and F is smooth from $\mathbb{R}_+ \times \mathbb{R}^d_x \times \mathbb{R}^N_u$ into $\mathbb{R}^N$.

We prove a general ill-posedness result in Sobolev spaces for the Cauchy problem for (1.1), under an assumption of a weak defect of hyperbolicity that describes the transition from hyperbolicity to ellipticity. This extends recent results of G. Métivier [16] and N. Lerner, Y. Morimoto and C.-J. Xu [12]. Here "well-posedness" is understood in the sense of Hadamard [4], meaning existence and regularity of a flow; "hyperbolicity", as discussed in Section 1.1, means reality of the spectrum of the principal symbol; and "ellipticity" corresponds to existence of non-real eigenvalues for the principal symbol.

We begin this introduction with a discussion of hyperbolicity and well-posedness (Section 1.1), then give three results: Theorem 1.2 describes ill-posedness of elliptic initial-value problems, while Theorems 1.3 and 1.6 are ill-posedness results for systems undergoing a transition from hyperbolicity to ellipticity. These results are illustrated in a series of examples in Section 1.5. Our main assumption (Assumption 2.1) and main result (Theorem 2.2) are stated in Sections 2.1 and 2.2.

1.1. Hyperbolicity as a necessary condition for well-posedness

Lax–Mizohata theorems, named after Peter Lax and Sigeru Mizohata, state that well-posed non-characteristic initial-value problems for first-order systems are necessarily hyperbolic, meaning that all eigenvalues of the principal symbol are real.

P. Lax's original result [10] is stated in a C^∞ framework, for linear equations, i.e. such that $A_j(t,x,u) \equiv A_j(t,x)$. Lax uses a relatively strong definition of well-posedness that includes continuous dependence not only on the data, but also on the source. This allows him in particular to consider WKB approximate solutions; the proof in [10] shows that in the non-hyperbolic case, if the eigenvalues are separated, the C^0 norms of high-frequency WKB solutions grow faster than the C^k norms of the datum and source, for any k. The separation assumption ensures that the eigenvalues are smooth, implying smoothness for the coefficients of the WKB cascade of equations. In the same C^∞ framework for linear equations but without assuming spectral separation, S. Mizohata [18] proved that existence, uniqueness and continuous dependence on the data cannot hold in the non-hyperbolic case.

Later S. Wakabayashi [26] and K. Yagdjian [27, 28] extended the analysis to the quasi-linear case, but it was only in 2005 that a precise description of the lack of regularity of the flow was given by Métivier: Theorem 3.2 in [16] states that in the case where the A_j are analytic, under the assumption that for some fixed vector $u^0 \in \mathbb{R}^N$ and some frequency $\xi^0 \in \mathbb{R}^d$ the principal symbol $\sum_{j=1}^d A_j(u^0)\xi^0_j$ is not hyperbolic, some analytical data uniquely generate analytical solutions, but the corresponding flow for (1.1) is not Hölder continuous from high Sobolev norms to L^2, locally around a Cauchy–Kovalevskaya solution issued from the constant datum u^0.

Métivier's result is a *long-time Cauchy–Kovalevskaya* result. Without loss of generality, assume indeed that $u^0 = 0$. Then Theorem 3.2 in [16] states that data that are small in high norms may generate solutions that are instantaneously large in low norms. To see this, assume in (1.1) the hyperbolic ansatz: $u(t,x) = \varepsilon v(t/\varepsilon, x/\varepsilon)$, where $\varepsilon > 0$. If we set $F \equiv 0$ for simplicity, and $\tau = t/\varepsilon$, $y = x/\varepsilon$, the equation for v is

$$\partial_\tau v + \sum_{j=1}^{d} A_j(\varepsilon\tau, \varepsilon y, \varepsilon v)\partial_{y_j} v = 0. \tag{1.2}$$

If all fluxes A_j are analytic in their arguments, the Cauchy–Kovalevskaya theorem ensures the existence and uniqueness of a solution v issued from an analytic datum $v(0,x)$, over an $O(1)$ time interval in the fast variable τ. What is more, by regularity of the coefficients of (1.2) with respect to ε, the solution v stays close, in analytical seminorms, to the solution w of the constant-coefficient system

$$\partial_\tau w + \sum_{j=1}^{d} A_j(0)\partial_{y_j} w = 0$$

over $O(1)$ time intervals. By Assumption on $A_j(0)$, the Fourier transform $\hat w(\tau, \xi^0)$ of w in the spectral direction ξ^0 grows like $e^{\tau C(\xi^0)}$ for some $C > 0$. This implies similar growth for $v(t/\varepsilon, \xi^0)$, and in turn growth like $\varepsilon e^{tC(\xi^0)/\varepsilon}$ for $\hat u(t, \xi^0)$, *but only on* $O(\varepsilon)$ *time intervals*, due to the initial rescaling in time. The content of Métivier's result is therefore that the solution v to (1.2) exists, and the growth persists, over "long", $O(|\log\varepsilon|)$ time intervals, so that the exponential amplification is effective.

In the scalar complex case, the results of N. Lerner, Y. Morimoto and C.-J. Xu [12] extended the analysis of Métivier to the situation where the symbol is initially hyperbolic, but hyperbolicity is instantaneously lost, in the sense that a characteristic root is real at $t = 0$, but leaves the real line at positive times. The main result of [12] states that such a weak defect of hyperbolicity implies a strong form of ill-posedness; the analysis is based on representations of solutions by the method of characteristics, following [15]. This argument does not carry over to systems, even in the case of a diagonal principal symbol, if the components of the solution are coupled through the lower-order term $F(u)$.

Our goal in this article is to extend the instability results of [12] on complex scalar equations to the case of quasi-linear first-order systems (1.1). In the process, we recover a version of the results of [16], with a method of proof that does not rely on analyticity.

1.2. On the local character of our assumptions and results

Our assumptions are local in nature. They bear on the germ, at a given point $(t_0, x_0, \xi_0) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$, representing time, position, and frequency, of the principal symbol evaluated at a given reference solution. Under these local assumptions, we prove local instabilities, which extend the aforementioned Lax–Mizohata theorems, and which roughly say that there are no local solutions possessing minimal smoothness with initial data taking values locally in an elliptic region. These local instabilities are independent of the global properties of the system (1.1). In particular, the system (1.1) may have formal conserved quantities; see for instance the compressible Euler equations (1.16) introduced in Section 1.5.

1.3. Transition from hyperbolicity to ellipticity

Our starting point is to assume that there exists a local smooth solution ϕ to (1.1) with a large Sobolev regularity:

$$\phi \in C^\infty([0, T_0], H^{s_1}(U)) \tag{1.3}$$

for some $T_0 > 0$, some open set $U \subset \mathbb{R}^d$, and some Sobolev regularity index $s_1 = 1+d/2+s_2$, where $s_2 > 0$ is large enough, depending on the parameters in our problem.[1] If the matrices A_j and the source F depend analytically on (t, x, u), then we can choose ϕ to be a Cauchy–Kovalevskaya solution. However, we do not use analyticity in the rest of the paper.

The linearized principal symbol at ϕ is

$$A(t, x, \xi) := \sum_{j=1}^{d} \xi_j A_j(t, x, \phi(t, x)). \tag{1.4}$$

The upcoming ill-posedness results are based on readily verifiable conditions bearing on the jet at $t = 0$ of the characteristic polynomial P of the principal symbol:

$$P(t, x, \xi, \lambda) := \det(\lambda \operatorname{Id} - A(t, x, \xi)). \tag{1.5}$$

Most of these conditions are stable under perturbations of the principal symbol, and all can be expressed in terms of the fluxes A_j and the initial datum $\phi(0)$. In particular, it is of key importance that the verification of these conditions does not require any knowledge of the behavior of the reference solution ϕ at positive times.

Also, it should be mentioned that our hypotheses do not require the computation of eigenvalues and are expressed explicitly in terms of derivatives of P given by (1.5) at initial time.

1.3.1. Hadamard instability. If (1.1) does possess a flow, how regular can we reasonably expect it to be? A good reference point is the regularity of the flow generated by a symmetric system. If for all j and all u, the matrices $A_j(u)$ are symmetric, then local-in-time solutions to the initial-value problem for (1.1) exist and are unique in H^s for $s > 1 + d/2$ [3, 7, 9]; moreover, given a ball $B_{H^s}(0, R) \subset H^s$, there is an associated existence time $T > 0$. The flow is Lipschitz $B_{H^s}(0, R) \cap H^{s+1} \to L^\infty([0, T], H^s)$, continuous $B_{H^s}(0, R) \to L^\infty([0, T], H^s)$, but not uniformly continuous $B_{H^s}(0, R) \to$

[1] We use regularity of ϕ in particular in the construction of the local solution operator—see Appendix D, specifically the proof of Lemma D.2, in which q_0 is the order of a Taylor expansion involving ϕ.

$L^\infty([0,T], H^s)$ in general [7]. Micro-locally symmetrizable systems also enjoy these properties [17].

Accordingly, ill-posedness will be understood as follows:

Definition 1.1. We will say that the initial-value problem for the system (1.1) is *ill-posed* in the vicinity of the reference solution ϕ satisfying (1.3) if for some $x_0 \in U$, given any parameters $m, \alpha, \delta > 0$, T such that

$$m \in \mathbb{R}, \qquad 1/2 < \alpha \leq 1, \qquad B(x_0, \delta) \subset U, \qquad 0 < T \leq T_0, \tag{1.6}$$

where U and T_0 are as in (1.3), there is no neighborhood $\mathcal{U}$ of $\phi(0)$ in $H^m(U)$ such that, for all $u(0) \in \mathcal{U}$, the system (1.1) has a solution $u \in L^\infty([0,T], W^{1,\infty}(B(x_0,\delta)))$ issued from $u(0)$ which satisfies

$$\sup_{\substack{u_0 \in \mathcal{U} \\ 0 \leq t \leq T}} \frac{\|u(t) - \phi(t)\|_{W^{1,\infty}(B(x_0,\delta))}}{\|u_0 - \phi(0)\|^\alpha_{H^m(U)}} < \infty. \tag{1.7}$$

Thus (1.1) is ill-posed near the reference solution ϕ if either data arbitrarily near $\phi(0)$ fail to generate trajectories, corresponding to absence of a solution, or if trajectories issued close to $\phi(0)$ deviate from ϕ, corresponding to absence of Hölder continuity for the solution operator. In the latter case, we note that:

- the deviation is relative to the initial closeness, so that ϕ is unstable in the sense of Hadamard, not in the sense of Lyapunov;
- the deviation is instantaneous: T is arbitrarily small, and it is localized: δ is arbitrarily small,
- the initial closeness is measured in a strong H^m norm, where m is arbitrarily large,[2] while the deviation is measured in a weaker $W^{1,\infty}$ norm, defined as $\|f\|_{W^{1,\infty}} = \|f\|_{L^\infty} + \|\nabla_x f\|_{L^\infty}$.

In our proofs of ill-posedness in the sense of Definition 1.1, we will always *assume* existence of a solution issued from a small perturbation of $\phi(0)$, and proceed to disprove (1.7).

Note that the flows of ill-posed problems in the sense of Definition 1.1 exhibit a lack of *Hölder* continuity. F. John [6] introduced a notion of "well-behaved" problem, weaker than well-posedness. In well-behaved problems, Cauchy data generate unique solutions, and, in restriction to balls in the $W^{M,\infty}$ topology, for some integer M, the flow is Hölder continuous in appropriate norms. The notions introduced in [6] were developed by H. Bahouri [1], who used sharp Carleman estimates.

The restriction to $\alpha > 1/2$ in Definition 1.1 is technical. Precisely, it comes from the fact that we prove ill-posedness by disproving (1.7), as indicated above. This gives weak bounds on the solution, which we use to bound the non-linear terms. Consider non-linear terms in (1.1) which are controlled by ℓ_0-homogeneous terms in u, with $\ell_0 \geq 2$, that is,

[2] That is, the only restriction on m is the Sobolev regularity of ϕ; we need, in particular, $m \leq s_1$ for (1.7) to make sense.

such that $\partial_u A_j = O(u^{\ell_0-2})$ and $\partial_u F = O(u^{\ell_0-1})$. These bounds hold if, for instance, $A_j(u)\partial_{x_j} u = u^{\ell_0-1}\partial_x u$ and $F(u) = u^{\ell_0}$, using scalar notation. Then the proof of our general result (Theorem 2.2) shows ill-posedness with $\alpha > 1/\ell_0$. (See indeed Lemma 3.16 and its proof, and note the constraint $2K' > K$ which appears at the end of the proof in Section 3.15.)

Finally, we point out that Definition 1.1 describes only the behavior of solutions which belong to $W^{1,\infty}$. This in particular excludes shocks, which are expected to form in finite time for systems (1.1), even in the case of smooth data. Shocks with jump across elliptic zones could exhibit some stability properties.

1.3.2. Initial ellipticity. Our first result states that the ellipticity condition

$$P(0,\omega_0) = 0, \qquad \omega_0 = (x_0, \xi_0, \lambda_0) \in U \times (\mathbb{R}^d \setminus \{0\}) \times (\mathbb{C} \setminus \mathbb{R}), \tag{1.8}$$

where P is the characteristic polynomial defined in (1.5), implies ill-posedness:

Theorem 1.2. *Under the ellipticity condition* (1.8)*, the Cauchy problem for system* (1.1) *is ill-posed in the vicinity of the reference solution ϕ, in the sense of Definition* 1.1.

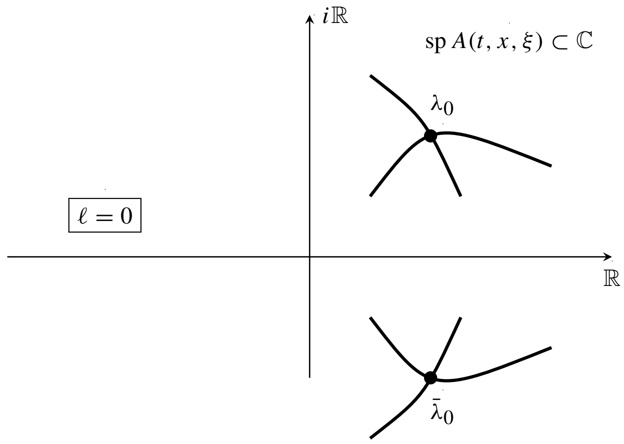

Fig. 1. In Theorem 1.2, corresponding to $\ell = 0$ in Assumption 2.1, the principal symbol at $(0, x_0, \xi_0)$ has non-real eigenvalues $\lambda_0, \bar\lambda_0$. These may correspond to coalescing points in the spectrum, for (t, x, ξ) near $(0, x_0, \xi_0)$.

Theorem 1.2 (proved in Section 4) states that hyperbolicity is a necessary condition for the well-posedness of the initial-value problem (1.1), and partially recovers Métivier's result[3]. An analogue to Theorem 1.2 in the high-frequency regime is given in [14], based on [24] just like our proof of Theorem 1.2; the main result of [14] precisely describes how resonances may induce local defects of hyperbolicity in strongly perturbed semilinear hyperbolic systems, and thus destabilize WKB solutions.

[3] Theorem 3.2 in [16] shows not only instability, but also existence and uniqueness, under assumption of analyticity for the fluxes, the source and the initial data.

1.3.3. Non-semisimple defect of hyperbolicity. We now turn to situations in which the initial principal symbol is hyperbolic:

$$P(0, x, \xi, \lambda) = 0 \quad \text{implies} \quad \lambda \in \mathbb{R}, \quad \text{for all } (x, \xi) \in U \times (\mathbb{R}^d \setminus \{0\}), \tag{1.9}$$

and aim to describe situations in which some roots of P are non-real for $t > 0$. Let

$$\Gamma := \{\omega = (x, \xi, \lambda) \in U \times (\mathbb{R}^d \setminus \{0\}) \times \mathbb{R} : P(0, \omega) = 0\},$$

By reality of the coefficients of P, non-real roots occur in conjugate pairs. In particular, eigenvalues must coalesce at $t = 0$ if we are to observe non-real eigenvalues for $t > 0$.

Let then $\omega_0 \in \Gamma$ be such that

$$\partial_\lambda P(0, \omega_0) = 0, \qquad \partial_\lambda^2 P(0, \omega_0) \neq 0. \tag{1.10}$$

The eigenvalue λ_0 of $A(0, x_0, \xi_0)$ thus has multiplicity two. Assume in addition that

$$(\partial_\lambda^2 P \partial_t P)(0, \omega_0) > 0. \tag{1.11}$$

The eigenvalues are continuous,[4] implying that condition (1.11) is open, meaning that if it holds at ω_0, then it holds at any nearby ω in Γ.

Theorem 1.3. *Assume that conditions* (1.10)–(1.11) *hold for some* $\omega_0 \in \Gamma$*, and the other eigenvalues of* $A(0, x_0, \xi_0)$ *are simple. Then the Cauchy problem for system* (1.1) *is ill-posed in the vicinity of the reference solution* ϕ*, in the sense of Definition* 1.1.

The conditions (1.10)–(1.11) are relevant, and, as far as we know, new, also in the linear case.

Van der Waals systems and Klein–Gordon–Zakharov systems illustrate Theorem 1.3 (see Sections 1.5, 7.3 and 7.4).

The proof of Theorem 1.3, given in Section 5, reveals that under (1.10)–(1.11), the eigenvalues that coalesce at $t = 0$ branch from the real axis. The branching time is typically not identically equal to $t = 0$ around (x_0, ξ_0); for (x, ξ) close to (x_0, ξ_0), it is equal to $t_\star(x, \xi) \geq 0$, with a smooth transition function $t_\star$. At $(t_\star(x, \xi), x, \xi)$ the branching eigenvalues are not time-differentiable, in particular not semisimple. Details are given in Section 5.1, in the proof of Theorem 1.3. Figure 3 shows the typical shape of the transition function. The elliptic domain is $\{t > t_\star\}$, and the hyperbolic domain is $\{t < t_\star\}$.

1.3.4. Semisimple defect of hyperbolicity. Time-differentiable defects of hyperbolicity of size two can be simply characterized in terms of P:

Proposition 1.4. *Let* $P(t, x, \xi, \lambda)$ *be the characteristic polynomial* (1.5) *of the principal symbol* $A(t, x, \xi)$ *as in* (1.4)*. Assume initial hyperbolicity* (1.9)*. Let* $\omega = (x, \xi, \lambda) \in \Gamma$*. If* $\partial_\lambda P(0, \omega) = 0 \neq \partial_\lambda^2 P(0, \omega)$*, then for the branches* λ *of eigenvalues of* A *which coalesce at* $(0, x, \xi)$*,*

$$\left.\begin{array}{r} \lambda(\cdot, x, \xi) \text{ is differentiable at } t = 0 \\ \Im m\, \partial_t \lambda(0, x, \xi) \neq 0 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} \partial_t P(0, \omega) = 0, \\ (\partial_{t\lambda}^2 P(0, \omega))^2 < (\partial_t^2 P \partial_\lambda^2 P)(0, \omega). \end{array}\right. \tag{1.12}$$

[4] By continuity of A and Rouché's theorem (see [8] or [25]).

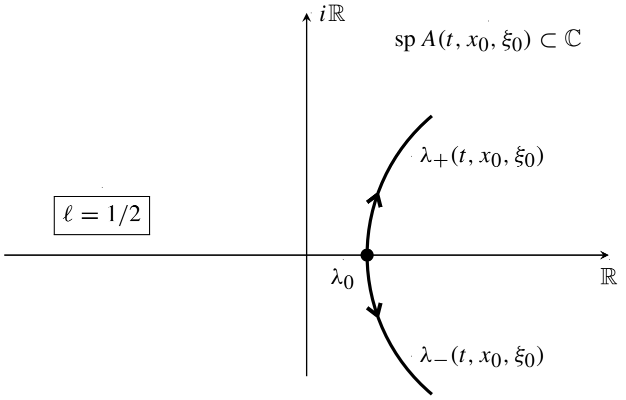

Fig. 2. In Theorem 1.3, corresponding to $\ell = 1/2$ in Assumption 2.1, a bifurcation occurs at $(0, x_0, \xi_0)$ in the spectrum of the principal symbol. The eigenvalues are not time-differentiable. The arrows indicate the direction of time.

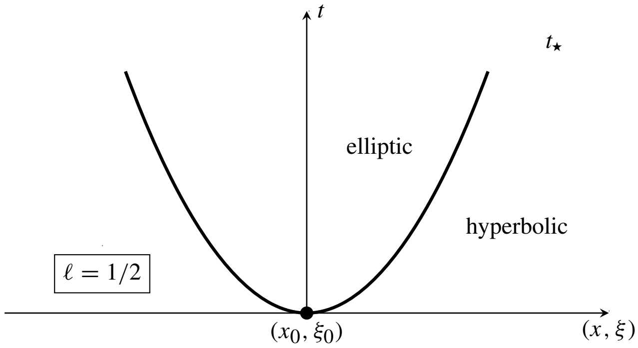

Fig. 3. In Theorem 1.3, the transition occurs at $t = t_\star(x, \xi) \geq 0$, near (x_0, ξ_0).

Proof. We assume $\lambda \in C^2$. The proof in the general case is postponed to Appendix A. For t in a neighborhood of 0, we have $P(t, x, \xi, \lambda(t, x, \xi)) \equiv 0$. Differentiating with respect to t, we find

$$\partial_t P(0, \omega) + \partial_t \lambda(0, x, \xi) \partial_\lambda P(0, \omega) = 0.$$

Since $\lambda(0, x, \xi)$ is real-valued, by reality of P, the derivatives $\partial_t P$ and $\partial_\lambda P$ are real. If we assume $\Im m\, \partial_t \lambda(0, x, \xi) \neq 0$, then $\partial_t P(0, \omega) = \partial_\lambda P(0, \omega) = 0$. Differentiating again with respect to t, we find

$$\partial_t^2 P(0, \omega) + 2\partial_t \lambda(0, x, \xi) \partial_{t\lambda}^2 P(0, \omega) + (\partial_t \lambda(0, x, \xi))^2 \partial_\lambda^2 P(0, \omega) = 0. \tag{1.13}$$

Equation (1.13), a second-order polynomial equation for $\partial_t \lambda(0, x, \xi)$, has non-real roots if and only if the second condition on the right-hand side of (1.12) holds. □

We now examine the situation in which a double and semisimple eigenvalue λ_0 belongs to a branch λ of double and semisimple eigenvalues at $t = 0$, which all satisfy conditions (1.12):

Hypothesis 1.5. For some $\omega_0 = (x_0, \xi_0, \lambda_0) \in \Gamma$ satisfying (1.10) and (1.12), and such that λ_0 is a semisimple eigenvalue of $A(0, x_0, \xi_0)$, and for all $\omega = (x, \xi, \lambda)$ in a neighborhood of ω_0 in Γ, we have

$$\partial_\lambda P(0, \omega) = \partial_t P(0, \omega) = 0,$$

and λ is a semisimple eigenvalue of $A(0, x, \xi)$.

Semisimplicity of an eigenvalue means simpleness as a root of the minimal polynomial.

Condition $(\partial^2_{t\lambda} P(\omega))^2 < (\partial_t^2 P \partial_\lambda^2 P)(\omega)$ is open; in particular, if it holds at $\omega_0 \in \Gamma$, it holds at all nearby $\omega \in \Gamma$. Thus under Hypothesis 1.5, conditions (1.10) and (1.12) hold in a neighborhood of ω_0 in Γ.

Theorem 1.6. *Assume that Hypothesis 1.5 holds, and that the other eigenvalues of $A(0, x_0, \xi_0)$ are simple. Then the Cauchy problem for system (1.1) is ill-posed in the vicinity of the reference solution ϕ, in the sense of Definition 1.1.*

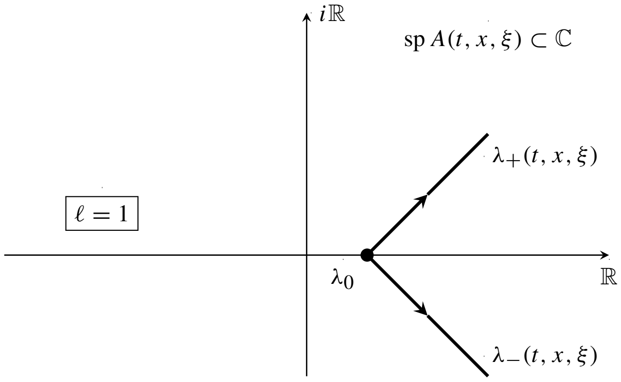

Fig. 4. In Theorem 1.6, corresponding to $\ell = 1$ in Assumption 2.1, a bifurcation occurs at $(0, x, \xi)$ in the spectrum of the principal symbol, for all (x, ξ) near (x_0, ξ_0). The eigenvalues are time-differentiable. The arrows indicate the direction of time.

An analogue to Theorem 1.6 in the high-frequency regime is the result of Y. Lu [13], in which it is shown how *higher-order resonances*, not present in the data, may destabilize precise WKB solutions. Under the assumptions of Theorem 1.6 and assuming analyticity of the coefficients, B. Morisse [19] proves existence, uniqueness and instability *in Gevrey spaces*, further extending G. Métivier's analysis [16].

Theorem 1.6 is illustrated by the Burgers systems of Sections 1.5 and 7.1.

1.4. Remarks

Taken together, our results assert that, for principal symbols with eigenvalues of multiplicity at most two, if one of the following holds:

(a) condition (1.8),
(b) conditions (1.10)–(1.11),
(c) Hypothesis 1.5,

then ill-posedness ensues.

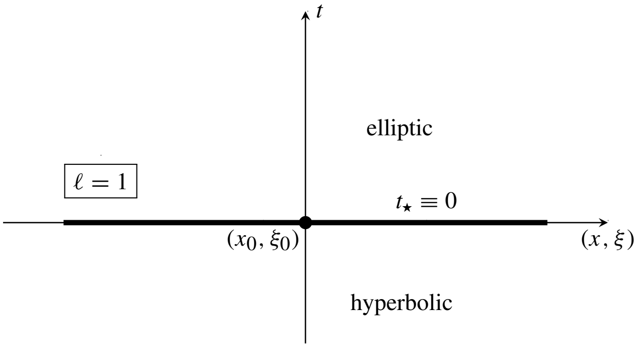

Fig. 5. In Theorem 1.6, the transition occurs at $t = 0$, uniformly near (x_0, ξ_0).

We note that condition (1.11) is stable by perturbation, and that conditions (1.10)–(1.11) are generically *necessary and sufficient* for occurrence of non-real eigenvalues in symbols that are initially hyperbolic. Indeed:

- non-real eigenvalues may occur only if the initial principal symbol has double eigenvalues, implying necessity of condition (1.10), and
- as shown by the proof of Theorem 1.3, the opposite sign $(\partial_\lambda^2 P \partial_t P)(0, \omega_0) < 0$ in condition (1.11) implies *real* eigenvalues for small $t > 0$.

Here *generically* means that the above discussion leaves out the degenerate case $\partial_t P(0, \omega_0) = 0$.

We consider the case $\partial_t P(0, \omega_0) = 0$ in Theorem 1.6. Note however that there is a significant gap between (b) and (c), the assumptions of Theorems 1.3 and 1.6. Indeed, while condition $\partial_t P = 0$ in Hypothesis 1.5 lies at the boundary of the case considered in Theorem 1.3, Hypothesis 1.5 describes a situation which is quite degenerate, since we ask for the closed conditions $\partial_\lambda P = 0$, $\partial_t P = 0$ (and also for semisimplicity) to hold *on a whole branch of eigenvalues* near λ_0.

Non-semisimple eigenvalues are typically not differentiable at the coalescing point, the canonical example being

$$\begin{pmatrix} 0 & 1 \\ \pm t^\alpha & 0 \end{pmatrix} \tag{1.14}$$

with $\alpha = 1$. The proof of Theorem 1.3 shows that the principal symbol at (t, x_0, ξ_0) can be reduced to (1.14), with $\alpha = 1$ and a negative sign, implying non-real and non-differentiable eigenvalues.

By contrast, semisimple eigenvalues admit one-sided directional derivatives (see for instance Chapter 2 of T. Kato's treatise [8], or [22, 25]). In particular, there is some redundancy in our assumptions of semisimplicity and condition (1.12).

We finally observe that our analysis extends somewhat beyond the framework of Theorems 1.2, 1.3 and 1.6. Consider for instance, in one space dimension, a smooth principal symbol of the form

$$\xi \begin{pmatrix} 0 & 1 \\ x^2 t - t^2 + t^3 a(x) & 0 \end{pmatrix} \quad \text{with eigenvalues } \lambda_\pm = \pm \xi (x^2 t - t^2 + t^3 a(x))^{1/2},$$

with $a(x) \in \mathbb{R}$, so that the eigenvalues are time-differentiable only at $x = 0$: conditions (1.12) hold only at $x = 0$. Semisimplicity does not hold at $(t, x) = (0, 0)$. Condition (1.11) does not hold at $(t, x, \lambda) = (0, 0, 0)$. However, by the implicit function theorem, eigenvalues cross at $(s(x), x)$ for a smooth s with $s(x) = x^2 + O(x^3)$. By inspection, condition (1.11) holds at $(s(x), x)$. Since x is arbitrarily small, Theorem 1.3 applies, yielding instability.

1.5. Examples

Burgers systems. Our first example is the family of Burgers-type systems in one space dimension

$$\partial_t \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} u_1 & -b(u)^2 u_2 \\ u_2 & u_1 \end{pmatrix} \partial_x \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = F(u), \tag{1.15}$$

in which $b > 0$ and $F = (F_1, F_2) \in \mathbb{R}^2$ are smooth. When $b(u)$ is not constant, the instability result for scalar equations in [12] does not directly apply. We show in particular that if F_2 is not initially zero, then Theorem 1.6 yields ill-posedness for the Cauchy problem for (1.15). Under the same condition $F_2(t = 0) \not\equiv 0$, Theorem 1.6 also applies to two-dimensional systems

$$\partial_t u + \begin{pmatrix} u_1 \partial_{x_1} & -b(u)^2 u_2 (\partial_{x_1} + \partial_{x_2}) \\ u_2 (\partial_{x_1} + \partial_{x_2}) & u_1 \partial_{x_1} \end{pmatrix} u = F(u).$$

Details are given in Sections 7.1 and 7.2.

Van der Waals gas dynamics. Our results also apply to the one-dimensional, isentropic Euler equations in Lagrangian coordinates

$$\begin{cases} \partial_t u_1 + \partial_x u_2 = 0, \\ \partial_t u_2 + \partial_x p(u_1) = 0, \end{cases} \tag{1.16}$$

with a Van der Waals equation of state, for which $p'(u_1) \le 0$ for some $u_1 \in \mathbb{R}$. We prove that if $(\phi_1, \phi_2)(t, x)$ is a smooth solution such that, for some $x_0 \in \mathbb{R}$,

$$\text{(i) } p'(\phi_1(0, x_0)) < 0 \quad \text{or} \quad \text{(ii) } p'(\phi_1(0, x_0)) = 0,\ p''(\phi_1(0, x_0)) \partial_x \phi_2(0, x_0) > 0,$$

then the initial-value problem for (1.16) is ill-posed in any neighborhood of ϕ. Condition (i) is an ellipticity assumption (under which Theorem 1.2 applies), and condition (ii) is an open condition on the boundary of the domain of hyperbolicity (under which Theorem 1.3 applies). System (1.16) has the formal conserved quantity $\int_{\mathbb{R}} (|v(t, x)|^2 + 2P(u(t, x)))\, dx$, where $P' = p$. As briefly discussed in Section 1.2, the instabilities evidenced here are local and neither preclude nor are contradicted by global stability properties of the system, such as formal conservation laws. This example is developed in Section 7.3.

Klein–Gordon–Zakharov systems. Our last class of examples is given by the following one-dimensional Klein–Gordon system coupled to wave equations with Zakharov-type non-linearities:

$$\begin{cases} \partial_t \begin{pmatrix} u \\ v \end{pmatrix} + \partial_x \begin{pmatrix} v \\ u \end{pmatrix} + \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} \partial_x \begin{pmatrix} n \\ m \end{pmatrix} = (n+1) \begin{pmatrix} v \\ -u \end{pmatrix}, \\ \partial_t \begin{pmatrix} n \\ m \end{pmatrix} + c\partial_x \begin{pmatrix} m \\ n \end{pmatrix} + \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} \partial_x \begin{pmatrix} u \\ v \end{pmatrix} = \partial_x \begin{pmatrix} 0 \\ u^2 + v^2 \end{pmatrix}. \end{cases} \tag{1.17}$$

The linear differential operator in (u, v) in the subsystem in (u, v) is a Klein–Gordon operator, with critical frequency scaled to 1. The linear differential operator in (n, m) in the subsystem in (n, m) is a wave operator, with acoustic velocity c. The source in $\partial_x(u^2 + v^2)$ is similar to the non-linearity in the Zakharov equation [20, 23]. Systems of the form (1.17) with $\alpha = 0$ are used to describe laser-matter interactions; in the high-frequency limit, they can be formally derived from the Maxwell–Euler equations [2]. We consider the case $|c| < 1$, corresponding to the physical situation of an acoustic velocity being smaller than the characteristic Klein–Gordon frequencies.

It was shown in [2] that for $\alpha = 0$ system (1.17) is conjugate via a non-linear change of variables to a semilinear system, implying in particular well-posedness in $H^s(\mathbb{R})$ for $s > 1/2$.

Here we show that if $\alpha \neq 0$ and $\phi = (u, v, n, m)$ is a smooth solution such that

$$u(0, x_0) = 0, \qquad v(0, x_0) = -\frac{c}{2\alpha}, \qquad \alpha c \partial_x u(0, x_0) > 0, \qquad \text{for some } x_0 \in \mathbb{R},$$

then Theorem 1.3 applies and the Cauchy problem for (1.17) is ill-posed in the vicinity of ϕ. This situation is analogous to the Turing instability, where 0 is a stable equilibrium point for both ordinary differential equations $X' = AX$ and $X' = BX$, but not for $X' = (A + B)X$.

We come back to this example in detail in Section 7.4.

2. Main assumption and result

Theorems 1.2, 1.3 and 1.6 can all be recast in the same framework, which we now present.

2.1. Bounds for the symbolic flow of the principal symbol

2.1.1. Degeneracy index and associated parameters. Let $\ell \in \{0, 1/2, 1\}$.[5] Define

$$h = \frac{1}{1+\ell} = \begin{cases} 1 & \text{if } \ell = 0, \\ 2/3 & \text{if } \ell = 1/2, \\ 1/2 & \text{if } \ell = 1, \end{cases} \qquad \zeta = \begin{cases} 0 & \text{if } \ell \in \{0, 1\}, \\ 1/3 & \text{if } \ell = 1/2. \end{cases}$$

The parameters h and ζ define our time, space and frequency scales.

[5] Theorem 1.2 corresponds to $\ell = 0$, Theorem 1.3 to $\ell = 1/2$, and Theorem 1.6 to $\ell = 1$.

2.1.2. The time transition function and the elliptic domain. Introduce a time transition function $t_\star$ such that

- if $\ell = 0$ or $\ell = 1$, then $t_\star \equiv 0$,
- if $\ell = 1/2$, then $t_\star$ depends smoothly on (x, ξ) and singularly on ε, and is slowly varying in x, in the sense that for some smooth function $\theta_\star$,

$$t_\star(\varepsilon, x, \xi) = \varepsilon^{-h}\theta_\star(\varepsilon^{1-h}x, \xi) \quad \text{with} \quad \theta_\star \geq 0,\ \theta_\star(0, \xi_0) = 0,\ \nabla_{x,\xi}\theta_\star(0, \xi_0) = 0. \tag{2.1}$$

Define then the elliptic domain[6] by

$$\mathcal{D} := \{(\tau; t, x, \xi) : t_\star(x, \xi) \leq \tau \leq t \leq T(\varepsilon),\ |x| \leq \delta,\ |\xi - \xi_0| \leq \delta\varepsilon^{\zeta}\}, \tag{2.2}$$

for some $\xi_0 \in \mathbb{R}^d \setminus \{0\}$ and some $\delta > 0$, with

$$T(\varepsilon)^{\ell+1} = T_\star|\log \varepsilon|, \qquad T_\star > 0. \tag{2.3}$$

2.1.3. The rescaled and advected principal symbol. We consider a reference solution ϕ satisfying (1.3), and the associated principal symbol (1.4). The rescaled and advected principal symbol is[7]

$$A_\star(\varepsilon, t, x, \xi) := \big(Q(A - \mu)Q^{-1}\big)\big(\varepsilon^h t,\ x_0 + \varepsilon^{1-h}x_\star(\varepsilon^h t, x, \xi),\ \xi_\star(\varepsilon^h t, x, \xi)\big) \tag{2.4}$$

for some $x_0 \in \mathbb{R}^d$, with $0 < h \leq 1$ as in (2.3), where, in $\mathcal{D}$,

- the symbol $\mu = \mu(t, x, \xi)$ is real and smooth, and $Q(t, x, \xi) \in \mathbb{C}^{N\times N}$ is smooth and pointwise invertible,
- the bicharacteristics $(x_\star, \xi_\star)$ solve

$$\partial_t \begin{pmatrix} x_\star \\ \xi_\star \end{pmatrix} = \begin{pmatrix} -\partial_\xi \mu \\ \varepsilon^{1-h}\partial_x \mu \end{pmatrix}(t, x_0 + \varepsilon^{1-h}x_\star, \xi_\star), \qquad \begin{pmatrix} x_\star \\ \xi_\star \end{pmatrix}(0, x, \xi) = \begin{pmatrix} x \\ \xi \end{pmatrix}. \tag{2.5}$$

We assume that the symbol $A_\star$ is block diagonal, and for its blocks $A_{\star j}$ we assume either the bound

$$\varepsilon^{h-1}|\partial_x^\alpha \partial_\xi^\beta A_{\star j}| \leq C_{\alpha\beta} < \infty \quad \text{for some } C_{\alpha\beta} > 0, \text{ in } \mathcal{D}, \text{ uniformly in } \varepsilon, \tag{2.6}$$

or the block structure

$$\varepsilon^{h-1}A_{\star j} = \begin{pmatrix} 0 & \varepsilon^{h-1}A_{\star j12} \\ \varepsilon^{1-h}A_{\star j21} & 0 \end{pmatrix} \tag{2.7}$$

with $|\partial_x^\alpha \partial_\xi^\beta A_{\star j12}| + |\partial_x^\alpha \partial_\xi^\beta A_{\star j21}| \leq C_{\alpha\beta}$ for some $C_{\alpha\beta} > 0$, in $\mathcal{D}$, uniformly in ε.

If $\ell = 0$, then $h = 1$. As a consequence, in the block diagonalization of $A_\star$ all blocks satisfy (2.6), by the assumed smoothness of the components of $A_\star$.

If $\ell = 1/2$, we assume that some block of $A_\star$ satisfies (2.7), and the other blocks of $A_\star$ satisfy (2.6).

If $\ell = 1$, we assume that all blocks of $A_\star$ satisfy (2.6).

[6] The elliptic domains corresponding to Theorems 1.3 and 1.6 are pictured in Figures 3 and 5.

[7] In the elliptic case, corresponding to Theorem 1.2, we have $\ell = 0$, $h = 1$, $Q \equiv \mathrm{Id}$, $\mu \equiv 0$, so that $(x_\star, \xi_\star) \equiv (x, \xi)$, and then $A_\star$ is simply $A_\star(\varepsilon, t, x, \xi) = A(\varepsilon t, x, \xi)$.

2.1.4. Symbolic flow and growth functions. The symbolic flow $S = S(\tau; t, x, \xi)$ of $A_\star$ is defined as the solution to the family of linear ordinary differential equations

$$\partial_t S + i\varepsilon^{h-1} A_\star(\varepsilon, t, x, \xi) S = 0, \qquad S(\tau; \tau) \equiv \mathrm{Id}. \tag{2.8}$$

In Section 2.1.3, we have assumed that $A_\star$ is block diagonal. Accordingly, the solution S to (2.8) is block diagonal, with blocks $S_{(1)}, S_{(2)}, \dots$.

Let $\gamma^\pm(x, \xi)$ be two continuous functions defined on $\{|x| \le \delta,\ |\xi - \xi_0| \le \delta\varepsilon^\zeta\}$ such that $\gamma^-(0, \xi_0) = \gamma^+(0, \xi_0)$, and

$$\mathbf{e}_{\gamma^\pm}(\tau; t, x, \xi) := \exp\Big(\gamma^\pm(x, \xi)\big((t - t_\star(x, \xi))_+^{\ell+1} - (\tau - t_\star(x, \xi))_+^{\ell+1}\big)\Big), \tag{2.9}$$

where $t_+ := \max(t, 0)$ and the time transition function $t_\star$ is defined in (2.1). We understand $\mathbf{e}_{\gamma^\pm}$ as *growth functions*,[8] measuring how fast the solution S to (2.8) is growing, as seen in Assumption 2.1 below. The associated $\gamma^\pm$ are *rates of growth*.

2.1.5. Bounds. We postulate bounds for S in the elliptic domain $\mathcal{D}$ in terms of the growth functions $\mathbf{e}_{\gamma^\pm}$:

Assumption 2.1. *For some $(x_0, \xi_0) \in U \times (\mathbb{R}^d \setminus \{0\})$, some ℓ, Q, μ, $\gamma^\pm$, and $t_\star$ as above, any $T_\star > 0$, for some $\delta > 0$, for $A_\star$ satisfying the structural assumptions of Section 2.1.3, for some $\varepsilon_0 > 0$ and all $0 < \varepsilon < \varepsilon_0$, the solution S to (2.8) satisfies:*

- *the lower bound, for some smooth family of unit vectors $\vec{e}(x) \in \mathbb{C}^N$, for $|x| < \delta$:*

$$\varepsilon^{-\zeta} \mathbf{e}_{\gamma^-}(0; T(\varepsilon), x, \xi_0) \lesssim |S(0; T(\varepsilon), x, \xi_0)\vec{e}(x)|, \tag{2.10}$$

- *the upper bound for the j-th diagonal block $S_{(j)}$ of S, for $(\tau, t, x, \xi) \in \mathcal{D}$:*

$$|S_{(j)}(\tau; t, x, \xi)| \lesssim \begin{pmatrix} 1 & \varepsilon^{-\zeta} \\ \varepsilon^\zeta & 1 \end{pmatrix} \mathbf{e}_{\gamma^+}(\tau; t, x, \xi). \tag{2.11}$$

In (2.10), the notation $a \lesssim b$, where a and b are functions of $(\varepsilon, \tau, t, x, \xi)$, is used to mean existence of a *uniform* bound

$$a(\varepsilon, \tau, t, x, \xi) \le C|\ln \varepsilon|^{C'} b(\varepsilon, \tau, t, x, \xi), \tag{2.12}$$

where $C, C' > 0$ do not depend on $(\varepsilon, \tau, t, x, \xi)$. This means in particular that powers of $|\ln \varepsilon|$ play the role of constants in our analysis. They are indeed destined to be absorbed by arbitrarily small powers of ε.

In (2.11), we use $\lesssim$ for matrices. Here we mean *blockwise* inequalities "modulo constants", in the sense of (2.12). That is, in (2.11) we assume that the j-th diagonal block of S itself has a block structure, with the top left block being bounded *entrywise* by $\mathbf{e}_{\gamma^+}$, the top right block being bounded *entrywise* by $\varepsilon^{-\zeta}\mathbf{e}_{\gamma^+}$, etc.

We further comment on Assumption 2.1 in Section 2.3.

[8] In the elliptic case, we have $\ell = 0$, $t_\star \equiv 0$, so that the growth functions are simply $\mathbf{e}_{\gamma^\pm} = e^{\gamma^\pm(t-\tau)}$.

2.2. *Hadamard instability*

The non-linear information contained in Assumption 2.1 on the symbolic flow of the principal symbol (1.4) translates into an instability result for the quasi-linear system (1.1):

Theorem 2.2. *Under Assumption 2.1, system (1.1) is ill-posed in the vicinity of the reference solution ϕ, in the sense of Definition 1.1.*

Theorem 2.2 states that either there exists no solution map, or the solution map fails to enjoy any Hölder-type continuity estimates. The proof of Theorem 2.2 is given in Section 3. Key ideas in the proof are sketched in Section 2.4.

Theorems 1.2, 1.3 and 1.6 all follow from Theorem 2.2.

2.3. *Comments on Assumption 2.1*

Our main assumption is flexible enough to cover the three different situations described in Theorems 1.2, 1.3 and 1.6. Before further commenting on its ingredients in Section 2.3.1 and its verification in Section 2.3.2, we point out two key features:

• Assumption 2.1 is *non-linear*. It bears on the whole system (1.1), not just the principal symbol. For instance, instability occurs for the Burgers systems of Section 1.5 under a condition bearing on the non-linear term F.

• Assumption 2.1 is *finite-dimensional*, in the sense that it postulates bounds for solutions to ordinary differential equations in a finite-dimensional setting. These are turned into bounds for solutions to partial differential equations via Theorem D.3. An informal discussion of the role of Theorem D.3 is given in Section 2.4.

2.3.1. *On the ingredients of Assumption 2.1*

• Our localization constraints in $(x, \xi) \in \mathbb{R}^{2d}$ respect the uncertainty principle. Indeed, we localize spatially in a box of size $\sim \varepsilon^{1-h}$. We localize in frequency in a box of size $\sim \varepsilon^{\zeta}$ around ξ_0 but then in the proof we use highly oscillating data and an ε^h-semiclassical quantization, so that frequencies $\varepsilon^h \xi$ are localized in a box of size ε^{ζ} around ξ_0, meaning a frequency localization in a box of size $\varepsilon^{\zeta-h}$. If $\ell = 0$ or $\ell = 1$, then $\zeta = 0$. The area of the (x, ξ)-box is then $\varepsilon^{2d(1-2h)} \geq 1$, since $h = 1$ or $h = 1/2$. If $\ell = 1/2$, then $h = 2/3$ and $\zeta = 1/3$. The area of the (x, ξ)-box is $\varepsilon^{2d(1-h+\zeta-h)} = 1$.

• The index ℓ measures the degeneracy of the defect of hyperbolicity. We have $\ell = 0$ in the case of initial ellipticity (Theorem 1.2), $\ell = 1/2$ in the case of a non-semisimple defect of hyperbolicity (Theorem 1.3) and $\ell = 1$ in the case of a semisimple defect of hyperbolicity (Theorem 1.6). The instability is recorded in $O(\varepsilon|\ln \varepsilon|)^{1/(1+\ell)}$ time for $O(1/\varepsilon)$ initial frequencies. In particular, the higher the degree of degeneracy, the longer we need to wait in order to record the instability.

• In the case $\ell > 0$, eigenvalues of the principal symbol are initially real (hyperbolicity). Instability occurs as (typically) a pair of eigenvalues branch from the real axis at $t = 0$.

The matrix Q should be understood as a change of basis, which includes a projection onto the space of bifurcating eigenvalues. The scalar μ corresponds to the real part of the bifurcating eigenvalues. Assumption 2.1 is formulated for the principal symbol evaluated along the bicharacteristics of μ.

• In the non-semisimple case, the defect of hyperbolicity is typically *not* uniform in (x, ξ). That is, if eigenvalues branch from the real axis at initial time at the distinguished point (x_0, ξ_0), then the branching will typically occur for later times $t_\star(x, \xi) > 0$ for (x, ξ) close to (x_0, ξ_0). This is clearly seen in Lemma 5.1, under the assumptions of Theorem 1.3, and pictured in Figure 3.

• The parameter γ^+ corresponds to an upper rate of growth. In the elliptic case, γ^+ is equal to the largest imaginary part in the initial spectrum, as seen in Section 4. In the case of a smooth defect of hyperbolicity, $\gamma^+ = \Im m\, \partial_t \lambda(0, x, \xi)$, where λ is a bifurcating eigenvalue, as seen in Section 6.

• In the case $\ell = 1/2$, the block structure (2.7) derives from a reduction of the principal symbol to normal form; see Sections 5.1 and 5.2 in the proof of Theorem 1.3.

• In the case $\ell = 1$, the block structure (2.6) reveals a cancellation, seen in (6.4) in the proof of Theorem 1.6.

• The smoothly varying direction $\vec{e}\,(x)$ along which the lower bound (2.10) holds is not necessarily an eigenvector of $A_\star$; see the discussion in Section 2.3.3 and Lemma 5.10.

2.3.2. On verification of Assumption 2.1. We give in Theorems 1.2, 1.3 and 1.6 sufficient conditions, expressed in terms of the spectrum of A and the jet of the characteristic polynomial of A at $t = 0$, for Assumption 2.1 to hold. These sufficient conditions are satisfied in particular by Burgers, Van der Waals, and Klein–Gordon–Zakharov systems (Section 7). These conditions bear only on the coefficients of the system (the differential operator and the source) and $\phi(0)$, the initial datum of the reference solution. In particular, we may in practice verify these conditions without having any knowledge of $\phi(t)$ for $t > 0$.

2.3.3. On spectral conditions describing the transition from hyperbolicity to ellipticity. Conditions (1.8), (1.10)–(1.11) and (1.12) are all expressed in terms of the characteristic polynomial of A. Their generalizations in the form of conditions (2.10) and (2.11) are expressed in terms of the symbolic flow of A. Our point here is to explain why conditions bearing on the *spectrum* of A do not seem to be appropriate. The discussion below also highlights three difficulties in the analysis of the case $\ell = 1/2$: the lack of smoothness of the eigenvalues of the principal symbol, the lack of uniformity of the transition time (in the sense that the function $t_\star$ does depend on (x, ξ)), and the lack of smoothness of the eigenvectors.

A simple way to express the fact that an eigenvalue λ of A branches from the real axis at $t = 0$ is

$$\lambda(0, x, \xi) \in \mathbb{R} \quad \text{for all } (x, \xi) \text{ near } (x_0, \xi_0), \quad \text{with} \quad \Im m\, \partial_t \lambda(0, x_0, \xi_0) > 0. \tag{2.13}$$

But then by reality of the coefficients of A, eigenvalues branch from $\mathbb{R}$ in pairs, so that $(0, x_0, \xi_0)$ is a branching point in the spectrum, and typically eigenvalues are *not* differentiable at a branching point, so that (2.13) is not general enough. For instance, the eigenvalues of the principal symbol for the one-dimensional compressible Euler equations

$$\begin{cases} \partial_t u + \partial_x v = 0, \\ \partial_t v + \partial_x p(u) = 0, \end{cases} \tag{2.14}$$

are

$$\lambda_\pm(t, x, \xi) = \pm\xi(p'(u(t, x)))^{1/2}.$$

For a Van der Waals equation of state, for which $p'(u) \leq 0$ for some $u \in \mathbb{R}$, a transition from hyperbolicity to ellipticity occurs for data $u(0, \cdot)$ satisfying

$$p'(u(0, x_0)) = 0, \qquad \partial_t(p'(u(0, x_0))_{|t=0} = -p''(u(0, x_0))\partial_x v(0, x_0) < 0. \tag{2.15}$$

The associated eigenvalues are $O(t^{1/2})$, in particular not time-differentiable at $t = 0$, so that condition (2.13) is not appropriate.

A way around this difficulty is to consider the integral growth condition

$$\int_0^t \Im m\, \lambda(\tau, x, \xi)\, d\tau = \gamma(t, x, \xi) t^{\ell+1}, \qquad \gamma(0, x_0, \xi_0) > 0, \tag{2.16}$$

for some $\ell \geq 0$ and some rate function γ that is continuous in (t, x, ξ) at $(0, x_0, \xi_0)$, and some local solution λ of $P = 0$. Condition (2.16) may be verified by using the Puiseux expansions of the eigenvalues at $t = 0$, such as in the Van der Waals example (for details on Puiseux expansions, see for instance [8, Chapter 2], or [25, Proposition 4.2]). There are, however, at least two serious problems with (2.16).

The first is that in (2.16), it is assumed that the loss of hyperbolicity occurs at $t = 0$ over a whole neighborhood of (x_0, ξ_0), which is typically *not* the case. Consider in this connection the preparation condition (2.15) for the datum. From the second condition in (2.15), we find by application of the implicit function theorem that in the vicinity of $(0, x_0)$ the set $\{p'(u) = 0\}$ is the graph of a smooth map $x \mapsto t_*(x)$. The transition curve $x \mapsto t_*(x)$, defined locally in a neighborhood of x_0, parameterizes the loss of hyperbolicity: for $t < t_*(x)$ we have $p'(u) > 0$, which implies $\Im m\, \lambda_\pm \equiv 0$, while for $t > t_*(x)$ we have $p'(u) < 0$, which implies $\Im m\, \lambda_\pm \neq 0$. On the curve $t = t_*(x)$, we have $p'(u) \equiv 0$, which implies that the eigenvalues coalesce: $\lambda_- = \lambda_+$. This means in particular that for (x, ξ) close to, and different from, (x_0, ξ_0) we should not expect the loss of hyperbolicity to be instantaneous as in (2.16), but rather to happen at time $t_*(x, \xi)$, and condition (2.16) should be replaced by

$$\int_{t_*(x,\xi)}^t \Im m\, \lambda(\tau, x, \xi)\, d\tau = \gamma(t, x, \xi)(t - t_*(x, \xi))^{\ell+1}, \qquad \gamma(0, x_0, \xi_0) > 0, \tag{2.17}$$

for some smooth time transition function $t_* \geq 0$ with $t_*(x_0, \xi_0) = 0$.

The second problem with (2.16), still present in (2.17), is that while failure of time-differentability of the eigenvalues is accounted for in (2.16), the associated lack of regularity of eigenvectors is not. For instance, in the Van der Waals system (2.14), the eigenvectors of the principal symbol are $e_\pm := (1, \pm(p'(u))^{1/2})$. In particular, under condition (2.15), the eigenvectors $e_\pm$ are not time-differentiable at $t = 0$. It is then unclear how to convert conditions (2.16) and (2.17) into growth estimates for corresponding system of partial differential equations. Indeed, for instance in the simpler case of ordinary differential equations, spectral estimates such as (2.16) or (2.17) are typically converted into growth estimates for solutions via projections onto spectral subspaces, an operation that requires smooth projections.

We conclude this discussion by sketching a way around the lack of regularity of eigenvectors. Going back to the Van der Waals example, consider the ordinary differential equations

$$\partial_t S + i\xi \begin{pmatrix} 0 & 1 \\ p'(u) & 0 \end{pmatrix} S = 0, \qquad S(\tau;\tau) \equiv \mathrm{Id},$$

parameterized by (x, ξ). Under condition (2.15), we have

$$p'(u(t,x)) = -\alpha(x)t + O(t^2)$$

for (t, x) close to $(0, x_0)$. Restricting for simplicity to the case $p'(u(t, x)) = -t$, we find that the entries (y, z) of a column of S satisfy the system of ordinary differential equations

$$y' + i\xi z = 0, \qquad z' - it\xi y = 0,$$

implying that y satisfies the Airy equation

$$y'' = t\xi^2 y,$$

for which sharp lower and upper bounds are known.

This motivates consideration, in Section 2.1, of the symbolic flow associated with the principal symbol A. An important issue is then the conversion of growth conditions for the symbolic flow into estimates for solutions to the system of partial differential equations. This is achieved via Theorem D.3.

2.4. On the proof of Theorem 2.2

We give here an informal account of key points in the proof of Theorem 2.2. The proof is in three parts: (1) preparation steps which transform the equation into the *prepared equation* (3.36)–(3.37), (2) the use of a Duhamel representation formula, (3) lower and upper bounds.

(1) We introduce a spatial scale h and write perturbation equations about the reference solution ϕ. We then block diagonalize the principal symbol (this is Q from Assumption 2.1), localize in space around the distinguished point x_0, factor out the real part of the branching eigenvalues (this is μ from Assumption 2.1) and change to a reference frame defined

by the bicharacteristics of μ. Finally, we operate a stiff localization in the elliptic domain $\mathcal{D}$ given by Assumption 2.1, and rescale time. The key point in these preparation steps is to carefully account for the *linear errors* in the principal symbol, which take the form of commutators. The resulting principal symbol is a perturbation of the symbol $A_\star(\varepsilon, t, x, \xi)$ defined in Assumption 2.1.

(2) Assumption 2.1 provides bounds for the flow of $A_\star$. As pointed out in Section 2.3, these bounds bear on solutions to ordinary differential equations in finite dimensions, in particular they are, at least theoretically, easier to verify than bounds bearing on spectra of differential operators. We use Theorem D.3, taken from [24] and proved in Appendix D, to convert these bounds into estimates for a solution to (1.1).

Consider a pseudo-differential Cauchy problem[9]

$$\partial_t u + \mathrm{op}_\varepsilon(\mathcal{A})u = g, \qquad u(0) = u_0 \in L^2, \tag{2.18}$$

where $\mathcal{A}$ is a symbol of order zero. Above, $\mathrm{op}_\varepsilon(\mathcal{A})$ denotes the ε^h-semiclassical quantization of the symbol $\mathcal{A}$, as defined in (3.1). Associated with the above Cauchy problem in infinite dimensions, consider the Cauchy problem in finite dimensions

$$\partial_t S + \mathcal{A}S = 0, \qquad S(\tau;\tau) = \mathrm{Id}.$$

Theorem D.3 asserts that if $S(\tau;t)$ and its (x,ξ)-derivatives grow in time like $\exp(\gamma t^{1+\ell})$, with rate $\gamma > 0$ and degeneracy index $\ell \geq 0$, then $\mathrm{op}_\varepsilon(S)$ furnishes a good approximation to a solution operator for $\partial_t + \mathrm{op}_\varepsilon(\mathcal{A})$, in $O(|\ln\varepsilon|)^{1/(1+\ell)}$ time. That is, the solution of (2.18) is given by

$$u(t) \simeq \mathrm{op}_\varepsilon(S(0;t))u_0 + \int_0^t \mathrm{op}_\varepsilon(S(\tau;t))g(\tau)\,d\tau. \tag{2.19}$$

(3) The preparation steps (see (1) above) reduce our problem to a system of the form (2.18). Via representation (2.19), upper and lower bounds for a solution u to (2.18) are easily derived from the bounds of Assumption 2.1, and from the *postulated* bound for the source g. In our proof, the source g includes in particular non-linear errors. Since we have no way of bounding solutions to (1.1) near ϕ (the impossibility of controlling the growth of solutions with respect to the initial data being precisely what we endeavor to prove), we *assume* a priori bounds for the solution. The compared growth of $\mathrm{op}_\varepsilon(S)u_0$ and the Duhamel term from (2.19) eventually provide a contradiction. Note that the a priori bound (see (3.9) in Section 3.5) is particularly weak, since we allow for arbitrarily large losses of derivatives.

We finally note that Gårding's inequality (see for instance [11, Theorem 1.1.26]) asserts that non-negativity of the symbol $\mathcal{A}$ implies semipositivity of the operator $\mathrm{op}_\varepsilon(\mathcal{A})$. This is the classical tool for converting bounds for symbols into estimates for the associated equations. It is shown in [24] how estimates derived from Gårding's inequality fail to be sharp in the non-self-adjoint case, as opposed to bounds based on Theorem D.3.

[9] Notation and results pertaining to pseudo-differential calculus are recalled in Appendix B.

3. Proof of Theorem 2.2

As discussed in Section 2.4, the proof decomposes into three parts:

(1) *Preparation* steps which transform the original equation (1.1) into the *prepared equation* (3.36)–(3.37). This step covers Sections 3.1 to 3.10.
(2) The use of a *Duhamel representation formula* in which the solution to the prepared equation is expressed as the sum of the "free" solution (defined as the action of an approximate solution operator on the datum) and a remainder (Sections 3.11 and 3.12).
(3) *Lower bounds* for the free solution and *upper bounds* for the remainder conclude the proof (Sections 3.13 to 3.15).

3.1. Initial perturbation

The goal is to prove ill-posedness, in the sense of Definition 1.1. Parameters m, α, δ, T are given, as in (1.6), and we endeavor to disprove (1.7). Define

$$\varphi_0(\varepsilon, x) := \Re e\big(\mathrm{op}_\varepsilon(Q_\varepsilon(0)^{-1})(e^{i(\cdot)\cdot\xi_0/\varepsilon^h}\theta\vec{e})(x)\big), \ \varepsilon > 0, \qquad h = \frac{1}{1+\ell}, \ \ell \geq 0,$$

where (x_0, ξ_0) is the distinguished point in the cotangent space given in Assumption 2.1, and

- $\mathrm{op}_\varepsilon(\cdot)$ denotes a pseudo-differential operator in ε^h-semiclassical quantization:

$$\mathrm{op}_\varepsilon(a)v := (2\pi)^{-d}\int_{\mathbb{R}^d} e^{ix\cdot\xi}a(t, x, \varepsilon^h\xi)\hat{v}(\xi)\,d\xi; \tag{3.1}$$

- $Q_\varepsilon(0) = Q(0, x_0 + \varepsilon^{1-h}x, \xi)$, with Q as in Assumption 2.1;
- the vector $\vec{e}$ is as in Assumption 2.1;
- the spatial cut-off $\theta \in C_c^\infty(\mathbb{R}^d)$ has support included in $B(0, \delta)$, and is such that $\theta \equiv 1$ in $B(0, 1/2)$.

Consider the following family of data, indexed by $\varepsilon > 0$:

$$u^\varepsilon(0, x) = \phi(0, x) + \varepsilon^K\varphi_0\left(\frac{x - x_0}{\varepsilon^{1-h}}\right) \tag{3.2}$$

where $\phi(0, x)$ is the datum for the background solution ϕ as in (1.3), and K will be chosen large enough so that $u^\varepsilon(0)$ is a small perturbation of $\phi(0)$ in H^m norm.

Theorem 2.2 is a consequence of the following result for the family of initial-value problems (1.1)–(3.2), indexed by ε:

Theorem 3.1. *Given the parameters defined in* (1.6)*, given a local solution ϕ of* (1.1) *satisfying* (1.3) *with s_1 large enough, under Assumption* 2.1*, if K is large enough then either*

- *for any T and δ with $0 < T \le T_0$ and $B(x_0, \delta) \subset U$ there is no $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, the initial-value problem for (1.1) with initial datum (3.2) has a solution in $L^\infty([0, T], W^{1,\infty}(B(x_0, \delta))$, or*
- *for some T and δ with $0 < T \le T_0$, $B(x_0, \delta) \subset U$, some $\varepsilon_0 > 0$, all $0 < \varepsilon < \varepsilon_0$, the initial-value problem for (1.1) with initial datum (3.2) has a solution u^ε in $L^\infty([0, T], W^{1,\infty}(B(x_0, \delta))$, and the solution satisfies*

$$\sup_{\substack{0<\varepsilon<\varepsilon_0 \\ 0\le t\le \varepsilon^h T(\varepsilon)}} \frac{\|u^\varepsilon(t) - \phi(t)\|_{W^{1,\infty}(B(x_0,\varepsilon^{1-h}\delta))}}{\|u^\varepsilon(0) - \phi(0)\|^\alpha_{H^m(U)}} = \infty \tag{3.3}$$

where $T(\varepsilon)$ is defined in (2.3), so that in particular $\varepsilon^h T(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Lemma 3.2. *Theorem 3.1 implies Theorem 2.2.*

Proof. We have

$$\varphi_0\left(\frac{x - x_0}{\varepsilon^{1-h}}\right) = \Re e\, e^{i(x-x_0)\xi_0/\varepsilon} \tilde{\varphi}\left(\frac{x - x_0}{\varepsilon^{1-h}}\right),$$

where $\tilde{\varphi} := \mathrm{op}_\varepsilon(Q(0, \cdot, \xi_0 + \cdot))(\theta \vec{e})$, hence

$$\left\| \varphi_0\left(\frac{x - x_0}{\varepsilon^{1-h}}\right) \right\|_{H^m(U)} \lesssim \varepsilon^{-m+(1-h)d/2}.$$

Let $K > m - (1 - h)d/2$. Then

$$\|u^\varepsilon(0, \cdot) - \phi(0, \cdot)\|_{H^m(U)} \lesssim \varepsilon^{K-m+(1-h)d/2} \xrightarrow{\varepsilon\to 0} 0.$$

Thus given a neighborhood $\mathcal{U}$ of $\phi(0)$ in $H^m(U)$, if ε is small enough then $u^\varepsilon(0)$ lies in $\mathcal{U}$.

If for some ε small enough, the initial-value problem (1.1), (3.2) does not have a solution, then this means ill-posedness in the sense of Definition 1.1. If there is a solution for any small ε, then (3.3) disproves (1.7), since the sequence $\varepsilon^h T(\varepsilon)$ converges to 0, and again this means ill-posedness in the sense of Definition 1.1. □

3.2. The posited solution and its avatars

We assume that for some $0 < T \le T_0$, some $\delta > 0$ with $B(x_0, \delta) \subset U$, some $\varepsilon_0 > 0$, and all $0 < \varepsilon < \varepsilon_0$, the Cauchy problem for (1.1) with initial datum (3.2) has a unique solution

$$u^\varepsilon \in L^\infty([0, T], W^{1,\infty}(B(x_0, \delta))).$$

Our goal is then to prove (3.3). For future reference, we list here the successive avatars of the solution that we will use in this proof:

$\dot{u}$	perturbation	$\dot{u} := (u^\varepsilon - \phi)(t, x_0 + \varepsilon^{1-h} x)$	(3.6)
$u^\flat$	spatial localization and projection	$u^\flat := \mathrm{op}_\varepsilon(Q_\varepsilon)(\theta \dot{u})$	(3.14)
$u_\star$	convection	$u_\star := M^\star(0; t) u^\flat$	(3.27)
v	stiff truncation and rescaling in time	$v := (\mathrm{op}_\varepsilon(\chi) u_\star)(\varepsilon^h t)$	(3.32)

3.3. Amplitude of the perturbation, limiting observation time and observation radius

The parameter K measures the size of the initial perturbation (3.2). We choose K to be large enough:

$$(2\alpha - 1)K > 2\alpha m + (1-\alpha)(1-h)d, \tag{3.4}$$

where m measures the loss of Sobolev regularity and α the loss of Hölder continuity of the flow (as seen in the target estimate (3.3)), and $h = 1/(1+\ell)$.

The parameter $T_\star$, defined in (2.3), measures the final observation time in the rescaled time frame. In the original time frame, the final observation time is $(\varepsilon T_\star |\ln \varepsilon|)^{1/(1+\ell)}$. We choose $T_\star$ to be large enough:

$$\gamma^-(0, \xi_0) T_\star > K, \tag{3.5}$$

depending on K and the lower rate of growth γ^- introduced in (2.9).

The parameter δ measures the radius of the observation ball $B(0, \delta)$ where the analysis takes place. (The radius is $\varepsilon^{1-h}\delta$ in the original spatial frame, and just δ in the rescaled spatial frame associated with $\dot u$; see Section 3.2 above and (3.6).) If Theorem 3.1 holds for a given value of δ, then it holds for any smaller radius. In particular, we may assume that the given value of δ is so small that the bounds of Assumption 2.1 hold on $|x| + |\xi - \xi_0| \le \delta$. In the final steps of our analysis (Sections 3.14 and 3.15), we will further choose δ to be small enough, depending on the growth functions $\gamma^\pm$ introduced in Assumption 2.1 and $T_\star$ (see condition (3.64) and the proof of Corollary 3.21).

3.4. The perturbation equations

Our analysis is local in t, x, ξ, with $0 \le t \le \varepsilon^h T(\varepsilon)$, $|x - x_0| \le \varepsilon^{1-h}\delta$ and $|\xi - \xi_0| \le \delta$, where $T(\varepsilon)$ is defined in (2.3), and $T_\star$ and δ are defined in Section 3.3.

The perturbation variable $\dot u$ is defined in a rescaled spatial frame by

$$\dot u(\varepsilon, t, x) := (u^\varepsilon - \phi)(t, x_0 + \varepsilon^{1-h} x) \qquad \text{with } h = 1/(1+\ell). \tag{3.6}$$

The equation for $\dot u$ is

$$\partial_t \dot u + \varepsilon^{-1} A(t,\ x_0 + \varepsilon^{1-h} x,\ \varepsilon^h \partial_x)\dot u + \dot B(\varepsilon, t, x)\dot u = \dot F, \tag{3.7}$$

where A is the 1-homogeneous principal symbol (1.4) and $\dot B$ is of order zero:

$$\dot B(\varepsilon, t, x)\dot u := \sum_j \big(\partial_u A_j(t, x_0 + \varepsilon^{1-h} x, \phi_\varepsilon)\, \dot u\big)\partial_{x_j}\phi_\varepsilon - \partial_u F(t, x_0 + \varepsilon^{1-h} x, \phi_\varepsilon)\dot u$$

with $\phi_\varepsilon := \phi(t, x_0 + \varepsilon^{1-h} x)$. In (3.7), the source $\dot F$ contains non-linear terms:

$$\dot F = G_0(\varepsilon, t, x, \dot u) \cdot (\dot u, \dot u) + \sum_{j=1}^d G_{1j}(\varepsilon, t, x, \dot u) \cdot (\dot u, \partial_{x_j} \dot u), \tag{3.8}$$

where $(u, v) \mapsto G_0(\varepsilon, t, x, \dot u) \cdot (u, v)$ and $(u, v) \mapsto G_{1j}(\varepsilon, t, x, \dot u) \cdot (u, v)$ are bilinear, defined as

$$G_0(\varepsilon, t, x, \dot u) := -\int_0^1 (1-\tau)\Big(\sum_{j=1}^d \partial_u^2 A_j(\phi_\varepsilon + \tau \dot u)\partial_{x_j}\phi_\varepsilon - \partial_u^2 F(\phi_\varepsilon + \tau \dot u)\Big)\, d\tau,$$

$$G_{1j}(\varepsilon, t, x, \dot u) := -\int_0^1 \partial_u A_j(\phi_\varepsilon + \tau \dot u)\, d\tau.$$

Above we have omitted the arguments $(t, x_0 + \varepsilon^{1-h} x)$ of $\partial_u^k A_j$ and $\partial_u^2 F$. In this proof, a perturbative analysis around ϕ at (x_0, ξ_0), we will treat $\dot F$ as a small source, and $\dot B$ as a small perturbation of the principal symbol.

3.5. *A priori bound*

The goal is to prove the instability estimate (3.3). We work by contradiction, as we assume that there exists $C > 0$ such that for all $t \in [0, \varepsilon^h T(\varepsilon)]$,

$$\|u^\varepsilon(t) - \phi(t)\|_{W^{1,\infty}(B(x_0, \varepsilon^{1-h}\delta))} \le C \|u^\varepsilon(0) - \phi(0)\|^\alpha_{H^m(U)},$$

uniformly in (ε, t), for $0 \le t \le \varepsilon^h T(\varepsilon) = (\varepsilon T_\star |\ln \varepsilon|)^{1/(1+\ell)}$. By choice of the initial datum (3.2), this implies (see the proof of Lemma 3.2)

$$\|u^\varepsilon(t) - \phi(t)\|_{W^{1,\infty}(B(x_0, \varepsilon^{1-h}\delta))} \le C \varepsilon^{\alpha(K - m + (1-h)d/2)}, \tag{3.9}$$

with a possibly different constant $C > 0$, for all $t \le \varepsilon^h T(\varepsilon)$. By definition of $\dot u$, this yields

$$\|\dot u(t)\|_{W^{1,\infty}(B(0,\delta))} \le C \varepsilon^{K'} \quad \text{for } t \le \varepsilon^h T(\varepsilon), \tag{3.10}$$

with

$$K' := \alpha(K - m) - (1-\alpha)(1-h)d/2. \tag{3.11}$$

By condition (3.4), $K' > K/2$.

3.6. *Uniform remainders*

The linear propagator in (3.7) will undergo many transformations in this proof, through linear changes of variables corresponding to projections, localizations, conjugations, and so on. Every change of variable produces error terms. We will henceforth denote by $\mathbf{R}_k$, for $k \in \mathbb{Z}$, any bounded family $\mathbf{R}_k(\varepsilon, t)$ in S^k, in the sense that

$$\sup_{\substack{0<\varepsilon<\varepsilon_0 \\ 0 \le t \le \varepsilon^h T(\varepsilon)}} \|\mathbf{R}_k(\varepsilon, t)\|_{k,r} < \infty \tag{3.12}$$

for r large enough, with $\|\cdot\|_{k,r}$ for symbols introduced in (B.2) of Appendix B. For $k = 0$, we say that a symbol belongs to $\mathbf{R}_0$ if either (3.12) holds with $k = 0$ or

$$\sup_{\substack{0<\varepsilon<\varepsilon_0 \\ 0 \le t \le \varepsilon^h T(\varepsilon)}} \sum_{|\alpha| \le d+1} \sup_{\xi \in \mathbb{R}^d} |\partial_x^\alpha \mathbf{R}_0(\varepsilon, t)|_{L^1(\mathbb{R}^d_x)} < \infty. \tag{3.13}$$

By Proposition B.1, the corresponding operators $\mathrm{op}_\varepsilon(\mathbf{R}_k)$ are bounded $H^k \to L^2$:

$$\|\mathbf{R}_k w\|_{L^2} \lesssim \|w\|_{\varepsilon,k}, \qquad \|w\|_{\varepsilon,s} := \|(1+|\varepsilon^h \xi|^2)^{s/2} \hat{w}(\xi)\|_{L^2(\mathbb{R}^d_\xi)},$$

uniformly in $0 < \varepsilon < \varepsilon_0$ and $0 \le t \le \varepsilon^h T(\varepsilon)$. The notation $\lesssim$ was introduced in (2.12).

3.7. *Spatial localization and projection*

The matrix-valued symbol $Q(x,\xi)$, introduced in Assumption 2.1, is smooth, locally defined and invertible around (x_0,ξ_0). As explained in Appendix C, we may extend Q smoothly to a globally defined symbol of order zero, which is globally invertible, with $Q^{-1} \in S^0$. In the following, we identify Q with its extension and let

$$u^\flat(t,x) := \mathrm{op}_\varepsilon(Q_\varepsilon)(\theta \dot{u}), \tag{3.14}$$

corresponding to a spatial localization followed by a micro-local change of basis. In (3.14), the function $\theta = \theta(x)$ is the spatial truncation introduced in Section 3.1, and we set

$$Q_\varepsilon(t,x,\xi) := Q(t, x_0 + \varepsilon^{1-h} x, \xi). \tag{3.15}$$

Here $\mathrm{op}_\varepsilon(\cdot)$ denotes a pseudo-differential operator in ε^h-semiclassical quantization, as in (3.1). Classical results on pseudo-differential calculus are gathered in Appendix B. In particular, $\mathrm{op}_\varepsilon(Q_\varepsilon)$ maps L^2 to L^2, uniformly in ε, so that

$$\|u^\flat\|_{L^2} \lesssim \|\theta \dot{u}\|_{L^2} \lesssim \|\dot{u}\|_{L^2(B(0,\delta))}. \tag{3.16}$$

We now deduce from equation (3.7) for $\dot{u}$ an equation for $u^\flat$, via the change of unknown (3.14). Here we note that the leading, first-order term in (3.7) is

$$A(t,\, x_0 + \varepsilon^{1-h} x,\, \varepsilon^h \partial_x) = \mathrm{op}_\varepsilon(i A_\varepsilon), \qquad A_\varepsilon := A(t, x_0 + \varepsilon^{1-h} x, \xi).$$

Thus the equation for $u^\flat$ is

$$\begin{aligned} &\partial_t u^\flat + \varepsilon^{-1} \mathrm{op}_\varepsilon(Q_\varepsilon)\, \mathrm{op}_\varepsilon(i A_\varepsilon)(\theta \dot{u}) + \mathrm{op}_\varepsilon(Q_\varepsilon)(\theta \dot{B} \dot{u}) - \mathrm{op}_\varepsilon((\partial_t Q)_\varepsilon)(\theta \dot{u}) \\ &\quad = \mathrm{op}_\varepsilon(Q_\varepsilon)(\theta \dot{F}) - \sum_{j=1}^d \mathrm{op}_\varepsilon(Q_\varepsilon)(A_j(\phi_\varepsilon) \dot{u} \partial_{x_j} \theta). \end{aligned}$$

At this point the goal is to express the terms in $\theta \dot{u}$ above in the form of terms in $u^\flat$, modulo small errors—that is, to approximately invert (3.14). This is done as follows.

By composition of pseudo-differential operators with slow x-dependence (Proposition B.3),

$$\mathrm{Id} = \mathrm{op}_\varepsilon(Q_\varepsilon^{-1})\, \mathrm{op}_\varepsilon(Q_\varepsilon) + \varepsilon\, \mathrm{op}_\varepsilon(\mathbf{R}_{-1}), \tag{3.17}$$

where $\mathbf{R}_{-1}$ is a uniform remainder in the sense of Section 3.6. With (3.17) we may thus express $\theta \dot{u}$ in terms of $u^\flat$:

$$\theta \dot{u} = \mathrm{op}_\varepsilon(Q_\varepsilon^{-1}) u^\flat + \varepsilon\, \mathrm{op}_\varepsilon(\mathbf{R}_{-1})(\theta \dot{u}). \tag{3.18}$$

Using inductively (3.17) and composition of pseudo-differential operators (Proposition B.2), we obtain

$$\theta\dot{u} = \mathrm{op}_\varepsilon(\mathbf{R}_0)u^\flat + \varepsilon^n \,\mathrm{op}_\varepsilon(\mathbf{R}_0)(\theta\dot{u}) \tag{3.19}$$

for n as large as allowed by the regularity of ϕ.

By (3.18), the first-order term in the above equation for $u^\flat$ is

$$\begin{aligned}\mathrm{op}_\varepsilon(Q_\varepsilon)\,\mathrm{op}_\varepsilon(iA_\varepsilon)(\theta\dot{u}) &= \mathrm{op}_\varepsilon(Q_\varepsilon)\,\mathrm{op}_\varepsilon(iA_\varepsilon)\,\mathrm{op}_\varepsilon(Q_\varepsilon^{-1})u^\flat \\ &\quad + \varepsilon\,\mathrm{op}_\varepsilon(Q_\varepsilon)\,\mathrm{op}_\varepsilon(iA_\varepsilon)\,\mathrm{op}_\varepsilon(\mathbf{R}_{-1})(\theta\dot{u}),\end{aligned}$$

implying, by Proposition B.3,

$$\mathrm{op}_\varepsilon(Q_\varepsilon)\,\mathrm{op}_\varepsilon(iA_\varepsilon)(\theta\dot{u}) = \mathrm{op}_\varepsilon(iQ_\varepsilon A_\varepsilon Q_\varepsilon^{-1})u^\flat + \varepsilon\,\mathrm{op}_\varepsilon(\mathbf{R}_0)(\theta\dot{u}).$$

Moreover, by (3.19), we may write

$$\varepsilon^h\big(\mathrm{op}_\varepsilon(Q_\varepsilon)\big(\theta\dot{B}\dot{u}\big) - \mathrm{op}_\varepsilon((\partial_t Q)_\varepsilon)(\theta\dot{u})\big) = \varepsilon^h\,\mathrm{op}_\varepsilon(B^\flat)u^\flat + \varepsilon^n\,\mathrm{op}_\varepsilon(\mathbf{R}_0)(\theta\dot{u}),$$

where $B^\flat \in \mathbf{R}_0$. From the above, the equation for $u^\flat$ becomes

$$\partial_t u^\flat + \varepsilon^{-1}\,\mathrm{op}_\varepsilon(iQ_\varepsilon A_\varepsilon Q_\varepsilon^{-1})u^\flat + \mathrm{op}_\varepsilon(B^\flat)u^\flat = F^\flat, \tag{3.20}$$

where

$$B^\flat \in \mathbf{R}_0, \qquad F^\flat := \mathrm{op}_\varepsilon(Q_\varepsilon)(\theta\dot{F}) - \sum_{j=1}^{d}\mathrm{op}_\varepsilon(Q_\varepsilon)(A_j(\phi_\varepsilon)\dot{u}\partial_{x_j}\theta) + \varepsilon^n\,\mathrm{op}_\varepsilon(\mathbf{R}_0)(\theta\dot{u}). \tag{3.21}$$

3.8. *Advected coordinates*

Let M be the flow of $\mathrm{op}_\varepsilon(i\mu_\varepsilon)$ (or rather, as argued at the beginning of Section 3.7, of $\mathrm{op}_\varepsilon(\tilde{\mu}_\varepsilon)$, where $\tilde{\mu}$ is a globally defined symbol extending μ) in the sense that

$$\partial_t M = \mathrm{op}_\varepsilon(i\mu_\varepsilon)M, \qquad M(\tau;\tau) \equiv \mathrm{Id},$$

where the symbol μ is introduced in Assumption 2.1, and μ_ε is defined from μ by rescaling space as in (3.15). Let $M^\star$ be the associated backward flow, defined by

$$\partial_\tau M^\star = -M^\star\,\mathrm{op}_\varepsilon(i\mu_\varepsilon), \qquad M^\star(\tau;\tau) \equiv \mathrm{Id}.$$

By hyperbolicity (reality and regularity of μ, and Proposition B.1), both M and $M^\star$ map L^2 to L^2, uniformly in ε, t, for $t \le \varepsilon^h T(\varepsilon)$. Egorov's lemma (see for instance [11, Theorem 4.7.8] or [21, Theorem 8.1]) states that

$$M^\star M = \mathrm{Id} + \varepsilon\,\mathrm{op}_\varepsilon(\mathbf{R}_{-1}), \qquad MM^\star = \mathrm{Id} + \varepsilon\,\mathrm{op}_\varepsilon(\mathbf{R}_{-1}), \tag{3.22}$$

where we recall that $\mathbf{R}_{-1}$ is a generic notation for bounded symbols of order -1 (see Section 3.6); in other words, the equalities in (3.22) really mean that both $M^\star M - \mathrm{Id}$ and

$MM^\star - \mathrm{Id}$ belong to the class of operators of the form $\varepsilon\,\mathrm{op}_\varepsilon(\mathbf{R}_{-1})$. By Egorov's lemma, given $a \in S^m$, we also have

$$M^\star\,\mathrm{op}_\varepsilon(a_\varepsilon)M = \mathrm{op}_\varepsilon(a_{\varepsilon\star}) + \varepsilon\,\mathrm{op}_\varepsilon(\mathbf{R}_{m-1}), \tag{3.23}$$

where $a_{\varepsilon\star}$ denotes the symbol a_ε evaluated along the bicharacteristic flow, in the following sense: for any symbol[10] b we denote

$$b_\star(t,x,\xi) := b(t, x_\star(t,x,\xi), \xi_\star(t,x,\xi)), \tag{3.24}$$

where $(x_\star, \xi_\star)$ is the bicharacteristic flow of μ_ε. What is more, the remainder $\mathbf{R}_{m-1}$ above has an expansion

$$\mathrm{op}_\varepsilon(\mathbf{R}_{m-1}) = \mathrm{op}_\varepsilon(a_{\star 1}) + \cdots + \varepsilon^n\,\mathrm{op}_\varepsilon(a_{\star n}) + \varepsilon^{n+1}\,\mathrm{op}_\varepsilon(\mathbf{R}_{m-n-1}), \tag{3.25}$$

for n as large as allowed by the regularity of μ, a and ϕ, where $a_{\star i} \in S^{m-i}$ has support included in the support of $a_\star$. Identities (3.23) and (3.25) also hold if M and $M^\star$ are interchanged, with backward bicharacteristics replacing forward bicharacteristics. By (3.22),

$$M^\star\,\mathrm{op}_\varepsilon(\mathbf{R}_0) = M^\star\,\mathrm{op}_\varepsilon(\mathbf{R}_0)(MM^\star + \varepsilon\,\mathrm{op}_\varepsilon(\mathbf{R}_{-1})),$$

implying, by (3.23),

$$M^\star\,\mathrm{op}_\varepsilon(\mathbf{R}_0) = \mathrm{op}_\varepsilon(\mathbf{R}_{0\star})M^\star + \varepsilon\big(M^\star\,\mathrm{op}_\varepsilon(\mathbf{R}_{-1}) + \mathrm{op}_\varepsilon(\mathbf{R}_{-1})M^\star\big),$$

and reasoning inductively we arrive at

$$M^\star\,\mathrm{op}_\varepsilon(\mathbf{R}_0) = \mathrm{op}_\varepsilon(\mathbf{R}_0)M^\star + \varepsilon^n\,\mathrm{op}_\varepsilon(\mathbf{R}_0), \tag{3.26}$$

where n is as large as allowed by the regularity of ϕ. The advected variable is defined as

$$u_\star := M^\star(0;t)u^\flat. \tag{3.27}$$

Then

$$\partial_t u_\star = M^\star(\partial_t - \mathrm{op}_\varepsilon(i\mu_\varepsilon))u^\flat,$$

and $\partial_t u^\flat$ is given by equation (3.20) for $u^\flat$. Using (3.22), we find

$$\begin{aligned} M^\star\,\mathrm{op}_\varepsilon\big(iQ_\varepsilon(A_\varepsilon - \mu_\varepsilon)Q_\varepsilon^{-1}\big)u^\flat &= \mathrm{op}_\varepsilon\big((iQ_\varepsilon(A_\varepsilon - \mu_\varepsilon)Q_\varepsilon^{-1})_\star\big)u_\star \\ &\quad + \varepsilon\,\mathrm{op}_\varepsilon(\mathbf{R}_0)u_\star - \varepsilon M^\star\,\mathrm{op}_\varepsilon(Q_\varepsilon A_\varepsilon Q_\varepsilon^{-1})\,\mathrm{op}_\varepsilon(\mathbf{R}_{-1})u^\flat, \end{aligned}$$

with notation (3.24). Thus, by (3.26),

$$\begin{aligned} M^\star\,\mathrm{op}_\varepsilon\big(iQ_\varepsilon(A_\varepsilon - \mu_\varepsilon)Q_\varepsilon^{-1}\big)u^\flat &= \mathrm{op}_\varepsilon\big((iQ_\varepsilon(A_\varepsilon - \mu_\varepsilon)Q_\varepsilon^{-1})_\star\big)u_\star \\ &\quad + \varepsilon\,\mathrm{op}_\varepsilon(\mathbf{R}_0)u_\star + \varepsilon^n\,\mathrm{op}_\varepsilon(\mathbf{R}_0)u^\flat. \end{aligned}$$

[10] Except for A, as seen in the definition of $A_\star$ in (2.4).

Moreover, in view of (3.26), the order-zero term $B^\flat$ in (3.20) contributes to the equation for $u_\star$ the terms

$$M^\star(0;t)\,\mathrm{op}_\varepsilon(B^\flat)u^\flat = \mathrm{op}_\varepsilon(B_\star)u_\star + \varepsilon^n\,\mathrm{op}_\varepsilon(\mathbf{R}_0)u^\flat, \qquad B_\star \in \mathbf{R}_0.$$

The equation in $u_\star$ thus becomes

$$\partial_t u_\star + \varepsilon^{-1}\,\mathrm{op}_\varepsilon\big((iQ_\varepsilon(A_\varepsilon - \mu_\varepsilon)Q_\varepsilon^{-1})_\star\big)u_\star + \mathrm{op}_\varepsilon(B_\star)u_\star = F_\star, \tag{3.28}$$

where

$$B_\star \in \mathbf{R}_0, \qquad F_\star := M^\star F^\flat + \varepsilon^n\,\mathrm{op}_\varepsilon(\mathbf{R}_0)u^\flat. \tag{3.29}$$

The symbol $(Q_\varepsilon(A_\varepsilon - \mu_\varepsilon)Q_\varepsilon^{-1})_\star$ equals $Q_\varepsilon(A_\varepsilon - \mu_\varepsilon)Q_\varepsilon^{-1}$ evaluated along the bicharacteristics of μ_ε, as defined in (3.24).

3.9. Frequency, space, and time truncation functions

We introduce frequency cut-offs $\chi_0^\flat$, $\tilde\chi_0$, χ_0, spatial cut-offs $\theta_0^\flat$, $\tilde\theta_0$, θ_0, and temporal cut-offs $\psi_0^\flat$, $\tilde\psi_0$, ψ_0. All are smooth and take values in [0, 1]. Given two cut-offs ψ_1 and ψ_2,

$$\psi_1 \prec \psi_2 \quad \text{means} \quad (1-\psi_2)\psi_1 \equiv 0. \tag{3.30}$$

Equivalently, $\psi_1 \prec \psi_2$ when $\psi_2 \equiv 1$ on the support of ψ_1.

The frequency cut-offs $\chi_0^\flat$, $\tilde\chi_0$, χ_0 are all assumed to be supported in the ball $\{|\xi| \le \delta\}$. All three are identically 1 on a neighborhood of $\xi = 0$. Furthermore, $\chi_0^\flat \prec \tilde\chi_0 \prec \chi_0$.

The spatial cut-offs $\theta_0^\flat$, $\tilde\theta_0$, θ_0 are all assumed to be supported in $\{|x| \le \delta\}$. All three are identically 1 on a neighborhood of $x = 0$. Moreover, $\theta_0^\flat \prec \tilde\theta_0 \prec \theta_0$. We assume in addition $\theta_0 \prec \theta$, where θ is the spatial cut-off of Section 3.7.

The temporal cut-offs $\psi_0^\flat$, $\tilde\psi_0$, ψ_0 are non-decreasing, supported in $\{t \ge -\delta\}$, and identically 1 in a neighborhood of $\{t \ge 0\}$. In particular, $\tilde\psi_0 \equiv 1$ on $\{t \ge -\delta/3\}$. We have $\psi_0^\flat \prec \tilde\psi_0 \prec \psi_0$. The truncation $\tilde\psi_0$ is pictured in Figure 6.

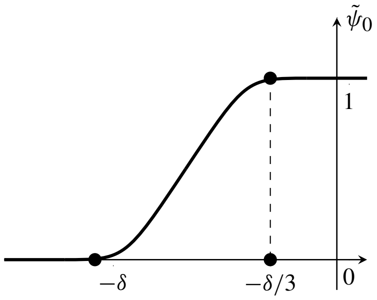

Fig. 6. The truncation function $\tilde\psi_0$.

Associated with these cut-offs, define

$$\chi(\varepsilon, t, x, \xi) := \chi_0\Big(\frac{\xi - \xi_0}{\varepsilon^\zeta}\Big)\theta_0(x)\psi_0(t - t_\star(\varepsilon, x, \xi)), \tag{3.31}$$

and define similarly $\chi^\flat$ and $\tilde{\chi}$ in terms of $\chi_0^\flat, \theta_0^\flat, \psi_0^\flat$ and $\tilde{\chi}_0, \tilde{\theta}_0, \tilde{\psi}_0$ respectively. In (3.31), the transition function $t_\star$ is defined in (2.1), which we reproduce here:

$$t_\star(\varepsilon, x, \xi) = \varepsilon^{-h}\theta_\star(\varepsilon^{1-h}x, \xi) \quad \text{with} \quad \theta_\star \geq 0, \ \theta_\star(0, \xi_0) = 0, \ \nabla_{x,\xi}\theta_\star(0, \xi_0) = 0.$$

Recall that the value of ζ is fixed in Assumption 2.1, depending on ℓ: $\zeta = 0$ if $\ell = 0$ or $\ell = 1$, and $\zeta = 1/3$ if $\ell = 1/2$.

Lemma 3.3. *The support of χ is a neighborhood of the elliptic domain $\mathcal{D}$ defined in* (2.2)*, and*

$$|\partial_x^\alpha \partial_\xi^\beta \chi| \lesssim \varepsilon^{-|\beta|\zeta},$$

where ζ is introduced in Section 2.1.1, in particular, $\zeta < h$. The same holds of course for the other truncations $\chi^\flat$ and $\tilde{\chi}$, which satisfy $\chi^\flat \prec \tilde{\chi} \prec \chi$.

Proof. For (t, x, ξ) to belong to the support of χ, we need to have simultaneously $|\xi - \xi_0| \leq \varepsilon^\zeta$, $|x| \leq \delta$, and $t_\star(\varepsilon, x, \xi) - \delta \leq t$. This defines a neighborhood of $\mathcal{D}$ (precisely, of the projection of $\mathcal{D}$ onto the (t, x, ξ) domain).

We may now assume that $t_\star$ is not identically zero, otherwise χ is not stiff. Then (see Section 2.1.1) $\ell = 1/2$, $h = 2/3$, $\zeta = 1/3$. By the Faà di Bruno formula

$$\partial_\xi^\beta\{\psi_0(t - t_\star(\varepsilon, x, \xi))\} = \sum_{\substack{1 \leq k \leq |\beta| \\ \beta_1 + \cdots + \beta_k = \beta}} C_{(\beta_k)}\psi_0^{(k)}(t - t_\star)\prod_{j=1}^{k}(\varepsilon^{-h}\partial_\xi^{\beta_j}\theta_\star),$$

where $C_{(\beta_k)}$ are positive constants. We note that $\partial_\xi^\alpha\theta_\star(\varepsilon, x, \xi) = O(\varepsilon^\zeta)$ if $|\alpha| = 1$ and ξ belongs to the support of χ_0, by assumption on $\theta_\star$, while $\partial_\xi^\alpha\theta_\star = O(1)$ if $|\alpha| \geq 2$.

Consider the case of a decomposition of β into a sum of β_j's of length one. By the above formula and the bound on $\nabla_\xi\theta_\star$, the corresponding bound is $\varepsilon^{-|\beta|(h-\zeta)}$.

If β is decomposed into $\beta_1 + \cdots + \beta_k$ with $|\beta_1| = 2$ and $|\beta_j| = 1$ for $j \geq 2$, then $k = |\beta| - 1$. The corresponding bound is $\varepsilon^{-h+(|\beta|-2)(h-\zeta)} \leq \varepsilon^{-|\beta|(h-\zeta)}$ as soon as $\zeta \leq h/2$, which holds true. It is now easy to verify that the decomposition of $|\beta|$ into sums of multi-indices of length one corresponds to the worst possible loss in powers of ε.

We turn to x-derivatives of $\psi_0(t - t_\star)$. By assumption, $\theta_\star$ is a function of $\varepsilon^{2(1-h)}(x, x)$ and $\varepsilon^{1-h}(x, \xi)$. Thus x-derivatives bring in either powers of $\varepsilon^{-h+2(1-h)} = 1$, since $h = 2/3$, or powers of $\varepsilon^{-h+1-h+\zeta} = 1$, since $\zeta = h/2 = 1/3$.

Thus $|\partial_x^\alpha\partial_\xi^\beta\psi_0(t - t_\star)| \lesssim \varepsilon^{-|\beta|(h-\zeta)}$ if $\ell = 1/2$. Considering finally the full truncation function χ, we observe that the term in χ_0 contributes the exact same loss per ξ-derivative, and the spatial truncation θ_0 gives no loss. □

Corollary 3.4. *The operator* $\mathrm{op}_\varepsilon(\chi)$ *maps* $L^2(\mathbb{R}^d)$ *to* $L^2(\mathbb{R}^d)$*, uniformly in ε, and so do* $\mathrm{op}_\varepsilon(\chi^\flat)$ *and* $\mathrm{op}_\varepsilon(\tilde{\chi})$.

Proof. Since χ is compactly supported in x, we may use pointwise bounds for $\partial_x^\alpha\chi$ and the bound (B.5) of Proposition B.1. The result then follows from Lemma 3.3. □

3.10. Localization in the elliptic zone and rescaling in time

We define

$$v := \mathrm{op}_\varepsilon(\tilde{\chi}(t))(u_\star(\varepsilon^h t)), \tag{3.32}$$

meaning that we first rescale time in $u_\star$ and then apply $\mathrm{op}_\varepsilon(\tilde{\chi})$ evaluated at t, where $\tilde{\chi}$ is defined just below (3.31). We now derive an equation for v, based on equation (3.28) for $u_\star$.

Consider first the leading, first-order term in (3.28). When evaluated at $\varepsilon^h t$, its symbol is precisely (for $A_\star$ see (2.4)) the rescaled and advected symbol for which Assumption 2.1 holds:

$$\big((i Q_\varepsilon(A_\varepsilon - \mu_\varepsilon)Q_\varepsilon^{-1})_\star\big)(\varepsilon^h t) = A_\star(\varepsilon, t, x, \xi).$$

Thus

$$\mathrm{op}_\varepsilon(\tilde{\chi})\Big(\mathrm{op}_\varepsilon\big((i Q_\varepsilon(A_\varepsilon - \mu_\varepsilon)Q_\varepsilon^{-1})_\star\big)u_\star\Big)(\varepsilon^h t) = \mathrm{op}_\varepsilon(\tilde{\chi})\,\mathrm{op}_\varepsilon(i A_\star)(u_\star(\varepsilon^h t)).$$

Similarly, denoting $B := B_\star(\varepsilon^h t)$, where $B_\star$ is the order-zero correction to the leading symbol which appears in (3.28),

$$\mathrm{op}_\varepsilon(\tilde{\chi})(\mathrm{op}_\varepsilon(B_\star)u_\star)(\varepsilon^h t) = \mathrm{op}_\varepsilon(\tilde{\chi})\,\mathrm{op}_\varepsilon(B)(u_\star(\varepsilon^h t)).$$

We now introduce a commutator:

$$\mathrm{op}_\varepsilon(\tilde{\chi})\,\mathrm{op}_\varepsilon(i A_\star + \varepsilon B)(u_\star(\varepsilon^h t)) = \mathrm{op}_\varepsilon(i A_\star + \varepsilon B)v + \tilde{\Gamma}(u_\star(\varepsilon^h t)),$$

where

$$\tilde{\Gamma} := [\mathrm{op}_\varepsilon(\tilde{\chi}), \mathrm{op}_\varepsilon(i A_\star + \varepsilon B)]. \tag{3.33}$$

By definition of v,

$$\partial_t v = \mathrm{op}_\varepsilon(\partial_t \tilde{\chi})(u_\star(\varepsilon^h t)) + \varepsilon^h\,\mathrm{op}_\varepsilon(\tilde{\chi})((\partial_t u_\star)(\varepsilon^h t)).$$

Together with (3.28), this implies

$$\partial_t v + \varepsilon^{h-1}\,\mathrm{op}_\varepsilon(i A_\star + \varepsilon B)v = \mathrm{op}_\varepsilon(\partial_t \tilde{\chi})(u_\star(\varepsilon^h t)) - \varepsilon^{h-1}\tilde{\Gamma}(u_\star(\varepsilon^h t)) + \varepsilon^h\,\mathrm{op}_\varepsilon(\tilde{\chi})F_\star.$$

We will handle the right-hand side as a remainder. The following lemma shows that we may introduce a truncation function in the above principal symbol.

Lemma 3.5. *For any bounded family $P(\varepsilon, t) \in S^1$, for χ defined in (3.31) and v defined in (3.32),*

$$\mathrm{op}_\varepsilon(P)v = \mathrm{op}_\varepsilon(\chi P)v + [\mathrm{op}_\varepsilon(P), \mathrm{op}_\varepsilon(\chi)]v + \varepsilon^n(\mathrm{op}_\varepsilon(\mathbf{R}_0)u_\star)(\varepsilon^h t), \tag{3.34}$$

where $\mathbf{R}_0$ is a uniform remainder in the sense of Section 3.6.

Proof. By definition of $\tilde{\chi}$ and χ, and Proposition B.2, for any $n' \in \mathbb{N}^*$,

$$\operatorname{op}_\varepsilon(\tilde{\chi}) = \operatorname{op}_\varepsilon(\chi\tilde{\chi}) = \operatorname{op}_\varepsilon(\chi)\operatorname{op}_\varepsilon(\tilde{\chi}) + \varepsilon^{n'h}\operatorname{op}_\varepsilon(R_{n'}(\chi,\tilde{\chi})),$$

and the remainder satisfies

$$\|\operatorname{op}_\varepsilon(R_{n'}(\chi,\tilde{\chi}))\|_{L^2\to\|\cdot\|_{\varepsilon,-n'}} \lesssim \|\partial_\xi^{n'}\chi\|_{0,C(d)}\|\partial_x^{n'}\tilde{\chi}\|_{0,C(d)}.$$

We use here the norms $\|\cdot\|_{m,r}$ for pseudo-differential symbols of order m, as defined in (B.2). By Lemma 3.3,

$$\|\partial_\xi^{n'}\chi\|_{0,C(d)} \lesssim \varepsilon^{-(n'+C(d))\zeta} \quad \text{and} \quad \|\partial_x^{n'}\tilde{\chi}\|_{0,C(d)} \le \varepsilon^{-C(d)\zeta}.$$

Since $\zeta < h$, we have $\varepsilon^{n'h-(n'+2C(d))(h-\zeta)} \le \varepsilon^n$ for any n, if n' is chosen large enough. Thus

$$\operatorname{op}_\varepsilon(\tilde{\chi}) = \operatorname{op}_\varepsilon(\chi)\operatorname{op}_\varepsilon(\tilde{\chi}) + \varepsilon^n\operatorname{op}_\varepsilon(\mathbf{R}_0), \tag{3.35}$$

where $\mathbf{R}_0$ is bounded for $t \le T(\varepsilon)$. This implies

$$\operatorname{op}_\varepsilon(P)v = \operatorname{op}_\varepsilon(P)\operatorname{op}_\varepsilon(\tilde{\chi})u_\star(\varepsilon^h t) = \operatorname{op}_\varepsilon(P)\operatorname{op}_\varepsilon(\chi)v + \varepsilon^n\operatorname{op}_\varepsilon(\mathbf{R}_0)(u_\star(\varepsilon^h t)).$$

Now

$$\operatorname{op}_\varepsilon(P)\operatorname{op}_\varepsilon(\chi) = \operatorname{op}_\varepsilon(\chi P) + [\operatorname{op}_\varepsilon(P), \operatorname{op}_\varepsilon(\chi)],$$

and (3.34) is proved, with a symbol $\mathbf{R}_0$ which is a uniform remainder in the sense of Section 3.6, meaning that we rescale in time the remainder which appears in (3.35). □

Applying Lemma 3.5 to $P = iA_\star + \varepsilon B$, we derive the final form of the equation satisfied by v:

$$\partial_t v + \varepsilon^{h-1}\operatorname{op}_\varepsilon(\chi(iA_\star + \varepsilon B))v = \varepsilon^h g, \tag{3.36}$$

where the source term g is defined in terms of the remainder $F_\star$ from (3.28)–(3.29):

$$\begin{aligned} g = {} & \varepsilon^{-1}(\Gamma v - \tilde{\Gamma}(u_\star(\varepsilon^h t))) + \varepsilon^{-h}\operatorname{op}_\varepsilon(\partial_t\tilde{\chi})(u_\star(\varepsilon^h t)) + \operatorname{op}_\varepsilon(\tilde{\chi})(F_\star(\varepsilon^h t)) \\ & + \varepsilon^n(\operatorname{op}_\varepsilon(\mathbf{R}_0)u_\star)(\varepsilon^h t), \end{aligned} \tag{3.37}$$

with Γ defined just like $\tilde{\Gamma}$ in (3.33), but with χ in place of $\tilde{\chi}$.

The derivation of (3.36)–(3.37) ends the first step of the proof of Theorem 2.2. Our goal now is to estimate the growth in time of the solution to (3.36) over the interval $[0, T(\varepsilon)]$, where $T(\varepsilon) = (T_\star|\ln\varepsilon|)^{1/(1+\ell)}$. For this, we will first derive an integral representation formula for v.

3.11. An integral representation formula

At this point we use the theory developed in Appendix D. Theorem D.3 gives an integral representation formula for the solution v to (3.36) issued from $v(0)$:

$$v = \mathrm{op}_\varepsilon(\Sigma(0;t)))v(0) + \varepsilon^h \int_0^t \mathrm{op}_\varepsilon(\Sigma(\tau;t))(\mathrm{Id} + \varepsilon\,\mathrm{op}_\varepsilon(\mathbf{R}_0))\big(g(\tau) + \varepsilon\,\mathrm{op}_\varepsilon(\mathbf{R}_0)v(0)\big)\,d\tau, \tag{3.38}$$

where $\mathbf{R}_0$ are uniform remainders, as defined in Section 3.6, and the approximate solution operator $\mathrm{op}_\varepsilon(\Sigma(s;t))$ is defined by

$$\Sigma = \sum_{q=0}^{q_0} \varepsilon^{hq} S_q, \tag{3.39}$$

where q_0 is large enough.[11] In (3.39), the leading term S_0 is defined for $0 \le \tau \le t \le T(\varepsilon)$ and all $(x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d$ by

$$\partial_t S_0 + \varepsilon^{h-1}\chi(iA_\star + \varepsilon B)(t,x,\xi)S_0 = 0, \qquad S_0(\tau;\tau) = \mathrm{Id}, \tag{3.40}$$

and the S_q for $q \ge 1$ are correctors, defined inductively as the solutions of

$$\begin{aligned} &\partial_t S_q + \varepsilon^{h-1}\chi(iA_\star + \varepsilon B)(t,x,\xi)S_q + \varepsilon^{h-1} \sum_{\substack{q_1+q_2=q\\ q_1>0}} (\chi(iA_\star + \varepsilon B)) \sharp_{q_1} S_{q_2}, \\ &S_q(\tau;\tau) \equiv 0, \end{aligned} \tag{3.41}$$

with $\sigma_1 \sharp_n \sigma_2 := (-i)^n (n!)^{-1} \sum_{|\alpha|=n} \partial_\xi^\alpha \sigma_1 \partial_x^\alpha \sigma_2$. From (3.40) and (3.41) we deduce the representation, for $q \ge 1$,

$$S_q(\tau;t) = -\varepsilon^{h-1} \int_\tau^t S_0(\tau';t) \sum_{\substack{q_1+q_2=q\\ q_1>0}} (\chi(iA_\star + \varepsilon B))(\tau',x,\xi)\sharp_{q_1} S_{q_2}(\tau;\tau')\,d\tau'. \tag{3.42}$$

In order to be able to exploit the representation (3.38), we need to check that Assumption D.1, under which Theorem D.3 holds, is satisfied. This is the object of the forthcoming section.

3.12. Bound on the solution operator

Recall that S, the symbolic flow of $i\varepsilon^{h-1}A_\star$, is defined in (2.8), and is assumed to satisfy the bounds of Assumption 2.1. The upper bound (2.11) in Assumption 2.1 is assumed to hold for S in the domain $\mathcal{D}$ defined in (2.2):

$$\mathcal{D} := \{(\tau;t,x,\xi) : t_\star(x,\xi) \le \tau \le t \le T(\varepsilon),\ |x| \le \delta,\ |\xi - \xi_0| \le \delta\varepsilon^\zeta\},$$

where the transition time $t_\star$ is defined in (2.1) and the final observation time $T(\varepsilon)$ is defined in (2.3).

[11] Depending on ζ, the final observation time $T_\star$ (see (2.3)), and the growth function γ—see Appendix D and in particular the proof of Lemma D.2.

The goal in this section is to show that the symbolic flow S_0, which is defined as the solution of (3.40), and the correctors S_q defined as the solutions to (3.41), satisfy the bounds of Assumption D.1. This will allow us to use the representation Theorem D.3, and will justify representation (3.38). This will also give a bound for the norm of the approximate solution operator $\mathrm{op}_\varepsilon(\Sigma)$ defined in (3.39).

We are looking for bounds for the correctors S_q and their derivatives. Consider the representation (3.42). Disregarding (x, ξ)-derivatives, we see in (3.42) that S_q appears as a time integral of a product $S_0(\chi(iA_\star + \varepsilon B))S_{q_2}$ with $q_2 < q$. We may recursively use (3.42) at rank q_2, and by induction $S_q(\tau; t)$ appears as a time integral of the form

$$\varepsilon^{q(h-1)} \int_{\tau \le \tau_1 \le \cdots \le \tau_q \le t} S_0(\tau_1; t)(\chi(iA_\star + \varepsilon B))(\tau_1) S_0(\tau_2; \tau_1)(\chi(iA_\star + \varepsilon B))(\tau_2) \cdots \\ \cdots S_0(\tau_n; \tau_{n-1})(\chi(iA_\star + \varepsilon B))(\tau_n) S_0(\tau; \tau_n)\, d\tau_1 \dots d\tau_n, \qquad (3.43)$$

in which there are q occurrences of $iA_\star + \varepsilon B$ and $q + 1$ occurrences of S_0. Note again that in (3.43) we ignore (x, ξ)-derivatives. From this, it can be seen that we need to

- derive bounds on S_0 and its (x, ξ)-derivatives; these will be deduced from the bounds of S postulated in Assumption 2.1;
- derive bounds for products of S_0 with $\chi(iA_\star + \varepsilon B)$; here the block structure assumption (2.6)–(2.7) from Assumption 2.1 will come in.

3.12.1. Product bounds for S. In a first step, we prove bounds for products of symbols involving the symbolic flow S of (2.8) and the rescaled and advected principal symbol $A_\star$ of (2.4). For $\alpha, \beta \in \mathbb{N}^{2d}$ and $0 \le \tau \le t$ we denote

$$S_{\alpha,\beta}(\tau; t) := \varepsilon^{h-1} S(\tau; t) \partial_x^\alpha \partial_\xi^\beta A_\star(\tau),$$

and for $(\alpha_i, \beta_i) \in \mathbb{N}^{2d}$ and $0 \le \tau \le \tau_n \le \tau_{n-1} \le \tau_1 \le t$ we consider the products

$$\mathbf{S}_n(\tau, \tau_1, \dots, \tau_n; t) := S_{\alpha_1,\beta_1}(\tau_1; t) S_{\alpha_2,\beta_2}(\tau_2; \tau_1) \cdots S_{\alpha_n,\beta_n}(\tau_n; \tau_{n-1}) S(\tau; \tau_n). \quad (3.44)$$

Lemma 3.6. *Under Assumption 2.1,*

$$|\mathbf{S}_n(\tau, \tau_1, \dots, \tau_n; t, x, \xi)| \lesssim \begin{pmatrix} 1 & \varepsilon^{-\zeta} \\ \varepsilon^{\zeta} & 1 \end{pmatrix} \mathbf{e}_{\gamma^+}(\tau; t, x, \xi)$$

for all $n \ge 1$ and all $\alpha_i, \beta_i \in \mathbb{N}^{2d}$, all $0 \le \tau \le \tau_n \le \cdots \le \tau_1 \le t$ with $(\tau; t, x, \xi) \in \mathcal{D}$, uniformly in ε. By $\lesssim$ we mean here entrywise inequalities modulo constants, for each block of $\mathbf{S}_n$, as described below (2.12).

Proof. If all blocks of $A_\star$ satisfy (2.6), then $\zeta = 0$, and the stated bound simply follows from the multiplicative nature of the growth function, namely the identity

$$\mathbf{e}_{\gamma^+}(\tau_1; t) \mathbf{e}_{\gamma^+}(\tau; \tau_1) = \mathbf{e}_{\gamma^+}(\tau; t). \qquad (3.45)$$

Suppose then that a block $A_{\star j}$ satisfies (2.7), and consider the associated component $S_{(j)}$ of the symbolic flow (2.8). Bound (2.11) from Assumption 2.1 and the cancellation observed in the product

$$\begin{pmatrix} 1 & \varepsilon^{h-1} \\ \varepsilon^{1-h} & 1 \end{pmatrix}\begin{pmatrix} 1 & \varepsilon^{h-1} \\ \varepsilon^{1-h} & 1 \end{pmatrix} = 2\begin{pmatrix} 1 & \varepsilon^{h-1} \\ \varepsilon^{1-h} & 1 \end{pmatrix} \tag{3.46}$$

imply, after omitting the index j,

$$|S(\tau_1; t)\partial_x^\alpha \partial_\xi A_\star(\tau_1) S(\tau; \tau_1)| \lesssim \begin{pmatrix} 1 & \varepsilon^{h-1} \\ \varepsilon^{1-h} & 1 \end{pmatrix} \mathbf{e}_{\gamma^+}(\tau; t). \tag{3.47}$$

The result follows from (3.45)–(3.47) by a straightforward induction. □

3.12.2. Product bounds for $\partial_x^\alpha S$. Next we show that spatial derivatives of S and products involving $\partial_x^\alpha S$ satisfy the upper bound (2.11) from Assumption 2.1. For $\alpha, \beta, \beta' \in \mathbb{N}^d$ and $0 \le \tau \le t$ we denote

$$\tilde{S}_{\alpha,\beta}(\tau; t) := \varepsilon^{h-1}\partial_x^\alpha S(\tau; t)\partial_x^\beta \partial_\xi^{\beta'} A_\star(\tau),$$

and for $\alpha_i \in \mathbb{N}^d$, $\beta_i \in \mathbb{N}^{2d}$ and $0 \le \tau \le \tau_n \le \tau_{n-1} \le \tau_1 \le t$ we set

$$\tilde{\mathbf{S}}_n(\tau, \tau_1, \dots, \tau_n; t) := \tilde{S}_{\alpha_1,\beta_1}(\tau_1; t)\tilde{S}_{\alpha_2,\beta_2}(\tau_2; \tau_1) \cdots \tilde{S}_{\alpha_n,\beta_n}(\tau_n; \tau_{n-1})\partial_x^\alpha S(\tau; \tau_n).$$

Lemma 3.7. *Under Assumption 2.1,*

$$|\partial_x^\alpha S| + |\tilde{\mathbf{S}}_n| \lesssim \begin{pmatrix} 1 & \varepsilon^{-\zeta} \\ \varepsilon^{\zeta} & 1 \end{pmatrix} \mathbf{e}_{\gamma^+}$$

for each block of the block diagonal matrices S and $\mathbf{S}_n$. The precise meaning of $\lesssim$ is described below (2.12).

Above, and often below, the time- and space-frequency arguments are omitted. In particular, the "interior" temporal arguments of $\tilde{\mathbf{S}}_n$, namely $\tau_n, \dots, \tau_1$, are omitted. It is implicit that the τ_i are constrained only by $\tau \le \tau_n \le \tau_{n-1} \le \dots \le \tau_1 \le t$, and that the α_i, β_i and $n \ge 1$ are arbitrary.

Proof of Lemma 3.7. We first prove by induction on $|\alpha|$ that $\partial_x^\alpha S$ enjoys the representation

$$\partial_x^\alpha S = \sum_{n \le |\alpha|} \int \mathbf{S}_n, \tag{3.48}$$

where $\mathbf{S}_n$ is defined in (3.44). By (3.48), we mean precisely

$$\partial_x^\alpha S(\tau; t) = \sum_{1 \le n \le |\alpha|} C_n \int_{\tau \le \tau_1 \le \dots \le \tau_n \le t} \mathbf{S}_n(\tau, \tau_1, \dots, \tau_n; t)\, d\tau_1 \dots d\tau_n, \tag{3.49}$$

with the constants C_n independent of (ε, t, x, ξ). In the following, whenever products, such as $\mathbf{S}_n$, are integrated in time, the integration variables are the "interior" variables, as described just above this proof. In order to prove (3.48) for $|\alpha| = 1$, we apply ∂_x^α to equation (2.8) for S:

$$\partial_t \partial_x^\alpha S + i\varepsilon^{h-1} A_\star \partial_x^\alpha S = -i\varepsilon^{h-1}(\partial_x^\alpha A_\star)S.$$

This implies the representation

$$\partial_x^\alpha S(\tau;t) = -i\varepsilon^{h-1} \int_\tau^t S(\tau';t)\partial_x^\alpha A_\star(\tau')S(\tau;\tau')\,d\tau', \qquad |\alpha| = 1,$$

which takes the form (3.48). For greater values of $|\alpha|$, we have similarly

$$\partial_x^\alpha S = -i\varepsilon^{h-1} \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ |\alpha_1|>0}} \int_\tau^t S\partial_x^{\alpha_1} A_\star \partial_x^{\alpha_2} S,$$

and the induction step is straightforward. From (3.48) the bound on $\partial_x^\alpha S$ follows by the bound on $\mathbf{S}_n$ in Lemma 3.6 and the multiplicative nature of $\mathbf{e}_{\gamma^+}$. Time integrals only contribute powers of $|\ln \varepsilon|$, which are invisible in $\lesssim$ estimates. Finally, from (3.48) we deduce (using the same notational convention as in (3.48))

$$\tilde{\mathbf{S}}_n = \sum_{n' \le |\alpha_1|+\cdots+|\alpha_n|+n+|\alpha|} \int \mathbf{S}_{n'}, \tag{3.50}$$

and Lemma 3.6 applies again. □

3.12.3. Bounds for the symbolic flow S_0. Here we bound S_0, the solution to (3.40). While S is the flow of $i\varepsilon^{h-1}A_\star$, the symbol S_0 is the flow of $\varepsilon^{h-1}\chi(iA_\star + \varepsilon B)$, where χ is a stiff truncation and B is a bounded symbol of order zero. We prove here that the upper bound (2.11) in Assumption 2.1 is stable under the perturbations induced by χ and B, in the sense that S_0 and its spatial derivatives satisfy the same upper bound as S.

In a first step, we consider the solution S_χ to

$$\partial_t S_\chi + \varepsilon^{h-1}\chi A_\star S_\chi = 0, \qquad S_\chi(\tau;\tau) \equiv \mathrm{Id}. \tag{3.51}$$

Associated with S_χ, we define products $\mathbf{S}_{\chi,n}$ involving $\partial_x^\alpha S_\chi$ just like $\tilde{\mathbf{S}}_n$ was defined as a product involving $\partial_x^\alpha S$, but with $\chi A_\star$ in place of $A_\star$; explicitly,

$$S_{\chi,\alpha,\beta}(\tau;t) := \varepsilon^{h-1}\partial_x^{\alpha_1} S_\chi(\tau;t)\partial_x^{\alpha_2}\partial_\xi^\beta(\chi A_\star(\tau)), \qquad \alpha = (\alpha_1,\alpha_2) \in \mathbb{N}^{2d},\ \beta \in \mathbb{N}^d,$$

and for $\alpha_i \in \mathbb{N}^{2d}$, $\beta_i \in \mathbb{N}^d$ and $0 \le \tau \le \tau_n \le \tau_{n-1} \le \tau_1 \le t$,

$$\begin{aligned}\mathbf{S}_{\chi,n}&(\tau,\tau_1,\ldots,\tau_n;t)\\ &:= S_{\chi,\alpha_1,\beta_1}(\tau_1;t)S_{\chi,\alpha_2,\beta_2}(\tau_2;\tau_1)\cdots S_{\chi,\alpha_n,\beta_n}(\tau_n;\tau_{n-1})\partial_x^\alpha S_\chi(\tau;\tau_n).\end{aligned}$$

Corollary 3.8. *The solution S_χ to (3.51) enjoys the bounds*

$$|\partial_x^\alpha S_\chi(\tau;t,x,\xi)| \lesssim \begin{pmatrix} 1 & \varepsilon^{-\zeta} \\ \varepsilon^{\zeta} & 1 \end{pmatrix} \mathbf{e}_{\gamma^+}(\tau;t,x,\xi), \tag{3.52}$$

$$|\mathbf{S}_{\chi,n}| \lesssim \varepsilon^{-|\beta|\zeta} \begin{pmatrix} 1 & \varepsilon^{-\zeta} \\ \varepsilon^{\zeta} & 1 \end{pmatrix} \mathbf{e}_{\gamma^+}(\tau;t,x,\xi), \tag{3.53}$$

for any $0 \le \tau \le t \le T(\varepsilon)$ and any (x,ξ).

Just like the bounds of Lemma 3.7, the bounds of Lemma 3.8 are understood entrywise, *for each block* of the block diagonal matrices S_χ and $\mathbf{S}_{\chi,n}$.

While the bound of Assumption 2.1 was stated over $\mathcal{D}$, the bound of Corollary 3.9 holds for any (x,ξ) and $0 \le \tau \le t \le T(\varepsilon)$, with $T(\varepsilon)$ defined in (2.3). This comes from the truncation χ in (3.51).

Proof of Corollary 3.8. There are five cases:

• If $\tau \le t \le t_\star - \delta$, then $\chi \equiv 0$ on $[\tau, t]$ (see the definition of χ in (3.31)), implying $S_\chi = \mathrm{Id}$.

• If $\tau \le t_\star - \delta \le t$, then by the properties of the flow and the previous case,

$$S_\chi(\tau;t) = S_\chi(\tau;t_\star)S_\chi(t_\star;t) = S_\chi(t_\star;t),$$

and we are reduced to the case $t_\star \le \tau \le t \le T(\varepsilon)$.

• If $t_\star \le \tau \le t \le T(\varepsilon)$, then $\chi \equiv 1$ on $[\tau;t]$, and $S_\chi = S$ by uniqueness.

• If $t_\star - \delta \le \tau \le t \le t_\star$, comparing equation (2.8) for S with equation (3.51) for S_χ we find the representation

$$S_\chi(\tau;t) = S(\tau;t) - \varepsilon^{h-1}\int_\tau^t S(\tau';t)(1-\chi)(\tau')A_\star(\tau')S_\chi(\tau;\tau')\,d\tau',$$

and applying ∂_x^α to both sides gives

$$\begin{aligned}&\partial_x^\alpha S_\chi(\tau;t)\\ &\quad= \partial_x^\alpha S(\tau;t) - \varepsilon^{h-1}\sum\int_\tau^t \partial_x^{\alpha_1}S(\tau';t)\partial_x^{\alpha_2}\big((1-\chi)(\tau')A_\star(\tau')\big)\partial_x^{\alpha_3}S_\chi(\tau;\tau')\,d\tau',\end{aligned} \tag{3.54}$$

where the sum is over all $\alpha_1+\alpha_2+\alpha_3=\alpha$. Since we are only interested in upper bounds, we have omitted multinomial constants $C_{\alpha_i} > 0$ in (3.54). We factor out the expected growth by letting $S_\chi^\flat := \mathbf{e}_{\gamma^+}^{-1}S_\chi$, $S^\flat := \mathbf{e}_{\gamma^+}^{-1}S$. Then, by the property (3.45) of the growth function, $\partial_x^\alpha S_\chi^\flat$ solves, with the summation sign omitted,

$$\partial_x^\alpha S_\chi^\flat(\tau;t) = \partial_x^\alpha S^\flat(\tau;t) - \varepsilon^{h-1}\int_\tau^t \partial_x^{\alpha_1}S^\flat(\tau';t)\partial_x^{\alpha_2}\big((1-\chi)(\tau')A_\star(\tau')\big)\partial_x^{\alpha_3}S_\chi^\flat(\tau;\tau')\,d\tau'. \tag{3.55}$$

Now given a matrix $M = \begin{pmatrix} m_{11} & m_{12}\\ m_{21} & m_{22}\end{pmatrix}$, we let

$$\underline{M} := \begin{pmatrix} m_{11} & \varepsilon^{\zeta}m_{12}\\ \varepsilon^{-\zeta}m_{21} & m_{22}\end{pmatrix}.$$

Then we have an identity analogous to (3.46):

$$\underline{M_1M_2} = \underline{M}_1\underline{M}_2. \tag{3.56}$$

In particular, if $\underline{M}_1$ and $\underline{M}_2$ are bounded in ε, then $\underline{M_1M_2}$ is bounded in ε. Thus from (3.55)–(3.56) we deduce

$$\partial_x^\alpha \underline{S}_\chi^\flat(\tau;t) = \partial_x^\alpha \underline{S}^\flat - \int_\tau^t \partial_x^{\alpha_1}\underline{S}^\flat(\tau';t)\partial_x^{\alpha_2}\big((1-\chi)(\tau')(\varepsilon^{h-1}\underline{A}_\star(\tau'))\big)\partial_x^{\alpha_3}\underline{S}_\chi^\flat(\tau;\tau')\,d\tau'.$$

The key is now that $\varepsilon^{h-1}\partial_x^\alpha((1-\chi)\underline{A}_\star)$ is uniformly bounded in ε. Indeed, by Lemma 3.3, spatial derivatives of the truncation χ are uniformly bounded. By the block conditions (2.6)–(2.7) and the definition of $\underline{A}_\star$ just above (3.56), the matrix $\varepsilon^{h-1}\underline{A}_\star$ is uniformly bounded in ε. Further, $\underline{S}^\flat$ is uniformly bounded, by Lemma 3.7. Thus we obtain the bound

$$|\partial_x^\alpha \underline{S}_\chi^\flat(\tau;t)| \leq C\left(1+\int_\tau^t |\partial_x^{\alpha_3}\underline{S}_\chi^\flat(\tau;\tau')|\,d\tau'\right)$$

with $|\alpha_3| \leq |\alpha|$, for some $C>0$, implying by Gronwall and a straightforward induction

$$|\partial_x^\alpha \underline{S}_\chi^\flat(\tau;t)| \leq Ce^{C(t-\tau)}, \qquad t_\star-\delta \leq \tau \leq t \leq t_\star,$$

which is good enough since $t-\tau \leq \delta$. Going back to S_χ, we find the bound (3.52).

• The same arguments apply in the remaining case $t_\star-\delta \leq \tau \leq t_\star \leq t$.

At this point, (3.52) is proved and we turn to (3.53). First consider products involving no spatial derivatives of S_χ, which we denote $\mathbf{S}_{\chi,n}$, for consistency with (3.44). Here we note that the proof of Lemma 3.6 uses only the upper bound (2.11) for S, via the cancellation (3.46). We may thus repeat that proof and derive a bound for $\mathbf{S}_{\chi,n}$. The only difference is that while $A_\star$ is uniformly bounded in ε, the truncation function χ is stiff in ξ, as seen in (3.31) and the definition of $t_\star$ in (2.1), and reflected in Lemma 3.3. This gives the bound (3.53) with $\mathbf{S}_{\chi,n}$ in place of $\tilde{\mathbf{S}}_{\chi,n}$.

Finally, spatial derivatives are well behaved, in the sense that they are bounded without loss in ε. Thus from the bound in $\mathbf{S}_{\chi,n}$ we derive the bound (3.53) exactly as in the proof of Lemma 3.7, via a representation formula identical to (3.50), with $\tilde{\mathbf{S}}_{\chi,n}$ and $\mathbf{S}_{\chi,n}$ in place of $\tilde{\mathbf{S}}_n$ and $\mathbf{S}_n$. □

Corollary 3.9. *The solution S_0 to (3.40) enjoys the bound*

$$|\partial_x^\alpha S_0(\tau;t,x,\xi)| \lesssim \varepsilon^{-\zeta}\mathbf{e}_{\gamma^+}(\tau;t,x,\xi)$$

for any $0 \leq \tau \leq t \leq T(\varepsilon)$ and any (x,ξ).

Proof. From equations (3.51) for $\tilde{S}_\chi$ and (3.40) for S_0, we deduce the representation

$$\partial_x^\alpha S_0(\tau;t) = \partial_x^\alpha S_\chi(\tau;t) + \varepsilon^h\int_\tau^t \partial_x^{\alpha_1}S_\chi(\tau';t)\partial_x^{\alpha_2}(\chi B(\tau'))\partial_x^{\alpha_3}S_0(\tau;\tau')\,d\tau', \quad (3.57)$$

with an implicit summation over $\alpha_1+\alpha_2+\alpha_3=\alpha$, and implicit multinomial constants $C_{\alpha_i}>0$. We factor out the expected growth before applying Gronwall's lemma, by letting $S_0^\flat := \mathbf{e}_{\gamma^+}^{-1}S_0$ and $S_\chi^\flat := \mathbf{e}_{\gamma^+}^{-1}S_\chi$. From Corollary 3.8, we know that $|\partial_x^\alpha S_\chi| \lesssim \varepsilon^{-\zeta}$. Moreover, from Lemma 3.3 and the fact that $B \in \mathbf{R}_0$, the symbol $\partial_x^{\alpha_2}(\chi B)$ is uniformly bounded. Thus from (3.57) we deduce

$$|\partial_x^\alpha S_0^\flat(\tau;t)| \lesssim \varepsilon^{-\zeta} + \varepsilon^{h-\zeta}\int_\tau^t |\partial_x^{\alpha_3}S_0^\flat(\tau;\tau')|\,d\tau', \qquad |\alpha_3| \leq |\alpha|.$$

We may now conclude the proof by Gronwall's lemma and a straightforward induction, since $T(\varepsilon)$ grows at most logarithmically and $h-\zeta>0$. □

3.12.4. Product bounds for S_0. The next step is to prove product bounds for $\partial_x^\alpha S_0$, as in Lemma 3.7. For this, for $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^{2d}$, $\beta \in \mathbb{N}^d$ and $0 \le \tau \le t$ we let

$$S_{0,\alpha,\beta}(\tau; t) := \varepsilon^{h-1} \partial_x^{\alpha_1} S_0(\tau; t) \partial_x^{\alpha_2} \partial_\xi^\beta (\chi A_\star(\tau)),$$

and for $\alpha_i \in \mathbb{N}^{2d}$, $\beta_i \in \mathbb{N}^d$ and $0 \le \tau \le \tau_n \le \tau_{n-1} \le \tau_1 \le t$ we write

$$\mathbf{S}_{0,n}(\tau, \tau_1, \ldots, \tau_n; t) := S_{0,\alpha_1,\beta_1}(\tau_1; t) S_{0,\alpha_2,\beta_2}(\tau_2; \tau_1) \cdots S_{0,\alpha_n,\beta_n}(\tau_n; \tau_{n-1}) \partial_x^\alpha S_0(\tau; \tau_n). \tag{3.58}$$

Corollary 3.10. *We have*

$$|\mathbf{S}_{0,n}| \lesssim \varepsilon^{-\zeta(1+|\beta|)} \mathbf{e}_{\gamma^+} \tag{3.59}$$

for any $0 \le \tau \le \tau_n \le \tau_{n-1} \le \cdots \le \tau_1 \le t \le T(\varepsilon)$*, any* (x, ξ)*, any* α_i*,* β_i *and any* $n \ge 1$*, with* $|\beta| = \sum_i |\beta_i|$.

Proof. First observe that for $n = 1$, if we used Corollary 3.9 directly, we would find the upper bound

$$\varepsilon^{h-1} |\partial_x^{\alpha_1} S_0(\tau'; t) \partial_x^{\alpha_2} A_\star(\tau') \partial_x^{\alpha_3} S_0(\tau; \tau')| \lesssim \varepsilon^{h-1-2\zeta} \mathbf{e}_{\gamma^+},$$

which is not good enough since $h - 1 - \zeta < 0$ if $\zeta = 1 - h$. Hence the need for more than the bound on $\partial_x^\alpha S_0$ from Corollary 3.9.

Second, note that the loss in (3.59) comes from ξ-derivatives applied to χ, exactly as in the proof of Corollary 3.8.

We are going to use the representations

$$\partial_x^\alpha S_0 = \partial_x^\alpha S_\chi + \sum_{1 \le k \le m-1} \int \mathbf{S}_{\chi,k}^B + \int \mathbf{S}_{\chi,0,m}^B, \qquad |\alpha| \ge 0, \tag{3.60}$$

for any $m \in \mathbb{N}$, with

$$\mathbf{S}_{\chi,k}^B := \varepsilon^{kh} (\partial_x^{\alpha_1} S_\chi \partial_x^{\alpha_1'} (\chi B)) \cdots (\partial_x^{\alpha_k} S_\chi \partial_x^{\alpha_k'} (\chi B)) \partial_x^{\alpha_{k+1}} S_\chi,$$

$$\mathbf{S}_{\chi,0,k}^B := \varepsilon^{kh} (\partial_x^{\alpha_1} S_\chi \partial_x^{\alpha_1'} (\chi B)) \cdots (\partial_x^{\alpha_k} S_\chi \partial_x^{\alpha_k'} (\chi B)) \partial_x^{\alpha_{k+1}} S_0.$$

In (3.60) we use compact notation as in (3.48). In particular, time arguments are implicit and form "chains" in the sense that in products $(S_\chi \partial_x^{\alpha_1'} \chi B)(S_\chi \partial_x^{\alpha_2'} \chi B)$, the time arguments of the first term are $(\tau; \tau_1)$ and those of the second term are $(\tau_1; \tau_2)$. The integrals in (3.60) are time integrals bearing on the "interior" variables of $\mathbf{S}_k^B$ and $\mathbf{S}_{0k}^B$; see the explicit definition of $\mathbf{S}$ in (3.44), and see how the compact notation of (3.48) is expanded into explicit notation in (3.49).

By Assumption 2.1 and Corollary 3.9,

$$|\mathbf{S}_{\chi,k}^B| + |\mathbf{S}_{\chi,0,k}^B| \lesssim \varepsilon^{k(h-\zeta)} \varepsilon^{-\zeta} \mathbf{e}_{\gamma^+}. \tag{3.61}$$

The representation (3.60) is proved by using (3.57) recursively $n - 1$ times.

From (3.60), we find $\mathbf{S}_{0,1} = \varepsilon^{h-1} \partial_x^{\alpha_1} S_0 \partial_x^{\alpha_2} \partial_\xi^\beta (\chi A_\star) \partial^{\alpha_3} S_0$ is a sum of nine terms:

- The term $\varepsilon^{h-1} \partial_x^{\alpha_1} S_\chi \partial_x^{\alpha_2} \partial_\xi^\beta (\chi A_\star) \partial_x^{\alpha_3} S_\chi = \mathbf{S}_{\chi,1}$ is bounded by using Corollary 3.8.

• For term II $= \varepsilon^{h-1}\mathbf{S}^B_{\chi,k}\partial_x^{\alpha_2}\partial_\xi^\beta(\chi A_\star)\partial_x^{\alpha_3}S_\chi$, we use Corollary 3.8 for the rightmost product, of the form $\varepsilon^{h-1}\partial_x^{\alpha'_{k+1}}S_\chi\partial_x^{\alpha_2}\partial_\xi^\beta(\chi A_\star)\partial_x^{\alpha_3}S_\chi$, and then bound separately the remaining k products of the form $\varepsilon^h\partial_x^{\alpha'_i}S\partial_x^{\alpha''_i}(\chi B)$. This gives $|\mathrm{II}| \lesssim \varepsilon^{k(h-\zeta)}\varepsilon^{-\zeta(1+|\beta|)}\mathbf{e}_{\gamma^+}$, and we use $\zeta < h$.

• Term III $= \varepsilon^{h-1}\mathbf{S}^B_{\chi,0,m}\partial_x^{\alpha_2}\partial_\xi^\beta(\chi A_\star)\partial_x^{\alpha_3}S_\chi$ cannot be handled by the same argument as II, since the rightmost product here has the form $\varepsilon^{h-1}\partial_x^{\alpha'_{m+1}}S_0\partial_x^{\alpha_2}\partial_\xi^\beta(\chi A_\star)\partial_x^{\alpha_3}S_\chi$. Here we use (3.61) with $k = m$ large, the bound $|\partial_x^{\alpha_2}\partial_\xi^\beta(\chi A_\star)| \lesssim \varepsilon^{-|\beta|\zeta}$, and the bound of Corollary 3.8 for $|\partial_x^{\alpha_3}S_\chi|$. There occurs a loss of $\varepsilon^{-\zeta}$, but this is compensated by an appropriate choice of m. Precisely, we find $|\mathrm{III}| \lesssim \varepsilon^{m(h-\zeta)-2\zeta-|\beta|\zeta}\mathbf{e}_{\gamma^+}$ and then choose m large enough, depending on h, so that $m(h-\zeta)-\zeta \geq 0$.

• Term IV $= \varepsilon^{h-1}\partial_x^{\alpha_1}S_\chi\partial_x^{\alpha_2}\partial_\xi^\beta(\chi A_\star)\mathbf{S}^B_{\chi,k}$ is symmetric to II. Here we isolate the leftmost product $\varepsilon^{h-1}\partial_x^{\alpha_1}S_\chi\partial_x^{\alpha_2}\partial_\xi^\beta(\chi A_\star)\partial_x^{\alpha'_1}S_\chi$.

• In term V $= \varepsilon^{h-1}\mathbf{S}^B_{\chi,k}\partial_x^{\alpha_2}\partial_\xi^\beta(\chi A_\star)\mathbf{S}^B_{\chi,k'}$, we use the cancellation of Corollary 3.8 for the term in the middle, $\varepsilon^{h-1}\partial_x^{\alpha_{k+1}}S_\chi\partial_x^{\alpha_2}\partial_\xi^\beta(\chi A_\star)\partial_x^{\alpha''_1}S_\chi$. The remaining terms contain $k+k'$ occurrences of S, and there is a prefactor $\varepsilon^{h(k+k')}$. Thus we obtain the bound $|\mathrm{V}| \lesssim \varepsilon^{(k+k')(h-\zeta)}\varepsilon^{-\zeta(1+|\beta|)}\mathbf{e}_{\gamma^+}$.

• The remaining terms all involve at least m factors, hence, by (3.61), have an $\varepsilon^{m(h-\zeta)}$ prefactor. We handle these terms just like III above.

From the above, we conclude that

$$|\mathbf{S}_{0,1}| \lesssim \varepsilon^{-\zeta(1+|\beta|)}\mathbf{e}_{\gamma^+},$$

where $|\beta|$ is the number of ξ-derivatives in $\tilde{\mathbf{S}}_{0,1}$.

The general case $n \geq 2$ is handled in exactly the same way. Via representations (3.60), products involving $\partial_x^\alpha S_0$, as in the statement of the corollary, are expanded into sums of products involving $\partial_x^{\alpha'}S_\chi$, and remainders which involve products with a large number of $\varepsilon^h B$ terms, and S_0 terms. These remainders are handled as term III above, using $h < \zeta$, hence $\varepsilon^{m(h-\zeta)}$ as small as needed for m large. We are then left with products involving only $\chi A_\star$, χB and spatial derivatives of S_χ. For these, we use Corollary 3.7 as we did above in the treatment of terms I and II (using the case $n \geq 2$ in Corollary 3.8, while we used only $n = 1$ in the above treatment of I and II). □

The final bound in these preparation steps involves products of S_0 with $\chi(iA_\star + \varepsilon B)$, as already met in (3.43). For $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^{2d}$, $\beta \in \mathbb{N}^d$ and $0 \leq \tau \leq t$ we let

$$S^B_{0,\alpha,\beta}(\tau; t) := \varepsilon^{h-1}\partial_x^{\alpha_1}S_0(\tau; t)\partial_x^{\alpha_2}\partial_\xi^\beta(\chi(iA_\star + \varepsilon B)(\tau)),$$

and for $\alpha_i \in \mathbb{N}^{2d}$, $\beta_i \in \mathbb{N}^d$ and $0 \leq \tau \leq \tau_n \leq \tau_{n-1} \leq \tau_1 \leq t$ we consider the products

$$\mathbf{S}^B_{0,n}(\tau, \tau_1, \ldots, \tau_n; t) := S^B_{0,\alpha_1,\beta_1}(\tau_1; t)S_{0,\alpha_2,\beta_2}(\tau_2; \tau_1)\cdots S^B_{0,\alpha_n,\beta_n}(\tau_n; \tau_{n-1})\partial_x^\alpha S_0(\tau; \tau_n). \tag{3.62}$$

Corollary 3.11. *We have*

$$|\mathbf{S}^B_{0,n}| \lesssim \varepsilon^{-\zeta(1+|\beta|)}\mathbf{e}_{\gamma^+}$$

for any $0 \leq \tau \leq \tau_n \leq \tau_{n-1} \leq \cdots \leq \tau_1 \leq t \leq T(\varepsilon)$*, any* (x,ξ)*, any* α_i*,* β_i*, and any* $n \geq 1$*, with* $|\beta| = \sum_i |\beta_i|$*.*

Proof. Developing the product, we find that $\mathbf{S}^B_{0,n}$ is a product of terms of the form $\mathbf{S}_{0,n'}$ of (3.58) and terms of the form $\varepsilon^h \partial_x^\alpha \partial_\xi^\beta \chi B$. By Corollary 3.10, $|\mathbf{S}_{0,n}| \lesssim \varepsilon^{-\zeta(1+|\beta|)}\mathbf{e}_{\gamma^+}$, where $|\beta|$ is the total number of ξ-derivatives that appear in $\mathbf{S}_{0,n}$. Moreover, $|\varepsilon^h \partial_x^\alpha \partial_\xi^\beta(\chi B)| \lesssim \varepsilon^{h-|\beta|\zeta}$. The result then follows from $\zeta < h$. □

3.12.5. Bounds on the correctors S_q*.* We are now ready to prove bounds on spatial derivatives of the correctors S_q introduced in (3.41):

Corollary 3.12. *For* $0 \leq \tau \leq t \leq T(\varepsilon)$ *and any* (x,ξ)*,*

$$|\partial_x^\alpha S_q| \lesssim \varepsilon^{-\zeta(1+q)}\mathbf{e}_{\gamma^+} \quad \textit{for } 0 \leq q \leq q_0.$$

Proof. For $q = 0$, Corollary 3.9 gives the desired bound. For $q = 1$, the corrector S_1 admits the representation (3.42), so that $\partial_x^\alpha S_1$ appears as a sum of terms

$$\varepsilon^{h-1}\int_\tau^t \partial_x^{\alpha_1} S_0(\tau';t)\partial_x^{\alpha_2}\partial_\xi^\kappa(\chi(iA_\star + \varepsilon B))(\tau')\partial_x^{\alpha_3+\kappa} S_0(\tau;\tau')\,d\tau',$$

where $|\kappa| = 1$ and $\alpha_1+\alpha_2+\alpha_3 = \alpha$. Corollary 3.10 applies and gives the desired bound. Consider now the representation (3.42) for S_q with any $q \geq 2$. In this representation, $\partial_x^\alpha S_q$ appears as a sum of terms

$$\varepsilon^{h-1}\int_\tau^t \partial_x^{\alpha_1} S_0(\tau';t)\partial_x^{\alpha_2}\partial_\xi^\kappa(\chi(iA_\star + \varepsilon B))(\tau')\partial_x^{\alpha_3+\kappa} S_{q'}(\tau;\tau')\,d\tau', \tag{3.63}$$

where $|\kappa| + q' = q$, $|\kappa| > 0$, and $\alpha_1 + \alpha_2 + \alpha_3 = \alpha$. We may recursively use the representation (3.42) in (3.63), so that $\partial_x^\alpha S_q$ appears as a time integral of terms $\mathbf{S}^B_{0,n}$ as in (3.62), with exactly q derivatives bearing on the ξ variables. The result then follows from Corollary 3.11. □

3.12.6. Bounds on the approximate solution operator $\mathrm{op}_\varepsilon(\Sigma)$*.* We arrive at a bound for the action of the approximate solution operator $\mathrm{op}_\varepsilon(\Sigma)$ defined in (3.39):

Corollary 3.13. *For* $0 \leq \tau \leq t \leq T(\varepsilon)$*,*

$$\|\mathrm{op}_\varepsilon(\Sigma(\tau;t))w\|_{L^2(B(x_0,\delta))} \lesssim \varepsilon^{-\zeta}\exp(\boldsymbol{\gamma}^+(t^{1+\ell} - \tau^{1+\ell}))\|w\|_{L^2(\mathbb{R}^d)},$$

where $\boldsymbol{\gamma}^+ := \max_{|x|\leq\delta,\,|\xi-\xi_0|\leq\delta}\gamma(x,\xi)$*.*

Proof. Let θ_1 be a spatial cut-off that is identically 1 on a neighborhood of $B(x_0,\delta)$. Then

$$\|\mathrm{op}_\varepsilon(\Sigma(\tau;t))w\|_{L^2(B(x_0,\delta))} \leq \|\mathrm{op}_\varepsilon(\theta_1\Sigma(\tau;t))w\|_{L^2(\mathbb{R}^d)}.$$

Now by Proposition B.1,

$$\|\mathrm{op}_\varepsilon(\theta_1 \Sigma(\tau;t))w\|_{L^2(\mathbb{R}^d)} \leq \sum_{|\alpha|\leq d+1} \sup_{\xi\in\mathbb{R}^d} \|\partial_x^\alpha(\theta_1\Sigma(\tau;t,\cdot,\xi))\|_{L^1(\mathbb{R}^d)}\|w\|_{L^2(\mathbb{R}^d)}.$$

By Corollary 3.10, for all $0 \leq q \leq q_0$ and $|\xi - \xi_0| \leq \delta$,

$$\varepsilon^{qh}\|\theta_1\partial_x^\alpha S_q(\tau;t,\cdot,\xi)\|_{L^1(\mathbb{R}^d)} \lesssim \varepsilon^{-\zeta(1+q)} \sup_{|x|+|\xi-\xi_0|\leq\delta} \mathbf{e}_{\gamma^+}(\tau;t,x,\xi).$$

The result then follows from $\zeta < h$ and the pointwise bound

$$\mathbf{e}_{\gamma^+}(\tau;t,x,\xi) \leq \exp(\boldsymbol{\gamma}^+(t^{1+\ell} - \tau^{1+\ell})) \quad \text{for } \ell \geq 0.$$ □

3.12.7. Lower bound for S_0. We verify that the lower bound (2.10) in Assumption 2.1 is stable by perturbation:

Lemma 3.14. *For δ and ε small enough, the flow S_0 satisfies the lower bound* (2.10) *from Assumption 2.1, that is,*

$$\varepsilon^{-\zeta}\mathbf{e}_{\gamma^-}(0;T(\varepsilon),x,\xi_0) \lesssim |S_0(0;T(\varepsilon),x,\xi_0)\vec{e}(x)|$$

for $|x| \leq \delta$ and $\vec{e}(x)$ as in Assumption 2.1.

Proof. First observe that $t_\star(\varepsilon,0,\xi_0) = 0$, so that the equations for S and S_χ coincide over the time interval $[\delta, T(\varepsilon)]$ at $(x,\xi) = (0,\xi_0)$. A simple perturbation argument, similar to the arguments developed in detail above, then implies that $S_\chi(0;T(\varepsilon),0,\xi_0)$ satisfies the lower bound (2.10). Next we use the representation (3.57) from the proof of Corollary 3.9, with $\alpha = 0$, which implies

$$|S_0\vec{e}| \geq |S_\chi\vec{e}| - \varepsilon^h \int_0^t |S_\chi(\tau';t)|\,|B(\tau')|\,|S_0(\tau;\tau')|\,d\tau'.$$

As argued above, we may use the lower bound (2.10) for S_χ; moreover, Corollaries 3.8 and 3.9 provide upper bounds for S_0 and S_χ. These yield the lower bound

$$|S_0(0;T(\varepsilon),x,\xi_0)\vec{e}| \gtrsim \varepsilon^{-\zeta}\mathbf{e}_{\gamma^-}(0;T(\varepsilon),x,\xi_0) - \varepsilon^{h-\zeta}\mathbf{e}_{\gamma^+}(0;T(\varepsilon),x,\xi_0).$$

Now at $(\tau;t,x,\xi) = (0;T(\varepsilon),x,\xi_0)$, for δ and ε small enough,

$$\mathbf{e}_{\gamma^+}\mathbf{e}_{\gamma^-}^{-1} \lesssim \exp\Big(\Big(\gamma^+(x,\xi_0) - \min_{|x|\leq\delta}\gamma^-(x,\xi_0)\Big)T(\varepsilon)^{1+\ell}\Big).$$

The constant

$$\delta_0 := \max_{|x|\leq\delta}\gamma^+(x,\xi_0) - \min_{|x|\leq\delta}\gamma^-(x,\xi_0)$$

is small for small δ, by continuity of $\gamma^\pm$ and the fact that $\gamma^+(0,\xi_0) = \gamma^-(0,\xi_0)$. In particular, we may choose δ depending on $T_\star$ defined in (2.3) and satisfying (3.5), so that

$$h - \zeta - T_\star\delta_0 > 0. \tag{3.64}$$

Thus for $|x| < \delta$ and ε small enough,

$$1 - \varepsilon^{h-\zeta}\mathbf{e}_{\gamma^+}\mathbf{e}_{\gamma^-}^{-1}(0;T(\varepsilon),x,\xi_0) \geq 1 - \varepsilon^{h-\zeta-T_\star\delta_0} \geq 1/2,$$

and the result follows. □

3.12.8. Bounds on $\partial_x^\alpha \partial_\xi^\beta S_q$. We finally give bounds on (x,ξ)-derivatives of S_0 and of the correctors S_q. This ends the verification of Assumption D.1.

Lemma 3.15. *For $0 \leq \tau \leq t \leq T(\varepsilon)$ and any (x,ξ),*

$$|\partial_x^\alpha \partial_\xi^\beta S_q| \lesssim \varepsilon^{-\zeta(1+|\beta|+q)} \mathbf{e}_{\gamma^+}. \tag{3.65}$$

Proof. From the representation

$$\partial_\xi^\beta S_0(\tau;t) = -\varepsilon^{h-1} \sum_{\substack{\beta_1+\beta_2=\beta \\ |\beta_1|>0}} \int_0^t S_0(\tau';t)\partial_\xi^{\beta_1}(\chi(iA_\star + \varepsilon B))(\tau')\partial_\xi^{\beta_2} S_0(\tau;\tau')\, d\tau'$$

and Corollary 3.11, we find by induction the bound (3.65) for $q = 0$. Moreover , applying $\partial_x^\alpha \partial_\xi^\beta$ to the equation for S_q, we find the representation

$$\partial_x^\alpha \partial_\xi^\beta S_q = -\varepsilon^{h-1} \int_\tau^t S_0\Big(\partial_x^{\alpha_1}\partial_\xi^{\beta_1}(\chi(iA_\star+\varepsilon B))\partial_x^{\alpha_2}\partial_\xi^{\beta_2} S_q + \partial_x^\alpha \partial_\xi^\beta \big((\chi(iA_\star+\varepsilon B))\sharp_{q_1} S_{q_2}\big)\Big),$$

with an implicit summation over $\alpha_1 + \alpha_2 = \alpha$, $\beta_1 + \beta_2 = \beta$, $q_1 + q_2 = q$ with $|\alpha_1|, |\beta_1|, q_1 > 0$. We use the above representation recursively, and find that $\partial_x^\alpha \partial_\xi^\beta S_q$ is a sum of terms $\mathbf{S}_{0,n}^B$, with a total number of ξ-derivatives equal to $|\beta| + q$. It now suffices to apply Corollary 3.11. □

3.12.9. Conclusion. Before moving on to the third and last part of the proof of Theorem 2.2, we recapitulate our arguments so far.

The bound $|\partial_x^\alpha \partial_\xi^\beta(\chi A_\star)| \lesssim \varepsilon^{-|\beta|\zeta}$ and Lemma 3.15 verify Assumption D.1. Thus at this point the integral representation (3.38) is justified, via Theorem D.3. We reproduce here equation (3.38):

$$v = \mathrm{op}_\varepsilon(\Sigma(0;t))v(0) + \varepsilon^h \int_0^t \mathrm{op}_\varepsilon(\Sigma(\tau;t))(\mathrm{Id} + \varepsilon\, \mathrm{op}_\varepsilon(\mathbf{R}_0))\big(g(\tau) + \varepsilon\, \mathrm{op}_\varepsilon(\mathbf{R}_0)v(0)\big)\, d\tau.$$

This ends the second part of the proof. Next we will prove a lower bound for v, in $L^2(B(0,\delta))$ norm. For this, we will bound from above the time-integrated term in (3.38), and bound from below the "free" solution $\mathrm{op}_\varepsilon(\Sigma(0;t))v(0)$. The bound from above rests on Corollary 3.13 bounding the action of $\mathrm{op}_\varepsilon(\Sigma)$, and on a bound for the source g, which is the object of the forthcoming Section 3.13. The bound from below is a consequence of Lemma 3.14.

3.13. Bound on the source term

Our goal in this section is to give an upper bound for the source term g from (3.36). The source g is defined in (3.37), which we reproduce here:

$$g = \varepsilon^{-1}(\Gamma v - \tilde{\Gamma}(u_\star(\varepsilon^h t))) + \varepsilon^{-h}\, \mathrm{op}_\varepsilon(\partial_t \tilde{\chi})(u_\star(\varepsilon^h t)) + \mathrm{op}_\varepsilon(\tilde{\chi})(F_\star(\varepsilon^h t))$$
$$+ \varepsilon^n(\mathrm{op}_\varepsilon(\mathbf{R}_0)u_\star)(\varepsilon^h t).$$

We let

$$g_0 := \mathrm{op}_\varepsilon(\tilde\chi)(F_\star(\varepsilon^h t)) + \varepsilon^n (\mathrm{op}_\varepsilon(\mathbf{R}_0)u_\star)(\varepsilon^h t),$$
$$g_\chi := \varepsilon^{-1}\big(\Gamma v - \tilde\Gamma(u_\star(\varepsilon^h t))\big) + \varepsilon^{-h}\,\mathrm{op}_\varepsilon(\partial_t\tilde\chi)(u_\star(\varepsilon^h t)),$$

so that $g = g_0 + g_\chi$. We first consider g_0.

Lemma 3.16. *We have*

$$\|\mathrm{op}_\varepsilon(\chi^\flat)g_0(t)\|_{L^2} \lesssim \|\dot u(\varepsilon^h t)\|^2_{W^{1,\infty}(B(0,\delta))} + \varepsilon^n \|\dot u(\varepsilon^h t)\|_{L^2(B(0,\delta))},$$

uniformly in $t \in [0, T(\varepsilon)]$, *for* n *as large as allowed by the regularity of* ϕ, *where the possibly stiff truncation* $\chi^\flat$ *is defined just below* (3.31), *in particular* $\chi^\flat \prec \tilde\chi$ *with the notation of* (3.30).

Proof. There are three types of terms in g_0: (a) a commutator coming from the spatial cut-off θ, (b) non-linear terms, (c) remainders of size $O(\varepsilon^n)$. The term $F_\star$ is defined in (3.29) in terms of $F^\flat$ defined in (3.21), which in turn is defined in terms of $\dot F$ defined in (3.8). We recall their definitions:

$$\dot F = G_0(\varepsilon, t, x, \dot u)\cdot(\dot u, \dot u) + \sum_{j=1}^d G_{1j}(\varepsilon, t, x, \dot u)\cdot(\dot u, \partial_{x_j}\dot u),$$
$$F^\flat = \mathrm{op}_\varepsilon(Q_\varepsilon)(\theta\dot F) - \sum_{j=1}^d \mathrm{op}_\varepsilon(Q_\varepsilon)(A_j(\phi_\varepsilon)\dot u\partial_{x_j}\theta) + \varepsilon^n\,\mathrm{op}_\varepsilon(\mathbf{R}_0)(\theta\dot u),$$
$$F_\star = M^\star F^\flat + \varepsilon^n\,\mathrm{op}_\varepsilon(\mathbf{R}_0)u^\flat.$$

(a) The commutator term is the term in $\partial_{x_j}\theta$ in $F^\flat$. This term is rendered small by the left action of $\mathrm{op}_\varepsilon(\chi^\flat)$. Indeed, commutators that arise from a localization step depend on derivatives p', where p is the localization symbol. If we further localize with $p^\flat$ such that $p^\flat$ is identically 1 on the support of p, then from $(1-p)p^\flat = 0$ we deduce $p^\flat p' \equiv 0$, and the associated commutator is arbitrarily small. This is made precise below.

Consider first the term

$$\mathrm{op}_\varepsilon(\chi^\flat)M^\star\,\mathrm{op}_\varepsilon(Q_\varepsilon)(A_j(\phi_\varepsilon)\dot u\partial_{x_j}\theta).$$

We start by approximately commuting $\mathrm{op}_\varepsilon(\chi^\flat)$ and $M^\star$, by use of (3.23) (with M in place of $M^\star$ and conversely): in view of Lemma 3.3,

$$\mathrm{op}_\varepsilon(\chi^\flat)M^\star = M^\star\,\mathrm{op}_\varepsilon(\chi^\flat_{(\star)}) + \varepsilon^{h-\zeta}\,\mathrm{op}_\varepsilon(\mathbf{R}_0), \tag{3.66}$$

where $\chi^\flat_{(\star)}$ denotes evaluation of $\chi^\flat$ along the backward characteristics of μ_ε. Note that we here use the definition (3.13) for $\mathbf{R}_0$. In particular, the characteristics depend on time through $\varepsilon^h t$. Expanding $\chi^\flat_{(\star)}$ in powers of $\varepsilon^h t$, for $t \le T(\varepsilon)$, we find terms that are all supported in the support of $\chi^\flat$, up to an $O(\varepsilon^n)$ remainder, with n as large as allowed by the regularity of ϕ. In conjunction with Proposition B.2, this implies

$$M^\star\,\mathrm{op}_\varepsilon(\chi^\flat_{(\star)})\,\mathrm{op}_\varepsilon(Q_\varepsilon) = \mathrm{op}_\varepsilon(\theta_0\mathbf{R}_0) + \varepsilon^n\,\mathrm{op}_\varepsilon(\mathbf{R}_0),$$

where θ_0 is defined in Section 3.10, in particular $(1-\theta)\theta_0 \equiv 0$. Now

$$\mathrm{op}_\varepsilon(\theta_0 \mathbf{R}_0)(A_j(\phi_\varepsilon)\dot u \partial_{x_j}\theta) = \mathrm{op}_\varepsilon(\theta_0\partial_{x_j}\theta\mathbf{R}_0)(A_j(\phi_\varepsilon)\dot u) + [\mathrm{op}_\varepsilon(\theta_0\mathbf{R}_0), \partial_{x_j}\theta](A_j(\phi_\varepsilon)\dot u),$$

and the first term on the right is identically zero, since $\theta_0\partial_{x_j}\theta \equiv 0$. For the second term, we use Proposition B.2: up to $O(\varepsilon^n)$, the operator involves products of θ_0 and its derivatives with derivatives of $\partial_{x_j}\theta$. These products are identically zero. Thus

$$\|M^\star \mathrm{op}_\varepsilon(\chi^\flat_{(\star)})\, \mathrm{op}_\varepsilon(Q_\varepsilon)(A_j(\phi_\varepsilon)\dot u)\|_{L^2} \lesssim \varepsilon^n \|\dot u\|_{L^2(B(0,\delta))}.$$

Now Egorov's lemma (see (3.23)), as used in (3.66), yields an expansion to arbitrary order, as in (3.25). The supports of the symbols that appear in this expansion share the property that we used for $\chi^\flat_{(\star)}$. We obtain

$$\|\mathrm{op}_\varepsilon(\chi^\flat)M^\star \mathrm{op}_\varepsilon(Q_\varepsilon)(A_j(\phi_\varepsilon)\dot u\partial_{x_j}\theta)\|_{L^2} \lesssim \varepsilon^n\|\dot u\|_{L^2(B(0,\delta))}.$$

(b) The non-linear terms are *local* in $\dot u$ and $\nabla_x \dot u$, so that

$$\theta(x)G_0(\varepsilon,t,x,\dot u)\cdot(\dot u,\dot u) \equiv \theta(x)G_0(\varepsilon,t,x,\theta^\sharp\dot u)\cdot(\theta^\sharp\dot u,\theta^\sharp\dot u),$$

where $\theta \prec \theta^\sharp$ (see notation (3.30)), with $\operatorname{supp}\theta^\sharp \subset B(0,\delta)$. A similar identity holds for G_1. Thus

$$\|\theta G_0(\dot u)\cdot(\dot u,\dot u)\|_{L^2} \le \|\theta G_0(\theta^\sharp\dot u)\cdot(\theta^\sharp\dot u,\theta^\sharp\dot u)\|_{L^2} \lesssim C(|\theta^\sharp\dot u|_{L^\infty})\|\theta^\sharp\dot u\|_{L^\infty}\|\theta^\sharp\dot u\|_{L^2}.$$

Since $\theta^\sharp$ has support in $B(0,\delta)$, we have $\|\theta^\sharp\dot u\|_{L^2} \lesssim \|\dot u\|_{L^2(B(0,\delta))}$, and the same in L^∞ norm. By the a priori bound (3.10), we have in particular $\|\dot u\|_{L^\infty(B(0,\delta))} \lesssim 1$, since $K'>0$. Thus

$$\|\theta G_0(\dot u)\cdot(\dot u,\dot u)\|_{L^2} \lesssim \|\dot u\|^2_{L^\infty(B(0,\delta))}.$$

A similar argument for G_1 yields an upper bound that involves $\|\dot u\|_{W^{1,\infty}(B(0,\delta))}$. We conclude that

$$\|\mathrm{op}_\varepsilon(\chi^\flat)M^\star \mathrm{op}_\varepsilon(Q_\varepsilon)(\theta\dot F)\|_{L^2} \lesssim \|\dot u\|^2_{W^{1,\infty}(B(0,\delta))}.$$

(c) The remainders of the form $\varepsilon^n \mathrm{op}_\varepsilon(\mathbf{R}_0)(\theta\dot u)$ and $\varepsilon^n\mathrm{op}_\varepsilon(\mathbf{R}_0)u^\flat$ in g contribute $\varepsilon^n\|\theta\dot u\|_{L^2}$ to the estimate for $\mathrm{op}_\varepsilon(\chi^\flat)g$, by definition of uniform remainders (Section 3.6), and the bound (3.16) on $u^\flat$. The same bound holds for the remainder in $u_\star$, since $\|u_\star\|_{L^2} \lesssim \|u^\flat\|_{L^2}$, by the properties of $M^\star$. We use in addition Corollary 3.4 for all these remainders. ◻

Corollary 3.17. *For any $P\in S^0$, for the source term g_0 defined just above the statement of Lemma 3.16, we have*

$$\|\mathrm{op}_\varepsilon(\chi^\flat)\,\mathrm{op}_\varepsilon(P)g_0\|_{L^2} \lesssim \varepsilon^{2K'}\|P\|_{0,C(K')}, \tag{3.67}$$

uniformly in $t\in[0,T(\varepsilon)]$, for some $C(K')>0$, where $\|\cdot\|_{0,r}$ is the norm in S^0 defined in (B.2). The constant K' is defined in (3.11) in terms of K, α, m and d.

Proof. By Lemma 3.16 and Proposition B.1,

$$\|\mathrm{op}_\varepsilon(P)\,\mathrm{op}_\varepsilon(\chi^\flat)g\|_{L^2} \lesssim \|P\|_{0,C(d)}(\|\dot u\|^2_{L^2(B(0,\delta))} + \varepsilon^n\|\dot u\|_{L^2(B(0,\delta))}),$$

and with (3.10) we obtain the upper bound (3.67) by taking $n = K'$. It remains to handle the commutator $[\mathrm{op}_\varepsilon(\chi^\flat), \mathrm{op}_\varepsilon(P)]$. Here we use Proposition B.2: modulo terms which are $O(\varepsilon^n)$, the symbol of the commutator is a sum of terms of the form $P_\alpha := \varepsilon^{h|\alpha|}(\partial_x^\alpha\chi^\flat\partial_\xi^\alpha P - \partial_\xi^\alpha\chi^\flat\partial_x^\alpha P)$. Since $\chi^\flat \prec \tilde\chi$ (with the notation introduced in (3.30)), we have $P_\alpha \equiv \tilde\chi P_\alpha$. Thus, by Proposition B.2 again,

$$\mathrm{op}_\varepsilon(P_\alpha)g = \mathrm{op}_\varepsilon(P_\alpha)\,\mathrm{op}_\varepsilon(\tilde\chi)g + \varepsilon^{h(|\alpha|+1)}\,\mathrm{op}_\varepsilon(Q_\alpha)g, \tag{3.68}$$

where the leading terms in Q_α have the same form as P_α. For the first term in (3.68) above, we use Corollary 3.16, as we may since $\tilde\chi \prec \chi$. We also use the fact that $\|\mathrm{op}_\varepsilon(P_\alpha)\|_{L^2\to L^2}$ is bounded uniformly in ε, by Lemma 3.3 and the bound (B.5). For the leading terms in Q_α, we use inductively (3.68), and arrive at (3.67). The Taylor expansion in the composition of operators needs to be carried out up to order $O(K')$, hence the dependence on $\|P\|_{0,C(K')}$ in the upper bound. □

Lemma 3.18. *For any $P \in S^0$ and for g_χ defined just above Lemma 3.16,*

$$\|\mathrm{op}_\varepsilon(\chi^\flat)\,\mathrm{op}_\varepsilon(P)g_\chi\|_{L^2} \lesssim \varepsilon^{K'+n}\|P\|_{0,n},$$

uniformly in $t \in [0, T(\varepsilon)]$, for n as large as allowed by the regularity of ϕ.

Proof. First consider the case $P = \mathrm{Id}$. Since derivatives of $\tilde\psi_0$ vanish identically on the support of $\psi_0^\flat$, by Proposition B.2 we have

$$\|\mathrm{op}_\varepsilon(\chi^\flat)\,\mathrm{op}_\varepsilon(\partial_t\tilde\chi)\|_{L^2\to L^2} \lesssim \varepsilon^{nh}\|\partial_\xi^n\chi^\flat\|_{0,C(d)}\|\partial_x^n\partial_t\tilde\chi\|_{0,C(d)}.$$

By Lemma 3.3, this implies

$$\|\mathrm{op}_\varepsilon(\chi^\flat)\,\mathrm{op}_\varepsilon(\partial_t\tilde\chi)\|_{L^2\to L^2} \lesssim \varepsilon^{nh}\varepsilon^{-(n+2C(d))\zeta},$$

and the above is arbitrarily small if n is large enough, since $\zeta < h$. The same argument holds for the other two terms in g_ψ. In the general case of $P \in S^0$, we may reason as in the proof of Corollary 3.17, that is, by first spelling out the composition $\mathrm{op}_\varepsilon(\psi^\flat)\,\mathrm{op}_\varepsilon(P)$ up to a large order, and then using the above. Remainders are small by the condition $\zeta < h$.

The above argument proves the bound

$$\|\mathrm{op}_\varepsilon(\psi^\flat)\,\mathrm{op}_\varepsilon(P)g_\psi\|_{L^2} \lesssim \varepsilon^n(\|u_\star(\varepsilon^h t)\|_{L^2} + \|v\|_{L^2}). \tag{3.69}$$

We finally use (B.4) to control $\|v\|_{L^2(\mathbb{R}^d)}$, and obtain, via Lemma 3.3,

$$\|v\|_{L^2(\mathbb{R}^d)} \lesssim \varepsilon^{-\zeta C(d)}\|u_\star(\varepsilon^h t)\|_{L^2(\mathbb{R}^d)}.$$

This loss is irrevelant since in (3.69) the integer n can be chosen to be very large. We conclude the proof by using the bound $\|u_\star\|_{L^2} \lesssim \|u^\flat\|_{L^2}$ and (3.16) for $u^\flat$. □

3.14. Lower bound for the free part of the solution

First we describe the time transition function and the truncation ψ_0 for frequencies close to ξ_0:

Lemma 3.19. *For $|x| \leq \delta$ and $|\xi| \leq \delta \varepsilon^{\zeta - h}$,*

$$0 \leq t_\star(\varepsilon, x, \xi_0 + \varepsilon^h \xi) \leq C\delta,$$

for some $C > 0$ independent of δ. In particular, for such (x, ξ),

$$\psi_0\big(-t_\star(\varepsilon, x, \xi_0 + \varepsilon^h \xi)\big) \equiv 1, \qquad \psi_0\big(T(\varepsilon) - t_\star(\varepsilon, x, \xi_0 + \varepsilon^h \xi)\big) \equiv 1,$$

and similarly for $\tilde{\psi}_0$ and $\psi_0^\flat$.

Proof. We may assume $\ell = 1/2$. We have

$$\begin{aligned} t_\star(\varepsilon, x, \xi_0 + \varepsilon^h \xi) &= \varepsilon^{-h} \theta_\star(\varepsilon^{1-h} x, \xi_0 + \varepsilon^h \xi) \\ &= \varepsilon^{-h} \theta_\star(\varepsilon^{1-h} x, \xi_0) + \int_0^1 \partial_\xi \theta_\star(\varepsilon^{1-h} x, \xi_0 + \varepsilon^h \tau \xi) \cdot \xi \, d\tau, \end{aligned}$$

and by the assumption on $\theta_\star$ defined in (2.1), for $|x| \leq \delta$ and $|\xi| \leq \delta \varepsilon^{-h+\zeta}$ we have

$$\begin{aligned} |\varepsilon^{-h} \theta_\star(\varepsilon^{1-h} x, \xi_0)| &\leq C \varepsilon^{-h+2(1-h)} \delta^2 = C\delta^2, \\ |\partial_\xi \theta_\star(\varepsilon^{1-h} x, \xi_0 + \varepsilon^h \tau \xi)| &\leq C(\varepsilon^{1-h}|x| + \varepsilon^h |\xi|) \leq C \varepsilon^\zeta \delta, \end{aligned}$$

for some $C > 0$ which does not depend on δ. This proves the bound on $t_\star$. In particular, for such (x, ξ) we have $-t_\star \geq -\delta/9$ and $T(\varepsilon) - t_\star \geq -\delta/9$, implying the result for ψ_0, by definition of ψ_0 (see for instance Figure 6). □

Based on the above result for $t_\star$ and $\tilde{\psi}_0$, we identify the leading term in the datum for v defined in (3.32):

Corollary 3.20. *We have*

$$\| v(0) - \varepsilon^K e^{i x \cdot \xi_0 / \varepsilon} \tilde{\theta}_0(x) \vec{e}(x) \|_{L^2} \lesssim \varepsilon^{K+h-\zeta}, \tag{3.70}$$

where $\tilde{\theta}_0$ is the spatial cut-off introduced in Section 3.9.

Proof. At this point the reader may find useful to jump back to Section 3.2. Since $M^\star_{|t=0} = \mathrm{Id}$, the datum for v is

$$v(0) = \varepsilon^K \operatorname{op}_\varepsilon(\tilde{\chi}(0)) \operatorname{op}_\varepsilon(Q_\varepsilon(0)) \Big(\theta \, \Re e \big(\operatorname{op}_\varepsilon(Q_\varepsilon(0)^{-1}) (e^{i(\cdot)\cdot \xi_0/\varepsilon^h} \theta \vec{e}) \big) \Big).$$

We may commute $\operatorname{op}_\varepsilon(Q_\varepsilon)$ and θ, since this produces an error that is $O(\varepsilon^{K+h})$ in L^2, thanks to Proposition B.2. Then in the datum we handle separately the oscillations in $+\xi_0/\varepsilon^h$ and the oscillations in $-\xi_0/\varepsilon^h$. We obtain

$$\varepsilon^K \operatorname{op}_\varepsilon(Q_\varepsilon(0)) \operatorname{op}_\varepsilon(Q_\varepsilon(0)^{-1}) (e^{i(\cdot)\cdot \xi_0/\varepsilon^h} \theta \vec{e}) = \varepsilon^K e^{i x \cdot \xi_0/\varepsilon^h} \theta \vec{e} + O(\varepsilon^{K+1}),$$

where $O(\cdot)$ denotes control in L^2. Thus the oscillation in ξ_0/ε^h contributes to $v(0)$ the term

$$\varepsilon^K \operatorname{op}_\varepsilon(\tilde\chi(0))(e^{ix\cdot\xi_0/\varepsilon^h}\theta^2\vec e\,) = \varepsilon^K e^{ix\cdot\xi_0/\varepsilon^h}\operatorname{op}_\varepsilon(\tilde\chi(0,x,\xi_0+\cdot))(\theta^2\vec e\,)$$

modulo terms that are $O(\varepsilon^{K+h})$. By definition of $\tilde\chi$ in Section 3.9,

$$\tilde\chi(0,x,\xi_0+\varepsilon^h\xi) = \tilde\chi_0(\varepsilon^{h-\zeta}\xi)\tilde\theta_0(x)\tilde\psi_0(-t_\star(\varepsilon,x,\xi_0+\varepsilon^h\xi)).$$

In the above we may use Lemma 3.19, since $\tilde\chi_0(\varepsilon^{h-\zeta}\xi)$ is non-zero only if $|\xi|\le\delta\varepsilon^{\zeta-h}$. Thus

$$\tilde\chi(0,x,\xi_0+\varepsilon^h\xi) = \chi_0(\varepsilon^{h-\zeta}\xi)\tilde\theta_0(x) = \tilde\theta_0(x)+\varepsilon^{h-\zeta}O(\xi).$$

We may expand the above up to an arbitrary power of $\varepsilon^{h-\zeta}$. The remainder is a stiff symbol in ξ, hence we will lose $\varepsilon^{-\zeta C(d)}$ in evaluating its operator norm (in accordance with Proposition B.1), but such a loss is irrevelant if the prefactor $\varepsilon^{n(h-\zeta)}$ is large enough. Also, the Taylor expansion brings out a large ξ^n prefactor, implying that the L^2 norm of the remainder depends on the high Sobolev norm $\|\tilde\theta_0\vec e\,\|_{H^n}$, but $\tilde\theta_0\in C_c^\infty$ and $\vec e$ is assumed to be smooth, hence this derivative loss is irrevelant as well. We thus obtain

$$\|\operatorname{op}_\varepsilon(\tilde\chi(0))(e^{ix\cdot\xi_0/\varepsilon^h}\theta^2\vec e\,) - e^{ix\cdot\xi_0/\varepsilon^h}\tilde\theta_0\theta^2\vec e\,\|_{L^2}\lesssim\varepsilon^{h-\zeta}.$$

By choice of θ_0, we have $\theta_0\prec\theta$. Thus $\tilde\theta_0\theta^2\equiv\tilde\theta_0$, and we have obtained the leading term $\varepsilon^K e^{ix\cdot\xi_0/\varepsilon^h}\tilde\theta_0(x)\vec e\,(x)$ as claimed in (3.70).

It remains to prove that the oscillation in $-\xi_0/\varepsilon^h$ has a small contribution to the datum. The leading term in the datum associated with this oscillation has the form

$$\varepsilon^K e^{-ix\cdot\xi_0/\varepsilon^h}\operatorname{op}_\varepsilon(\tilde\chi(-\xi_0+\cdot)P_\varepsilon)\tilde\theta,\qquad \tilde\theta := \overline{\theta\vec e}\,,$$

where $P_\varepsilon\in S^0$, uniformly in ε. The key is then that $\operatorname{op}_\varepsilon(\chi(-\xi_0+\cdot)P_\varepsilon)\tilde\theta$, which is smooth and supported in $B(0,\delta)$ (because $\tilde\theta_0$ is supported in $B(0,\delta)$) is pointwise bounded by $\|\mathcal F\tilde\theta\|_{L^1(|\xi|\ge c/\varepsilon^h)}$, with $c=2|\xi_0|-\varepsilon^\zeta\delta/2>0$. This L^1 norm is arbitrarily small, since $\mathcal F\tilde\theta$ belongs to the Schwartz class. Spatial derivatives are handled in the same way. □

Corollary 3.21. *For v defined in (3.32) and for ε and δ small enough,*

$$\|\operatorname{op}_\varepsilon(\chi^\flat(T(\varepsilon)))\operatorname{op}_\varepsilon(S_0(0;T(\varepsilon)))v(0)\|_{L^2(B(0,\delta))}\gtrsim\varepsilon^{K-\zeta}\exp(\boldsymbol\gamma^- T(\varepsilon)^{\ell+1}),$$

where $\boldsymbol\gamma^- := \min_{|x|\le\delta}\gamma^-(x,\xi_0)$ and $\chi^\flat$ is introduced in Section 3.9.

Proof. According to Corollary 3.20, the datum is

$$v(0,x)=\varepsilon^K\tilde v(0,x)+O(\varepsilon^{K+h-\zeta}),\qquad \tilde v(0,x)=e^{ix\cdot\xi_0/\varepsilon^h}\tilde\theta_0(x)\vec e(x).$$

We have

$$\operatorname{op}_\varepsilon(S_0(0;T(\varepsilon)))\tilde v(0)(x)=e^{ix\cdot\xi_0/\varepsilon^h}S_0(0;T(\varepsilon),x,\xi_0)\tilde\theta_0\,\vec e(x)+\varepsilon^h V_0,\tag{3.71}$$

with

$$V_0 := e^{ix\cdot(\xi+\xi_0/\varepsilon^h)} \sum_{|\alpha|=1} \mathrm{op}_\varepsilon\Big(\int_0^1 (\partial_\xi^\alpha S_0)\big(0; T(\varepsilon), x, \xi_0+\tau(\cdot)\big)\, d\tau\Big)(\partial_x^\alpha(\tilde\theta_0\vec{e}))(x).$$

We now apply $\mathrm{op}_\varepsilon(\chi^\flat(T(\varepsilon)))$ to (3.71). The leading term is

$$e^{ix\cdot\xi_0/\varepsilon^h}\,\mathrm{op}_\varepsilon(\chi^\flat(T(\varepsilon), x, \xi_0+\cdot))\big(S_0(0; T(\varepsilon), \cdot, \xi_0)\tilde\theta_0\vec{e}\big).$$

By definition of $\chi^\flat$ in Section 3.10,

$$\chi^\flat(T(\varepsilon), x, \xi_0+\varepsilon^h\xi) = \chi_0^\flat(\varepsilon^{h-\zeta}\xi)\theta_0^\flat(x)\psi_0^\flat\big(T(\varepsilon) - t_\star(\varepsilon, x, \xi_0+\varepsilon^h\xi)\big).$$

We use Lemma 3.19, as in the proof of Corollary 3.20:

$$\chi^\flat(T(\varepsilon), x, \xi_0+\varepsilon^h\xi) = \chi_0^\flat(\varepsilon^{h-\zeta}\xi)\theta_0^\flat(x) = \theta_0^\flat(x)\Big(1+\varepsilon^{h-\zeta}\int_0^1 \partial_\xi\chi_0^\flat(\varepsilon^{h-\zeta}\tau\xi)\cdot\xi\, d\tau\Big).$$

The leading term now is

$$U_0 = e^{ix\cdot\xi_0/\varepsilon^h} S_0(0; T(\varepsilon), x, \xi_0)\theta_0^\flat(x)\vec{e}(x),$$

since $\theta_0^\flat \prec \tilde\theta_0$. And since $\theta_0^\flat \equiv 1$ in a neighborhood of 0, we may use Lemma 3.14, which states that for ε and δ small enough and $|x| < \delta$,

$$\varepsilon^{-\zeta}\mathbf{e}_{\gamma^-}(0; T(\varepsilon), x, \xi_0) \lesssim |S_0(0; T(\varepsilon), x, \xi_0)\vec{e}(x)|.$$

Consider the lower growth function $\mathbf{e}_{\gamma^-}$, as defined in (2.9), at $(x,\xi) = (x,\xi_0)$. It involves $t_\star(\varepsilon, x, \xi_0)$. By Lemma 3.19, $0 \le t_\star(\varepsilon, x, \xi_0) \le C\delta$. Thus

$$\varepsilon^{-\zeta}\exp(\boldsymbol{\gamma}^- T(\varepsilon)^{1+\ell}) \lesssim \varepsilon^{-\zeta}\mathbf{e}_{\gamma^-}(0; T(\varepsilon), x, \xi_0),$$

uniformly in $|x| \le \delta$, with $\boldsymbol{\gamma}^-$ defined in the statement of this corollary. Hence we obtain a lower bound for the $L^2(B(0,\delta))$ norm of the leading term of the free solution as desired.

It remains to bound from above the terms we ignored so far. The first involves the remainder in the datum, which is $O(\varepsilon^{K+h-\zeta})$ in L^2 norm. By Corollaries 3.4 and 3.13, the action of $\mathrm{op}_\varepsilon(S_0)$ on this remainder is controlled by $\varepsilon^{h-\zeta}\varepsilon^{K-\zeta}\exp(\boldsymbol{\gamma}^+T(\varepsilon)^{1+\ell})$.

The other terms are $\varepsilon^{K+h}\,\mathrm{op}_\varepsilon(\chi^\flat(T(\varepsilon)))V_0$ and $\varepsilon^{K+h-\zeta}W_0$, where

$$W_0 := e^{ix\cdot\xi_0/\varepsilon^h}\theta_0^\flat(x)\sum_{|\alpha|=1}\mathrm{op}_\varepsilon\Big(\int_0^1 \partial_\xi^\alpha\chi_0^\flat(\varepsilon^{-\zeta}\tau(\cdot))\, d\tau\Big)\big(\partial_x^\alpha(S_0(0; T(\varepsilon), \cdot, \xi_0)\tilde\theta_0\vec{e})\big)(x).$$

By Corollary 3.4, Lemma 3.15 and Proposition B.1,

$$\varepsilon^{K+h}\|\mathrm{op}_\varepsilon(\chi^\flat(T(\varepsilon)))V_0\|_{L^2(\mathbb{R}^d)} \lesssim \varepsilon^{h-\zeta}\varepsilon^{K-\zeta}\exp(\boldsymbol{\gamma}^+T(\varepsilon)^{1+\ell}).$$

For W_0, we observe that $\mathrm{op}_\varepsilon(\partial_\xi^\alpha\chi_0^\flat)$ is a Fourier multiplier, bounded in $L^2 \to L^2$ norm by the maximum of its symbol, equal to a constant independent of ε. We may then bound $\partial_x^\alpha S_0$ by using Corollary 3.9. Thus $\varepsilon^{K+h-\zeta} W_0$ is controlled in L^2 just like $\varepsilon^{K+h} \operatorname{op}_\varepsilon(\chi^\flat(T(\varepsilon))) V_0$.

Summing up, we have obtained a lower bound of the form

$$\varepsilon^{K-\zeta} \exp(\boldsymbol{\gamma}^- T(\varepsilon)^{\ell+1})\big(1 - \varepsilon^{h-\zeta} \exp((\boldsymbol{\gamma}^+ - \boldsymbol{\gamma}^-) T(\varepsilon)^{1+\ell})\big).$$

By definition of $T(\varepsilon)$ in (2.3),

$$\varepsilon^{h-\zeta} \exp((\boldsymbol{\gamma}^+ - \boldsymbol{\gamma}^-) T(\varepsilon)^{1+\ell}) = \varepsilon^{h-\zeta-(\boldsymbol{\gamma}^+-\boldsymbol{\gamma}^-)T_\star}.$$

Given $T_\star$, since $h - \zeta > 0$, we may choose δ small enough for $\boldsymbol{\gamma}^+ - \boldsymbol{\gamma}^-$ to be so small that $h - \zeta - (\boldsymbol{\gamma}^+ - \boldsymbol{\gamma}^-)T_\star$ is strictly positive. The result follows if ε is small enough. □

3.15. Endgame

We apply $\operatorname{op}_\varepsilon(\chi^\flat(T(\varepsilon)))$ to the left of the representation formula (3.38) for v at $t = T(\varepsilon)$, with $\chi^\flat$ defined in Section 3.10, and prove that the contribution of the initial datum dominates the time-integrated Duhamel term. This eventually provides a contradiction to the assumed a priori bound (3.9), and concludes the proof.

In view of (3.38), we find

$$\|\operatorname{op}_\varepsilon(\chi^\flat(T(\varepsilon))) v(T(\varepsilon))\|_{L^2(B(0,\delta))} \geq \mathrm{I} - (\mathrm{II} + \mathrm{III}).$$

The leading term is

$$\mathrm{I} = \operatorname{op}_\varepsilon(\chi^\flat(T(\varepsilon))) \operatorname{op}_\varepsilon(S_0(0; T(\varepsilon))) v(0).$$

This term is bounded from below in $L^2(B(0,\delta))$ norm by Corollary 3.21:

$$\|\mathrm{I}\|_{L^2(B(0,\delta))} \gtrsim \varepsilon^{K-\zeta} \exp(\boldsymbol{\gamma}^- T(\varepsilon)^{\ell+1}).$$

The error term in the contribution of the datum is

$$\mathrm{II} := \sum_{q=1}^{q_0} \varepsilon^{hq} \operatorname{op}_\varepsilon(\chi^\flat(T(\varepsilon))) \operatorname{op}_\varepsilon(S_q(0; T(\varepsilon))) v(0).$$

We control II by Corollary 3.4 (action of $\operatorname{op}_\varepsilon(\chi^\flat)$ in L^2), Lemma 3.15 (bounds for S_q and their derivatives) and (B.5). This gives

$$\|\mathrm{II}\|_{L^2(B(0,\delta))} \lesssim \varepsilon^{K+h-\zeta} \exp(\boldsymbol{\gamma}^+ T(\varepsilon)^{\ell+1}).$$

The Duhamel term is

$$\mathrm{III} = \varepsilon^h \int_0^{T(\varepsilon)} \operatorname{op}_\varepsilon(\chi^\flat(T(\varepsilon))) \operatorname{op}_\varepsilon(\Sigma(\tau; T(\varepsilon)))(\mathrm{Id} + \varepsilon \operatorname{op}_\varepsilon(\mathbf{R}_0))(g + \varepsilon \operatorname{op}_\varepsilon(\mathbf{R}_0) v(0))\, dt'.$$

We bound III by using Corollary 3.17 and Lemma 3.18, in which we choose $n = K'$:

$$\|\mathrm{III}\|_{L^2(B(0,\delta))} \lesssim \varepsilon^{h-\zeta} \exp(\boldsymbol{\gamma}^+ T(\varepsilon)^{1+\ell})(\varepsilon^{2K'} + \varepsilon^{1+K}).$$

Above, we have used the bounds of Corollary 3.12 to control the norm $\|\Sigma\|_{0,C(K')}$, which appears in the upper bound of Corollary 3.17. Since $2K' > K$ by (3.4), we have obtained

$$\|\mathrm{op}_\varepsilon(\chi^\flat)v(T(\varepsilon))\|_{L^2(B(0,\delta))} \gtrsim \varepsilon^{K-\zeta}\exp(\boldsymbol{\gamma}^- T(\varepsilon)^{\ell+1}) - \varepsilon^{K+h-\zeta}\exp(\boldsymbol{\gamma}^+ T(\varepsilon)^{\ell+1}).$$

We can now conclude the proof just as that of Corollary 3.21. Precisely, rewriting the lower bound as

$$\|\mathrm{op}_\varepsilon(\chi^\flat)v(T(\varepsilon))\|_{L^2(B(0,\delta))} \gtrsim \varepsilon^{K-\zeta}\exp(\boldsymbol{\gamma}^- T(\varepsilon)^{\ell+1})(1-\varepsilon^{h-(\boldsymbol{\gamma}^+-\boldsymbol{\gamma}^-)T_\star}),$$

and choosing δ small enough that $h-(\boldsymbol{\gamma}^+-\boldsymbol{\gamma}^-)T_\star > 0$ (recall that $\boldsymbol{\gamma}^+$ is the local maximum of the rate function γ from Assumption 2.1 and $\boldsymbol{\gamma}^-$ is the local minimum of the lower rate function γ^-), we find for ε small enough the lower bound

$$\|\mathrm{op}_\varepsilon(\chi^\flat)v(T(\varepsilon))\|_{L^2(B(0,\delta))} \geq C|\ln\varepsilon|^*\varepsilon^{K-\zeta-\boldsymbol{\gamma}^- T_\star}$$

for some $C > 0$ independent of ε, where $|\ln\varepsilon|^*$ is some power of $|\ln\varepsilon|$. By choice of $T_\star$ in (3.5), we have $K-\zeta-\boldsymbol{\gamma}^- T_\star < 0$ if δ is small enough. Hence we get a lower bound which blows up as $\varepsilon \to 0$, contradicting the a priori bound (3.10).

4. Proof of Theorem 1.2: initial ellipticity

We are going to verify that, under the assumption (1.8) of initial ellipticity, Assumption 2.1 holds with parameters

$$\ell = 0, \quad h = 1, \quad \zeta = 0, \quad \mu \equiv 0, \quad t_\star \equiv 0.$$

Then Theorem 1.2 is a consequence of Theorem 2.2.

4.1. Block decomposition

Let $\lambda_0, \lambda_1, \ldots, \lambda_p$ be the spectrum of A at $(0, x_0, \xi_0)$. The ellipticity condition (1.8) states that one of the λ_j is non-real. By reality of A, complex eigenvalues come in conjugate pairs. In particular, at least one of the λ_j has strictly positive imaginary part. We may assume $\Im m\,\lambda_0 > 0$ and

$$\Im m\,\lambda_0 > \max_{1\leq j\leq p} \Im m\,\lambda_j. \tag{4.1}$$

Let m be the algebraic multiplicity of λ_0 in the spectrum, and $E_0(t,x,\xi)$ the generalized eigenspace associated with the family $\lambda_{0,1}(t,x,\xi),\ldots,\lambda_{0,m}(t,x,\xi)$ of (possibly non-distinct) eigenvalues of A which coalesce at $(0,x_0,\xi_0)$ with value λ_0, that is, $\lambda_{0,j}(0,x_0,\xi_0) = \lambda_0$. Let $E_1(t,x,\xi)$ be the direct sum of the other generalized eigenspaces. Let Q_0 be the projector onto E_0 parallel to E_1. The $\lambda_{0,j}$ may not be smooth, but are continuous (see for instance [25, Proposition 1.1]), and Q_0 is smooth (see for instance [8] or [25, Proposition 2.1]), and determines a smooth change of basis $Q(t,x,\xi)$ in which A is block diagonal:

$$QAQ^{-1} =: \begin{pmatrix} A_{(0)} & 0 \\ 0 & A_{(1)} \end{pmatrix}. \tag{4.2}$$

We focus on the block $A_{(0)}$ associated with the eigenvalues $\lambda_{0,j}$. The symbolic flow accordingly splits into $S_{(0)}$, $S_{(1)}$, where $S_{(0)}$ solves

$$\partial_t S_{(0)} + iA_{(0)}(\varepsilon t, x_0 + x, \xi)S_{(0)} = 0, \qquad S_{(0)}(\tau;\tau) = \mathrm{Id}.$$

By (4.1) and a repeat of the arguments below, the component $S_{(1)}$ of the symbolic flow is seen to grow no faster than $S_{(0)}$ near $(0,\xi_0)$.

4.2. *Reduction to upper triangular form at the distinguished point*

Let now P be a *constant* (independent of t,x,ξ) change of basis to upper triangular form of $A_{(0)}(0,x_0,\xi_0)$, and Q_μ be the diagonal matrix

$$Q_\mu = \mathrm{diag}(1, \mu^{-1}, \mu^{-2}, \dots, \mu^{1-m}).$$

The parameter μ will be chosen small enough below. Then

$$Q_\mu P A_{(0)}(0,x_0,\xi_0) P^{-1} Q_\mu^{-1} = \lambda_0 \,\mathrm{Id} + \mu J,$$

where J is upper triangular, bounded in μ, with zeros on the diagonal. By a Taylor expansion of $A_{(0)}(\varepsilon t, x_0 + x, \xi)$ in (t,x,ξ), we observe that

$$iA_{(0)}(\varepsilon t, x_0 + x, \xi) = P^{-1} Q_\mu^{-1}\big(i\lambda_0 \,\mathrm{Id} + i\mu J + B(\varepsilon, t, x, \xi)\big) P Q_\mu,$$

where the Taylor remainder B has the form

$$B = \varepsilon t B_1(\varepsilon, t, x, \xi) + (x, \xi - \xi_0)\cdot B_2(\varepsilon, t, x, \xi)$$

with B_1 and B_2 bounded, uniformly in ε, in the domain

$$|x| \le \delta, \qquad |\xi - \xi_0| \le \delta, \qquad t \le T_\star |\ln \varepsilon|. \tag{4.3}$$

Let

$$\tilde S(\tau;t) := \exp(i(t-\tau)\lambda_0) P Q_\mu S_{(0)}(\tau;t). \tag{4.4}$$

Then $\tilde S$ solves

$$\partial_t \tilde S + (i\mu J + B)\tilde S = 0, \qquad \tilde S(\tau;\tau) = P Q_\mu. \tag{4.5}$$

4.3. Bounds for the symbolic flow

Consider the Hermitian matrix

$$\Re e(i\mu J + B) := \frac{1}{2}\big((i\mu J + B) + (i\mu J + B)^*\big).$$

Its eigenvalues λ are semisimple, and vanish at $(\mu, t, x, \xi) = (0, 0, 0, \xi_0)$, hence satisfy, for (t, x, ξ) in the domain (4.3), the bound (see for instance [25, Corollary 3.4])

$$|\lambda| \leq c_0(\mu + \varepsilon T_\star |\ln \varepsilon| + |x| + |\xi - \xi_0|) =: \gamma_0(\varepsilon, \mu, t, x, \xi)$$

for some $c_0 > 0$ independent of $\varepsilon, \mu, t, x, \xi$, for (t, x, ξ) in (4.3) and (ε, μ) small enough. Thus

$$-\gamma_0 \,\mathrm{Id} \leq \Re e(i\mu J + B) \leq \gamma_0 \mathrm{Id}. \tag{4.6}$$

Let

$$S_\pm := \exp\big(\pm(t - \tau)\gamma_0(\varepsilon, \mu, t, x, \xi)\big)\tilde{S}.$$

From (4.5) we deduce, for any fixed vector $\vec{e} \in \mathbb{C}^m$,

$$\frac{1}{2}\,\Re e\,(\partial_t S_\pm \vec{e}, S_\pm \vec{e})_{\mathbb{C}^m} + \big((\Re e(i\mu J + B) \pm \gamma_0)S_\pm \vec{e}, S_\pm \vec{e}\big)_{\mathbb{C}^m} = 0.$$

By (4.6), this implies, for $\vec{e}$ unitary,

$$|P Q_\mu \vec{e}\,| e^{-(t-\tau)\gamma_0} \leq |\tilde{S}(\tau; t)\vec{e}\,| \leq |P Q_\mu| e^{(t-\tau)\gamma_0}. \tag{4.7}$$

Back to $S_{(0)}$ via (4.4), we now have

$$|S_{(0)}(\tau; t, x, \xi)| \leq |P Q_\mu|\,|(P Q_\mu)^{-1}| e^{(t-\tau)(\Im m\, \lambda_0 + \gamma_0)}.$$

We now choose $\mu = |\ln \varepsilon|^{-1}$ and let

$$\gamma^+ := \Im m\, \lambda_0 + c_0(|x| + |\xi - \xi_0|).$$

We have obtained, for ε small enough,

$$|S_{(0)}(\tau; t)| \leq |\ln \varepsilon|^\star e^{(t-\tau)\gamma^+},$$

corresponding to the upper bound (2.11). Finally, since $|P Q_\mu \vec{e}\,| = c_1 \mu^{1-m}$ for some $c_1 > 0$ and $\vec{e} = (0, 0, \ldots, 0, 1) \in \mathbb{C}^m$, we deduce from (4.7) the lower bound

$$|S_{(0)}(\tau; t)\vec{e}\,| \gtrsim e^{(t-\tau)\gamma^-}, \qquad \gamma^- = \Im m\, \lambda_0 - c_0(|x| + |\xi - \xi_0|),$$

corresponding to the lower bound (2.10).

5. Proof of Theorem 1.3: non-semisimple defect of hyperbolicity

It suffices to verify that, under the assumptions of Theorem 1.3, Assumption 2.1 holds with parameters

$$\ell = 1/2, \qquad h = 2/3, \qquad \zeta = 1/3.$$

Then Theorem 1.3 is a consequence of Theorem 2.2. We may assume initial hyperbolicity (1.9), since otherwise Theorem 1.2 applies.

5.1. The branching eigenvalues

Let $\omega_0 := (x_0, \lambda_0, \xi_0)$. By assumption,

$$P(0, \omega_0) = 0, \quad \partial_\lambda P(0, \omega_0) = 0, \quad \partial_t P(0, \omega_0) \neq 0, \quad \partial_\lambda^2 P(\omega_0) \neq 0. \tag{5.1}$$

By the second and fourth conditions in (5.1) and the implicit function theorem, there exists a smooth function $\mu_\star$ with $\mu_\star(0, x_0, \xi_0) = \lambda_0$ such that $\partial_\lambda P = 0$ is equivalent to $\lambda = \mu_\star(t, x, \xi)$ for (t, x, ξ) close to $(0, x_0, \xi_0)$.

By the first three conditions in (5.1) and the implicit function theorem, there is a smooth $\tau_\star$ with $\tau_\star(x_0, \xi_0) = 0$ such that $P(\mu_\star) = 0$ is locally equivalent to $t = \tau_\star(x, \xi)$.

We now use the above implicitly defined functions $\mu_\star$ and $\tau_\star$ to describe the spectrum of A near $(t, x, \xi, \lambda) = (0, \omega_0)$.

Lemma 5.1. *In a neighborhood of* $(0, \omega_0)$*, we have* $P = 0$ *if and only if*

$$(\lambda - \mu(x, \xi))^2 = -(t - \tau_\star(x, \xi))e(t, x, \xi, \lambda), \tag{5.2}$$

where $\mu(x, \xi) := \mu_\star(\tau_\star(x, \xi), x, \xi)$ *and* e *is smooth and satisfies* $e(0, x_0, \xi_0, \lambda_0) > 0$.

Proof. Given t close to 0 and (x, ξ, λ) close to ω_0, we expand the characteristic polynomial:

$$\begin{aligned} P(t, x, \xi, \lambda) &= P\big(\tau_\star(x, \xi), x, \xi, \mu_\star(\tau_\star(x, \xi)), x, \xi\big) \\ &\quad + (t - \tau_\star(x, \xi))e_1(t, x, \xi) + (\lambda - \mu(x, \xi))^2 e_2(t, x, \xi, \lambda) \\ &= (t - \tau_\star(x, \xi))e_1(t, x, \xi) + (\lambda - \mu(x, \xi))^2 e_2(t, x, \xi, \lambda), \end{aligned} \tag{5.3}$$

since $P(\tau_\star, \cdot, \mu_\star(\tau_\star)) \equiv 0$, with

$$e_1(t, x, \xi) := \int_0^1 (\partial_t P)\big((1 - \tau)\tau_\star(x, \xi) + \tau t, x, \mu(x, \xi)\big)\, d\tau,$$

$$e_2(t, x, \xi, \lambda) := \int_0^1 (1 - \tau)(\partial_\lambda^2 P)\big(t, x, \xi, (1 - \tau)\mu(x, \xi) + \tau\lambda\big)\, d\tau.$$

We let $e := e_1 e_2^{-1}$. Then $e(0, x_0, \xi_0, \lambda_0) > 0$, as a consequence of the definition of e_1 and e_2 and condition (1.11). The result follows from (5.3). □

Equation (5.2) describes a pair of eigenvalues branching at $t = \tau_\star(x, \xi)$ from the real axis, with imaginary parts growing like $(t - \tau_\star)^{1/2}$. The time curve $t = \tau_\star(x, \xi)$ is the boundary between the hyperbolic region $t < \tau_\star(x, \xi)$ in which the eigenvalues are real, and the elliptic region $t > \tau_\star(x, \xi)$ in which the eigenvalues are not real and where we expect to record an exponential growth for the symbolic flow. In the introduction, Figure 3 pictures the hyperbolic and elliptic zones in the (t, x, ξ) domain near $(0, x_0, \xi_0)$.

We note that under the assumptions of Theorem 1.3, the above defined time transition function $\tau_\star$ satisfies

$$\tau_\star \geq 0, \quad \tau_\star'(x_0, \xi_0) = 0, \quad \tau_\star''(x_0, \xi_0) \geq 0, \tag{5.4}$$

where $\tau_\star'(x_0, \xi_0)$ is the differential at (x_0, ξ_0), and $\tau_\star''(x_0, \xi_0)$ the Hessian. Indeed, if the first condition in (5.4) were violated, then hyperbolicity would not hold at $t = 0$, contradicting (1.9). Thus 0 is a global mininum for $\tau_\star$, and (5.4) ensues.

Remark 5.2. Note that under the assumption $\partial_\lambda^2 P \partial_t P < 0$, eigenvalues stay real for small $t > 0$, by Lemma 5.1.

5.2. *Change of basis*

By Lemma 5.1, the eigenvalues $\lambda_\pm(t,x,\xi)$ of $A(t,x,\xi) - \mu(x,\xi)\,\mathrm{Id}$ satisfy

$$\lambda_\pm(t,x,\xi)^2 = -(t - \tau_\star(x,\xi))e\big(t,x,\xi,\mu(x,\xi) + \lambda_\pm(t,x,\xi)\big), \tag{5.5}$$

where e is smooth in all its arguments, and $e(0,x_0,\xi_0,\mu(x_0,\xi_0)) > 0$ with $\mu(x_0,\xi_0) = \lambda_0$. From (5.5) and continuity of e, we deduce that λ_- and λ_+ are purely imaginary, hence $\lambda_+ + \lambda_- = 0$, since the matrix $A - \mu\,\mathrm{Id}$ has real coefficients. From (5.5), we also deduce the bound, for some $C > 0$, locally around $(0, x_0, \xi_0)$,

$$|\lambda_\pm| \leq C|t - t_\star|^{1/2},$$

which we may plug back in (5.5) and deduce, by regularity of e,

$$\lambda_\pm = \pm i(t - \tau_\star)^{1/2} e(t,x,\xi,\mu(x,\xi))^{1/2} + O(t - \tau_\star). \tag{5.6}$$

We now reduce A to canonical form:

Lemma 5.3. *There exists a smooth change of basis Q such that locally around* $(0, x_0, \xi_0)$,

$$Q(t,x,\xi)\big(A(t,x,\xi) - \mu(x,\xi)\,\mathrm{Id}\big)Q(t,x,\xi)^{-1} = \begin{pmatrix} A_{(0)} & 0 \\ 0 & A_{(1)} \end{pmatrix},$$

where

$$A_{(0)} = \begin{pmatrix} 0 & 1 \\ -(t - \tau_\star)e_0 + O(t - \tau_\star)^{3/2} & 0 \end{pmatrix},$$

and $A_{(1)} \in \mathbb{C}^{(N-2)\times(N-2)}$ is smooth. In the bottom left entry of $A_{(0)}$, the function $\tau_\star$ is the time transition function introduced just above Lemma 5.1, and we denote $e_0(t,x,\xi) = e(t,x,\xi,\mu(t,x,\xi))$ with e as in Lemma 5.1.

Proof. We may smoothly block diagonalize $A - \mu$, for instance as described in Section 4.1. The block associated with λ_0 is size two, since the multiplicity of λ_0 is 2 (see (1.10)). Thus for some smooth $\tilde{Q}$, $\tilde{Q}(A - \mu)\tilde{Q}^{-1} = \begin{pmatrix} B_0 & 0 \\ 0 & A_{(1)} \end{pmatrix}$, where B_0 is a 2×2 matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ with smooth entries a_{ij}. The spectrum of B_0 is $\{\lambda_-, \lambda_+\}$, where $\lambda_\pm$ satisfy (5.6). Since, as noted above, $\lambda_- = -\lambda_+$, the trace of B_0 is zero, that is, $a_{22} = -a_{11}$. Moreover, $a_{21}a_{12} \neq 0$ at $(0, x_0, \xi_0)$. Indeed, if $a_{21}a_{12}(0, x_0, \xi_0) = 0$, the spectrum of B_0 would be smooth in time, contradicting (5.2). Without loss of generality, we assume $a_{21}(0, x_0, \xi_0) \neq 0$. Then $Q_0 = \begin{pmatrix} 0 & -a_{21}^{-1} \\ 1 & -a_{21}^{-1}a_{11} \end{pmatrix}$ is a smooth change of basis such that $Q_0 B_0 Q_0^{-1} = \begin{pmatrix} 0 & 1 \\ \star & 0 \end{pmatrix}$. The bottom left of B_0 is the product $\lambda_-\lambda_+$ of its eigenvalues. By (5.6), we find that $\lambda_-\lambda_+ = -(t - \tau_\star)e_0 + O(t - \tau_\star)$, and the result holds with $Q = \begin{pmatrix} Q_0 & 0 \\ 0 & \mathrm{Id}_{\mathbb{C}^{N-2}} \end{pmatrix}\tilde{Q}$. □

5.3. The symbolic flow

Our goal is to prove the bounds of Assumption 2.1 for the solution S to the ordinary differential equation

$$\partial_t S(\tau;t) + i\varepsilon^{-1/3} A_\star(t) S(\tau;t) = 0, \qquad S(\tau;\tau) \equiv \mathrm{Id},$$

where $A_\star$ is defined by (2.4), which we recall:

$$A_\star(t) = \big(Q(A - \mu\,\mathrm{Id})Q^{-1}\big)\big(\varepsilon^{2/3}t, x_0 + \varepsilon^{1/3}x_\star(\varepsilon^{2/3}t, x, \xi), \xi_\star(\varepsilon^{2/3}t, x, \xi)\big).$$

Recall that $h = 2/3$ and $\zeta = 1/3$ here. The change of basis Q is given by Lemma 5.3, the real part of the branching eigenvalues μ is introduced in Lemma 5.1, and $(x_\star, \xi_\star)$ is the bicharacteristic flow, solving (2.5), which we reproduce here:

$$\partial_t x_\star = -\partial_\xi \mu(t, x_0 + \varepsilon^{1/3}x_\star, \xi_\star), \qquad \partial_t \xi_\star = \varepsilon^{1/3}\partial_x \mu(t, x_0 + \varepsilon^{1/3}x_\star, \xi_\star). \tag{5.7}$$

The block decomposition of A given by Lemma 5.3 induces a block decomposition of $A_\star$. We focus on the top left block $A_{\star(0)}$ in $A_\star$. As per Lemma 5.3, its bottom left entry involves the function

$$\tau_{\star\star}(\varepsilon, t, x, \xi) := \tau_\star\big(x_0 + \varepsilon^{1/3}x_\star(\varepsilon^{2/3}t, x, \xi), \xi_\star(\varepsilon^{2/3}t, x, \xi)\big). \tag{5.8}$$

The bicharacteristic flow (5.7) satisfies

$$x_0 + \varepsilon^{1/3}x_\star(\varepsilon^{2/3}t, x, \xi) = x_0 + \varepsilon^{1/3}x + O(\varepsilon t), \qquad \xi_\star(\varepsilon^{2/3}t, x, \xi) = \xi + O(\varepsilon), \tag{5.9}$$

uniformly in (x, ξ) with $|x| + |\xi - \xi_0| \le \delta$. Here $O(\varepsilon)$ refers to a uniform $\lesssim$ bound. In particular for t bounded from above by a power of $|\ln\varepsilon|$, we have $O(\varepsilon t) = O(\varepsilon)$. Thus $\tau_{\star\star}$ defined in (5.8) satisfies

$$\tau_{\star\star} = \theta_\star(\varepsilon^{1/3}x, \xi) + O(\varepsilon), \qquad \theta_\star(x, \xi) := \tau_\star(x_0 + x, \xi).$$

By (5.4), we see that $\theta_\star$ as defined above satisfies the conditions of equation (2.1). In accordance with (2.1), we let $t_\star(\varepsilon, t, x, \xi) := \varepsilon^{-2/3}\theta_\star(\varepsilon^{1/3}x, \xi)$. Then

$$\varepsilon^{-1/3}(\varepsilon^{2/3}t - \tau_{\star\star}) = \varepsilon^{1/3}(t - \varepsilon^{-2/3}\theta_\star(\varepsilon^{1/3}x, \xi)) + O(\varepsilon^{2/3}) = \varepsilon^{1/3}(t - t_\star) + O(\varepsilon^{2/3}), \tag{5.10}$$

uniformly in (x, ξ) with $|x| + |\xi - \xi_0| \le \delta$.

Lemma 5.4. *The top left block $A_{\star(0)}$ of $A_\star$ in the block decomposition of Lemma 5.3 satisfies*

$$\varepsilon^{-1/3}A_{\star(0)} = \begin{pmatrix} 0 & \varepsilon^{-1/3} \\ -\varepsilon^{1/3}(t - t_\star)f_0 + O(\varepsilon^{2/3}) & 0 \end{pmatrix}$$

with $f_0(\varepsilon, x, \xi) := e_0(0, x_0 + \varepsilon^{1/3}x, \xi)$, so that $f_0 > 0$ for (x, ξ) near $(0, \xi_0)$.

Proof. The bottom left entry of $\varepsilon^{-1/3}A_{\star(0)}$ involves the function $e_0 = e(\mu)$ evaluated in the time-rescaled and advected frame. In view of (5.9), by regularity of e and μ,

$$e_0\big(\varepsilon^{2/3}t, x_0 + \varepsilon^{1/3}x_\star(\varepsilon^{2/3}t, x, \xi), \xi_\star(\varepsilon^{2/3}t, x, \xi)\big) = e_0(0, x_0 + \varepsilon^{1/3}x, \xi) + O(\varepsilon^{2/3}).$$

The bottom left entry of $\varepsilon^{-1/3}A_{\star(0)}$ also involves $t-\tau_\star$ and $(t-\tau_\star)^{3/2}$ in the time-rescaled and advected frame. In view of (5.10), these functions contribute to $\varepsilon^{-1/3}A_{\star(0)}$

$$\varepsilon^{-1/3}(\varepsilon^{2/3}t - \tau_{\star\star}) = \varepsilon^{1/3}(t - t_\star) + O(\varepsilon^{2/3})$$

and $\varepsilon^{-1/3}(\varepsilon^{2/3}t - \tau_{\star\star})^{3/2} = O(\varepsilon^{2/3})$. We may invoke Lemma 5.3 to conclude the proof. □

By Lemma 5.4, the flow $S_{(0)}$ of $i\varepsilon^{-1/3}A_{\star(0)}$ solves

$$\partial_t S_{(0)} + i\varepsilon^{-1/3}\begin{pmatrix} 0 & \varepsilon^{-1/3} \\ -\varepsilon^{1/3}(t-t_\star)f_0 & 0 \end{pmatrix} S_{(0)} = \varepsilon^{2/3} C S_{(0)}, \quad S_{(0)}(\tau;\tau) = \mathrm{Id}, \tag{5.11}$$

where $C := C(\varepsilon, t, x, \xi) = \left(\begin{smallmatrix} 0 & 0 \\ c & 0 \end{smallmatrix}\right)$ with $|\partial_x^\alpha \partial_\xi^\beta c| \lesssim 1$ for $0 \leq t \leq T(\varepsilon)$ and $|x| + |\xi - \xi_0| \leq \delta$. The coefficient $f_0 = f_0(\varepsilon, x, \xi)$ satisfies $f_0 > 0$ for $|x| + |\xi - \xi_0| \leq \delta$.

5.4. *Reduction to a perturbed Airy equation*

Let

$$D(x,\xi) := \begin{pmatrix} -i\varepsilon^{1/3} f_0(x,\xi)^{1/3} & 0 \\ 0 & 1 \end{pmatrix},$$

so that D is well-defined and invertible on $|x| < \delta$, $|\xi - \xi_0| \leq \delta$, and

$$Z(\tau; t) := D S_{(0)}(f_0^{-1/3}\tau + t_\star\,;\, f_0^{-1/3}t + t_\star), \tag{5.12}$$

where f_0, $t_\star$, D, $S_{(0)}$ and Z all depend on (x, ξ). For future use, we note that

$$D(x,\xi)\begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} D(x,\xi)^{-1} = \begin{pmatrix} z_{11} & -i(\varepsilon f_0)^{1/3} z_{12} \\ i(\varepsilon f_0)^{-1/3} z_{21} & z_{22} \end{pmatrix}, \tag{5.13}$$

$$D(x,\xi)^{-1}\begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} D(x,\xi) = \begin{pmatrix} z_{11} & i(\varepsilon f_0)^{-1/3} z_{12} \\ -i(\varepsilon f_0)^{1/3} z_{21} & z_{22} \end{pmatrix}. \tag{5.14}$$

Lemma 5.5. *On $|x| \leq \delta$, $|\xi - \xi_0| \leq \delta$, the map Z satisfies the perturbed Airy equation*

$$Z' + \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix} Z = \varepsilon^{1/3}\tilde{C} Z, \qquad Z(\tau;\tau) = D, \tag{5.15}$$

where $\tilde{C} := (DCD^{-1})(f_0^{-1/3}t + t_\star)$ with C as in (5.11).

Proof. It suffices to use (5.13) and observe that, by Lemma 5.4,

$$\frac{i\varepsilon^{-1/3}}{f_0(x,\xi)^{1/3}} D(x,\xi) A_{\star(0)}\left(\frac{t}{f_0(x,\xi)^{1/3}} + t_\star,\ x,\ \xi\right) D(x,\xi)^{-1} = \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix} - \varepsilon^{1/3}\tilde{C}. \quad \square$$

From the above, we will deduce lower and upper bounds for $S_{(0)}$ by comparison with the vector Airy function $\mathbf{Z}$ defined as the solution of

$$\mathbf{Z}' + \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix} \mathbf{Z} = 0, \qquad \mathbf{Z}(\tau;\tau) = \mathrm{Id}. \tag{5.16}$$

5.5. *Bounds for the Airy function*

We will use (5.15) to show that the symbolic flow grows in time like the Airy function, for which the following is known (see for instance [5, Chapter 7.6]).

Lemma 5.6 (Airy equation). *Let* Ai *be the inverse Fourier transform of* $e^{i\xi^3/3}$, *and* $j = e^{2i\pi/3}$. *The functions* Ai, Ai$(j\,\cdot)$ *form a basis of solutions of the ordinary differential equation* $y'' = ty$, *and they satisfy*

$$\begin{aligned}
\mathrm{Ai}(t) &= \frac{1}{2\sqrt{\pi}} e^{-(2/3)t^{3/2}} t^{-1/4}(1 + O(t^{-3/2})), && t \to \infty,\\
\mathrm{Ai}(-t) &= \frac{1}{\sqrt{\pi}} t^{-1/4}\big(\sin\big(\tfrac{2}{3}t^{3/2} + \pi/4\big) + O(t^{-3/2})\big), && t \to \infty,\\
\mathrm{Ai}(jt) &= \frac{1}{2\sqrt{\pi}} e^{-i\pi/6} e^{(2/3)t^{3/2}} t^{-1/4}(1 + O(t^{-3/2})), && t \to \infty,\\
\mathrm{Ai}(-jt) &= \frac{1}{2\sqrt{\pi}} e^{i\pi/6} e^{(2/3)it^{3/2}} t^{-1/4}(1 + O(t^{-3/2})), && t \to \infty.
\end{aligned}$$

From the above lemma, we deduce uniform bounds for the time derivative Ai$'$:

$$e^{(2/3)t^{3/2}}|\mathrm{Ai}'(t)| + e^{-(2/3)t^{3/2}}|\mathrm{Ai}'(jt)| + |\mathrm{Ai}'(-t)| + |\mathrm{Ai}'(-jt)| \le C(1+t)^{1/4}$$

for some $C > 0$ and all $t \ge 0$. By Lemma 5.6, the solution to (5.16) is

$$\mathbf{Z}(\tau;t) = \frac{1}{W(\tau)} \begin{pmatrix} -j\mathrm{Ai}'(j\tau)\mathrm{Ai}(t) + \mathrm{Ai}'(\tau)\mathrm{Ai}(jt) & -\mathrm{Ai}(j\tau)\mathrm{Ai}(t) + \mathrm{Ai}(\tau)\mathrm{Ai}(jt) \\ j\mathrm{Ai}'(j\tau)\mathrm{Ai}'(t) - j\mathrm{Ai}'(\tau)\mathrm{Ai}'(jt) & \mathrm{Ai}(j\tau)\mathrm{Ai}'(t) - j\mathrm{Ai}(\tau)\mathrm{Ai}'(jt) \end{pmatrix},$$

where W is the Wronskian, satisfying

$$W(\tau) := \mathrm{Ai}(j\tau)\mathrm{Ai}'(\tau) - j\mathrm{Ai}'(j\tau)\mathrm{Ai}(\tau) \equiv \frac{1}{4\pi}(-\sqrt{3} + i).$$

The bounds for Ai and Ai$'$ imply the upper bound, for $0 \le \tau \le t$,

$$|\mathbf{Z}(\tau;t)| \le C(1+|\tau|)^{1/4}(1+|t|)^{1/4} e_{\mathrm{Ai}}(\tau;t), \tag{5.17}$$

and the lower bound

$$\left|(1\ \ 0)\mathbf{Z}(0;t)\begin{pmatrix}0\\1\end{pmatrix}\right| \geq c e_{\mathrm{Ai}}(0;t) \tag{5.18}$$

for some $c > 0$ independent of τ, t, where the growth function e_{Ai} is defined by

$$e_{\mathrm{Ai}}(\tau;t) = \exp\left(\tfrac{2}{3}(t_+^{3/2} - \tau_+^{3/2})\right), \qquad x_+ := \max(x,0). \tag{5.19}$$

We note that e_{Ai} is multiplicative:

$$e_{\mathrm{Ai}}(\tau;t')e_{\mathrm{Ai}}(t';t) = e_{\mathrm{Ai}}(\tau;t) \qquad \text{for all } \tau, t', t'. \tag{5.20}$$

Remark 5.7. If we assumed $\partial_\lambda^2 P \partial_t P < 0$, then we would have to consider the Airy condition for negative times. Lemma 5.6 would then yield polynomial bounds for the symbolic flow.

5.6. *Bounds for the symbolic flow*

Let

$$\Theta(t,x,\xi) := f_0(x,\xi)^{1/3}(\tau - t_\star(\varepsilon,x,\xi)). \tag{5.21}$$

Our goal is to verify the bounds of Assumption 2.1 for $S_{(0)}$ in the elliptic domain $\mathcal{D}$ defined in (2.2). We recall the definition of $\mathcal{D}$:

$$\mathcal{D} := \{(\tau;t,x,\xi) : t_\star(\varepsilon,x,\xi) \leq \tau \leq t \leq T(\varepsilon),\ |x| \leq \delta,\ |\xi - \xi_0| \leq \delta\varepsilon^{1/3}\}.$$

Lemma 5.8. *In $\mathcal{D}$,*

$$|D^{-1}\mathbf{Z}(\Theta(\tau);\Theta(t))D| \lesssim \begin{pmatrix}1 & \varepsilon^{-1/3}\\ \varepsilon^{1/3} & 1\end{pmatrix} e_{\mathrm{Ai}}(\Theta(\tau);\Theta(t)),$$

with e_{Ai} defined in (5.19).

Above, $\lesssim$ means *entrywise* inequality "modulo constants", as defined in (2.12).

Proof. In $\mathcal{D}$ we have $0 \leq \Theta(\tau) \leq \Theta(t) \lesssim \Theta(T(\varepsilon))$. The bound (5.17) states that $|\mathbf{Z}(\Theta)| \lesssim e_{\mathrm{Ai}}(\Theta)$. Then (5.14) implies the result. □

From Lemma 5.8 we now derive bounds for $S_{(0)}$. Given that $S_{(0)}$ is expressed in terms of Z defined in (5.12) and that Z is a perturbation of $\mathbf{Z}$, we are in a situation very much like the one of Section 3.12. Accordingly, the proof of the following corollary borrows from Section 3.12, in particular from the proofs of Corollaries 3.9 and 3.8.

Corollary 5.9. *The flow $S_{(0)}$ of $i\varepsilon^{-1/3}A_{\star(0)}$, which solves (5.11), satisfies*

$$|S_{(0)}| \lesssim \begin{pmatrix}1 & \varepsilon^{-1/3}\\ \varepsilon^{1/3} & 1\end{pmatrix} e_{\mathrm{Ai}}(\Theta). \tag{5.22}$$

Proof. By Lemma 5.5 and the definition (5.16) of $\mathbf{Z}$,

$$Z(\tau;t) = \mathbf{Z}(\tau;t)D + \varepsilon^{1/3}\int_\tau^t \mathbf{Z}(t';t)\tilde{C}(t')Z(\tau;t')\,dt'.$$

By the definition (5.12) of Z, we have $S_{(0)} = D^{-1}Z(\Theta)$. Thus

$$S_{(0)}(\tau;t) = D^{-1}\mathbf{Z}(\Theta(\tau);\Theta(t))D + \varepsilon^{2/3}\int_{\Theta(\tau)}^{\Theta(t)} D^{-1}\mathbf{Z}(t';\Theta(t))\tilde{C}(t')Z(\Theta(\tau);t')\,dt'.$$

Since $\tilde{C}$ is defined in Lemma 5.5 to be $(D^{-1}CD)(f_0^{-1/3}t + t_\star)$, we obtain

$$\begin{aligned} S_{(0)}(\tau;t) &= D^{-1}\mathbf{Z}(\Theta(\tau);\Theta(t))D \\ &\quad + \varepsilon^{2/3}\int_{\Theta(\tau)}^{\Theta(t)} D^{-1}\mathbf{Z}(t';\Theta(t))DC(f_0^{-1/3}t' + t_\star)S_{(0)}(\tau; f_0^{-1/3}t' + t_\star)\,dt'. \end{aligned}$$

The change of variable $t' = \Theta(\tau')$ corresponding to $\tau' = f_0^{-1/3}(t' + t_\star)$ transforms the above integral into

$$\varepsilon^{2/3} f_0^{1/3}\int_\tau^t D^{-1}\mathbf{Z}(\Theta(\tau');\Theta(t))DC(\tau')S_{(0)}(\tau;\tau')\,d\tau'.$$

We now factor out the expected growth in view of applying Gronwall's lemma, as we did before in the proof of Corollary 3.9: we let

$$S^\flat_{(0)} := e_{\mathrm{Ai}}(\Theta)^{-1}S_{(0)} \quad \text{and} \quad \mathbf{Z}^\flat(\tau';t) := e_{\mathrm{Ai}}(\Theta)^{-1}D^{-1}\mathbf{Z}(\Theta(\tau');\Theta(t))D.$$

By the multiplicative property (5.20) of the growth function e_{Ai}, we find

$$S^\flat_{(0)}(\tau;t) = \mathbf{Z}^\flat(\tau;t) + \varepsilon^{2/3} f_0\int_\tau^t \mathbf{Z}^\flat(\tau';t)C(\tau')S^\flat_{(0)}(\tau';t)\,d\tau'.$$

We now rescale the top right and bottom left entries, as we consider the equation for $\underline{S}^\flat_{(0)}$, with notation introduced just above (3.56). In view of (3.56),

$$\underline{S}^\flat_{(0)}(\tau;t) = \underline{\mathbf{Z}}^\flat(\tau;t) + \varepsilon^{1/3} f_0\int_\tau^t \underline{\mathbf{Z}}^\flat(\tau';t)(\varepsilon^{1/3}\underline{C}(\tau'))\underline{S}^\flat_{(0)}(\tau';t)\,d\tau'. \tag{5.23}$$

We have $\varepsilon^{1/3}\underline{C}(t) = O(t)$, and t is bounded by some power of $|\ln\varepsilon|$ in $\mathcal{D}$. Lemma 5.8 implies that $|\underline{\mathbf{Z}}^\flat| \lesssim 1$. Hence Gronwall's lemma implies the bound $|\underline{S}^\flat_{(0)}(\tau;t)| \lesssim 1$, which corresponds precisely to (5.22). □

Lemma 5.10. *For some universal constant $c_0 > 0$,*

$$\left|(1\ \ 0)S_{(0)}\begin{pmatrix}0\\1\end{pmatrix}\right| \geq c_0\,\varepsilon^{-1/3}e_{\mathrm{Ai}}(\Theta).$$

Proof. Consider the representation (5.23). We focus on the top right entry. The lower bound (5.18) for the vector Airy function states that the top right entry of $\mathbf{Z}$ is bounded from below by e_{Ai}. By (5.14), this implies

$$\left|(1 \;\; 0) D^{-1} \mathbf{Z} D \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right| \geq c_0 \varepsilon^{-1/3} e_{\mathrm{Ai}} \tag{5.24}$$

for some $c_0 > 0$ independent of τ, t. Borrowing notation from the proof of Corollary 5.9, this means that the top right entry of $\underline{\mathbf{Z}}^\flat$ is bounded away from zero, uniformly in time. We know from Corollary 5.9 that $|\underline{S}^\flat_{(0)}| \lesssim 1$ and $|\underline{\mathbf{Z}}^\flat| \lesssim 1$. Thus from (5.23) and (5.24) we deduce the result, since $t \lesssim 1$ in $\mathcal{D}$. □

We observe that, for e_{Ai} defined in (5.19) and Θ defined in (5.21),

$$e_{\mathrm{Ai}}(\Theta) \equiv \mathbf{e}_\gamma \quad \text{with} \quad \gamma(x, \xi) := \tfrac{2}{3} f_0(x, \xi)^{1/2}, \quad t_\star = \varepsilon^{-2/3} \tau_\star(x_0 + \varepsilon^{1/3} x, \xi),$$

where $\tau_\star$ is given by the implicit function theorem in Section 5.1. Hence Corollary 5.9 and Lemma 5.10 yield the bounds of Assumption 2.1 for $S_{(0)}$ with $\gamma^+ = \gamma^- = \gamma$ and with $\vec{e}$ equal to the constant vector $\binom{0}{1}$.

In order to complete the verification of Assumption 2.1, and thus conclude the proof of Theorem 1.3, it remains to show that the other components of the symbolic flow do not grow faster than $S_{(0)}$. This follows directly from the simplicity hypothesis in Theorem 1.3. Indeed, by that hypothesis, we may smoothly diagonalize the other component $A_{\star(1)}$ of $A_\star$ near $(0, x_0, \xi_0)$ (use for instance [25, Corollary 2.2]). The eigenvalues of $A_{\star(1)}$ are real near $(0, x_0, \xi_0)$. The equation for the symbolic flow of $A_{\star(1)}$ splits into scalar differential equations, with purely imaginary coefficients. Thus the symbolic flow of $A_{\star(1)}$ is bounded.

6. Proof of Theorem 1.6: smooth defect of hyperbolicity

It suffices to verify that, under the assumptions of Theorem 1.6, Assumption 2.1 holds with parameters

$$\ell = 1, \quad h = 1/2, \quad \zeta = 0, \quad \mu = \Re e\, \lambda_\pm, \quad t_\star \equiv 0,$$

where $\lambda_\pm$ are the bifurcating eigenvalues, as given by Proposition 1.4.

6.1. Block decomposition

As in the proof of Theorem 1.2, we may smoothly block diagonalize A by a change of basis $Q(t, x, \xi)$, for small t and (x, ξ) close to (x_0, ξ_0). Then identity (4.2) holds, and we focus on the block $A_{(0)}$, of size two, such that

$$\operatorname{sp} A_{(0)}(0, x_0, \xi_0) = \{\lambda_0\}, \qquad \lambda_0 \in \mathbb{R},$$

where (x_0, ξ_0, λ_0) are the coordinates of $\omega_0 \in \Gamma$ which appears in Hypothesis 1.5. By Hypothesis 1.5 and Proposition 1.4, the eigenvalues $\lambda_\pm$ of $A_{(0)}$ branch from the real axis at $t = 0$, for all (x, ξ) in a neighborhood of (x_0, ξ_0). We define μ to be the real part of these eigenvalues. The corresponding equation for the symbolic flow is

$$\partial_t S_{(0)} + \varepsilon^{-1/2}(A_{(0)} - \mu)\big(\varepsilon^{1/2}t, x_0 + \varepsilon^{1/2}x_\star(\varepsilon^{1/2}t, x, \xi), \xi_\star(\varepsilon^{1/2}t, x, \xi)\big)S_{(0)} = 0, \quad (6.1)$$

where $(x_\star, \xi_\star)$ are the bicharacteristics of μ.

6.2. Time regularity and cancellation

By Proposition 1.4, the eigenvalues $\lambda_\pm$ are differentiable in time, at $t = 0$ and for all (x, ξ) near (x_0, ξ_0). Indeed, Hypothesis 1.5 implies that conditions (1.12) are satisfied in a whole neighborhood of (x_0, ξ_0). We may thus write

$$\lambda_\pm(\varepsilon^{1/2}t, x_0 + \varepsilon^{1/2}x_\star, \xi_\star) - \mu(0, x_0 + \varepsilon^{1/2}x_\star, \xi_\star) = i\varepsilon^{1/2}t\tilde{\lambda}_\pm(\varepsilon, t, x, \xi) + o(\varepsilon^{1/2}), \quad (6.2)$$

uniformly in $t = O(|\ln \varepsilon|)$ and (x, ξ) close to (x_0, ξ_0), where $(x_\star, \xi_\star)$ is evaluated at $(\varepsilon^{1/2}t, x, \xi)$, and where

$$\tilde{\lambda}_\pm(\varepsilon, 0, x, \xi) = \partial_t \Im m\, \lambda_\pm(0, x_0 + \varepsilon^{1/2}x_\star(0, x, \xi), \xi_\star(0, x, \xi)) \in \mathbb{R}. \quad (6.3)$$

Consider the 2×2 matrix $A_{(0)}(0, x, \xi)$. It has one semisimple eigenvalue $\mu(0, x, \xi)$ (the semisimplicity is part of Hypothesis 1.5). Thus

$$A_{(0)}(0, x, \xi) = \mu(0, x, \xi)\mathrm{Id}.$$

In particular, by regularity of the entries of A,

$$\varepsilon^{-1/2}A_{(0)}(\varepsilon^{1/2}t, x_0 + \varepsilon^{1/2}x_\star, \xi_\star) = t\tilde{A}_{(0)}(\varepsilon, 0, x, \xi) + \varepsilon^{1/2}t^2 B(\varepsilon, t, x, \xi), \quad (6.4)$$

where B is uniformly bounded for ε close to 0, $t = O(|\ln \varepsilon|^*)$ and (x, ξ) close to (x_0, ξ_0). Thus equation (6.1) takes the form

$$\partial_t S_{(0)} + t\tilde{A}_{(0)}(\varepsilon, 0, x, \xi)S_{(0)} = \varepsilon^{1/2}t^2 B(\varepsilon, t, x, \xi)S_{(0)}. \quad (6.5)$$

The key cancellation that takes place in (6.4) has transformed the equation for $S_{(0)}$ into an autonomous equation with a small, linear, time-dependent perturbation. The eigenvalues of $\tilde{A}_{(0)}$ are $\tilde{\lambda}_\pm(\varepsilon, 0, x, \xi)$ from (6.2)–(6.3). These eigenvalues are distinct by Proposition 1.4.

6.3. Bounds for the symbolic flow

The solution S to

$$\partial_t S + it\tilde{A}_{(0)}(\varepsilon, 0, x, \xi)S = 0, \qquad S(\tau; \tau) = \mathrm{Id},$$

is

$$S(\tau; t) = \exp\big(-i\tilde{A}_{(0)}(\varepsilon, 0, x, \xi)(t^2 - \tau^2)/2\big).$$

The eigenvalues of $\tilde{A}_{(0)}$, being distinct, are smooth in (ε, x, ξ) (see for instance [25, Corollary 2.2]). In particular,

$$\tilde{\lambda}_\pm(\varepsilon, 0, x, \xi) = \tilde{\lambda}_\pm(0, 0, x, \xi) + O(\varepsilon) = \Im m\, \partial_t\lambda_\pm(0, x_0, \xi_\star(0, x, \xi))$$

locally uniformly in (x, ξ). Let λ_+ be the eigenvalue with positive imaginary part, and

$$\gamma(x, \xi) := \frac{1}{2}\tilde{\lambda}_\pm(0, 0, x, \xi) = \frac{1}{2}\,\Im m\, \partial_t \lambda_+(0, x_0, \xi_\star(0, x, \xi)).$$

Then

$$|S(\tau; t, x, \xi)| \lesssim \exp(\gamma(x, \xi)(t^2 - \tau^2)), \tag{6.6}$$

and since $\tilde{A}_{(0)}$ is smoothly diagonalizable, for some smoothly varying vector $\vec{e}(x, \xi)$ we have

$$|S(\tau; t, x, \xi)\vec{e}(x, \xi)| \gtrsim \exp(\gamma(x, \xi)(t^2 - \tau^2)). \tag{6.7}$$

Perturbation arguments already encountered in Section 3.12 (specifically, in the proof of Corollary 3.9) show that the bounds (6.6)–(6.7) for S yield similar bounds for the symbolic flow $S_{(0)}$ solving (6.5). These bounds imply the upper and lower bounds (2.11) and (2.10) from Assumption 2.1.

For the other components of the flow, we use the simplicity assumption in Theorem 1.6, as we did in the last paragraph of Section 5.6 in the proof of Theorem 1.3.

7. Examples

7.1. One-dimensional Burgers systems

The 2×2, one-dimensional Burgers system

$$\partial_t \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} u_1 & -b(u)^2 u_2 \\ u_2 & u_1 \end{pmatrix} \partial_x \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = F(u_1, u_2), \tag{7.1}$$

where F and b are smooth and real-valued, has a complex structure if b is constant. In the case $b \equiv 1$, $F \equiv (0, 1)$, a strong instability result for the Cauchy–Kovalevskaya solution issued from $(u_1^0, 0)$, where u_1^0 is analytic and real-valued, was proved in [12].

We assume $b > 0$, and the existence of a local smooth solution $\phi = (\phi_1, \phi_2)$. The principal symbol is

$$A(t, x, \xi) = \xi \begin{pmatrix} \phi_1 & -b(\phi)^2 \phi_2 \\ \phi_2 & \phi_1 \end{pmatrix}.$$

Without loss of generality, we let $\xi = 1$. The eigenvalues and eigenvectors are

$$\lambda_\pm = \phi_1 \pm i\phi_2 b(\phi), \qquad e_\pm = \frac{1}{(1 + b(\phi)^2)^{1/2}} \begin{pmatrix} \pm i b(\phi) \\ 1 \end{pmatrix}.$$

The characteristic polynomial is

$$P = (\lambda - \phi_1)^2 + b(\phi)^2 \phi_2^2.$$

Initial ellipticity. If $\phi_2(0, x_0) \neq 0$ for some $x_0 \in \mathbb{R}$, then the principal symbol is elliptic at $t = 0$, and Theorem 1.2 applies.

Smooth defect of hyperbolicity. Consider the case $\phi_2(0,x) \equiv 0$. We cannot observe a defect of hyperbolicity as in Theorem 1.3, since the eigenvalues are smooth in time. Via Proposition 1.4, we see that Theorem 1.6 holds as soon as

$$F_2(\phi(0,x_0)) \neq 0 \quad \text{for some } x_0 \in \mathbb{R}. \tag{7.2}$$

If $b(u) = b(u_2)$, then (7.1) is a system of conservation laws

$$\partial_t u_1 + \partial_x f_1(u) = F_1(u), \qquad \partial_t u_2 + \partial_x f_2(u) = F_2(u),$$

with fluxes

$$f_1(u) = \tfrac{1}{2}u_1^2 - \int_0^{u_2} y b(y)^2 \, dy, \qquad f_2(u) = u_1 u_2.$$

If, for instance, $F(u) = (0, u_1^2)$ and $b(u_2) = 1 + u_2^2$, then the system is ill-posed for all data.

7.2. *Two-dimensional Burgers systems*

Consider the following family of 2×2 systems in $\mathbb{R}^2$:

$$\partial_t u + \begin{pmatrix} u_1 \partial_{x_1} & -b(u)^2 u_2 (\partial_{x_2} + \partial_{x_1}) \\ u_2(\partial_{x_1} + \partial_{x_2}) & u_1 \partial_{x_1} \end{pmatrix} u = F(u).$$

We assume $b > 0$, and the existence of a local smooth solution $\phi = (\phi_1, \phi_2)$. The principal symbol is

$$A = \begin{pmatrix} \xi_1 \phi_1 & -(\xi_1 + \xi_2) b(\phi)^2 \phi_2 \\ (\xi_1 + \xi_2)\phi_2 & \xi_1 \phi_1 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$\lambda_\pm = \xi_1 \phi_1 \pm i(\xi_1 + \xi_2)\phi_2 b(\phi), \qquad e_\pm = \frac{1}{(1 + b(\phi)^2)^{1/2}} \begin{pmatrix} \pm i b(\phi) \\ 1 \end{pmatrix}.$$

Initial ellipticity. If $\phi_2(0,x_0) \neq 0$ for some $x_0 \in \mathbb{R}^2$, then the principal symbol is initially elliptic at any $(\xi_1, \xi_2) \in \mathbb{S}^1$ such that $\xi_1 + \xi_2 \neq 0$.

Smooth defect of hyperbolicity. Consider the case $\phi_2(0,x) \equiv 0$. By Proposition 1.4, the assumptions of Theorem 1.6 are satisfied under condition (7.2).

7.3. *Van der Waals gas dynamics*

The compressible Euler equations in one space dimension, in Lagrangian coordinates, are

$$\begin{cases} \partial_t u_1 + \partial_x u_2 = 0, \\ \partial_t u_2 + \partial_x p(u_1) = 0. \end{cases}$$

We assume that the smooth pressure law p satisfies the Van der Waals condition

$$p'(u_1) \le 0 \quad \text{for some } u_1 \in \mathbb{R},$$

and we assume existence of a smooth solution $\phi = (\phi_1, \phi_2)$. The principal symbol at $\xi = 1$ is

$$A = \begin{pmatrix} 0 & 1 \\ p'(\phi_1) & 0 \end{pmatrix}.$$

The eigenvalues are $\lambda_\pm = (p'(\phi_1))^{1/2}$. *Initial ellipticity.* If $p'(\phi_1(0, x_0)) < 0$ for some $x_0 \in \mathbb{R}$, then Theorem 1.2 applies.

Non-semisimple defect of hyperbolicity. If $p'(\phi_1(0, x)) \ge 0$ for all x (initial hyperbolicity) and $p'(\phi_1(0, x_0)) = 0$ for some x_0 (coalescence of two eigenvalues), and if

$$p''(\phi_1(0, x_0))\partial_x \phi_2(0, x_0) > 0,$$

then condition (1.11) holds, and Theorem 1.3 applies.

7.4. *Klein–Gordon–Zakharov systems*

Consider the family of systems in one space dimension

$$\begin{cases} \partial_t \begin{pmatrix} u \\ v \end{pmatrix} + \partial_x \begin{pmatrix} v \\ u \end{pmatrix} + \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} \partial_x \begin{pmatrix} n \\ m \end{pmatrix} = (n+1) \begin{pmatrix} v \\ -u \end{pmatrix}, \\ \partial_t \begin{pmatrix} n \\ m \end{pmatrix} + c\partial_x \begin{pmatrix} m \\ n \end{pmatrix} + \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} \partial_x \begin{pmatrix} u \\ v \end{pmatrix} = \partial_x \begin{pmatrix} 0 \\ u^2 + v^2 \end{pmatrix}, \end{cases} \tag{7.3}$$

indexed by $\alpha \in \mathbb{R}$ and $c \in \mathbb{R} \setminus \{-1, 1\}$. We assume existence of a smooth solution $\phi = (u, v, n, m)$. The principal symbol at $\xi = 1$ is

$$A = \begin{pmatrix} 0 & 1 & \alpha & 0 \\ 1 & 0 & 0 & 0 \\ \alpha & 0 & 0 & c \\ -2u & -2v & c & 0 \end{pmatrix}. \tag{7.4}$$

The case $\alpha = 0$. The principal symbol is block diagonal, and there are four distinct eigenvalues $\{\pm 1, \pm c\}$. This implies that (7.3) is strictly hyperbolic, hence locally well-posed in H^s for $s > 3/2$ (see for instance [17, Theorem 7.3.3]). It was observed in [2] that for $c \notin \{-1, 1\}$ and $\alpha = 0$, system (7.3) is conjugate to a *semilinear* system, which implies a sharper existence result:

Proposition 7.1 ([2, Section 2.2]). *If* $c \notin \{-1, 1\}$ *and* $\alpha = 0$*, the system* (7.3) *is locally well-posed in* $H^s(\mathbb{R})$ *for* $s > 1/2$.

Proof. The change of variables

$$\begin{aligned} (\tilde{u}, \tilde{v}) &= (u + v, u - v), \\ (\tilde{n}, \tilde{m}) &= \left(n + m - \frac{1}{1-c}\tilde{u}^2 - \frac{1}{1+c}\tilde{v}^2, n - m - \frac{1}{1+c}\tilde{u}^2 - \frac{1}{1-c}\tilde{v}^2 \right), \end{aligned}$$

transforms (7.3) into a system for $\tilde U := (\tilde u, \tilde v, \tilde n, \tilde m)$:

$$\partial_t \tilde U + \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c \\ 0 & 0 & c & 0 \end{pmatrix} \partial_x \tilde U = (n+1) \begin{pmatrix} -\tilde v \\ \tilde u \\ -2(1-c)^{-1}\tilde u \tilde v \\ -2(1+c)^{-1}\tilde u \tilde v \end{pmatrix}. \tag{7.5}$$

System (7.5), being symmetric hyperbolic and semilinear, is locally well-posed in $H^s(\mathbb{R})$ for $s > 1/2$. □

The case $\alpha \neq 0$. By Proposition 7.1, system (7.3) takes the form of a symmetric perturbation of a well-posed system. The characteristic polynomial of the principal symbol (7.4) at $\xi = 1$ is

$$P(t, x, \lambda) = (\lambda^2 - c^2)(\lambda^2 - 1) - \alpha^2\lambda^2 + 2\alpha c(v + u\lambda).$$

Consider an initial datum for $(u(0), v(0), n(0), m(0))$ such that, for some $x_0 \in \mathbb{R}$,

$$u(0, x_0) = 0, \qquad v(0, x_0) = -\frac{c}{2\alpha}, \qquad \alpha c \partial_x u(0, x_0) > 0. \tag{7.6}$$

The first two conditions in (7.6) imply that at $\omega_0 = (x_0, 1, 0)$,

$$P(0, \omega_0) = \partial_\lambda P(0, \omega_0) = 0.$$

The third condition in (7.6) implies

$$(\partial_t P \partial_\lambda^2 P)(0, \omega_0) = \big(2\alpha c \partial_t v(0, x_0)\big)(-1 - c^2 - \alpha^2) = \big(2\alpha c \partial_x u(0, x_0)\big)(1 + c^2 + \alpha^2) > 0,$$

so that the third condition in (7.6) implies (1.11). Theorem 1.3 thus asserts instability of the Cauchy problem for (7.3) in the vicinity of any smooth solution ϕ satisfying (7.6) at $t = 0$.

In particular, for any given $\alpha_0 > 0$, we can find initial data, depending on α_0, such that (7.3) with $\alpha = 0$ is well-posed, whereas (7.3) with $\alpha = \alpha_0$ is ill-posed. Such initial data are $O(1/\alpha_0)$ in $L^\infty(\mathbb{R})$.

Appendix A. Proof of Proposition 1.4

The principal symbol can be block diagonalized, with a 2×2 block A_0 with double real eigenvalue λ_0 at $(0, x, \xi)$, and an $(N-2) \times (N-2)$ block which does not admit λ_0 as an eigenvalue at $(0, x, \xi)$. Throughout this proof, (x, ξ) are fixed and omitted in the arguments. The characteristic polynomial of A factorizes into $P = P_0 P_1$, where $P_1(0, \omega_0) \neq 0$ and P_0, P_1 have real coefficients. We may concentrate on P_0:

$$P_0(\lambda) = \lambda^2 - \lambda \operatorname{tr} A_0 + \det A_0.$$

The eigenvalues $\lambda_\pm$ of A_0 at (t, x, ξ) are

$$\lambda_\pm(t) = \tfrac{1}{2} \operatorname{tr} A_0(t) \pm \tfrac{1}{2}\Delta(t)^{1/2}, \qquad \Delta(t) := (\operatorname{tr} A_0)^2 - 4 \det A_0. \tag{A.1}$$

By assumption, these eigenvalues coalesce at $t = 0$, so that $\Delta(0) = 0$. The goal is then to prove the equivalence (1.12).

If the left side of (1.12) holds, then $\Delta(t) = -\alpha t^2 + O(t^3)$ with $\alpha > 0$. Thus $\partial_t \Delta(0) = 0$; on the other hand,

$$\begin{aligned}\partial_t \Delta(0) &= 2 \operatorname{tr} A_0(0) \partial_t \operatorname{tr} A_0(0) - 4\partial_t \det A_0(0) = 4\lambda_0 \partial_t \operatorname{tr} A_0(0) - 4\partial_t \det A_0(0) \\ &= -4(\partial_t P_0)(0).\end{aligned}$$

Moreover, $\partial_t^2 \Delta(0) < 0$; on the other hand,

$$\partial_t^2 \Delta(0) = 2(\partial_t \operatorname{tr} A_0(0))^2 + 2 \operatorname{tr} A_0(0) \partial_t^2 \operatorname{tr} A_0(0) - 4\partial_t^2 \det A_0(0),$$

implying, since $\operatorname{tr} A_0(0) = 2\lambda_0$,

$$\begin{aligned}\partial_t^2 \Delta(0) &= 2(\partial_t \operatorname{tr} A_0(0))^2 + 4\lambda_0 \partial_t^2 \operatorname{tr} A_0(0) - 4\partial_t^2 \det A_0(0) \\ &= 2(\partial_t \partial_\lambda P_0)^2 - 2\partial_\lambda^2 P_0 \partial_t^2 P_0(0),\end{aligned}$$

which gives indeed $(\partial_{t\lambda}^2 P)^2 < \partial_t^2 P \partial_\lambda^2 P$ at $t = 0$.

The converse implication is proved in the same way: the right side of (1.12) implies $\partial_t \Delta(0) = 0$, $\partial_t^2 \Delta(0) < 0$, as shown above, and this implies that the eigenvalues in (A.1) are differentiable and leave the real axis at $t = 0$.

Appendix B. Symbols and operators

Pseudo-differential operators in ε^h-semiclassical quantization are defined by

$$\operatorname{op}_\varepsilon(a)u := (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} a(x, \varepsilon^h \xi) \hat{u}(\xi)\, d\xi, \qquad \varepsilon, h > 0. \tag{B.1}$$

Here $h = 1/(1+\ell)$, as in Assumption 2.1, and a is a classical symbol of order m: $a \in S^m$, for some $m \in \mathbb{R}$, that is, a smooth map in (x, ξ), with values in a finite-dimensional space, such that

$$\|a\|_{m,r} := \sup_{\substack{|\alpha|,|\beta| \le r \\ (x,\xi)\in\mathbb{R}^{2d}}} \langle \xi \rangle^{|\beta|-m} |\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| < \infty \tag{B.2}$$

A family $\|\cdot\|_{\varepsilon,s}$ of ε-dependent norms is defined by

$$\|u\|_{\varepsilon,s} := \|\langle \varepsilon^h \xi \rangle^{s/2} \hat{u}(\xi)\|_{L^2(\mathbb{R}^d_\xi)}, \qquad s \in \mathbb{R}, \qquad \langle \cdot \rangle := (1 + |\cdot|^2)^{1/2}.$$

Introducing dilations (d_ε) such that $(d_\varepsilon u)(x) = \varepsilon^{hd/2} u(\varepsilon^h x)$, we observe that

$$\|d_\varepsilon u\|_{H^s} = \|u\|_{\varepsilon,s}, \qquad \operatorname{op}_\varepsilon(a) = d_\varepsilon^{-1} \operatorname{op}(\tilde{a}) d_\varepsilon, \qquad \tilde{a}(x, \xi) := a(\varepsilon^h x, \xi). \tag{B.3}$$

Proposition B.1. *Given $m \in \mathbb{R}$ and $a \in S^m$,*

$$\|\mathrm{op}_\varepsilon(a)u\|_{L^2} \lesssim \|a\|_{m,C(d)}\|u\|_{\varepsilon,-m} \tag{B.4}$$

for all $u \in H^{-m}$, with some $C(d) > 0$ depending only on d. If $m = 0$, then

$$\|\mathrm{op}_\varepsilon(a)u\|_{L^2} \lesssim \sum_{0\le|\alpha|\le d+1} \sup_{\xi\in\mathbb{R}^d} \|\partial_x^\alpha a(\cdot,\xi)\|_{L^1(\mathbb{R}^d_x)}\|u\|_{L^2}. \tag{B.5}$$

Proof. By use of dilations (B.3), we observe that $\mathrm{op}_\varepsilon(a)u = \mathrm{op}_1(\langle\xi\rangle^{-m}\tilde{a})\langle D\rangle^m d_\varepsilon u$. Then (B.4) with any $C(d) > [d/2] + 1$ follows for instance from [11, Theorem 1.1.4 and its proof], and (B.5) is proved in [5, Vol. 3, Theorem 18.8.1]. □

Proposition B.2. *Given $a_1 \in S^{m_1}$, $a_2 \in S^{m_2}$, and $n \in \mathbb{N}$,*

$$\mathrm{op}_\varepsilon(a_1)\,\mathrm{op}_\varepsilon(a_2) = \sum_{q=0}^{n} \varepsilon^{hq}\,\mathrm{op}_\varepsilon(a_1 \sharp_q a_2) + \varepsilon^{h(n+1)}\,\mathrm{op}_\varepsilon(R_{n+1}(a_1,a_2)),$$

where

$$a_1 \sharp_q a_2 = \sum_{|\alpha|=q} \frac{(-i)^{|\alpha|}}{\alpha!}\partial_\xi^\alpha a_1 \partial_x^\alpha a_2, \tag{B.6}$$

and $R_{n+1}(a_1,a_2) \in S^{m_1+m_2-(n+1)}$ satisfies

$$\|\mathrm{op}_\varepsilon(R_{n+1}(a_1,a_2))u\|_{L^2} \lesssim \|\partial_\xi^n a_1\|_{m_1,C(d)}\|\partial_x^n a_2\|_{m_2,C(d)}\|u\|_{\varepsilon,m_1+m_2-n-1},$$

with $C(d) > 0$ depending only on d, for all $u \in H^{m_1+m_2-n-1}$.

Proof. Use for instance [11, Theorem 1.1.20, Lemma 4.1.2 and Remark 4.1.4] and (B.3). □

Specializing to symbols with a slow x-dependence, we obtain:

Proposition B.3. *Given $a_1 \in S^{m_1}$ and $a_2 \in S^{m_2}$, if a_2 depends on x through $\varepsilon^{1-h}x$, then*

$$\left\|\big(\mathrm{op}_\varepsilon(a_1)\,\mathrm{op}_\varepsilon(a_2) - \mathrm{op}_\varepsilon(a_1a_2)\big)u\right\|_{\varepsilon,s} \lesssim \varepsilon\|a_1\|_{m_1,C(d)}\|a_2\|_{m_2,C(d)}\|u\|_{\varepsilon,s+m_1+m_2-1}.$$

Appendix C. On extending locally defined symbols

Our assumptions are local in (x,ξ) around (x_0,ξ_0). Accordingly the symbols Q and μ that appear in Assumption 2.1 are defined (after a change of spatial frame) only locally around $(0,\xi_0)$. We explain here how to extend the locally defined family of invertible matrices $Q(x,\xi)$ into an element of S^0 with an inverse (in the sense of matrices) which belongs to S^0.

The spectrum of $Q(0,\xi_0)$ is a finite subset of $\mathbb{C}$. In particular, we can find $\alpha \in \mathbb{R}$ such that the spectrum is contained in $\mathbb{C} \setminus e^{i\alpha}\mathbb{R}_-$. By continuity of the spectrum, for all (x,ξ) close enough to $(0,\xi_0)$, the spectrum of $Q(x,\xi)$ does not intersect the half-line $e^{i\alpha}\mathbb{R}_-$.

Let $\delta > 0$ be such that this property holds true over $B_\delta = B(0,\delta) \times B(\xi_0,\delta)$. We may then define the logarithm of the matrix $e^{-i\alpha}Q$ in B_δ by

$$\mathrm{Log}(e^{-i\alpha}Q) = \int_0^1 (e^{-i\alpha}Q - \mathrm{Id})\big((1-t)\,\mathrm{Id} + te^{-i\alpha}Q\big)^{-1}\,dt,$$

and the notation Log is justified by the identity

$$\exp\mathrm{Log}(e^{-i\alpha}Q) = e^{-i\alpha}Q \quad \text{ in } B_\delta. \tag{C.1}$$

Let $\sigma(x,\xi)$ be a smooth cut-off in $C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ such that $0 \le \sigma(x,\xi) \le 1$, $\sigma \equiv 1$ on a neighborhood of (x_0,ξ_0), and the support of σ is included in $B_{\delta/2}$. Let

$$R(x,\xi) = \sigma(x,\xi)\,\mathrm{Log}(e^{-i\alpha}Q(x,\xi)) + (1-\sigma(x,\xi))\,\mathrm{Id} \quad \text{ in } B_\delta.$$

We may extend smoothly R by setting $R \equiv \mathrm{Id}$ on the complement of B_δ in $\mathbb{R}^{2d}$. Then for all $(x,\xi) \in \mathbb{R}^{2d}$, the matrix

$$\tilde{Q}(x,\xi) = \exp R(x,\xi)$$

is smooth and invertible. Moreover,

$$\inf_{\mathbb{R}^{2d}} \det \tilde{Q} > 0.$$

Indeed, the infimum over the closed ball $\bar{B}_\delta$ is positive, by compactness and continuity, and the determinant is constantly e^N outside $\bar{B}_\delta$. Thus the norms $|\tilde{Q}(x,\xi)|$ and $|\tilde{Q}(x,\xi)^{-1}|$ are globally bounded over $\mathbb{R}^{2d}$. Since $\tilde{Q}$ is constant outside a compact set, this implies $\tilde{Q} \in S^0$ and $\tilde{Q}^{-1} \in S^0$. Finally, by (C.1) and the definition of the cut-off σ,

$$\tilde{Q}(x,\xi) = e^{-i\alpha}Q(x,\xi) \quad \text{ for } (x,\xi) \text{ close to } (0,\xi_0).$$

Thus $e^{i\alpha}\tilde{Q}$ is an appropriate extension of Q.

Appendix D. An integral representation formula

We adapt to the present context an integral representation formula introduced in [24]. Consider the following initial value problem, posed in the time interval $[0, T(\varepsilon)]$ with $T(\varepsilon) := (T_\star|\ln\varepsilon|)^{1/(1+\ell)}$ for some $T_\star > 0$ and some $\ell \ge 0$:

$$\partial_t u + \mathrm{op}_\varepsilon(\mathcal{A})u = g, \qquad u(0) = u_0, \tag{D.1}$$

where $\mathcal{A} = \mathcal{A}(\varepsilon, t)$ belongs to S^0 for all $\varepsilon > 0$ and all $t \le T(\varepsilon)$. Recall that $\mathrm{op}_\varepsilon(\cdot)$ denotes ε^h-semiclassical quantization of operators, as defined in (B.1). The parameter h belongs to $(0,1]$. The datum u_0 belongs to L^2, and the source g is in $C^0([0,T(\varepsilon)], L^2(\mathbb{R}^d))$. Denote by S_0 the flow of $-\mathcal{A}$, defined for $0 \le \tau \le t \le T(\varepsilon)$ by

$$\partial_t S_0(\tau;t) + \mathcal{A}S_0(\tau;t) = 0, \qquad S_0(\tau;\tau) = \mathrm{Id}.$$

For some $q_0 \in \mathbb{N}^*$, denote by $\{S_q\}_{1\le q\le q_0}$ the solution to the triangular system of linear ordinary differential equations

$$\partial_t S_q + \mathcal{A} S_q + \sum_{\substack{q_1+q_2=q\\ q_1>0}} \mathcal{A} \sharp_{q_1} S_{q_2} = 0, \qquad S_q(\tau;\tau) = 0, \tag{D.2}$$

with $\sharp_q$ introduced in (B.6).

Assumption D.1. *The symbol $\mathcal{A}$ is compactly supported in x, uniformly in ε, t, ξ, and*

$$\langle \xi\rangle^{|\beta|} |\partial_x^\alpha \partial_\xi^\beta \mathcal{A}(\varepsilon, t, x, \xi)| \lesssim \varepsilon^{-|\beta|\zeta},$$
$$|\partial_x^\alpha \partial_\xi^\beta S_q(\varepsilon, \tau; t, x, \xi)| \lesssim \varepsilon^{-\zeta(1+|\beta|+q)} \exp(\gamma t^{\ell+1}).$$

for some $0 \le \zeta < h$, some $\gamma > 0$, all x, ξ and all $t \le T(\varepsilon)$.

Denote $\Sigma := \sum_{0\le q\le q_0} \varepsilon^{qh} S_q$. Then $\mathrm{op}_\varepsilon(\Sigma)$ is an approximate solution operator for (D.1):

Lemma D.2. *Under Assumption D.1, if q_0 is large enough, depending on ζ, h, γ and $T_\star$, then*

$$\partial_t \,\mathrm{op}_\varepsilon(\Sigma) + \mathrm{op}_\varepsilon(\mathcal{A})\,\mathrm{op}_\varepsilon(\Sigma) = \rho, \tag{D.3}$$

where ρ satisfies, for $0 \le \tau \le t \le T(\varepsilon)$ and all $u \in L^2(\mathbb{R}^d)$,

$$\|\rho(\tau;t)u\|_{L^2} \lesssim \varepsilon \|u\|_{L^2}. \tag{D.4}$$

Proof. By Proposition B.2,

$$\mathrm{op}_\varepsilon(\mathcal{A})\,\mathrm{op}_\varepsilon(S_q) = \mathrm{op}_\varepsilon(\mathcal{A}S_q) + \sum_{1\le q'\le q_0} \varepsilon^{q'h}\,\mathrm{op}_\varepsilon(\mathcal{A}\sharp_{q'} S_q) + \varepsilon^{(q_0+1)h}\,\mathrm{op}_\varepsilon(R_{q_0+1}(\mathcal{A}, S_q)),$$

and summing over $0 \le q \le q_0$ we obtain

$$\mathrm{op}_\varepsilon(\mathcal{A})\,\mathrm{op}_\varepsilon(\Sigma) = \mathrm{op}_\varepsilon(\mathcal{A}\Sigma) + \sum_{\substack{0\le q_2\le q_0\\ 1\le q_1\le q_0}} \varepsilon^{(q_1+q_2)h}\,\mathrm{op}_\varepsilon(\mathcal{A}\,\sharp_{q_1} S_{q_2}) + \varepsilon^{(q_0+1)h} R,$$

where

$$R := \sum_{0\le q\le q_0} \varepsilon^{qh}\,\mathrm{op}_\varepsilon(R_{q_0+1}(\mathcal{A}, S_q)).$$

Further, by the definition (D.2) of the correctors,

$$-\partial_t\,\mathrm{op}_\varepsilon(\Sigma) = \mathrm{op}_\varepsilon(\mathcal{A}\Sigma) + \sum_{\substack{1\le q_1+q_2\le q_0\\ q_1>0}} \varepsilon^{(q_1+q_2)h}\,\mathrm{op}_\varepsilon(\mathcal{A}\,\sharp_{q_1} S_{q_2}).$$

Comparing this with the above, we find that (D.3) holds with

$$-\rho := \sum_{\substack{q_0+1\le q_1+q_2\le 2q_0\\ 1\le q_1\le q_0\\ 0\le q_2\le q_0}} \varepsilon^{(q_1+q_2)h}\,\mathrm{op}_\varepsilon(\mathcal{A}\,\sharp_{q_1} S_{q_2}) + \varepsilon^{(q_0+1)h} R. \tag{D.5}$$

Since $\mathcal{A}$ is compactly supported in x, so are the correctors S_q for $q \geq 1$, and the derivatives of S_0. Thus under Assumption D.1,

$$|\partial_x^\alpha (\mathcal{A} \sharp_{q_1} S_{q_2})|_{L^1(\mathbb{R}^d_x)} \lesssim \varepsilon^{-\zeta(1+q_1+q_2)} \exp(\gamma t^{1+\ell}),$$

uniformly in ξ. Hence, by Proposition B.1,

$$\|\mathrm{op}_\varepsilon(\mathcal{A} \sharp_{q_1} S_{q_2})\|_{L^2 \to L^2(\mathbb{R}^d)} \lesssim \varepsilon^{-\zeta(1+q_1+q_2)} \exp(\gamma t^{1+\ell}).$$

Thus the $L^2 \to L^2$ norm of the first term in the right-hand side of (D.5) is controlled by

$$\sum_{\substack{q_0+1 \leq q_1+q_2 \leq 2q_0 \\ 1 \leq q_1 \leq q_0 \\ 0 \leq q_2 \leq q_0}} \varepsilon^{(q_1+q_2)h} \varepsilon^{-\zeta(1+q_1+q_2)} \exp(\gamma t^{1+\ell}) \lesssim \varepsilon^{(h-\zeta)(q_0+1)-\zeta-\gamma T_\star}$$

over the interval $[0, T(\varepsilon)]$, implying the desired bound as soon as

$$(h-\zeta)(q_0+1) \geq 1 + \zeta + \gamma T_\star.$$

Moreover, by Proposition B.2,

$$\|\mathrm{op}_\varepsilon(R_{q_0+1}(\mathcal{A}, S_q))\|_{L^2 \to L^2} \lesssim \|\partial_\xi^{q_0+1} \mathcal{A}\|_{0,C(d)} \|\partial_x^{q_0+1} S_q\|_{0,C(d)},$$

and by Assumption D.1,

$$\|\partial_\xi^{q_0+1} \mathcal{A}\|_{C(d)} \lesssim \varepsilon^{-(q_0+1+C(d))\zeta}, \qquad \|\partial_x^{q_0+1} S_q\|_{C(d)} \lesssim \varepsilon^{-\zeta(1+q+C(d))} \exp(\gamma t^{\ell+1}).$$

This gives control of the $L^2 \to L^2$ norm of the second term in the right-hand side of (D.5) by

$$\varepsilon^{(q_0+1)h} \sum_{0 \leq q \leq q_0} \varepsilon^{qh} \varepsilon^{-(q_0+1+C(d))\zeta-(1+q+C(d))\zeta} \exp(\gamma t^{\ell+1}).$$

We conclude that (D.4) holds if q_0 satisfies

$$(h-\zeta)(q_0+1) \geq 1 + \gamma T_\star + 2C(d)\zeta,$$

which can be achieved since in Assumption D.1 we postulated $\zeta < h$. □

Theorem D.3. *Under Assumption D.1, the initial value problem (D.1) has a unique solution $u \in C^0([0, T(\varepsilon)], L^2(\mathbb{R}^d))$, given by*

$$u = \mathrm{op}_\varepsilon(\Sigma(0;t))u_0 + \int_0^t \mathrm{op}_\varepsilon(\Sigma(t';t))(\mathrm{Id} + \varepsilon R_1(t'))\big(g(t') + \varepsilon R_2(t')u_0\big)\, dt', \quad \text{(D.6)}$$

where R_1 and R_2 are bounded: for all $v \in L^2$,

$$\|R_1(t)v\|_{L^2} + \|R_2(t)v\|_{L^2} \lesssim \|v\|_{L^2}, \quad \text{(D.7)}$$

uniformly in ε and $t \in [0, T(\varepsilon)]$.

Proof. Let $h \in L^\infty([0, T(\varepsilon)], L^2(\mathbb{R}^d))$. By Lemma D.2, the map u defined by

$$u := \mathrm{op}_\varepsilon(\Sigma(0; t))u_0 + \int_0^t \mathrm{op}_\varepsilon(\Sigma(t'; t))h(t')\,dt'$$

solves (D.1) if and only if, for all t,

$$((\mathrm{Id} + \rho_0)h)(t) = g - \rho(0; t)u_0, \tag{D.8}$$

where ρ_0 is the linear integral operator

$$C^0([0, T(\varepsilon)], L^2) \ni v \mapsto \left(t \mapsto \int_0^t \rho(\tau; t)v(\tau)\,d\tau\right) \in C^0([0, T(\varepsilon)], L^2).$$

By (D.4),

$$\sup_{0 \le t \le T(\varepsilon)} \|(\rho_0 v)(t)\|_{L^2} \lesssim \varepsilon \sup_{0 \le t \le T(\varepsilon)} \|v(t)\|_{L^2}.$$

Thus $\mathrm{Id} + \rho_0$ is invertible in the Banach algebra of bounded linear operators acting on $C^0([0, T(\varepsilon)], L^2(\mathbb{R}^d))$. This provides a solution h to (D.8), and we obtain the representation (D.6) with $R_1 := \varepsilon^{-1}((\mathrm{Id} + \rho_0)^{-1} - \mathrm{Id})$ and $R_2 = -\rho(0; \cdot)$. The bound (D.7) is a consequence of (D.4). Uniqueness follows from Cauchy–Lipschitz, since $\mathcal{A} \in S^0$. □

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