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Minkowski valuations and generalized valuations

Dedicated to Prof. Erwin Lutwak on the occasion of his seventieth birthday

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Abstract. A convolution representation of continuous translation-invariant and $SO(n)$ -equivariant Minkowski valuations is established. This is based on a new classification of translation-invariant generalized spherical valuations. As applications, Crofton and kinematic formulas for Minkowski valuations are obtained.

Keywords. Minkowski valuations, generalized valuations, kinematic formula, Crofton formula

1. Introduction

A valuation on convex bodies (non-empty compact convex sets) is a *finitely additive* function. More precisely, let \mathcal{K}^n denote the space of convex bodies in \mathbb{R}^n endowed with the Hausdorff metric. A map $\phi : \mathcal{K}^n \rightarrow \mathcal{A}$ with values in an Abelian semigroup \mathcal{A} is a *valuation* if

$$\phi(K) + \phi(L) = \phi(K \cup L) + \phi(K \cap L)$$

whenever $K \cup L$ is convex. As a generalization of the notion of measure and as the crucial ingredient in Dehn's solution of Hilbert's third problem, scalar valuations (where $\mathcal{A} = \mathbb{R}$ or \mathbb{C}) have long played a central role in convex and discrete geometry (see [39] or [56, Chapter 6]). The most famous classical result on valuations is the celebrated characterization of rigid-motion-invariant valuations by Hadwiger [33] (which was slightly improved later by Klain [37]).

Theorem 1 ([33, 37]). *The intrinsic volumes V_0, V_1, \dots, V_n form a basis of the vector space of all continuous scalar valuations on \mathcal{K}^n which are translation- and $SO(n)$ -invariant.*

Hadwiger's characterization theorem had a transformative effect on integral geometry. It not only allowed for almost effortless proofs of the principal and more general kinematic formulas (see, e.g., [39]) but also made the importance of precise descriptions of classes

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of invariant valuations evident. Still to this day, Theorem 1 often serves as a starting point for the classification of invariant scalar valuations (see, e.g., [4, 10, 13, 31, 43]) and, more generally, equivariant tensor valued valuations, where $\mathcal{A} = \text{Sym}^k \mathbb{R}^n$ (see [3, 9, 35, 67]). These results in turn were critical for the tremendous progress in integral geometry of recent years (see [5, 16, 17, 20, 35, 66] and the references therein).

In 1974 Schneider [54, 55] first investigated valuations where $\mathcal{A} = \mathcal{K}^n$, and addition on \mathcal{K}^n is the usual Minkowski addition. In a more recent influential article, Ludwig [41] coined the name *Minkowski valuations* for such maps and started a line of research concerned with the classification of Minkowski valuations intertwining *linear* transformations [1, 2, 30, 40, 42, 61, 65].

The recent results on Minkowski valuations which are equi- or contravariant with respect to linear transformations show that they often form convex cones generated by *finitely* many valuations, such as the projection or difference body operators. In contrast, the cone of translation-invariant and $\text{SO}(n)$ -equivariant Minkowski valuations is *infinite-dimensional*. This is one reason why no full analogue of Theorem 1 for Minkowski valuations has been obtained yet, except for dimension $n = 2$, where Schneider [55] already established such a result. We therefore assume throughout that $n \geq 3$.

About a decade ago, Kiderlen [36] and the first author [59] were the first to obtain convolution representations of translation-invariant and $\text{SO}(n)$ -equivariant *continuous* Minkowski valuations. However, their results were limited to valuations of degree 1 and $n - 1$, respectively, where a map Φ from \mathcal{K}^n to \mathcal{K}^n (or \mathbb{R}) is said to have *degree* i if $\Phi(\lambda K) = \lambda^i \Phi K$ for $K \in \mathcal{K}^n$ and $\lambda > 0$. The convolution of measures on \mathbb{S}^{n-1} used in [36] and [59] is induced from the group $\text{SO}(n)$ by identifying \mathbb{S}^{n-1} with the homogeneous space $\text{SO}(n)/\text{SO}(n - 1)$ (see Section 2).

Under additional *smoothness* assumptions, the first author [60] and both the present authors [62] extended the results from [36] and [59] to the remaining (non-trivial) degrees $i \in \{2, \dots, n - 2\}$ when the Minkowski valuations are *even*. (McMullen [47] showed that only integer degrees $0 \leq i \leq n$ can occur.) However, the techniques employed in [60] or [62] were not suited to describe merely continuous Minkowski valuations, which is the goal since the 1970s.

In this article we establish a precise description of all *continuous* Minkowski valuations which are translation-invariant and $\text{SO}(n)$ -equivariant without any further assumptions on the parity or degree of the valuations. As we explain in Section 5, our main theorem generalizes and implies all previously obtained convolution representations of Minkowski valuations intertwining rigid motions.

In order to state our main result, recall that a convex body $K \in \mathcal{K}^n$ is uniquely determined by its support function $h_K(u) = \max\{u \cdot x : x \in K\}$ for $u \in \mathbb{S}^{n-1}$, and let $\mathcal{M}_o(\mathbb{S}^{n-1})$ and $C_o(\mathbb{S}^{n-1})$ denote the spaces of signed finite Borel measures and continuous functions on \mathbb{S}^{n-1} , respectively, having their center of mass at the origin. If G denotes a group acting on a set X , then as usual X^G denotes the set of all elements in X which are G -invariant. We also emphasize that throughout the article we use the spherical Lebesgue measure to identify functions on \mathbb{S}^{n-1} (whether smooth or continuous) with absolutely continuous measures on the sphere. In particular, equation (1.1) below has to be understood in this sense.

Theorem 2. *If $\Phi : \mathcal{K}^n \rightarrow \mathcal{K}^n$ is a continuous Minkowski valuation which is translation-invariant and $\text{SO}(n)$ -equivariant, then there exist unique constants $c_0, c_n \geq 0$, measures $\mu_i \in \mathcal{M}_0(\mathbb{S}^{n-1})^{\text{SO}(n-1)}$, $1 \leq i \leq n-2$, and a function $f_{n-1} \in C_0(\mathbb{S}^{n-1})^{\text{SO}(n-1)}$ such that*

$$h_{\Phi K} = c_0 + \sum_{i=1}^{n-2} S_i(K, \cdot) * \mu_i + S_{n-1}(K, \cdot) * f_{n-1} + c_n V_n(K) \quad (1.1)$$

for every $K \in \mathcal{K}^n$.

The Borel measures $S_i(K, \cdot)$, $1 \leq i \leq n-1$, on \mathbb{S}^{n-1} are Aleksandrov's *area measures* (see, e.g., [56]) associated with $K \in \mathcal{K}^n$. If K is sufficiently smooth and has positive curvature, then each $S_i(K, \cdot)$ is absolutely continuous with respect to spherical Lebesgue measure and its density is (up to a constant) given by the i th elementary symmetric function of the principal radii of curvature of K .

For $n \leq 4$, we show in Section 5 that if Φ has degree 1 or 2, then the measures μ_1 or μ_2 , respectively, are in fact absolutely continuous with a density in $L^2(\mathbb{S}^{n-1})$. However, this is no longer true in general when $n > 4$.

The proof of Theorem 2 is based on new techniques involving translation-invariant *generalized valuations* which were only recently introduced by Alesker and Faifman [10] (see also [15]). Generalized valuations are related to smooth valuations in the same way as generalized functions are related to smooth functions. More precisely, let \mathbf{Val}_i^∞ , $0 \leq i \leq n$, denote the space of *smooth* translation-invariant scalar valuations of degree i endowed with the Gårding topology which makes it a Fréchet space (see Section 3 for details). The space $\mathbf{Val}_i^{-\infty}$ of (translation-invariant) *generalized valuations* of degree i is defined by

$$\mathbf{Val}_i^{-\infty} := (\mathbf{Val}_{n-i}^\infty)^* \otimes \text{Dens}(\mathbb{R}^n) \quad (1.2)$$

endowed with the weak topology. Here, $\text{Dens}(\mathbb{R}^n)$ denotes the 1-dimensional space of densities on \mathbb{R}^n . However, since throughout the article we fix a Euclidean structure on \mathbb{R}^n , and thus a choice of Lebesgue measure, we obtain an isomorphism $\text{Dens}(\mathbb{R}^n) \cong \mathbb{R}$ which we use to identify $\mathbf{Val}_i^{-\infty}$ with the topological dual $(\mathbf{Val}_{n-i}^\infty)^*$.

As part of his far reaching reconceptualization of integral geometry, Alesker [6] discovered a continuous *non-degenerate* bilinear pairing

$$\langle \cdot, \cdot \rangle : \mathbf{Val}_i^\infty \times \mathbf{Val}_{n-i}^\infty \rightarrow \mathbb{R}$$

for $0 \leq i \leq n$ (see also Section 3). The induced *Poincaré duality map*

$$\text{pd} : \mathbf{Val}_i^\infty \rightarrow (\mathbf{Val}_{n-i}^\infty)^* = \mathbf{Val}_i^{-\infty}$$

is therefore continuous, injective and has dense image with respect to the weak topology. This was the motivation for definition (1.2) and shows that $\mathbf{Val}_i^{-\infty}$ can be seen as a completion of \mathbf{Val}_i^∞ with respect to the weak topology. Alesker [8] also proved that the Poincaré duality map admits a unique continuous extension to the space \mathbf{Val}_i of *continuous* translation-invariant valuations of degree i . Thus, just like smooth and continuous functions or, more generally, signed Borel measures can be identified with subclasses of

generalized functions (compare Section 2), we can use the Poincaré duality map in the following to *identify* the spaces \mathbf{Val}_i^∞ or \mathbf{Val}_i , respectively, with certain dense subspaces of $\mathbf{Val}_i^{-\infty}$.

It was first observed in [60] that a translation-invariant and $\mathrm{SO}(n)$ -equivariant continuous Minkowski valuation Φ is uniquely determined by a scalar valuation $\varphi \in \mathbf{Val}_i^{\mathrm{SO}(n-1)}$. In turn, the space $\mathbf{Val}_i^{\mathrm{SO}(n-1)}$ consists of *spherical* valuations.

Spherical (generalized) valuations correspond to *spherical representations* of $\mathrm{SO}(n)$ (see Section 3 for details). Let $\mathbf{Val}_i^{\infty, \mathrm{sph}}$ and $\mathbf{Val}_i^{-\infty, \mathrm{sph}}$ denote the subspaces of smooth and generalized spherical valuations, respectively, and let $C_0^{-\infty}(\mathbb{S}^{n-1})$ denote the space of generalized functions on \mathbb{S}^{n-1} which vanish on restrictions of linear functions to \mathbb{S}^{n-1} . Our second main result, which is critical for the proof of Theorem 2 but also of independent interest, is the following classification of (generalized) spherical valuations.

Theorem 3. *Suppose that $1 \leq i \leq n - 1$.*

(a) *The map $E_i : C_0^\infty(\mathbb{S}^{n-1}) \rightarrow \mathbf{Val}_i^{\infty, \mathrm{sph}}$ defined by*

$$(E_i f)(K) = \int_{\mathbb{S}^{n-1}} f(u) dS_i(K, u)$$

is an $\mathrm{SO}(n)$ -equivariant isomorphism of topological vector spaces which admits a unique extension by continuity in the weak topologies to an isomorphism

$$\tilde{E}_i : C_0^{-\infty}(\mathbb{S}^{n-1}) \rightarrow \mathbf{Val}_i^{-\infty, \mathrm{sph}}.$$

(b) *The space $\mathbf{Val}_i^{\mathrm{SO}(n-1)}$ is contained in $\tilde{E}_i(\mathcal{M}_0(\mathbb{S}^{n-1}))$ if $i \leq n - 2$ and in $\tilde{E}_i(C_0(\mathbb{S}^{n-1}))$ if $i = n - 1$.*

Theorem 3(a) for $i = 1$ was recently proved by Alesker [11, Appendix]. Theorem 3(b) for $i = n - 1$ follows from a classical result of McMullen [48].

Characterizations of Minkowski valuations, in particular, earlier versions of Theorem 2, have had far reaching implications for isoperimetric type inequalities (see, e.g., [2, 11, 32, 44–46, 61]). Motivated by a recent important Crofton type formula for the identity map of Goodey and Weil [26], we show in the final section of this paper how Theorem 2 can be applied to obtain a general Crofton formula for continuous Minkowski valuations which generalizes the result from [26] and an earlier result of this type from [62]. Using our new Crofton formula and Hadwiger's general integral geometric theorem, a consequence of Theorem 1 (cf. [58, p. 173]), we can then also state a kinematic formula for Minkowski valuations.

2. Preliminaries

In this section we first recall basic notions from Riemannian geometry and compute several quantities in cylindrical coordinates on \mathbb{S}^{n-1} which will be needed in the proof of Theorem 3(b). Next, we collect background material from representation theory and harmonic analysis, in particular, about the convolution of measures on \mathbb{S}^{n-1} and its relation

to the theory of spherical harmonics. We also recall some well known facts about generalized functions on \mathbb{S}^{n-1} and the definition of Berg's functions used in the solution of the classical Christoffel problem.

Although in this article we are mainly concerned with the Euclidean unit sphere \mathbb{S}^{n-1} in \mathbb{R}^n and the Lie group $\text{SO}(n)$ of proper rotations of \mathbb{R}^n , let us first consider a general smooth manifold M . We denote by $C^\infty(M)$ the space of all smooth functions on M equipped with the Fréchet space topology of uniform convergence of any finite number of derivatives on each compact subset of M . For a Banach space X , the Fréchet space $C^\infty(M, X)$ of all infinitely differentiable functions on M with values in X is defined similarly.

If in addition M is compact and endowed with a Riemannian metric, then the C^k norm $\|f\|_{C^k}$ of a function $f \in C^k(M)$ is defined by (see, e.g., [52, p. 301])

$$\|f\|_{C^k} = \sum_{j=0}^k \max_M |\nabla^j f|, \quad (2.1)$$

where ∇ denotes the covariant derivative with respect to the given Riemannian metric and $|\nabla^j f|$ is the (Euclidean) norm of the tensor field $\nabla^j f$.

Among other quantities, we compute in the following example the C^2 norm of an $\text{SO}(n-1)$ -invariant function on \mathbb{S}^{n-1} more explicitly. This will be useful later on for the proof of Theorem 3(b).

Example 2.1. In this article we use $\bar{e} \in \mathbb{S}^{n-1}$ to denote an arbitrary but fixed point (the pole) of the sphere and we write $\text{SO}(n-1)$ for the stabilizer of \bar{e} in $\text{SO}(n)$. Clearly, every $u \in \mathbb{S}^{n-1} \setminus \{-\bar{e}, \bar{e}\}$ can be uniquely written in the form

$$u = t\bar{e} + \sqrt{1-t^2}v \quad (2.2)$$

for some $t \in (-1, 1)$ and $v \in \mathbb{S}_{\bar{e}}^{n-2} = \{w \in \mathbb{S}^{n-1} : \bar{e} \cdot w = 0\}$. In the cylindrical coordinates (2.2), the (standard) metric tensor ϱ on \mathbb{S}^{n-1} is given by

$$\varrho = \frac{1}{1-t^2} dt \otimes dt + (1-t^2) dv \otimes dv, \quad (2.3)$$

where $dv \otimes dv$ is the metric tensor on $\mathbb{S}_{\bar{e}}^{n-2}$.

Let $\Delta_{\mathbb{S}}$ denote the Laplacian (or Laplace–Beltrami operator) on \mathbb{S}^{n-1} ; recall that, for $f, g \in C^2(\mathbb{S}^{n-1})$, we have

$$\int_{\mathbb{S}^{n-1}} f(u) \Delta_{\mathbb{S}} g(u) du = \int_{\mathbb{S}^{n-1}} g(u) \Delta_{\mathbb{S}} f(u) du. \quad (2.4)$$

Using (2.3), one can easily obtain the following expression for the Laplacian in cylindrical coordinates (cf. [12, Proposition 2.6]):

$$\Delta_{\mathbb{S}} = \frac{1}{\sqrt{|\varrho|}} \partial_i (\sqrt{|\varrho|} \varrho^{ij} \partial_j) = (1-t^2) \frac{\partial^2}{\partial t^2} - (n-1)t \frac{\partial}{\partial t} + \frac{1}{1-t^2} \bar{\Delta}_{\mathbb{S}}, \quad (2.5)$$

where $\bar{\Delta}_{\mathbb{S}}$ denotes the Laplacian on $\mathbb{S}_{\bar{e}}^{n-2}$.

Now let $f \in C^2(\mathbb{S}^{n-1})$ be $\text{SO}(n-1)$ -invariant, that is, in the cylindrical coordinates (2.2) the function f depends only on t . Then, by (2.5), we have

$$\Delta_{\mathbb{S}} f = (1-t^2) \frac{\partial^2 f}{\partial t^2} - (n-1)t \frac{\partial f}{\partial t}. \quad (2.6)$$

Moreover, a straightforward computation, using (2.3) again, yields

$$|\nabla f|^2 = (1-t^2) \left(\frac{\partial f}{\partial t} \right)^2, \quad (2.7)$$

$$|\nabla^2 f|^2 = (n-2) \left(t \frac{\partial f}{\partial t} \right)^2 + \left((1-t^2) \frac{\partial^2 f}{\partial t^2} - t \frac{\partial f}{\partial t} \right)^2. \quad (2.8)$$

We turn now to representations of Lie groups. First recall that since $\text{SO}(n)$ is compact, all its irreducible representations are finite-dimensional, and the equivalence classes of irreducible complex representations of $\text{SO}(n)$ are uniquely determined by their highest weights (see, e.g., [18]). These highest weights, in turn, can be indexed by $\lfloor n/2 \rfloor$ -tuples $(\lambda_1, \dots, \lambda_{\lfloor n/2 \rfloor})$ of integers such that

$$\begin{cases} \lambda_1 \geq \dots \geq \lambda_{\lfloor n/2 \rfloor} \geq 0 & \text{for odd } n, \\ \lambda_1 \geq \dots \geq \lambda_{n/2-1} \geq |\lambda_{n/2}| & \text{for even } n. \end{cases} \quad (2.9)$$

A notion of particular importance for our purposes is that of smooth vectors of an infinite-dimensional representation of a Lie group.

Definition. Let ρ be a continuous representation of a Lie group G on a Banach space X . An element $x \in X$ is called a *smooth vector* if the map $z_x : G \rightarrow X$ defined by $z_x(\vartheta) = \rho(\vartheta)x$ is infinitely differentiable. The subspace of all smooth vectors in X is denoted by X^∞ .

It is well known (cf. [68, Section 4.4]) that the subspace X^∞ is a G -invariant and dense subset of X . Moreover, the map $X^\infty \rightarrow C^\infty(G, X)$ given by $x \mapsto z_x$ leads to an identification of X^∞ with a closed subspace of $C^\infty(G, X)$. Hence, we can endow X^∞ with the relative topology induced by $C^\infty(G, X)$. This topology on X^∞ is called the *Gårding topology* and turns X^∞ into a Fréchet space. An important property of the Gårding topology on X^∞ is that the restriction of the representation of G to X^∞ is *continuous*.

Two more basic facts about smooth vectors are contained in the next lemma.

Lemma 2.2. *Let G be a Lie group.*

- (a) *If ρ and τ are continuous representations of G on Banach spaces X and Y and $T : X \rightarrow Y$ is a continuous and G -equivariant linear map, then $T(X^\infty) \subseteq Y^\infty$ and the restriction $T : X^\infty \rightarrow Y^\infty$ is continuous.*
- (b) *If H is a closed subgroup of G , then the smooth vectors of the left regular representation of G on $C(G/H)$ are precisely the smooth functions on G/H , that is, $(C(G/H))^\infty = C^\infty(G/H)$.*

In this article, we are specifically interested in spherical representations of $SO(n)$ with respect to $SO(n - 1)$.

Definition. Let G be a compact Lie group and H a closed subgroup of G . A representation ρ of G on a vector space X is called *spherical* with respect to H if there exists an H -invariant non-zero $x \in X$, that is, $\rho(\vartheta)x = x$ for every $\vartheta \in H$.

For the following two important facts about spherical representations (see, e.g., [64, p. 17]), we consider the left regular representation of G on the Hilbert space $L^2(G/H)$ of square-integrable functions on the homogeneous space G/H .

Theorem 2.3. Let G be a compact Lie group and H a closed subgroup of G .

- (i) Every subrepresentation of $L^2(G/H)$ is spherical with respect to H .
- (ii) Every irreducible representation of G which is spherical with respect to H is isomorphic to a subrepresentation of $L^2(G/H)$.

Example 2.4. The space $L^2(\mathbb{S}^{n-1})$ is an orthogonal sum of $SO(n)$ irreducible subspaces,

$$L^2(\mathbb{S}^{n-1}) = \bigoplus_{k \geq 0} \mathcal{H}_k^n,$$

where \mathcal{H}_k^n is the space of spherical harmonics of dimension n and degree k . It is well known that the highest weights corresponding to the spaces \mathcal{H}_k^n are the $\lfloor n/2 \rfloor$ -tuples $(k, 0, \dots, 0)$. Since \mathbb{S}^{n-1} is diffeomorphic to the homogeneous space $SO(n)/SO(n - 1)$, it follows from Theorem 2.3 that every irreducible representation of $SO(n)$ which is spherical with respect to $SO(n - 1)$ is isomorphic to one of the spaces \mathcal{H}_k^n , and thus their highest weights are of the form $(k, 0, \dots, 0)$, $k \geq 0$.

For the discussion of Theorem 2 and its applications to integral geometry, we need some more background from the theory of spherical harmonics (see, e.g., [28]). Let $N(n, k)$ denote the dimension of the space \mathcal{H}_k^n and recall that

$$N(n, k) = \frac{n + 2k - 2}{n + k - 2} \binom{n + k - 2}{n - 2} = O(k^{n-2}) \quad \text{as } k \rightarrow \infty. \tag{2.10}$$

Using $\pi_k : L^2(\mathbb{S}^{n-1}) \rightarrow \mathcal{H}_k^n$ to denote the orthogonal projection, we can write

$$f \sim \sum_{k=0}^{\infty} \pi_k f \tag{2.11}$$

for the (condensed) Fourier expansion of $f \in L^2(\mathbb{S}^{n-1})$. Recall that the Fourier series in (2.11) converges to f in the L^2 norm.

In the theory of spherical harmonics, a function or measure on \mathbb{S}^{n-1} which is $SO(n - 1)$ -invariant is often called *zonal*. The subspace of zonal functions in \mathcal{H}_k^n is 1-dimensional for every $k \geq 0$ and spanned by the function $u \mapsto P_k^n(u \cdot \bar{e})$, where $P_k^n \in C([-1, 1])$ denotes the *Legendre polynomial* of dimension n and degree k . Since the

spaces \mathcal{H}_k^n are orthogonal, it is not difficult to show that any zonal function $f \in L^2(\mathbb{S}^{n-1})$ admits a series expansion of the form

$$f \sim \sum_{k=0}^{\infty} \frac{N(n, k)}{\omega_n} a_k^n[f] P_k^n(\cdot \cdot \bar{e}), \tag{2.12}$$

where ω_n denotes the surface area of the n -dimensional Euclidean unit ball and

$$a_k^n[f] = \omega_{n-1} \int_{-1}^1 f(t) P_k^n(t) (1 - t^2)^{(n-3)/2} dt. \tag{2.13}$$

Here, we have again used the cylindrical coordinates (2.2) to identify the zonal function f with a function on $[-1, 1]$.

Now we turn to (formal) Fourier expansions of measures and, more generally, distributions on \mathbb{S}^{n-1} . To this end, first recall that a *distribution* on \mathbb{S}^{n-1} is a continuous linear functional on $C^\infty(\mathbb{S}^{n-1})$. Since \mathbb{S}^{n-1} is compact, every distribution ν on \mathbb{S}^{n-1} is of *finite order*, that is, there exist $k \geq 0$ and $C > 0$ such that

$$|\nu(f)| \leq C \|f\|_{C^k} \tag{2.14}$$

for every $f \in C^\infty(\mathbb{S}^{n-1})$. The *order* of ν is the smallest k such that (2.14) holds.

Distributions on a general smooth manifold M are often also called generalized densities, and $C^{-\infty}(|\Lambda|(M))$ is used to denote the space of distributions on M . However, the choice of a Riemannian metric on M induces an isomorphism between the space of distributions and the space of *generalized functions* on M , usually denoted by $C^{-\infty}(M)$ (cf. [29]). Throughout this article, when $M = \mathbb{S}^{n-1}$, we always make use of this identification, and write $C^{-\infty}(\mathbb{S}^{n-1})$ for the space of distributions on \mathbb{S}^{n-1} equipped with the topology of weak convergence. The canonical bilinear pairing on $C^\infty(\mathbb{S}^{n-1}) \times C^{-\infty}(\mathbb{S}^{n-1})$ will be denoted by $\langle \cdot, \cdot \rangle$.

Since every (signed) Borel measure μ on \mathbb{S}^{n-1} defines a distribution ν_μ by

$$\langle f, \nu_\mu \rangle = \int_{\mathbb{S}^{n-1}} f(u) d\mu(u), \quad f \in C^\infty(\mathbb{S}^{n-1}),$$

we will use the continuous linear injection $\mu \mapsto \nu_\mu$ to identify $\mathcal{M}(\mathbb{S}^{n-1})$ with a subspace of $C^{-\infty}(\mathbb{S}^{n-1})$. Clearly, this subspace consists precisely of the distributions of order 0. In the same way, the spaces $C^\infty(\mathbb{S}^{n-1})$, $C(\mathbb{S}^{n-1})$, and $L^2(\mathbb{S}^{n-1})$ can be identified with (dense) subspaces of $C^{-\infty}(\mathbb{S}^{n-1})$ and we have

$$C^\infty(\mathbb{S}^{n-1}) \subseteq C(\mathbb{S}^{n-1}) \subseteq L^2(\mathbb{S}^{n-1}) \subseteq \mathcal{M}(\mathbb{S}^{n-1}) \subseteq C^{-\infty}(\mathbb{S}^{n-1}). \tag{2.15}$$

A natural action of $\text{SO}(n)$ on $C^{-\infty}(\mathbb{S}^{n-1})$ is defined as follows: For $\vartheta \in \text{SO}(n)$ and $\nu \in C^{-\infty}(\mathbb{S}^{n-1})$, we set

$$\langle f, \vartheta \nu \rangle = \langle \vartheta^{-1} f, \nu \rangle, \quad f \in C^\infty(\mathbb{S}^{n-1}). \tag{2.16}$$

Note that if ν is a measure on \mathbb{S}^{n-1} , then $\vartheta\nu$ is just the image measure of ν under the rotation ϑ ; moreover, (2.16) is also consistent with the left regular representation of $\text{SO}(n)$ on the spaces $C^\infty(\mathbb{S}^{n-1})$, $C(\mathbb{S}^{n-1})$, and $L^2(\mathbb{S}^{n-1})$.

In order to extend the orthogonal projection $\pi_k : L^2(\mathbb{S}^{n-1}) \rightarrow \mathcal{H}_k^n$ to $C^{-\infty}(\mathbb{S}^{n-1})$, note that π_k is self-adjoint. In particular, $\langle f, \pi_k g \rangle = \langle \pi_k f, g \rangle$ for all $f \in C^\infty(\mathbb{S}^{n-1})$ and $g \in L^2(\mathbb{S}^{n-1})$. In view of (2.15), it is therefore consistent to define $\pi_k \nu$ for ν in $C^{-\infty}(\mathbb{S}^{n-1})$ as the distribution given by

$$\langle f, \pi_k \nu \rangle = \langle \pi_k f, \nu \rangle, \quad f \in C^\infty(\mathbb{S}^{n-1}).$$

From this, it follows (cf. [49, p. 38]) that indeed $\pi_k \nu \in \mathcal{H}_k^n$ for every $k \in \mathbb{N}$.

Next let us discuss the convolution of measures and distributions on \mathbb{S}^{n-1} . Recall that the convolution $\sigma * \mu$ of signed measures σ, μ on $\text{SO}(n)$ can be defined by

$$\int_{\text{SO}(n)} f(\vartheta) d(\sigma * \mu)(\vartheta) = \int_{\text{SO}(n)} \int_{\text{SO}(n)} f(\eta\theta) d\sigma(\eta) d\mu(\theta), \quad f \in C(\text{SO}(n)).$$

In other words, $\sigma * \mu$ is the pushforward of the product measure $\sigma \otimes \mu$ by the group multiplication $m : \text{SO}(n) \times \text{SO}(n) \rightarrow \text{SO}(n)$, that is, $\sigma * \mu = m_*(\sigma \otimes \mu)$. Since $\text{SO}(n)$ is compact, this definition can be readily extended to distributions by replacing the product measure with the tensor product of distributions (see, e.g., [34, p. 128]).

The identification of \mathbb{S}^{n-1} with the homogeneous space $\text{SO}(n)/\text{SO}(n-1)$ can now be used to identify $C^{-\infty}(\mathbb{S}^{n-1})$ with the right $\text{SO}(n-1)$ -invariant distributions on $\text{SO}(n)$. Via this correspondence, the convolution of distributions on $\text{SO}(n)$ induces a convolution product on $C^{-\infty}(\mathbb{S}^{n-1})$ as follows: Let $\pi : \text{SO}(n) \rightarrow \mathbb{S}^{n-1}$, $\pi(\eta) = \eta\bar{e}$, be the canonical projection. Then the convolution of distributions $\delta, \nu \in C^{-\infty}(\mathbb{S}^{n-1})$ is defined by

$$\delta * \nu = \pi_* m_*(\pi^* \delta \otimes \pi^* \nu),$$

where π_* and π^* denote the pushforward and pullback by π , respectively.

The convolution product defined in this way has the following well-known continuity property (see, e.g., [34, Chapter 6]).

Lemma 2.5. *If $\nu_j \in C^{-\infty}(\mathbb{S}^{n-1})$, $j \in \mathbb{N}$, converge weakly to $\nu \in C^{-\infty}(\mathbb{S}^{n-1})$, then $\lim_{j \rightarrow \infty} \delta * \nu_j = \delta * \nu$ weakly for every $\delta \in C^{-\infty}(\mathbb{S}^{n-1})$.*

Zonal measures play an essential role for spherical convolution. For later use we state here the explicit expressions for the convolution of a function $h \in C(\mathbb{S}^{n-1})$ and a measure $\sigma \in \mathcal{M}(\mathbb{S}^{n-1})$ with a zonal measure $\mu \in \mathcal{M}(\mathbb{S}^{n-1})$ and a zonal function $f \in C(\mathbb{S}^{n-1})$, respectively:

$$(h * \mu)(\bar{\eta}) = \int_{\mathbb{S}^{n-1}} h(\eta u) d\mu(u) \quad \text{and} \quad (\sigma * f)(\bar{\eta}) = \int_{\mathbb{S}^{n-1}} f(\eta^{-1}u) d\sigma(u), \quad (2.17)$$

where for $\eta \in \text{SO}(n)$, we write $\pi(\eta) = \bar{\eta} \in \mathbb{S}^{n-1}$.

Note that for signed measures σ, μ on \mathbb{S}^{n-1} and every $\vartheta \in \text{SO}(n)$, we have $(\vartheta\sigma) * \mu = \vartheta(\sigma * \mu)$; further, by (2.17), the convolution of zonal measures on \mathbb{S}^{n-1} is Abelian.

Moreover, from the identification of a zonal measure μ on \mathbb{S}^{n-1} with a measure on $[-1, 1]$ and the well-known Funk–Hecke Theorem, it follows (cf. [59]) that the Fourier expansion of $\sigma * \mu$ is given by

$$\sigma * \mu \sim \sum_{k=0}^{\infty} a_k^n[\mu] \pi_k \sigma, \tag{2.18}$$

where the numbers $a_k^n[\mu]$ are defined by (2.13).

Like convolution on \mathbb{R}^n , spherical convolution can be used to approximate a given measure or distribution on \mathbb{S}^{n-1} by smooth functions. To this end, let $B_j(\bar{e})$, $j \in \mathbb{N}$, denote the open geodesic ball of radius $1/j$ centered at $\bar{e} \in \mathbb{S}^{n-1}$. A sequence of non-negative zonal functions $\zeta_j \in C^\infty(\mathbb{S}^{n-1})$, $j \in \mathbb{N}$, is called a *spherical approximate identity* if for each j ,

$$\int_{\mathbb{S}^{n-1}} \zeta_j(u) \, du = 1 \quad \text{and} \quad \text{supp } \zeta_j \subseteq B_j(\bar{e}). \tag{2.19}$$

For a proof of the following auxiliary result, we refer to [27] or [53, Chapter 6].

Lemma 2.6. *If $\zeta_j \in C^\infty(\mathbb{S}^{n-1})$, $j \in \mathbb{N}$, is a spherical approximate identity, then*

- (i) $\lim_{j \rightarrow \infty} g * \zeta_j = g$ uniformly for every $g \in C(\mathbb{S}^{n-1})$;
- (ii) $\lim_{j \rightarrow \infty} \nu * \zeta_j = \nu$ weakly for every $\nu \in C^{-\infty}(\mathbb{S}^{n-1})$.

In the final part of this section, we turn to the Christoffel problem and its solution by Berg [12]. First recall that spherical harmonics are eigenfunctions of the Laplacian $\Delta_{\mathbb{S}}$, more precisely, for $Y_k \in \mathcal{H}_k^n$,

$$\Delta_{\mathbb{S}} Y_k = -k(k + n - 2) Y_k. \tag{2.20}$$

Like the orthogonal projection π_k , the Laplacian $\Delta_{\mathbb{S}}$ is self-adjoint. Thus, it is consistent to define $\Delta_{\mathbb{S}} \nu$ for $\nu \in C^{-\infty}(\mathbb{S}^{n-1})$ as the distribution given by

$$\langle f, \Delta_{\mathbb{S}} \nu \rangle = \langle \Delta_{\mathbb{S}} f, \nu \rangle, \quad f \in C^\infty(\mathbb{S}^{n-1}).$$

In particular, by (2.15), $\Delta_{\mathbb{S}}$ can now be applied to support functions of not necessarily smooth convex bodies. This is important for us, since the first-order area measure $S_1(K, \cdot)$ of $K \in \mathcal{K}^n$ and its support function h_K are related by a *linear* differential operator \square_n :

$$S_1(K, \cdot) = h_K + \frac{1}{n-1} \Delta_{\mathbb{S}} h_K =: \square_n h_K. \tag{2.21}$$

From (2.20) and the definition of \square_n , it follows that for $f \in C^\infty(\mathbb{S}^{n-1})$, the spherical harmonic expansion of $\square_n f$ is given by

$$\square_n f \sim \sum_{k=0}^{\infty} \frac{(1-k)(k+n-1)}{n-1} \pi_k f. \tag{2.22}$$

Hence, the kernel of $\square_n : C^\infty(\mathbb{S}^{n-1}) \rightarrow C^\infty(\mathbb{S}^{n-1})$ is given by \mathcal{H}_1^n , that is, it consists precisely of the restrictions of linear functions on \mathbb{R}^n to \mathbb{S}^{n-1} . Let

$$C_0^\infty(\mathbb{S}^{n-1}) = \{f \in C^\infty(\mathbb{S}^{n-1}) : \pi_1 f = 0\}$$

and define $C_0^{-\infty}(\mathbb{S}^{n-1})$ in the same way. Then $\square_n : C_0^\infty(\mathbb{S}^{n-1}) \rightarrow C_0^\infty(\mathbb{S}^{n-1})$ is an $\text{SO}(n)$ -equivariant isomorphism of topological vector spaces.

The classical Christoffel problem consists in finding necessary and sufficient conditions for a Borel measure on \mathbb{S}^{n-1} to be the first-order area measure of a convex body. A solution was obtained by Berg [12] by providing an explicit inversion formula for the operator \square_n . In order to state this result, we define for a function g on $[-1, 1]$, an associated zonal function \check{g} on \mathbb{S}^{n-1} by

$$\check{g}(u) = g(u \cdot \bar{e}), \quad u \in \mathbb{S}^{n-1}.$$

Theorem 2.7 ([12]). *For every $n \geq 2$ there exists a unique C^∞ function g_n on $(-1, 1)$ such that the associated zonal function \check{g}_n is in $L^1(\mathbb{S}^{n-1})$ and*

$$a_1^n[g_n] = 0, \quad a_k^n[g_n] = \frac{n-1}{(1-k)(k+n-1)}, \quad k \neq 1. \tag{2.23}$$

It follows from (2.22), (2.18), and (2.23) that for every $f \in C_0^\infty(\mathbb{S}^{n-1})$,

$$f = (\square_n f) * \check{g}_n. \tag{2.24}$$

In the final section, we need a generalization of (2.24) that can be deduced from [26, Theorem 4.3] and was independently proved in [11]:

Proposition 2.8. *For every $j \in \{2, \dots, n\}$, the integral transform*

$$T_{g_j} : C_0^\infty(\mathbb{S}^{n-1}) \rightarrow C_0^\infty(\mathbb{S}^{n-1}), \quad f \mapsto f * \check{g}_j,$$

is an isomorphism.

Proposition 2.8 and (2.24) give rise to the following:

Definition. For $j \in \{2, \dots, n\}$, let $\square_j : C_0^\infty(\mathbb{S}^{n-1}) \rightarrow C_0^\infty(\mathbb{S}^{n-1})$ denote the linear operator which is inverse to the integral transform T_{g_j} .

3. Smooth and generalized valuations

We now turn to the background material on translation-invariant scalar and convex body valued valuations. In particular, we recall the definitions of smooth and generalized (spherical) valuations as well as the Alesker–Poincaré duality map.

If G is a group of affine transformations on \mathbb{R}^n , a valuation ϕ is called G -invariant if $\phi(gK) = \phi(K)$ for all $K \in \mathcal{K}^n$ and every $g \in G$. Let \mathbf{Val} denote the vector space of continuous translation-invariant scalar valued valuations on \mathbb{R}^n . It was first proved by McMullen [47] that

$$\mathbf{Val} = \bigoplus_{0 \leq i \leq n} \mathbf{Val}_i, \tag{3.1}$$

where $\mathbf{Val}_i \subseteq \mathbf{Val}$ denotes the subspace of valuations (homogeneous) of degree i .

Recall that a map $\Phi : \mathcal{K}^n \rightarrow \mathcal{K}^n$ is called a *Minkowski valuation* if

$$\Phi K + \Phi L = \Phi(K \cup L) + \Phi(K \cap L)$$

whenever $K \cup L$ is convex and addition on \mathcal{K}^n is Minkowski addition. We denote by \mathbf{MVal} the set of continuous translation-invariant Minkowski valuations, and we write \mathbf{MVal}_i , $0 \leq i \leq n$, for its subset of Minkowski valuations of degree i .

More general than Minkowski valuations, we also consider valuations with values in $C(\mathbb{S}^{n-1})$, that is, maps $F : \mathcal{K}^n \rightarrow C(\mathbb{S}^{n-1})$, $K \mapsto F_K$, such that

$$F_K + F_L = F_{K \cup L} + F_{K \cap L}$$

whenever $K \cup L$ is convex. Let \mathbf{CVal} denote the vector space of all such valuations which are continuous and translation-invariant and, as before, let \mathbf{CVal}_i , $0 \leq i \leq n$, denote its subspace of valuations of degree i .

Note that any Minkowski valuation $\Phi \in \mathbf{MVal}$ induces a valuation $F^\Phi \in \mathbf{CVal}$ by $F_K^\Phi = h_{\Phi K}$, and Φ is $\mathrm{SO}(n)$ -equivariant if and only if F^Φ is $\mathrm{SO}(n)$ -equivariant. Via the map $\Phi \mapsto F^\Phi$, the set \mathbf{MVal} can be identified with an infinite-dimensional convex cone in \mathbf{CVal} .

Clearly, a valuation $F \in \mathbf{CVal}$ is uniquely determined by the family of valuations $\varphi_u \in \mathbf{Val}$, $u \in \mathbb{S}^{n-1}$, defined by $\varphi_u(K) = F_K(u)$ for $K \in \mathcal{K}^n$. If in addition F is $\mathrm{SO}(n)$ -equivariant, then for $\eta \in \mathrm{SO}(n)$ and $\bar{\eta} = \eta\bar{e} \in \mathbb{S}^{n-1}$,

$$\varphi_{\bar{\eta}}(K) = F_K(\eta\bar{e}) = (\eta^{-1}F_K)(\bar{e}) = F_{\eta^{-1}K}(\bar{e}) = \varphi_{\bar{e}}(\eta^{-1}K).$$

Thus, an $\mathrm{SO}(n)$ -equivariant $F \in \mathbf{CVal}$ is uniquely determined by a single $\mathrm{SO}(n-1)$ -invariant valuation $\varphi_{\bar{e}} \in \mathbf{Val}$. In fact, there is a one-to-one correspondence between the subspace of $\mathrm{SO}(n-1)$ -invariant valuations in \mathbf{Val} and the subspace of $\mathrm{SO}(n)$ -equivariant valuations in \mathbf{CVal} . This observation leads to the following:

Definition. Suppose that $F \in \mathbf{CVal}$ is $\mathrm{SO}(n)$ -equivariant. The $\mathrm{SO}(n-1)$ -invariant valuation $\varphi \in \mathbf{Val}$ defined by

$$\varphi(K) = F_K(\bar{e}), \quad K \in \mathcal{K}^n,$$

is called the *associated real valued valuation* of $F \in \mathbf{CVal}$.

The following collection of examples and results on *homogeneous* valuations will be useful for later reference.

Examples 3.1. (a) It is a trivial fact that \mathbf{Val}_0 is 1-dimensional and spanned by the Euler characteristic V_0 . (Recall that $V_0(K) = 1$ for every $K \in \mathcal{K}^n$.) From this observation, it follows that $\Phi_0 \in \mathbf{MVal}_0$ if and only if there exists an $L_0 \in \mathcal{K}^n$ such that $\Phi_0 K = L_0$ for every $K \in \mathcal{K}^n$. If Φ_0 is also $\mathrm{SO}(n)$ -equivariant, then $L_0 = c_0 B$, where B denotes the Euclidean unit ball in \mathbb{R}^n and $c_0 \geq 0$.

Hadwiger [33, p. 79] proved that also \mathbf{Val}_n is 1-dimensional and spanned by the ordinary volume V_n . From this one can easily deduce that $\Phi_n \in \mathbf{MVal}_n$ if and only if

there exists an $L_n \in \mathcal{K}^n$ such that $\Phi_n K = L_n V(K)$ for every $K \in \mathcal{K}^n$. If Φ_n is also $\text{SO}(n)$ -equivariant, then $L_n = c_n B$ for some $c_n \geq 0$.

(b) It was first proved by Spiegel [63] that if $\psi \in \mathbf{Val}_1$, then

$$\psi(K + L) = \psi(K) + \psi(L)$$

for all $K, L \in \mathcal{K}^n$. Using this Minkowski additivity, Goodey and Weil [24] obtained a description of valuations in \mathbf{Val}_1 , refined by Kiderlen [36]. In order to state their result, recall that any $f \in C^\infty(\mathbb{S}^{n-1})$ can be written as a difference of two support functions $f = h_{K_f} - h_{r_f B}$, where $r_f \geq 0$ (see, e.g., [56, Lemma 1.7.8]). Now for $\psi \in \mathbf{Val}_1$, let $\nu_\psi \in C_0^{-\infty}(\mathbb{S}^{n-1})$ be given by

$$\langle f, \nu_\psi \rangle = \psi(K_f) - \psi(r_f B), \quad f \in C^\infty(\mathbb{S}^{n-1}), \quad (3.2)$$

which is well defined by the Minkowski additivity of ψ . Moreover, since r_f depends continuously on f in the C^2 norm, the distribution $\nu_\psi \in C_0^{-\infty}(\mathbb{S}^{n-1})$ is of order at most 2.

This allows one to conclude that if $\psi \in \mathbf{Val}_1$, then there exists a unique ν_ψ in $C_0^{-\infty}(\mathbb{S}^{n-1})$ of order at most 2, which can be extended to the vector space spanned by support functions, such that

$$\psi(K) = \langle h_K, \nu_\psi \rangle$$

for every $K \in \mathcal{K}^n$. Also observe that $\psi \mapsto \nu_\psi$ is continuous as a map from \mathbf{Val}_1 to $C_0^{-\infty}(\mathbb{S}^{n-1})$. Using this description of valuations in \mathbf{Val}_1 , Kiderlen [36] proved that if $\Phi_1 \in \mathbf{MVal}_1$ is $\text{SO}(n)$ -equivariant, then there exists a unique zonal $\nu_1 \in C_0^{-\infty}(\mathbb{S}^{n-1})$ of order at most 2 such that

$$h_{\Phi_1 K} = h_K * \nu_1 \quad (3.3)$$

for every $K \in \mathcal{K}^n$. From (3.3) and a straightforward generalization of (2.18), it follows that for every $\text{SO}(n)$ -equivariant $\Phi_1 \in \mathbf{MVal}_1$, there exists a uniquely determined sequence of real numbers $a_k^n[\Phi_1]$, $k \geq 0$, such that $a_1^n[\Phi_1] = 0$ and

$$\pi_k h_{\Phi_1 K} = a_k^n[\Phi_1] \pi_k h_K \quad (3.4)$$

for every $K \in \mathcal{K}^n$ and $k \geq 0$. In fact, (3.4) was already proved in 1974 by Schneider [54], who also showed that for every $k \neq 1$,

$$|a_k^n[\Phi_1]| \leq a_0^n[\Phi_1]. \quad (3.5)$$

We note that a precise description of the cone of zonal distributions in $C_0^{-\infty}(\mathbb{S}^{n-1})$ which generate a Minkowski valuation by (3.3) is still open. However, Kiderlen [36] showed that this cone contains all non-negative zonal measures on \mathbb{S}^{n-1} . More precisely, if $\mu_1 \in \mathcal{M}_0(\mathbb{S}^{n-1})$ is zonal and non-negative, then

$$h_{\Psi_1 K} = h_K * \mu_1, \quad K \in \mathcal{K}^n,$$

defines an $\text{SO}(n)$ -equivariant Minkowski valuation in \mathbf{MVal}_1 .

(c) A classification of continuous translation-invariant scalar valued valuations of degree $n - 1$ was obtained by McMullen [48]. It states that $\phi \in \mathbf{Val}_{n-1}$ if and only if there exists a unique $f \in C_0(\mathbb{S}^{n-1})$ such that

$$\phi(K) = \int_{\mathbb{S}^{n-1}} f(u) dS_{n-1}(K, u)$$

for every $K \in \mathcal{K}^n$. Applying McMullen's result to associated real valued valuations, the first author [59] obtained the following description of $\mathrm{SO}(n)$ -equivariant Minkowski valuations of degree $n - 1$: If $\Phi_{n-1} \in \mathbf{MVal}_{n-1}$ is $\mathrm{SO}(n)$ -equivariant, then there exists a unique zonal function $f_{n-1} \in C_0(\mathbb{S}^{n-1})$ such that

$$h_{\Phi_{n-1}K} = S_{n-1}(K, \cdot) * f_{n-1} \quad (3.6)$$

for every $K \in \mathcal{K}^n$. As in the case of Minkowski valuations of degree 1, a precise description of the cone of zonal functions in $C_0(\mathbb{S}^{n-1})$ which generate a Minkowski valuation by (3.6) is still open (see [59] for more information).

(d) Several important Minkowski valuations arise from data about sections or projections of convex bodies and are therefore objects of intensive research in geometric tomography (see, e.g., [21, 30, 36, 40, 41]). Of particular interest for us are the *normalized mean section operators* $M_j \in \mathbf{MVal}_{n+1-j}$, $2 \leq j \leq n$, introduced by Goodey and Weil [25]. In [26, Theorem 4.4], they showed that for $K \in \mathcal{K}^n$,

$$h_{M_j K} = q_{n,j} S_{n+1-j}(K, \cdot) * \check{g}_j, \quad (3.7)$$

where g_j is the j th Berg function and the constant $q_{n,j}$ is given by

$$q_{n,j} = \frac{j-1}{2\pi(n+1-j)} \frac{\kappa_{j-1}\kappa_{j-2}\kappa_{n-j}}{\kappa_{j-3}\kappa_{n-2}}.$$

Here, κ_i is the i -dimensional volume of the i -dimensional Euclidean unit ball.

A simple consequence of (3.1) is that the space \mathbf{Val} becomes a Banach space when endowed with the norm

$$\|\phi\| = \sup\{|\phi(K)| : K \subseteq B\}.$$

The natural *continuous* action of the general linear group $\mathrm{GL}(n)$ on the Banach space \mathbf{Val} is for $A \in \mathrm{GL}(n)$ given by

$$(A\phi)(K) = \phi(A^{-1}K), \quad \phi \in \mathbf{Val}, K \in \mathcal{K}^n.$$

The notion of smooth vectors of a continuous representation now gives rise to the notion of smooth valuations, first introduced by Alesker [5].

Definition. A valuation $\phi \in \mathbf{Val}$ is called *smooth* if the map $\mathrm{GL}(n) \rightarrow \mathbf{Val}$ defined by $A \mapsto A\phi$ is infinitely differentiable.

Note that smooth valuations are precisely the smooth vectors of the natural representation of $\mathrm{GL}(n)$ on \mathbf{Val} . We therefore write \mathbf{Val}^∞ for the Fréchet space of smooth translation-invariant valuations endowed with the Gårding topology (see Section 2). We denote by

\mathbf{Val}_i^∞ the subspace of smooth valuations of degree i . By the general properties of smooth vectors discussed in Section 2, \mathbf{Val}_i^∞ is a dense $GL(n)$ -invariant subspace of \mathbf{Val}_i , and from (3.1) one can deduce that

$$\mathbf{Val}^\infty = \bigoplus_{0 \leq i \leq n} \mathbf{Val}_i^\infty.$$

The one-to-one correspondence between $SO(n)$ -equivariant valuations in \mathbf{CVal} and $SO(n - 1)$ -invariant valuations in \mathbf{Val} now motivates the following.

Definition. Let $F \in \mathbf{CVal}$ be $SO(n)$ -equivariant and let $\varphi \in \mathbf{Val}$ be the associated real valued valuation of F . We define the *norm* of F by

$$\|F\| = \sup\{|\varphi(K)| : K \subseteq B\}. \tag{3.8}$$

Furthermore, we call F *smooth* if φ is smooth.

While it is easy to see that McMullen’s decomposition result (3.1) implies that

$$\mathbf{CVal} = \bigoplus_{0 \leq i \leq n} \mathbf{CVal}_i, \tag{3.9}$$

it was recently proved by Parapatits and the second author [51] that, in general, a Minkowski valuation $\Phi \in \mathbf{MVal}$ need not be a sum of homogeneous Minkowski valuations $\Phi_i \in \mathbf{MVal}_i$. However, from (3.9) one can still deduce a decomposition result for translation-invariant Minkowski valuations (cf. [57]), which we state here under the additional assumption of $SO(n)$ -equivariance.

Lemma 3.2. *If $\Phi \in \mathbf{MVal}$ is $SO(n)$ -equivariant, then there exist unique $c_0, c_n \geq 0$ and $SO(n)$ -equivariant valuations $F_i \in \mathbf{CVal}_i$, $1 \leq i \leq n - 1$, such that*

$$h_{\Phi K} = c_0 + \sum_{i=1}^{n-1} F_{i,K} + c_n V(K)$$

for every $K \in \mathcal{K}^n$. Moreover, if Φ is smooth, then each F_i is also smooth.

The space of $SO(n)$ -equivariant valuations in \mathbf{CVal} endowed with the norm (3.8) becomes a Banach space in which smooth valuations form a dense subspace. However, it is a priori not clear that an $SO(n)$ -equivariant Minkowski valuation in \mathbf{MVal} can be approximated by smooth ones. We will prove this in Section 5.

It is well known that for any $\phi \in \mathbf{Val}$ and $K \in \mathcal{K}^n$, McMullen’s decomposition (3.1) implies that the function $t \mapsto \phi(K + tB)$ is a polynomial of degree at most n . This, in turn, gives rise to a derivation operator $\Lambda : \mathbf{Val} \rightarrow \mathbf{Val}$, defined by

$$(\Lambda\phi)(K) = \left. \frac{d}{dt} \right|_{t=0} \phi(K + tB). \tag{3.10}$$

Using (3.10) it is not difficult to show that if $\phi \in \mathbf{Val}_i$, then $\Lambda\phi \in \mathbf{Val}_{i-1}$, that Λ is continuous, $SO(n)$ -equivariant, and that Λ maps smooth valuations to smooth ones.

The advantage of working with smooth translation-invariant valuations instead of merely continuous ones is that the space \mathbf{Val}^∞ admits more algebraic structure. For example, the following hard Lefschetz type theorem for the operator Λ was proved by Alesker [5] for even valuations and by Bernig and Bröcker [14] for general valuations.

Theorem 3.3 ([5, 14]). *Suppose that $n/2 < i \leq n$. Then $\Lambda^{2i-n} : \mathbf{Val}_i^\infty \rightarrow \mathbf{Val}_{n-i}^\infty$ is an isomorphism.*

Recently, Parapatits and the first author [50] proved that for any $\Phi \in \mathbf{MVal}$, there exist $\Phi^{(j)} \in \mathbf{MVal}$, $0 \leq j \leq n$, such that

$$\Phi(K + tB) = \sum_{j=0}^n t^{n-j} \Phi^{(j)}(K)$$

for every $K \in \mathcal{K}^n$ and $t \geq 0$. This Steiner type formula shows that the operator Λ from (3.10) has a natural analogue for Minkowski valuations $\Lambda : \mathbf{MVal} \rightarrow \mathbf{MVal}$.

Definition. For $\Phi \in \mathbf{MVal}$, define $\Lambda\Phi \in \mathbf{MVal}$ by

$$h_{(\Lambda\Phi)(K)}(u) = \left. \frac{d}{dt} \right|_{t=0} h_{\Phi(K+tB)}(u), \quad u \in \mathbb{S}^{n-1}.$$

Note that if $\Phi \in \mathbf{MVal}_i$ is $SO(n)$ -equivariant, then so is $\Lambda\Phi \in \mathbf{MVal}_{i-1}$. Moreover, if $\varphi \in \mathbf{Val}_i$ is the associated real valued valuation of Φ , then $\Lambda\varphi \in \mathbf{Val}_{i-1}$ is associated with $\Lambda\Phi$. In particular, if Φ is smooth, then so is $\Lambda\Phi$.

Another important structural property of smooth valuations is the existence of a *continuous* bilinear product, discovered by Alesker [6],

$$\cdot : \mathbf{Val}^\infty \times \mathbf{Val}^\infty \rightarrow \mathbf{Val}^\infty, \quad (\phi, \psi) \mapsto \phi \cdot \psi.$$

Endowed with the Alesker product, \mathbf{Val}^∞ becomes an associative and commutative algebra with unit given by the Euler characteristic, which is graded by the degree of homogeneity, that is,

$$\mathbf{Val}_i^\infty \cdot \mathbf{Val}_j^\infty \subseteq \mathbf{Val}_{i+j}^\infty. \quad (3.11)$$

Recall that \mathbf{Val}_n is 1-dimensional and spanned by V_n , our fixed Lebesgue measure on \mathbb{R}^n . If $V_n^* \in \mathbf{Val}_n^*$ is the unique element such that $\langle V_n, V_n^* \rangle = 1$, then it follows from (3.11) that for every $0 \leq i \leq n$,

$$\langle \cdot, \cdot \rangle : \mathbf{Val}_i^\infty \times \mathbf{Val}_{n-i}^\infty \rightarrow \mathbb{R}, \quad (\phi, \psi) \mapsto \langle \phi \cdot \psi, V_n^* \rangle, \quad (3.12)$$

defines a continuous bilinear pairing between smooth valuations of complementary degree. Moreover, Alesker [6] proved that this pairing is *non-degenerate*.

Definition. The space of translation-invariant *generalized valuations* of degree i with $0 \leq i \leq n$ is defined as the topological dual

$$\mathbf{Val}_i^{-\infty} = (\mathbf{Val}_{n-i}^\infty)^*$$

endowed with the weak topology.

Since the pairing (3.12) is non-degenerate, the Poincaré duality map defined by

$$\text{pd} : \mathbf{Val}_i^\infty \rightarrow \mathbf{Val}_i^{-\infty}, \quad \langle \text{pd} \phi, \psi \rangle = \langle \phi, \psi \rangle, \tag{3.13}$$

is continuous, injective and has dense image with respect to the weak topology. Moreover, a result of Alesker [8, Proposition 8.1.2] directly implies another property which will be crucial for our purposes:

Proposition 3.4. *For $0 \leq i \leq n$, the Poincaré duality map $\text{pd} : \mathbf{Val}_i^\infty \rightarrow \mathbf{Val}_i^{-\infty}$ admits a unique continuous extension to \mathbf{Val}_i .*

In the following, we will use the Poincaré duality map to identify the spaces \mathbf{Val}_i^∞ and \mathbf{Val}_i with dense subspaces of $\mathbf{Val}_i^{-\infty}$. Hence, we have the inclusions

$$\mathbf{Val}_i^\infty \subseteq \mathbf{Val}_i \subseteq \mathbf{Val}_i^{-\infty}.$$

Next we recall a recent result of Alesker, Bernig and the first author [9] on the decomposition of the vector spaces of translation invariant (generalized) valuations into $\text{SO}(n)$ -irreducible subspaces.

Theorem 3.5 ([9]). *For $0 \leq i \leq n$, the spaces \mathbf{Val}_i^∞ , \mathbf{Val}_i , and $\mathbf{Val}_i^{-\infty}$ are multiplicity free under the action of $\text{SO}(n)$. Moreover, the highest weights of the $\text{SO}(n)$ -irreducible subspaces in either of them are precisely given by the tuples $(\lambda_1, \dots, \lambda_{\lfloor n/2 \rfloor})$ satisfying (2.9) and the following additional conditions:*

- (i) $\lambda_j = 0$ for $j > \min\{i, n - i\}$; (ii) $|\lambda_j| \neq 1$ for $1 \leq j \leq \lfloor n/2 \rfloor$; (iii) $|\lambda_2| \leq 2$.

We now use the notion of spherical representations (see Section 2) to define spherical valuations.

Definition. For $0 \leq i \leq n$, the subspaces $\mathbf{Val}_i^{\text{sph}}$, $\mathbf{Val}_i^{\infty, \text{sph}}$, and $\mathbf{Val}_i^{-\infty, \text{sph}}$ of translation invariant continuous, smooth, and generalized spherical valuations of degree i are defined as the closures (in the respective topologies) of the direct sum of all $\text{SO}(n)$ -irreducible subspaces in \mathbf{Val}_i , \mathbf{Val}_i^∞ , and $\mathbf{Val}_i^{-\infty}$, respectively, which are spherical with respect to $\text{SO}(n - 1)$.

By Theorems 3.5 and 2.3 (see also Example 2.4), $\mathbf{Val}_i^{-\infty, \text{sph}}$ is the annihilator of the closure of the direct sum of all $\text{SO}(n)$ -irreducible subspaces in $\mathbf{Val}_{n-i}^\infty$ with highest weights not of the form $(k, 0, \dots, 0)$, $k \geq 0$. Consequently, Theorem 2.3(b) implies the following critical fact:

Proposition 3.6. *Every $\text{SO}(n - 1)$ -invariant (generalized) valuation in \mathbf{Val}_i , \mathbf{Val}_i^∞ , or $\mathbf{Val}_i^{-\infty}$, where $0 \leq i \leq n$, is spherical.*

Examples 3.7. (a) It follows from Theorem 3.5 or Example 3.1(b) that

$$\mathbf{Val}_1 = \mathbf{Val}_1^{\text{sph}}, \quad \mathbf{Val}_1^\infty = \mathbf{Val}_1^{\infty, \text{sph}}, \quad \mathbf{Val}_1^{-\infty} = \mathbf{Val}_1^{-\infty, \text{sph}}.$$

(b) It follows from Theorem 3.5 or Example 3.1(c) that

$$\mathbf{Val}_{n-1} = \mathbf{Val}_{n-1}^{\text{sph}}, \quad \mathbf{Val}_{n-1}^\infty = \mathbf{Val}_{n-1}^{\infty, \text{sph}}, \quad \mathbf{Val}_{n-1}^{-\infty} = \mathbf{Val}_{n-1}^{-\infty, \text{sph}}.$$

4. Auxiliary results about smooth valuations

In this section we begin with the proof of Theorem 3(a). As a corollary, we obtain a version of Theorem 2 for smooth Minkowski valuations. We also determine an explicit expression for the pairing (3.12) when one of the valuations is spherical. This will be needed in the next section to complete the proof of Theorem 3(a).

Theorem 4.1. *For $1 \leq i \leq n - 1$, the map $E_i : C_0^\infty(\mathbb{S}^{n-1}) \rightarrow \mathbf{Val}_i^{\infty, \text{sph}}$ defined by*

$$(E_i f)(K) = \int_{\mathbb{S}^{n-1}} f(u) dS_i(K, u) \tag{4.1}$$

is an $\text{SO}(n)$ -equivariant isomorphism of topological vector spaces.

Proof. Clearly, the maps $E_i : C_0(\mathbb{S}^{n-1}) \rightarrow \mathbf{Val}_i$ given by (4.1) are linear and $\text{SO}(n)$ -equivariant for every $i \in \{1, \dots, n - 1\}$. If $N(K)$ denotes the normal cycle of a convex body K , then

$$E_i f(K) = c_i \int_{N(K)} f(y) \sum_{\pi} \text{sgn}(\pi) y_{\pi_1} dy_{\pi_2} \wedge \dots \wedge dy_{\pi_{n-i}} \wedge dx_{\pi_{n-i+1}} \wedge \dots \wedge dx_{\pi_n},$$

where c_i is a constant and the sum ranges over all permutations of $\{1, \dots, n\}$ [17]. This shows that the maps $E_i : C_0^\infty(\mathbb{S}^{n-1}) \rightarrow \mathbf{Val}_i^\infty$ are well defined and continuous.

Since differences of area measures of order i of convex bodies in \mathcal{K}^n are dense in the set of all signed finite Borel measures on \mathbb{S}^{n-1} with centroid at the origin (see, e.g., [56, p. 477]), the maps E_i are also injective. Consequently, by Schur’s lemma and Example 2.4, $E_i(\mathcal{H}_k^n)$, $k \neq 1$, is an $\text{SO}(n)$ -irreducible subspace of \mathbf{Val}_i^∞ of highest weight $(k, 0, \dots, 0)$. By Theorem 3.5 and the definition of the spaces $\mathbf{Val}_i^{\infty, \text{sph}}$, it follows that $\mathbf{U}_i := E_i(C_0^\infty(\mathbb{S}^{n-1}))$ is a dense subspace of $\mathbf{Val}_i^{\infty, \text{sph}}$. By the open mapping theorem, it remains to show that \mathbf{U}_i is closed.

First, let $i = n - 1$. In this case, E_{n-1} can be rewritten as a $\text{GL}(n)$ -equivariant map $\Psi : C(\mathbb{P}_+^n(\mathbb{R}^n), \mathcal{L}) \rightarrow \mathbf{Val}_{n-1}$ mapping continuous sections of a certain line bundle over the manifold of cooriented linear hyperplanes of \mathbb{R}^n to valuations (see [8, Section 2] for details). As an immediate consequence of Lemma 2.2 and the Casselman–Wallach theorem [19], the image of $\Psi : C^\infty(\mathbb{P}_+^n(\mathbb{R}^n), \mathcal{L}) \rightarrow \mathbf{Val}_{n-1}^\infty$, which coincides with the image of $E_{n-1} : C_0^\infty(\mathbb{S}^{n-1}) \rightarrow \mathbf{Val}_{n-1}^\infty$, is closed.

Next, recall that the area measures satisfy the Steiner type formula

$$S_i(K + tB, \cdot) = \sum_{j=0}^i t^{i-j} \binom{i}{j} S_j(K, \cdot)$$

for every $K \in \mathcal{K}^n$ and $t \geq 0$. Thus, for $f \in C_0(\mathbb{S}^{n-1})$ and $i \geq 2$, we have

$$(\Lambda E_i f)(K) = \frac{d}{dt} \Big|_{t=0} \int_{\mathbb{S}^{n-1}} f(u) dS_i(K + tB, u) = i(E_{i-1} f)(K). \tag{4.2}$$

In particular, the restriction of the derivation operator Λ to \mathbf{U}_i is injective for $2 \leq i \leq n - 1$ and $\Lambda(\mathbf{U}_i) = \mathbf{U}_{i-1}$. Since Λ is a linear $\text{SO}(n)$ -equivariant operator, its kernel is an $\text{SO}(n)$ -invariant subspace. Consequently, the restriction of Λ to $\text{cl } \mathbf{U}_i = \mathbf{Val}_i^{\infty, \text{sph}}$ is injective as

well. By Theorem 3.3, $\Lambda^{n-2} : \mathbf{Val}_{n-1}^\infty \rightarrow \mathbf{Val}_1^\infty$ is an $\mathrm{SO}(n)$ -equivariant isomorphism. But $\mathbf{U}_{n-1} = \mathbf{Val}_{n-1}^\infty$ by what we have proved above. Therefore,

$$\mathbf{U}_1 = \Lambda^{n-2}(\mathbf{U}_{n-1}) = \mathbf{Val}_1^\infty.$$

In particular, $\mathbf{U}_1 = \Lambda^{i-1}(\mathbf{U}_i)$ is closed. Hence,

$$\mathbf{U}_i = (\Lambda^{i-1})^{-1}\mathbf{U}_1 \cap \mathbf{Val}_i^\infty$$

is closed, and therefore $\mathbf{U}_i = \mathbf{Val}_i^{\infty, \mathrm{sph}}$. □

Using Theorem 4.1, we can now prove the following Hadwiger type result for smooth $\mathrm{SO}(n)$ -equivariant valuations with values in $C(\mathbb{S}^{n-1})$.

Theorem 4.2. *A map $F : \mathcal{K}^n \rightarrow C(\mathbb{S}^{n-1})$ is a smooth translation-invariant and $\mathrm{SO}(n)$ -equivariant valuation if and only if there exist unique $c_0, c_n \in \mathbb{R}$ and $\mathrm{SO}(n-1)$ -invariant $f_i \in C_0^\infty(\mathbb{S}^{n-1})$, $1 \leq i \leq n-1$, such that*

$$F_K = c_0 + \sum_{i=1}^{n-1} S_i(K, \cdot) * f_i + c_n V_n(K) \tag{4.3}$$

for every $K \in \mathcal{K}^n$.

Proof. Let $F \in \mathbf{CVal}$ be $\mathrm{SO}(n)$ -equivariant and smooth. From (3.9), we deduce that there exist smooth and $\mathrm{SO}(n)$ -equivariant valuations $F_i \in \mathbf{CVal}_i$, $0 \leq i \leq n$, such that

$$F = F_0 + \dots + F_n.$$

The classifications of valuations from \mathbf{Val}_0 and \mathbf{Val}_n described in Example 3.1(a) imply that there exist unique $c_0, c_n \in \mathbb{R}$ such that $F_{0,K} = c_0$ and $F_{n,K} = c_n V(K)$ for every $K \in \mathcal{K}^n$. It remains to show that for every $\mathrm{SO}(n)$ -equivariant smooth $F_i \in \mathbf{CVal}_i$, $1 \leq i \leq n-1$, there exists a unique $\mathrm{SO}(n-1)$ -invariant $f_i \in C_0^\infty(\mathbb{S}^{n-1})$ such that for every $K \in \mathcal{K}^n$,

$$F_{i,K} = S_i(K, \cdot) * f_i. \tag{4.4}$$

To this end, let $\varphi_i \in \mathbf{Val}_i^\infty$ denote the associated real valued valuation of F_i and recall that, by definition, φ_i is $\mathrm{SO}(n-1)$ -invariant. Therefore, by Proposition 3.6, $\varphi_i \in \mathbf{Val}_i^{\infty, \mathrm{sph}}$. Consequently, by Theorem 4.1, there exists a unique $f_i \in C_0^\infty(\mathbb{S}^{n-1})$ such that

$$\varphi_i(K) = \int_{\mathbb{S}^{n-1}} f_i(u) dS_i(K, u). \tag{4.5}$$

Moreover, the $\mathrm{SO}(n-1)$ -invariance of φ_i implies that f_i is also $\mathrm{SO}(n-1)$ -invariant. Hence, by the definition of φ_i and (2.17), we obtain

$$F_{i,K}(\bar{\eta}) = \varphi_i(\eta^{-1}K) = \int_{\mathbb{S}^{n-1}} f_i(\eta^{-1}u) dS_i(K, u) = (S_i(K, \cdot) * f_i)(\bar{\eta}),$$

where for $\eta \in \mathrm{SO}(n)$ we set, as before, $\bar{\eta} = \eta\bar{e} \in \mathbb{S}^{n-1}$.

Conversely, let $F \in \mathbf{CVal}$ be defined by (4.3). Then, by the properties of spherical convolution, F is $\mathrm{SO}(n)$ -equivariant and the associated real valued valuation φ_i of its homogeneous component of degree $i \in \{1, \dots, n-1\}$ is given by (4.5). By Theorem 4.1, φ_i is smooth, and thus F is smooth. \square

Since any smooth $\mathrm{SO}(n)$ -equivariant Minkowski valuation $\Phi \in \mathbf{MVal}$ induces a smooth $\mathrm{SO}(n)$ -equivariant valuation $F^\Phi \in \mathbf{CVal}$, as a direct consequence of Lemma 3.2 and Theorem 4.2 we obtain the following:

Corollary 4.3. *If $\Phi : \mathcal{K}^n \rightarrow \mathcal{K}^n$ is a smooth Minkowski valuation which is translation-invariant and $\mathrm{SO}(n)$ -equivariant, then there exist unique $c_0, c_n \geq 0$ and $\mathrm{SO}(n-1)$ -invariant $f_i \in C_0^\infty(\mathbb{S}^{n-1})$, $1 \leq i \leq n-1$, such that*

$$h_{\Phi K} = c_0 + \sum_{i=1}^{n-1} S_i(K, \cdot) * f_i + c_n V_n(K) \quad (4.6)$$

for every $K \in \mathcal{K}^n$.

Corollary 4.3 under the additional assumption that the Minkowski valuation Φ is even was recently obtained by the authors [62] using a different approach.

Finally, we require a generalization of a formula of Bernig and Hug [17] for the pairing (3.12) of spherical valuations. To this end, let $a : \mathbb{S}^{n-1} \rightarrow \mathbb{S}^{n-1}$ denote the antipodal map given by $a(u) = -u$, $u \in \mathbb{S}^{n-1}$, and recall from Example 3.1(b) that any $\psi \in \mathbf{Val}_1$ determines a unique $v_\psi \in C_0^{-\infty}(\mathbb{S}^{n-1})$, defined by (3.2).

Proposition 4.4. *Let $1 \leq i \leq n-1$. For $\phi_i \in \mathbf{Val}_i$ and $f \in C_0^\infty(\mathbb{S}^{n-1})$, we have*

$$\langle \phi_i, E_{n-i} f \rangle = \frac{(n-i)!}{(n-1)!} \langle f \circ a, v_{\Lambda^{i-1} \phi_i} \rangle. \quad (4.7)$$

Proof. Since both pairings in (4.7) are jointly continuous and bilinear, we may assume that $f \in \mathcal{H}_k^n$ for some $k \geq 0$, $k \neq 1$, and that ϕ_i belongs to an $\mathrm{SO}(n)$ -irreducible subspace $\Gamma_\lambda \subseteq \mathbf{Val}_i$ of highest weight $\lambda = (\lambda_1, \dots, \lambda_{\lfloor n/2 \rfloor})$. In particular, ϕ_i is smooth.

Next, note that $\Lambda^{i-1} \phi_i \in \mathbf{Val}_1^\infty = \mathbf{Val}_1^{\infty, \mathrm{sph}}$ (cf. Examples 3.7). Therefore, by Theorem 4.1, there exists a smooth function $h \in C_0^\infty(\mathbb{S}^{n-1})$ (in fact, $h \in \mathcal{H}_m^n$ for some $m \in \mathbb{N}$) such that

$$(\Lambda^{i-1} \phi_i)(K) = i! \int_{\mathbb{S}^{n-1}} h(u) dS_1(K, u). \quad (4.8)$$

The normalizing coefficient $i!$ is chosen for convenience, as will become clear below. From (2.21), the fact that \square_n is self-adjoint, and definition (3.2), it follows that (4.8) can be rewritten in the following way:

$$(\Lambda^{i-1} \phi_i)(K) = i! \int_{\mathbb{S}^{n-1}} h_K(u) \square_n h(u) du = \langle h_K, v_{\Lambda^{i-1} \phi_i} \rangle.$$

In particular, $v_{\Lambda^{i-1} \phi_i} = i! \square_n h$.

Since the pairing (3.12) is biinvariant under the (simultaneous) action of $SO(n)$, and the spaces \mathcal{H}_k^n are self-dual as $SO(n)$ -modules, the restriction of the Poincaré duality map to Γ_λ defines a linear $SO(n)$ -equivariant map from Γ_λ to \mathcal{H}_k^n , that is,

$$\text{pd}|_{\Gamma_\lambda} \in \text{Hom}_{SO(n)}(\Gamma_\lambda, \mathcal{H}_k^n).$$

Since both Γ_λ and \mathcal{H}_k^n are $SO(n)$ -irreducible, it follows from Schur’s lemma that $\text{pd}|_{\Gamma_\lambda}$ and thus the left hand side of (4.7), can only be non-zero when Γ_λ and \mathcal{H}_k^n are isomorphic, that is, when $(\lambda_1, \dots, \lambda_{\lfloor n/2 \rfloor}) = (k, 0, \dots, 0)$. Similarly, the right hand side of (4.7) can only be non-zero if $(\lambda_1, \dots, \lambda_{\lfloor n/2 \rfloor}) = (k, 0, \dots, 0)$. We may therefore assume that ϕ_i is spherical. But if $\phi_i \in \mathbf{Val}_i^{\infty, \text{sph}}$, then, by (4.8) and (4.2), we have

$$\phi_i(K) = \int_{\mathbb{S}^{n-1}} h(u) dS_i(K, u) = E_i h.$$

In this case, it follows from [17, Proposition 4.11] that

$$\langle E_i h, E_{n-i} f \rangle = \frac{(n-i)!i!}{(n-1)!} \int_{\mathbb{S}^{n-1}} h(u) \square_n f(-u) du. \tag{4.9}$$

Using again the fact that \square_n is self-adjoint and $v_{\Lambda^{i-1}\phi_i} = i! \square_n h$, we obtain

$$\langle \phi_i, E_{n-i} f \rangle = \frac{(n-i)!}{(n-1)!} \langle f \circ a, v_{\Lambda^{i-1}\phi_i} \rangle,$$

which is the desired relation. □

5. Proof of the main results

We also discuss a more precise version of Theorem 2 for homogeneous Minkowski valuations in dimensions $n \leq 4$. At the end of the section we include a result on approximation of continuous Minkowski valuations by smooth ones.

We begin with the following slightly more precise version of Theorem 3(a).

Theorem 5.1. *For $1 \leq i \leq n - 1$, the isomorphism $E_i : C_0^\infty(\mathbb{S}^{n-1}) \rightarrow \mathbf{Val}_i^{\infty, \text{sph}}$ admits a unique extension by continuity in the weak topologies to an isomorphism*

$$\tilde{E}_i : C_0^{-\infty}(\mathbb{S}^{n-1}) \rightarrow \mathbf{Val}_i^{-\infty, \text{sph}}.$$

That is, the diagram

$$\begin{array}{ccc} C_0^\infty(\mathbb{S}^{n-1}) & \xrightarrow{E_i} & \mathbf{Val}_i^{\infty, \text{sph}} \\ \downarrow & & \downarrow \text{pd} \\ C_0^{-\infty}(\mathbb{S}^{n-1}) & \xrightarrow{\tilde{E}_i} & \mathbf{Val}_i^{-\infty, \text{sph}} \end{array}$$

commutes and the vertical maps have dense image.

Proof. First recall that $\mathbf{Val}_i^{-\infty, \text{sph}}$ is the annihilator of the subspace spanned by all $\text{SO}(n)$ -irreducible subspaces of $\mathbf{Val}_{n-i}^\infty$ which are non-spherical. Hence, using Theorem 4.1, we can define a map $\tilde{E}_i : C_0^{-\infty}(\mathbb{S}^{n-1}) \rightarrow \mathbf{Val}_i^{-\infty, \text{sph}}$ by

$$\langle \tilde{E}_i v, \phi_{n-i} \rangle = \frac{i!}{(n-1)!} \langle v, (\square_n \circ a^* \circ E_1^{-1} \circ \Lambda^{n-i-1}) \phi_{n-i} \rangle, \tag{5.1}$$

where $\phi_{n-i} \in \mathbf{Val}_{n-i}^\infty$ and a^* denotes the pullback by the antipodal map. Since \square_n and a^* are self-adjoint, (4.9) shows that \tilde{E}_i continuously extends E_i .

Since E_i is $\text{SO}(n)$ -equivariant and injective, \tilde{E}_i is also injective. To prove that \tilde{E}_i is surjective, let $\xi \in \mathbf{Val}_i^{-\infty, \text{sph}}$ be given and note that $\xi \circ E_{n-i} \in C_0^{-\infty}(\mathbb{S}^{n-1})$. If we set

$$v = \frac{(n-1)!}{(n-i)!i!} \xi \circ E_{n-i} \circ a^* \circ \square_n^{-1} \in C_0^{-\infty}(\mathbb{S}^{n-1})$$

and use the fact that, by (4.2),

$$(E_1^{-1} \circ \Lambda^{n-i-1}) E_{n-i} f = (n-i)! f,$$

then, by (5.1),

$$\langle \tilde{E}_i v, E_{n-i} f \rangle = \langle \xi, E_{n-i} f \rangle$$

for all $f \in C_0^\infty(\mathbb{S}^{n-1})$, that is, $\tilde{E}_i v = \xi$. Clearly, the map \tilde{E}_i^{-1} thus defined is continuous. \square

In the next lemma, which is crucial for the proof of Theorem 3(b), and in all that follows, the letter C will denote a constant that can be different from one line to the next and that depends only on the dimension n .

Lemma 5.2. *There exists a constant $C > 0$ such that*

$$\|f\|_{C^2} \leq C \|\square_n f\|_{C^0}$$

for every $\text{SO}(n-1)$ -invariant $f \in C_0^2(\mathbb{S}^{n-1})$.

Proof. For $q \in \mathbb{R}$, we consider the linear differential operator $D_q : C^2(\mathbb{S}^{n-1}) \rightarrow C(\mathbb{S}^{n-1})$ defined by

$$D_q f = \Delta_{\mathbb{S}} f + qf.$$

Note that D_q is $\text{SO}(n)$ -equivariant and $D_{n-1} = (n-1)\square_n$. Moreover, by (2.20), the operator D_q is injective and has dense image for every $q \neq k(k+n-2)$, $k \geq 0$. If $q = k(k+n-2)$ for some $k \geq 0$, then the kernel of D_q is \mathcal{H}_k^n .

First, we show that there exists a constant $C > 0$ such that for $k = 1, 2$,

$$|\nabla^k f|_0 := \max_{\mathbb{S}^{n-1}} |\nabla^k f| \leq C(\|f\|_{C^0} + \|D_q f\|_{C^0}) \tag{5.2}$$

for every $\text{SO}(n-1)$ -invariant $f \in C^2(\mathbb{S}^{n-1})$, and therefore, by (2.1),

$$\|f\|_{C^2} \leq C(\|f\|_{C^0} + \|D_q f\|_{C^0}). \tag{5.3}$$

Using the cylindrical coordinates (2.2) and expressions (2.7) and (2.8), we see that in order to prove (5.2), it suffices to prove that

$$|\partial_t f|_0 := \sup_{(-1,1)} \left| \frac{\partial f}{\partial t} \right| \leq C(\|f\|_{C^0} + \|D_q f\|_{C^0}), \tag{5.4}$$

$$|(1-t^2)\partial_{tt} f|_0 := \sup_{(-1,1)} \left| (1-t^2) \frac{\partial^2 f}{\partial t^2} \right| \leq C(\|f\|_{C^0} + \|D_q f\|_{C^0}), \tag{5.5}$$

for every $SO(n-1)$ -invariant $f \in C^2(\mathbb{S}^{n-1})$. Since $(1-t^2)\partial_{tt} f = \Delta_{\mathbb{S}} f + (n-1)t\partial_t f$ by (2.6), it follows that (5.5) is actually an immediate consequence of (5.4) and the definition of D_q . Thus, to prove (5.2), we only have to show that (5.4) holds for every $SO(n-1)$ -invariant $f \in C^2(\mathbb{S}^{n-1})$.

Let $f \in C^2(\mathbb{S}^{n-1})$ be $SO(n-1)$ -invariant. Since $|\nabla f|$ and $|\nabla^2 f|$ are bounded on \mathbb{S}^{n-1} , it follows from (2.8) that $\partial_t f$ is bounded on $(-1, 1)$. Assume that $|\partial_t f|$ attains its maximum at $t_0 \in (-1, 1)$. Since, by (2.6),

$$\Delta_{\mathbb{S}} f = (1-t^2)\partial_{tt} f - (n-1)t\partial_t f = (1-t^2)^{1-(n-1)/2} \partial_t \left((1-t^2)^{(n-1)/2} \partial_t f \right),$$

it follows from the definition of D_q that

$$\begin{aligned} (1-t_0^2)^{(n-1)/2} \partial_t f(t_0) &= \int_{-1}^{t_0} \partial_t \left((1-t^2)^{(n-1)/2} \partial_t f(t) \right) dt \\ &= \int_{-1}^{t_0} (1-t^2)^{(n-1)/2-1} (D_q f(t) - qf(t)) dt, \end{aligned}$$

and hence

$$|(1-t_0^2)^{(n-1)/2} \partial_t f(t_0)| \leq C(\|f\|_{C^0} + \|D_q f\|_{C^0}). \tag{5.6}$$

This shows that we may assume that $|t_0| \geq \alpha$ for some fixed $\alpha > 0$, as otherwise (5.6) implies (5.4). But since $\partial_{tt} f(t_0) = 0$, we conclude from (2.6) that

$$-(n-1)t_0 \partial_t f(t_0) = D_q f(t_0) - qf(t_0),$$

which also yields $|\partial_t f(t_0)| \leq C(\|f\|_{C^0} + \|D_q f\|_{C^0})$. Hence, we have shown that

$$|\partial_t f|_0 \leq \max \left\{ C(\|f\|_{C^0} + \|D_q f\|_{C^0}), \limsup_{t \rightarrow \pm 1} |\partial_t f(t)| \right\} \tag{5.7}$$

and it remains to bound $\limsup_{t \rightarrow \pm 1} |\partial_t f(t)|$ in terms of $\|f\|_{C^0}$ and $\|D_q f\|_{C^0}$. To do this, note that, by (2.6), $\partial_t f$ is a *bounded* solution on $(-1, 1)$ of the differential equation

$$y'(t) - \frac{(n-1)t}{1-t^2} y(t) = \frac{D_q f(t) - qf(t)}{1-t^2}.$$

All solutions of this equation are given by

$$y(t) = (1-t^2)^{-(n-1)/2} \left(\int_{-1}^t \frac{D_q f(s) - qf(s)}{(1-s^2)^{1-(n-1)/2}} ds + c \right),$$

where $c \in \mathbb{R}$. Since $\partial_t f$ is bounded, we must have

$$\partial_t f(t) = (1 - t^2)^{-(n-1)/2} \int_{-1}^t \frac{D_q f(s) - qf(s)}{(1 - s^2)^{1-(n-1)/2}} ds, \tag{5.8}$$

and

$$\int_{-1}^1 \frac{D_q f(s) - qf(s)}{(1 - s^2)^{1-(n-1)/2}} ds = 0. \tag{5.9}$$

Consequently,

$$\begin{aligned} \limsup_{t \rightarrow 1} |\partial_t f(t)| &= \limsup_{t \rightarrow 1} (1 - t^2)^{-(n-1)/2} \left| \int_t^1 \frac{D_q f(s) - qf(s)}{(1 - s^2)^{1-(n-1)/2}} ds \right| \\ &\leq \limsup_{t \rightarrow 1} \frac{1 - t}{1 - t^2} \|D_q f - qf\|_{C^0} \leq C(\|f\|_{C^0} + \|D_q f\|_{C^0}). \end{aligned}$$

Similarly, $\limsup_{t \rightarrow -1} |\partial_t f(t)| \leq C(\|f\|_{C^0} + \|D_q f\|_{C^0})$, which, by (5.7), completes the proof of (5.4), and thus of (5.2) and (5.3).

Next, assume that $q < 0$ and note that at a point x_0 where $|f(x_0)|$ is maximal, we have $\text{sign } f(x_0) = -\text{sign } \Delta_{\mathbb{S}} f(x_0)$. Therefore, there exists a $C > 0$ such that

$$\|f\|_{C^0} \leq C \|D_q f\|_{C^0}$$

for every $f \in C^2(\mathbb{S}^{n-1})$. Consequently, we deduce from (5.3) that

$$\|f\|_{C^2} \leq C \|D_q f\|_{C^0}$$

for every $\text{SO}(n - 1)$ -invariant $f \in C^2(\mathbb{S}^{n-1})$. From this, together with the fact that $D_q : C^2(\mathbb{S}^{n-1}) \rightarrow C(\mathbb{S}^{n-1})$ is injective and has dense image, we conclude that

$$D_q^{-1} : C(\mathbb{S}^{n-1})^{\text{SO}(n-1)} \rightarrow C^2(\mathbb{S}^{n-1})^{\text{SO}(n-1)} \hookrightarrow C(\mathbb{S}^{n-1})^{\text{SO}(n-1)}$$

exists and is bounded. Recall that $C(\mathbb{S}^{n-1})^{\text{SO}(n-1)}$ denotes the Banach subspace of all $\text{SO}(n - 1)$ -invariant functions in $C(\mathbb{S}^{n-1})$, and $C^2(\mathbb{S}^{n-1})^{\text{SO}(n-1)}$ is defined similarly. Moreover, the Arzelà–Ascoli theorem implies that D_q^{-1} is compact.

Now, choose $m > n - 1$ and set $q = n - m - 1 < 0$. Applying the Fredholm alternative (see, e.g., [22, Theorem 5.3]) to $D_q^{-1} : C_0(\mathbb{S}^{n-1})^{\text{SO}(n-1)} \rightarrow C_0(\mathbb{S}^{n-1})^{\text{SO}(n-1)}$ shows that either

$$f + mD_q^{-1} f = 0 \tag{5.10}$$

has a non-trivial solution $f \in C_0(\mathbb{S}^{n-1})^{\text{SO}(n-1)}$, or

$$f + mD_q^{-1} f = D_q^{-1} h \tag{5.11}$$

has a solution for every $h \in C_0(\mathbb{S}^{n-1})^{\text{SO}(n-1)}$. In the latter case, the operator $(\text{Id} + mD_q^{-1})^{-1}$ is bounded. However, since

$$D_q(f + mD_q^{-1} f) = \Delta_{\mathbb{S}} f + (n - 1)f = 0$$

for $f \in C_0(\mathbb{S}^{n-1})^{\text{SO}(n-1)}$ implies that $f = 0$, equation (5.10) has no non-trivial solution in $C_0(\mathbb{S}^{n-1})^{\text{SO}(n-1)}$, and thus (5.11) is solvable for every $h \in C_0(\mathbb{S}^{n-1})^{\text{SO}(n-1)}$, that is, $h = \Delta_{\mathbb{S}} f + (n - 1)f$ is solvable for every $h \in C_0(\mathbb{S}^{n-1})^{\text{SO}(n-1)}$ and

$$\|f\|_{C^0} = \|(\text{Id} + mD_q^{-1})^{-1}D_q^{-1}h\|_{C^0} \leq C\|D_q^{-1}h\|_{C^0} \leq C\|h\|_{C^0} = C\|D_{n-1}f\|_{C^0}.$$

Combining this with (5.3) for $q = n - 1$ and recalling that $D_{n-1} = (n - 1)\square_n$ completes the proof of the lemma. \square

We remark that Lemma 5.2 without the assumption of $\text{SO}(n - 1)$ -invariance does not hold in general.

Using Lemma 5.2 and Proposition 4.4, we can now complete the proof of Theorem 3(b).

Proof of Theorem 3(b). Let $\phi_i \in \mathbf{Val}_i^{\text{SO}(n-1)}$, $1 \leq i \leq n - 1$, and recall that, by Proposition 3.6, every $\text{SO}(n - 1)$ -invariant valuation is spherical. Hence, by Proposition 3.4, we can use the Poincaré duality map to identify ϕ_i with a generalized valuation from $\mathbf{Val}_i^{-\infty, \text{sph}}$. By Proposition 4.4,

$$\langle \phi_i, E_{n-i}f \rangle = \frac{(n - i)!}{(n - 1)!} \langle f \circ a, v_{\Lambda^{i-1}\phi_i} \rangle$$

for $f \in C_0^\infty(\mathbb{S}^{n-1})$. Since $\Lambda^{i-1}\phi_i$ is 1-homogeneous, $v_{\Lambda^{i-1}\phi_i} \in C_0^{-\infty}(\mathbb{S}^{n-1})$ is of order at most 2 (cf. Example 3.1(b)). Hence, $\phi_i \circ E_{n-i} \in C_0^{-\infty}(\mathbb{S}^{n-1})$ defines an $\text{SO}(n - 1)$ -invariant distribution of order at most 2.

At the same time, by Theorem 5.1, $\phi_i = \tilde{E}_i\gamma$ for a unique $\gamma \in C_0^{-\infty}(\mathbb{S}^{n-1})$, and since ϕ_i is $\text{SO}(n - 1)$ -invariant, so is γ . We want to show that γ is of order 0, and thus in fact a measure. To this end, first note that, by (5.1),

$$\phi_i \circ E_{n-i} = \frac{(n - i)!i!}{(n - 1)!} \gamma \circ \square_n \circ a^*.$$

Now, for $f \in C_0^\infty(\mathbb{S}^{n-1})$, let

$$\bar{f} = \int_{\text{SO}(n-1)} \vartheta f \, d\vartheta = \delta_{\bar{e}} * f$$

denote the $\text{SO}(n - 1)$ -rotational symmetrization of f . Clearly, $\|\bar{f}\|_{C^0} \leq \|f\|_{C^0}$. Moreover, it is not difficult to show (cf. [53, Theorem 6.30]) that the $\text{SO}(n - 1)$ -invariance of γ implies $\gamma(f) = \gamma(\bar{f})$. Consequently, using Lemma 5.2, we obtain

$$|\gamma(f)| = |\gamma(\bar{f})| = C|(\phi_i \circ E_{n-i} \circ a^*)(\square_n^{-1}\bar{f})| \leq C\|\square_n^{-1}\bar{f}\|_{C^2} \leq C\|\bar{f}\|_{C^0} \leq C\|f\|_{C^0},$$

that is, γ is of order 0 and therefore a measure.

In the case $i = n - 1$, it follows from the result of McMullen [48] described in Example 3.1(c) that in fact $\phi_i \in \tilde{E}_i(C_0(\mathbb{S}^{n-1}))$. \square

In the same way as Theorem 4.1 implies Corollary 4.3, we can now use Theorem 3(b) and an approximation argument to deduce Theorem 2.

Proof of Theorem 2. By Lemma 3.2, we have to show that for every $\text{SO}(n)$ -equivariant $F_i \in \mathbf{CVal}_i$, $1 \leq i \leq n - 1$, there exist unique $\text{SO}(n - 1)$ -invariant measures μ_i in $\mathcal{M}_0(\mathbb{S}^{n-1})$, $1 \leq i \leq n - 2$, and an $\text{SO}(n - 1)$ -invariant function $f_{n-1} \in C_0(\mathbb{S}^{n-1})$, such that for $1 \leq i \leq n - 2$,

$$F_{i,K} = S_i(K, \cdot) * \mu_i, \tag{5.12}$$

$$F_{n-1,K} = S_{n-1}(K, \cdot) * f_{n-1}, \tag{5.13}$$

for every $K \in \mathcal{K}^n$.

Since (5.13) can be proved, using Theorem 3(b), in exactly the same way as (4.4) was deduced from Theorem 4.1, we only explain the proof of (5.12) here. First, let $K \in \mathcal{K}^n$ be such that $h_K \in C^\infty(\mathbb{S}^{n-1})$ and K has positive curvature. Then the area measure $S_i(K, \cdot)$ of K is absolutely continuous with respect to spherical Lebesgue measure with a smooth density function $s_i(K, \cdot) \in C_0^\infty(\mathbb{S}^{n-1})$ (see, e.g., [56, Chapter 2.5]). We want to show that if $\varphi_i \in \mathbf{Val}_i^{\text{sph}}$ denotes the $\text{SO}(n - 1)$ -invariant associated real valued valuation of F_i , $1 \leq i \leq n - 2$, then there exists a unique $\text{SO}(n - 1)$ -invariant $\mu_i \in \mathcal{M}_0(\mathbb{S}^{n-1})$ such that

$$\varphi_i(K) = \int_{\mathbb{S}^{n-1}} s_i(K, u) d\mu_i(u). \tag{5.14}$$

To this end, note that, by Theorem 3, there exists a unique $\text{SO}(n - 1)$ -invariant μ_i in $\mathcal{M}_0(\mathbb{S}^{n-1})$ such that $\varphi_i = \tilde{E}_i \mu_i$. Moreover, a result of Bernig and Faifman [15, p. 11] implies that

$$\varphi_i(K) = \langle \varphi_i, \psi_{n-i}^K \rangle, \tag{5.15}$$

where $\psi_{n-i}^K \in \mathbf{Val}_{n-i}^\infty$ is given by the mixed volume

$$\psi_{n-i}^K(L) = \binom{n}{i} V(L[n - i], -K[i]).$$

Now, let $f_{i,j} \in C_0^\infty(\mathbb{S}^{n-1})$, $j \in \mathbb{N}$, be a sequence of smooth functions which converges weakly to μ_i . Then, by (5.15) and Proposition 4.4, we have

$$\varphi_i(K) = \lim_{j \rightarrow \infty} \langle \tilde{E}_i f_{i,j}, \psi_{n-i}^K \rangle = \frac{i!}{(n - 1)!} \lim_{j \rightarrow \infty} \langle f_{i,j} \circ a, v_{\Lambda^{n-i-1} \psi_{n-i}^K} \rangle.$$

Using the definitions of ψ_{n-i}^K and Λ it is not difficult to show that

$$(\Lambda^{n-i-1} \psi_{n-i}^K)(L) = \frac{n!}{i!} V(L, B[n - i - 1], -K[i]) = \frac{(n - 1)!}{i!} \int_{\mathbb{S}^{n-1}} h_L(u) dS_i(-K, u).$$

Thus, using $dS_i(-K, u) = s_i(K, -u) du$ and the definition of $v_{\Lambda^{n-i-1} \psi_{n-i}^K}$, we obtain

$$\varphi_i(K) = \lim_{j \rightarrow \infty} \int_{\mathbb{S}^{n-1}} s_i(K, u) f_{i,j}(u) du = \int_{\mathbb{S}^{n-1}} s_i(K, u) d\mu_i(u),$$

which completes the proof of (5.14).

From the definition of φ_i , (5.14), and (2.17), we now obtain

$$F_{i,K}(\bar{\eta}) = \varphi_i(\eta^{-1}K) = \int_{\mathbb{S}^{n-1}} s_i(K, \eta u) d\mu_i(u) = (s_i(K, \cdot) * \mu_i)(\bar{\eta}).$$

Since both sides depend continuously on K , (5.12) follows from the fact that convex bodies with smooth support functions and positive curvature are dense in \mathcal{K}^n . \square

The following consequence of Theorem 2 for homogeneous Minkowski valuations includes a slight improvement for dimensions $n \leq 4$ which we deduce from the existence of the derivation operator $\Lambda : \mathbf{MVal} \rightarrow \mathbf{MVal}$ and the estimate (3.5).

Corollary 5.3. *Let $\Phi_i : \mathcal{K}^n \rightarrow \mathcal{K}^n$ be a continuous, translation-invariant, and $\text{SO}(n)$ -equivariant Minkowski valuation of degree $i \in \{0, \dots, n\}$.*

- (i) *If $i = 0$, then $\Phi_0 K = c_0 B$ for some $c_0 \geq 0$ and every $K \in \mathcal{K}^n$.*
- (ii) *If $1 \leq i \leq n - 2$, then there exists a unique $\text{SO}(n - 1)$ -invariant $\mu_i \in \mathcal{M}_0(\mathbb{S}^{n-1})$ such that $h_{\Phi_i K} = S_i(K, \cdot) * \mu_i$ for every $K \in \mathcal{K}^n$.*
- (iii) *If $i = n - 1$, then there exists a unique $\text{SO}(n - 1)$ -invariant $f_{n-1} \in C_0(\mathbb{S}^{n-1})$ such that $h_{\Phi_{n-1} K} = S_{n-1}(K, \cdot) * f_{n-1}$ for every $K \in \mathcal{K}^n$.*
- (iv) *If $i = n$, then $\Phi_n K = c_n V_n(K)B$ for some $c_n \geq 0$ and every $K \in \mathcal{K}^n$.*

Moreover, if $n = 3$ or $n = 4$, then the measures μ_i , $i = 1, 2$, from (ii) are absolutely continuous with respect to spherical Lebesgue measure with densities in $L^2_0(\mathbb{S}^{n-1})$.

Proof. (i)–(iv) are direct consequences of Theorem 2, so we only have to prove the absolute continuity of μ_i , $i = 1, 2$, for $n \leq 4$. To this end, first note that $\Lambda^{i-1}\Phi_i \in \mathbf{MVal}_1$ is $\text{SO}(n)$ -equivariant, and if $\varphi_i \in \mathbf{Val}_i$ is the associated real valued valuation of Φ_i , then $\Lambda^{i-1}\varphi_i \in \mathbf{Val}_1$ is associated with $\Lambda^{i-1}\Phi_i$. Thus, it follows easily from (ii), (4.2), (2.21) and the fact that multiplier transformations commute that for every $K \in \mathcal{K}^n$,

$$h_{\Lambda^{i-1}\Phi_i K} = i!S_1(K, \cdot) * \mu_i = i!h_K * \square_n \mu_i. \tag{5.16}$$

Hence, the distribution determined by $\Lambda^{i-1}\varphi_i \in \mathbf{Val}_1$ (cf. Example 3.1(b)) is given by $i!\square_n \mu_i$. Since μ_i is $\text{SO}(n - 1)$ -invariant, so is $i!\square_n \mu_i$, and it follows from (2.12) and (2.22) that the Fourier expansion of $i!\square_n \mu_i$ is given by

$$i!\square_n \mu_i \sim i! \sum_{k=0}^{\infty} \frac{N(n, k)}{\omega_n} \frac{(1 - k)(k + n - 1)}{n - 1} a_k^n[\mu_i] P_k^n(\cdot, \bar{e}).$$

Therefore, using (5.16), (2.18), and (3.5), we deduce that there exists an absolute constant $C > 0$ such that for every $k \geq 2$,

$$|a_k^n[\mu_i]| \leq C \frac{i!(n - 1)}{(k - 1)(k + n - 1)}. \tag{5.17}$$

But since $(N(n, k)/\omega_n)^{1/2} P_k^n(\cdot, \bar{e})$ forms an orthonormal sequence in $L^2(\mathbb{S}^{n-1})$ (see, e.g., [28, p. 84]) and, by (2.10), $N(n, k) = O(k^{n-2})$ as $k \rightarrow \infty$, we see that

$$\mu_i \sim \sum_{k=0}^{\infty} \frac{N(n, k)}{\omega_n} a_k^n[\mu_i] P_k^n(\cdot, \bar{e})$$

converges in $L^2(\mathbb{S}^{n-1})$ as long as $n \leq 4$. \square

Corollary 5.3(iii) was previously obtained by the first author [59], as already explained in Example 3.1(c). The case $i = 1$ of Corollary 5.3(ii) can be reformulated as follows (cf. (5.16)): There exists a unique $\text{SO}(n - 1)$ -invariant $\mu_1 \in \mathcal{M}_o(\mathbb{S}^{n-1})$ such that for every $K \in \mathcal{K}^n$,

$$h_{\Phi_1 K} = h_K * \square_n \mu_1. \quad (5.18)$$

Comparing (5.18) with the corresponding result (3.3) of Kiderlen [36] shows that we have slightly improved the latter by proving that the $\text{SO}(n - 1)$ -invariant distribution $\nu_1 \in C_0^{-\infty}(\mathbb{S}^{n-1})$ of order at most 2 determined by Φ_1 is always of the form $\nu_1 = \square_n \mu_1$ for some $\text{SO}(n - 1)$ -invariant $\mu_1 \in \mathcal{M}_o(\mathbb{S}^{n-1})$.

Note that the estimate (5.17) is not strong enough to deduce that μ_i is absolutely continuous in higher dimensions, as can be seen for example from the spherical Radon (or Minkowski–Funk) transform $\mathbf{R} : C(\mathbb{S}^{n-1}) \rightarrow C(\mathbb{S}^{n-1})$ defined by

$$\mathbf{R}f = f * \mu_{\mathbb{S}^{n-2}},$$

where $\mu_{\mathbb{S}^{n-2}} \in \mathcal{M}(\mathbb{S}^{n-1})$ is uniformly concentrated on $\mathbb{S}^{n-1} \cap \bar{e}^\perp$. Clearly, $\mu_{\mathbb{S}^{n-2}}$ is $\text{SO}(n - 1)$ -invariant but not absolutely continuous with respect to spherical Lebesgue measure. However, $|a_k^n[\mu_{\mathbb{S}^{n-2}}]| = O(k^{1-n/2})$ as $k \rightarrow \infty$ [28, Lemma 3.4.8].

Finally, we remark that Corollary 5.3(ii) does not leave much room for improvement since the zonal functions \check{g}_j associated with Berg's functions are not continuous on \mathbb{S}^{n-1} for $n, j \geq 3$ and they do not lie in $L_0^2(\mathbb{S}^{n-1})$ but merely in $L_0^1(\mathbb{S}^{n-1})$ for $n, j \geq 5$. However, they are generating functions of the normalized mean section operators \mathbf{M}_j as described in Example 3.1(d).

We conclude this section with a result on approximation of continuous Minkowski valuations by smooth ones which generalizes a corresponding result for *even* Minkowski valuations of the first author [60] and will be useful in the last section.

Corollary 5.4. *Every continuous, translation-invariant, and $\text{SO}(n)$ -equivariant Minkowski valuation can be approximated uniformly on compact subsets of \mathcal{K}^n by smooth, translation-invariant, and $\text{SO}(n)$ -equivariant Minkowski valuations.*

Proof. Let $\Phi \in \mathbf{MVal}$ be $\text{SO}(n)$ -equivariant and let

$$h_{\Phi K} = c_0 + \sum_{i=1}^{n-2} S_i(K, \cdot) * \mu_i + S_{n-1}(K, \cdot) * f_{n-1} + c_n V_n(K) \quad (5.19)$$

be the convolution representation of Φ according to Theorem 2. We define a sequence $\Phi^j \in \mathbf{MVal}$, $j \in \mathbb{N}$, of $\text{SO}(n)$ -equivariant Minkowski valuations by

$$h_{\Phi^j K} = h_{\Phi K} * \zeta_j, \quad K \in \mathcal{K}^n,$$

where ζ_j , $m \in \mathbb{N}$, is a spherical approximate identity. Note that since $\zeta_j \geq 0$, Φ^j is well defined by the result of Kiderlen [36] described at the end of Example 3.1(b). Using (2.19) and the $\text{SO}(n)$ -equivariance of Φ , it is easy to show that Φ^j converges to Φ on compact subsets (cf. the proof of [60, Theorem 6.5]).

It remains to show that the Minkowski valuations Φ^j are smooth, that is, the associated real valued valuations $\varphi^j \in \mathbf{Val}^{\text{sp}}$ are smooth. To this end note that by the linearity of convolution, the fact that $\mu_i * \zeta_j, f_{n-1} * \zeta_j \in C^\infty(\mathbb{S}^{n-1})$, and (2.17), we have

$$\begin{aligned} \varphi^j(K) &= c_0 + \sum_{i=1}^{n-2} \int_{\mathbb{S}^{n-1}} (\mu_i * \zeta_j)(u) dS_i(K, u) \\ &\quad + \int_{\mathbb{S}^{n-1}} (f_{n-1} * \zeta_j)(u) dS_{n-1}(K, u) + c_n V_n(K). \end{aligned}$$

Thus, an application of Theorem 4.1 completes the proof. □

6. Integral geometry of Minkowski valuations

In this final section we apply Theorem 2 to establish a Crofton formula for continuous, translation-invariant, and $\text{SO}(n)$ -equivariant Minkowski valuations. Combining this with Hadwiger’s general kinematic formula allows us also to deduce a kinematic formula for such Minkowski valuations.

We begin by recalling the classical *Crofton formula* (see, e.g., [39, p. 124]) for intrinsic volumes: For $0 \leq i, j \leq n$ and $K \in \mathcal{K}^n$, we have

$$\int_{\text{AGr}_{n-i,n}} V_j(K \cap E) d\sigma_{n-i}(E) = \begin{bmatrix} i+j \\ j \end{bmatrix} V_{i+j}(K). \tag{6.1}$$

Here, $\text{AGr}_{i,n}$ denotes the affine Grassmannian of i planes in \mathbb{R}^n , and σ_i is the rigid-motion-invariant measure on $\text{AGr}_{i,n}$ normalized in such a way that the set of planes having non-empty intersection with the Euclidean unit ball in \mathbb{R}^n has measure

$$\begin{bmatrix} n \\ i \end{bmatrix} \kappa_{n-i} := \binom{n}{i} \frac{\kappa_n}{\kappa_i}.$$

The Crofton formula (6.1) is intimately related to the *general kinematic formula*: For $0 \leq j \leq n$ and $K, L \in \mathcal{K}^n$, we have

$$\int_{\overline{\text{SO}(n)}} V_j(K \cap gL) dg = \sum_{i=0}^{n-j} \begin{bmatrix} i+j \\ j \end{bmatrix} \begin{bmatrix} n \\ i \end{bmatrix}^{-1} V_{i+j}(K) V_{n-i}(L), \tag{6.2}$$

where $\overline{\text{SO}(n)} = \text{SO}(n) \times \mathbb{R}^n$ (see [39, 58] for more information).

The obvious connection between (6.1) and (6.2) is just a special case of *Hadwiger’s general integral geometric theorem* [58, p. 173], which states in the translation-invariant case that for every $\phi \in \mathbf{Val}$ and $K, L \in \mathcal{K}^n$, we have

$$\int_{\overline{\text{SO}(n)}} \phi(K \cap gL) dg = \sum_{i=0}^n V_{n-i}(L) \begin{bmatrix} n \\ i \end{bmatrix}^{-1} \int_{\text{AGr}_{n-i,n}} \phi(K \cap E) d\sigma_{n-i}(E). \tag{6.3}$$

Applying (6.3) to the real valued associated valuation of an $\text{SO}(n)$ -equivariant Minkowski valuation $\Phi \in \mathbf{MVal}$ immediately yields the following kinematic formula for such Minkowski valuations.

Corollary 6.1. *If $\Phi \in \mathbf{MVal}$ is $\text{SO}(n)$ -equivariant, then*

$$\int_{\text{SO}(n)} h_{\Phi(K \cap gL)}(u) dg = \sum_{i=0}^n V_{n-i}(L) \begin{bmatrix} n \\ i \end{bmatrix}^{-1} \int_{\text{AGr}_{n-i,n}} h_{\Phi(K \cap E)}(u) d\sigma_{n-i}(E) \quad (6.4)$$

for every $K, L \in \mathcal{K}^n$ and $u \in \mathbb{S}^{n-1}$.

Note that the sum on the right hand side of (6.4) is again the support function of a convex body. Thus, it remains to determine the Crofton integral in (6.4). In view of Lemma 3.2 and Theorem 2, this is accomplished by our final result.

Theorem 6.2. *Suppose that $1 \leq j \leq n - 2$ and $1 \leq i \leq n - j - 1$. If $F_j \in \mathbf{CVal}_j$ is $\text{SO}(n)$ -equivariant and, for $K \in \mathcal{K}^n$, given by*

$$F_{j,K} = S_j(K, \cdot) * \mu$$

for some (uniquely determined) $\text{SO}(n - 1)$ -invariant measure $\mu \in \mathcal{M}_0(\mathbb{S}^{n-1})$, then

$$\int_{\text{AGr}_{n-i,n}} F_{j,K \cap E} d\sigma_{n-i}(E) = q_{n,i,j} S_{i+j}(K, \cdot) * (\mu * \square_{n-j+1} \check{g}_{n-i-j+1}), \quad (6.5)$$

where $q_{n,i,j} = \frac{2^i}{i! \kappa_i} \prod_{k=j}^{i+j-1} c_{n,k}$ with

$$c_{n,k} = \frac{k(n-k-1)(n-k+1)\kappa_{n-k-2}^2 \kappa_{n-k+1} \kappa_k}{2(n-k)(k+1)\kappa_{n-k-3} \kappa_{n-k}^2 \kappa_{k-1}}.$$

Proof. Consider the isomorphism $\Theta_j : C_0^\infty(\mathbb{S}^{n-1}) \rightarrow C_0^\infty(\mathbb{S}^{n-1})$ defined by

$$\Theta_j f = c_{n,j} \square_{n-j+1} f * \check{g}_{n-j} = c_{n,j} f * \square_{n-j+1} \check{g}_{n-j}.$$

Here and in (6.5), $\square_k \check{g}_l$ is to be understood in the sense of distributions, where we use the canonical extension of the self-adjoint operator \square_k to $C_0^{-\infty}(\mathbb{S}^{n-1})$.

Now, let us first assume that F_j is smooth, that is, μ is absolutely continuous with respect to spherical Lebesgue measure with density $f \in C_0^\infty(\mathbb{S}^{n-1})$. In this case it was proved by the authors [62, Theorem 6.3] that

$$\int_{\text{AGr}_{n-1,n}} F_{j,K \cap E} d\sigma_{n-1}(E) = S_{j+1}(K, \cdot) * \Theta_j f. \quad (6.6)$$

In order to obtain from this the more general formula (6.5), we use the following well known relation (which can be proved by induction using Crofton’s formula; see, e.g., [39, p. 124]):

$$\begin{aligned} & \int_{\text{AGr}_{n-i,n}} f(E) d\sigma_{n-i}(E) \\ &= \frac{2^i}{i! \kappa_i} \int_{\text{AGr}_{n-1,n}} \cdots \int_{\text{AGr}_{n-1,n}} f(E_1 \cap \cdots \cap E_i) d\sigma_{n-1}(E_1) \cdots d\sigma_{n-1}(E_i) \end{aligned}$$

for every Borel measurable $f \in L^1(\text{AGr}_{n-i,n})$. Comparing this with (6.6), we obtain

$$\int_{\text{AGr}_{n-i,n}} F_{j,K \cap E} d\sigma_{n-i}(E) = \frac{2^i}{i!k_i} S_{i+j}(K, \cdot) * \Theta_{i+j-1} \cdots \Theta_{j+1} \Theta_j f. \quad (6.7)$$

Next, note that if $\tau_{\bar{e}} = \delta_{\bar{e}} - \pi_1 \delta_{\bar{e}} \in \mathcal{M}_0(\mathbb{S}^{n-1})$, where $\delta_{\bar{e}}$ is the Dirac measure at $\bar{e} \in \mathbb{S}^{n-1}$, then, by (2.18), $f * \tau_{\bar{e}} = f$ for every $f \in C_0^\infty(\mathbb{S}^{n-1})$. But since $\square_k \check{g}_k = \tau_{\bar{e}}$, (6.5) follows from (6.7) and the definition of Θ_j .

In order to establish (6.5) in the general case, where F_j is merely continuous, we use a spherical approximate identity ζ_k , $k \in \mathbb{N}$, (instead of repeating the arguments from the proof of [62, Theorem 6.3]) to define $F_{j,K}^k = F_{j,K} * \zeta_k$ for every $K \in \mathcal{K}^n$. Then $F_j^k \in \mathbf{CVal}_j$ is $\text{SO}(n)$ -equivariant and smooth, and by what we have already shown and the fact that multiplier transformations commute,

$$\int_{\text{AGr}_{n-i,n}} F_{j,K \cap E}^k d\sigma_{n-i}(E) = q_{n,i,j} S_{i+j}(K, \cdot) * (\mu * \square_{n-j+1} \check{g}_{n-i-j+1}) * \zeta_k.$$

Letting now $k \rightarrow \infty$, we obtain (6.5) from Lemmas 2.5 and 2.6. \square

We conclude with the remark that equivalent forms of Theorem 6.2 were obtained very recently, independently, and using different approaches by Bernig and Hug [17] and Goodey, Hug, and Weil [23].

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References

- [1] Abarodia, J.: Difference bodies in complex vector spaces. *J. Funct. Anal.* **263**, 3588–3603 (2012) [Zbl 1262.52012](#) [MR 2984076](#)
- [2] Abarodia, J., Bernig, A.: Projection bodies in complex vector spaces. *Adv. Math.* **227**, 830–846 (2011) [Zbl 1217.52009](#) [MR 2793024](#)
- [3] Alesker, S.: Continuous rotation invariant valuations on convex sets. *Ann. of Math. (2)* **149**, 977–1005 (1999) [Zbl 0941.52002](#) [MR 1709308](#)
- [4] Alesker, S.: Description of translation invariant valuations on convex sets with solution of P. McMullen’s conjecture. *Geom. Funct. Anal.* **11**, 244–272 (2001) [Zbl 0995.52001](#) [MR 1837364](#)
- [5] Alesker, S.: Hard Lefschetz theorem for valuations, complex integral geometry, and unitarily invariant valuations. *J. Differential Geom.* **63**, 63–95 (2003) [Zbl 1073.52004](#) [MR 2015260](#)
- [6] Alesker, S.: The multiplicative structure on polynomial continuous valuations. *Geom. Funct. Anal.* **14**, 1–26 (2004) [Zbl 1072.52011](#) [MR 2053598](#)
- [7] Alesker, S.: Hard Lefschetz theorem for valuations and related questions of integral geometry. In: *Geometric Aspects of Functional Analysis*, Lecture Notes in Math. 1850, Springer, Berlin, 9–20 (2004) [Zbl 1070.52007](#) [MR 2087146](#)
- [8] Alesker, S.: A Fourier type transform on translation invariant valuations on convex sets. *Israel J. Math.* **181**, 189–294 (2011) [Zbl 1219.52009](#) [MR 2773042](#)

- [9] Alesker, S., Bernig, A., Schuster, F. E.: Harmonic analysis of translation invariant valuations. *Geom. Funct. Anal.* **21**, 751–773 (2011) [Zbl 1228.53088](#) [MR 2827009](#)
- [10] Alesker, S., Faifman, D.: Convex valuations invariant under the Lorentz group. *J. Differential Geom.* **98**, 183–236 (2014) [Zbl 1312.52007](#) [MR 3238311](#)
- [11] Berg, A., Parapatits, L., Schuster, F. E., Weberndorfer, M.: Log-concavity properties of Minkowski valuations. [arXiv:1411.7891](#) (2014)
- [12] Berg, C.: Corps convexes et potentiels sphériques. *Mat.-Fys. Medd. Danske Vid. Selsk.* **37**, 64 pp. (1969) [Zbl 0181.38303](#) [MR 0254789](#)
- [13] Bernig, A.: A Hadwiger-type theorem for the special unitary group. *Geom. Funct. Anal.* **19**, 356–372 (2009) [Zbl 1180.53076](#) [MR 2545241](#)
- [14] Bernig, A., Bröcker, L.: Valuations on manifolds and Rumin cohomology. *J. Differential Geom.* **75**, 433–457 (2007) [Zbl 1117.58005](#) [MR 2301452](#)
- [15] Bernig, A., Faifman, D.: Generalized translation invariant valuations and the polytope algebra. *Adv. Math.* **290**, 36–72 (2016) [Zbl 1336.52019](#) [MR 3451918](#)
- [16] Bernig, A., Fu, J. H. G.: Hermitian integral geometry. *Ann. of Math. (2)* **173**, 907–945 (2011) [Zbl 1230.52014](#) [MR 2776365](#)
- [17] Bernig, A., Hug, D.: Kinematic formulas for tensor valuations. *J. Reine Angew. Math.* **736**, 141–191 (2018) [Zbl 06847720](#) [MR 3769988](#)
- [18] Bröcker, T., tom Dieck, T.: Representations of Compact Lie Groups. *Grad. Texts in Math.* 98. Springer, New York (1985) [Zbl 0874.22001](#) [MR 1410059](#)
- [19] Casselman, W.: Canonical extensions of Harish-Chandra modules to representations of G . *Canad. J. Math.* **41**, 385–438 (1989) [Zbl 0702.22016](#) [MR 1013462](#)
- [20] Fu, J. H. G.: Structure of the unitary valuation algebra. *J. Differential Geom.* **72**, 509–533 (2006) [Zbl 1096.52003](#) [MR 2219942](#)
- [21] Gardner, R. J.: *Geometric Tomography*. 2nd ed., Cambridge Univ. Press, Cambridge (2006) [Zbl 1102.52002](#) [MR 2251886](#)
- [22] Gilbarg, D., Trudinger, N. S.: *Elliptic Partial Differential Equations of Second Order*. Reprint of the 1998 ed., *Classics Math.*, Springer, Berlin (2001) [Zbl 1042.35002](#) [MR 1814364](#)
- [23] Goodey, P., Hug, D., Weil, W.: Kinematic formulas for area measures. *Indiana Univ. Math. J.* **66**, 997–1018 (2017) [Zbl 1376.52005](#)
- [24] Goodey, P., Weil, W.: Distributions and valuations. *Proc. London Math. Soc.* **49**, 504–516 (1984) [Zbl 0526.52004](#) [MR 0759301](#)
- [25] Goodey, P., Weil, W.: The determination of convex bodies from the mean of random sections. *Math. Proc. Cambridge Philos. Soc.* **112**, 419–430 (1992) [Zbl 0772.52005](#) [MR 1171176](#)
- [26] Goodey, P., Weil, W.: Sums of sections, surface area measures and the general Minkowski problem. *J. Differential Geom.* **97**, 477–514 (2014) [Zbl 1301.52009](#) [MR 3263512](#)
- [27] Grinberg, E., Zhang, G.: Convolutions, transforms, and convex bodies. *Proc. London Math. Soc. (3)* **78**, 77–115 (1999) [Zbl 0974.52001](#) [MR 1658156](#)
- [28] Groemer, H.: *Geometric Applications of Fourier Series and Spherical Harmonics*. Cambridge Univ. Press, Cambridge (1996) [Zbl 0877.52002](#) [MR 1412143](#)
- [29] Guillemin, V., Sternberg, S.: *Geometric Asymptotics*. *Math. Surveys* 14, Amer. Math. Soc., Providence, RI (1977) [Zbl 0364.53011](#) [MR 0516965](#)
- [30] Haberl, C.: Minkowski valuations intertwining the special linear group. *J. Eur. Math. Soc.* **14**, 1565–1597 (2012) [Zbl 1270.52018](#) [MR 2966660](#)
- [31] Haberl, C., Parapatits, L.: The centro-affine Hadwiger theorem. *J. Amer. Math. Soc.* **27**, 685–705 (2014) [Zbl 1319.52006](#) [MR 3194492](#)
- [32] Haberl, C., Schuster, F. E.: General L_p affine isoperimetric inequalities. *J. Differential Geom.* **83**, 1–26 (2009) [Zbl 1185.52005](#) [MR 2545028](#)

- [33] Hadwiger, H.: Vorlesungen über Inhalt, Oberfläche und Isoperimetrie. Springer, Berlin (1957) [Zbl 0078.35703](#) [MR 1012775](#)
- [34] Hörmander, L.: The Analysis of Linear Partial Differential Operators. I. Distribution Theory and Fourier Analysis. Grundlehren Math. Wiss. 256, Springer, Berlin (1983) [Zbl 0521.35001](#) [MR 0717035](#)
- [35] Hug, D., Schneider, R., Schuster, R.: The space of isometry covariant tensor valuations. *Algebra i Analiz* **19**, 194–224 (2007) (in Russian) [Zbl 1134.52008](#) [MR 2319515](#)
- [36] Kiderlen, M.: Blaschke- and Minkowski-endomorphisms of convex bodies. *Trans. Amer. Math. Soc.* **358**, 5539–5564 (2006) [Zbl 1111.52007](#) [MR 2238926](#)
- [37] Klain, D. A.: A short proof of Hadwiger’s characterization theorem. *Mathematika* **42**, 329–339 (1995) [Zbl 0835.52010](#) [MR 1376731](#)
- [38] Klain, D. A.: Even valuations on convex bodies. *Trans. Amer. Math. Soc.* **352**, 71–93 (2000) [Zbl 0940.52002](#) [MR 1487620](#)
- [39] Klain, D. A., Rota, G.-C.: Introduction to Geometric Probability. Cambridge Univ. Press, Cambridge (1997) [Zbl 0896.60004](#) [MR 1608265](#)
- [40] Ludwig, M.: Projection bodies and valuations. *Adv. Math.* **172**, 158–168 (2002) [Zbl 1019.52003](#) [MR 1942402](#)
- [41] Ludwig, M.: Minkowski valuations. *Trans. Amer. Math. Soc.* **357**, 4191–4213 (2005) [Zbl 1077.52005](#) [MR 2159706](#)
- [42] Ludwig, M.: Minkowski areas and valuations. *J. Differential Geom.* **86**, 133–161 (2010) [Zbl 1215.52004](#) [MR 2772547](#)
- [43] Ludwig, M., Reitzner, M.: A classification of $SL(n)$ invariant valuations. *Ann. of Math. (2)* **172**, 1219–1267 (2010) [Zbl 1223.52007](#) [MR 2680490](#)
- [44] Lutwak, E., Yang, D., Zhang, G.: L_p affine isoperimetric inequalities. *J. Differential Geom.* **56**, 111–132 (2000) [Zbl 1034.52009](#) [MR 1863023](#)
- [45] Lutwak, E., Yang, D., Zhang, G.: Orlicz projection bodies. *Adv. Math.* **223**, 220–242 (2010) [Zbl 05643962](#) [MR 2563216](#)
- [46] Lutwak, E., Yang, D., Zhang, G.: Orlicz centroid bodies. *J. Differential Geom.* **84**, 365–387 (2010) [Zbl 1206.49050](#) [MR 2652465](#)
- [47] McMullen, P.: Valuations and Euler-type relations on certain classes of convex polytopes. *Proc. London Math. Soc.* **35**, 113–135 (1977) [Zbl 0353.52001](#) [MR 0448239](#)
- [48] McMullen, P.: Continuous translation-invariant valuations on the space of compact convex sets. *Arch. Math. (Basel)* **34**, 377–384 (1980) [Zbl 0424.52003](#) [MR 0593954](#)
- [49] Morimoto, M.: Analytic Functionals on the Sphere. *Transl. Math. Monogr.* 178, Amer. Math. Soc., Providence, RI (1998) [Zbl 0922.46040](#) [MR 1641900](#)
- [50] Parapatits, L., Schuster, F. E.: The Steiner formula for Minkowski valuations. *Adv. Math.* **230**, 978–994 (2012) [Zbl 1252.52011](#) [MR 2921168](#)
- [51] Parapatits, L., Wannerer, T.: On the inverse Klain map. *Duke Math. J.* **162**, 1895–1922 (2013) [Zbl 1280.52003](#) [MR 3090780](#)
- [52] Petersen, P.: Riemannian Geometry. 2nd ed., *Grad. Texts in Math.* 171, Springer, New York (2006) [Zbl 06520113](#) [MR 3469435](#)
- [53] Rudin, W.: Functional Analysis. 2nd ed., McGraw-Hill, New York (1991) [Zbl 0867.46001](#) [MR 1157815](#)
- [54] Schneider, R.: Equivariant endomorphisms of the space of convex bodies. *Trans. Amer. Math. Soc.* **194**, 53–78 (1974) [Zbl 0287.52004](#) [MR 0353147](#)
- [55] Schneider, R.: Bewegungsäquivalente, additive und stetige Transformationen konvexer Bereiche. *Arch. Math. (Basel)* **25**, 303–312 (1974) [Zbl 0286.52003](#) [MR 0344999](#)

- [56] Schneider, R.: *Convex Bodies: The Brunn–Minkowski Theory*. 2nd ed., Encyclopedia Math. Appl. 151, Cambridge Univ. Press, Cambridge (2013) [Zbl 1287.52001](#) [MR 3155183](#)
- [57] Schneider, R., Schuster, F. E.: Rotation equivariant Minkowski valuations. *Int. Math. Res. Notices* **2006**, art. ID 72894, 20 pp. [Zbl 1127.52017](#) [MR 2272092](#)
- [58] Schneider, R., Weil, W.: *Stochastic and Integral Geometry*. Probability Appl. (New York), Springer, Berlin (2008) [Zbl 1175.60003](#) [MR 2455326](#)
- [59] Schuster, F. E.: Convolutions and multiplier transformations of convex bodies. *Trans. Amer. Math. Soc.* **359**, 5567–5591 (2007) [Zbl 1156.52003](#) [MR 2327043](#)
- [60] Schuster, F. E.: Crofton measures and Minkowski valuations. *Duke Math. J.* **154**, 1–30 (2010) [Zbl 1205.52004](#) [MR 2668553](#)
- [61] Schuster, F. E., Wannerer, T.: $GL(n)$ contravariant Minkowski valuations. *Trans. Amer. Math. Soc.* **364**, 815–826 (2012) [Zbl 1246.52009](#) [MR 2846354](#)
- [62] Schuster, F. E., Wannerer, T.: Even Minkowski valuations. *Amer. J. Math.* **137**, 1651–1683 (2015) [Zbl 1336.52020](#) [MR 3432270](#)
- [63] Spiegel, W.: Zur Minkowski-Additivität bestimmter Eikörperabbildungen. *J. Reine Angew. Math.* **286/287**, 164–168 (1976) [Zbl 0333.52003](#) [MR 0430965](#)
- [64] Takeuchi, M.: *Modern Spherical Functions*. Transl. Math. Monogr. 135, Amer. Math. Soc., Providence, RI (1994) [Zbl 0830.43002](#) [MR 1280269](#)
- [65] Wannerer, T.: $GL(n)$ equivariant Minkowski valuations. *Indiana Univ. Math. J.* **60**, 1655–1672 (2011) [Zbl 1270.52019](#) [MR 2997003](#)
- [66] Wannerer, T.: The module of unitarily invariant area measures. *J. Differential Geom.* **96**, 141–182 (2014) [Zbl 1296.53149](#) [MR 3161388](#)
- [67] Wannerer, T.: Integral geometry of unitary area measures. *Adv. Math.* **263**, 1–44 (2014) [Zbl 1296.53150](#) [MR 3239133](#)
- [68] Warner, G.: *Harmonic Analysis on Semi-Simple Lie Groups I*. Springer, Berlin (1972) [Zbl 0265.22020](#) [MR 0498999](#)