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The equational theory of the weak Bruhat order on finite symmetric groups

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Abstract. It is well known that the weak Bruhat order on the symmetric group on a finite number n of letters is a lattice, denoted by $\mathsf{P}(n)$ and often called the *permutohedron on n letters*, of which the *Tamari lattice* $\mathsf{A}(n)$ is a lattice retract. The *equational theory* (or *word problem*) of a class of lattices is the set of all lattice identities satisfied by all members of that class. Our main results imply, as particular cases, the following.

Theorem I. *The equational theories of all* $\mathsf{P}(n)$ *and of all* $\mathsf{A}(n)$ *are both decidable.*

Theorem I. *There exists a lattice identity that holds in all* $\mathsf{P}(n)$, *but fails in a certain* 3,338-*element lattice.*

Theorem III. *The equational theory of all extended permutohedra, on arbitrary* (*possibly infinite*) *posets, is trivial.*

The proofs of Theorems I and II involve reductions of algebraic statements to certain tiling properties of finite chains.

Keywords. Lattice, identity, weak order, permutohedron, Cambrian lattice, Tamari lattice, monadic second-order logic, decidability, score, bounded homomorphic image, subdirectly irreducible, splitting lattice, splitting identity, polarized measure, sub-tensor product, box product, dismantlable lattice

1. Introduction

1.1. Motivation

The last two decades have seen a surge in the investigation of the interactions between the combinatorial structure of Coxeter groups, hyperplane arrangements, and related structures, and their lattice-theoretical properties (Reading [60]; for a survey and many additional references, see Reading [61, 62]).

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Finite symmetric groups with their weak Bruhat ordering (also called *Coxeter lattices of type A*) are the *permutohedra* introduced by Guilbaud and Rosenstiehl [27]. Their subdirectly irreducible quotients are the *Cambrian lattices of type A*, particular cases of which are the *Tamari lattices* (Friedman and Tamari [17]), and which all have geometric realizations as *associahedra* (Hohlweg, Lange, and Thomas [32]). One of the most natural problems arising when considering a given class of algebraic structures is to determine its *word problem*, or, using the universal algebraic equivalent formulation, its *equational theory* (see Sections 1.2–1.4).

The aim of the present paper is to solve that problem completely (first stated in our paper [64]) for Coxeter lattices, and Cambrian lattices, of type A. This is achieved by Theorems I (decidability) and II (existence of a nontrivial identity) stated in the Abstract. An attempt to generalize those results to "extended permutohedra" on arbitrary posets (i.e., partially ordered sets) has led us to Theorem III.

In order to prove Theorems I and II, we reduce the satisfaction of a given lattice identity in a Cambrian lattice of type A to a certain tiling problem on a finite chain. Theorem I then follows from Büchi's decidability theorem for the monadic second-order theory `MSO` of the successor function on the natural numbers. It can be extended to any class of Cambrian lattices of type A with `MSO`-definable set of orientations.

Although a general formalization of the above-mentioned tiling properties may appear cumbersome (see Section 5), some special cases turn out with rather appealing combinatorial descriptions (see Appendix A).

1.2. Some background

For a positive integer n, the *permutohedron on n letters*, denoted by $\mathsf{P}(n)$ throughout the paper, is the set $\mathfrak{S}_n$ of all permutations of the finite set $[n] = \{1, \dots, n\}$ endowed with the *weak Bruhat ordering* (Guilbaud and Rosenstiehl [27], Björner [3]): comparing two permutations amounts to comparing their inversion sets. This ordering turns out to be a *lattice* (see Section 3 for more details and generalizations), meaning that any two permutations $\boldsymbol{x}, \boldsymbol{y} \in \mathfrak{S}_n$ have a least upper bound $\boldsymbol{x} \vee \boldsymbol{y}$ and a greatest lower bound $\boldsymbol{x} \wedge \boldsymbol{y}$.

Lattice terms are formed by starting with a set of "variables" and closing under the binary operations $\vee$ and $\wedge$. A *lattice identity* is a formula of the form $p = q$ for lattice terms p and q. The *equational theory* of a class $\mathcal{K}$ of lattices is the set of all lattice identities that hold in every member of $\mathcal{K}$. A *lattice variety* is the class of all lattices satisfying a given set of identities (Grätzer [21], Jipsen and Rose [37, 38]). In our paper [64] we stated the following problem.

Problem. Is the equational theory of all permutohedra decidable? Is there a nontrivial lattice identity holding in all permutohedra?

By "nontrivial" we mean not satisfied in all lattices (or, equivalently, in all free lattices). It has been known since Skolem [71] (reprinted in [72], see also Whitman [76], Freese and Nation [16, p. 30], Freese, Ježek, and Nation [14, Ch. I]) that the equational theory of all lattices, or equivalently the word problem in free lattices, is decidable. In our paper [64] we could settle the analogue, for *Tamari lattices* (known since Björner and Wachs [4] to be *lattice retracts* of permutohedra), of the second part of the problem above, by constructing an infinite sequence of lattice identities, the *Gazpacho identities*, holding in all Tamari lattices. Furthermore, we proved there that the permutohedron $\mathsf{P}(4)$ fails at least one Gazpacho identity, thus proving that the equational theory of all Tamari lattices *properly* contains that of all permutohedra. Nevertheless we could, at that time, neither achieve decidability of the equational theory of all Tamari lattices (or permutohedra), nor find a nontrivial identity holding in all permutohedra (that last part proving the trickiest of all).

As a side remark, let us mention that the group-theoretical analogue of the problem above has a well known solution, established in Iwasawa [36], using a result from Magnus [49]: *every free group embeds into a product of finite symmetric groups. Consequently, a nontrivial group word cannot vanish identically on all finite symmetric groups*.

1.3. Cousins of Theorem I: word problems in lattice-based structures

Most of the known decidability results for the word problem (or equational theory) in lattice-based structures are formulated for varieties. As mentioned above, the word problem in lattices is decidable. The decidability of the word problem in distributive lattices goes back to Dedekind [10] and Skolem [70]. Freese [13] proved that the word problem in the free modular lattice on five generators is undecidable; Herrmann [31] improved this result to four generators. Precursors of those works can be found in Hutchinson [33], Lipshitz [48], Hutchinson and Czédli [34] (the latter dealing with submodule lattices of modules). For a more complete discussion, with many additional references, we refer the reader to Jipsen and Rose [38, §1.3].

Adding a unary operation symbol $'$ for orthocomplementation, we get *ortholattices*, for which the decidability of the uniform word problem was established by Goldblatt [20] (see also Bruns [5, (4.1)]). Adding a binary operation symbol for the Heyting implication, Gentzen [18] established the decidability of the word problem for the structures nowadays known as *Heyting algebras* and widely studied.

Decidability results and existence of nontrivial identities are related by McKinsey's classical argument [54]: for instance, if a variety is defined by finitely many identities and generated by its finite members, then its equational theory is decidable. Nonetheless, as we shall briefly demonstrate in our next section, nontrivial identities enjoy a life of their own.

1.4. Cousins of Theorem II: hidden identities in lattice-based structures

Let us present a small sample of situations where a class of (often lattice-based) algebraic structures satisfies new unexpected identities, leading to important subsequent developments in the study of those structures.

Starting with lattice structures, the best known example is probably given by the *Arguesian identity*, originating in Schützenberger [68]. A statement of that identity can be found in any textbook of lattice theory: see for example Grätzer [21, p. 368]. This identity is stronger than the modular identity, and it is a lattice-theoretical form of a statement of classical geometry, namely Desargues' Theorem. It gave rise to huge developments in lattice theory, establishing connections with other topics such as combinatorics, representation theory, and logic. In all the situations encountered, the satisfaction of an identity was shown to be equivalent to a combinatorial, or geometrical, statement. Lattices of submodules of modules, or, more generally, lattices of commuting equivalence relations, often called *linear lattices*, were proved by Jónsson [39] to satisfy the Arguesian identity. Freese and Jónsson [15] extended that result to arbitrary congruence-modular varieties.

Jónsson [40] proved a partial converse of his result of [39], namely that *every complemented Arguesian lattice is linear*. The case of noncomplemented lattices was settled with the construction of nonlinear Arguesian lattices by Haiman [29, 30]. Haiman [28] also proved that *the class of all linear lattices is not finitely axiomatizable*. For an overview of related results and problems, see Kung and Yan [47]. Freese [12] found an identity holding in all finite modular lattices but not in all modular lattices.

Moving again to ortholattices, it was realized long ago that the lattice Sub H of all closed subspaces of an infinite-dimensional Hilbert space H, although failing modularity, satisfies the *orthomodular identity* $x \vee y = x \vee ((x \vee y) \wedge x')$ (Kalmbach [42]). The question whether Sub H satisfies any further identity not following from orthomodularity was settled in 1975 by Alan Day with his *orthoarguesian identity* (Greechie [26] and Godowski and Greechie [19]). Since then many other identities have been found for Sub H: see, in particular, Megill and Pavičić [55] and their subsequent papers.

Straying off lattices and moving to rings, we enter the huge subject of *rings with polynomial identities*, of which a fundamental prototype is the Amitsur–Levitzki Theorem [1], stating an identity holding in all matrix rings of given order over any field.

If we decree (somewhat arbitrarily) that properties like modularity stand on the bright side of the moon, then the lattices dealt with in the present paper, mainly permutohedra, would rather fit on the dark side. (A collection of results concerning identities in nonmodular varieties appears in Jipsen and Rose [37, Ch. 4].) An important highlight in that direction was Caspard's result [7] that permutohedra are all *bounded homomorphic images of free lattices*, so they belong to the class $\mathbf{B}_{\text{fin}}$ of Section 2.4, whose modular (or orthomodular) members are all distributive. Caspard's result was later extended to all finite Coxeter lattices (i.e., finite Coxeter groups with the weak order) by Caspard, Le Conte de Poly-Barbut, and Morvan [8]; then to further lattices of regions arising from hyperplane arrangements by Reading [59]; and also to "extended permutohedra" arising from posets, graphs, semilattices, and various classes of closure spaces in our works [65, 66, 67].

To our knowledge, the present paper is the first extensive (and complete) scrutiny of hidden identities in a combinatorially defined class of lattices on the dark side.

For a fascinating, though a bit outdated, survey on equational logic, see Taylor [73]. An elementary exposition of hidden identities, presented to high school students in March 2014 (Coutances, France) and undergraduate students in December 2017 at Garware College (Pune, India), can be found on the second author's Web page at https://wehrungf.users.lmno.cnrs.fr/fichiers/GAR17_Fred.pdf.

1.5. Organization of the paper

Let us recall the statements of our main theorems one by one.

Theorem I. *The equational theories of all permutohedra* $\mathsf{P}(n)$ *and of all Tamari lattices* $\mathsf{A}(n)$ *are both decidable.*

A far more general version of Theorem I is stated in Theorem 7.8. This statement involves Reading's *Cambrian lattices of type A* (Reading [60]), which turn out to be the quotients of the permutohedra by their minimal meet-irreducible congruences (Santocanale and Wehrung [64, Corollary 6.10]) and thus they generate the same lattice variety as the permutohedra (Lemma 3.1). The statement of Theorem 7.8 is sufficiently general to imply Theorem I trivially.

The first key ingredient of the proof of Theorem 7.8 originates in Reading's [60, Theorem 3.5], implying that the dual of a Cambrian lattice is Cambrian, and stated for Cambrian lattices of type A by Santocanale and Wehrung [64, Corollary 6.11]. In Section 4 we describe that duality via an "orthogonality relation" $\perp_U$ between intervals of the original chain. In Section 5 (culminating in Lemma 5.5) we relate the evaluation of lattice polynomials in Cambrian lattices to new combinatorial objects that we call *half-scores*, which encode certain tilings of finite chains. By combining that result with the duality from Section 4, we are thus able to relate, in Lemma 6.3, the failure of a lattice identity in a Cambrian lattice to new combinatorial objects called *scores*. Finally, in Section 7, we translate the statements about scores to monadic second-order logic of one successor $\mathtt{MSO}$. By using a famous decidability theorem due to Büchi (Theorem 7.1), we are able to reach the desired conclusion, namely Theorem 7.8.

However, the algorithm given by Büchi's Theorem, although theoretically sound, is at least one exponential away from any even remote hope for implementation, even for such uncomplicated lattices as the $\mathsf{B}(m,n)$ (Section 2.5). In particular, this algorithm is of no help for deciding even simple lattice identities. We show, in Appendix A, a combinatorial statement, involving objects called (m,n)*-scores*, describing the membership problem of the lattice $\mathsf{B}(m,n)$ in the variety generated by a Cambrian lattice $\mathsf{A}_U(E)$ (where E is a finite chain and $U \subseteq E$). This description involves certain tiling properties of the chain E.

Theorem II. *There exists a lattice identity that holds in all* $\mathsf{P}(n)$*, but fails in a certain* 3,338*-element lattice.*

Somewhat paradoxically, it turns out that proving Theorem II requires far more ingenuity than for Theorem I. The 3,338-element lattice L involved in Theorem II is constructed via a variant of Fraser's semilattice tensor product from [11] called *complete tensor product* by Wille [77], and *box product* by Grätzer and Wehrung [23]. (The two concepts, although not equivalent in general, are equivalent for finite lattices.) The lattice L, represented in Figure B.1, is given as the box product of the lattices N_5 (Figure 2.1) and $\mathsf{B}(3,2)$ (Section 2.5). Box products, and, more generally, sub-tensor products of lattices, are presented in Section 8.

The identity in question in Theorem II is the so-called *splitting identity* θ_L of L, which turns out to be the weakest identity failing for L (Section 2.4). The identity θ_L can be constructed explicitly (McKenzie [53, §6], Freese, Ježek, and Nation [14, Corollary 2.76]). In the present case, this task would probably take up of a whole book. Fortunately, we do not need to undergo such an ordeal, and we resort instead to an "identity-free" description of lattice varieties in Section 9. The main objects of study in that section are called

EA-duets; they consist of a join-homomorphism and a meet-homomorphism subject to a few simple conditions. The proof of the expanded version of Theorem II, namely Theorem 10.1, relies mostly on the description of the box product $L = \mathsf{N}_5 \mathbin{\square} \mathsf{B}(3,2)$ as a *sub-tensor product* (Definition 8.2). The only specificity of the box product, compared to other sub-tensor products, that we use in the proof of Theorem 10.1, is that it enables us to state that L is a *splitting lattice* (Section 2.4). It is plausible that the method used in Section 10 could be extended to arbitrary sub-tensor products of N_5 and $\mathsf{B}(3,2)$, but we would then lose the simplification brought by EA-duets, which would bring considerable unwieldiness to the argument.

Then the question of the extension of Theorem II to more general "permutohedra" arises naturally. There are many such constructions. We shall focus on the one from our paper [65], which yields the "extended permutohedron" $\mathsf{R}(E)$ on a poset E (Section 11), which turns out to be the Dedekind–MacNeille completion of a "generalized permutohedron" introduced by Pouzet et al. [58].

Theorem III. *The equational theory of all extended permutohedra, on arbitrary* (*possibly infinite*) *posets, is trivial.*

In fact, we prove in Theorem 11.6 a much stronger result: *every finite meet-semidistributive lattice embeds into* $\mathsf{R}(E)$ *for some countable poset* E. *Furthermore, the poset* E *can be taken to be a directed union of finite dismantlable lattices.* Theorem III is then a simple consequence of that result (Corollary 11.8).

2. Notation and terminology

We shall mainly follow the notation and terminology from standard references on lattice theory such as Grätzer [21], Freese, Ježek, and Nation [14], and Jipsen and Rose [37].

2.1. Basic concepts

We shall denote by $[n]$ the set $\{1,\dots,n\}$, endowed with its standard ordering. The *dual poset* P^{op} of a poset P has the same universe as P and opposite ordering (i.e., $x \leq^{\mathrm{op}} y$ if $y \leq x$). We say that P is *bounded* if it has both a least and a largest element, denoted by 0_P and 1_P, respectively, or 0 and 1 if P is understood. For $a \leq b$ in P and $X \subseteq P$, we set

$$\begin{aligned} P\downarrow X &= \{p \in P \mid p \leq x \text{ for some } x \in X\} \quad\text{and}\quad P\downarrow a = P\downarrow\{a\},\\ P\uparrow X &= \{p \in P \mid p \geq x \text{ for some } x \in X\} \quad\text{and}\quad P\uparrow a = P\uparrow\{a\},\\ [a,b] &= \{p \in P \mid a \leq p \leq b\}, \qquad]a,b] = \{p \in P \mid a < p \leq b\}, \qquad \text{etc.} \end{aligned}$$

An element a is a *lower cover* of an element b if $a < b$ and $]a,b[= \varnothing$. A map $f\colon P \to Q$ between posets is *isotone* (resp., *antitone*) if $x \leq y$ implies $f(x) \leq f(y)$ (resp., $f(y) \leq f(x)$), for all $x, y \in P$.

We denote by $\operatorname{Con} L$ the lattice of all congruences of a lattice L, and by $\operatorname{Con_c} L$ the $(\vee, 0)$-semilattice of all compact (i.e., finitely generated) congruences of L. Whenever

$a, b \in L$, we denote by $\mathrm{con}(a, b)$, or $\mathrm{con}_L(a, b)$ if L needs to be specified, the least congruence $\boldsymbol{\theta}$ of L such that $(a, b) \in \boldsymbol{\theta}$.

A lattice L is *subdirectly irreducible* if it has a least nonzero congruence, which is then called the *monolith* of L.

An element p in a lattice L is

- *completely join-irreducible* if $p = \bigvee X$ implies that $p \in X$, for all $X \subseteq L$;
- *join-irreducible* if $p = \bigvee X$ implies that $p \in X$, for all finite $X \subseteq L$;
- *completely join-prime* if $p \leq \bigvee X$ implies that $p \in L \downarrow X$, for all $X \subseteq L$;
- *join-prime* if $p \leq \bigvee X$ implies that $p \in L \downarrow X$, for all finite $X \subseteq L$.

If p is completely join-irreducible, then it has a unique lower cover, which will be denoted by p_*. In finite lattices, join-irreducibility and join-primeness are equivalent to their complete versions. Meet-irreducibility and meet-primeness are the duals of join-irreducibility and join-primeness, respectively. We denote by $\mathrm{Ji}\, L$ (resp., $\mathrm{Mi}\, L$) the set of all join-irreducible (resp., meet-irreducible) elements of L.

We shall often write lattice identities as *lattice inclusions* $p \leq q$ (which is indeed equivalent to the identity $p \vee q = q$) for lattice terms p and q. We denote by $\mathbf{Var}(\mathcal{K})$ the variety generated by a class $\mathcal{K}$ of lattices, and we write $\mathbf{Var}(K)$ instead of $\mathbf{Var}(\{K\})$.

2.2. *Semidistributivity*

A lattice L is *meet-semidistributive* if the implication

$$x \wedge z = y \wedge z \;\Rightarrow\; x \wedge z = (x \vee y) \wedge z$$

holds for all $x, y, z \in L$. *Join-semidistributivity* is defined dually. A lattice is *semidistributive* if it is both join-semidistributive and meet-semidistributive.

For a completely join-irreducible element p in a lattice L, we denote by $\kappa(p)$, or $\kappa_L(p)$ if L needs to be specified, the largest $u \in L$, if it exists, such that $p_* \leq u$ and $p \nleq u$. We shall occasionally use the following easy fact (Freese, Ježek, and Nation [14, Lemma 2.57]): for all p, x in a lattice L such that p is completely join-irreducible and $\kappa_L(p)$ exists,

$$x \leq \kappa_L(p) \text{ iff } p \nleq p_* \vee x. \tag{2.1}$$

If p is *completely join-prime*, then $\kappa(p)$ is defined, and it is also the largest $u \in L$ such that $p \nleq u$.

A finite lattice L is meet-semidistributive iff $\kappa(p)$ exists for every $p \in \mathrm{Ji}\, L$ (Freese, Ježek, and Nation [14, Theorem 2.56]). If, in addition, L is semidistributive, then the assignment $p \mapsto \kappa_L(p)$ defines a bijection from $\mathrm{Ji}\, L$ onto $\mathrm{Mi}\, L$ (Freese, Ježek, and Nation [14, Corollary 2.55]).

2.3. *Join-dependency and congruences*

For more details about the material of this section, see Freese, Ježek, and Nation [14]. The *join-dependency relation*, among join-irreducible elements in a finite lattice L, denoted by D (or D_L if L needs to be specified), is defined, on pairs (p,q) of join-irreducible elements, by

$$p\ D\ q \quad \text{if} \quad \bigl(p \neq q \text{ and } (\exists x)(p \leq q \vee x \text{ and } p \nleq q_* \vee x)\bigr).$$

Denote by $\trianglelefteq_L$ the reflexive, transitive closure of the join-dependency relation D_L and set $\operatorname{con}(p) = \operatorname{con}_L(p) = \operatorname{con}_L(p_*, p)$ whenever $p \in \operatorname{Ji} L$. The following is contained in Freese, Ježek, and Nation [14, Lemma 2.36]:

$$p \trianglelefteq_L q \text{ iff } \operatorname{con}_L(p) \subseteq \operatorname{con}_L(q), \qquad \text{for all } p, q \in \operatorname{Ji} L. \tag{2.2}$$

2.4. Bounded homomorphic images of free lattices

For more details about the material of this section, see Freese, Ježek, and Nation [14]. A surjective homomorphism $h\colon K \twoheadrightarrow L$ between lattices is *lower bounded* (resp., *bounded*) if $h^{-1}\{y\}$ has a least element (resp., both a least and a largest element) whenever $y \in L$. Denote by $\mathbf{LB}_{\text{fin}}$ the class of all finite lower bounded homomorphic images of free lattices, and by $\mathbf{B}_{\text{fin}}$ the class of all finite bounded homomorphic images of free lattices.[1] A lattice L belongs to $\mathbf{B}_{\text{fin}}$ iff L and L^{op} both belong to $\mathbf{LB}_{\text{fin}}$. It follows from [14, Corollary 2.39] that a finite lattice L belongs to $\mathbf{LB}_{\text{fin}}$ iff its join-dependency relation D_L has no cycle. Every member of $\mathbf{B}_{\text{fin}}$ is semidistributive. Of the lattices M_3 and N_5 represented in Figure 2.1, the former does not belong to $\mathbf{LB}_{\text{fin}}$, while the latter belongs to $\mathbf{B}_{\text{fin}}$. The labeling of N_5 introduced in Figure 2.1 will be used in Section 10.

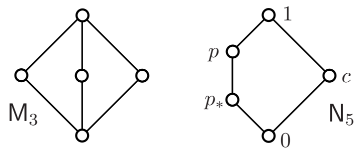

Fig. 2.1. The lattices M_3 and N_5.

A lattice K is *splitting* if there is a largest lattice variety $\mathcal{C}_K$ such that $K \notin \mathcal{C}_K$. Necessarily, $\mathcal{C}_K = \{L \mid K \notin \mathbf{Var}(L)\}$ and $\mathcal{C}_K$ is defined by a single identity θ_K, called the *splitting identity* of K (depending not only on K, but on a given generating subset of K). Since a lattice L fails θ_K iff $K \in \mathbf{Var}(L)$, it follows from Jónsson's Lemma that K has the smallest size among all lattices not satisfying θ_K. The splitting lattices are exactly the finite subdirectly irreducible members of $\mathbf{B}_{\text{fin}}$ (McKenzie [53, §5] or Freese, Ježek, and Nation [14, §II.6]). The lattice N_5 is splitting, with monolith $\operatorname{con}(p)$. An algorithm to compute the splitting identity of a finite splitting lattice is given in [14, §II.6].

[1] To the great puzzlement of many people, bounded homomorphic images of free lattices are often called *bounded lattices*. In the present paper, we revert to the original usage, by just defining bounded lattices as those with both a least and a largest element.

2.5. The lattices $\mathsf{B}(m,n)$

Following the notation introduced in Santocanale and Wehrung [64], for all positive integers m and n, we denote by $\mathsf{B}(m,n)$ the lattice obtained from the Boolean lattice with $m+n$ atoms $a_1,\dots,a_m,b_1,\dots,b_n$ by adding a new element q above $a=\bigvee_{i=1}^{m} a_i$ such that $q<a\vee b_j$ whenever $1\le j\le n$. In particular, q is join-irreducible with lower cover $q_*=a$. The lattice $\mathsf{B}(m,n)$ is splitting, with monolith $\operatorname{con}(q)$. We set $\mathbf{a}=\{a_1,\dots,a_m\}$ and $\mathbf{b}=\{b_1,\dots,b_n\}$. Observe that $\mathsf{B}(1,1)=\mathsf{N}_5$.

The join-prime elements in the lattices N_5 and $\mathsf{B}(3,2)$ are exactly the atoms, that is, p_*,c for N_5 and a_1,a_2,a_3,b_1,b_2 for $\mathsf{B}(3,2)$. The join-irreducible elements in those lattices, represented in Figure 2.2, are the atoms together with p (for N_5) and q (for $\mathsf{B}(3,2)$). This labeling will be further put to use in Section 10.

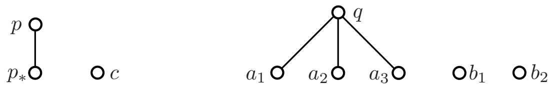

Fig. 2.2. The join-irreducible elements of N_5 (left) and $\mathsf{B}(3,2)$ (right).

We will later need the following easily verified equations, valid in the lattice $\mathsf{B}(3,2)$ whenever $\{i,j\}=\{1,2\}$ and $k,l\in\{1,2,3\}$:

$$b_j=(q_*\vee b_j)\wedge(b_1\vee b_2), \tag{2.3}$$

$$a_k=(a_k\vee b_i)\wedge(q_*\vee b_j), \tag{2.4}$$

$$a_k\vee a_l=(a_k\vee a_l\vee b_i)\wedge(q_*\vee b_j). \tag{2.5}$$

3. Permutohedra and Cambrian lattices of type A

We set $\boldsymbol{\delta}_E=\{(p,q)\in E\times E\mid p<q\}$ for any poset E. That is, $\boldsymbol{\delta}_E$ is the *strict ordering* associated to E. As in our papers [64, 65], we denote by $\operatorname{cl}(\boldsymbol{a})$ the transitive closure of any subset $\boldsymbol{a}$ of $\boldsymbol{\delta}_E$, and we set $\operatorname{int}(\boldsymbol{a})=\boldsymbol{\delta}_E\setminus\operatorname{cl}(\boldsymbol{\delta}_E\setminus\boldsymbol{a})$. Define

$\mathsf{P}(E)=\{\boldsymbol{a}\subseteq\boldsymbol{\delta}_E\mid \boldsymbol{a}=\operatorname{cl}(\boldsymbol{a})=\operatorname{int}(\boldsymbol{a})\}$, the *permutohedron on* E,

$\mathsf{R}(E)=\{\boldsymbol{a}\subseteq\boldsymbol{\delta}_E\mid \boldsymbol{a}=\operatorname{cl}\operatorname{int}(\boldsymbol{a})\}$, the *extended permutohedron on* E,

both endowed with set containment. Although $\mathsf{P}(E)$ may not be a lattice for an arbitrary poset E, it is always a lattice if E is a so-called *square-free poset* (Pouzet et al. [58], Santocanale and Wehrung [65]). By definition, E is square-free if it does not contain any copy of the four-element Boolean poset. For example, every chain is square-free.

On the other hand, $\mathsf{R}(E)$ is always a lattice, which turns out to be the Dedekind–MacNeille completion of $\mathsf{P}(E)$. The join, in $\mathsf{R}(E)$, of a family $(\boldsymbol{a}_i\mid i\in I)$ is always the transitive closure of the union of the $\boldsymbol{a}_i$ [65].

For a positive integer n, the lattice $\mathsf{R}([n])=\mathsf{P}([n])$, simply denoted by $\mathsf{P}(n)$, was first considered by Guilbaud and Rosenstiehl [27]; it turns out to be isomorphic to the symmetric group on n letters endowed with its *weak Bruhat ordering* (see for example

Bennett and Birkhoff [2, §5]). We refer to Björner [3] for the definition of the weak Bruhat ordering in Coxeter groups of any type.

For an arbitrary poset E, we prove in [65] that the completely join-irreducible elements of $\mathsf{R}(E)$ all belong to $\mathsf{P}(E)$, and they are exactly the sets of the form

$$\langle a,b\rangle_U = \bigl\{(x,y) \in (\{a\} \cup U^{\mathsf{c}}) \times (\{b\} \cup U) \bigm| a \leq x < y \leq b\bigr\}, \tag{3.1}$$

where $(a,b) \in \boldsymbol{\delta}_E$, $U \subseteq E$, and $U^{\mathsf{c}} = E \setminus U$. For notational convenience, we shall also set $\langle a,a\rangle_U = \emptyset$. Notice that $\langle a,b\rangle_U = \langle a,b\rangle_V$ iff $U \cap]a,b[= V \cap]a,b[$. Any subset U of E defines the set $\mathsf{D}_U(E)$ of all $\boldsymbol{a} \subseteq \boldsymbol{\delta}_E$ such that both conditions

$$\bigl(x < y < z \text{ and } (x,z) \in \boldsymbol{a} \text{ and } y \in U\bigr) \Rightarrow (x,y) \in \boldsymbol{a},$$
$$\bigl(x < y < z \text{ and } (x,z) \in \boldsymbol{a} \text{ and } y \notin U\bigr) \Rightarrow (y,z) \in \boldsymbol{a}$$

are satisfied for all $x,y,z \in E$. The set $\mathsf{A}_U(E)$ of all transitive members of $\mathsf{D}_U(E)$ is contained in $\mathsf{P}(E)$. We shall also write $\mathsf{A}(E) = \mathsf{A}_E(E)$. We prove in [65] that $\mathsf{A}_U(E)$ is a sublattice of $\mathsf{P}(E) = \mathsf{R}(E)$ whenever E is square-free (this turns out to characterise the square-freeness of E). Furthermore, the meet in $\mathsf{A}_U(E)$ is always the set-theoretical intersection. Whenever $(a,b) \in \boldsymbol{\delta}_E$, the set $\langle a,b\rangle_U$ defined in (3.1) is the least element $\boldsymbol{x}$ of $\mathsf{A}_U(E)$, with respect to containment, such that $(a,b) \in \boldsymbol{x}$. It is completely join-irreducible in $\mathsf{R}(E)$, with lower cover

$$(\langle a,b\rangle_U)_* = \langle a,b\rangle_U \setminus \{(a,b)\}, \tag{3.2}$$

and both $\langle a,b\rangle_U$ and $(\langle a,b\rangle_U)_*$ also belong to $\mathsf{A}_U(E)$. In case n is a positive integer and $E = [n]$, we shall write $\mathsf{A}_U(n)$ instead of $\mathsf{A}_U([n])$.

As discussed in Santocanale and Wehrung [64, §6], the lattices $\mathsf{A}_U(n)$ are exactly the *Cambrian lattices of type A*, with index n, introduced by Reading [60]. As established in [64, Proposition 6.7 and Corollary 6.10], the $\mathsf{A}_U(n)$ are exactly the quotients of $\mathsf{P}(n)$ by its minimal meet-irreducible congruences, and $\mathsf{P}(n)$ is a subdirect product of all the $\mathsf{A}_U(n)$ for $U \subseteq [n]$. In particular, we record the following lemma.

Lemma 3.1. *The class of all permutohedra $\mathsf{P}(n)$, for n a positive integer, and the class of all Cambrian lattices of type A, generate the same lattice variety.*

4. Dualities between Cambrian lattices of type A

Throughout this section we fix a finite chain E and a subset U of E. As usual, we set $U^{\mathsf{c}} = E \setminus U$. We proved in [64, Corollary 6.11] that the lattices $\mathsf{A}_U(E)$ and $\mathsf{A}_{U^{\mathsf{c}}}(E)$ are dually isomorphic. In the present section we shall give a more precise version of that result.

For each join-irreducible $\boldsymbol{p} \in \mathsf{A}_U(E)$, we set $\kappa_U(\boldsymbol{p}) = \kappa_{\mathsf{A}_U(E)}(\boldsymbol{p})$, the largest $\boldsymbol{u} \in \mathsf{A}_U(E)$, necessarily meet-irreducible, such that $\boldsymbol{p}_* \subseteq \boldsymbol{u}$ and $\boldsymbol{p} \nsubseteq \boldsymbol{u}$.

For $(a,b), (c,d) \in \boldsymbol{\delta}_E$, let $(a,b) \sim_U (c,d)$ hold if $\langle a,b\rangle_U \cap \langle c,d\rangle_{U^{\mathsf{c}}} \neq \emptyset$, and let $(a,b) \perp_U (c,d)$ hold if $(a,b) \sim_U (c,d)$ does not hold, that is, $\langle a,b\rangle_U \cap \langle c,d\rangle_{U^{\mathsf{c}}} = \emptyset$.

Say that a closed interval $[u,v]$ is *nontrivial* if $u < v$.

Lemma 4.1. *$(a,b) \sim_U (c,d)$ iff $[a,b] \cap [c,d]$ is a nontrivial interval $[u,v]$ and $(u,v) \in \langle a,b\rangle_U \cap \langle c,d\rangle_{U^{\mathsf{c}}}$. Furthermore, if $(a,b) \sim_U (c,d)$, then $\langle a,b\rangle_U \cap \langle c,d\rangle_{U^{\mathsf{c}}}$ is exactly the singleton $\{(u,v)\}$.*

Proof. If $[a,b] \cap [c,d] = [u,v]$ with $(u,v) \in \langle a,b\rangle_U \cap \langle c,d\rangle_{U^{\mathsf{c}}}$, then, by the definition of $\sim_U$, we get $(a,b) \sim_U (c,d)$. Conversely, suppose that $(a,b) \sim_U (c,d)$ and let $(x,y) \in \langle a,b\rangle_U \cap \langle c,d\rangle_{U^{\mathsf{c}}}$. Setting $u = \max\{a,c\}$ and $v = \min\{b,d\}$, we find that

$$u \leq x < y \leq v,$$

while

$$\begin{aligned} x &\in (\{a\} \cup U^{\mathsf{c}}) \cap (\{c\} \cup U) \\ &= (\{a\} \cap \{c\}) \cup (\{a\} \cap U) \cup (\{c\} \cap U^{\mathsf{c}}), \\ y &\in (\{b\} \cup U) \cap (\{d\} \cup U^{\mathsf{c}}) \\ &= (\{b\} \cap \{d\}) \cup (\{b\} \cap U^{\mathsf{c}}) \cup (\{d\} \cap U). \end{aligned}$$

There are nine cases to consider, for example $x = a = c$ and $y = b \in U^{\mathsf{c}}$ with $b < d$; in each of those cases, $(x,y) = (u,v)$. □

Definition 4.2. Let E be a finite chain, and let $x, y \in E$ be such that $x < y$. A *subdivision* of the interval $[x,y] \subseteq E$ is a subset P of $[x,y]$ containing the pair $\{x,y\}$. We shall often write such a subdivision in the form $x = z_0 < z_1 < \cdots < z_n = y$, where $P = \{z_0, z_1, \ldots, z_n\}$. Then we set

$$\operatorname{cvs}(P) = \{(z_i, z_{i+1}) \mid 0 \leq i < n\}.$$

Lemma 4.3. *For all $(x,y), (a,b) \in \boldsymbol{\delta}_E$, $(x,y) \in \kappa_U(\langle a,b\rangle_U)$ iff $(x,y) \perp_U (a,b)$.*

Proof. We prove the contrapositive statement. Suppose first that $(x,y) \notin \kappa_U(\langle a,b\rangle_U)$, that is, $\langle a,b\rangle_U \subseteq (\langle a,b\rangle_U)_* \vee \langle x,y\rangle_U$, in other words $(a,b) \in (\langle a,b\rangle_U)_* \vee \langle x,y\rangle_U$. There exists a subdivision $a = c_0 < c_1 < \cdots < c_n = b$ such that each (c_k, c_{k+1}) belongs to $(\langle a,b\rangle_U)_* \cup \langle x,y\rangle_U$. We may assume that n is least possible. Since $(a,b) \notin (\langle a,b\rangle_U)_*$, we deduce that $(c_k, c_{k+1}) \in \langle x,y\rangle_U$ for some $k \in [0, n-1]$. By the minimality of n, either $c_k = a$, or $k > 0$ and $(c_{k-1}, c_k) \in (\langle a,b\rangle_U)_*$. In the latter case, $c_k \in U$. In any case, $c_k \in U \cup \{a\}$. Symmetrically, $c_{k+1} \in U^{\mathsf{c}} \cup \{b\}$, whence $(c_k, c_{k+1}) \in \langle a,b\rangle_{U^{\mathsf{c}}}$. Therefore, (c_k, c_{k+1}) belongs to $\langle x,y\rangle_U \cap \langle a,b\rangle_{U^{\mathsf{c}}}$, so $(x,y) \sim_U (a,b)$.

Suppose, conversely, that $(x,y) \sim_U (a,b)$ and let $(u,v) \in \langle x,y\rangle_U \cap \langle a,b\rangle_{U^{\mathsf{c}}}$. Since $(u,v) \in \langle a,b\rangle_{U^{\mathsf{c}}}$, both (a,u) and (v,b) belong to the union of $(\langle a,b\rangle_U)_*$ with the diagonal. Since $(u,v) \in \langle x,y\rangle_U$, it follows that $\langle a,b\rangle_U \subseteq (\langle a,b\rangle_U)_* \vee \langle x,y\rangle_U$, thus $(x,y) \notin \kappa_U(\langle a,b\rangle_U)$. □

Set $\varphi(\boldsymbol{x}) = \{(i,j) \in \boldsymbol{\delta}_E \mid \boldsymbol{x} \cap \langle i,j\rangle_{U^{\mathsf{c}}} = \emptyset\}$ for every $\boldsymbol{x} \in \mathsf{A}_U(E)$. Notice that $\varphi(\boldsymbol{x}) = \{(i,j) \in \boldsymbol{\delta}_E \mid (u,v) \perp_U (i,j) \text{ whenever } (u,v) \in \boldsymbol{x}\}$. It is trivial that $\varphi(\boldsymbol{x})$ belongs to $\mathsf{D}_{U^{\mathsf{c}}}(E)$. Furthermore, $\boldsymbol{x}^{\mathsf{c}}$ is transitive, and $(i,j) \in \varphi(\boldsymbol{x})$ iff $\langle i,j\rangle_{U^{\mathsf{c}}} \subseteq \boldsymbol{x}^{\mathsf{c}}$, hence, if (i,j) and (j,k) both belong to $\varphi(\boldsymbol{x})$, then

$$\langle i,k\rangle_{U^{\mathsf{c}}} \subseteq \langle i,j\rangle_{U^{\mathsf{c}}} \vee \langle j,k\rangle_{U^{\mathsf{c}}} \subseteq \boldsymbol{x}^{\mathsf{c}},$$

that is, $(i,k) \in \varphi(\boldsymbol{x})$, and so $\varphi(\boldsymbol{x})$ is transitive. Therefore, $\varphi(\boldsymbol{x}) \in \mathsf{A}_{U^{\mathsf{c}}}(E)$, and $\varphi(\boldsymbol{x})$ is the largest $\boldsymbol{y} \in \mathsf{A}_{U^{\mathsf{c}}}(E)$ such that $\boldsymbol{x} \cap \boldsymbol{y} = \emptyset$.

Symmetrically, for every $\boldsymbol{y} \in \mathsf{A}_{U^c}(E)$, $\psi(\boldsymbol{y}) = \{(i, j) \in \boldsymbol{\delta}_E \mid \langle i, j\rangle_U \cap \boldsymbol{y} = \emptyset\}$ is the largest $\boldsymbol{x} \in \mathsf{A}_U(E)$ such that $\boldsymbol{x} \cap \boldsymbol{y} = \emptyset$.

Proposition 4.4. *The maps φ and ψ are mutually inverse dual isomorphisms between $\mathsf{A}_U(E)$ and $\mathsf{A}_{U^c}(E)$.*

Proof. The maps φ and ψ are both antitone, thus, by symmetry, it suffices to prove that $\psi \circ \varphi = \mathrm{id}_{\mathsf{A}_U(E)}$. It is obvious that $(\psi \circ \varphi)(\boldsymbol{c})$ contains $\boldsymbol{c}$ whenever $\boldsymbol{c} \in \mathsf{A}_U(E)$, so it suffices to prove that $(\psi \circ \varphi)(\boldsymbol{c})$ is contained in $\boldsymbol{c}$. Furthermore, it suffices to establish this fact in case $\boldsymbol{c}$ is meet-irreducible, that is, $\boldsymbol{c} = \kappa_U(\langle a, b\rangle_U)$ for some $(a, b) \in \boldsymbol{\delta}_E$.

Let $(x, y) \in (\psi \circ \varphi)(\boldsymbol{c})$; it is easily argued that this condition is equivalent to

$$\big(\forall (i, j) \in \langle x, y\rangle_U\big)\big(\boldsymbol{c} \cap \langle i, j\rangle_{U^c} \neq \emptyset\big). \tag{4.1}$$

Suppose that $(x, y) \notin \boldsymbol{c} = \kappa_U(\langle a, b\rangle_U)$. By Lemma 4.3, $(x, y) \sim_U (a, b)$, that is, there exists $(i, j) \in \langle x, y\rangle_U \cap \langle a, b\rangle_{U^c}$. By (4.1), there exists $(u, v) \in \boldsymbol{c} \cap \langle i, j\rangle_{U^c}$. Since $(i, j) \in \langle a, b\rangle_{U^c}$, we get $(u, v) \in \langle a, b\rangle_{U^c}$. Thus, both (a, u) and (v, b) belong to the union of $(\langle a, b\rangle_U)_*$ with the diagonal, and since $(u, v) \in \boldsymbol{c}$, it follows that (a, b) belongs to $(\langle a, b\rangle_U)_* \vee \boldsymbol{c} = (\langle a, b\rangle_U)_* \vee \kappa_U(\langle a, b\rangle_U) = \kappa_U(\langle a, b\rangle_U)$, a contradiction. □

Notation 4.5. Denote[2] by $\psi_U : \mathsf{A}_{U^c}(E) \to \mathsf{A}_U(E)^{\mathrm{op}}$ the map denoted by ψ above.

It follows from the definition of φ that $\varphi = \psi_{U^c}$. Hence, by Proposition 4.4, ψ_U is a dual isomorphism from $\mathsf{A}_{U^c}(E)$ onto $\mathsf{A}_U(E)$, with inverse ψ_{U^c}. Whenever $\boldsymbol{y} \in \mathsf{A}_{U^c}(E)$, $\psi_U(\boldsymbol{y})$ is the largest $\boldsymbol{x} \in \mathsf{A}_U(E)$ such that $\boldsymbol{x} \cap \boldsymbol{y} = \emptyset$.

As an immediate consequence of Lemma 4.3, we obtain

$$\psi_U(\langle a, b\rangle_{U^c}) = \kappa_U(\langle a, b\rangle_U) \quad \text{for all } (a, b) \in \boldsymbol{\delta}_E. \tag{4.2}$$

5. Half-scores and alternating words

Throughout this section we shall fix a finite set $\Omega = \{\mathsf{z}_1, \dots, \mathsf{z}_\ell\}$ (the "variables") of cardinality a positive integer ℓ, and we shall denote by $\mathrm{T}_{\mathcal{L}}(\Omega)$ the set of lattice terms whose variables belong to Ω. Elements of $\mathrm{T}_{\mathcal{L}}(\Omega)$ are generated from variables in Ω by applying the binary symbols $\wedge$ and $\vee$. The *rank* of a term is defined in a standard way, so that the rank of each subterm of a given term is (strictly) smaller than the rank of the term.

Let $p \in \mathrm{T}_{\mathcal{L}}(\Omega)$; the set $\mathrm{Cov}(p)$ of *canonical join-covers* of p is inductively defined as follows:

- If $p = \mathsf{z}_i \in \Omega$, then we set $\mathrm{Cov}(\mathsf{z}_i) = \{\{\mathsf{z}_i\}\}$.
- If $p = p_0 \vee p_1$, then we set $\mathrm{Cov}(p) = \{\{p_0, p_1\}\}$.
- If $p = p_0 \wedge p_1$, then we set $\mathrm{Cov}(p) = \mathrm{Cov}(p_0) \cup \mathrm{Cov}(p_1)$.

[2] Strictly speaking, we should write something like $\psi_{E,U}$ instead of just ψ_U; however, E will always be clear from the context.

A number of induction proofs will be based on the simple observation that for every $p \in \mathrm{T}_{\mathcal{L}}(\Omega) \setminus \Omega$, $\mathrm{Cov}(p)$ is a nonempty finite set of nonempty finite subsets of $\mathrm{T}_{\mathcal{L}}(\Omega)$ whose elements all have smaller rank than p has. This observation will be used implicitly throughout the text.

In the following lemma we consider terms formed using joins and meets indexed by nonempty subsets. This is achieved as usual, by considering the identity as the unary meet and unary join and otherwise by coding indexed joins (resp. meets) by an arbitrary parenthesizing of the binary join (resp. meet) operator. The lemma is proved by a straightforward induction argument.

Lemma 5.1. *The identity* $p = \bigwedge_{C \in \mathrm{Cov}(p)} \bigvee C$ *holds in every lattice for all* $p \in \mathrm{T}_{\mathcal{L}}(\Omega)$. *In particular,* $p \leq \bigvee C$ *is a valid lattice inclusion whenever* $C \in \mathrm{Cov}(p)$.

Definition 5.2. An *alternating word* on a term p in $\mathrm{T}_{\mathcal{L}}(\Omega)$ is a finite sequence $\alpha = (C_0, p_1, C_1, \dots, p_n, C_n)$, where n is a nonnegative integer and the following conditions hold:

(i) C_0 is the one-element set $\{p\}$.
(ii) $p_j \notin \Omega$ and $C_j \in \mathrm{Cov}(p_j)$ whenever $1 \leq j \leq n$.
(iii) $p_{j+1} \in C_j$ whenever $0 \leq j < n$.

We set $C_\alpha = C_n$. Let $\mathrm{Alt}(p)$ be the set of all alternating words on p. For $\alpha, \beta \in \mathrm{Alt}(p)$, let $\alpha \sqsubset \beta$ hold if α is a proper prefix of β.

Observe that the definition above implies that if $n > 0$, then $p_1 = p$. Furthermore, $\mathrm{Alt}(p)$ is finite. An example of an alternating word on the term $p = ((\mathsf{z}_1 \wedge \mathsf{z}_2) \vee \mathsf{z}_3) \wedge \mathsf{z}_4$, with $n = 2$, is given by

$$\alpha = (\{p\}, p, \{\mathsf{z}_1 \wedge \mathsf{z}_2, \mathsf{z}_3\}, \mathsf{z}_1 \wedge \mathsf{z}_2, \{\mathsf{z}_2\}).$$

We shall denote by $\alpha \frown \beta$ the concatenation of words α and β.

Definition 5.3. Let E be a finite chain (with at least two elements) and let $p \in \mathrm{T}_{\mathcal{L}}(\Omega)$. Denote by $\perp$ any object outside $\mathrm{T}_{\mathcal{L}}(\Omega)$ (thought of as the "undefined" symbol). A *half p-score on E* is a family $\vec{P} = ((P_\alpha, \tau_\alpha) \mid \alpha \in \mathrm{Alt}(p))$ satisfying the following conditions:

(i) $P_\alpha \subseteq E$ is a subdivision of the interval $[0_E, 1_E]$ (see Definition 4.2) and $\tau_\alpha \colon \mathrm{cvs}(P_\alpha) \to C_\alpha \cup \{\perp\}$ (the *valuation* of index α), for every $\alpha \in \mathrm{Alt}(p)$.
(ii) $P_{(\{p\})} = \{0_E, 1_E\}$ and $\tau_{(\{p\})}(0_E, 1_E) = p$.
(iii) For all $\alpha \in \mathrm{Alt}(p)$, all $(x, y) \in \mathrm{cvs}(P_\alpha)$, all $q = \tau_\alpha(x, y) \notin \Omega \cup \{\perp\}$, and all $C \in \mathrm{Cov}(q)$, the pair $\{x, y\}$ is contained in $P_{\alpha \frown (q,C)}$, and $\tau_{\alpha \frown (q,C)}(u, v) \in C$ whenever $(u, v) \in \mathrm{cvs}(P_{\alpha \frown (q,C)} \cap [x, y])$.

Example 5.4. Consider the term $p = \mathsf{z}_1 \vee \mathsf{z}_2$. Then $\mathrm{Alt}(p) = \{\alpha, \beta\}$, where $\alpha = (\{\mathsf{z}_1 \vee \mathsf{z}_2\})$ and $\beta = (\{\mathsf{z}_1 \vee \mathsf{z}_2\}, \mathsf{z}_1 \vee \mathsf{z}_2, \{\mathsf{z}_1, \mathsf{z}_2\})$; observe that $\alpha \sqsubset \beta$. The half p-scores on a nontrivial finite chain E are the pairs $\vec{P} = ((P_\alpha, \tau_\alpha), (P_\beta, \tau_\beta))$, where $P_\alpha = \{0_E, 1_E\} \subseteq P_\beta \subseteq E$, $\tau_\alpha \colon \{(0_E, 1_E)\} \to \{\mathsf{z}_1 \vee \mathsf{z}_2, \perp\}$, $\tau_\beta \colon \mathrm{cvs}(P_\beta) \to \{\mathsf{z}_1, \mathsf{z}_2, \perp\}$, and whenever $\tau_\alpha(0_E, 1_E) = \mathsf{z}_1 \vee \mathsf{z}_2$, $\tau_\beta(u, v) \in \{\mathsf{z}_1, \mathsf{z}_2\}$ for all $(u, v) \in \mathrm{cvs}(P_\beta)$.

Moving to a more graphical, though less formal, level, we can observe that the upper three rows and the lower three rows of Figure A.1 are both special cases of a suitable modification of the concept of half-score. For example, the upper three rows make up a half-score for the "term" $\bigwedge_{i=1,2,3}(a_i \vee b_1 \vee b_2 \vee b_3)$, with finitary (as opposed to just binary) joins and meets allowed.

For a half p-score $\vec{P}$ as above, we shall set

$$P_\alpha[q] = \{(x, y) \in \operatorname{cvs}(P_\alpha) \mid \tau_\alpha(x, y) = q\} \quad \text{whenever } \alpha \in \operatorname{Alt}(p) \text{ and } q \in C_\alpha. \tag{5.1}$$

The main lemma of this section, relating half p-scores and evaluations of lattice terms in Cambrian lattices $\mathsf{A}_U(E)$, is the following.

Lemma 5.5. *Let p be a lattice term on Ω, let E be a finite chain, let $U \subseteq E$, and let $\boldsymbol{a}_1, \dots, \boldsymbol{a}_\ell \in \mathsf{A}_U(E)$. The following are equivalent:*

(i) $(0_E, 1_E) \in p(\boldsymbol{a}_1, \dots, \boldsymbol{a}_\ell)$*, where $p(\boldsymbol{a}_1, \dots, \boldsymbol{a}_\ell)$ is evaluated within $\mathsf{A}_U(E)$.*
(ii) *There exists a half p-score $\vec{P}$ on E such that*

$$P_\alpha[\mathsf{z}_i] \subseteq \boldsymbol{a}_i \quad \textit{whenever } \alpha \in \operatorname{Alt}(p) \textit{ and } i \in [\ell].$$

From now on we shall use the abbreviation $\vec{\boldsymbol{a}} = (\boldsymbol{a}_1, \dots, \boldsymbol{a}_\ell)$.

Proof. (i)⇒(ii). We construct the finite subsets P_α of E and the valuations $\tau_\alpha \colon \operatorname{cvs}(P_\alpha) \to C_\alpha \cup \{\bot\}$, with $P_{(\{p\})} = \{0_E, 1_E\}$ and $\tau_{(\{p\})}(0_E, 1_E) = p$, subject to the following induction hypothesis (relative to the strict ordering $\sqsubset$ of $\operatorname{Alt}(p)$):

$$P_\alpha[q] \subseteq q(\vec{\boldsymbol{a}}) \quad \text{whenever } q \in C_\alpha. \tag{5.2}$$

The statement (5.2) holds at $\alpha = (\{p\})$ by assumption (i). Suppose that P_α and τ_α are constructed in such a way that (5.2) holds at α. The finite sequence $\beta = \alpha \frown (q, C)$ belongs to $\operatorname{Alt}(p)$ whenever $q \in C_\alpha \setminus \Omega$ and $C \in \operatorname{Cov}(q)$. Let $(x, y) \in P_\alpha[q]$. By our induction hypothesis, (x, y) belongs to $q(\vec{\boldsymbol{a}})$, thus (by Lemma 5.1) to $\bigvee_{r \in C} r(\vec{\boldsymbol{a}})$, and therefore there exists a subdivision $P_\beta^{x,y}$ of $[x, y]$ such that

$$\operatorname{cvs}(P_\beta^{x,y}) \subseteq \bigcup_{r \in C} r(\vec{\boldsymbol{a}}). \tag{5.3}$$

Set

$$P_\beta = \{0_E, 1_E\} \cup \bigcup (P_\beta^{x,y} \mid (x, y) \in P_\alpha[q]). \tag{5.4}$$

Observe that $P_\beta \cap [x, y] = P_\beta^{x,y}$ whenever $(x, y) \in \operatorname{cvs}(P_\alpha)$. Now let $(u, v) \in \operatorname{cvs}(P_\beta)$. If $(u, v) \in \operatorname{cvs}(P_\beta^{x,y})$ for some (necessarily unique) $(x, y) \in P_\alpha[q]$, it follows from (5.3) that there exists $r \in C$ such that $(u, v) \in r(\vec{\boldsymbol{a}})$; define $\tau_\beta(u, v)$ to be any such r. In all other cases, that is, when there is no $(x, y) \in P_\alpha[q]$ such that $(u, v) \in \operatorname{cvs}(P_\beta^{x,y})$, we set $\tau_\beta(u, v) = \bot$. By construction, the induction hypothesis (5.2) still holds at β. The family of all pairs (P_α, τ_α) is therefore a half p-score on E, and moreover it satisfies (5.2) whenever $\alpha \in \operatorname{Alt}(p)$. By applying (5.2) to the case where $q = \mathsf{z}_i$, we get the condition (ii).

(ii)⇒(i). We again prove the statement (5.2), this time by downward $\sqsubset$-induction on $\alpha \in \mathrm{Alt}(p)$. Let $\alpha \in \mathrm{Alt}(p)$ and suppose that (5.2) holds at every $\beta \in \mathrm{Alt}(p)$ with $\alpha \sqsubset \beta$. Let $q \in C_\alpha$ and let $(x, y) \in P_\alpha[q]$; we must prove that $(x, y) \in q(\vec{\boldsymbol{a}})$. If $q \in \Omega$, then this follows from assumption (ii). Suppose from now on that $q \notin \Omega$. Let $C \in \mathrm{Cov}(q)$. The finite sequence $\beta = \alpha \frown (q, C)$ belongs to $\mathrm{Alt}(p)$. Since $\vec{P}$ is a half p-score, $\{x, y\}$ is contained in P_β and $\tau_\beta(u, v) \in C$ whenever $(u, v) \in \mathrm{cvs}(P_\beta \cap [x, y])$. Set $r = \tau_\beta(u, v)$; it follows from our induction hypothesis that $(u, v) \in r(\vec{\boldsymbol{a}})$. This holds for all $(u, v) \in \mathrm{cvs}(P_\beta \cap [x, y])$, whence $(x, y) \in \bigvee_{r\in C} r(\vec{\boldsymbol{a}})$. As this is true for all $C \in \mathrm{Cov}(q)$, and the meet in $\mathsf{A}_U(E)$ is intersection, we get

$$(x, y) \in \bigwedge_{C\in\mathrm{Cov}(q)} \bigvee_{r\in C} r(\vec{\boldsymbol{a}}).$$

By Lemma 5.1, this means that $(x, y) \in q(\vec{\boldsymbol{a}})$, thus completing the proof of the induction step for (5.2). By applying (5.2) to $\alpha = (\{p\})$, we get the desired conclusion. □

6. Scores and lattice inclusions

In this section we fix a set $\Omega = \{\mathsf{z}_i \mid i \in [\ell]\}$ of cardinality a positive integer ℓ.

We leave to the reader the straightforward proof of the following lemma.

Lemma 6.1. *Let F be an interval in a chain E. Then $\mathsf{A}_{U\cap F}(F)$ is a lattice retract of $\mathsf{A}_U(E)$, with retraction defined by*

$$\pi\colon \mathsf{A}_U(E) \to \mathsf{A}_{U\cap F}(F), \qquad \boldsymbol{x} \mapsto \boldsymbol{x} \cap \boldsymbol{\delta}_F.$$

In the context of Lemma 6.1, we shall call π the *projection map* from $\mathsf{A}_U(E)$ onto $\mathsf{A}_{U\cap F}(F)$.

From now on we shall denote by q^{op} the dual of a term $q \in \mathrm{T}_{\mathcal{L}}(\Omega)$, that is, q with meets and joins interchanged.

Definition 6.2. Let $p, q \in \mathrm{T}_{\mathcal{L}}(\Omega)$, let E be a finite chain, and let $U \subseteq E$. A (p, q, U)-*score on E* is a pair $(\vec{P}, \vec{Q})$, where

$$\vec{P} = ((P_\alpha, \mu_\alpha) \mid \alpha \in \mathrm{Alt}(p)) \qquad \text{is a half } p\text{-score on } E,$$
$$\vec{Q} = ((Q_\beta, \nu_\beta) \mid \beta \in \mathrm{Alt}(q^{\mathrm{op}})) \qquad \text{is a half } q^{\mathrm{op}}\text{-score on } E,$$

and the following condition holds:

$$\text{Whenever } i \in [\ell],\ \alpha \in \mathrm{Alt}(p),\ \beta \in \mathrm{Alt}(q^{\mathrm{op}}),\ (x, y) \in P_\alpha[\mathsf{z}_i],\ (u, v) \in Q_\beta[\mathsf{z}_i],$$
$$\text{the condition } (x, y) \perp_U (u, v) \text{ holds.} \qquad (6.1)$$

We refer to Section 4 for the definition of the binary relation $\perp_U$ and the isomorphism $\psi = \psi_U\colon \mathsf{A}_{U^{\mathsf{c}}}(E) \to \mathsf{A}_U(E)^{\mathrm{op}}$. The notation $P_\alpha[q]$ is defined in (5.1).

The following crucial lemma states the equivalence between a negated lattice inclusion $p \not\leq q$ in $\mathsf{A}_U(E)$ and the existence of a (p, q, U)-score on E.

Recall that, in the statement and proof of the following lemma, $\psi_U\colon \mathsf{A}_{U^{\mathsf{c}}}(E) \to \mathsf{A}_U(E)^{\mathrm{op}}$ is the canonical isomorphism defined in Section 4.

Lemma 6.3. *Let $p, q \in \mathrm{T}_{\mathcal{L}}(\Omega)$, let E be a finite chain, and let U be a subset of E. The following are equivalent:*

(i) *There are $\boldsymbol{a}_1, \dots, \boldsymbol{a}_\ell \in \mathsf{A}_U(E)$ such that $p(\vec{\boldsymbol{a}}) \nsubseteq q(\vec{\boldsymbol{a}})$.*
(ii) *There are $\boldsymbol{a}_1, \dots, \boldsymbol{a}_\ell \in \mathsf{A}_U(E)$ such that $(0_E, 1_E) \in p(\vec{\boldsymbol{a}}) \cap \psi_U^{-1}(q(\vec{\boldsymbol{a}}))$.*
(iii) *There exists a (p, q, U)-score on E.*

Proof. (iii)⇒(ii). We set $\boldsymbol{a}_i = \bigvee(\langle x, y\rangle_U \mid \alpha \in \mathrm{Alt}(p) \text{ and } (x, y) \in P_\alpha[\mathsf{z}_i])$ for $i \in [\ell]$. It follows from Lemma 5.5[(ii)⇒(i)] that $(0_E, 1_E) \in p(\vec{\boldsymbol{a}})$.

We must prove that $(0_E, 1_E) \in q^{\mathrm{op}}(\psi_U^{-1}\vec{\boldsymbol{a}})$. By Lemma 5.5[(ii)⇒(i)], it suffices to prove that $Q_\beta[\mathsf{z}_i] \subseteq \psi_U^{-1}(\boldsymbol{a}_i)$ whenever $i \in [\ell]$ and $\beta \in \mathrm{Alt}(q^{\mathrm{op}})$. Let $(u, v) \in Q_\beta[\mathsf{z}_i]$. We must prove that $\boldsymbol{a}_i \subseteq \psi_U(\langle u, v\rangle_{U^{\mathsf{c}}})$, that is, $(x, y) \perp_U (u, v)$ whenever $\alpha \in \mathrm{Alt}(p)$ and $(x, y) \in P_\alpha[\mathsf{z}_i]$. However, this follows from the definition of a score.

(ii)⇒(i). Suppose that $p(\vec{\boldsymbol{a}}) \subseteq q(\vec{\boldsymbol{a}})$ and set $\boldsymbol{b} = q(\vec{\boldsymbol{a}})$. Then $\langle 0_E, 1_E\rangle_U \subseteq \boldsymbol{b}$ and $\langle 0_E, 1_E\rangle_{U^{\mathsf{c}}} \subseteq \psi_U^{-1}(\boldsymbol{b})$. The second containment can be written $\boldsymbol{b} \subseteq \psi_U(\langle 0_E, 1_E\rangle_{U^{\mathsf{c}}})$. It follows that $\langle 0_E, 1_E\rangle_U \subseteq \psi_U(\langle 0_E, 1_E\rangle_{U^{\mathsf{c}}})$, that is, $\langle 0_E, 1_E\rangle_U \subseteq \kappa_U(\langle 0_E, 1_E\rangle_U)$ (use (4.2)), a contradiction.

(i)⇒(iii). Pick a minimal interval F of E such that $(0_F, 1_F) \in p(\vec{\boldsymbol{a}}) \setminus q(\vec{\boldsymbol{a}})$, and using the projection homomorphism $\pi \colon \mathsf{A}_U(E) \twoheadrightarrow \mathsf{A}_{U\cap F}(F)$ (Lemma 6.1), set $\boldsymbol{a}'_i = \pi(\boldsymbol{a}_i)$ for each $i \in [\ell]$. Let $V = U \cap F$ and $V^{\mathsf{c}} = F \setminus V$. Observe that $(0_F, 1_F) \in p(\vec{\boldsymbol{a}}') \setminus q(\vec{\boldsymbol{a}}')$ (within $\mathsf{A}_V(F)$); thus F has at least two elements.

Suppose that $\langle 0_F, 1_F\rangle_{V^{\mathsf{c}}} \nsubseteq \psi_V^{-1}(q(\vec{\boldsymbol{a}}'))$. Then $q(\vec{\boldsymbol{a}}') \nsubseteq \psi_V(\langle 0_F, 1_F\rangle_{V^{\mathsf{c}}})$. By (4.2), it follows that $q(\vec{\boldsymbol{a}}') \nsubseteq \kappa_V(\langle 0_F, 1_F\rangle_V)$. Hence, by (2.1),

$$\langle 0_F, 1_F\rangle_V \subseteq (\langle 0_F, 1_F\rangle_V)_* \vee q(\vec{\boldsymbol{a}}'). \tag{6.2}$$

Since $\langle 0_F, 1_F\rangle_V \subseteq p(\vec{\boldsymbol{a}}')$, every $(x, y) \in (\langle 0_F, 1_F\rangle_V)_*$ belongs to $p(\vec{\boldsymbol{a}}')$; moreover, since $(x, y) \neq (0_F, 1_F)$ for every such (x, y), we obtain, by projecting onto $[x, y]$ as in the paragraph above and by the minimality assumption on $[0_F, 1_F]$, the relation $(x, y) \in q(\vec{\boldsymbol{a}}')$; hence we get $(\langle 0_F, 1_F\rangle_V)_* \subseteq q(\vec{\boldsymbol{a}}')$. Therefore, by (6.2), we obtain $\langle 0_F, 1_F\rangle_V \subseteq q(\vec{\boldsymbol{a}}')$, hence $(0_F, 1_F) \in q(\vec{\boldsymbol{a}}')$, a contradiction; this proves that $\langle 0_F, 1_F\rangle_{V^{\mathsf{c}}} \subseteq \psi_V^{-1}(q(\vec{\boldsymbol{a}}'))$.

Since $(0_F, 1_F) \in p(\vec{\boldsymbol{a}}')$, it follows from Lemma 5.5[(i)⇒(ii)] that there exists a half p-score

$$\vec{P} = ((P_\alpha, \mu_\alpha) \mid \alpha \in \mathrm{Alt}(p))$$

on F such that

$$P_\alpha[\mathsf{z}_i] \subseteq \boldsymbol{a}'_i \quad \text{whenever } \alpha \in \mathrm{Alt}(p) \text{ and } i \in [\ell]. \tag{6.3}$$

Similarly, since $(0_F, 1_F) \in \psi_V^{-1}(q(\vec{\boldsymbol{a}}')) = q^{\mathrm{op}}(\psi_V^{-1}\vec{\boldsymbol{a}}')$, there exists a half q^{op}-score

$$\vec{Q} = ((Q_\beta, \nu_\beta) \mid \beta \in \mathrm{Alt}(q^{\mathrm{op}}))$$

on F such that

$$Q_\beta[\mathsf{z}_i] \subseteq \psi_V^{-1}(\boldsymbol{a}'_i) \quad \text{whenever } \beta \in \mathrm{Alt}(q^{\mathrm{op}}) \text{ and } i \in [\ell]. \tag{6.4}$$

Let $i \in [\ell]$, $\alpha \in \mathrm{Alt}(p)$, $\beta \in \mathrm{Alt}(q^{\mathrm{op}})$, $(x, y) \in P_\alpha[\mathsf{z}_i]$, and $(u, v) \in Q_\beta[\mathsf{z}_i]$. By (6.3) and (6.4), it follows that $(x, y) \in \boldsymbol{a}'_i$ and $(u, v) \in \psi_V^{-1}(\boldsymbol{a}'_i)$, that is, $\langle x, y\rangle_V \subseteq \boldsymbol{a}_i$ and $\langle u, v\rangle_{V^{\mathrm{c}}} \subseteq \psi_V^{-1}(\boldsymbol{a}_i)$. By the definition of the map ψ_V (Section 4), it follows that $\langle x, y\rangle_V \cap \langle u, v\rangle_{V^{\mathrm{c}}} = \emptyset$, that is, $(x, y) \perp_V (0_F, 1_F)$. Therefore, $(\vec{P}, \vec{Q})$ is a (p, q, V)-score on F.

It remains to extend the (p, q, V)-score $(\vec{P}, \vec{Q})$ on F to a (p, q, U)-score on E. To this end let $\xi : F \to E$ be the map extending the identity on $F \setminus \{0_F, 1_F\}$ such that $\xi(0_F) = 0_E$ and $\xi(1_F) = 1_E$. Observe that ξ is an order-embedding. The proof of the following claim is a straightforward application of Lemma 4.1.

Claim. *Let $(x, y), (u, v) \in \boldsymbol{\delta}_F$. Then $(x, y) \perp_V (u, v)$ iff $(\xi(x), \xi(y)) \perp_U (\xi(u), \xi(v))$.*

Now we set $P'_\alpha = \xi(P_\alpha)$ and $\mu'_\alpha(\xi(x), \xi(y)) = \mu_\alpha(x, y)$, for all $\alpha \in \mathrm{Alt}(p)$ and all $(x, y) \in \mathrm{cvs}(P_\alpha)$. Likewise, we set $Q'_\beta = \xi(Q_\beta)$ and $\nu'_\beta(\xi(x), \xi(y)) = \nu_\beta(x, y)$, for all $\beta \in \mathrm{Alt}(q^{\mathrm{op}})$ and all $(x, y) \in \mathrm{cvs}(Q_\beta)$. It is straightforward to verify that the families

$$\vec{P}' = ((P'_\alpha, \mu'_\alpha) \mid \alpha \in \mathrm{Alt}(p)), \qquad \vec{Q}' = ((Q'_\beta, \nu'_\beta) \mid \beta \in \mathrm{Alt}(q^{\mathrm{op}}))$$

are a half p-score and a half q^{op}-score on E, respectively. Furthermore, by the Claim above, $(\vec{P}', \vec{Q}')$ satisfies (6.1), so it is a (p, q, U)-score on E. □

7. Expressing scores within monadic second-order logic: proving Theorem I

We consider the monadic second-order language `MSO` of one successor (Büchi [6]). We denote by $u, v, w, x, y, \dots$ the variables of the first-order language (s) consisting of one unary function symbol s. In addition to that language, `MSO` has a binary relation symbol $\in$, second-order variables $U, V, W, X, Y, \dots$, and new atomic formulas $t \in X$, where t is a term of the first-order language (s) and X is a second-order variable. The formulas of `MSO` are obtained by closing the atomic formulas under propositional connectives and quantification both on first- and second-order variables. The standard model of `MSO` is (ω, s), where s is the successor function on the set ω of all nonnegative integers. The satisfaction by (ω, s) of a formula of `MSO` is defined inductively on the complexity of the formula, in a standard fashion. The following fundamental result is due to Büchi [6].

Theorem 7.1 (Büchi's Theorem). *The theory* `S1S` *consisting of all statements of* `MSO` *valid in (ω, s) is decidable (i.e., recursive).*

By Büchi's Theorem, in order to decide the validity of a statement θ (in any mathematical field), it suffices to find a statement $\tilde{\theta}$ of `MSO` which is equivalent to θ (i.e., θ holds iff the structure (ω, s) satisfies $\tilde{\theta}$), and then apply Büchi's decision procedure to $\tilde{\theta}$. A standard fact, which we shall use repeatedly, is that the binary relations $x < y$ and $x \leq y$ on ω are both `MSO`-definable, respectively by the statements

$$(\exists X)\big((\forall z)(z \in X \Rightarrow \mathsf{s}(z) \in X) \;\wedge\; y \in X \;\wedge\; \neg(x \in X)\big), \tag{7.1}$$

$$x < y \;\vee\; x = y. \tag{7.2}$$

Let $\Omega = \{\mathsf{z}_i \mid i \in [\ell]\}$ be a set of cardinality a positive integer ℓ, and let $p \in \mathrm{T}_{\mathcal{L}}(\Omega)$. In order to be able to code half p-scores (Definition 5.3) in `MSO`, a necessary preliminary step is to describe such objects by finite collections of subsets of ω.

For the P_α nothing needs to be done (they are already sets of integers).

For a subset P of ω, the set $\mathrm{cvs}(P)$ of all covers in P (Definition 4.2) is in one-to-one correspondence with the set P^* defined to be P if P has no largest element, and to be $P \setminus \{\max P\}$ otherwise. Hence, for a finite set C, a map $\tau\colon \mathrm{cvs}(P) \to C$ can be described by the collection of all subsets $P_c = \{x \in P^* \mid (\exists y)(\tau(x, y) = c)\}$, where $c \in C$. Accordingly, we set the following definition.

Definition 7.2. The *code* of a half p-score $\vec{P}$ as above is the family

$$((P_\alpha, P_{\alpha,q}) \mid \alpha \in \mathrm{Alt}(p),\ q \in C_\alpha \cup \{\bot\}),$$

where we set

$$P_{\alpha,q} = \bigl\{x \in P_\alpha \bigm| (\exists y)\bigl((x, y) \in \mathrm{cvs}(P_\alpha) \text{ and } \tau_\alpha(x, y) = q\bigr)\bigr\}.$$

Since the code of a half p-score is a finite family of sets of integers (viz. the P_α and the $P_{\alpha,q}$), its entries can be used as parameters for MSO formulas.

Lemma 7.3. *The statement saying that a given family*

$$\vec{P} = ((P_\alpha, P_{\alpha,q}) \mid \alpha \in \mathrm{Alt}(p) \text{ and } q \in C_\alpha \cup \{\bot\})$$

is the code of a half p-score on an interval $[u, v]$ of ω is equivalent to an MSO *statement.*

Proof. Axiom (i) of Definition 5.3, with 0_E replaced by u and 1_E by v, can be expressed by the conjunction of $u < v$ and the following statements:

$$P_\alpha \subseteq [u, v] \qquad \text{for } \alpha \in \mathrm{Alt}(p), \tag{7.3}$$

$$P^*_\alpha = \bigcup_{q \in C_\alpha \cup \{\bot\}} P_{\alpha,q} \qquad \text{for } \alpha \in \mathrm{Alt}(p), \tag{7.4}$$

$$P_{\alpha,q} \cap P_{\alpha,r} = \varnothing \qquad \text{for } \alpha \in \mathrm{Alt}(p) \text{ and distinct } q, r \in C_\alpha. \tag{7.5}$$

The statement (7.3) is equivalent to the MSO formula

$$\bigwedge_{\alpha \in \mathrm{Alt}(p)} (\forall x)\bigl(x \in P_\alpha \Rightarrow (u \leq x \ \wedge\ x \leq v)\bigr).$$

Now the statement "$(x, y) \in \mathrm{cvs}(P_\alpha)$" is equivalent to the following MSO formula:

$$x \in P_\alpha \ \wedge\ y \in P_\alpha \ \wedge\ x < y \ \wedge\ (\forall z)\neg(x < z \ \wedge\ z < y \ \wedge z \in P_\alpha).$$

(The symbols $\wedge$ and $\neg$ stand for conjunction and negation, respectively. The quotes in what follows will mean that we are replacing the statement $(x, y) \in \mathrm{cvs}(P)$ by its MSO equivalent found previously, so we are reminded that the job is already done for that statement.)

This implies immediately that (7.4) is equivalent to the conjunction of the following two MSO statements:

$$\bigwedge_{\alpha \in \mathrm{Alt}(p),\ q \in C_\alpha \cup \{\bot\}} \bigl(x \in P_{\alpha,q} \Rightarrow (\exists y)\text{“}(x, y) \in \mathrm{cvs}(P_\alpha)\text{”}\bigr),$$

$$\bigwedge_{\alpha \in \mathrm{Alt}(p)} (\forall x)(\forall y)\Bigl(\text{“}(x, y) \in \mathrm{cvs}(P_\alpha)\text{”} \Rightarrow \bigvee_{q \in C_\alpha \cup \{\bot\}} x \in P_{\alpha,q}\Bigr)$$

(following the usual convention, $\bigwedge$ and $\bigvee$ stand for conjunction and disjunction over a given index set, respectively). The translation of (7.5) to an MSO statement is even more straightforward.

Axiom (ii) of Definition 5.3 can be expressed by the statement

$$u \in P_{(\{p\}),p} \ \wedge \ (\forall x)\bigl(x \in P_{(\{p\})} \Leftrightarrow (x = u \ \vee \ x = v)\bigr).$$

Finally, axiom (iii) of Definition 5.3 is equivalent to the conjunction, over all (α, q, C) with $\alpha \in \mathrm{Alt}(p)$, $q \in C_\alpha \setminus \Omega$, and $C \in \mathrm{Cov}(q)$, of the statements

$$(\forall x)(\forall y)\Bigl(\bigl(\text{“}(x, y) \in \mathrm{cvs}(P_\alpha)\text{”} \ \wedge \ x \in P_{\alpha,q}\bigr) \Rightarrow \bigl(x \in P_{\alpha^\frown(q,C)} \ \wedge \ y \in P_{\alpha^\frown(q,C)} \ \wedge \ \vartheta_{\alpha,q,C}(x, y)\bigr)\Bigr),$$

where $\vartheta_{\alpha,q,C}(x, y)$ is the statement

$$(\forall u)(\forall v)\Bigl(\bigl(\text{“}(u, v) \in \mathrm{cvs}(P_{\alpha^\frown(q,C)})\text{”} \wedge x \leq u \ \wedge \ v \leq y\bigr) \Rightarrow \bigvee_{r \in C}\bigl(u \in P_{\alpha^\frown(q,C),r}\bigr)\Bigr).$$

This concludes the proof. □

Now we formulate the following analogue of Definition 7.2 for scores.

Definition 7.4. Let $p, q \in \mathrm{T}_{\mathcal{L}}(\Omega)$. Consider families

$$\dot{P} = ((P_\alpha, P_{\alpha,r}) \mid \alpha \in \mathrm{Alt}(p) \text{ and } r \in C_\alpha \cup \{\bot\}), \tag{7.6}$$

$$\dot{Q} = ((Q_\beta, Q_{\beta,s}) \mid \beta \in \mathrm{Alt}(q^{\mathrm{op}}) \text{ and } s \in C_\beta \cup \{\bot\}). \tag{7.7}$$

The pair $(\dot{P}, \dot{Q})$ is the *code for a* (p, q, U)*-score* if $\dot{P}$ is the code of a half p-score $\vec{P}$, $\dot{Q}$ is the code of a half q^{op}-score $\vec{Q}$, and $(\vec{P}, \vec{Q})$ is a (p, q, U)-score.

The analogue of Lemma 7.3 for scores is the following.

Lemma 7.5. *The statement saying that a pair* $(\dot{P}, \dot{Q})$ *is the code of a* (p, q, U)*-score on an interval* $[u, v]$ *of* ω *is equivalent to an* MSO *statement.*

Proof. Let $\dot{P}$ and $\dot{Q}$ be given by (7.6) and (7.7). By Lemma 7.3, the statements that $\dot{P}$ and $\dot{Q}$ are codes of a half p-score and a half q^{op}-score on $[u, v]$, respectively, are equivalent to MSO formulas.

Next, the relations $(x, y) \in \langle x', y' \rangle_U$ and $(x, y) \in \langle x', y' \rangle_{U^{\mathrm{c}}}$ are, respectively, equivalent to the following MSO formulas:

$$x' \leq x \ \wedge \ x < y \ \wedge \ y \leq y' \wedge (x = x' \ \vee \ \neg(x \in U)) \ \wedge \ (y = y' \ \vee \ y \in U),$$
$$x' \leq x \ \wedge \ x < y \ \wedge \ y \leq y' \wedge (x = x' \ \vee \ x \in U) \ \wedge \ (y = y' \ \vee \ \neg(y \in U)).$$

From this we can deduce the following MSO equivalent of $(x_0, y_0) \perp_U (x_1, y_1)$:

$$\neg(\exists x, y)\bigl(x < y \ \wedge \ \text{“}(x, y) \in \langle x_0, y_0 \rangle_U\text{”} \ \wedge \ \text{“}(x, y) \in \langle x_1, y_1 \rangle_{U^{\mathrm{c}}}\text{”}\bigr).$$

Therefore, an $\mathtt{MSO}$ equivalent of the statement (6.1) is the conjunction, over all $i \in [\ell]$, $\alpha \in \mathrm{Alt}(p)$, and $\beta \in \mathrm{Alt}(q^{\mathrm{op}})$, of the following formulas:

$$(\forall x_0)(\forall y_0)(\forall x_1)(\forall y_1)\Big(\big(\text{“}(x_0, y_0) \in \mathrm{cvs}(P_\alpha)\text{”} \ \wedge \ \text{“}(x_1, y_1) \in \mathrm{cvs}(Q_\beta)\text{”} \\ \wedge \ x_0 \in P_{\alpha, \mathsf{z}_i} \ \wedge \ x_1 \in Q_{\beta, \mathsf{z}_i}\big) \Rightarrow \text{“}(x_0, y_0) \perp_U (x_1, y_1)\text{”}\Big). \qquad \square$$

Lemma 7.6. *Let $p, q \in \mathrm{T}_{\mathcal{L}}(\Omega)$. The statement, depending on two first-order variables x and y and a second-order predicate U, saying that $\mathsf{A}_U([x, y])$ satisfies the lattice inclusion $p \leq q$ is equivalent to an* $\mathtt{MSO}$ *statement.*

Proof. By Lemma 6.3, $\mathsf{A}_U([x, y])$ does not satisfy the lattice inclusion $p \leq q$ iff there is a (p, q, U)-score on $[x, y]$. Now the existence of a score can be expressed via existential quantification, over all second-order predicates P_α, $P_{\alpha,r}$, Q_β, $Q_{\beta,s}$, of the $\mathtt{MSO}$ formula, obtained from Lemma 7.5, that expresses being a (p, q, U)-score. Therefore, the following formula is equivalent to $\mathsf{A}_U([x, y])$ not satisfying $p \leq q$:

$$(\exists \dot{P})(\exists \dot{Q})\big(\text{“}(\dot{P}, \dot{Q}) \text{ is the code of a } (p, q, U)\text{-score on } [x, y]\text{”}\big),$$

where, in an obvious sense, $\exists \dot{P}$ stands for a string of quantifiers of the form $\exists P_\alpha$ or $\exists P_{\alpha,r}$ for $\alpha \in \mathrm{Alt}(p)$ and $r \in C_\alpha \cup \{\perp\}$ (and similarly for $\exists \dot{Q}$). □

Definition 7.7. An *orientation* is a triple (u, v, U), where $u, v \in \omega$, $u < v$, and $U \subseteq [u, v]$.

We can now state an expanded form of Theorem I.

Theorem 7.8. *Let $\mathfrak{U}$ be an* $\mathtt{MSO}$*-definable set of orientations. Then the equational theory of all lattices $\mathsf{A}_U([x, y])$, where $(x, y, U) \in \mathfrak{U}$, is decidable.*

Proof. For all $(x, y, U) \in \mathfrak{U}$, the Cambrian lattice $\mathsf{A}_U([x, y])$ satisfies the lattice inclusion $p \leq q$ iff the following $\mathtt{MSO}$ formula $\theta_{p,q}$ (obtained from the proof of Lemma 7.6) is in $\mathtt{S1S}$:

$$(\forall x)(\forall y)(\forall U)\big(\text{“}(x, y, U) \in \mathfrak{U}\text{”} \Rightarrow \text{“}\mathsf{A}_U([x, y]) \text{ satisfies the inclusion } p \leq q\text{”}\big).$$

Further, the assignment $(p, q) \mapsto \theta_{p,q}$ is given by an effectively computable procedure, that is, it is recursive. The desired conclusion follows from Theorem 7.1. □

Defining $\mathfrak{U}$ as the set of all (x, y, U) with $x < y$ and $U \subseteq [x, y]$, we obtain the equational theory of all Cambrian lattices of type A, which, by Lemma 3.1, is identical to the equational theory of all permutohedra.

Corollary 7.9. *The equational theory of all permutohedra lattices is decidable.*

By defining $\mathfrak{U}$ as the set of all triples (x, y, U) with $U = [x, y]$, we obtain the following.

Corollary 7.10. *The equational theory of all Tamari lattices is decidable.*

8. Tensor products and box products

Sections 8–10 will be mainly devoted to the proof of Theorem II, more precisely Theorem 10.1, showing that the equational theory of all permutohedra is nontrivial. We shall show that every Cambrian lattice of type A satisfies the splitting identity of the lattice $\mathsf{N}_5 \,\square\, \mathsf{B}(3,2)$; in this section we give the background and the tools for constructing and handling that lattice.

Our presentation originates from the *tensor product of* $(\vee, 0)$*-semilattices* considered by Grätzer, Lakser, and Quackenbush [22], which is a variant of Fraser's tensor product of join-semilattices considered in [11].

Definition 8.1. Let A and B be $(\vee, 0)$-semilattices. A *bi-ideal of* $A \times B$ is a lower subset I of $A \times B$ (endowed with the componentwise ordering), containing the subset

$$0_{A,B} = (A \times \{0_B\}) \cup (\{0_A\} \times B),$$

such that whenever $(a, b_0) \in I$ and $(a, b_1) \in I$, then $(a, b_0 \vee b_1) \in I$, and similarly with the roles of A and B reversed. The $(\vee, 0)$-semilattice $A \otimes B$ of all compact elements of $A \mathbin{\overline{\otimes}} B$ is called the *tensor product* of the $(\vee, 0)$-semilattices A and B.

The following elements of $A \mathbin{\overline{\otimes}} B$ deserve special attention:

— The *pure tensors* $a \otimes b = 0_{A,B} \cup \{(x, y) \in A \times B \mid x \leq a \text{ and } y \leq b\}$ for $(a, b) \in A \times B$. In particular, $0_{A,B} = 0_A \otimes 0_B$.
— The *mixed tensors* $(a \otimes b') \cup (a' \otimes b)$ for $a \leq a'$ in A and $b \leq b'$ in B.
— The *boxes* $a \,\square\, b = \{(x, y) \in A \times B \mid x \leq a \text{ or } y \leq b\}$.

Clearly, the inequalities $a \otimes b \leq a \,\square\, b'$ and $a \otimes b \leq a' \,\square\, b$ hold whenever $a, a' \in A$ and $b, b' \in B$. In fact, $a \otimes b = (a \,\square\, 0_B) \cap (0_A \,\square\, b)$. Notice also that if a and b are both nonzero, then $a \otimes b \leq a' \otimes b'$ iff $a \leq a'$ and $b \leq b'$. While pure tensors and mixed tensors always belong to $A \otimes B$ (in particular, $(a \otimes b') \cup (a' \otimes b)$ is really the *join* of $a \otimes b'$ and $a' \otimes b$), the box $a \,\square\, b$ may not belong to $A \otimes B$. However, if A and B both have a unit element, then $a \,\square\, b = (a \otimes 1_B) \cup (1_A \otimes b)$ is a mixed tensor, thus it belongs to $A \otimes B$.

If A and B are finite lattices, then $A \otimes B = A \mathbin{\overline{\otimes}} B$ is a finite lattice as well. In the infinite case, $A \otimes B$ may not be a lattice. For example, if $\mathsf{F}(3)$ denotes the free lattice on three generators, then $\mathsf{M}_3 \otimes \mathsf{F}(3)$ is not a lattice (Grätzer and Wehrung [24]). The following comes from Grätzer and Wehrung [25, Definition 4.1].

Definition 8.2. For $(\vee, 0)$-semilattices A and B, a subset C of $A \otimes B$ is a *sub-tensor product* if it contains all mixed tensors, is closed under nonempty finite intersection, and is a lattice under set inclusion. We say that C is *capped* if every member of C is a finite union of pure tensors.

If A and B are both finite, then every sub-tensor product is, trivially, capped. Grätzer and Wehrung [25] posed the problem whether $A \otimes B$ a lattice implies that $A \otimes B$ is a capped tensor product, for any lattices A and B with zero. This problem appeared to be difficult, and was finally solved, with a sophisticated counterexample, by Chornomaz [9].

A key property of sub-tensor products, with trivial proof, is the following.

Lemma 8.3. *Let A and B be lattices with zero, let C be a sub-tensor product of A and B, and let $a \in A$. Then the map $(B \to C,\ x \mapsto a \otimes x)$ is a zero-preserving lattice homomorphism.*

While even in the finite case, the ordinary tensor product $A \otimes B$ will not be satisfactory for our current purposes, a variant called *box product* will do the trick. The box product is an analogue, for lattices that are not necessarily complete, of Wille's tensor product of concept lattices [77]. Although the two concepts are, for finite lattices, equivalent, we found the box product presentation and results from Grätzer and Wehrung [23] more suited to our approach, heavily relying on join-coverings in our lattices.

The box product of A and B behaves well only in case both lattices A and B are bounded.[3] The following result is contained in Grätzer and Wehrung [23, Proposition 2.9 and Lemma 3.8].

Proposition 8.4. *Let A and B be bounded lattices. The set $A \Box B$ of all intersections of the form $\bigcap_{i=1}^{n}(a_i \Box b_i)$, for n a nonnegative integer, $a_1, \dots, a_n \in A$, and $b_1, \dots, b_n \in B$, is a lattice under set-theoretical inclusion, called the* box product *of A and B. Furthermore, $A \Box B$ is a capped sub-tensor product of A and B.*

Let $A = \mathsf{N}_5$ and $B = \mathsf{B}(3,2)$. By combining Lemma 8.3, Proposition 8.4, and the equations (2.3)–(2.5), we immediately obtain the following equations, valid in the lattice $\mathsf{N}_5 \Box \mathsf{B}(3,2)$ whenever $\{i, j\} = \{1, 2\}$ and $k, l \in \{1, 2, 3\}$:

$$c \otimes b_j = \big(c \otimes (q_* \vee b_j)\big) \wedge \big(c \otimes (b_1 \vee b_2)\big), \tag{8.1}$$

$$c \otimes a_k = \big(c \otimes (a_k \vee b_i)\big) \wedge \big(c \otimes (q_* \vee b_j)\big), \tag{8.2}$$

$$c \otimes (a_k \vee a_l) = \big(c \otimes (a_k \vee a_l \vee b_i)\big) \wedge \big(c \otimes (q_* \vee b_j)\big). \tag{8.3}$$

The behavior of capped tensor products with respect to congruences will be especially important to us. The following is a consequence of Lemma 5.3 and Theorem 2 in Grätzer and Wehrung [25].

Proposition 8.5. *Let A and B be lattices with zero and let C be a capped sub-tensor product of A and B. Then there exists a unique lattice isomorphism ε from $(\mathrm{Con_c}\, A) \otimes (\mathrm{Con_c}\, B)$ onto $\mathrm{Con_c}\, C$ such that*

$$\varepsilon\big(\mathrm{con}_A(a, a') \otimes \mathrm{con}_B(b, b')\big) = \mathrm{con}_C\big((a \otimes b') \cup (a' \otimes b), a' \otimes b'\big) \quad \textit{whenever } a \le a' \textit{ in } A \textit{ and } b \le b' \textit{ in } B. \tag{8.4}$$

From now on we shall abuse notation and write $\boldsymbol{\alpha} \otimes \boldsymbol{\beta}$ instead of $\varepsilon(\boldsymbol{\alpha} \otimes \boldsymbol{\beta})$ whenever $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in (\mathrm{Con_c}\, A) \times (\mathrm{Con_c}\, B)$. With this abuse of notation, the formula (8.4) becomes

$$\mathrm{con}_A(a, a') \otimes \mathrm{con}_B(b, b') = \mathrm{con}_C\big((a \otimes b') \cup (a' \otimes b), a' \otimes b'\big) \quad \text{whenever } a \le a' \text{ in } A \text{ and } b \le b' \text{ in } B. \tag{8.5}$$

3 The box product $A \Box B$ is a precursor of the further "lattice tensor product" construction $A \boxtimes B$, which may be defined even in some unbounded cases. This will not be pursued here.

Lemma 8.6. *The following statements hold for any sub-tensor product C of finite lattices A and B:*

(i) *The join-irreducible elements of C are exactly the $p \otimes q$, where $p \in \mathrm{Ji}\, A$ and $q \in \mathrm{Ji}\, B$. Furthermore, $(p \otimes q)_* = (p_* \otimes q) \cup (p \otimes q_*)$.*

(ii) *The join-prime elements of C are exactly the $p \otimes q$, where p and q are join-prime in A and B, respectively.*

Proof. The (easy) first part of (i) is contained in Wehrung [75, Lemma 7.2].

(ii) It is an easy exercise to verify that if $p \otimes q$ is join-prime, then so are p and q. Conversely, suppose that p and q are both join-prime. The box $H = \kappa_A(p) \,\square\, \kappa_B(q)$ belongs to C, and $p \otimes q \nsubseteq H$. Let $I \in C$ be such that $p \otimes q \nsubseteq I$, and suppose that $I \nsubseteq H$. There exists $(x, y) \in I \setminus H$. By the definition of H, $x \nleq \kappa_A(p)$ and $y \nleq \kappa_B(q)$, that is, $p \leq x$ and $q \leq y$, so $(p, q) \in x \otimes y \subseteq I$, a contradiction. Therefore, H is the largest element of C not containing $p \otimes q$. □

A simple application of Proposition 8.5 and Lemma 8.6 yields, with the notational convention introduced in (8.5), the formula

$$\mathrm{con}_C(p \otimes q) = \mathrm{con}_A(p) \otimes \mathrm{con}_B(q) \quad \text{for all } p \in \mathrm{Ji}\, A \text{ and all } q \in \mathrm{Ji}\, B, \tag{8.6}$$

whenever C is a sub-tensor product of finite lattices A and B.

Lemma 8.7. *The following statements hold for any capped sub-tensor product C of lattices A and B with zero:*

(i) *If A and B are both subdirectly irreducible, then so is C. Furthermore, if $\mathrm{con}_A(p)$ is the monolith of A and $\mathrm{con}_B(q)$ is the monolith of B, then $\mathrm{con}_C(p \otimes q)$ is the monolith of C.*

(ii) *If A and B both belong to $\mathbf{LB}_{\mathrm{fin}}$, then so does C.*

(iii) *If A and B both belong to $\mathbf{B}_{\mathrm{fin}}$, then so does $A \,\square\, B$. Further, $\kappa_{A \square B}(p \otimes q) = \kappa_A(p) \,\square\, \kappa_B(q)$ for all $p \in \mathrm{Ji}\, A$ and all $q \in \mathrm{Ji}\, B$.*

(iv) *If A and B are both splitting, then so is $A \,\square\, B$.*

Proof. (i) (see also Wille [77, Corollary 15]) It follows from Proposition 8.5 that if $\boldsymbol{\alpha}$ is the monolith of A and $\boldsymbol{\beta}$ is the monolith of B, then $\boldsymbol{\alpha} \otimes \boldsymbol{\beta}$ is the monolith of C. If $\boldsymbol{\alpha} = \mathrm{con}_A(p)$ and $\boldsymbol{\beta} = \mathrm{con}_B(q)$, then $\boldsymbol{\alpha} \otimes \boldsymbol{\beta} = \mathrm{con}_C(p \otimes q)$ (use (8.6)).

(ii) Since the relations $\trianglelefteq_A$ and $\trianglelefteq_B$ are both antisymmetric, it follows from (8.6) and (2.2) that $\trianglelefteq_C$ is also antisymmetric.

(iii) By (ii) together with Theorems 2.56 and 2.64 of Freese, Ježek, and Nation [14], it suffices to prove the second statement. Let $H = \bigcap_{i<n} (a_i \,\square\, b_i)$ with $p \otimes q \nsubseteq H$ (i.e., $(p, q) \notin H$) and $(p \otimes q)_* \subseteq H$. There exists $i < n$ such that $(p, q) \notin a_i \,\square\, b_i$, that is, $p \nleq a_i$ and $q \nleq b_i$. By Lemma 8.6(i), (p_*, q) and (p, q_*) both belong to $a_i \,\square\, b_i$. It follows that $p_* \leq a_i$ and $q_* \leq b_i$, hence $a_i \leq \kappa_A(p)$ and $b_i \leq \kappa_B(q)$. Therefore, $H \subseteq a_i \,\square\, b_i \subseteq \kappa_A(p) \,\square\, \kappa_B(q)$.

(iv) follows trivially from (i) and (iii) above. □

Denote by $\lambda(L)$ (resp., $\mu(L)$) the cardinality of $\mathrm{Ji}\, L$ (resp., $\mathrm{Mi}\, L$) for any finite lattice L. It follows from Freese, Ježek, and Nation [14, Theorem 2.40] that L belongs to $\mathbf{LB}_{\mathrm{fin}}$

iff $\lambda(L) = \lambda(\mathrm{Con}\, L)$, and [14, Theorem 2.67] shows that L belongs to $\mathbf{B}_{\mathrm{fin}}$ iff $\lambda(L) = \mu(L) = \lambda(\mathrm{Con}\, L)$. While Lemma 8.7(ii) trivially implies that ($A \in \mathbf{LB}_{\mathrm{fin}}$ and $B \in \mathbf{LB}_{\mathrm{fin}}$) implies that $A \otimes B \in \mathbf{LB}_{\mathrm{fin}}$, the analogous result for $\mathbf{B}_{\mathrm{fin}}$ does not hold in general. For example, $\mathsf{N}_5 \otimes \mathsf{N}_5$ has nine join-irreducible elements and ten meet-irreducible elements (for the union $(p \otimes p_*) \cup (p_* \otimes p) \cup (c \otimes c)$ is meet-irreducible, but it is not a box), thus it does not belong to $\mathbf{B}_{\mathrm{fin}}$. Hence, neither (iii) nor (iv) in Lemma 8.7, stated for the box product $A \boxtimes B$, can be extended to arbitrary capped sub-tensor products, even in the finite case.

9. Tight EA-duets of maps

In the present section we shall introduce an "equation-free" view of lattice varieties, to a great extent inspired by McKenzie [53]. This will enable us to prove Theorem II without needing to write huge equations.

Following Keimel and Lawson [44], a *Galois adjunction* between posets K and L is a pair (f, h) of maps, where $f\colon K \to L$ and $h\colon L \to K$, such that

$$f(x) \leq y \iff x \leq h(y) \qquad \text{for all } (x, y) \in K \times L.$$

In such a case, each of the maps f and h is uniquely determined by the other. We say that f is the *lower adjoint of* h and h is the *upper adjoint of* f.

Definition 9.1. Let K and L be lattices. A pair (f, g) of maps from K to L is an *EA-duet*[4] if there are a sublattice H of L and a surjective lattice homomorphism $h\colon H \twoheadrightarrow K$ such that f is the lower adjoint of h and g is the upper adjoint of h.

Lemma 9.2. *Let K and L be lattices and let $f, g\colon K \to L$. Then (f, g) is an EA-duet iff f is a join-homomorphism, g is a meet-homomorphism, and*

$$f(x) \leq g(y) \iff x \leq y \qquad \text{whenever } x, y \in K. \tag{9.1}$$

Proof. If (f, g) is an EA-duet with respect to $h\colon H \twoheadrightarrow K$, then it is straightforward to verify that f is a join-homomorphism and g is a meet-homomorphism. Furthermore, $f \leq g$, so $x \leq y$ implies $f(x) \leq g(y)$, and conversely, for all $x, y \in K$, $f(x) \leq g(y)$ implies that $x = hf(x) \leq hg(y) = y$.

Conversely, suppose that f is a join-homomorphism, g is a meet-homomorphism, and (9.1) holds. We set

$$H = \bigcup_{x \in K} [f(x), g(x)]. \tag{9.2}$$

For $y \in H$, let $x_0, x_1 \in K$ be such that $y \in [f(x_0), g(x_0)] \cap [f(x_1), g(x_1)]$. From $f(x_0) \leq y \leq g(x_1)$ and our assumptions it follows that $x_0 \leq x_1$. Likewise, $x_1 \leq x_0$, whence $x_0 = x_1$. This entitles us to define a map $h\colon H \to K$ by the rule

$$h(y) = \text{unique } x \in K \text{ such that } f(x) \leq y \leq g(x) \qquad \text{for each } y \in H. \tag{9.3}$$

4 After the soprano Aloysia Weber (1760–1839) and the bass Édouard de Reske (1853–1917), moreover following the categorical logic notation $\exists_h$ and $\forall_h$ for the left and right adjoint of h, respectively. Calling "scores" the main objects of Section 6 and Appendix A stays in line with that musical spirit.

Observe in particular that $h \circ f = h \circ g = \mathrm{id}_K$ (so h is surjective). Furthermore, $f \circ h \leq \mathrm{id}_H \leq g \circ h$. It is also easily seen that h is isotone. Therefore the previous relations determine h as the upper adjoint of f and as the lower adjoint of g; it follows that h preserves all the meets and joins that exist in H. Hence it remains to show that H is a sublattice of L. If $x_i \in H$ and $x_i = h(y_i)$ for $i \in \{0, 1\}$, then

$$f(x_0 \wedge x_1) \leq f(x_0) \wedge f(x_1) \leq y_0 \wedge y_1 \leq g(x_0) \wedge g(x_1) = g(x_0 \wedge x_1)$$

(because g is a meet-homomorphism), whence $y_0 \wedge y_1 \in H$. The proof that $y_0 \vee y_1 \in H$ is similar. □

Remark 9.3. It is an easy exercise to verify that in the context of Lemma 9.2 above, the sublattice H of L and the homomorphism $h\colon H \twoheadrightarrow K$ are uniquely determined, by the formulas (9.2) and (9.3), respectively.

Our next lemma involves the relation of weak projectivity $\Rightarrow$. Let us first recall some basic definitions (Grätzer [21, §III.1]). We define binary relations $\overset{\text{up}}{\twoheadrightarrow}$ and $\overset{\text{dn}}{\twoheadrightarrow}$ on the collection of all closed intervals of a lattice L by setting

$$[a, b] \overset{\text{up}}{\twoheadrightarrow} [c, d] \quad \text{if} \quad a \leq c \text{ and } d = b \vee c,$$

$$[a, b] \overset{\text{dn}}{\twoheadrightarrow} [c, d] \quad \text{if} \quad d \leq b \text{ and } c = a \wedge d,$$

whenever $a \leq b$ and $c \leq d$ in L. Furthermore, we denote by $\Rightarrow$ (the relation of *weak projectivity*) the transitive closure of the union of $\overset{\text{up}}{\twoheadrightarrow}$ and $\overset{\text{dn}}{\twoheadrightarrow}$.

Lemma 9.4. *Let K and L be lattices and let $f, g\colon K \to L$ be such that f is a join-homomorphism, g is a meet-homomorphism, and $f \leq g$ (with respect to the componentwise ordering). Let $a \leq b$ and $c \leq d$ in K. If $[a, b] \Rightarrow [c, d]$ and $f(b) \leq g(a)$, then $f(d) \leq g(c)$.*

Proof. It suffices to settle the cases $[a, b] \overset{\text{up}}{\twoheadrightarrow} [c, d]$ and $[a, b] \overset{\text{dn}}{\twoheadrightarrow} [c, d]$. In the former case,

$$f(d) = f(b \vee c) = f(b) \vee f(c) \leq g(a) \vee g(c) = g(c).$$

The case where $[a, b] \overset{\text{dn}}{\twoheadrightarrow} [c, d]$ is dual. □

Definition 9.5. A pair (u, v) of elements in a lattice L is

— *critical* if $\mathrm{con}(u \wedge v, u)$ is the monolith of L,
— *prime critical* if it is critical and $u \wedge v$ is a lower cover of u.

The following lemma gives a convenient characterization of EA-duets defined on a subdirectly irreducible lattice with a prime critical pair.

Lemma 9.6. *Let K and L be lattices, with K subdirectly irreducible, and let $u, v \in K$ with $u \nleq v$. Let $f\colon K \to L$ be a join-homomorphism and let $g\colon K \to L$ be a meet-homomorphism with $f \leq g$. If (f, g) is an EA-duet, then $f(u) \nleq g(v)$. Furthermore, if (u, v) is prime critical, then the converse holds.*

Proof. If (f, g) is an EA-duet, then, by Lemma 9.2, the condition (9.1) is satisfied, thus $u \nleq v$ implies that $f(u) \nleq g(v)$.

Suppose, conversely, that (u, v) is prime critical and $f(u) \nleq g(v)$. If (f, g) is not an EA-duet, then there are $x, y \in K$ such that $f(x) \leq g(y)$ and $x \nleq y$. Since $f(x) \leq g(x)$ and g is a meet-homomorphism, we infer that $f(x) \leq g(x \wedge y)$. Since $x \wedge y < x$, the congruence $\mathrm{con}(x \wedge y, x)$ is nonzero, thus it contains the monolith $\mathrm{con}(u \wedge v, u)$ of K. Since $u \wedge v$ is a lower cover of u, Grätzer's [21, Theorem 230] shows that the relation $[x \wedge y, x] \Rightarrow [u \wedge v, u]$ holds. Since $f(u) \nleq g(u \wedge v)$, Lemma 9.4 implies that $f(x) \nleq g(x \wedge y)$, a contradiction. □

From now on until the end of this section we fix lattices K and L of finite length.

Lemma 9.7. *The following are equivalent:*

(i) *K is a homomorphic image of a sublattice of L.*
(ii) *There exists an EA-duet (f, g) of maps from K to L.*

Proof. (i)⇒(ii). By assumption, there are a sublattice H of L and a surjective homomorphism $h \colon H \twoheadrightarrow K$. Since L has finite length, the lower adjoint (resp., upper adjoint) f (resp., g) of h are both well-defined. By definition, they form an EA-duet.

(ii)⇒(i) follows trivially from Lemma 9.2. □

For every map $f \colon K \to L$, the pointwise supremum $f^{\vee}$ of all join-homomorphisms below f (for the componentwise ordering) is itself a join-homomorphism, and thus it is the largest join-homomorphism below f. We denote it by $f^{\vee}$. Likewise, $f^{\wedge}$ is the least meet-homomorphism above f for the componentwise ordering. In particular, $f^{\vee} \leq f \leq f^{\wedge}$.

Definition 9.8. A pair (f, g) of maps from K to L is *tight* if $f = g^{\vee}$ and $g = f^{\wedge}$.

In particular, if (f, g) is tight, then f is a join-homomorphism, g is a meet-homomorphism, and $f \leq g$.

Lemma 9.9. *For every pair (f, g) of maps from K to L such that f is a join-homomorphism, g is a meet-homomorphism, and $f \leq g$, there exists a tight pair $(\overline{f}, \overline{g})$ such that $f \leq \overline{f} \leq \overline{g} \leq g$. If (f, g) is an EA-duet, then so is $(\overline{f}, \overline{g})$.*

Proof. Since $f \leq g$ and g is a meet-homomorphism, we get $f \leq f^{\wedge} \leq g$. Now, since f is a join-homomorphism, we get $f = f^{\vee} \leq f^{\wedge\vee} \leq f^{\wedge} \leq g$, so it suffices to prove that the pair $(f^{\wedge\vee}, f^{\wedge})$ is tight, for which we shall argue that $f^{\wedge} = f^{\wedge\vee\wedge}$:

$$f^{\wedge\vee\wedge} \leq f^{\wedge\wedge} = f^{\wedge}, \qquad f^{\wedge\vee\wedge} \geq f^{\vee\wedge} = f^{\wedge},$$

the last equality following from the assumption that $f^{\vee} = f$.

Finally, if (f, g) satisfies (9.1), then so does $(\overline{f}, \overline{g})$, since $f \leq \overline{f} \leq \overline{g} \leq g$. □

By applying Lemmas 9.2 and 9.9, we immediately obtain the following corollary.

Corollary 9.10. *The following are equivalent:*

(i) *L is a homomorphic image of a sublattice of K.*
(ii) *There is an EA-duet of maps from K to L.*
(iii) *There exists a tight EA-duet of maps from K to L.*

Although the two components of a tight pair may not be identical, we shall see that they agree on join-prime or meet-prime elements (Corollary 9.12). In order to see this, the key lemma is the following.

Lemma 9.11. *Let $g\colon K \to L$ be an isotone map. Then $g(0) = g^{\vee}(0)$. Furthermore, $g(p) = g^{\vee}(p)$ for any join-prime element p of K.*

Proof. Whenever p is join-prime, the map $f\colon K \to L$ defined by

$$f(x) = \begin{cases} g(p) & \text{if } p \leq x, \\ g(0_K) & \text{otherwise,} \end{cases} \qquad \text{for all } x \in K,$$

is a join-homomorphism. (If there is no join-prime, define $f(x) = g(0_K)$ everywhere.) From the assumption that g is isotone it follows that $f \leq g$, thus $f \leq g^{\vee}$. Hence, $g(0_K) = f(0_K) \leq g^{\vee}(0_K) \leq g(0_K)$ and $g(p) = f(p) \leq g^{\vee}(p) \leq g(p)$. □

Corollary 9.12. *Let (f, g) be a tight EA-duet of maps from K to L. Then f and g agree on all elements of K that are either 0_K, 1_K, join-prime, or meet-prime.*

Proof. Apply Lemma 9.11 to $g\colon K \to L$ and $f\colon K^{\mathrm{op}} \to L^{\mathrm{op}}$. □

10. An identity for all permutohedra: proving Theorem II

Throughout this section we shall use the labelings of the join-irreducible elements of N_5 and $B = \mathsf{B}(3,2)$ introduced in Figure 2.2. Further, we shall set $L = \mathsf{N}_5 \,\square\, B$. Since N_5 and $\mathsf{B}(3,2)$ are both splitting lattices, Lemma 8.7 shows that L is also splitting. This section will be devoted to the proof of the following more precise form of Theorem II.

Theorem 10.1. *Every permutohedron $\mathsf{P}(n)$ satisfies the splitting identity θ_L of L.*

Brute force calculation, based on the `Mace4` component of McCune's wonderful `Prover9 -Mace4` software [51], shows that L has 3,338 elements, so θ_L, although failing in L, holds in all lattices with at most 3,337 elements (see Section 2.4).

Towards a contradiction, assume that not every permutohedron satisfies the splitting identity of L. By Lemma 3.1, there are a finite chain E and a subset U of E such that $\mathsf{A}_U(E)$ does not satisfy the splitting identity of L, that is, L belongs to the lattice variety generated by $\mathsf{A}_U(E)$. Since L is subdirectly irreducible and $\mathsf{A}_U(E)$ is finite, it follows from Jónsson's Lemma (Jónsson [41], Jipsen and Rose [37, Ch. 1, Corollary 1.7]) that L is a homomorphic image of a sublattice of $\mathsf{A}_U(E)$. By Corollary 9.10, there is a tight EA-duet (f, g) of maps from L to $\mathsf{A}_U(E)$.

Since $\mathrm{con}_{\mathsf{N}_5}(p)$ is the monolith of N_5 and $\mathrm{con}_B(q)$ is the monolith of B, Lemma 8.7(i) shows that $\mathrm{con}_L(p \otimes q)$ is the monolith of L. Hence the pair $(p \otimes q, \kappa_L(p \otimes q))$ is prime critical in L (Definition 9.5). Further, it follows from Lemma 8.7(iii) that $\kappa_L(p \otimes q) = \kappa_{\mathsf{N}_5}(p) \,\square\, \kappa_B(q) = p_* \,\square\, q_*$. By Lemma 9.6, (f, g) being an EA-duet means that f is a join-homomorphism, g is a meet-homomorphism, $f \leq g$, and $f(p \otimes q) \nsubseteq g(p_* \,\square\, q_*)$.

Take E of least possible cardinality and pick a pair

$$(u, v) \in f(p \otimes q) \setminus g(p_* \mathbin{\square} q_*).$$

It is easy to verify that the projection $\pi \colon \mathsf{A}_U(E) \to \mathsf{A}_{U \cap [u,v]}([u, v])$, $\boldsymbol{a} \mapsto \boldsymbol{a} \cap \boldsymbol{\delta}_{[u,v]}$, is a lattice homomorphism. Furthermore, the maps $f' = \pi \circ f$ and $g' = \pi \circ g$ are, respectively, a join-homomorphism and a meet-homomorphism from L to $\mathsf{A}_{U \cap [u,v]}([u, v])$ with $f' \leq g'$ and $(u, v) \in f'(p \otimes q) \setminus g'(p_* \mathbin{\square} q_*)$. In particular, (f', g') is also an EA-duet. By the minimality assumption on E, u and v are the least and the largest element of E, respectively. Hence, we may assume that $E = [N]$ for some positive integer N with $(1, N) \in f(p \otimes q) \setminus g(p_* \mathbin{\square} q_*)$, and that N is least possible.

Lemma 10.2. *Let $(x, y) \in \langle 1, N \rangle_U$. If $(x, y) \in f(c \otimes q)$, then $(x, y) \in g(0)$.*

Proof. From $(x, y) \in \langle 1, N \rangle_U$ and $\langle 1, N \rangle_U \subseteq f(p \otimes q)$ it follows that $(x, y) \in f(p \otimes q)$, thus also $(x, y) \in g(p \otimes q)$, since $f \leq g$. From $(x, y) \in f(c \otimes q)$ we get $(x, y) \in g(c \otimes q)$. Since g is a meet-homomorphism and $(p \otimes q) \wedge (c \otimes q) = (p \wedge c) \otimes q = 0 \otimes q = 0$, (x, y) belongs to $g(p \otimes q) \wedge g(c \otimes q) = g((p \otimes q) \wedge (c \otimes q)) = g(0)$. □

Let $(x, y) \in f(c \otimes q)$. Whenever $j \in \{1, 2\}$, the inequality $q \leq a_1 \vee a_2 \vee a_3 \vee b_j$ (within B) entails $c \otimes q \leq (c \otimes a_1) \vee (c \otimes a_2) \vee (c \otimes a_3) \vee (c \otimes b_j)$ (within L), thus there exists a subdivision $x = z_0^j < z_1^j < \cdots < z_{n_j}^j = y$ such that

$$\text{whenever } 0 \leq i < n_j, \text{ there exists } d \in \mathbf{a} \cup \{b_j\} \text{ such that } (z_i^j, z_{i+1}^j) \in f(c \otimes d). \tag{10.1}$$

Denote by $\nu_j(x, y)$ the least possible value of n_j. Our main lemma is the following.

Lemma 10.3. *$f(c \otimes q)$ is contained in $g(c \otimes q_*)$.*

Proof. Let $(x, y) \in f(c \otimes q)$; we argue by induction on $y - x$ that $(x, y) \in g(c \otimes q_*)$. Consider subdivisions $(z_i^j \mid 0 \leq i \leq n_j)$ of $[x, y]$ satisfying (10.1) with $n_j = \nu_j(x, y)$. Set $S_j = \{(z_i^j, z_{i+1}^j) \mid 0 \leq i < n_j\}$ and $Z_j = \{z_i^j \mid 0 \leq i \leq n_j\}$, for each $j \in \{1, 2\}$.

Suppose first that $n_j = 1$ for some $j \in \{1, 2\}$, say $n_1 = 1$. It follows from (10.1) that $(x, y) \in f(c \otimes d)$ for some $d \in \mathbf{a} \cup \{b_1\}$. Hence $(x, y) \in g(c \otimes d)$. Since $(x, y) \in f(c \otimes q) \subseteq g(c \otimes q)$ and g is a meet-homomorphism, and since $d \wedge q \leq q_*$, it follows that (x, y) belongs to $g((c \otimes d) \wedge (c \otimes q)) = g(c \otimes (d \wedge q)) \subseteq g(c \otimes q_*)$, and we are done. Therefore we can suppose that $n_j > 1$ for every $j \in \{1, 2\}$.

Claim 1. *There is no i such that $0 \leq i < n_j$, $z_i^j \notin U$, and $z_{i+1}^j \in U$.*

Proof of Claim. Suppose that $0 \leq i < n_j$ with $z_i^j \notin U$ and $z_{i+1}^j \in U$. It follows that $(z_i^j, z_{i+1}^j) \in \langle 1, N \rangle_U$. Without loss of generality, we can suppose that $i > 0$; let then $d \in \mathbf{a} \cup \{b_j\}$ be such that $(z_{i-1}^j, z_i^j) \in f(c \otimes d)$. Recall that $(x, y) \in f(c \otimes q)$, thus $(z_i^j, z_{i+1}^j) \in f(c \otimes q)$ as well; by using Lemma 10.2, we get $(z_i^j, z_{i+1}^j) \in g(0)$, and thus a fortiori $(z_i^j, z_{i+1}^j) \in g(c \otimes d)$. Since (f, g) is a tight pair and $c \otimes d$ is join-prime in L, we deduce, using Lemma 9.11, that $f(c \otimes d) = g(c \otimes d)$, thus $(z_i^j, z_{i+1}^j) \in f(c \otimes d)$, and so $(z^j_{i-1}, z^j_{i+1}) \in f(c \otimes d)$, and the subdivision

$$x = z^j_0 < \cdots < z^j_{i-1} < z^j_{i+1} < \cdots < z^j_{n_j} = y$$

fulfils the same purpose as Z_j while it has length $n_j - 1$, in contradiction with the minimality of n_j. $\square_{\text{Claim 1}}$

Claim 1 means that $Z_j \setminus \{x, y\}$ consists of a (possibly empty) bunch of elements of U, followed by a (possibly empty) bunch of elements of U^{c}. This can be formally expressed by saying that for each $j \in \{1, 2\}$, there exists a unique integer $m_j \in [0, n_j - 1]$ such that

$$z^j_i \in U \text{ whenever } 0 < i \leq m_j \quad \text{and} \quad z^j_i \notin U \text{ whenever } m_j + 1 \leq i < n_j.$$

To ease notation, we shall from now on set $x_j = z^j_{m_j}$ and $y_j = z^j_{m_j+1}$ whenever $j \in \{1, 2\}$. We shall also set

$$\Delta = \{(t, t) \mid t \in [N]\}.$$

Claim 2. *Suppose that (x_j, y_j) belongs to $f(c \otimes a_k)$ for some $j \in \{1, 2\}$ and some $k \in \{1, 2, 3\}$. Then $(x, y) \in g(c \otimes q_*)$.*

Proof of Claim. From $(x, y) \in f(c \otimes q)$, $x \leq x_j \leq y$, and $x_j \in \{x\} \cup U$ it follows that $(x, x_j) \in f(c \otimes q) \cup \Delta$. Likewise, $(y_j, y) \in f(c \otimes q) \cup \Delta$. By our induction hypothesis (on $y - x$), it follows that (x, x_j) and (y_j, y) both belong to $g(c \otimes q_*) \cup \Delta$. Furthermore, from $a_k \leq q_*$ it follows that $c \otimes a_k \leq c \otimes q_*$, thus

$$(x_j, y_j) \in f(c \otimes a_k) \subseteq f(c \otimes q_*) \subseteq g(c \otimes q_*).$$

Since (x, y) is contained in $\langle x, x_j \rangle_U \vee \langle x_j, y_j \rangle_U \vee \langle y_j, y \rangle_U$, we are done. $\square_{\text{Claim 2}}$

From now on until the end of the proof of Lemma 10.3, we shall thus assume that $(x_j, y_j) \notin f(c \otimes a_k)$ whenever $j \in \{1, 2\}$ and $k \in \{1, 2, 3\}$. By (10.1), the only remaining possibility is that $(x_j, y_j) \in f(c \otimes b_j)$ for each $j \in \{1, 2\}$.

If $\{i, j\} = \{1, 2\}$ and $x_i \leq x_j$, define the *left fin* of S_j to be (x_j, x_j) if $x_i = x_j$, and to be the unique $(u, v) \in S_j$ such that $u \leq x_i < v$ if $x_i < x_j$. Necessarily, $\{u, v\} \subseteq U$.

Symmetrically, the *right fin* of S_j, defined in case $y_j \leq y_i$, is (y_j, y_j) if $y_i = y_j$, and the unique $(u, v) \in S_j$ such that $u < y_i \leq v$ if $y_j < y_i$. Necessarily, $\{u, v\} \subseteq U^{\mathsf{c}}$.

Observe that any (left or right) fin of S_j belongs to $S_j \cup \Delta$.

Claim 3. *The following statements hold whenever $\{i, j\} = \{1, 2\}$.*

(i) *If $x_i \leq x_j$, then the left fin (u, v) of S_j belongs to $f(c \otimes a_k) \cup \Delta$ for some $k \in \{1, 2, 3\}$; furthermore, $v = x_j$.*
(ii) *If $y_j \leq y_i$, then the right fin (u, v) of S_j belongs to $f(c \otimes a_k) \cup \Delta$ for some $k \in \{1, 2, 3\}$; furthermore, $u = y_j$.*

Proof of Claim. We prove (i); the proof of (ii) is symmetric. The case where $x_i = x_j$ is trivial, so we shall suppose that $x_i < x_j$; hence $u \leq x_i < v \leq x_j$.

Suppose first that $(u, v) \notin f(c \otimes a_k)$ for any $k \in \{1, 2, 3\}$. It follows from (10.1) that $(u, v) \in f(c \otimes b_j)$. Since $u \le x_i < v$ and $x_i \in U$, it follows that

$$(u, x_i) \in f(c \otimes b_j) \cup \Delta. \tag{10.2}$$

Moreover, from $\{u, x_j, y_j\} \subseteq Z_j$ and $u < x_j < y_j$ it follows that

$$(u, x_j) \in f\big(c \otimes (q_* \vee b_j)\big) \quad \text{and} \quad (u, y_j) \in f\big(c \otimes (q_* \vee b_j)\big). \tag{10.3}$$

Now we argue by considering several cases. In all cases, the key point here is to prove that $(u, y_j) \in f(c \otimes (b_1 \vee b_2))$.

Case 1: $y_i \le y_j$. This case is illustrated with the two diagrams in Figure 10.1. In this figure and all the subsequent ones, the notation $\overrightarrow{z}$ reminds us that $z \in \{x\} \cup U^{\mathsf{c}}$, while the notation $\overleftarrow{z}$ reminds us that $z \in \{y\} \cup U$.

(a) If $x_j \le y_i$, then since $y_i \le y_j \le y$, $y_i \in \{y\} \cup U^{\mathsf{c}}$, and $(x_j, y_j) \in f(c \otimes b_j)$, we get

$$(y_i, y_j) \in f(c \otimes b_j) \cup \Delta. \tag{10.4}$$

(b) If $y_i < x_j$, then, since $y_i \notin U$ and $x_j \in U$, we get $(y_i, x_j) \in \langle 1, N\rangle_U$; thus, since $(y_i, x_j) \in \langle x, y\rangle_U \subseteq f(c \otimes q)$, Lemma 10.2 yields $(y_i, x_j) \in g(0)$, and thus a fortiori $(y_i, x_j) \in g(c \otimes b_j)$. Since $c \otimes b_j$ is join-prime, it follows from Lemma 9.11 that $(y_i, x_j) \in f(c \otimes b_j)$. Since $(x_j, y_j) \in f(c \otimes b_j)$, (10.4) follows again.

Hence, (10.4) is valid in any case. Now it follows from $(x_i, y_i) \in f(c \otimes b_i)$, together with (10.2) and (10.4), that $(u, y_j) \in f(c \otimes (b_1 \vee b_2))$, thus $(u, y_j) \in g(c \otimes (b_1 \vee b_2))$. By applying the meet-homomorphism g to (8.1) and by using (10.3), we see that (u, y_j) belongs to

$$\begin{aligned} g\big(c \otimes (q_* \vee b_j)\big) \wedge g\big(c \otimes (b_1 \vee b_2)\big) & \\ &= g\Big(\big(c \otimes (q_* \vee b_j)\big) \wedge \big(c \otimes (b_1 \vee b_2)\big)\Big) \\ &= g(c \otimes b_j) \\ &= f(c \otimes b_j) \qquad \text{(use Lemma 9.11 again)}. \end{aligned}$$

It follows that the subdivision obtained from Z_j by removing all the elements of $Z_j \cap\,]u, y_j[$ (in particular x_j) fulfils the same purpose as Z_j, contrary to the minimality of n_j.

Case 2: $y_j < y_i$ (see Figure 10.2). From $x_i < x_j < y_i$, $x_j \in U$, and $(x_i, y_i) \in f(c \otimes b_i)$ it follows that $(x_i, x_j) \in f(c \otimes b_i)$. By (10.2) together with $(x_j, y_j) \in f(c \otimes b_j)$, we get $(u, y_j) \in f(c \otimes (b_1 \vee b_2))$, thus $(u, y_j) \in g(c \otimes (b_1 \vee b_2))$. By applying the meet-homomorphism g to (8.1) and by using (10.3), it follows again, as in Case 1, that $(u, y_j) \in f(c \otimes b_j)$, which leads to the same contradiction as above.

We have proved that $(u, v) \in f(c \otimes a_k)$ for some $k \in \{1, 2, 3\}$. Since $x \le u \le x_i < v$ and $x_i \in \{x\} \cup U$, it follows that

$$(u, x_i) \in f(c \otimes a_k) \cup \Delta. \tag{10.5}$$

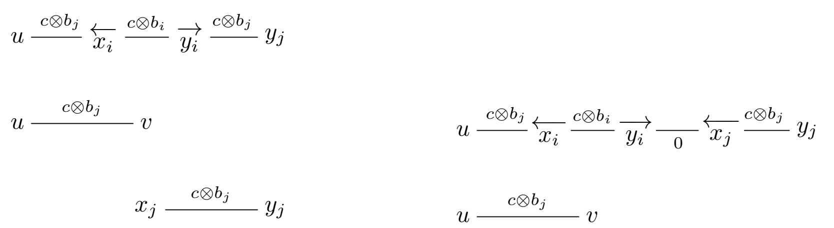

Fig. 10.1. Cases 1(a) (left) and 1(b) (right) in the proof of $(u, v) \in f(c \otimes a_k) \cup \Delta$ in Claim 3.

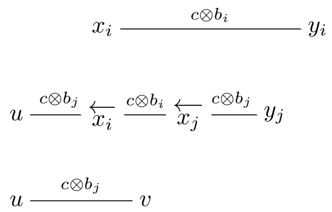

Fig. 10.2. Case 2 in the proof of $(u, v) \in f(c \otimes a_k) \cup \Delta$ in Claim 3.

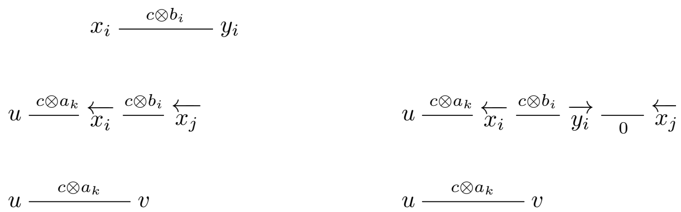

Fig. 10.3. Cases 1 (left) and 2 (right) in the proof of $v = x_j$ in Claim 3.

Now we must prove that $v = x_j$. We argue by separating cases. In all cases, the key point is to show that $(u, x_j) \in f(c \otimes (a_k \vee b_i))$; see Figure 10.3.

Case 1: $x_j \leq y_i$. From $(x_i, y_i) \in f(c \otimes b_i)$, $x_i < x_j \leq y_i$, and $x_j \in U$ it follows that $(x_i, x_j) \in f(c \otimes b_i)$. Hence, by (10.5), it follows that $(u, x_j) \in f(c \otimes (a_k \vee b_i))$, thus $(u, x_j) \in g(c \otimes (a_k \vee b_i))$. By using (10.3) and by applying the meet-homomorphism g to (8.2), it follows that $(u, x_j) \in g(c \otimes a_k)$, thus, by Lemma 9.11, $(u, x_j) \in f(c \otimes a_k)$. It follows that the subdivision obtained by removing from Z_j all the elements of $Z_j \cap \,]u, x_j[$ fulfils the same purpose as Z_j; whence, by the minimality of Z_j, we get $v = x_j$.

Case 2: $y_i < x_j$. Then $(y_i, x_j) \in \langle 1, N\rangle_U$. Since $(y_i, x_j) \in \langle x, y\rangle_U \subseteq f(c \otimes q)$, Lemma 10.2 yields $(y_i, x_j) \in g(0)$, and thus a fortiori $(y_i, x_j) \in g(c \otimes b_i)$; hence, by Lemma 9.11, $(y_i, x_j) \in f(c \otimes b_i)$. Since $(x_i, y_i) \in f(c \otimes b_i)$, (10.5) implies that $(u, x_j) \in f(c \otimes (a_k \vee b_i))$. The conclusion $v = x_j$ is then obtained as in Case 1.

This completes the proof of Claim 3. $\square_{\text{Claim 3}}$

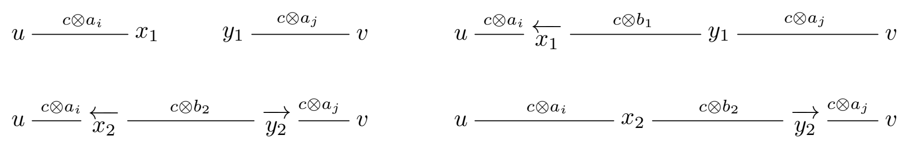

Fig. 10.4. Final cases in the proof of Lemma 10.3: Case 1 (left) and Case 2 (right).

In order to finish the proof of Lemma 10.3, we argue by separating cases, according to the relative positions of the intervals $[x_1, y_1]$ and $[x_2, y_2]$. By symmetry, there are two cases to consider (see Figure 10.4).

Case 1: $[x_1, y_1] \subseteq [x_2, y_2]$. Denote by (u, x_1) and (y_1, v) the left fin and the right fin of S_1, respectively (Claim 3). In particular, $u \leq x_2 \leq x_1 < y_1 \leq y_2 \leq v$. Furthermore, by Claim 3, there are $i, j \in \{1, 2, 3\}$ such that $(u, x_1) \in f(c \otimes a_i) \cup \Delta$ and $(y_1, v) \in f(c \otimes a_j) \cup \Delta$. From $x \leq u \leq x_2 \leq x_1$, $(u, x_1) \in f(c \otimes a_i) \cup \Delta$, and $x_2 \in \{x\} \cup U$ it follows that $(u, x_2) \in f(c \otimes a_i) \cup \Delta$. Symmetrically, $(y_2, v) \in f(c \otimes a_j) \cup \Delta$. Since $(x_2, y_2) \in f(c \otimes b_2) \cup \Delta$, it follows that $(u, v) \in f(c \otimes (a_i \vee a_j \vee b_2))$. On the other hand, from $\{u, v\} \subseteq Z_1$ and $u < v$ it follows that $(u, v) \in f(c \otimes (q_* \vee b_1))$. Since $f \leq g$ and by applying the meet-homomorphism g to (8.3), it follows that $(u, v) \in g(c \otimes (a_i \vee a_j))$, whence $(u, v) \in g(c \otimes q_*)$. Now, by the induction hypothesis, (x, u) and (v, y) both belong to the set $g(c \otimes q_*) \cup \Delta$, whence $(x, y) \in g(c \otimes q_*)$.

Case 2: $x_1 < x_2$ and $y_1 < y_2$. Denote by (u, x_2) the left fin of S_2 and by (y_1, v) the right fin of S_1 (Claim 3). It follows from Claim 3 that there are $i, j \in \{1, 2, 3\}$ such that $(u, x_2) \in f(c \otimes a_i) \cup \Delta$ and $(y_1, v) \in f(c \otimes a_j) \cup \Delta$. From $u \leq x_1 < x_2$, $(u, x_2) \in f(c \otimes a_i) \cup \Delta$, and $x_1 \in U$ it follows that $(u, x_1) \in f(c \otimes a_i) \cup \Delta$. Since $(x_1, y_1) \in f(c \otimes b_1)$ and $(y_1, v) \in f(c \otimes a_j)$, it thus follows that $(u, v) \in f(c \otimes (a_i \vee a_j \vee b_1))$. A similar proof, using this time the subdivision $u < x_2 < y_2 \leq v$, yields the relation $(u, v) \in f(c \otimes (a_i \vee a_j \vee b_2))$. Since $f \leq g$, by applying the meet-homomorphism g to (8.3) we see that $(u, v) \in g(c \otimes (a_i \vee a_j))$. We conclude that $(x, y) \in g(c \otimes q_*)$ as in Case 1.

This concludes the proof of Lemma 10.3. □

The conclusion of Lemma 10.3, together with $c \otimes q \nsubseteq c \otimes q_*$, implies that (f, g) cannot be an EA-duet. This contradiction concludes the proof of Theorem 10.1.

It is plausible that a more detailed argument, based on the same idea, would show that no sub-tensor product of $\mathsf{N}_5 \otimes \mathsf{B}(3, 2)$ belongs to the variety generated by all $\mathsf{P}(n)$. There would be some difficulties in checking this; for instance, other sub-tensor products (different from the box product) are no longer splitting lattices.

11. Permutohedra on locally dismantlable lattices: proving Theorem III

The present section will deal with the *extended permutohedron* $\mathsf{R}(E)$ on a (possibly infinite) poset E, as introduced in Santocanale and Wehrung [65] (Section 3), and prove that those $\mathsf{R}(E)$ satisfy no nontrivial lattice identity. The posets in question will actually be lattices of a very special kind.

Definition 11.1. A lattice L is

— *dismantlable* (Rival [63], Kelly and Rival [46]) if it is finite and every sublattice of L with at least three elements has an element which is *doubly irreducible*, that is, both meet- and join-irreducible;
— *locally dismantlable* if every finite subset of L is contained in a dismantlable sublattice of L.

A poset S is a *subposet* of a poset T if S is contained in T and the inclusion mapping of S into T is an order-embedding.

Definition 11.2. A poset T is a *segment extension* of a subposet S if there is a nonempty finite chain C of T, with extremities $x = \min C$ and $y = \max C$, such that

(i) $C \cap S = \{x, y\}$ and $C \cup S = T$;
(ii) $(s \leq x \Leftrightarrow s \leq y)$ and $(s \geq x \Leftrightarrow s \geq y)$, whenever $s \in S \setminus \{x, y\}$.

The proof of the following lemma is straightforward.

Lemma 11.3. *The following statements hold for any segment extension T of a poset S:*

(i) *If S is a lattice, then so is T. Furthermore, S is a sublattice of T.*
(ii) *If S is a dismantlable lattice, then so is T.*

The following definition is mainly taken from Santocanale and Wehrung [64, §10].

Definition 11.4. Let S be a poset and let L be a lattice.

- A map $\mu\colon \boldsymbol{\delta}_S \to L$ is an *L-valued polarized measure on S* if $\mu(x, y) \leq \mu(x, z) \leq \mu(x, y) \vee \mu(y, z)$ whenever $x < y < z$ in S.
- A *refinement problem* for a polarized measure μ is a quadruple (x, y, a_0, a_1), where $(x, y) \in \boldsymbol{\delta}_S$ and $a_0, a_1 \in L$, such that $\mu(x, y) \leq a_0 \vee a_1$.
- A *solution* of the refinement problem above is a subdivision $x = z_0 < z_1 < \cdots < z_n = y$ in S such that each $\mu(z_i, z_{i+1})$ is contained in some a_j.

The main lemma of this section is the following.

Lemma 11.5. *Let S be a finite poset, let $u < v$ in S, let L be a finite meet-semidistributive lattice, let $\mu\colon \boldsymbol{\delta}_S \to L$ be a polarized measure, and let $a_0, a_1 \in L$ be such that $\mu(u, v) \leq a_0 \vee a_1$. Then there are a finite segment extension T of S and a polarized measure $\nu\colon \boldsymbol{\delta}_T \to L$ extending μ such that:*

(i) *The refinement problem $\nu(u, v) \leq a_0 \vee a_1$ can be solved in T.*
(ii) *If the range of μ does not contain zero, then neither does the range of ν.*

Proof. As the conclusion is trivial if $\mu(u, v) \leq a_j$ for some $j < 2$ (take $T = S$ and $\nu = \mu$), we shall assume that $\mu(u, v) \nleq a_j$ for all $j < 2$. In particular, both a_0 and a_1 are nonzero; furthermore, it is ruled out that $\mu(u, v) \wedge a_j = 0$ for each $j < 2$, for then we would infer, by the meet-semidistributivity of L, that $\mu(u, v) = \mu(u, v) \wedge (a_0 \vee a_1) = 0$, a contradiction. Hence we may assume that $\mu(u, v) \wedge a_0$ is nonzero.

An intuitive description of what follows is that we first attach an infinite copy of the chain ω of all nonnegative integers to S between u and v; then we show that all large enough members of that ω are redundant, so we get rid of them.

We shall also use the convention $\mu(x, x) = 0$ for each $x \in S$. We shall set $\varepsilon(n) = n \bmod 2$ for each integer n, and we shall endow the cartesian product $(S \downarrow u) \times \omega$ with the partial ordering $\leq^*$ defined by

$$(x, k) \leq^* (y, l) \ \Leftrightarrow\ (y \leq x \text{ and } k \leq l) \quad \text{whenever } (x, k), (y, l) \in (S \downarrow u) \times \omega.$$

We define, by $\leq^*$-induction, a map $f : (S \downarrow u) \times \omega \to L$ by the rule

$$f(x, 0) = \mu(x, u), \tag{11.1}$$

$$f(x, k+1) = \bigwedge\bigl(\mu(x, t) \vee f(t, k+1) \mid t \in \,]x, u]\bigr) \wedge \bigl(f(x, k) \vee a_{\varepsilon(k)}\bigr) \wedge \mu(x, v), \tag{11.2}$$

for each $(x, k) \in (S \downarrow u) \times \omega$. As usual, empty meets are identified with the top element of L.

Claim 1. *The inequality $f(x, k) \leq \mu(x, y) \vee f(y, k)$ holds for all $x < y$ in $S \downarrow u$ and all $k < \omega$.*

Proof of Claim. We argue by induction on k. The conclusion holds for $k = 0$ because μ is a polarized measure. If the statement holds at k, then setting $t = y$ in the meet in the defining equation (11.2), we obtain $f(x, k+1) \leq \mu(x, y) \vee f(y, k+1)$. $\square_{\text{Claim 1}}$

Claim 2. $\mu(x, u) \leq f(x, k) \leq \mu(x, v)$ *for each* $(x, k) \in (S \downarrow u) \times \omega$.

Proof of Claim. The inequality $f(x, k) \leq \mu(x, v)$ is trivial. For $\mu(x, u) \leq f(x, k)$, we argue by $\leq^*$-induction on (x, k). The result is trivial for $k = 0$. Suppose that it holds at every pair $\leq^*$-smaller than $(x, k+1)$. For each $t \in \,]x, u]$, it follows from the induction hypothesis that $\mu(t, u) \leq f(t, k+1)$, thus $\mu(x, u) \leq \mu(x, t) \vee \mu(t, u) \leq \mu(x, t) \vee f(t, k+1)$. Furthermore, by the induction hypothesis, $\mu(x, u) \leq f(x, k)$, whence $\mu(x, u) \leq f(x, k) \vee a_{\varepsilon(k)}$. Recalling also that $\mu(x, u) \leq \mu(x, v)$, we see that the result follows immediately from equation (11.2) defining $f(x, k+1)$. $\square_{\text{Claim 2}}$

Claim 3. *The inequality $f(x, k) \leq f(x, k+1)$ holds for each $(x, k) \in (S \downarrow u) \times \omega$.*

Proof of Claim. We argue by downward induction on x. For each $t \in \,]x, u]$, it follows from the induction hypothesis that $f(t, k) \leq f(t, k+1)$, thus, by Claim 1, $f(x, k) \leq \mu(x, t) \vee f(t, k) \leq \mu(x, t) \vee f(t, k+1)$. Since $f(x, k) \leq f(x, k) \vee a_{\varepsilon(k)}$, the result follows immediately from (11.2). $\square_{\text{Claim 3}}$

By Claim 3 and as L and $S \downarrow u$ are both finite, there exists $m \in \omega \setminus \{0\}$ such that

$$(\forall x \in S \downarrow u)(\forall k \geq m \text{ in } \omega)\bigl(f(x, k) = f(x, m)\bigr).$$

For the rest of the proof of Lemma 11.5 we shall fix the integer m. Set $g(x) = f(x, m)$ for each $x \in S \downarrow u$.

Claim 4. *The equality $g(x) = \mu(x, v)$ holds for each $x \in S \downarrow u$.*

Proof of Claim. We argue by (downward) induction on x. For each $t \in]x, u]$, it follows from the induction hypothesis that $g(t) = \mu(t, v)$, thus $\mu(x, t) \vee g(t) \geq \mu(x, v)$. Therefore, by applying (11.2) to $k \in \{m + 1, m + 2\}$, we obtain

$$\begin{aligned} g(x) &= \bigwedge\big(\mu(x, t) \vee g(t) \mid t \in]x, u]\big) \wedge \big(g(x) \vee a_{\varepsilon(k)}\big) \wedge \mu(x, v) \\ &= \big(g(x) \vee a_{\varepsilon(k)}\big) \wedge \mu(x, v). \end{aligned}$$

Hence, by using the meet-semidistributivity of L, we obtain

$$g(x) = (g(x) \vee a_0 \vee a_1) \wedge \mu(x, v). \tag{11.3}$$

Now, by Claim 2, $g(x) \vee a_0 \vee a_1 \geq \mu(x, u) \vee \mu(u, v) \geq \mu(x, v)$, thus, by (11.3), $g(x) = \mu(x, v)$. □Claim 4

Now we fix new symbols $t_1, \dots, t_{m-1}$ and we set $T = S \cup \{t_1, \dots, t_{m-1}\}$ with $u < t_1 < \cdots < t_{m-1} < v$. Furthermore, we extend the ordering of S to T by letting ($s \leq t_i \Leftrightarrow s \leq u$) and ($t_i \leq s \Leftrightarrow v \leq s$), whenever $s \in S$.

We extend the map μ to a map $\nu\colon \boldsymbol{\delta}_T \to L$ by setting

$$\nu(x, t_k) = f(x, k) \quad \text{for } (x, k) \in (S \downarrow u) \times [1, m[\,, \tag{11.4}$$

$$\nu(t_k, t_l) = \bigvee(a_{\varepsilon(i)} \mid k \leq i < l) \quad \text{for } 1 \leq k < l < m, \tag{11.5}$$

$$\nu(t_k, y) = \bigvee(a_{\varepsilon(i)} \mid k \leq i < m) \vee \mu(v, y) \quad \text{for } (k, y) \in [1, m[\times (S \uparrow v). \tag{11.6}$$

Verifying that ν is a polarized measure amounts to verifying the following statements:

- $\mu(x, y) \leq f(x, k) \leq \mu(x, y) \vee f(y, k)$ for all $x < y$ in $S \downarrow u$ and all $k \in [1, m[$. This follows trivially from Claims 1 and 2.
- $f(x, k) \leq f(x, l) \leq f(x, k) \vee \nu(t_k, t_l)$ for all $x \in S \downarrow u$ and all $k < l$ in $[1, m[$. The first inequality follows from Claim 3. For $l = k + 1$, the second inequality follows trivially from (11.2) and (11.5), while for $l \geq k + 2$, it follows from (11.5) together with the case of $l = k + 1$.
- $f(x, k) \leq \mu(x, y) \leq f(x, k) \vee \nu(t_k, y)$ for all $(x, y) \in (S \downarrow u) \times (S \uparrow v)$ and all $k \in [1, m[$. The first inequality follows from Claim 2 together with $\mu(x, v) \leq \mu(x, y)$. To prove the second, we separate cases. If $k \leq m - 2$, then, as $\mu(u, v) \leq a_0 \vee a_1$,

$$\begin{aligned} f(x, k) \vee \nu(t_k, y) &= f(x, k) \vee a_0 \vee a_1 \vee \mu(v, y) \\ &\geq \mu(x, u) \vee \mu(u, v) \vee \mu(v, y) \quad \text{(by Claim 2)} \\ &\geq \mu(x, y), \end{aligned}$$

  and we are done. If $k = m - 1$, then

$$\begin{aligned} f(x, k) \vee \nu(t_k, y) &= f(x, k) \vee a_{\varepsilon(k)} \vee \mu(v, y) \\ &\geq f(x, k + 1) \vee \mu(v, y) \quad \text{(use (11.2))} \\ &= g(x) \vee \mu(v, y) \\ &= \mu(x, v) \vee \mu(v, y) \quad \text{(by Claim 4)} \\ &\geq \mu(x, y), \end{aligned}$$

  and we are done again.

- $\nu(t_k, t_l) \leq \nu(t_k, y) \leq \nu(t_k, t_l) \vee \nu(t_l, y)$ for all $k < l$ in $[1, m[$ and all $y \in S \uparrow v$. This follows immediately from (11.5) and (11.6).
- $\nu(t_k, x) \leq \nu(t_k, y) \leq \nu(t_k, x) \vee \mu(x, y)$ for all $k \in [1, m[$ and all $x < y$ in $S \uparrow v$. This follows immediately from (11.6).

Hence we have proved that ν is a polarized measure. By construction, the refinement problem $\nu(u, v) \leq a_0 \vee a_1$ can be solved in T.

Now suppose that the range of μ does not contain the zero of L (provided the latter exists). In order to prove that ν satisfies the same statement and recalling that $a_i \neq 0$ for $i < 2$, it will be enough to prove that $f(x, k)$ is nonzero for every $x \in S \downarrow u$ and every positive integer k. By Claim 2, if $f(x, k) = 0$, then $\mu(x, u) = 0$ (remember the convention $\mu(u, u) = 0$), thus $x = u$, and so, by Claim 3, $f(u, 1) = 0$, that is, using (11.2), $a_0 \wedge \mu(u, v) = 0$, which we have ruled out from the beginning. □

This brings us to the main result of this section, involving the extended permutohedron $\mathsf{R}(E)$ and its meet-subsemilattice $\mathsf{A}(E)$ (Section 3). From now on, by "countable" we will always mean "at most countable".

Theorem 11.6. *Let L be a finite meet-semidistributive lattice. There are a countable, locally dismantlable lattice E together with a zero-preserving lattice embedding $\varphi\colon L \hookrightarrow \mathsf{R}(E)$ with range contained into $\mathsf{A}(E)$. In particular, φ is also a zero-preserving lattice embedding from L into $\mathsf{A}(E)$.*

Proof. If we endow the finite set $E_0 = L \setminus \{0\}$ with any strict well-ordering, the map $\mu_0\colon \boldsymbol{\delta}_{E_0} \to L$, $(x, y) \mapsto x$, is a polarized measure with nonzero values. Having defined a polarized measure $\mu_n\colon \boldsymbol{\delta}_{E_n} \to L$ with nonzero values, and with E_n a dismantlable lattice, a straightforward iteration of Lemma 11.5, invoking Lemma 11.3 for the preservation of dismantlability, yields a dismantlable extension E_{n+1} of E_n and a polarized measure $\mu_{n+1}\colon \boldsymbol{\delta}_{E_{n+1}} \to L$ with nonzero values, extending μ_n, such that every refinement problem for μ_n is solved by μ_{n+1}.

The union μ of all μ_n is an L-valued polarized measure on the countable, locally dismantlable lattice $E = \bigcup_{n \in \omega} E_n$. It has nonzero values, and every refinement problem for μ has a solution. The map φ defined on L by the rule

$$\varphi(a) = \{(x, y) \in \boldsymbol{\delta}_E \mid \mu(x, y) \leq a\} \quad \text{for all } a \in L$$

takes its values in $\mathsf{A}(E)$. As the meet in $\mathsf{A}(E)$ is intersection, φ is a meet-homomorphism to $\mathsf{A}(E)$; as $\mathsf{A}(E)$ is a meet-subsemilattice of $\mathsf{R}(E)$, φ is also a meet-homomorphism to $\mathsf{R}(E)$. Since μ takes nonzero values, φ is zero-preserving. Moreover, since μ solves all its own refinement problems and since the join in $\mathsf{R}(E)$ is the transitive closure of the union, the definition of φ implies immediately that φ is a join-homomorphism to $\mathsf{R}(E)$. Finally, notice that φ is also a join-homomorphism to $\mathsf{A}(E)$; indeed, while the join in $\mathsf{A}(E)$ is not in general the transitive closure of the union, the fact that $\varphi(a_0 \vee a_1)$ belongs to $\mathsf{A}(E)$ forces it to be the join $\varphi(a_0) \vee \varphi(a_1)$ within $\mathsf{A}(E)$.

Finally, since μ extends μ_0, its range is $L \setminus \{0\}$; hence φ is one-to-one. □

Corollary 11.7. *Every free lattice embeds, as a sublattice, into* $\mathsf{R}(E)$ *for some locally dismantlable lattice* E *via a map with range contained in* $\mathsf{A}(E)$.

Proof. A well known result by Day (see Freese, Ježek, and Nation [14, Theorem 2.84]) states that every free lattice embeds into a direct product of members of $\mathbf{B}_{\mathrm{fin}}$. Since every member of $\mathbf{B}_{\mathrm{fin}}$ is meet-semidistributive, it follows from Theorem 11.6 that every free lattice embeds into a product $\prod_{i\in I}\mathsf{R}(E_i)$ for a collection $(E_i \mid i \in I)$ of locally dismantlable lattices E_i. If we fix a strict well-ordering $\lhd$ on I, the disjoint union $E = \bigcup_{i\in I}(\{i\}\times E_i)$ endowed with the lexicographical ordering (i.e., $(i,x) \leq (j,y)$ if either $i \lhd j$ or ($i = j$ and $x \leq y$)) is locally dismantlable, and $\prod_{i\in I}\mathsf{R}(E_i)$ embeds into $\mathsf{R}(E)$ via $(\boldsymbol{x}_i \mid i \in I) \mapsto \bigcup_{\in I}\boldsymbol{x}_i$. The latter assignment maps $\prod_{i\in I}\mathsf{A}(E_i)$ into $\mathsf{A}(E)$. □

In particular, we get the following more precise form of Theorem III.

Corollary 11.8. *There is no nontrivial lattice-theoretical identity satisfied by all* $\mathsf{R}(E)$ (*resp.,* $\mathsf{A}(E)$) *for* E *a countable, locally dismantlable lattice.*

Remark 11.9. Every subposet E of a poset F induces a $(\wedge,1)$-homomorphism $\pi_E^F\colon \mathsf{A}(F) \to \mathsf{A}(E)$, $\boldsymbol{x} \mapsto \boldsymbol{x} \cap \boldsymbol{\delta}_E$. This map preserves all directed joins. Now let $E = \bigcup_{n\in\omega} E_n$ be an increasing union of finite dismantlable lattices E_n. It is obvious that $\mathsf{A}(E)$, together with the maps $\pi_{E_n}^E$, is the inverse limit, in the category of all $(\wedge,1)$-semilattices, of the $\mathsf{A}(E_n)$. Now it can be proved that this implies that $\mathsf{A}(E)$ belongs to the lattice variety generated by all $\mathsf{A}(E_n)$. Hence we can strengthen part of the statement of Corollary 11.8 as follows: *the lattices* $\mathsf{A}(E)$*, for* E *ranging over all finite dismantlable lattices, do not satisfy any nontrivial lattice identity*.

However, for a subposet E of a poset F, the assignment $\boldsymbol{x} \mapsto \boldsymbol{x} \cap \boldsymbol{\delta}_E$ does not necessarily map $\mathsf{R}(F)$ to $\mathsf{R}(E)$, so the argument above does not extend to $\mathsf{R}(E)$.

Remark 11.10. The locally dismantlable lattice E in Theorem 11.6 is obtained by means of successive segment extensions. Such extensions usually create squares. It can therefore be asked whether a better construction would lead to an embedding of every lattice from $\mathbf{B}_{\mathrm{fin}}$ into some $\mathsf{P}(E)$ with E square-free. This is actually impossible, because if E is square-free, then $\mathsf{P}(E)$ is a subdirect product of permutohedra (Santocanale and Wehrung [67, Exercices 8.4–8.6]).

12. Discussion

Our results raise a whole array of new questions.

12.1. How far can we go?

Extending a result by Sekanina [69], the three papers of Iturrioz [35], Katrnoška [43], and Mayet [50] established simultaneously that every orthoposet can be obtained as the poset of all clopen (closed and open) subsets in some closure space; hence the ortholattices of clopen sets satisfy no nontrivial identity. Nevertheless, setting restrictions on the closure space (P, φ) brings restrictions to the corresponding lattice $\mathrm{Reg}(P, \varphi)$ of *regular closed* subsets (the closures of open sets). For example, we prove in [66] that if (P, φ) is a *finite convex geometry*, then $\mathrm{Reg}(P, \varphi)$ is *pseudocomplemented*. We do not know whether there is a nontrivial lattice identity satisfied by $\mathrm{Reg}(P, \varphi)$ for every *finite* convex geometry (P, φ). In view of Theorem III (see Corollary 11.8), this sounds improbable. Then the possibility arises that every class of closure spaces (P, φ) would yield an identity for all the corresponding $\mathrm{Reg}(P, \varphi)$. Particular instances of that question, along with natural variants, would be the following:

(1) Is it the case that for every positive integer d there exists a nontrivial lattice identity satisfied by the extended permutohedron $\mathsf{R}(E)$ for every finite poset E of order-dimension at most d? Note that there are finite dismantlable posets of arbitrarily large order-dimension (Kelly [45]).
(2) Can every finite Coxeter lattice be embedded into some $\mathsf{P}(n)$? (We know that this holds for Coxeter lattices of type B.) Does it at least belong to the variety generated by all $\mathsf{P}(n)$?
(3) Similar questions can be asked for the various classes of "permutohedra" considered in our papers [65, 66]: most notably, lattices of regular closed subsets constructed from *semilattices*, *graphs*, *hyperplane arrangements*.

12.2. Finitely based subvarieties of the variety generated by all permutohedra

Denote by $\mathcal{P}$ the variety generated by all permutohedra. Is it decidable whether the class of all lattices satisfying a given lattice identity is contained in $\mathcal{P}$? Since the variety generated by a given finite lattice can be defined by a single identity (McKenzie [52]), this would solve the other question whether a given finite lattice belongs to $\mathcal{P}$. Those questions arise, for instance, for the lattices $\mathsf{B}(m, n)$ (see Section 2.5, and also Appendix A where we give a combinatorial equivalent of the corresponding question), or for Nation's identity β'_1 from [56, p. 537] (since $\mathsf{N}_5 \square \mathsf{B}(3, 2)$ satisfies β'_2, we do not need to try other β'_n). In particular, we know from [64] that $\mathsf{B}(3, 3)$ and all $\mathsf{B}(n, 2)$ belong to $\mathcal{P}$, but we do not know whether $\mathsf{B}(4, 3)$ belongs to $\mathcal{P}$ (see Appendix A). A related question is whether the variety $\mathcal{P}$ can be defined by finitely many lattice identities (equivalently, by a single lattice identity).

12.3. Varieties and quasivarieties of ortholattices

Recall that a *quasi-identity* is a formula of the form

$$(\forall \vec{x})\Big(\big(p_1(\vec{x}) = q_1(\vec{x}) \text{ and } \cdots \text{ and } p_n(\vec{x}) = q_n(\vec{x})\big) \Rightarrow p(\vec{x}) = q(\vec{x})\Big),$$

where all p, q, p_i, q_i are terms. It is known (Section 1.3) that the set of all quasi-identities satisfied by all ortholattices is decidable. Can Theorem I be extended to permutohedra viewed as *ortholattices*, that is, lattices with an additional unary operation symbol for complementation? Can Theorem I be extended to quasi-identities?

Of course, the questions asked in Sections 12.2–12.3 can be extended similarly.

12.4. Tractability of the algorithm

While the equational theory of all permutohedra, respectively Tamari lattices, is decidable (Corollaries 7.9 and 7.10), the implied algorithms are totally intractable, even for very simple identities. We do not know whether there is any tractable algorithm for those problems. The algorithms rely on Büchi's Theorem [6] for `S1S`; the complexity of deciding `MSO` statements is determined by the automata-theoretical constructions corresponding to logical operations (Thomas [74, §3] or Perrin and Pin [57, Ch. 1]).

A. An example: (m, n)-scores on a finite chain

It is interesting to see what becomes of the decidability results established in Section 7 for concrete lattice identities. A blunt application of Theorem 7.1 to the translation obtained in Section 6, via scores, of negated lattice inclusions looks quite hopeless from a practical viewpoint.

However, in some cases it is possible to express a negated lattice inclusion in a way which, even if it falls short of yielding any practical implementation, produces nonetheless a rather transparent combinatorial description. We choose to illustrate this here for the splitting identity of the lattice $\mathsf{B}(m, n)$ described in Section 2.5.

Definition A.1. Let E be a chain and let $U \subseteq E$. A pair $(x, y) \in \boldsymbol{\delta}_E$ is

— a *valley* of (E, U) if $x \in \{0_E\} \cup U^{\mathsf{c}}$ and $y \in \{1_E\} \cup U$;
— a *peak* of (E, U) if $x \in \{0_E\} \cup U$ and $y \in \{1_E\} \cup U^{\mathsf{c}}$;
— a *slope* of (E, U) if it is neither a peak nor a valley.

Definition A.2. Let E be a finite chain, let $U \subseteq E$, and let m and n be positive integers. An *(m, n)-score on E with respect to U* is a triple $\boldsymbol{\tau} = (\vec{B}, \vec{A}, \tau)$ such that:

- $\vec{B} = (B_1, \dots, B_n)$, where each B_j is a subdivision of E. We call the B_j the *Basso subdivisions* of $\boldsymbol{\tau}$ and we set $\mathrm{cvs}(\vec{B}) = \bigcup_{j=1}^{n} \mathrm{cvs}(B_j)$.
- $\vec{A} = (A_1, \dots, A_m)$, where each A_i is a subdivision of E. We call the A_i the *Alto subdivisions* of $\boldsymbol{\tau}$ and we set $\mathrm{cvs}(\vec{A}) = \bigcup_{i=1}^{m} \mathrm{cvs}(A_i)$.
- $\tau\colon\ \mathrm{cvs}(\vec{A}) \cup \mathrm{cvs}(\vec{B}) \to \mathbf{a} \cup \mathbf{b}$, and the following conditions hold:
  - (ScA) Let $i \in [m]$ and let $(x, y) \in \mathrm{cvs}(A_i)$. Then $\tau(x, y) \in \{a_i\} \cup \mathbf{b}$; moreover, if (x, y) is a valley of (E, U), then $\tau(x, y) = a_i$.
  - (ScB) Let $j \in [n]$ and let $(x, y) \in \mathrm{cvs}(B_j)$. Then $\tau(x, y) \in \{b_j\} \cup \mathbf{a}$; moreover, if (x, y) is a peak of (E, U), then $\tau(x, y) = b_j$.
  - (Comp) Let $(x, y) \in \mathrm{cvs}(\vec{B})$ and let $(x', y') \in \mathrm{cvs}(\vec{A})$. Then $(x, y) \sim_U (x', y')$ (see Section 4) implies that $\tau(x, y) = \tau(x', y')$.

The terminology *Basso* and *Alto* follows the commonly used notation (β, α) for the pair consisting of the lower and upper adjoints of a lattice homomorphism (Freese, Ježek, and Nation [14]). It is also adjusted to the notation b_j, a_i for the atoms of $\mathsf{B}(m, n)$ (Figure 2.2).

The following result translates the membership problem, of the lattice $\mathsf{B}(m, n)$ in the lattice variety generated by $\mathsf{A}_U(E)$, in terms of certain tiling properties of the chain E.

This result is not too hard to obtain via a combination of the methods of Sections 6 and 9. We do not include a proof here.

Theorem A.3. *The following statements are equivalent for all positive integers m and n and every subset U in a finite chain E:*

(i) $\mathsf{B}(m,n)$ *belongs to the lattice variety generated by* $\mathsf{A}_U(E)$.
(ii) $\mathsf{A}_U(E)$ *does not satisfy the splitting identity of* $\mathsf{B}(m,n)$.
(iii) *There exists an EA-duet of maps from* $\mathsf{B}(m,n)$ *to* $\mathsf{A}_U(E)$.
(iv) *There exists an* (m,n)*-score on* E *with respect to* U.

We proved in [64, §12] that $\mathsf{B}(3,3)$ belongs to the lattice variety generated by $\mathsf{A}_U(12)$, where $U = \{5,6,9,10,11\}$. The corresponding score is represented in Figure A.1. The circled vertices correspond to the elements of the chain [12], while the labels on the edges are the corresponding values of τ. The notation $\overrightarrow{x}$ means that $x \notin U$, while $\overleftarrow{x}$ means that $x \in U$.

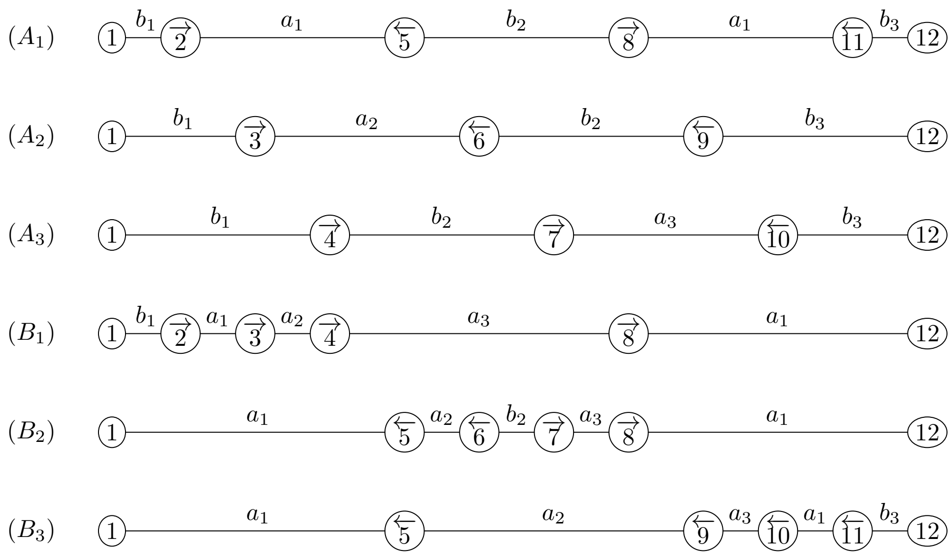

Fig. A.1. A $(3,3)$-score on [12] with respect to $U = \{5,6,9,10,11\}$.

We do not know whether all $\mathsf{B}(m,n)$ belong to the lattice variety generated by all permutohedra, even in the particular case where $m = 4$ and $n = 3$. (This question is also related to Section 12.2.) Equivalently, we do not know whether there are a positive integer N, a subset U of $[N]$, and a $(4,3)$-score on $[N]$ with respect to U. Although the algorithm given by Büchi's Theorem certainly makes it possible to settle that question in principle (for fixed m and n), the time and space requirements of that particular assignment ($m = 4$ and $n = 3$) are far too large.

B. Choir in the cathedral: a portrait view of $\mathsf{N}_5 \,\square\, \mathsf{B}(3,2)$

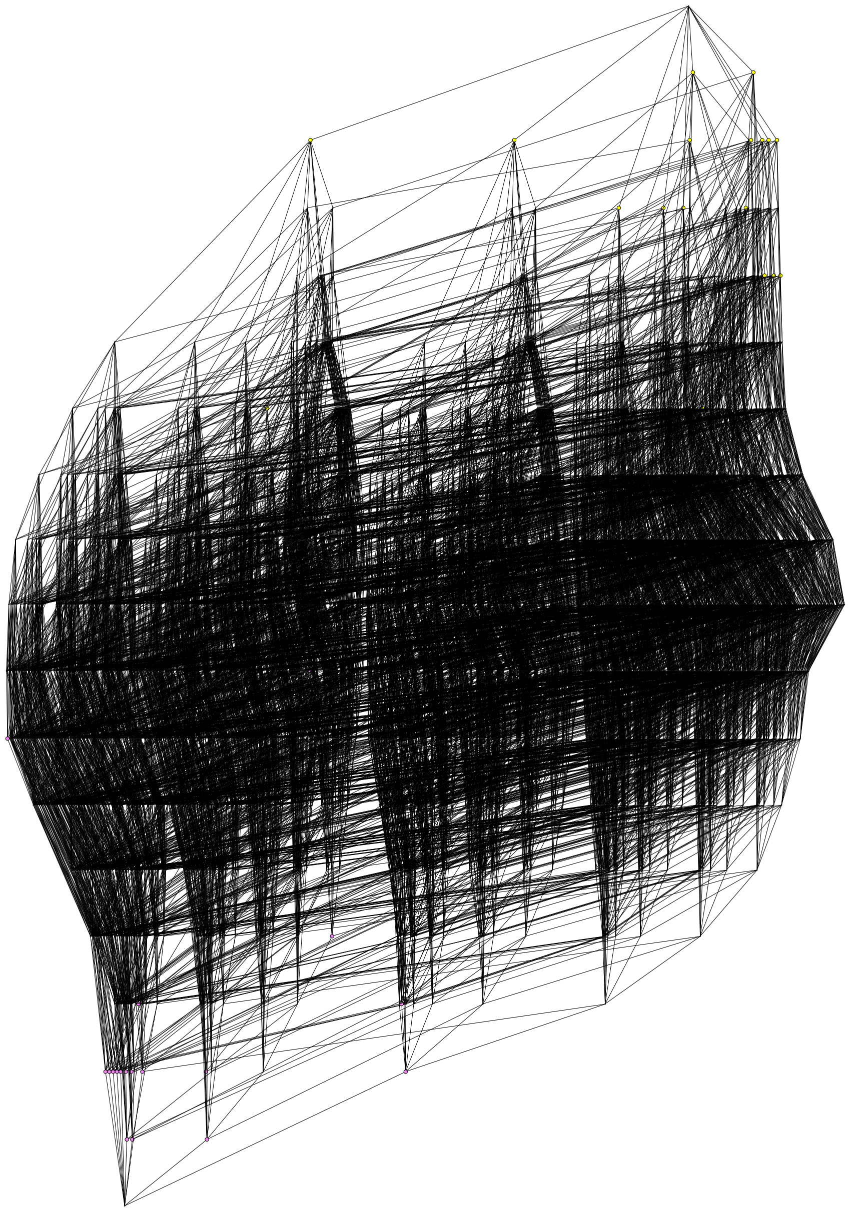

Fig. B.1. The lattice $\mathsf{N}_5 \,\square\, \mathsf{B}(3,2)$ (portrait).

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