J. Eur. Math. Soc. 20, 2005–2062

DOI 10.4171/JEMS/805

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Statistical properties of quadratic polynomials with a neutral fixed point

Received July 28, 2015

Abstract. We describe the statistical properties of the dynamics of the quadratic polynomials $P_\alpha(z) = e^{2\pi\alpha\mathbf{i}}z + z^2$ on the complex plane, with α of high type. In particular, we show that these maps are uniquely ergodic on their measure-theoretic attractors, and the unique invariant probability is a physical measure describing the statistical behaviour of typical orbits in the Julia set. This confirms a conjecture of Pérez-Marco on the unique ergodicity of hedgehog dynamics, in this class of maps.

Keywords. Small divisors, statistical behaviour, unique ergodicity, hedgehog dynamics, small cycles

Introduction

In this paper we are interested in the asymptotic distribution of the orbits of quadratic polynomials

$$P_\alpha(z) = e^{2\pi\alpha\mathbf{i}}z + z^2$$

acting on the complex plane, for irrational values of α.

When considering conservative dynamical systems (i.e., those preserving a smooth density), the Ergodic Theorem ensures the existence of a basic statistical description (stationarity) of typical (with respect to Lebesgue measure) orbits, and the initial focus of analysis tends to be the nature of the ergodic decomposition of Lebesgue measure. However, for non-conservative dynamical systems, stationarity is far from ensured in principle. A nice situation emerges when one can identify *physical measures* μ, which describe the behaviour of large subsets of the phase space, in the sense that their basins (the set of orbits whose Birkhoff averages of any continuous observable are given by the spatial average with respect to μ) have positive Lebesgue measure. Ideally, one would like to be able to describe the behaviour of almost every orbit using one of only finitely many

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Mathematics Subject Classification (2010): Primary 37F50; Secondary 28D05, 30D05, 37F25, 37A50

physical measures: indeed, that this arises frequently is the main content of the famous Palis conjecture [Pal00].

In the case of rational maps, the analysis is rather simple in the Fatou set (in particular, all orbits exhibit stationary behaviour), but the behaviour in the Julia set is much less well understood. When the Julia set is not the whole Riemann sphere (e.g., for polynomial maps), a general theorem of Lyubich [Lyu83] ensures that the ω-limit set of almost every orbit is contained in the *postcritical* set $\mathcal{PC}$, defined as the closure of the forward orbits of all critical values contained in the Julia set, so it is of importance to understand the asymptotic distribution of orbits in the postcritical set. Particularly, if all orbits in $\mathcal{PC}$ admit a common asymptotic distribution μ (i.e., the dynamics restricted to the postcritical set is uniquely ergodic), then almost every orbit in the Julia set must have this same asymptotic distribution, and μ will be a physical measure provided the Julia set has positive Lebesgue measure.

Of course, in many cases the Julia sets of quadratic polynomials have zero Lebesgue measure, and this is true in particular for almost every irrational α [PZ04]. However, it has been shown by Buff and Chéritat [BC06, BC12] that for some irrational values of α the Julia set of P_α has positive Lebesgue measure. Their analysis depends on a renormalization approach introduced by Inou and Shishikura [IS06] to control the postcritical set of P_α whenever α is of *high type*: $\alpha \in HT_N$ for N suitably large. Here the class HT_N is defined in terms of the modified continued fraction expansion of α as the set of all

$$\alpha = a_{-1} + \cfrac{\varepsilon_0}{a_0 + \cfrac{\varepsilon_1}{a_1 + \cfrac{\varepsilon_2}{a_2 + \cdots}}}$$

with $a_{-1} \in \mathbb{Z}$, $\varepsilon_i = \pm 1$ and $a_i \geq N \geq 2$, $i \geq 0$.

A systematic approach to study the dynamics of the quadratics P_α using Inou–Shishikura renormalization has been laid out by Cheraghi [Che10]. A new analytic technique is introduced in [Che13] to prove optimal estimates on the changes of coordinates (Fatou coordinates) that appear in the renormalization. These have led to numerous important results on the dynamics of the quadratics P_α with α of high type. In particular, the combination of [Che10] and [Che13] provides a detailed geometric description of the postcritical sets $\mathcal{PC}(P_\alpha)$ (implying zero area), and explains the topological behaviour of the orbits of these maps. Here, we make use of the Inou–Shishikura renormalization as well as of the optimal estimates on the changes of coordinates established in [Che10, Che13] in order to describe the asymptotic distribution of the orbits of P_α.

Main Theorem 0.1 (unique ergodicity). *There exists $N \geq 2$ such that for $\alpha \in HT_N$, $P_\alpha : \mathcal{PC}(P_\alpha) \to \mathcal{PC}(P_\alpha)$ is uniquely ergodic.*

For the rest of this section, we fix N as in the previous theorem.

Corollary 0.2. *For every $\alpha \in HT_N$, if $J(P_\alpha)$ has positive Lebesgue measure then the map $P_\alpha : J(P_\alpha) \to J(P_\alpha)$ admits a unique physical measure, which describes the behaviour of almost every orbit in $J(P_\alpha)$.*

While the use of the Inou–Shishikura renormalization operator of course restricts the set of parameters that we can address, it is believed that some renormalization operator with similar qualitative features should be available through the whole class of irrational numbers, so the arguments developed here might eventually be applied to the general case. Irrational rotations of dynamically significant types such as bounded type, Brjuno, non-Brjuno, Herman, non-Herman, and Liouville are present in HT_N. Less speculatively, considering the class HT_N is enough to conclude (due to the work of Buff–Chéritat) that Corollary 0.2 is indeed meaningful, in the sense that it applies non-trivially to some quadratic polynomials.

In order to identify precisely the unique invariant measure on the postcritical set, we need to discuss in more detail the dynamics of the maps P_α near the origin. One can identify two basic cases with fundamentally different behaviour, distinguished by whether zero belongs to $\mathcal{PC}(P_\alpha)$ or not, which turns out to depend on the local dynamics of P_α at zero. The map P_α is called *linearizable* at zero if there exists a local conformal change of coordinate ϕ fixing zero and conjugating P_α to the rotation of angle $2\pi\alpha$ around zero: $R_\alpha \circ \phi = \phi \circ P_\alpha$ on the domain of ϕ, where $R_\alpha(z) = e^{2\pi\alpha\mathbf{i}}z$. It was a nontrivial problem to determine the values of α for which P_α is linearizable. We refer the interested reader to [Mil06] for its rich history, and only need to mention that the answer to this problem depends on the arithmetic nature of α. For Lebesgue almost every $\alpha \in [0, 1]$ such a linearization exists, and on the other hand, for a generic choice of $\alpha \in [0, 1]$, P_α is not linearizable. When P_α is linearizable at zero, the maximal domain on which P_α is conjugate to a rotation is called the *Siegel disk* of P_α.

By a result of Mañé [Mañ87], the orbit of the critical point is recurrent, and when P_α is linearizable, this orbit accumulates on the whole boundary of the Siegel disk, while when P_α is not linearizable, the orbit accumulates on zero. In either case, this allows us to construct natural invariant measures supported in the postcritical set: in the linearizable case, one takes the harmonic measure on the boundary of the Siegel disk, viewed from zero, while in the non-linearizable case, one takes the Dirac mass at the origin. It is worth pointing out that Buff and Chéritat construct examples of Julia sets of positive Lebesgue measure in both linearizable and non-linearizable cases, so there are indeed physical measures of both types.

Theorem 0.1 implies that there is no periodic point in $\mathcal{PC}(P_\alpha)$, except possibly zero. Here, we prove an interesting property of the dynamics of linearizable maps, a counterpart of the *small cycle property* of non-linearizable quadratics obtain by Yoccoz [Yoc95b]. That is, when P_α is not linearizable at 0, every neighbourhood of 0 contains infinitely many periodic cycles of P_α.

Main Theorem 0.3 (asymptotic cycles). *When P_α is linearizable at 0 for some $\alpha \in HT_N$, every neighbourhood of the Siegel disk of P_α contains infinitely many periodic cycles of P_α.*

We also prove

Main Theorem 0.4. *For every $\alpha \in HT_N$, $P_\alpha : \mathcal{PC}(f) \to \mathcal{PC}(f)$ is one-to-one.*

Our results can also be seen as providing insight into the dynamics of so-called "hedgehogs". Let f be a holomorphic germ defined on a neighbourhood of zero with $f(0) = 0$ and $f'(0) = e^{2\pi\alpha\mathbf{i}}$ for an irrational α. Pérez-Marco [PM97] introduced local invariant sets for f, called *Siegel compacta*, that were used to study the local dynamics of f at zero. More precisely, he proves that for f as above and a Jordan domain $U \ni 0$ such that f and its inverse are one-to-one on a neighbourhood of the closure of U, there exists a unique compact connected invariant set K with $0 \in K \subseteq \overline{U}$ and $K \cap \partial U \neq \emptyset$. A distortion property of the iterates of f on U is translated into the unique ergodicity of $f : \partial K \to \partial K$, which was conjectured to hold in this generality. When f is not linearizable at zero, K has no interior ($\partial K = K$) and is called the *hedgehog* of f on U. In Section 4, we show that the boundary of every Siegel compactum of P_α must be either an invariant curve in the Siegel disk of P_α or a subset of $\mathcal{PC}(P_\alpha)$. Hence, we the following partial result on this conjecture.[1]

Main Theorem 0.5 (hedgehog dynamics). *For every $\alpha \in HT_N$ and every Siegel compacta K of P_α, the map $P_\alpha : \partial K \to \partial K$ is uniquely ergodic.*

Besides the quadratic polynomials, all results of this paper also apply to maps in the Inou–Shishikura class (see Section 1.1 for the definition) with rotation numbers of high type. In particular, one infers the appropriately interpreted statements for a large class of rational maps.

It is worth noting that $\mathcal{PC}(P_\alpha)$ may have a complicated topology, such as being non-locally connected. In [Che17], the second author establishes a complete topological description of $\mathcal{PC}(P_\alpha)$ for $\alpha \in HT_N$. That is, $\mathcal{PC}(P_\alpha)$ is either a closed Jordan curve, a one-sided hairy circle, or a Cantor bouquet.

The result of Inou and Shishikura [IS06] and the analytic technique introduced in [Che13] have led to recent major advances on the dynamics of quadratic polynomials, and their numerous applications are still being harvested. They have been used to confirm a fine relation between the sizes of the Siegel disks and the arithmetic of the rotation α in [CC15], and have resulted in a breakthrough on the local connectivity of the Mandelbrot set [CS15] (see also [CP17]). Most of the current paper is devoted to analysing the delicate relation between the arithmetic of α and the geometry of the renormalization tower. Indeed, once we carry out this analysis, the proofs of the above theorems only occupy two to three pages each. We expect our analysis of this interaction will help answering the remaining questions concerning the dynamics of the quadratic polynomials P_α.

1. Near-parabolic renormalization and an invariant class

1.1. Inou–Shishikura class

Consider the cubic polynomial $P(z) = z(1+z)^2$. We have $P(0) = 0$ and $P'(0) = 1$. Also, P has a critical point at $\mathrm{cp}_P = -1/3$ which is mapped to the critical value at $\mathrm{cv}_P = -4/27$, and another critical point at -1 which is mapped to zero.

[1] The conjecture was announced at several workshops around 1995. It has been mistakenly reported in [Yoc99, Section 3.4] that it has been proved in full generality by R. Pérez-Marco. However, our communication with him confirms that there was never any proof of this statement.

Consider the filled-in ellipse

$$E = \left\{ x + \mathbf{i}y \in \mathbb{C} \;\middle|\; \left(\frac{x + 0.18}{1.24}\right)^2 + \left(\frac{y}{1.04}\right)^2 \le 1 \right\},$$

and let

$$U = g(\hat{\mathbb{C}} \setminus E), \quad \text{where} \quad g(z) = -\frac{4z}{(1+z)^2}. \tag{1.1}$$

The domain U contains 0 and cp_P, but not the other critical point of P at -1.

Following [IS06], we define the class of maps

$$\mathcal{IS} = \left\{ h = P \circ \varphi^{-1} \colon U_h \to \mathbb{C} \;\middle|\; \begin{array}{l} \varphi \colon U \to U_h \text{ is univalent onto, } \varphi(0) = 0,\ \varphi'(0) = 1, \\ \text{and } \varphi \text{ has a quasi-conformal extension onto } \mathbb{C} \end{array} \right\}.$$

Every map h in $\mathcal{IS}$ has a fixed point of multiplier one at 0, and a unique critical point at $\mathrm{cp}_h = \varphi(-1/3) \in U_h$ with $\mathrm{cv}_h = -4/27$. Elements of $\mathcal{IS}$ have the same covering structure as the one of P on U.

For $\alpha \in \mathbb{R}$, let R_α denote the rotation of angle α about zero: $R_\alpha(z) = e^{2\pi\alpha\mathbf{i}}z$. By precomposing the elements of $\mathcal{IS}$ with rotations R_α, $\alpha \in \mathbb{R}$, we define the classes of maps

$$\mathcal{IS}_\alpha = \{h \circ R_\alpha \mid h \in \mathcal{IS}\}.$$

Let us also normalize the quadratic family to the form

$$Q_\alpha(z) = e^{2\pi\alpha\mathbf{i}}z + \tfrac{27}{16}e^{4\pi\alpha\mathbf{i}}z^2,$$

so that it has a fixed point of multiplier $e^{2\pi\alpha\mathbf{i}}$ at zero, and its critical value is $-4/27$.

Consider a holomorphic map $h\colon \operatorname{Dom} h \to \mathbb{C}$, where $\operatorname{Dom} h$ denotes the domain of definition (always assumed to be open) of h. Given a compact set $K \subset \operatorname{Dom} h$ and an $\varepsilon > 0$, a neighbourhood of h (in the compact-open topology) is defined as the set of holomorphic maps $g\colon \operatorname{Dom} g \to \mathbb{C}$ such that $K \subset \operatorname{Dom} g$ and $|g(z) - h(z)| < \varepsilon$ for all $z \in K$. Then a sequence $h_n\colon \operatorname{Dom} h_n \to \mathbb{C}$, $n = 1, 2, \ldots$, *converges* to h if for every neighbourhood of h defined as above, h_n is contained in that neighbourhood for sufficiently large n. Note that the maps h_n are not necessarily defined on the same set.

Every $h \in \mathcal{IS}_\alpha$, with $h(z) = f_0(e^{2\pi\alpha\mathbf{i}}z)$ for some $f_0 \in \mathcal{IS}$ and $\alpha \in \mathbb{R}$, fixes 0 with multiplier $h'(0) = e^{2\pi\alpha\mathbf{i}}$. Provided α is small enough and non-zero, h has a non-zero fixed point in $\operatorname{Dom} h$, denoted by σ_h, that has split from 0 at $\alpha = 0$. The fixed point σ_h depends continuously on f_0 and α, with asymptotic expansion $\sigma_h = -4\pi\alpha\mathbf{i}/f_0''(0) + o(\alpha)$ as $\alpha \to 0$. Clearly, $\sigma_h \to 0$ as $\alpha \to 0$. Indeed, the choices of the domain U and the polynomial P guarantee (using the area theorem) that $f_0''(0)$ is uniformly bounded away from 0.

Lemma 1.1 ([IS06]). *The set $\{h''(0) \mid h \in \mathcal{IS}\}$ is relatively compact in $\mathbb{C} \setminus \{0\}$.*

We summarize the basic local dynamics of maps in $\mathcal{IS}_\alpha$, for small α, in the following theorem. See Figure 1.

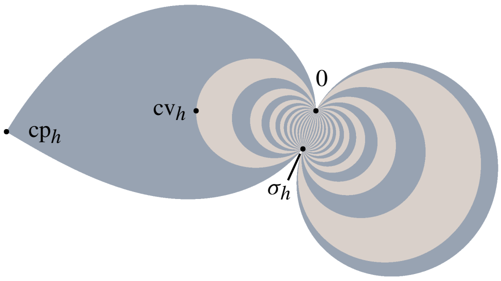

Fig. 1. The domain $\mathcal{P}_h$ and the special points associated to some $h \in \mathcal{IS}_\alpha$. The map Φ_h sends each coloured croissant to an infinite vertical strip of width one.

Theorem 1.2 (Inou–Shishikura [IS06]). *There exists $\alpha_* > 0$ such that for every map $h: U_h \to \mathbb{C}$ in $\mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $\alpha \in (0, \alpha_*]$, there exist a Jordan domain $\mathcal{P}_h \subset U_h$ and a univalent map $\Phi_h: \mathcal{P}_h \to \mathbb{C}$ satisfying the following properties:*

(a) *The domain $\mathcal{P}_h$ is bounded by piecewise smooth curves and is compactly contained in U_h. Moreover, cp_h, 0, and σ_h belong to the boundary of $\mathcal{P}_h$, while cv_h belongs to the interior of $\mathcal{P}_h$.*
(b) *$\Phi_h(\mathcal{P}_h)$ contains the set $\{w \in \mathbb{C} \mid \mathrm{Re}\, w \in (0, 1]\}$.*
(c) *$\mathrm{Im}\, \Phi_h(z) \to +\infty$ as $p_h \ni z \to 0$, and $\mathrm{Im}\, \Phi_h(z) \to -\infty$ as $\mathcal{P}_h \ni z \to \sigma_h$.*
(d) *Φ_h satisfies the Abel functional equation on $\mathcal{P}_h$, that is,*

$$\Phi_h(h(z)) = \Phi_h(z) + 1 \qquad \textit{whenever } z \textit{ and } h(z) \textit{ belong to } \mathcal{P}_h.$$

(e) *The map Φ_h satisfying the above properties is unique, once normalized by setting $\Phi_h(\mathrm{cp}_h) = 0$. Moreover, the normalized map Φ_h depends continuously on h.*

In the above theorem and in the following statements, if $h = Q_\alpha$, the U_h is taken to be $\mathbb{C}$.

The class $\mathcal{IS}$ is denoted by $\mathcal{F}_1$ in [IS06]. The properties listed in the above theorem follow from [IS06, Theorem 2.1 and Main Theorems 1 and 3]. We state some crucial geometric properties of the domains $\mathcal{P}_h$ in the following proposition. See [Che10, Prop. 1.4] or [BC12, Prop. 12] for different proofs of it.

Proposition 1.3 ([Che10], [BC12]). *There exist $\alpha'_* > 0$ and positive integers $\mathbf{k}, \mathbf{k}'$ such that for every map $h: U_h \to \mathbb{C}$ in $\mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $\alpha \in (0, \alpha'_*]$, the domain $\mathcal{P}_h \subset U_h$ in the above theorem may be chosen to satisfy the additional properties:*

(a) *there exists a continuous branch of argument defined on $\mathcal{P}_h$ such that*

$$\max_{w, w' \in \mathcal{P}_h} |\arg(w) - \arg(w')| \le 2\pi \mathbf{k}',$$

(b) *$\Phi_h(\mathcal{P}_h) = \{w \in \mathbb{C} \mid 0 < \mathrm{Re}\, w < \alpha^{-1} - \mathbf{k}\}$.*

The map $\Phi_h: \mathcal{P}_h \to \mathbb{C}$ obtained in the above theorem is called the *perturbed Fatou coordinate*, or the *Fatou coordinate* for short, of h. In this paper, by this coordinate we mean the map $\Phi_h: \mathcal{P}_h \to \mathbb{C}$, where $\mathcal{P}_h$ satisfies the extra properties in the above proposition. See Figure 1.

1.2. Near-parabolic renormalization

Let $h\colon U_h \to \mathbb{C}$ be in $\mathcal{IS}_\alpha \cup \{Q_\alpha\}$, with $\alpha \in (0, \alpha'_*]$, and let $\Phi_h\colon \mathcal{P}_h \to \mathbb{C}$ denote the Fatou coordinate of h defined in the previous section. Define

$$\begin{aligned} \mathcal{C}_h &= \{z \in \mathcal{P}_h \mid 1/2 \le \operatorname{Re}\Phi_h(z) \le 3/2,\ -2 < \operatorname{Im}\Phi_h(z) \le 2\}, \\ \mathcal{C}_h^\sharp &= \{z \in \mathcal{P}_h \mid 1/2 \le \operatorname{Re}\Phi_h(z) \le 3/2,\ 2 \le \operatorname{Im}\Phi_h(z)\}. \end{aligned} \tag{1.2}$$

By definition, $\mathrm{cv}_h \in \operatorname{int}(\mathcal{C}_h)$ and $0 \in \partial(\mathcal{C}_h^\sharp)$. We also assume that α is small enough that $1/\alpha - \mathbf{k} \ge 3/2$ (see (1.8)). See Figure 2.

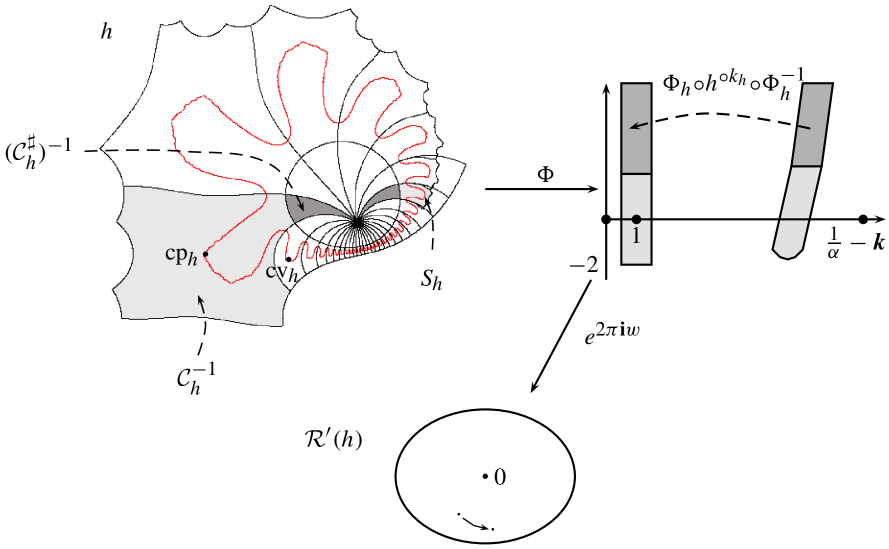

Fig. 2. The figure shows the sets $\mathcal{C}_h, \mathcal{C}_h^\sharp, \dots, \mathcal{C}_h^{-k_h}, (\mathcal{C}_h^\sharp)^{-k_h}$, and the sector S_h. The induced map $\Phi_h \circ h^{\circ k_h} \circ \Phi_h^{-1}$ projects via $e^{2\pi \mathbf{i} w}$ to a well-defined map $\mathcal{R}(h)$ on a neighbourhood of 0. The amoeba curve around 0 is a large number of iterates of cp_h under h.

Assume for a moment that there exists a positive integer k_h, depending on h, such that the following four properties hold:

- For every integer k with $1 \le k \le k_h$, there exists a unique connected component of $h^{-k}(\mathcal{C}_h^\sharp)$ which is compactly contained in $\operatorname{Dom} h$ and contains 0 on its boundary. We denote this component by $(\mathcal{C}_h^\sharp)^{-k}$.
- For every integer k with $1 \le k \le k_h$, there exists a unique connected component of $h^{-k}(\mathcal{C}_h)$ which has non-empty intersection with $(\mathcal{C}_h^\sharp)^{-k}$, and is compactly contained in $\operatorname{Dom} h$. This component is denoted by $\mathcal{C}_h^{-k}$.
- The sets $\mathcal{C}_h^{-k_h}$ and $(\mathcal{C}_h^\sharp)^{-k_h}$ are contained in[2]

$$\{z \in \mathcal{P}_h \mid 1/2 < \operatorname{Re}\Phi_h(z) < \langle 1/\alpha \rangle - \mathbf{k} - 1/2\}.$$

[2] The notation $\langle r \rangle$ stands for the integer closest to $r \in \mathbb{R}$.

- The maps $h\colon \mathcal{C}_h^{-k} \to \mathcal{C}_h^{-k+1}$, for $2 \le k \le k_h$, and $h\colon (\mathcal{C}_h^\sharp)^{-k} \to (\mathcal{C}_h^\sharp)^{-k+1}$, for $1 \le k \le k_h$, are univalent. The map $h\colon \mathcal{C}_h^{-1} \to \mathcal{C}_h$ is a degree two proper branched covering.

Let k_h denote the smallest positive integer for which the above conditions hold, and define

$$S_h = \mathcal{C}_h^{-k_h} \cup (\mathcal{C}_h^\sharp)^{-k_h}.$$

Consider the map

$$E_h = \Phi_h \circ h^{\circ k_h} \circ \Phi_h^{-1}\colon \Phi_h(S_h) \to \mathbb{C}. \tag{1.3}$$

By the functional equation in Theorem 1.2(d), $E_h(z+1) = E_h(z) + 1$ when both z and $z+1$ belong to the boundary of $\Phi_h(S_h)$. Hence, E_h projects via $z = \frac{-4}{27} e^{2\pi \mathbf{i} w}$ to a map $\mathcal{R}(h)$ defined on a set containing a punctured neighbourhood of 0. However, zero is a removable singularity of this map, and one can see that $\mathcal{R}(h)$ must be of the form

$$z \mapsto e^{2\pi \frac{-1}{\alpha} \mathbf{i}} z + O(z^2)$$

near zero. The map $\mathcal{R}(h)$, restricted to the interior of $\frac{-4}{27} e^{2\pi \mathbf{i}(\Phi_h(S_h))}$, is called the *near-parabolic renormalization* of h. We may simply refer to the near-parabolic renormalization as *renormalization* for short.[3] Note that Φ_h maps the critical value of h to 1, and the projection $w \mapsto \frac{-4}{27} e^{2\pi \mathbf{i} w}$ maps integers to $-4/27$. Thus, the critical value of $\mathcal{R}(h)$ is $-4/27$. See Figure 2.

Define

$$V = P^{-1}\big(B\big(0, \tfrac{4}{27} e^{4\pi}\big)\big) \setminus ((-\infty, -1] \cup B) \tag{1.4}$$

where B is the component of $P^{-1}\big(B\big(0, \frac{4}{27} e^{-4\pi}\big)\big)$ containing -1. By an explicit calculation (see [IS06, Prop. 5.2]) one can see that the closure of U is contained in the interior of V. See Figure 3.

The following theorem [IS06, Main Thm. 3] guarantees that the above definition of renormalization $\mathcal{R}$ can be carried out for certain perturbations of maps in $\mathcal{IS}$. In particular, this implies the existence of k_h satisfying the four properties needed for the definition of renormalization.

Theorem 1.4 (Inou–Shishikura). *There exists a constant $\alpha^* > 0$ such that if $h \in \mathcal{IS}_\alpha$ with $\alpha \in (0, \alpha^*]$, then $\mathcal{R}(h)$ is well-defined and belongs to the class $\mathcal{IS}_{-1/\alpha}$. That is, there exists a univalent map $\psi : U \to \mathbb{C}$ with $\psi(0) = 0$ and $\psi'(0) = 1$ such that*

$$\mathcal{R}(h)(z) = P \circ \psi^{-1}(e^{\frac{-2\pi}{\alpha}\mathbf{i}} z), \qquad \forall z \in \psi(U) \cdot e^{\frac{2\pi}{\alpha}\mathbf{i}}.$$

Furthermore, $\psi : U \to \mathbb{C}$ extends to a univalent map on V.

Similarly, for $\alpha \in (0, \alpha^*]$, $\mathcal{R}(Q_\alpha)$ is well-defined and belongs to $\mathcal{IS}_{-1/\alpha}$.

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[3] Inou and Shishikura give a somewhat different definition of this renormalization operator using slightly different regions $\mathcal{C}_h$ and $\mathcal{C}_h^\sharp$. However, the resulting maps $\mathcal{R}(h)$ are the same modulo their domains of definition. More precisely, there is a natural extension of Φ_h onto the sets $\mathcal{C}_h^{-k} \cup (\mathcal{C}_h^\sharp)^{-k}$, for $0 \le k \le k_h$, such that each set $\Phi_h(\mathcal{C}_h^{-k} \cup (\mathcal{C}_h^\sharp)^{-k})$ is contained in the union

$$D_{-k}^\sharp \cup D_{-k} \cup D''_{-k} \cup D'_{-k+1} \cup D_{-k+1} \cup D_{-k+1}^\sharp$$

in the notation of [IS06, Section 5.A].

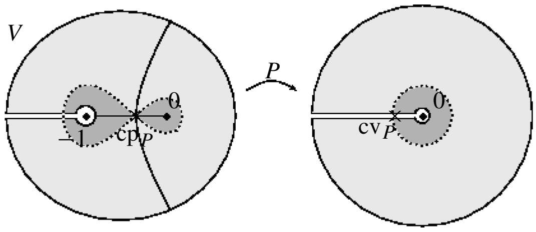

Fig. 3. A schematic representation of the polynomial P; its domain and its range. Similar colours and line styles are mapped on one another.

For $h \in \mathcal{IS}_\alpha$ or $h = Q_\alpha$ with $\alpha \in [-\alpha^*, 0)$, the conjugate map $\hat{h} = s \circ h \circ s$, where $s(z) = \bar{z}$ is complex conjugation, satisfies $\hat{h}(0) = 0$ and $\hat{h}'(0) = e^{-2\pi\alpha\mathbf{i}}$. Since the class $\mathcal{IS}$ is invariant under conjugation by s, $\hat{h}$ belongs to $\mathcal{IS}_{-\alpha} \cup \{Q_{-\alpha}\}$. In particular, by the above theorems the near-parabolic renormalization of $\hat{h}$ is defined. Through this, we extend the domain of definition of $\mathcal{R}$ to contain the maps $h \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$, $\alpha \in [-\alpha^*, 0)$.

The near-parabolic renormalization and a small variation of Theorem 1.4 are extended to cover unisingular holomorphic maps in [Ché14], and for cubic maps in [Yan15]. A numerical study of the parabolic renormalization of Inou–Shishikura has been carried out in [LY11].

1.3. Fatou coordinates

In this section we state some basic properties of the Fatou coordinates that will be used throughout this paper. One may refer to [Che10] for detailed arguments. We say that a smooth curve $\gamma\colon (0,1) \to \mathbb{C} \setminus \{0\}$ *lands at* 0 if $\gamma(t) \to 0$ as $t \to 0$. Moreover, we say that γ lands at 0 *at a well-defined angle* if there is a branch of arg defined on $\gamma(t)$ for all small values of t and $\lim_{t\to 0} \arg \gamma(t)$ exists. The next proposition follows from the estimates on Fatou coordinates in [Che10]. As it is not stated in that paper, we present a proof in Section 3.3.

Proposition 1.5. *For all $h \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $0 < |\alpha| \le \alpha^*$ and every $r \in [0, 1/\alpha - \mathbf{k}]$, the curve $\Phi_h^{-1}(\mathbb{R} + r\mathbf{i})$ lands at 0 at a well-defined angle.*

It follows from the above proposition that for all $h \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $0 < |\alpha| \le \alpha^*$ the sector S_h is bounded by piecewise smooth curves, two of which land at 0 at well-defined angles. Then one can see that for every $h \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $0 < |\alpha| \le \alpha^*$,

$$k_h \ge 2. \tag{1.5}$$

On the other hand, we have the following upper bound on k_h.

Proposition 1.6 ([Che10]). *There exists $\mathbf{k}'' \in \mathbb{Z}$ such that $k_h \le \mathbf{k}''$ for every h in $\mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $0 < |\alpha| \le \alpha^*$.*

Proposition 1.7. *There is a constant $\delta > 0$ such that for every h in $\mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $0 < |\alpha| \le \alpha^*$ and every $z \in \bigcup_{i=0}^{k_h} h^{\circ i}(S_h) \cup \mathcal{P}_h$, we have $|z| \le 1/\delta$, $B_\delta(z) \subset \operatorname{Dom} h$, and $B_\delta(\mathcal{C}_h^{-k_h}) \subset \operatorname{Dom} h \setminus \{0\}$.*

Proof. Fix α satisfying the hypothesis. According to Inou–Shishikura [IS06], for every $h \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ the set $\bigcup_{i=0}^{k_h} h^{\circ i}(S_h) \cup \mathcal{P}_h$ is compactly contained in $\operatorname{Dom} h$. Indeed, they show [IS06, Section 5.N, the value of η] that one may use the set

$$\{z \in \mathcal{P}_h \mid 1/2 \le \operatorname{Re} \Phi_h(z) \le 3/2, -13 \le \operatorname{Im} \Phi_h(z) \le 2\}$$

in place of $\mathcal{C}_h$ (considered here) to define the near-parabolic renormalization of h. That is, the corresponding preimages of this (larger) set are defined and contained in $\operatorname{Dom} h$. In particular, the preimages of the set $\mathcal{C}_h \cup \mathcal{C}_h^\sharp$, up to k_h, are compactly contained in $\operatorname{Dom} h$. Similarly, for every h, the set $\mathcal{C}_h^{-k_h}$ is compactly contained in $\operatorname{Dom} h \setminus \{0\}$. By the pre-compactness of the class $\mathcal{IS}_\alpha$ and the continuous dependence of Φ_h on h, we conclude that there are constants δ and c satisfying the conclusion of the lemma for h in $\mathcal{IS}_\alpha \cup \{Q_\alpha\}$. It remains to see what happens as $\alpha \to 0$.

As $h \in \bigcup_{\alpha\in(0,\alpha^*]} \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ tends to some map $h_0 \in \mathcal{IS}_0 \cup \{Q_0\}$, the Fatou coordinate Φ_h tends to the attracting and repelling Fatou coordinates of h_0. According to Inou and Shishikura [IS06], the sets $\mathcal{C}_{h_0}$, $\mathcal{C}_{h_0}^\sharp$, and their corresponding preimages are defined for all $h_0 \in \mathcal{IS}_0 \cup \{Q_0\}$, with S_{h_0} contained in the domain of the repelling Fatou coordinate of h_0. Moreover, these domains are compactly contained in the domain of h_0. Then, one defines the (parabolic) renormalization of h_0 as in the previous section. By the work of Inou–Shishikura, the Fatou coordinate and the renormalization depend continuously on the map $h \in \bigcup_{\alpha\in[-\alpha^*,\alpha^*]} \mathcal{IS}_\alpha \cup \{Q_\alpha\}$. This implies that there are positive constants δ and c satisfying the properties in the lemma. As this is the only place where we use the (parabolic) renormalization of maps in $\mathcal{IS}_0 \cup \{Q_0\}$, we do not explain this further and refer the reader to [IS06] for more details. □

Let $h \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $0 < |\alpha| < \alpha^*$. Define

$$\mathcal{D}'_h := \Phi_h(\mathcal{P}_h) \cup \bigcup_{j=0}^{k_h+\langle 1/\alpha\rangle-\mathbf{k}-2} (\Phi_h(S_h) + j).$$

Using the dynamics of h we may extend the domain of definition of Φ_h^{-1}.

Lemma 1.8. *The map $\Phi_h^{-1} \colon \Phi_h(\mathcal{P}_h) \to \mathcal{P}_h$ extends to a holomorphic map*

$$\Phi_h^{-1} \colon \mathcal{D}'_h \to \Big(\bigcup_{i=0}^{k_h} h^{\circ i}(S_h) \cup \mathcal{P}_h\Big) \setminus \{0\}$$

such that for all $w \in \mathbb{C}$ with $w, w+1 \in \mathcal{D}'_h$ we have

$$\Phi_h^{-1}(w+1) = h \circ \Phi_h^{-1}(w).$$

Proof. If $z \in S_h$, then for all integers j with $0 \le j \le k_h + \langle 1/\alpha\rangle - \mathbf{k} - 2$, $h^{\circ j}(z)$ is defined and belongs to $\operatorname{Dom} h$. Indeed, since h is near-parabolic renormalizable, the iterates $z, h(z), \ldots, h^{\circ k_h}(z)$ are defined and

$$h^{\circ k_h}(z) \in \mathcal{P}_h, \qquad \operatorname{Re} \Phi_h(h^{\circ k_h}(z)) \in [1/2, 3/2].$$

Then it follows from the conjugacy relation for Φ_h and Proposition 1.3 that for all j with $k_h \le j \le k_h + \langle 1/\alpha \rangle - \mathbf{k} - 2$, $h^{\circ j}(z)$ is defined and belongs to $\mathcal{P}_h$.

Define the map Φ_h^{-1} on $\mathcal{D}'_h$ as

$$\Phi_h^{-1}(w) = \begin{cases} \Phi_h^{-1}(w) & \text{if } 0 < \operatorname{Re} w < \alpha^{-1} - \mathbf{k}, \\ h^{\circ j} \circ \Phi_h^{-1}(w - j) & \text{if } w \in \Phi_h(S_h) + j. \end{cases} \tag{1.6}$$

The conjugacy relation in Theorem 1.2(d) implies that this is a well-defined holomorphic map on $\mathcal{D}'_h$, and satisfies the desired functional equation on $\mathcal{D}'_h$. However, Φ_h^{-1} is not univalent on $\mathcal{D}'_h$. $\square$

Define

$$\mathbb{E}\mathrm{xp}(\zeta) = \tfrac{-4}{27} e^{2\pi \mathbf{i} \zeta}, \quad \mathbb{E}\mathrm{xp}\colon \mathbb{C} \to \mathbb{C} \setminus \{0\}.$$

Then we may lift the maps $\Phi_h^{-1}\colon \mathcal{D}'_h \to \mathbb{C} \setminus \{0\}$ and $s \circ \Phi_h^{-1}\colon \mathcal{D}'_h \to \mathbb{C} \setminus \{0\}$ under the covering map $\mathbb{E}\mathrm{xp}\colon \mathbb{C} \to \mathbb{C} \setminus \{0\}$, that is, there is a map $\chi_h\colon \mathcal{D}'_h \to \mathbb{C}$ such that

$$\forall w \in \mathcal{D}_h, \quad \begin{cases} \mathbb{E}\mathrm{xp} \circ \chi_h(w) = \Phi_h^{-1}(w) & \text{if } \alpha \in (0, 1/2), \\ \mathbb{E}\mathrm{xp} \circ \chi_h(w) = s \circ \Phi_h^{-1}(w) & \text{if } \alpha \in (-1/2, 0). \end{cases}$$

Each χ_h is either holomorphic or anti-holomorphic. The lift χ_h is determined up to translations by integers.

$$\begin{tikzcd} & \mathbb{C} \arrow[d, "\mathbb{E}\mathrm{xp}"] \\ \mathcal{D}'_h \arrow[ur, "\chi_h"] \arrow[r, "s\circ\Phi_h^{-1}"'] & \mathbb{C}\setminus\{0\} \end{tikzcd} \qquad \begin{tikzcd} & \mathbb{C} \arrow[d, "\mathbb{E}\mathrm{xp}"] \\ \mathcal{D}'_h \arrow[ur, "\chi_h"] \arrow[r, "\Phi_h^{-1}"'] & \mathbb{C}\setminus\{0\} \end{tikzcd}$$

Fig. 4. The lift χ_h depends on the sign of α.

Proposition 1.9. *There exists an integer $\hat{\mathbf{k}}$ such that for every $h \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $0 < |\alpha| \le \alpha^*$ and any choice of the lift χ_h we have*

$$\sup\{|\operatorname{Re} w - \operatorname{Re} w'| \mid w, w' \in \chi_h(\mathcal{D}'_h)\} \le \hat{\mathbf{k}}.$$

Proof. By Proposition 1.3, there is a uniform bound on the total spiral of the set $\mathcal{P}_h$ about zero. By the precompactness of the class $\mathcal{IS}_\alpha$, this implies that there is a uniform bound on the total spiral of each set $h^{\circ i}(S_h)$, for $0 \le i \le k_h$, about zero. (See the proof of Proposition 1.7 for further details.) In other words, the lifts of these sets under $\mathbb{E}\mathrm{xp}$ have uniformly bounded horizontal width. Combined with the uniform bound on k_h in Proposition 1.6, this implies the existence of a constant $\hat{\mathbf{k}}$ satisfying the conclusion of the lemma. $\square$

Since $\Phi_h^{-1}(\mathcal{D}'_h)$ is contained in the image of h, for any choice of the lift χ_h and every $w \in \chi_h(\mathcal{D}'_h)$ we must have $\operatorname{Im} w > -2$. Hence, by the above proposition, there is a choice of χ_h, denoted by $\chi_{h,0}$, such that

$$\chi_{h,0}(\mathcal{D}'_h) \subset \{w \in \mathbb{C} \mid 1 \le \operatorname{Re} w \le \hat{\mathbf{k}} + 2, \operatorname{Im} w > -2\}.$$

Define

$$\mathcal{D}_h := \Phi_h(\mathcal{P}_h) \cup \bigcup_{j=0}^{k_h+\hat{\mathbf{k}}+\mathbf{k}+2} (\Phi_h(S_h) + j).$$

Lemma 1.10. *For every $h \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $0 < |\alpha| \le \min\{\alpha^*, 1/(2\mathbf{k} + \hat{\mathbf{k}} + 4.5)\}$,*

$$\mathcal{D}_h \subset \mathcal{D}'_h.$$

Proof. The condition on α guarantees that $\langle 1/\alpha \rangle \ge 2\mathbf{k} + \hat{\mathbf{k}} + 4$. Hence,

$$k_h + \hat{\mathbf{k}} + \mathbf{k} + 2 \le k_h + \langle 1/\alpha \rangle - \mathbf{k} - 2,$$

and by the definitions of the sets $\mathcal{D}_h$ and $\mathcal{D}'_h$, the inclusion follows. □

Remark 1.11. Although the above lemma easily follows from the definitions and the condition imposed on α, it has been emphasized to make it clear where the high type condition is used in this paper. That is, to extend Φ_h^{-1} on $\mathcal{D}_h$, we need that the map h can be iterated at least $k_h + \hat{\mathbf{k}} + \mathbf{k} + 2$ times on S_h. This is crucial if one wishes to prove the main theorems of this article for all rotation numbers using an invariant class under a similar renormalization operator. Note that the constants $\mathbf{k}$ and $\hat{\mathbf{k}}$ depend only on the class of maps invariant under the renormalization.

1.4. Modified continued fractions and infinitely renormalizable maps

For irrational $\alpha \in \mathbb{R}$, define

$$\alpha_0 = d(\alpha, \mathbb{Z}) \quad \text{and} \quad \alpha_{i+1} = d(1/\alpha_i, \mathbb{Z}) \quad \text{for } i \ge 0, \tag{1.7}$$

where d denotes the Euclidean distance on $\mathbb{R}$. Then, choose $a_{-1} \in \mathbb{Z}$ with $\alpha - a_{-1} \in (-1/2, +1/2)$, and $a_i \in \mathbb{Z}$ with

$$1/\alpha_i - a_i \in (-1/2, +1/2) \quad \text{for } i = 0, 1, 2, \dots.$$

Define $\varepsilon_0 = 1$ if $\alpha - a_{-1} \in (0, 1/2)$, and $\varepsilon_0 = -1$ if $\alpha - a_{-1} \in (-1/2, 0)$. Similarly, for $i = 0, 1, 2, \dots$, let

$$\varepsilon_{i+1} = \begin{cases} 1 & \text{if } 1/\alpha_i - a_i \in (0, 1/2), \\ -1 & \text{if } 1/\alpha_i - a_i \in (-1/2, 0). \end{cases}$$

Note that for all $i \ge 0$, $\alpha_i \in (0, 1/2)$ and $a_i \ge 2$. It follows that α is given as the infinite continued fraction

$$\alpha = a_{-1} + \cfrac{\varepsilon_0}{a_0 + \cfrac{\varepsilon_1}{a_1 + \cfrac{\varepsilon_2}{a_2 + \dots}}}.$$

Recall the class HT_N of high type numbers defined in the introduction. Fix an integer

$$N \ge 1/\alpha^* + 1/2.$$

Using the formula $\alpha_{i-1} = 1/(a_{i-1} + \varepsilon_i \alpha_i)$, $\alpha \in HT_N$ implies that for all $i \geq 0$ we have $\alpha_i \in (0, \alpha^*]$. We also need to assume the constant N is large enough that

$$N \geq 2\mathbf{k} + \hat{\mathbf{k}} + 4.5. \tag{1.8}$$

This guarantees that for every $n \geq 0$, we have $\alpha_n \leq 1/(2\mathbf{k} + \hat{\mathbf{k}} + 4.5)$, needed in Lemma 1.10.

Let $\alpha \in HT_N$ and $f_\alpha \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$. Define

$$f_0 = \begin{cases} f_\alpha & \text{if } \varepsilon_0 = +1, \\ s \circ f_\alpha \circ s & \text{if } \varepsilon_0 = -1, \end{cases}$$

where $s(z) = \overline{z}$ denotes complex conjugation. Now, f_0 has asymptotic rotation α_0 in $(0, \alpha^*]$ at 0. Then, by Theorem 1.4, we may inductively define the sequence of maps

$$f_{n+1} = \begin{cases} \mathcal{R}(f_n) & \text{if } \varepsilon_{n+1} = -1, \\ s \circ \mathcal{R}(f_n) \circ s & \text{if } \varepsilon_{n+1} = +1. \end{cases}$$

Let $U_n = U_{f_n}$ denote the domain of definition of f_n for $n \geq 0$. It follows that, for every $n \geq 1$,

$$f_n\colon U_n \to \mathbb{C}, \quad f_n \in \mathcal{IS}_{\alpha_n}, \quad f_n(0) = 0, \quad f_n'(0) = e^{2\pi\alpha_n \mathbf{i}}.$$

The reason for considering the above notion of continued fraction instead of the standard one is that the set of high type numbers in this expansion is strictly bigger than the set of high type numbers in the standard expansion. It is the nature of near-parabolic renormalization that makes this notion of continued fraction more suitable to work with.

2. Symbolic dynamics near the attractor

2.1. Changes of coordinates

Recall the sequence of maps f_n, $n \geq 0$, defined in Section 1.4. In this section we shall define some changes of coordinates between the dynamic planes of these maps. Because of the complex conjugation s that appears in the definition of f_n, extra care is needed in defining these changes of coordinates.

For $n \geq 0$, let $\Phi_n = \Phi_{f_n}$ denote the Fatou coordinate of $f_n\colon U_n \to \mathbb{C}$ defined on the set $\mathcal{P}_n = \mathcal{P}_{f_n}$ (see Theorem 1.2, Proposition 1.3 and the definition after them).

For every $n \geq 0$, let $\mathcal{C}_n$ and $\mathcal{C}_n^\sharp$ denote the corresponding sets for f_n defined in (1.2) (i.e., replace h by f_n). Denote by k_n the smallest positive integer with

$$S_n^0 = \mathcal{C}_n^{-k_n} \cup (\mathcal{C}_n^\sharp)^{-k_n} \subset \{z \in \mathcal{P}_n \mid 1/2 < \operatorname{Re} \Phi_n(z) < a_n - \mathbf{k} - 1/2\}.$$

By definition, the critical value of f_n is contained in $f_n^{\circ k_n}(S_n^0)$. For each $n \geq 0$ we define

$$\mathcal{D}_n := \Phi_n(\mathcal{P}_n) \cup \bigcup_{j=0}^{k_n + \mathbf{k} + \hat{\mathbf{k}} + 2} (\Phi_n(S_n^0) + j).$$

With $h = f_n$ in Section 1.3 and Lemma 1.10, we have holomorphic maps

$$\Phi_n^{-1} \colon \mathcal{D}_n \to \operatorname{Dom} f_n \setminus \{0\}$$

such that

$$\Phi_n^{-1}(w+1) = f_n \circ \Phi_n^{-1}(w)$$

for all w and $w+1$ in $\mathcal{D}_n$.

We denote the corresponding lifts $\chi_{f_n,0}$ by $\chi_{n,0}$. Then, for $n \geq 1$,

$$\forall w \in \mathcal{D}_n, \qquad \begin{cases} \mathbb{E}\mathrm{xp} \circ \chi_{n,0}(w) = \Phi_n^{-1}(w) & \text{if } \varepsilon_n = -1, \\ \mathbb{E}\mathrm{xp} \circ \chi_{n,0}(w) = s \circ \Phi_n^{-1}(w) & \text{if } \varepsilon_n = +1, \end{cases}$$

and

$$\chi_{n,0}(\mathcal{D}_n) \subset \{w \in \mathbb{C} \mid 1 \leq \operatorname{Re} w \leq \hat{\mathbf{k}} + 2, \operatorname{Im} w > -2\} \subset \Phi_{n-1}(\mathcal{P}_{n-1}). \tag{2.1}$$

Each $\chi_{n,0}$ is either holomorphic or anti-holomorphic, depending on the sign of ε_n. Define $\chi_{n,i} = \chi_{n,0} + i$ for $i \in \mathbb{Z}$. The condition (1.8) on α implies that for all $n \geq 1$ and all integers i with $0 \leq i \leq a_{n-1}$, we have

$$\chi_{n,i} \colon \mathcal{D}_n \to \mathcal{D}_{n-1}. \tag{2.2}$$

Indeed, we show in the next lemma that a stronger form of inclusion holds.

Lemma 2.1. *There exists a constant $\delta_0 > 0$, depending only on the class $\mathcal{IS}$, such that for every $n \geq 1$ and every i with $0 \leq i \leq a_{n-1}$, we have*

$$\forall w \in \chi_{n,i}(\mathcal{D}_n), \qquad B_{\delta_0}(w) \subset \mathcal{D}_{n-1}.$$

Proof. By (2.1), $\chi_{n,0}(\mathcal{D}_n) \subset \mathcal{D}_{n-1}$. By Proposition 1.7, $B_\delta(\Phi_n^{-1}(\mathcal{D}_n))$ is contained in the domain of f_n and $\mathbb{E}\mathrm{xp} \circ \Phi_{n-1}(S_{n-1}^0) = \operatorname{Dom} f_n$. This implies that there is a uniform constant δ_0 such that

$$B_{\delta_0}(\chi_{n,0}(\mathcal{D}_n)) \subset \Phi_{n-1}(S_{n-1}^0) + \mathbb{Z}.$$

On the other hand, by the first inclusion in (2.1) and the lower bound on k_n in (1.5), for all $n \geq 1$, all integers i with $0 \leq i \leq a_n$, and all $w \in \chi_{n,i}(\mathcal{D}_n)$ we must have

$$1 \leq \operatorname{Re} w \leq \hat{\mathbf{k}} + 2 + a_n \leq \hat{\mathbf{k}} + k_n + a_n < (a_n - \mathbf{k} - 1/2) + k_n + \mathbf{k} + \hat{\mathbf{k}} + 1/2.$$

This finishes the proof of the proposition by making α_0 less than or equal to $1/2$. □

For $n \geq 1$, define $\psi_n \colon \mathcal{P}_n \to \mathcal{P}_{n-1}$ as

$$\psi_n = \Phi_{n-1}^{-1} \circ \chi_{n,0} \circ \Phi_n.$$

Each ψ_n is a holomorphic or anti-holomorphic map, depending on the sign of ε_n, and extends continuously to $0 \in \partial \mathcal{P}_n$ by mapping it to 0. Define the compositions

$$\begin{aligned} \Psi_1 &= \psi_1 \colon \mathcal{P}_1 \to \mathcal{P}_0, \\ \Psi_2 &= \psi_1 \circ \psi_2 \colon \mathcal{P}_2 \to \mathcal{P}_0, \\ \Psi_n &= \psi_1 \circ \cdots \circ \psi_n \colon \mathcal{P}_n \to \mathcal{P}_0 \quad \text{for } n \geq 3. \end{aligned}$$

These are holomorphic or anti-holomorphic maps, depending on the sign of $(-1)^n \varepsilon_1 \dots \varepsilon_n$.

For every $n \geq 0$ and $i \geq 2$, define the sectors

$$S_n^1 = \psi_{n+1}(S_{n+1}^0) \subset \mathcal{P}_n, \qquad S_n^i = \psi_{n+1} \circ \cdots \circ \psi_{n+i}(S_{n+i}^0) \subset \mathcal{P}_n \quad \text{for } i \geq 2.$$

All these sectors contain 0 on their boundaries. For the reader's convenience, the lower index of each map ψ_n, Ψ_n, $\chi_{n,i}$ determines the level of its domain of definition, that is, for example, ψ_n is defined on a set that is on the dynamic plane of f_n. Similarly, the set S_n^i is contained in the dynamic plane of f_n (and is at depth i). However, we shall mainly work with S_i^0 and S_0^i for $i \geq 0$.

There are two collections of changes of coordinates ψ_n and Ψ_n, for $n \geq 1$, as well as $\chi_{n,i}$, for $n \geq 1$ and $0 \leq i \leq a_n$, that function in parallel. The former are more convenient for the combinatorial study of the dynamics of the map using the tower of maps f_n. This is presented in Sections 2.2 and 2.3. The latter set of changes of coordinates are more suitable for the analytic aspects of the associated problems. This analysis appears in Section 3. The relations between the two collections are discussed in Section 2.4.

2.2. *Orbit relations*

By the definition of renormalization, iterating $\mathcal{R}(h)$ once corresponds to iterating h several times, via the changes of coordinates between the dynamic planes of h and $\mathcal{R}(h)$. This is made more precise in the next lemma.

Lemma 2.2. *Let $n \geq 0$ and $z \in \mathcal{P}_n$ be a point with $w = \mathbb{E}\mathrm{xp} \circ \Phi_n(z) \in \operatorname{Dom} \mathcal{R}(f_n)$. There exists an integer ℓ_z with $1 \leq \ell_z \leq a_n - \mathbf{k} + k_n - 1/2$ such that*

- *the orbit $z, f_n(z), f_n^{\circ 2}(z), \dots, f_n^{\circ \ell_z}(z)$ is defined, and $f_n^{\circ \ell_z}(z) \in \mathcal{P}_n$;*
- $\mathbb{E}\mathrm{xp} \circ \Phi_n(f_n^{\circ \ell_z}(z)) = \mathcal{R}(f_n)(w)$.

Proof. Since $w \in \operatorname{Dom} \mathcal{R}(f_n)$, $\mathcal{R}(f_n)(w)$ is defined. By the definition of renormalization, there are $\zeta \in \Phi_n(S_n^0)$ and $\zeta' \in \Phi_n(\mathcal{C}_n \cup \mathcal{C}_n^\sharp)$ such that

$$\mathbb{E}\mathrm{xp}(\zeta) = w, \quad \mathbb{E}\mathrm{xp}(\zeta') = \mathcal{R}(f_n)(w), \quad \zeta' = \Phi_n \circ f_n^{\circ k_n} \circ \Phi_n^{-1}(\zeta).$$

Since $\mathbb{E}\mathrm{xp}(\Phi_n(z)) = w$, there exists an integer ℓ with

$$-k_n + 1 \leq \ell \leq a_n - \mathbf{k} - 1/2$$

such that $\Phi_n(z) + \ell = \zeta$.

By the functional equation in Theorem 1.2(d), we have

$$\zeta' = \Phi_n \circ f_n^{\circ k_n} \circ \Phi_n^{-1}(\zeta) = \Phi_n \circ f_n^{\circ k_n} \circ \Phi_n^{-1}(\Phi_n(z) + \ell) = \Phi_n \circ f_n^{\circ k_n + \ell}(z).$$

Letting $\ell_z = k_n + \ell$, we have

$$1 \leq \ell_z \leq k_n + a_n - \mathbf{k} - 1/2, \qquad f_n^{\circ \ell_z}(z) = \Phi_n^{-1}(\zeta') \in \mathcal{P}_n,$$
$$\mathbb{E}\mathrm{xp} \circ \Phi_n(f_n^{\circ \ell_z}(z)) = \mathbb{E}\mathrm{xp} \circ \Phi_n(\Phi_n^{-1}(\zeta')) = \mathbb{E}\mathrm{xp}(\zeta') = \mathcal{R}(f_n)(w). \qquad \square$$

On the practical side, if $\varepsilon_n = -1$, then $\mathcal{R}(f_n) = f_{n+1}$ and the above lemma holds for f_{n+1} instead of $\mathcal{R}(f_n)$. If $\varepsilon_n = +1$, then one may replace $\mathcal{R}(f_n)$ by f_{n+1} and w by $s(w)$ in the above lemma.

It should be clear that there are many choices for ℓ_z in the above lemma. We need precise formulas relating the number of iterates on consecutive renormalization levels. This may be achieved by imposing some condition on the beginning and termination point of an orbit of f_n that reduces to one iterate of $\mathcal{R}(f_n)$. Here, we require an orbit of f_n to start and terminate in $\psi_{n+1}(\mathcal{P}_{n+1})$. Then combining these relations for several values of n, we relate appropriate iterates of f_0 to one iterate of f_n, through the change of coordinates Ψ_n.

Let R_α denote the rotation of angle $2\pi\alpha$ about 0: $R_\alpha(z) = e^{2\pi\alpha i} z$, $z \in \mathbb{C}$. The *closest return times* of R_α are a sequence of positive integers $q_0 < q_1 < q_2 < \cdots$ defined as follows. Let $q_0 = 1$, $q_1 = a_0$, and for $i \geq 1$, q_{i+1} is the smallest integer greater than q_{i-1} such that

$$|R_\alpha^{\circ q_{i+1}}(1) - 1| < |R_\alpha^{\circ q_i}(1) - 1|.$$

Define

$$\mathcal{P}'_n = \{w \in \mathcal{P}_n \mid 0 < \operatorname{Re} \Phi_n(w) < \alpha_n^{-1} - \mathbf{k} - 1\}.$$

We have $f_n(\mathcal{P}'_n) \subset \mathcal{P}_n$.

Lemma 2.3. *For every $n \geq 1$ we have*

(a) $f_{n-1}^{\circ a_{n-1}} \circ \psi_n(w) = \psi_n \circ f_n(w)$ *for every* $w \in \mathcal{P}'_n$,
(b) $f_{n-1}^{\circ(k_n a_{n-1}+1)} \circ \psi_n(w) = \psi_n \circ f_n^{\circ k_n}(w)$ *for every* $w \in S_n^0$.

Lemma 2.4. *For every $n \geq 1$ we have*

(a) $f_0^{\circ q_n} \circ \Psi_n(w) = \Psi_n \circ f_n(w)$ *for every* $w \in \mathcal{P}'_n$,
(b) $f_0^{\circ(k_n q_n + q_{n-1})} \circ \Psi_n(w) = \Psi_n \circ f_n^{\circ k_n}(w)$ *for every* $w \in S_n^0$,
(c) *similarly, for every $m < n-1$, $f_n \colon \mathcal{P}'_n \to \mathcal{P}_n$ and $f_n^{\circ k_n} \colon S_n^0 \to \mathcal{C}_n \cup \mathcal{C}_n^\sharp$ are conjugate to some iterates of f_m on $\psi_{m+1} \circ \cdots \circ \psi_n(\mathcal{P}_n)$.*

To find the correct number of iterates in the above lemmas, we compare the maps f_n near 0 to the rotations of angle $2\pi\alpha_n$ about 0. That is, the relations hold near zero, and hence must hold on the region where the equations are defined. For a detailed proof of the above two lemmas one may refer to [Che10] or [Che13].

2.3. Petals covering the postcritical set

For every $n \geq 1$, we define

$$I_a^n = \bigcup_{i=0}^{k_n + a_n - \mathbf{k} - 2} f_0^{\circ(i q_n)}(S_0^n), \quad I_b^n = f_0^{\circ q_{n-1}}(I_a^n).$$

Note that by (1.5) we have $k_n \geq 2$ for all $n \geq 1$. Set

$$I^n = I_a^n \cup I_b^n.$$

The set I^n defined through the above mechanism has some crucial dynamical properties. We investigate these in the remainder of this section.

Lemma 2.5. *For every $n \geq 1$, the sets I_a^n, I_b^n, and I^n are connected subsets of $\mathbb{C}$. Moreover, each of them is bounded by piecewise smooth curves two of which land at 0 at well-defined angles.*

Proof. By Proposition 1.5, the set S_n^0 is bounded by piecewise smooth curves two of which land at 0 at well-defined angles. Moreover, one of these boundary curves is mapped to the other by f_n. Combining this with Lemma 2.4, we conclude that $S_0^n = \Psi_n(S_n^0)$ is bounded by piecewise smooth curves two of which land at 0 at well-defined angles, and one boundary curve is mapped to the other by $f_0^{\circ q_n}$. In particular, each $f_0^{\circ i q_n}(S_0^n)$ is defined and is a connected set. Moreover, the consecutive sets $f_0^{\circ i q_n}(S_0^n)$ and $f_0^{\circ (i+1) q_n}(S_0^n)$ share a boundary curve. This implies that I_a^n is a connected set bounded by piecewise smooth curves landing at 0 at well-defined angles. Clearly, $I_b^n = f_0^{\circ q_{n-1}}(I_a^n)$ must enjoy the same properties.

It remains to show that I^n is a connected set. There is a point z in the interior of S_n^0 (sufficiently close to 0) and an integer i with $k_n \leq i \leq a_n + k_n - \mathbf{k} - 2$ such that $w = f_n^{\circ i}(z) \in S_n^0$. The points $z' = \Psi_n(z)$ and $w' = \Psi_n(w)$ belong to S_0^n. Using Lemma 2.4, we conclude that $f_0^{\circ(q_{n-1}+i q_n)}(z') = w'$. Thus,

$$w' \in S_0^n \subset I_a^n, \qquad w' \in f_0^{\circ q_{n-1}} \circ f_0^{\circ i q_n}(S_0^n) \subset I_b^n.$$

This implies that I_a^n and I_b^n intersect, and so their union is connected. □

For $n \geq 1$, define

$$\Upsilon^n = \bigcup_{i=0}^{q_n - 1} f_0^{\circ i}(I^n) \cup \{0\}.$$

The sets S_0^n are bounded by piecewise smooth curves two of which land at zero at well-defined angles. Comparing f_0 with the rotation of angle $2\pi\alpha_0$ near 0, one can verify that Υ^n contains a neighbourhood of 0.

When $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ is linearizable at 0, $\Delta(f)$ denotes the Siegel disk of f centred at 0, and when it is not linearizable at 0 we define $\Delta(f)$ as the empty set.

Some similarly defined unions of sectors, denoted by Ω_0^n, have been studied in [Che10, Che13] in detail. Indeed, each Υ^n involves $q_n - 1$ more iterates of S_0^n than the set Ω_0^n. This modification is made to achieve some nice combinatorial features that were not available through the sets Ω_0^n. It is proved in [Che10, Propositions 2.4 and 4.9] that the sets Ω_0^n form a nest of domains shrinking to $\mathcal{PC}(f_0) \cup \Delta(f_0)$. The same arguments may be repeated here to prove this result for the domains Υ^n. We shall skip repeating these arguments here as they are not the main focus of this paper, but state them in the next two propositions for reference. (Alternatively, one can see that for every $n \geq 2$, $\Upsilon^{n+1} \subseteq \Omega_0^n \subseteq \Upsilon^{n-1}$, and therefore $\bigcap_{n \geq 1} \Upsilon^n = \bigcap_{n \geq 1} \Omega_0^n$.)

Recall that $\mathcal{PC}(f_j)$ denotes the postcritical set of f_j.

Proposition 2.6. *Let $\alpha \in HT_N$. Then*

(a) *for every $n \geq 0$,*

$$\mathcal{PC}(f_n) \subset \bigcup_{i=0}^{k_n+a_n-\mathbf{k}-2} f_n^{\circ i}(S_n^0) \cup \{0\};$$

(b) *for every $n \geq 1$,*

$$\mathcal{PC}(f_0) \subseteq \Upsilon^n.$$

Proposition 2.7. *Assume that $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ is linearizable at 0 and $\alpha \in HT_N$. Then, for every $n_0 \geq 1$ we have*

$$\bigcap_{n \geq n_0} \Upsilon^n = \Delta(f) \cup \mathcal{PC}(f)$$

and

$$\partial\Delta(f) \subset \mathcal{PC}(f).$$

Proposition 2.8. *For every $n \geq 1$ we have*

$$\mathcal{PC}(f_0) \cap \Psi_n(\mathcal{P}_n) = \Psi_n(\mathcal{PC}(f_n) \cap \mathcal{P}_n).$$

Proof. Recall that cv_n denotes the critical value of f_n and $\Phi_n(\mathrm{cv}_n) = 1$. As $\mathbb{E}\mathrm{xp}(1) = \mathrm{cv}_n$ for all n, it follows from the definition of ψ_n that there is a non-negative integer j_n with $\psi_n(\mathrm{cv}_n) = f_{n-1}^{\circ j_n}(\mathrm{cv}_{n-1})$.

First we show that for every $n \geq 1$ we have

$$\mathcal{PC}(f_{n-1}) \cap \psi_n(\mathcal{P}_n) = \psi_n(\mathcal{PC}(f_n) \cap \mathcal{P}_n).$$

Fix $z \in \psi_n(\mathcal{P}_n)$ and $w \in \mathcal{P}_n$ with $\psi_n(w) = z$. It is enough to show that $z \in \mathcal{PC}(f_{n-1})$ if and only if $w \in \mathcal{PC}(f_n)$.

Assume that $w \in \mathcal{PC}(f_n)$. As $\mathcal{P}_n$ is an open set, there is an increasing sequence of positive integers n_i, for $i \geq 0$, such that the iterates $f_n^{\circ n_i}(\mathrm{cv}_n)$ belong to $\mathcal{P}_n$ and converge to w. Then, one infers from Lemma 2.4 that there is an increasing sequence of positive integers m_i such that $f_{n-1}^{\circ m_i}(\psi_n(\mathrm{cv}_n)) = \psi_n(f_n^{\circ n_i}(\mathrm{cv}_n))$. Thus, $f_{n-1}^{\circ(j_n+m_i)}(\mathrm{cv}_{n-1}) = f_{n-1}^{\circ m_i}(\psi_n(\mathrm{cv}_n))$ converges to z. That is, $z \in \mathcal{PC}(f_{n-1})$.

Let $x_0 = \mathrm{cv}_n, x_1, x_2, \ldots$ denote the (ordered) points in the orbit of cv_n that are in $\mathcal{P}_n$. Define the sequence of positive integers l_i, for $i \geq 0$, so that $x_{i+1} = f_n^{\circ l_i}(x_i)$. By Proposition 2.6(a), for every $i \geq 0$, either $l_i = 1$ or $2 \leq l_i \leq k_n$. For every i with $l_i = 1$ we have $\psi_n(x_{i+1}) = f_{n-1}^{\circ a_{n-1}}(\psi_n(x_n))$, and by the definition of renormalization, a_{n-1} is the smallest positive integer s with $f_{n-1}^{\circ s}(\psi_n(x_n)) \in \psi_n(\mathcal{P}_n)$. That is, all intermediate iterates are outside $\psi_n(\mathcal{P}_n)$. Similarly, when some $l_i \geq 2$ then the intermediate iterates $f_n^{\circ 1}(x_i)$, $\ldots$, $f_n^{\circ l_i-1}(x_i)$ are outside $\mathcal{P}_n$. This implies that

$$\psi_n(x_{i+1}) = f_{n-1}^{\circ(l_i a_{n-1}+1)}(\psi_n(x_i)) \in \psi_n(\mathcal{P}_n),$$

and $l_i a_{n-1} + 1$ is the smallest positive integer s with $f_{n-1}^{\circ s}(\psi_n(x_i)) \in \psi_n(\mathcal{P}_n)$.

Now assume that $z \in \mathcal{PC}(f_{n-1})$. As $\psi_n(\mathcal{P}_n)$ is an open neighbourhood of z, by the above paragraph, there is an increasing sequence of integers n_i such that $\psi_n(x_{n_i}) \to z$. Therefore, $x_{n_i} \to w$. That is, $w \in \mathcal{PC}(f_n)$.

The equality in the proposition holds for $n = 1$ by the above equation. Assume that

$$\mathcal{PC}(f_0) \cap \Psi_{n-1}(\mathcal{P}_{n-1}) = \Psi_{n-1}(\mathcal{PC}(f_{n-1}) \cap \mathcal{P}_{n-1}) \tag{2.3}$$

(the induction hypothesis). As each $\Psi_n \colon \mathcal{P}_n \to \mathcal{P}_0$ is univalent and $\Psi_n = \Psi_{n-1} \circ \psi_n$, we obtain

$$\begin{aligned}
\mathcal{PC}(f_0) \cap \Psi_n(\mathcal{P}_n) &= \mathcal{PC}(f_0) \cap \big(\Psi_{n-1}(\mathcal{P}_{n-1}) \cap \Psi_{n-1}(\psi_n(\mathcal{P}_n))\big) \\
&= (\mathcal{PC}(f_0) \cap \Psi_{n-1}(\mathcal{P}_{n-1})) \cap \Psi_{n-1}(\psi_n(\mathcal{P}_n)) \\
&= \Psi_{n-1}(\mathcal{PC}(f_{n-1}) \cap \mathcal{P}_{n-1}) \cap \Psi_{n-1}(\psi_n(\mathcal{P}_n)) \quad \text{(induction hypothesis)} \\
&= \Psi_{n-1}(\mathcal{PC}(f_{n-1}) \cap \psi_n(\mathcal{P}_n)) \\
&= \Psi_{n-1}(\psi_n(\mathcal{PC}(f_n) \cap \mathcal{P}_n)) \quad ((2.3)) \\
&= \Psi_n(\mathcal{PC}(f_n) \cap \mathcal{P}_n).
\end{aligned}$$

This finishes the proof of the proposition. □

Lemma 2.9. *For every $n \geq 1$, $f_0^{\circ q_n}(\mathcal{PC}(f_0) \cap I_b^n) \subset I^n$.*

Proof. Fix $z \in \mathcal{PC}(f_0) \cap I_b^n$. By the definition of I_b^n, z is in $f_0^{\circ q_{n-1}} \circ f_0^{\circ i q_n}(S_0^n)$ for some i with $0 \leq i \leq k_n + a_n - \mathbf{k} - 2$. We consider two cases.

If $i < k_n + a_n - \mathbf{k} - 2$, then

$$f_0^{\circ q_n}(z) \in f_0^{\circ q_{n-1}} \circ f_0^{\circ (i+1) q_n}(S_0^n) \subset I_b^n \subset I^n.$$

If $i = k_n + a_n - \mathbf{k} - 2$, then by Lemma 2.4, z is in

$$\begin{aligned}
f_0^{\circ q_{n-1}} \circ f_0^{\circ (k_n + a_n - \mathbf{k} - 2) q_n}(S_0^n) &= f_0^{\circ (a_n - \mathbf{k} - 2) q_n} \circ f_0^{\circ k_n q_n + q_{n-1}}(S_0^n) \\
&= f_0^{\circ (a_n - \mathbf{k} - 2) q_n} \circ \Psi_n \circ f_n^{\circ k_n}(S_n^0) \\
&= \Psi_n \circ f_n^{\circ (k_n + a_n - \mathbf{k} - 2)}(S_n^0).
\end{aligned}$$

On the other hand, since $z \in \mathcal{PC}(f_0)$ and $f_n^{\circ (k_n + a_n - \mathbf{k} - 2)}(S_n^0) \subset \mathcal{P}_n$, by Proposition 2.8 there is

$$w \in f_n^{\circ (k_n + a_n - \mathbf{k} - 2)}(S_n^0) \cap \mathcal{PC}(f_n) \quad \text{with} \quad \Psi_n(w) = z.$$

In particular,

$$\operatorname{Re} \Phi_n(w) \in [a_n - \mathbf{k} - 3/2, a_n - \mathbf{k} - 1/2].$$

As $w \in \mathcal{PC}(f_n)$, by Proposition 2.6 its forward orbit remains in the union of the sectors $f_n^{\circ i}(S_n^0)$ for $0 \leq i \leq k_n + a_n - \mathbf{k} - 2$. Hence, by the mapping property of f_n on these sectors in the definition of the renormalization, we must have $w \in \bigcup_{j=0}^{k_n - 1} f_n^{\circ j}(S_n^0) \cap \mathcal{P}_n$. Then, by Lemma 2.4, $\Psi_n(w) \in \bigcup_{j=0}^{k_n - 1} f_0^{\circ (j q_n)}(S_0^n)$. Therefore,

$$f_0^{\circ q_n}(z) = f_0^{\circ q_n}(\Psi_n(w)) \in \bigcup_{j=1}^{k_n} f_0^{\circ (j q_n)}(S_0^n) \subset I_a^n \subset I^n. \qquad \square$$

Lemma 2.10. *For every $n \geq 1$ and every i with $0 \leq i \leq q_{n-1} - 1$ we have*

$$f_0^{\circ i}(f_0^{\circ (m q_n)}(S_0^n)) \subset f_0^{\circ (q_n - q_{n-1} + i)}(I_b^n), \qquad \text{where } m = k_n + a_n - \mathbf{k} - 2.$$

Proof. It is enough to prove the inclusion for $i = 0$. We may rearrange the iterates as in

$$f_0^{\circ(mq_n)}(S_0^n) = f_0^{\circ(q_n - q_{n-1})}(f_0^{\circ q_{n-1}} \circ f_0^{\circ((m-1)q_n)}(S_0^n)).$$

On the other hand, as $f_0^{\circ((m-1)q_n)}(S_0^n) \subset I_a^n$, we have $f_0^{\circ q_{n-1}} \circ f_0^{\circ((m-1)q_n)}(S_0^n) \subset I_b^n$. □

The next proposition is the ultimate statement in this section we need for the proofs of the main results of this paper.

Proposition 2.11. *For every $\alpha \in HT_N$ and every integer $n \geq 1$ we have the following. For every non-zero $z \in \mathcal{PC}(f_0)$ there is an integer ℓ with $0 \leq \ell \leq q_n - 1$ such that*

(a) $f_0^{\circ j}(z) \in f_0^{\circ(\ell+j)}(I^n)$ *for all j with $0 \leq j \leq q_n - \ell - 1$;*

(b) $f_0^{\circ j}(z) \in f_0^{\circ(j-q_n+\ell)}(I^n)$ *for all j with $q_n - \ell \leq j \leq q_n - 1$.*

Proof. By Proposition 2.6, z is in Υ^n. We may rewrite this set as

$$\begin{aligned}
\Upsilon_n \setminus \{0\} = \bigcup_{i=0}^{q_n-1} f_0^{\circ i}(I_a^n \cup I_b^n) &= \bigcup_{i=0}^{q_n-1} f_0^{\circ i}(I_a^n) \cup \bigcup_{i=0}^{q_n-1} f_0^{\circ i}(I_b^n) \\
&= \bigcup_{i=0}^{q_{n-1}-1} f_0^{\circ i}(I_a^n) \cup \bigcup_{i=0}^{q_n-1} f_0^{\circ i}(I_b^n) \\
&= I_a^n \cup \bigcup_{i=1}^{q_{n-1}-1} f_0^{\circ i}(I_a^n) \cup \bigcup_{i=0}^{q_n-1} f_0^{\circ i}(I_b^n).
\end{aligned}$$

Now, we consider four cases.

1) If $z \in I_a^n$, we let $\ell = 0$. Here, $z \in I_a^n \subset I^n$, and therefore we have part (a) in the proposition. There is no j satisfying (b).

2) If $z \in f_0^{\circ i}(I_b^n)$ for some i with $0 \leq i \leq q_n - 1$, we let $\ell = i$. Then $z \in f_0^{\circ \ell}(I_b^n) \subset f_0^{\circ \ell}(I^n)$, and hence (a) holds. On the other hand, since $z \in f_0^{\circ \ell}(I_b^n)$, by Lemma 2.9 we have $f_0^{\circ(q_n-\ell)}(z) \in f_0^{\circ q_n}(I_b^n) \subset I^n$. This implies (b).

It remains to prove the proposition for $z \in \bigcup_{i=1}^{q_{n-1}-1} f_0^{\circ i}(I_a^n)$. Fix i with $z \in f_0^{\circ i}(I_a^n)$. We consider two cases.

3) If $z \in f_0^{\circ i}(f_0^{\circ(mq_n)}(S_0^n))$ for some m with $0 \leq m \leq k_n + a_n - \mathbf{k} - 3$, then with $\ell = i$, we have $z \in f_0^{\circ \ell}(I_a^n) \subset f_0^{\circ \ell}(I^n)$. This implies (a).

On the other hand,

$$f_0^{\circ(q_n-\ell)}(z) \in f_0^{\circ q_n}(f_0^{\circ(mq_n)}(S_0^n)) = f_0^{\circ((m+1)q_n)}(S_0^n) \subset I_a^n \subset I^n,$$

which yields (b).

4) If $z \in f_0^{\circ i}(f_0^{\circ(mq_n)}(S_0^n))$ for $m = k_n + a_n - \mathbf{k} - 2$, we let $\ell = q_n - q_{n-1} + i$. By Lemma 2.10, $z \in f_0^{\circ \ell}(I_b^n) \subset f_0^{\circ \ell}(I^n)$, and hence (a) holds. On the other hand, by Lemma 2.9, $z \in f_0^{\circ \ell}(I_b^n)$ implies that

$$f_0^{\circ(q_n-\ell)}(z) \in f_0^{\circ q_n}(I_b^n) \subset I^n,$$

giving (b). □

2.4. Lifts versus iterates

In this section we give an alternative definition of the sectors I^n, and their forward iterates, in terms of the lifts $\chi_{n,i}$. We give an alternative definition in Proposition 2.13 which makes later arguments simpler.

Lemma 2.12. *For every $n \geq 1$ and every integer i with $0 \leq i \leq k_n + a_n - \mathbf{k} - 2$ we have*

(a) $\Phi_{n-1} \circ f_{n-1}^{\circ(ia_{n-1})}(S^1_{n-1}) \subset \{w \in \mathbb{C} \mid -2 < \operatorname{Im} w, 0 \leq \operatorname{Re} w \leq \hat{\mathbf{k}} + 2\}$;

(b) $\Phi_{n-1} \circ f_{n-1}^{\circ(ia_{n-1}+1)}(S^1_{n-1}) \subset \{w \in \mathbb{C} \mid -2 < \operatorname{Im} w, 1 \leq \operatorname{Re} w \leq \hat{\mathbf{k}} + 3\}$;

(c) $f_0^{\circ(iq_n)}(S^n_0) = \Phi_0^{-1} \circ \chi_{1,0} \circ \chi_{2,0} \circ \cdots \circ \chi_{n-1,0} \circ \Phi_{n-1} \circ f_{n-1}^{\circ(ia_{n-1})}(S^1_{n-1})$;

(d) $f_0^{\circ(iq_n+q_{n-1})}(S^n_0) = \Phi_0^{-1} \circ \chi_{1,0} \circ \chi_{2,0} \circ \cdots \circ \chi_{n-1,0} \circ \Phi_{n-1} \circ f_{n-1}^{\circ(ia_{n-1}+1)}(S^1_{n-1})$.

Proof. (a) Fix $n \geq 1$. We consider two cases:

Case 1: Assume that i satisfies $0 \leq i \leq k_n$. First, by an inductive argument we show that for all $w \in S^0_n$,

$$f_{n-1}^{\circ(ia_{n-1})} \circ \psi_n(w) = \Phi_{n-1}^{-1} \circ \chi_{n,0}(\Phi_n(w) + i). \tag{2.4}$$

Note that for all $w \in S^0_n$ we have $\Phi_n(w) + i \in \mathcal{D}_n$, and hence the right hand side of (2.4) is defined. Moreover, the right hand side is either a holomorphic or an anti-holomorphic function of w.

By definition, $\psi_n = \Phi_{n-1}^{-1} \circ \chi_{n,0} \circ \Phi_n$, and hence (2.4) holds for $i = 0$.

Assume that (2.4) holds for all integers less than or equal to some $0 \leq i < k_n$. We wish to show that it holds for $i + 1$. For $|w|$ small enough, the left hand side is defined for $i + 1$. Moreover, for w on a smooth curve on ∂S^0_n landing at 0 there is w' on ∂S^0_n with $f_n(w') = w$. By Lemma 2.3 for w', we obtain

$$\begin{aligned} f_{n-1}^{\circ((i+1)a_{n-1})} \circ \psi_n(w') &= f_{n-1}^{\circ(ia_{n-1})} \circ \psi_n(w) \\ &= \Phi_{n-1}^{-1} \circ \chi_{n,0}(\Phi_n(w) + i) = \Phi_{n-1}^{-1} \circ \chi_{n,0}(\Phi_n(w') + 1 + i). \end{aligned}$$

That is, (2.4) holds for $i + 1$ on a curve landing at 0 on S^0_n. Hence, by uniqueness of analytic continuation, it must hold for $w \in S^0_n$ close to 0. On the other hand, by the open mapping property of holomorphic and anti-holomorphic maps, the set of points on which the equality holds forms an open and closed subset of S^0_n. This implies that the equality must hold on S^0_n.

Equation (2.4) and $S^1_{n-1} = \psi_n(S^0_n)$ imply that $\Phi_{n-1} \circ f_{n-1}^{\circ(ia_{n-1})}(S^1_{n-1})$ is contained in $\chi_{n,0}(\mathcal{D}_n)$. Combining this with the inclusion in (2.1), we deduce (a) for these values of i.

Case 2: Assume that i satisfies $k_n \leq i \leq k_n + a_n - \mathbf{k} - 2$. For all such i we have $f_n^{\circ i}(w) \in \mathcal{P}_n$, and by Lemma 2.3,

$$f_{n-1}^{\circ(ia_{n-1}+1)}(\psi_n(w)) = \psi_n(f_n^{\circ i}(w)) = \Phi_{n-1}^{-1} \circ \chi_{n-1,0} \circ \Phi_n(f_n^{\circ i}(w)).$$

Therefore, it follows from (2.1) that

$$\begin{aligned}\Phi_{n-1}(f_{n-1}^{\circ(ia_{n-1}+1)}(\psi_n(w))) \subset \chi_{n-1,0}(\Phi_n(\mathcal{P}_n)) \subset \chi_{n-1,0}(\mathcal{D}_n)\\ \subset \{w \in \mathbb{C} \mid -2 < \operatorname{Im} w,\, 1 \leq \operatorname{Re} w \leq \hat{\mathbf{k}} + 2\}.\end{aligned}$$

Hence, the relation $\Phi_{n-1} \circ f_{n-1}(z) = \Phi_{n_1}(z) + 1$ implies (a) for these values of i (i.e. 1 is replaced by 0).

(b) By the restriction in (1.8) on the rotation, we have $\hat{\mathbf{k}} + 3 \leq \alpha_n^{-1} - \mathbf{k} - 1$. This implies that the sets $f_{n-1}^{\circ(ia_{n-1})}(S_{n-1}^1)$ and $f_{n-1}^{\circ(ia_{n-1}+1)}(S_{n-1}^1)$ are contained in $\mathcal{P}'_{n-1}$. As Φ_{n-1} conjugates f_{n-1} to the translation by 1 on $\mathcal{P}'_{n-1}$, (b) follows from (a).

(c) Recall the changes of coordinate $\psi_n = \Phi_{n-1} \circ \chi_{n,0} \circ \Phi_n$, and their compositions Ψ_n, for $n \geq 1$. It follows from (a) that for every $w \in S_{n-1}^1$, $\Phi_{n-1} \circ f_{n-1}^{\circ(ia_{n-1})}(w)$ is contained in the domain of $\chi_{n-1,0}$. Thus, $\Psi_{n-1} \circ f_{n-1}^{\circ(ia_{n-1})}(w)$ is defined. On the other hand, by Lemma 2.4(b), for every $w \in S_0^n$, $f_0^{\circ k_n q_n + q_{n-1}}(w)$ is defined and $\operatorname{Re} f_n^{\circ k_n}(w) \in [1/2, 3/2]$. In particular, for every i with $0 \leq i \leq k_n$, $f_0^{\circ(iq_n)}(w)$ is defined. Then Lemma 2.4(a) implies that for all $w \in S_0^n$ and all i with $k_n \leq i \leq a_n + k_n - \mathbf{k} - 2$, $f_0^{\circ(iq_n)}(w)$ is defined.

We want to show that

$$f_0^{\circ(iq_n)} \circ \Psi_n(w) = \Psi_{n-1} \circ f_{n-1}^{\circ(ia_{n-1})} \circ \psi_n(w), \qquad w \in S_n^0,$$

for all i as in the lemma. The set S_n^0 is bounded by piecewise smooth curves two of which land at 0 where one boundary curve is mapped to another by f_n. By Lemma 2.3, S_{n-1}^1 is bounded by piecewise smooth curves, one of which is mapped to another by $f_{n-1}^{a_{n-1}}$. Similarly, by Lemma 2.4 the set S_0^n is bounded by piecewise smooth curves, one of which is mapped to another by $f_0^{\circ q_n}$. These imply that the above equation, for each i, is valid on a boundary curve of S_n^0. Hence the above equation holds on a curve on the boundary of S_n^0. By the uniqueness of holomorphic and anti-holomorphic continuations, the above equation must hold on the connected set S_n^0.

(d) By (b), the set $\Phi_{n-1} \circ f_{n-1}^{\circ(ia_{n-1}+1)}(S_{n-1}^1)$ is contained in the domain of $\chi_{n-1,0}$. Thus, the right hand side of the equality is defined. On the other hand, since $f_{n-1}^{\circ(ia_{n-1}+1)}(S_{n-1}^1)$ is contained in $\mathcal{P}'_{n-1}$, the equality follows from (c) and the conjugacy relation in Lemma 2.4(a). □

Define

$$J_{n-1} = \bigcup_{j=0}^{1} \bigcup_{i=0}^{k_n + a_n - \mathbf{k} - 2} f_{n-1}^{\circ(ia_{n-1}+j)}(S_{n-1}^1), \qquad n \geq 1.$$

By Lemma 2.12, for all $n \geq 1$, we have

$$\Phi_{n-1}(J_{n-1}) \subset \{w \in \mathbb{C} \mid -2 < \operatorname{Im} w,\ 1 \leq \operatorname{Re} w \leq \hat{\mathbf{k}} + 3\} \subset \mathcal{D}_{n-1}, \tag{2.5}$$

and

$$I^n = \Phi_0^{-1} \circ \chi_{1,0} \circ \chi_{2,0} \circ \cdots \circ \chi_{n-1,0} \circ \Phi_{n-1}(J_{n-1}) = \Psi_{n-1}(J_{n-1}).$$

Proposition 2.13. *For every $n \geq 1$ and every i with $0 \leq i \leq q_n$, there are integers i_j with $0 \leq i_j \leq a_{j-1}$, for $1 \leq j \leq n$, such that*

$$\forall w \in J_{n-1},\ f_0^{\circ i}(\Psi_{n-1}(w)) = \Phi_0^{-1} \circ \chi_{1,i_1} \circ \chi_{2,i_2} \circ \cdots \circ \chi_{n-1,i_{n-1}}(\Phi_{n-1}(w) + i_n). \quad (2.6)$$

In particular,

$$f_0^{\circ i}(I^n) = \Phi_0^{-1} \circ \chi_{1,i_1} \circ \chi_{2,i_2} \circ \cdots \circ \chi_{n-1,i_{n-1}}(J_{n-1} + i_n).$$

Proof. The latter statement follows from the former. For the former, first we need to show that both sides of (2.6) are defined. Fix $n \geq 1$. By (2.5) and (1.8), for all integers i_n with $0 \leq i_n \leq a_{n-1}$, we have

$$\Phi_{n-1}(J_{n-1}) + i_n \subset \mathcal{D}_{n-1}.$$

Hence, $\chi_{n-1,i_{n-1}}$ is defined on $\Phi_{n-1}(J_{n-1}) + i_n$. Now,

$$\chi_{n-1,i_{n-1}}(\Phi_{n-1}(J_{n-1}) + i_n) \subset \chi_{n-1,i_{n-1}}(\mathcal{D}_{n-1}).$$

Combining the above inclusion with (2.1), we conclude that for all integers i_{n-1} with $0 \leq i_{n-1} \leq a_{n-2}$,

$$\chi_{n-1,i_{n-1}}(\Phi_{n-1}(J_{n-1}) + i_n) \subseteq \mathcal{D}_{n-2}.$$

Therefore,

$$\chi_{n-2,i_{n-2}} \circ \chi_{n-1,i_{n-1}}(\Phi_{n-1}(J_{n-1}) + i_n)$$

is defined. Continuing, one infers that the right hand side of (2.6) is defined. On the other hand, since f_{n-1} may be iterated at least a_{n-1} times on J_{n-1}, and $J_{n-1} \subset \mathcal{P}_{n-1}$, Lemma 2.4 implies that f_0 may be iterated at least $a_{n-1}q_{n-1} + q_{n-2} = q_n$ times on $\Psi_{n-1}(J_{n-1}) = I^n$. Thus, for all such i, the left hand side of (2.6) is defined.

By the conjugacy property of the coordinates Φ_m^{-1} on $\mathcal{D}_m$, each composition on the right hand side of (2.6) corresponds to some non-negative iterate of f_0 on the left hand side. Comparing f_0 with the rotation R_{α_0} near zero, one can see that for all given i there are integers i_j as in the proposition such that the equality holds near 0. By uniqueness of analytic continuation, it must hold on the connected set J_{n-1}. □

Remark 2.14. All the lemmas and propositions in Sections 2.3 and 2.4 hold for all maps $f \in \mathcal{IS}_\alpha$ with $\alpha \in HT_N$. That is, f need not be a quadratic polynomial.

3. Geometry of the renormalization tower and the arithmetic of α

3.1. Nearby sectors

Recall the sequence of numbers α_i, $i \geq 0$, defined in (1.7). Let $\beta_0 = 1$ and $\beta_k = \prod_{i=1}^k \alpha_i$, $k \geq 1$. The irrational number α is called a *Brjuno number* if

$$\sum_{j=0}^{\infty} \beta_{j-1} \log \alpha_j^{-1} < +\infty.$$

An equivalent characterization of Brjuno numbers in terms of best rational approximants of α is

$$\sum_{j=0}^{\infty} q_j^{-1} \log q_{j+1} < +\infty.$$

For a detailed study of the Brjuno function see [MMY01]. By the Siegel–Brjuno–Yoccoz theorem [Sie42, Brj71, Yoc95b] the quadratic map Q_α is linearizable at 0 if and only if α is a Brjuno number. This optimality result has been extended to the maps in $\mathcal{IS}_\alpha$, $\alpha \in HT_N$, in [Che10] (see the Remark 3.3). See also [Yoc95a, Rog98, Zak99, Gey01, Zha11].

In this section we analyse the sizes of the sectors $f_0^{\circ i}(I^n)$, $0 \le i \le q_{n-1}$, defined in Section 2.3. Our ultimate goal is to prove the following two propositions.

For $\delta > 0$ let $B_\delta(0)$ denote the open disk of radius δ centred at 0. For every $n \ge 1$ and $\delta > 0$ define

$$G(n, \delta) = \{i \in \mathbb{Z} \mid 0 \le i \le q_n - 1,\ f_0^{\circ i}(I^n) \subset B_\delta(0)\}.$$

Let $|X|$ denote the cardinality of a set X.

Proposition 3.1. *Let α be a non-Brjuno number in HT_N. Then, for every $\delta > 0$,*

$$\limsup_{n\to\infty} \frac{|G(n, \delta)|}{q_n} = 1.$$

Indeed, we state and prove a stronger statement in Proposition 3.19. However, as we show in Section 4, the above statement is enough to derive unique ergodicity (stated in the introduction) for non-Brjuno values of α.

For Brjuno values of α, let $\Delta(f_i)$ denote the Siegel disk of f_i, $i \ge 0$. The δ-neighbourhood of $\Delta(f_0)$ is denoted by $B_\delta(\Delta(f_0))$. Given $n \in \mathbb{N}$ and $\delta > 0$, define

$$H(n, \delta) = \{i \in \mathbb{Z} \mid 0 \le i \le q_n - 1,\ f_0^{\circ i}(I^n) \subset B_\delta(\Delta(f_0))\}.$$

Proposition 3.2. *For every Brjuno $\alpha \in HT_N$ we have the following:*

(a) *for every $\delta > 0$,*

$$\limsup_{n\to\infty} \frac{|H(n, \delta)|}{q_n} = 1;$$

(b) *for every $\varepsilon > 0$ there are $n_0 \in \mathbb{Z}$ and $\delta_0 > 0$ such that for every $n \ge n_0$ and every $\delta < \delta_0$ we have*

$$\forall k \in H(n, \delta), \qquad \operatorname{diam}(f_0^{\circ k}(I^n) \setminus \Delta(f_0)) \le \varepsilon.$$

When δ is small, the sets $H(n, \delta)$ may be empty for small values of n, but eventually they become non-empty by (a) of the above proposition.

The remainder of this section is devoted to the proofs of the above propositions. The argument has two flavours: an arithmetic part and an analytic part. Readers interested in the proofs of unique ergodicity using the above propositions may safely skip the rest of this section and go directly to Section 4.

Remark 3.3. In [Che10] it is proved that for every $\delta > 0$ the sets $G(n, \delta)$ and $H(n, \delta)$ are non-empty for large values of n. The analysis presented in this paper uses an improved distortion estimate on the Fatou coordinates Φ_n that has been established in [Che13], but was not available at the time of writing [Che10].

3.2. An arithmetic lemma

Consider the sequence of numbers α_i, $i \geq 0$, and let

$$\beta_0 = 1, \qquad \beta_k = \prod_{i=1}^{k} \alpha_i, \qquad k \geq 1.$$

Fix $B \in \mathbb{R}$ and define

$$\begin{aligned} &B_{k,k} = -2, \qquad k = 0, 1, 2, \dots, \\ &B_{k,i-1} = \alpha_i B_{k,i} + \log \alpha_i^{-1} - B, \qquad 1 \leq i \leq k. \end{aligned} \tag{3.1}$$

For $T \geq 0$ and integers $l \geq 0$ define

$$\mathcal{L}(\alpha, T, l) = \{k \in \{l+1, l+2, \dots\} \mid B_{k,i} \geq T\alpha_i^{-1} \text{ for } l < i < k\}.$$

By definition, $l + 1 \in \mathcal{L}(\alpha, T, l)$ as there is no condition to be satisfied. Hence, every $\mathcal{L}(\alpha, T, l)$ is a non-empty set.

Lemma 3.4. *For every irrational α and every $B \geq 0$,*

(a) *for every $k \geq 0$,*

$$B_{k,0} \leq \sum_{i=1}^{k} \beta_{i-1} \log \alpha_i^{-1} \leq B_{k,0} + 2B + 2;$$

(b) *if α is non-Brjuno, then for every $T \in \mathbb{R}$, the set $\mathcal{L}(\alpha, T, 0)$ has infinite cardinality;*

(c) *if α is a Brjuno number such that for some $T \geq 0$ and every $l \geq 0$, the set $\mathcal{L}(\alpha, T, l)$ has finite cardinality, then*

$$\liminf_{j\to\infty} \lim_{m\to\infty} (B_{m,j} - T\alpha_j^{-1}) < +\infty.$$

Items (a) and (b) of the above lemma, when one starts with $B_{k,k} = 0$ instead of -2 in (3.1) and $B = T$, appear in [Yoc95b, Section 1.6]. The same argument works here as well. For completeness, and due to their use in (c), we present a proof of these items here.

Proof. (a) First, by reverse induction we will show that for every $k \geq 1$,

$$B_{k,i-1} = -2\beta_{i-1}^{-1}\beta_k + \beta_{i-1}^{-1} \sum_{j=i}^{k} \beta_{j-1}(\log \alpha_j^{-1} - B), \quad 1 \leq i \leq k. \tag{3.2}$$

For $i = k$ the formula becomes

$$B_{k,k-1} = -2\beta_{k-1}^{-1}\beta_k + \beta_{k-1}^{-1}\beta_{k-1}(\log \alpha_k^{-1} - B) = -2\alpha_k + \log \alpha_k^{-1} - B,$$

which is valid by (3.1). Now assume that (3.2) holds for $i = m + 1$. Then, by (3.1),

$$\begin{aligned} B_{k,m-1} &= \alpha_m B_{k,m} + \log \alpha_m^{-1} - B \\ &= \alpha_m \Big(-2\beta_m^{-1}\beta_k + \beta_m^{-1} \sum_{j=m+1}^{k} \beta_{j-1}(\log \alpha_j^{-1} - B) \Big) + \log \alpha_m^{-1} - B \\ &= -2\beta_{m-1}^{-1}\beta_k + \beta_{m-1}^{-1} \sum_{j=m}^{k} \beta_{j-1}(\log \alpha_j^{-1} - B), \end{aligned}$$

which finishes the proof of the induction step.

In particular, for $i = 1$ formula (3.2) becomes

$$B_{k,0} = -2\beta_k + \sum_{j=1}^{k} \beta_{j-1} \log \alpha_j^{-1} - B \sum_{j=1}^{k} \beta_{j-1}.$$

On the other hand, since each α_i is in $(0, 1/2)$, we have

$$\sum_{j=1}^{k} \beta_{j-1} \leq \sum_{j=1}^{\infty} (1/2)^{j-1} = 2. \tag{3.3}$$

The above two relations prove the first part of the lemma.

(b) Assume on the contrary that $\mathcal{L}(\alpha, T, 0)$ has finite cardinality for some $T \in \mathbb{R}$. Let k_0 denote the largest element in $\mathcal{L}(\alpha, T, 0)$.

Given an integer $N_0 > k$, define the integers $N_0 > N_1 > \cdots > N_r \leq k_0$ according to

$$B_{N_{l-1},N_l} < T\alpha_{N_l}^{-1} \quad \text{for } l = 1, \ldots, r.$$

By (3.2) for $k = N_{l-1}$ and $i = N_l + 1$, the above inequality implies that

$$\sum_{j=N_l+1}^{N_{l-1}} \beta_{j-1}(\log \alpha_j^{-1} - B) = \beta_{N_l} B_{N_{l-1},N_l} + 2\beta_{N_{l-1}} \leq T\beta_{N_l-1} + 2\beta_{N_{l-1}}.$$

Adding the above sums together for $l = 1, \ldots, r$, we obtain

$$\sum_{j=N_r+1}^{N_0} \beta_{j-1} \log \alpha_j^{-1} \leq 2 \sum_{l=1}^{r} \beta_{N_{l-1}} + T \sum_{l=1}^{r} \beta_{N_l-1} + B \sum_{j=N_r+1}^{N_0} \beta_{j-1}.$$

As $N_0 \to \infty$, using the bound in (3.3), we conclude that

$$\sum_{j=k_0+1}^{\infty} \beta_{j-1} \log \alpha_j^{-1} \leq 2(2 + T + B),$$

contradicting our assumption on the type of α.

(c) Define $\ell_0 = 0$ and

$$\ell_{j+1} = \max \mathcal{L}(\alpha, T, \ell_j) \quad \text{for } j \geq 0.$$

By definition, $\ell_{k+1} \geq \ell_k + 1$, and hence $\lim_{k \to +\infty} \ell_k = +\infty$.

Since α is a Brjuno number, by (3.2) and the uniform bound in (3.3), for every $i \geq 0$, $\lim_{m\to+\infty} B_{m,i}$ exists and is finite. We claim that for every $k \geq 0$,

$$\lim_{m\to+\infty} B_{m,\ell_k} - T\alpha_{\ell_k}^{-1}$$

is uniformly bounded from above independently of k. This will imply (c).

First we will show that

$$\forall j \geq 0, \qquad B_{\ell_{j+1},\ell_j} < T\alpha_{\ell_j}^{-1}. \tag{3.4}$$

Indeed, otherwise for some $j \geq 0$ we have

$$B_{\ell_{j+1},\ell_j} \geq T\alpha_{\ell_j}^{-1} \geq -2 = B_{\ell_j,\ell_j}.$$

Then, by (3.1) and the definition of ℓ_j, for all i with $\ell_{j-1} < i < \ell_j$ we must have

$$B_{\ell_{j+1},i} \geq B_{\ell_j,i} \geq T\alpha_i^{-1}.$$

However, this contradicts the choice of $\ell_j = \max \mathcal{L}(\alpha, T, \ell_{j-1})$ since $\ell_{j+1} > \ell_j$ and satisfies the inequality $B_{\ell_{j+1},i} > T\alpha_i^{-1}$ for all $\ell_{j-1} < i < \ell_{j+1}$.

Fix $k \in \mathbb{N}$ and let $n > k$. Recall that $B_{k,k} = -2$ for $k \geq 0$. By (3.4), for every j with $k + 1 \leq j \leq n$ we have

$$|B_{\ell_j,\ell_{j-1}} - B_{\ell_{j-1},\ell_{j-1}}| < T\alpha_{\ell_{j-1}}^{-1} + 2.$$

Then, recursively multiplying the above inequality by α_i and then adding and subtracting $\log \alpha_i^{-1} - B$ within the absolute value, for $i = \ell_{j-1}, \dots, \ell_k + 1$, we arrive at

$$|B_{\ell_j,\ell_k} - B_{\ell_{j-1},\ell_k}| < (T\alpha_{\ell_{j-1}}^{-1} + 2)\alpha_{\ell_{j-1}}\alpha_{\ell_{j-1}-1}\dots\alpha_{\ell_k+1} = T\beta_{\ell_{j-1}-1}\beta_{\ell_k}^{-1} + 2\beta_{\ell_{j-1}}\beta_{\ell_k}^{-1}.$$

Then, by the triangle inequality,

$$\begin{aligned}
|B_{\ell_n,\ell_k} - B_{\ell_k,\ell_k}| &\leq \sum_{j=k+1}^{n} |B_{\ell_j,\ell_k} - B_{\ell_{j-1},\ell_k}| \\
&\leq \sum_{j=k+1}^{n} (T\beta_{\ell_{j-1}-1}\beta_{\ell_k}^{-1} + 2\beta_{\ell_{j-1}}\beta_{\ell_k}^{-1}) \\
&\leq T\alpha_{\ell_k}^{-1} + T\sum_{j=k+2}^{\infty} (\beta_{\ell_{j-1}-1}\beta_{\ell_k}^{-1}) + 2\sum_{j=0}^{\infty} \beta_j \\
&\leq T\alpha_{\ell_k}^{-1} + 2T + 4.
\end{aligned}$$

It follows that for every $n > k$,

$$B_{\ell_n,\ell_k} - T\alpha_{\ell_k}^{-1} \leq 2T + 4 + 2.$$

If $m \notin \{\ell_k, \ell_{k+1}, \ell_{k+2}, \dots\}$, choose n with $\ell_{n-1} < m < \ell_n$. By the definition of ℓ_n,

$$B_{\ell_n,m} \geq T\alpha_m^{-1} > B_{m,m}.$$

This implies that

$$B_{m,\ell_k} - T\alpha_{\ell_k}^{-1} \leq B_{\ell_n,\ell_k} - T\alpha_{\ell_k}^{-1} \leq 2T + 6.$$

All in all, we have shown that for all $m \geq \ell_k$ we have the uniform bound

$$B_{m,\ell_k} - T\alpha_{\ell_k}^{-1} \leq 2(T+2)+2.$$

This finishes the proof of (c). □

3.3. Estimates on Fatou coordinates

To control the geometry of the sectors $f_0^{\circ i}(I^n)$ for $0 \leq i \leq q_n - 1$ through Proposition 2.13, we need some estimates on the Fatou coordinates. In this section we assemble two crucial estimates, a global one and an infinitesimal one, that we need for the analysis in this paper.

Recall the Fatou coordinate $\Phi_h : \mathcal{P}_h \to \mathbb{C}$ of a map $h \in \mathcal{IS}_\alpha$, $\alpha \in (0, \alpha_*]$, from Theorem 1.2. We denote the non-zero fixed point of h that lies on the boundary of $\mathcal{P}_h$ by σ_h. Consider the covering map

$$T_h(w) = \frac{\sigma_h}{1 - e^{-2\pi\alpha w \mathbf{i}}} : \mathbb{C} \to \hat{\mathbb{C}} \setminus \{0, \sigma_h\}. \tag{3.5}$$

The map T_h commutes with translation by $1/\alpha$.

The connected components of $T_h^{-1}(\mathcal{P}_h)$ are simply connected sets that are disjoint from $\mathbb{Z}/\alpha$. Moreover, the projection of each such component onto the imaginary axis covers the whole imaginary axis. In particular, we denote by $\tilde{\mathcal{P}}_h$ the connected component of $T_h^{-1}(\mathcal{P}_h)$ that separates 0 from $1/\alpha$. We may lift the map $\Phi_h^{-1} : \mathcal{D}_h \to \mathbb{C} \setminus \{0, \sigma_h\}$ to a map $L_h : \mathcal{D}_h \to \mathbb{C}$ such that

$$T_h \circ L_h(\zeta) = \Phi_h^{-1}(\zeta), \quad \forall \zeta \in \mathcal{D}_h. \tag{3.6}$$

The above lifts are determined up to translation by an element of $\mathbb{Z}/\alpha$. We choose the one that maps $\Phi_h(\mathcal{P}_h)$ to $\tilde{\mathcal{P}}_h$. In other words, L_h is the unique extension of $T_h \circ \Phi_h$, mapping $\Phi_h(\mathcal{P}_h)$ to $\tilde{\mathcal{P}}_h$.

Recall the lift

$$\chi_{h,0} : \mathcal{D}_h \to \mathbb{C}$$

defined in Section 1.3. We need to control the derivative of this map.

One may lift h via T_h to obtain a holomorphic map F_h defined on $L_h(\mathcal{D}_h)$. Indeed, the lift is univalent on $\tilde{\mathcal{P}}_h$ and has a univalent extension onto a larger set. The map F_h on $\tilde{\mathcal{P}}_h$ is close to translation by 1, with explicit estimates that can be worked out using classical distortion estimates on h. It follows that L_h^{-1} conjugates F_h to translation by 1, and it is a classical non-trivial problem to prove estimates on L_h and in particular $|L_h' - 1|$, depending on the bounds on $|F_h(w) - (w+1)|$. One may refer to [Yoc95b, Shi98, Shi00, Che10, Che13] for further details. Then, through the factorization of Φ_h^{-1} to $L_h \circ T_h$ one obtains estimates on Φ_h. The following estimates are the finest estimates known to us. They have been established in [Che10] and [Che13].

Proposition 3.5 ([Che10]). *For every $D \in \mathbb{R}$ there exists $M > 0$ such that for all h in $\mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $0 < \alpha \leq \alpha^*$ we have*

$$\forall \zeta \in \mathcal{D}_h \text{ with } \operatorname{Im} \zeta \geq D/\alpha, \quad |L_h(\zeta) - \zeta| \leq M \log(1 + 1/\alpha).$$

Proposition 3.6 ([Che13]). *For all $D > 0$ there exists $M > 0$ such that for all h in $\mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $0 < \alpha \le \alpha_*$ the following holds. For all $\zeta \in \mathcal{D}_h$ satisfying $\operatorname{Im} \zeta \ge D/\alpha$ we have*

$$|\chi'_{h,0}(\zeta) - \alpha| \le M\alpha e^{-2\pi\alpha \operatorname{Im} \zeta}.$$

Remark 3.7. Proposition 3.5 is stated as Proposition 5.15 in [Che10]. In the notation of [Che10], $x_h \ge y_h$ (by definition), and by [Che10, Proposition 5.14], $y_h \ge \alpha^{-1} - \mathbf{k}$. That is, the inequality in Proposition 3.5 is proved for $\zeta \in \Phi_h(\mathcal{P}_h)$. On the other hand, for $\zeta \in \mathcal{D}_h \setminus \Phi_h(\mathcal{P}_h)$ one uses the equation $F_h \circ L_h(\zeta) = L_h(\zeta + 1)$ and the uniform bounds on F_h to bound $L_h(\zeta) - \zeta$. Since the number of iterates of F_h involved is uniformly bounded (by $\mathbf{k}'' + \mathbf{k} + \hat{\mathbf{k}} + 2$), the estimate also holds on $\mathcal{D}_h$. Note that the condition $\alpha < \alpha_2$ in that proposition is already incorporated in Proposition 1.3 of this paper under the constant α'_*. Hence, we do not impose any further condition on α here.

Proposition 3.6 stated above is proved in [Che13, Proposition 3.3]. Indeed, the latter proves a stronger statement where the dependence of M on D is established and the inequality holds on a larger domain. The latter part of Proposition 3.6 follows from the proof of [Che13, Proposition 3.3].

Proof of Proposition 1.5. Fix $r \in [0, 1/\alpha - \mathbf{k}]$. The curve $t \mapsto \Phi_h^{-1}(t + r\mathbf{i})$ lands at 0 at a well-defined angle as $t \to +\infty$ if and only if $\lim_{t\to+\infty} \operatorname{Re} \chi_{h,0}(t + r\mathbf{i})$ exists and is finite. We use Proposition 3.6 with $D = 1$ to obtain a constant M_1 and the estimate on the derivative of $\chi_{h,0}$ above the horizontal line $1/\alpha$. For all $t_1 > t_2 \ge 1/\alpha$ we have

$$|\operatorname{Re} \chi_{h,0}(t_1 + r\mathbf{i}) - \operatorname{Re} \chi_{h,0}(t_2 + r\mathbf{i})| \le \int_{t_2}^{t_1} M_1 \alpha e^{-2\pi\alpha t}\, dt \le \frac{M_1}{2\pi} e^{-2\pi\alpha t_2}.$$

Since $e^{-2\pi\alpha t_2} \to 0$ as $t_2 \to +\infty$, we conclude that $\operatorname{Re} \chi_{h,0}(t + r\mathbf{i})$ tends to a finite limit as $t \to +\infty$. ◻

Recall the sets $\mathcal{D}_n$ defined in Section 2.1. Let $\rho_n(z)|dz|$ denote the Poincaré metric on $\mathcal{D}_n$, i.e. the hyperbolic metric of constant curvature -1. The changes of coordinates $\chi_{n,i} : \mathcal{D}_n \to \mathcal{D}_{n-1}$ have the following nice property with respect to these metrics.

Lemma 3.8. *There exists a constant $\rho \in (0, 1)$ such that for every $n \ge 1$ and all integers i with $0 \le i \le a_i$, we have $\|\chi'_{n,i}\| \le \rho$, where the norm is calculated with respect to the hyperbolic metrics on $\mathcal{D}_n$ and $\mathcal{D}_{n-1}$.*

The above lemma appears in [Che10], and is also proved here for the reader's convenience.

Proof. Let $\tilde{\rho}_n(z)|dz|$ denote the Poincaré metric on $\chi_{n,i}(\mathcal{D}_n)$. We may decompose the map $\chi_{n,i} : (\mathcal{D}_n, \rho_n) \to (\mathcal{D}_{n-1}, \rho_{n-1})$ as follows:

$$(\mathcal{D}_n, \rho_n) \xrightarrow{\chi_{n,i}} (\chi_{n,i}(\mathcal{D}_n), \tilde{\rho}_n) \xhookrightarrow{\text{inc.}} (\mathcal{D}_{n-1}, \rho_{n-1}).$$

By the Schwarz–Pick Lemma, the first map in the above chain is non-expanding. Hence, it is enough to show that the inclusion map is uniformly contracting in the respective metrics. For this, we use Lemma 2.1, which provides us with $\delta > 0$, independent of n and i, such that the δ-neighbourhood of $\chi_{n,i}(\mathcal{D}_n)$ is contained in $\mathcal{D}_{n-1}$.

To prove the uniform contraction, fix ξ_0 in $\chi_{n,i}(\mathcal{D}_n)$, and consider the map

$$H(\xi) := \xi + \frac{\delta(\xi - \xi_0)}{\xi - \xi_0 + 2\hat{\mathbf{k}} + 1}, \qquad \xi \in \chi_{n,i}(\mathcal{D}_n).$$

By Proposition 1.9, for every $\xi \in \chi_{n,i}(\mathcal{D}_n)$ we have $|\mathrm{Re}(\xi - \xi_0)| \leq \hat{\mathbf{k}}$ (note that $\mathcal{D}_h \subset \mathcal{D}'_h$). This implies that $|\xi - \xi_0| < |\xi - \xi_0 + 2\hat{\mathbf{k}} + 1|$, and hence $|H(\xi) - \xi| < \delta$. In particular, H is a holomorphic map from $\chi_{n,i}(\mathcal{D}_n)$ into $\mathcal{D}_{n-1}$. By the Schwarz–Pick Lemma, H is non-expanding. In particular, at $H(\xi_0) = \xi_0$ we obtain

$$\rho_{n-1}(\xi_0)|H'(\xi_0)| = \rho_{n-1}(\xi_0)\left(1 + \frac{\delta}{2\hat{\mathbf{k}} + 1}\right) \leq \hat{\rho}_n(\xi_0).$$

That is,

$$\rho_{n-1}(\xi_0) \leq \left(\frac{2\hat{\mathbf{k}} + 1}{2\hat{\mathbf{k}} + 1 + \delta}\right)\hat{\rho}_n(\xi_0).$$

As ξ_0 was arbitrary, this finishes the proof of the lemma. □

3.4. Geometry of sectors on level n

First we need the following basic property of the maps in $\mathcal{IS}_\alpha \cup \{Q_\alpha\}$.

Lemma 3.9. *There exists a constant $C_1 > 0$ such that for all $\alpha \in (0, \alpha^*]$ and $h \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ we have*

$$C_1^{-1}\alpha \leq |\sigma_h| \leq C_1\alpha.$$

Proof. The non-zero fixed point of Q_α has the form

$$(1 - e^{2\pi\alpha\mathbf{i}})\tfrac{16}{27}e^{-4\pi\alpha\mathbf{i}}.$$

Thus, there is an explicit constant C_1 satisfying the inequalities for the map Q_α. Below, we assume that $h \in \mathcal{IS}_\alpha$.

For every $h \in \mathcal{IS}_\alpha$, $0 \leq \alpha \leq \alpha^*$, there is a holomorphic map u_h, defined on the domain of h, such that

$$h(z) = z + z(z - \sigma_h)u_h(z).$$

The map u_h depends continuously on h. Consider the set

$$A = \{u_h(0) \mid h \in \mathcal{IS}_\alpha,\ 0 \leq \alpha \leq \alpha^*\}.$$

For $h \in \mathcal{IS}_\alpha$ with $\alpha \neq 0$, 0 is a simple fixed point of h and hence $u_h(0) \neq 0$. For $h \in \mathcal{IS}_0$, we have $u_h(0) = h''(0)/2$ where $|h''(0)|$ is uniformly bounded from above and away from 0, by Lemma 1.1. By the precompactness of the class $\bigcup_{\alpha\in[0,\alpha^*]}\mathcal{IS}_\alpha$, the set A is compactly contained in $\mathbb{C} \setminus \{0\}$.

Differentiating the equation $h(z) = z + z(z - \sigma_h)u_h(z)$ at 0 we obtain

$$\sigma_h = \frac{1 - e^{2\pi\alpha\mathbf{i}}}{u_h(0)}.$$

Combining this with the uniform bounds from the previous paragraph finishes the proof. □

For $\alpha \in (0, 1/2)$ define

$$X_\alpha = \{\zeta \in \mathbb{C} \mid \operatorname{Im} \zeta \geq -2, (1/\alpha)^{2/3} \leq \operatorname{Re} \zeta \leq 1/\alpha - (1/\alpha)^{2/3}\}. \tag{3.7}$$

Let M be the constant from Proposition 3.5 for $D = -1$, and define

$$\hat{X}_\alpha = \left\{ w \in \mathbb{C} \;\middle|\; \operatorname{Im} w \geq 2M \log \alpha,\ \tfrac{1}{2}(1/\alpha)^{2/3} \leq \operatorname{Re} w \leq 1/\alpha - \tfrac{1}{2}(1/\alpha)^{2/3} \right\}.$$

Choose $\delta_1 > 0$ such that

$$\delta_1 \leq 1/8, \qquad \delta_1 \leq (1/\mathbf{k})^{3/2},$$

and for all $\alpha \in (0, \delta_1]$ we have

$$\begin{aligned} &-2 - M \log(1 + 1/\alpha) \geq -2M \log(1/\alpha), \\ &\tfrac{1}{2}(1/\alpha)^{2/3} \leq (1/\alpha)^{2/3} - M \log(1 + 1/\alpha), \\ &1/\alpha - (1/\alpha)^{2/3} + M \log(1 + 1/\alpha) \leq 1/\alpha - \tfrac{1}{2}(1/\alpha)^{2/3}. \end{aligned} \tag{3.8}$$

Lemma 3.10. *For all $h \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $\alpha \in (0, \delta_1]$, we have $L_h(X_\alpha) \subset \hat{X}_\alpha$.*

Proof. First note that since $\alpha < 1/8$, both sets X_α and $\hat{X}_\alpha$ are non-empty. Also, since $\alpha \leq (1/\mathbf{k})^{3/2}$, the set X_α is contained in $\Phi_h(\mathcal{P}_h)$, and hence L_h^{-1} is defined on X_h.

Now we use Proposition 3.5 with $D = -1$. Note that for every $\zeta \in X_\alpha$,

$$\operatorname{Re} \zeta \in (0, 1/\alpha - \mathbf{k}) \quad \text{and} \quad \operatorname{Im} \zeta \geq -2 \geq -1/\alpha.$$

Hence the inequality of Proposition 3.5 holds at all points in X_α, with the constant M. In particular, the inequality implies that for all $\zeta \in X_\alpha$,

$$(1/\alpha)^{2/3} - M \log(1 + 1/\alpha) \leq \operatorname{Re} L_h(\zeta) \leq 1/\alpha - (1/\alpha)^{2/3} + M \log(1 + 1/\alpha)$$

and

$$\operatorname{Im} L_h(\zeta) \geq -2 - M \log(1 + 1/\alpha).$$

Now the lemma follows from the conditions imposed on δ_1 in (3.8). □

Lemma 3.11. *There exists $C > 0$ such that for all $h \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $\alpha \in (0, \delta_1]$ the following holds. For every $z \in \mathbb{C}$ with $\mathbb{E}\mathrm{xp}(z) \in T_h(\hat{X}_\alpha)$, we have*

$$\operatorname{Im} z \geq \frac{1}{3\pi} \log \frac{1}{\alpha} - C.$$

Proof. First note that for all $w \in \hat{X}_\alpha$,

$$-2\pi + \pi \alpha^{1/3} \leq \operatorname{Im}(-2\pi \alpha w \mathbf{i}) \leq -\pi \alpha^{1/3}.$$

This implies that

$$|\arg(e^{-2\pi \alpha w \mathbf{i}})| \leq \pi \alpha^{1/3}.$$

Using the inequality $\sin x \geq 2x/\pi$ on the interval $(0, \pi/2)$, as $\alpha \leq 1/8$, we conclude that

$$|1 - e^{-2\pi \alpha w \mathbf{i}}| \geq \sin(\pi \alpha^{1/3}) \geq 2\alpha^{1/3}.$$

On the other hand, by Lemma 3.9, $|\sigma_h| \leq C_1 \alpha$. Therefore, for all $w \in \hat{X}_\alpha$, the above inequality implies that

$$|T_h(w)| \leq \frac{C_1}{2} \alpha^{2/3}.$$

Assume that $z \in \mathbb{C}$ and $\mathbb{E}\mathrm{xp}(z) \in T_h(\hat{X}_\alpha)$. By the definition of $\mathbb{E}\mathrm{xp}$, the above inequality implies that

$$\operatorname{Im} z \geq \frac{1}{2\pi} \log \frac{8}{27C_1} + \frac{1}{2\pi} \frac{2}{3} \log \frac{1}{\alpha}. \qquad \square$$

Note that the statement of the above lemma holds similarly for $z \in \mathbb{C}$ with $\mathbb{E}\mathrm{xp}(z) \in s \circ T_h(\hat{X}_\alpha)$.

Lemma 3.12. *There exists $C' > 0$ such that for all $h \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $\alpha \in (0, \delta_1]$,*

$$\frac{1}{2\pi} \log \frac{1}{\alpha} - C' \leq \operatorname{Im} \chi_{h,0}\left(\frac{1}{2\alpha} - 2\mathbf{i}\right) \leq \frac{1}{2\pi} \log \frac{1}{\alpha} + C'.$$

Proof. Let $\zeta_0 = 1/(2\alpha) - 2\mathbf{i}$. Note that $\zeta_0 \in \Phi_h(\mathcal{P}_h) \subset \operatorname{Dom} L_h$ for $\alpha \leq 1/(2\mathbf{k})$. Using Proposition 3.5 with $D = -1$, we know that

$$|L_h(\zeta_0) - \zeta_0| \leq M \log(1 + 1/\alpha).$$

On the other hand, by the choice of δ_1, $M \log(1 + 1/\alpha) \leq 1/(3\alpha)$. It follows that

$$-2 - \frac{1}{3\alpha} \leq \operatorname{Im} L_h(\zeta_0) \leq \frac{1}{3\alpha}, \qquad \frac{1}{6\alpha} \leq \operatorname{Re} L_h(\zeta_0) \leq \frac{5}{6\alpha}.$$

Then

$$\frac{-5\pi}{3} \leq \operatorname{Im}(-2\pi\alpha L_h(\zeta_0)\mathbf{i}) \leq \frac{-\pi}{3}, \qquad \frac{-2\pi}{3} \leq \operatorname{Re}(-2\pi\alpha L_h(\zeta_0)\mathbf{i}) \leq 4\pi\alpha + \frac{2\pi}{3}.$$

Therefore,

$$1 \leq |1 - e^{-2\pi\alpha L_h(\zeta_0)\mathbf{i}})| \leq 1 + e^{7\pi/6}.$$

Combining the above bounds with the bounds on the size of σ_h in Lemma 3.9, we obtain

$$\frac{1}{C_1} \frac{1}{1 + e^{7\pi/6}} \alpha \leq |T_h(L_h(\zeta_0))| \leq C_1 \alpha.$$

Finally, as $\mathbb{E}\mathrm{xp}(\chi_{h,0}(\zeta_0)) = T_h(L_h(\zeta_0))$, we conclude that

$$\frac{1}{2\pi} \log \frac{1}{\alpha} - C' \leq \operatorname{Im} \chi_{h,0}(\zeta_0) \geq \frac{1}{2\pi} \log \frac{1}{\alpha} + C',$$

for some constant C' depending only on C_1. $\square$

In the following proposition, C' is the constant obtained in the above lemma.

Proposition 3.13. *There is $\delta_0 > 0$ such that for all $h \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $\alpha \in (0, \delta_0]$,*

$$\forall \zeta \in X_\alpha, \quad \operatorname{Im} \chi_{h,0}(\zeta) \geq \frac{1}{2} \operatorname{Im} \chi_h\left(\frac{1}{2\alpha} - 2\mathbf{i}\right) + C'.^{4} \tag{3.9}$$

[4] Here, C' may be replaced by any constant, with δ_0 depending on that constant.

Proof. There is $\delta_0 < \delta_1$ such that for all $\alpha \in (0, \delta_0]$ we have

$$\frac{1}{12\pi} \log \frac{1}{\alpha} \geq C + 3C'/2.$$

For all ζ in X_α, by Lemma 3.10, $L_h(\zeta)$ belongs to $\hat{X}_\alpha$. Then, by Lemma 3.11 ,

$$\operatorname{Im} \chi_{h,0}(\zeta) \geq \frac{1}{2\pi} \frac{2}{3} \log \frac{1}{\alpha} - C.$$

On the other hand, by Lemma 3.12,

$$\operatorname{Im} \chi_{h,0}\left(\frac{1}{2\alpha} - 2\mathbf{i}\right) \leq \frac{1}{2\pi} \log \frac{1}{\alpha} + C'.$$

Thus, we need

$$\frac{1}{2\pi} \frac{2}{3} \log \frac{1}{\alpha} - C \geq \frac{1}{2}\left(\frac{1}{2\pi} \log \frac{1}{\alpha} + C'\right) + C',$$

which is guaranteed by $\alpha \in (0, \delta_0]$. □

Lemma 3.14. *We have*

$$\lim_{\alpha \to 0^+} \frac{\sup_{w, w' \in X_\alpha} |\operatorname{Re} w - \operatorname{Re} w'|}{1/\alpha} = 1.$$

Proof. This is straightforward, since

$$\sup_{w, w' \in X_\alpha} |\operatorname{Re} w - \operatorname{Re} w'| \geq 1/\alpha - (1/\alpha)^{2/3} - (1/\alpha)^{2/3}$$

and

$$\lim_{\alpha \to 0^+} \frac{1/\alpha - 2(1/\alpha)^{2/3}}{1/\alpha} = 1.$$ □

3.5. The heights of chains in the tower

Recall the maps $\chi_{n,0} = \chi_{f_n} : \mathcal{D}_n \to \mathcal{D}_{n-1}$ for $n \geq 1$. Given $n \geq 1$ and a point $\zeta_n \in \mathcal{D}_n$, define

$$\mathcal{C}_n(\zeta_n) = \{\langle \zeta_j \rangle_{j=n}^0 \mid \forall j \text{ with } 0 \leq j \leq n-1,\ \zeta_j \in \mathcal{D}_j \text{ and } \zeta_j \in \chi_{j+1,0}(\zeta_{j+1}) + \mathbb{Z}\}.$$

That is, members of $\mathcal{C}_n(\zeta_n)$ are (ordered) sequences of length $n+1$ starting with ζ_n and mapping an element to the next element in the sequence by translation by some $\chi_{j,0}$. Given $\tau = \langle \zeta_n, \zeta_{n-1}, \ldots, \zeta_0 \rangle$ in $\mathcal{C}_n(\zeta_n)$, for every $k \in \{n, n-1, \ldots, 0\}$ we define $\tau_k = \zeta_k$. We aim to study the behaviour of the sequences $\langle \operatorname{Im} \tau_j \rangle_{j=n}^0$, for τ in $\mathcal{C}_n(t)$, as t varies in $\mathcal{D}_n$. This heavily depends on the arithmetic of α.

For Brjuno values of α, let $\Delta(f_j)$, for $j \geq 0$, denote the Siegel disk of f_j. Then define

$$\tilde{\Delta}(f_j) = \Phi_j(\Delta(f_j) \cap \mathcal{P}_j) \quad \text{for } j \geq 0.$$

Also, recall the bi-sequence $B_{n,j}$ defined in Section 3.2.

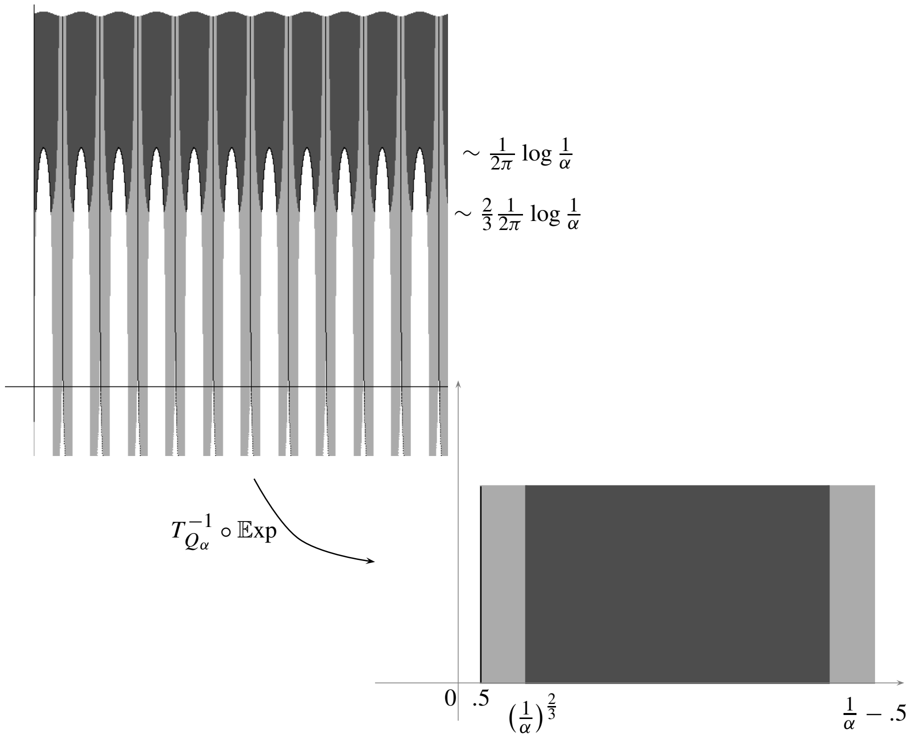

Fig. 5. The lower part shows a dark grey rectangle $[(1/\alpha)^{2/3}, 1/\alpha - (1/\alpha)^{2/3}] + \mathbf{i}[0, .5/\alpha]$, and two light grey rectangles $[.5, 1/\alpha)^{2/3}] + \mathbf{i}[0, .5/\alpha]$ and $[1/\alpha - (1/\alpha)^{2/3}, 1/\alpha - .5] + \mathbf{i}[0, .5/\alpha]$. The top figure shows some of the lifts of these rectangles under $T_{Q_\alpha}^{-1} \circ \mathbb{E}\mathrm{xp}$. The dark grey rectangle lifts to periodic regions above the approximate height $\frac{1}{2\pi} \frac{2}{3} \log \frac{1}{\alpha}$, up to an additive constant. The lift of the line segment between $(.5, 0)$ and $(.5, .5/\alpha)$ lifts to the black curves separating different lifts. Here, $\alpha = 1/1000$ and the choice of .5 is only to make the pictures clearer.

Proposition 3.15. *There are constants B and M in $\mathbb{R}$ satisfying the following. Let $B_{k,i}$ be the bi-sequence defined in (3.1) with the constant B, and let $\zeta_n = 1/(2\alpha_n) - 2\mathbf{i}$. Then*

(a) *for all $\alpha \in HT_N$ and every integer $n \geq 1$ there exists $\tau \in \mathcal{C}_n(\zeta_n)$ such that for all j with $0 \leq j \leq n-1$,*

$$B_{n,j} \leq \operatorname{Im} \tau_j \leq B_{n,j} + M;$$

(b) *if α is a Brjuno number, for every $j \geq 0$ we have*

$$\left| \lim_{n\to\infty} B_{n,j} - \max_{w \in \partial \tilde{\Delta}(f_j)} \operatorname{Im} w \right| < M.$$

Proof. (a) By the condition (1.8) on the rotation numbers α_n we have

$$\frac{1}{\alpha_n} - \mathbf{k} - \frac{1}{2\alpha_n} \geq \frac{\hat{\mathbf{k}}}{2} + \frac{5}{2} \geq 3.$$

Hence,

$$\zeta_n = \frac{1}{2\alpha_n} - 2\mathbf{i} \in \Phi_n(\mathcal{P}_n) \subset \mathcal{D}_n, \qquad \forall n \geq 1. \tag{3.10}$$

Inductively define the sequence ζ_j, for $n-1 \leq j \leq 0$, such that

$$\zeta_j \in \chi_{j+1,0}(\zeta_{j+1}) + \mathbb{Z} \quad \text{and} \quad \operatorname{Re} \zeta_j \in \left[\frac{1}{2\alpha_j} - \frac{1}{2}, \frac{1}{2\alpha_j} + \frac{1}{2}\right].$$

The above condition on the rotation implies that $\zeta_j \in \mathcal{D}_j$ for all j with $0 \leq j \leq n-1$. In particular, $\tau = \langle \zeta_j \rangle_{j=n}^0$ belongs to $\mathcal{C}_n(\zeta_n)$.

Note that for every $j \geq 0$, the set $\Phi_j^{-1}(\mathcal{D}_j)$ is contained in the image of f_j, because $f_j(\bigcup_{i=0}^{k_j} f_j^{\circ i}(S_j^0) \cup \mathcal{P}_j)$ covers this set. By the definition of renormalization, the image of each f_j, $j \geq 1$, is equal to the disk of radius $e^{4\pi} 4/27$ about zero. In particular, $\Phi_n^{-1}(\zeta_n)$ is contained in the image of f_n. This implies that $\operatorname{Im} \zeta_{n-1} \geq -2$. Continuing inductively, we conclude that $\operatorname{Im} \zeta_j \geq -2$ for all j with $0 \leq j \leq n$. In particular, for all such j,

$$\operatorname{Im} \zeta_j \geq -1/\alpha.$$

Let L_j denote the map L_{f_j} introduced in (3.6). By the above inequality, we may apply Proposition 3.5 with $D = -1$ to obtain $M_1 \in \mathbb{R}$ such that for $0 \leq j \leq n$,

$$L_j(\zeta_j) \in B(\zeta_j, M_1 \log(1 + 1/\alpha_j)).$$

Then, by a basic estimate there is a constant M_2 depending only on M_1 such that for $0 \leq j \leq n$,

$$M_2^{-1} e^{-2\pi\alpha_j \operatorname{Im} \zeta_j} \leq |1 - e^{-2\pi\alpha_j \mathbf{i} L_j(\zeta_j)}| \leq M_2 e^{-2\pi\alpha_j \operatorname{Im} \zeta_j}.$$

Now, the uniform bounds in Lemma 3.9 imply that there exists a constant M_3 depending only on M_2 and C_1 such that for $0 \leq j \leq n$,

$$M_3 \alpha_j e^{-2\pi\alpha_j \operatorname{Im} \zeta_j} \leq |\Phi_j^{-1}(\zeta_j)| \leq M_3 \alpha_j e^{-2\pi\alpha_j \operatorname{Im} \zeta_j}.$$

As $\mathbb{E}\mathrm{xp}\, \zeta_{j-1} = \Phi_j^{-1}(\zeta_j)$ or $\mathbb{E}\mathrm{xp}\, \zeta_{j-1} = s \circ \Phi_j^{-1}(\zeta_j)$, an explicit estimate on $\mathbb{E}\mathrm{xp}$ implies that there is $M_4 \geq 0$ such that for $0 \leq j \leq n$,

$$\alpha_j \operatorname{Im} \zeta_j + \log \frac{1}{\alpha_{j-1}} - M_4 \leq \operatorname{Im}(\zeta_{j-1}) \leq \alpha_j \operatorname{Im} \zeta_j + \log \frac{1}{\alpha_{j-1}} + M_4. \tag{3.11}$$

Note that the constant M_4 depends only on the class $\mathcal{IS}$.

Let $B = M_4$, that is, the constant used in the definition of the bi-sequence $B_{k,j}$. By reverse induction (from $j = n$ to $j = 0$) we now show that

$$B_{n,j} \leq \operatorname{Im} \zeta_j \leq B_{n,j} + 4M_4\Big(1 + \sum_{i=j}^{n} \prod_{l=i+1}^{n} \alpha_{l-1}\Big). \tag{3.12}$$

For $j = n$, (3.12) reduces to $B_{n,n} = -2 \le \operatorname{Im} \zeta_n \le -2$ (there is no sum), which holds by the definition of ζ_n. Assume that (3.12) holds for j. Multiplying both sides of (3.12) by α_j, then adding $\log(1/\alpha_j)$ and subtracting B we come up with

$$\alpha_j B_{n,j} + \log \frac{1}{\alpha_{j-1}} - B \le \alpha_j \operatorname{Im} \zeta_j + \log \frac{1}{\alpha_{j-1}} - B$$

and

$$\alpha_j \operatorname{Im} \zeta_j + \log \frac{1}{\alpha_{j-1}} - B \le \alpha_j B_{n,j} + \log \frac{1}{\alpha_{j-1}} - B + 4\alpha_j M_4 \Big(1 + \sum_{i=j}^{n} \beta_i^{-1} \beta_n\Big).$$

By the definition of the bi-sequence and (3.11), the first inequality above provides

$$B_{n,j-1} \le \alpha_j \operatorname{Im} \zeta_j + \log \frac{1}{\alpha_{j-1}} - B \le \operatorname{Im} \zeta_{j-1},$$

and similarly the second inequality gives

$$\begin{aligned}
\operatorname{Im}(\zeta_{j-1}) &\le \alpha_j \operatorname{Im} \zeta_j + \log \frac{1}{\alpha_{j-1}} + M_4 \\
&\le \alpha_j \operatorname{Im} \zeta_j + \log \frac{1}{\alpha_{j-1}} - B + 2M_4 \\
&\le B_{n,j-1} + 4\alpha_j M_4 \Big(1 + \sum_{i=j}^{n} \beta_i^{-1} \beta_n\Big) + 2M_4 \\
&= B_{n,j-1} + 4M_4 \Big(\alpha_j + \sum_{i=j-1}^{n} \beta_i^{-1} \beta_n\Big) + 2M_4 \\
&\le B_{n,j-1} + 4M_4 \Big(1 + \sum_{i=j-1}^{n} \beta_i^{-1} \beta_n\Big).
\end{aligned}$$

In the last inequality we have used $\alpha_j + 1/2 \le 1$. This finishes the proof of the induction step.

Finally, since $\alpha_j \in (0, 1/2)$ for all j, we have

$$\begin{aligned}
4M_4 \Big(1 + \sum_{i=j}^{n} \beta_{i-1}^{-1} \beta_{n-1}\Big) &\le 4M_4 \bigg(1 + \sum_{i=j}^{n} \Big(\frac{1}{2}\Big)^{n-i}\bigg) \\
&\le 4M_4 \bigg(1 + \sum_{i=0}^{\infty} \Big(\frac{1}{2}\Big)^{i}\bigg) \le 12M_4.
\end{aligned}$$

The above bound combined with (3.12) implies (a), that is, we define $M = 12M_4$.

(b) By (3.1), for every $j \ge 0$, $\lim_{n\to\infty} B_{n,j}$ exists and

$$\Big| \lim_{n\to\infty} B_{n,j} - \beta_j^{-1} \sum_{i=j}^{\infty} \beta_i \log \alpha_{i+1}^{-1} \Big| \le B\beta_j^{-1} \sum_{i=j}^{\infty} \beta_i \le 2B.$$

Each f_j, $j \geq 0$, belongs to the class $\mathcal{IS}_{\alpha_j}$ and must be univalent on $B(0, 1/12)$. That is because the polynomial P is univalent on $B(0, 1/3)$ and by the 1/4-theorem, $\varphi(B(0, 1/3))$ contains $B(0, 1/12)$. By the classical result of Yoccoz [Yoc95b], there is a uniform constant $C > 0$ (independent of j) such that the ball of radius $C \exp(-\beta_j^{-1} \sum_{i=j}^{\infty} \beta_i \log \alpha_{i+1}^{-1})$ about 0 is contained in $\Delta(f_{j+1})$. Since $\Delta(f_{j+1})$ lifts under $\mathbb{E}\mathrm{xp}$ to the set $\tilde{\Delta}(f_j) + \mathbb{Z}$, we have

$$\max_{w \in \partial \tilde{\Delta}(f_j)} \operatorname{Im} w \leq \frac{1}{2\pi} \log \frac{4}{27C} + \frac{1}{2\pi} \beta_j^{-1} \sum_{i=j}^{\infty} \beta_i \log \alpha_{i+1}^{-1}.$$

Indeed, one may recover the lower bound of Yoccoz from the estimates in (a), which we briefly sketch here. In (a) we have chosen the branches of the lifts $\chi_{j,0} + \mathbb{Z}$ (the ones at the centre) to obtain the highest possible imaginary parts. That is, all other choices lead to smaller imaginary parts. This implies that all the sets Υ_n, similarly defined for f_j, contain the ball of radius $C'' \exp(-\beta_j^{-1} \sum_{i=j}^{\infty} \beta_i \log \alpha_{i+1}^{-1})$ centred at 0, for some uniform constant C''. Since Proposition 2.7 holds for all maps f_j, with appropriately defined Υ^n, one concludes that $\Delta(f_j)$ must contain the disk of that radius.

On the other hand, we need to prove an upper bound on the size of the biggest ball that fits into the Siegel disk of f_j in terms of the above infinite series. This upper bound is proved in [Che10]; we briefly present the argument below for the reader's convenience. Another proof of the upper bound, which only works when f_j is a quadratic polynomial, is given in [BC04].

Fix $j \geq 0$ and let $n > j$. Let $\tau \in \mathcal{C}_n(\zeta_n)$ denote the sequence obtained in (a). Consider the sequence $\tau' \in \mathcal{C}_n(\langle 1/(2\alpha_n) \rangle)$ defined along the same branches determining τ; that is, for each i with $1 \leq i \leq n$, if $\zeta_{i-1} = \chi_{i,l}(\zeta_i)$ for some integer l then $\tau'_{i-1} = \chi_{i,l}(\tau'_i)$. The point $\Phi_n^{-1}(\tau'_n)$ is in the forward orbit of the critical point of f_n. It follows from the proof of Proposition 2.13 that $\Phi_j^{-1}(\tau'_j)$ is in the forward orbit of the critical point of f_j. In particular, $\Phi_j^{-1}(\tau'_j)$ is not in the Siegel disk of f_j. By definition, $\tau'_j \notin \tilde{\Delta}(f_j)$.

By the uniform contraction in Lemma 3.8 and the uniform inclusion in Lemma 2.1, we conclude that provided n is sufficiently large, $|\operatorname{Im} \zeta_j - \operatorname{Im} \tau'_j| \leq 1$. Thus, by (a), $\operatorname{Im} \tau'_j \geq B_{n,j} - 1$. (Alternatively, one may repeat the estimates in the proof of (a) for the sequence τ' to obtain estimates as in (3.12) for τ'.) Combining this with the previous paragraph, we obtain

$$\max_{w \in \partial \tilde{\Delta}(f_j)} \operatorname{Im} w \geq \operatorname{Im} \tau'_j \geq B_{n,j} - 1.$$

In particular,

$$\max_{w \in \partial \tilde{\Delta}(f_j)} \operatorname{Im} w - \lim_{n \to \infty} B_{n,j} \geq -1.$$

This finishes the proof of (b). □

By Proposition 3.6 with $D = 1$ there is a constant M_1 such that the inequality in that proposition holds. In particular, for every $D \geq 1$, the inequalities hold with the constant M_1. This implies that we may choose $D \geq 1$ such that with the corresponding constant M from the proposition, for every $\alpha \in (0, 1/2)$ we have

$$\int_{D/\alpha}^{\infty} M\alpha e^{-2\pi\alpha t}\, dt = \frac{M}{2\pi} e^{-2\pi D} \leq 1/2, \tag{3.13}$$

$$\int_{0}^{1/\alpha+\mathbf{k}''+\mathbf{k}+\hat{\mathbf{k}}+2} M\alpha e^{-2\pi D}\, dt = M\alpha e^{-2\pi D}(1/\alpha + \mathbf{k}'' + \mathbf{k} + \hat{\mathbf{k}} + 2) \leq 1/2. \tag{3.14}$$

Lemma 3.16. *With the constant D obtained above we have the following. If for some positive integers $m < n$ there exists $\tau \in \mathcal{C}_n(1/(2\alpha_n) - 2\mathbf{i})$ satisfying*

$$\operatorname{Im} \tau_j \geq \frac{D+2}{\alpha_j}, \quad \forall j \text{ with } m \leq j \leq n-1,$$

then for every $\tau' \in \mathcal{C}_n(1/(2\alpha_n) - 2\mathbf{i})$ we have

$$|\operatorname{Im} \tau'_j - \operatorname{Im} \tau_j| \leq 2, \quad \forall j \text{ with } m-1 \leq j \leq n-1.$$

Proof. Let $\tau \in \mathcal{C}_n(1/(2\alpha_n) - 2\mathbf{i})$ satisfy the hypotheses, and $\tau' \in \mathcal{C}_n(1/(2\alpha_n) - 2\mathbf{i})$ be arbitrary. By induction we will show that for $0 \leq j \leq n-1$ we have

$$|\operatorname{Im} \tau'_j - \operatorname{Im} \tau_j| \leq 1 + \sum_{k=j+1}^{n-1} \beta_j^{-1}\beta_k. \tag{3.15}$$

First note that since $\alpha_i \in (0, 1/2)$ for all $i \geq 0$, we have

$$1 + \sum_{k=j+1}^{n-1} \beta_j^{-1}\beta_k \leq 1 + \sum_{k=1}^{n-1} \beta_k \leq 1 + \sum_{k=1}^{\infty} \frac{1}{2^k} \leq 2.$$

For $j = n-1$, we have $\tau'_{n-1} \in \chi_{n,0}(\zeta_n) + \mathbb{Z}$, and hence $\operatorname{Im} \tau'_{n-1} = \operatorname{Im} \tau_{n-1}$. Therefore, (3.15) holds for $j = n-1$.

Assume (3.15) holds for some j. By the assumption in the lemma we have $\operatorname{Im} \tau_j \geq (D+2)/\alpha_j$, and so by the induction assumption,

$$\operatorname{Im} \tau'_j \geq (D+2)/\alpha_j - 2 \geq D/\alpha_j.$$

Let γ_j be a piecewise smooth curve in $\mathcal{D}_j$ that lies above the horizontal line passing through $D\mathbf{i}/\alpha_j$ and connects τ_j to τ'_j. We may choose γ_j to consist of a horizontal line segment of length at most $1/\alpha_j + \mathbf{k}'' + \mathbf{k} + \hat{\mathbf{k}} + 2$ and a vertical line segment of length at most 2. Then

$$\begin{aligned} |\operatorname{Im}(\tau_{j-1} - \tau'_{j-1})| &= \left|\operatorname{Im} \int_{\gamma_j} \chi'_{j,0}\, dz\right| \leq \operatorname{Im} \int_{\gamma_j} |\chi'_{j,0} - \alpha_j|\, dz + \left|\operatorname{Im} \int_{\gamma_j} \alpha_j\, dz\right| \\ &\leq 1/2 + 1/2 + \alpha_j |\operatorname{Im}(\tau_j - \tau'_j)|, \end{aligned}$$

where in the last inequality we have used (3.13) and (3.14). Combining this with the induction hypothesis we obtain (3.15) for $j-1$. □

3.6. *Proof of Proposition 3.1*

Recall the set J_n defined in Section 2.4 and the set X_{α_n} defined in Section 3.4. Consider the set

$$G_n = \{j \in \mathbb{Z} \mid f_n^{\circ j}(J_n) \subset \mathcal{P}_n,\ \Phi_n(f_n^{\circ j}(J_n)) \subset X_{\alpha_n}\}.$$

Lemma 3.17. *For every $\varepsilon > 0$, there exists $\delta > 0$ such that if $\alpha_n \le \delta$ for some $n \ge 1$, then*

$$\frac{|G_n|}{a_n + 1} > 1 - \varepsilon. \tag{3.16}$$

Proof. Recall that the set $\Phi_n(J_n)$ satisfies (2.5). Hence, by Proposition 1.3, for every positive integer $j \le a_n - \mathbf{k} - \hat{\mathbf{k}} - 3$, $\Phi_n(J_n) + j$ is contained in $\Phi_n(\mathcal{P}_n)$. In particular, for all such j, $f_n^{\circ j}(J_n)$ is defined and is contained in $\mathcal{P}_n$.

On the other hand, for all integers j such that

$$(1/\alpha_n)^{2/3} \le j + 1 \quad \text{and} \quad \hat{\mathbf{k}} + 2 + j \le (1/\alpha_n) - (1/\alpha_n)^{2/3},$$

we have $\Phi_n(J_n) + j \subset X_{\alpha_n}$. Therefore, all such integers j are in G_n. Since the number of integers in a closed interval of length l is at least $l - 1$, we conclude that

$$\begin{aligned} |G_n| &\ge (1/\alpha_n) - (1/\alpha_n)^{2/3} - \hat{\mathbf{k}} - 2 - (1/\alpha_n)^{2/3} - 1 - 1 \\ &\ge (1/\alpha_n) - 2(1/\alpha_n)^{2/3} - \hat{\mathbf{k}} - 4. \end{aligned}$$

Since

$$\frac{(1/\alpha_n) - 2(1/\alpha_n)^{2/3} - \hat{\mathbf{k}} - 4}{\langle 1/\alpha_n\rangle + 1} \to 1 \quad \text{as } \alpha_n \to 0,$$

the statement of the lemma follows. □

Recall the notation $\beta_0 = 1$ and $\beta_k = \prod_{i=1}^{k} \alpha_i$ for $k \ge 1$. Also, recall the bi-sequence $B_{n,j}$, defined in 3.1. Consider the sets

$$W_j^n = \left\{w \in \mathcal{D}_j \;\middle|\; \operatorname{Im} w \ge B_{n,j} - \tfrac{1}{2}\beta_j^{-1}\beta_{n-1}\log(1/\alpha_n) - 4\right\}.$$

Recall the set $\mathcal{L}(\alpha, T, l)$ defined in Section 3.2. In the following proposition, D is the constant satisfying (3.13) and (3.14) (or any larger constant).

Proposition 3.18. *Let $T \in \mathbb{R}$ be a constant satisfying*

$$T \ge (D+2)\frac{4\pi}{4\pi - 1}, \tag{3.17}$$

and m be a positive integer such that $\mathcal{L}(\alpha, T, m)$ is not empty. Then, for all $n \in \mathcal{L}(\alpha, T, 0)$ and for all integers l_i, for $m + 1 \le i \le n$ with $0 \le l_i \le a_{i-1}$, we have

$$\chi_{m+1,l_{m+1}} \circ \chi_{m+2,l_{m+2}} \circ \cdots \circ \chi_{n,l_n}(X_{\alpha_n}) \subset W_m^n.$$

Proof. For any $w_n \in X_{\alpha_n}$, define a sequence $w_n, w_{n-1}, \dots, w_m$ by

$$w_{j-1} = \chi_{j,l_j}(w_j) \quad \text{for } m+1 \le j \le n.$$

As $n \in \mathcal{L}(\alpha, T, m)$, by Proposition 3.15(a), there exists $\tau \in \mathcal{C}_n(1/(2\alpha_n) - 2\mathbf{i})$ with

$$\operatorname{Im} \tau_j \ge B_{n,j} \ge T/\alpha_j \quad \text{for } m+1 \le j \le n-1. \tag{3.18}$$

Let $\tau' \in \mathcal{C}_n(1/(2\alpha_n) - 2\mathbf{i})$ be the sequence defined along the same branches as $\langle w_j \rangle$, that is,

$$\tau'_{j-1} = \chi_{j,l_j}(\tau'_j) \quad \text{for } m+1 \le j \le n.$$

By Lemma 3.16, we must have

$$\operatorname{Im} \tau'_j \ge B_{n,j} - 2 \quad \text{for } m \le j \le n-1. \tag{3.19}$$

By an inductive procedure we now show that for $m \le j \le n-1$,

$$\operatorname{Im} w_j \ge \operatorname{Im} \tau'_j - \frac{1}{4\pi} \beta_j^{-1} \beta_{n-1} \log \frac{1}{\alpha_n} - \beta_j^{-1} \sum_{k=j}^{\infty} \beta_k. \tag{3.20}$$

By definition,

$$\operatorname{Im} \tau'_{n-1} = \operatorname{Im} \tau_{n-1} = \operatorname{Im} \chi_{n,0}\left(\frac{1}{2\alpha_n} - 2\mathbf{i}\right).$$

For $w_n \in X_{\alpha_n}$, by Lemma 3.12 and Proposition 3.13, we have

$$\begin{aligned} \operatorname{Im} w_{n-1} - \operatorname{Im} \tau'_{n-1} &\ge \frac{1}{2} \operatorname{Im} \tau'_{n-1} + 2C' - \operatorname{Im} \tau'_{n-1} \\ &\ge -\frac{1}{2} \frac{1}{2\pi} \log \frac{1}{\alpha_n} - \frac{C'}{2} + C' \\ &\ge -\frac{1}{4\pi} \log \frac{1}{\alpha_n}. \end{aligned}$$

Therefore, (3.20) holds for $j = n-1$.

Assume we have (3.20) for some j. We want to prove it for $j-1$. By the hypothesis and (3.19),

$$\begin{aligned} \operatorname{Im} w_j &\ge \operatorname{Im} \tau'_j - \frac{1}{4\pi} \beta_j^{-1} \beta_{n-1} \log \frac{1}{\alpha_n} - \beta_j^{-1} \sum_{k=j}^{\infty} \beta_k && ((3.20)) \\ &\ge (B_{n,j} - 2) - \frac{1}{4\pi} \beta_j^{-1} \beta_{n-1} \log \frac{1}{\alpha_n} - 2 && ((3.19) \text{ and } (3.3)) \\ &\ge B_{n,j} - \frac{1}{4\pi} \beta_j^{-1} \beta_{n-1} \log \frac{1}{\alpha_n} - 4 \\ &\ge \left(1 - \frac{1}{4\pi}\right) B_{n,j} - 4 && ((3.2)) \\ &\ge \left(1 - \frac{1}{4\pi}\right) \frac{T}{\alpha_j} - 4 \ge \frac{D}{\alpha_j} && ((3.18) \text{ and } (3.17)) \end{aligned}$$

By the above inequality and (3.17)–(3.19), the points w_j and τ_j lie above the horizontal line $\operatorname{Im} w = D/\alpha_j$. On the other hand, since they belong to the image of the same lift $\chi_{j+1,l_{i+1}}$, by (2.1) their real parts differ at most by $\hat{\mathbf{k}}$. We may choose a piecewise horizontal-vertical curve γ_j connecting w_j to τ'_j that lies above the horizontal line $\operatorname{Im} w = D/\alpha_j$. Hence,

$$w_{j-1} - \tau'_{j-1} = \int_{\gamma_j} \chi'_{j,l_j} = \int_{\gamma_j} (\chi'_{j,l_j} - \alpha_j) + \int_{\gamma_j} \alpha_j.$$

By (3.13) and (3.14), $|\int_{\gamma_j} (\chi'_{j,l_j} - \alpha_j)|$ is uniformly bounded by 1. By taking the imaginary part of the above equation, and using (3.20) for j, we obtain (3.20) for $j-1$.

Finally, (3.19), (3.20) for $j = m$, as well as (3.3), imply the desired lower bound on $\operatorname{Im} w_m$; that is, w_m belongs to W^n_m. □

Now we are ready to prove the following stronger statement that implies Proposition 3.1.

Proposition 3.19. *For every non-Brjuno $\alpha \in HT_N$ and every $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ there exists a sequence of real numbers δ_i, $i \geq 1$, converging to zero such that*

$$\limsup_{n\to\infty} \frac{|G(n,\delta_n)|}{q_n} = 1.$^{[5]}$$

Proof. The proof is just by putting the above proposition and the lemma together. Define

$$\delta_{n+1} = \sup_{w\in W^n_0} |\Phi_0^{-1}(w)|, \qquad n \geq 0.$$

Since α is a non-Brjuno number,

$$\lim_{n\to+\infty} \big(B_{n,0} - \tfrac{1}{2}\alpha_1 \ldots \alpha_{n-1} \log(1/\alpha_n) - 4\big) = +\infty.$$

This implies that $\delta_n \to 0$ as $n \to +\infty$.

Let $\varepsilon > 0$. We need to find $j \in \mathbb{N}$ such that $\delta_j \leq \varepsilon$ and $|G(j,\delta_j)|/q_j \geq 1-\varepsilon$. Let δ be the constant obtained in Lemma 3.17 for this ε.

By Lemma 3.4, if α is a non-Brjuno number, for any $T \in \mathbb{R}$ the set $\mathcal{L}(\alpha, T, 0)$ is non-empty. For $T > 0$, let $n = \min \mathcal{L}(\alpha, T, 0)$, where n depends on T. By the definition of $\mathcal{L}(\alpha, T, 0)$, if T tends to $+\infty$, then n tends to $+\infty$ and α_n tends to 0. Indeed, if $n \in \mathcal{L}(\alpha, T, 0)$ we must have $B_{n,1} \geq T/\alpha_1$ where $B_{n,1}$ is a finite number. Thus, as T gets larger, n must be larger. On the other hand, $n \in \mathcal{L}(\alpha, T, 0)$ requires $B_{n,n-1} \geq T\alpha_{n-1}^{-1}$, which by definition of the bi-sequence implies that

$$-2\alpha_n + \log \alpha_n^{-1} - B \geq T/\alpha_{n-1}.$$

Hence, as T tends to infinity, α_n must tend to 0. Therefore, we may choose $T > 0$ so large that it satisfies (3.17) and $n = \mathcal{L}(\alpha, T, 0)$ is large enough that $\delta_n \geq \varepsilon$ and $\alpha_n \in (0, \delta)$.

[5] It is likely that one can replace the lim sup by lim in this proposition, but we do not need it here.

By Lemma 3.17, we have $|G_n|/q_n \geq 1 - \varepsilon$. And by definition, for all $j \in G_n$, $\Phi_n(f_n^{\circ j}(J_n)) \subset X_{\alpha_n}$. Then, by Proposition 3.18 with $m = 0$, for $j \in G_n$, $\Phi_n(f_n^{\circ j}(J_n))$ lifts to a set in W_0^n under all possible lifts in the renormalization tower. By Proposition 2.13, for every $i \in G_n$ and every integer k with $iq_n \leq k < (i+1)q_n$, $f_0^{\circ k}(J_n) \subset B_{\delta_{n+1}}(0)$. Hence,

$$\frac{|G(n+1, \delta_{n+1})|}{q_{n+1}} \geq \frac{|G_n| q_n}{q_{n+1}} \geq \frac{|G_n| q_n}{(a_n + 1) q_n} \geq 1 - \varepsilon.$$

As ε was arbitrary, this finishes the proof of the proposition. □

Remark 3.20. One may extract an alternative proof of Proposition 3.1 from the above analysis. The steps in the above argument are set up to be used for the argument in the linearizable case. But, roughly speaking, an alternative proof may go as follows. Given $\varepsilon > 0$ choose D large enough so that the points $w \in \mathcal{D}_0$ with $\operatorname{Im} w \geq D/\alpha_0$ map into $B_\varepsilon(0)$ under Φ_0^{-1}. Then, choose $T \in \mathbb{R}$ that satisfies (3.17). The set $\mathcal{L}(\alpha, T, 0)$ has infinite cardinality, hence we may let $n \to \infty$ within it. We start with $\operatorname{Im} w_{n-1} \geq T/(2\alpha_{n-1})$. Then, inductively, using the five-line inequality in the proof of Proposition 3.18 to obtain the lower bound on $\operatorname{Im} w_j$, we show that $\operatorname{Im} w_0 \geq D/\alpha_0$.

3.7. Proof of Proposition 3.2

Recall that for $m \geq 0$, $\Delta(f_m)$ denotes the Siegel disk of f_m, and $\tilde{\Delta}(f_m)$ denotes the image of $\Delta(f_m) \cap \mathcal{P}_m$ under Φ_m. We consider the following two scenarios:

$\mathscr{A}$: for every $T \in \mathbb{R}$ there exist $m_0 \in \mathbb{N}$ and infinitely many integers $m > m_0$ such that there is $\tau \in \mathcal{C}_m(1/(2\alpha_m) - 2\mathbf{i})$ satisfying $\operatorname{Im} \tau_j \geq T/\alpha_j$ for all integers j with $m_0 + 1 \leq j \leq m - 1$.

$\mathscr{B}$: there exist real constants T and C as well as infinitely many integers m such that for all $w \in \mathcal{D}_m$ with $\operatorname{Im} w \geq T/\alpha_m + C$ we have $\Phi_m^{-1}(w) \in \Delta(f_m)$.

Lemma 3.21. *For every irrational $\alpha \in HT_N$, at least one of $\mathscr{A}$ and $\mathscr{B}$ holds.*

Proof. By Proposition 3.15(a), there are constants M and B such that the bi-sequence $B_{n,i}$ defined in Section 3.2 using B satisfies the following: for every $n \geq 1$ and every j with $0 \leq j \leq n-1$ we have $B_{n,j} \leq \operatorname{Im} \tau_j \leq B_{n,j} + M$. Recall the set $\mathcal{L}(\alpha, T, l)$ defined in Section 3.2.

Assume that $\mathscr{A}$ does not hold. That means there is $T_0 \in \mathbb{R}$ such that for every $m_0 \in \mathbb{N}$ there is $m' \in \mathbb{N}$ such that for every $m \geq m'$ and every $\tau \in \mathcal{C}_m(1/(2\alpha_m) - 2\mathbf{i})$ there is an integer j with $m - 1 \leq j \leq m_0 + 1$ such that $\operatorname{Im} \tau_j < T_0/\alpha_j$. In particular, $\mathscr{A}$ does not hold for any constant larger than T_0. Below we assume that $T_0 > 0$.

By the first paragraph above, for some choice of $\tau \in \mathcal{C}_m(1/(2\alpha_m) - 2\mathbf{i})$, $B_{n,j} \leq \operatorname{Im} \tau_j$. Then, by the second paragraph above, this implies that $\mathcal{L}(\alpha, T_0, l)$ is finite for every $l \in \mathbb{N}$. Combining this with Lemma 3.4(c), we conclude that there exists a sequence of integers $\ell_0, \ell_1, \ell_2, \ldots$, tending to $+\infty$, such that

$$\sup_{j \geq 0} \left(\lim_{m \to \infty} B_{m,\ell_j} - T_0 \alpha_{\ell_j}^{-1} \right) < +\infty.$$

Then, by Proposition 3.15(b), we obtain

$$\sup_{j\geq 0}\Big(\max_{w\in\partial(\Phi_j(\Delta(f_j)))} \operatorname{Im} w - T\alpha_{\ell_j}^{-1}\Big) < +\infty.$$

This implies that $\mathscr{B}$ must hold. □

Lemma 3.22. *Assume that $\alpha \in HT_N$ satisfies $\mathscr{A}$, and T is a real constant satisfying (3.17) and*

$$T \geq M+3, \tag{3.21}$$

where M is the constant in Proposition 3.15. Let $m_0 \in \mathbb{N}$ and the infinite set of integers A be such that for all $m \in A$ there is $\tau \in \mathcal{C}_m(1/(2\alpha_m) - 2\mathbf{i})$ satisfying $\operatorname{Im}\tau_j \geq T/\alpha_j$ for all integers j with $m-1 \leq j \leq m_0+1$. Then

$$\liminf_{m\in A,\, m\to+\infty} \alpha_m = 0.$$

Proof. First we show that if some $\alpha \in HT_N$ satisfies $\mathscr{A}$, then

$$\liminf_{i\to+\infty} \alpha_i = 0. \tag{3.22}$$

By $\mathscr{A}$, for every $T \in \mathbb{R}$ there are positive integers m_0 and $m \geq m_0+2$, and τ in $\mathcal{C}_m(1/(2\alpha_m) - 2\mathbf{i})$, such that $\operatorname{Im}\tau_{m-1} \geq T/\alpha_{m-1}$. Then, by Lemma 3.12,

$$T \leq \frac{T}{\alpha_{m-1}} \leq \operatorname{Im}\tau_{m-1} \leq \frac{1}{2\pi}\log\frac{1}{\alpha_m} + C'.$$

That is, for every $T > 0$ there is $m \in \mathbb{N}$ such that $\log\alpha_m^{-1} \geq 2\pi(T - C')$. This implies (3.22).

Let m_0 and A be as in the lemma. If $\mathbb{N}\setminus A$ has finite cardinality, then the statement of the lemma follows from (3.22). Below we assume that $\mathbb{N}\setminus A$ hs infinite cardinality.

Assume for contradiction that

$$\eta = \inf\{\alpha_i \mid i \in A\} > 0.$$

Let

$$\mu = \min\left\{\frac{\eta}{2}, T\left(\frac{1}{2\pi}\log\frac{1}{\eta} + C'\right)^{-1}\right\}.$$

By (3.22), there is $n \in \mathbb{N}\setminus A$ such that $\alpha_n < \mu$ and $n \geq m_0+2$. Then, let m be the smallest element of A greater than n. Let n' be the largest integer in $[n,m]$ such that $\alpha_{n'} < \mu$. Since $\alpha_m \geq \eta \geq 2\mu$, we have $n' < m$. By definition, $\alpha_i \geq \mu$ for all i with $n'+1 \leq i \leq m$.

First we note that $m \geq n'+2$. Indeed, by Lemma 3.12,

$$\frac{T}{\alpha_{m-1}} \leq \operatorname{Im}\tau_{m-1} \leq \frac{1}{2\pi}\log\frac{1}{\alpha_m} + C' \leq \frac{1}{2\pi}\log\frac{1}{\eta} + C'.$$

By the definition of μ, this implies that $\alpha_{m-1} \geq \mu$. Since $\alpha_{n'} < \mu$, it follows that $n' \neq m-1$.

Let B and M be the constants in Proposition 3.15, and $B_{n,i}$ be the bi-sequence defined with the constant B. By Proposition 3.15(a), there is τ' in $C_m(1/(2\alpha_m) - 2\mathbf{i})$ such that $\operatorname{Im}\tau'_j \le B_{m,j} + M$ for all j with $0 \le j \le m-1$. Then, by Lemma 3.16,

$$|\operatorname{Im}\tau_j - \operatorname{Im}\tau'_j| \le 2 \quad \text{for } m_0 + 1 \le j \le m.$$

Putting the above paragraphs together, we obtain

$$\begin{aligned}\frac{T}{\alpha_{n'}} &\le \operatorname{Im}\tau_{n'} \le \operatorname{Im}\tau'_{n'} + 2 \le B_{m,n'} + M + 2\\ &\le -2\beta_{n'}^{-1}\beta_m + \beta_{n'}^{-1}\sum_{j=n'+1}^{m}\beta_{j-1}\left(\log\frac{1}{\alpha_j} - B\right) + M + 2\\ &\le \left(\log\frac{1}{\mu} - B\right)\sum_{j=0}^{m-n'-1}\left(\frac{1}{2}\right)^j + M + 2 \le 2\log\frac{1}{\mu} + M + 2.\end{aligned}$$

Thus, as $2\log x < x$ for all $x \in (2, +\infty)$, the above inequality implies that

$$T \le \alpha_{n'}\left(2\log\frac{1}{\mu} + M + 2\right) < 2\mu\log\frac{1}{\mu} + M + 2 < M + 3.$$

This contradicts the choice of T. □

Remark 3.23. In Lemma 3.22, since the lim inf is zero along any infinite set A on which $\mathscr{A}$ holds for the constant T, one may conclude that indeed the lim inf can be replaced by a lim. However, we have stated the least information enough to prove Proposition 3.2.

Proof of Proposition 3.2 when $\mathscr{A}$ holds. The argument is similar to the one for nonlinearizable maps. By Proposition 3.15(b), there is a constant B such that with the corresponding bi-sequence $B_{n,j}$,

$$H = \sup_{j\in\mathbb{N}}\left|\lim_{n\to\infty} B_{n,j} - \max_{w\in\partial\tilde{\Delta}(f_j)}\operatorname{Im} w\right| < +\infty.$$

Fix $\delta > 0$. Choose $\delta' > 0$ such that if $w \in B_{\delta'}(\tilde{\Delta}(f_0)) \cap \mathcal{D}_0$ then $\Phi_0^{-1}(w) \in B_\delta(\Delta(f_0))$. Recall that by Lemma 2.1, for every $m \ge 0$ and $0 \le j \le a_m$ the δ_0-neighbourhood of $\chi_{m+1,j}(\mathcal{D}_{m+1})$ is contained in $\mathcal{D}_m$. An elementary estimate shows that the Poincaré metric $\rho_m|dw|$ on $\mathcal{D}_m$ satisfies $\rho_m(w) \le 2/d(w, \partial\mathcal{D}_m)$. This implies that for (every such m and j, as well as) every $w \in \chi_{m+1,j}(\mathcal{D}_{m+1})$ with

$$\operatorname{Im} w > \max_{w\in\partial\tilde{\Delta}(f_m)}\operatorname{Im} w - H - 6$$

there is $w' \in \partial\tilde{\Delta}(f_m)$ such that the hyperbolic distance between w and w' in $\mathcal{D}_m$ is uniformly bounded from above by $2(H+6)/\delta_0$. Then, by Lemma 3.8, there is m'_0 such that for every $m \ge m'_0$, every $w \in \chi_{m+1,j}(\mathcal{D}_{m+1})$ with

$$\operatorname{Im} w \ge \max_{w\in\partial\tilde{\Delta}(f_m)}\operatorname{Im} w - H - 6$$

and all integers l_j, $1 \le j \le m$, with $0 \le l_j \le a_{j-1}$, we have

$$\chi_{1,l_1} \circ \chi_{2,l_2} \circ \cdots \circ \chi_{m,l_m}(w) \in B_{\delta'}(\tilde{\Delta}(f_0)) \cap \mathcal{D}_0.$$

Let T be a constant satisfying (3.17) and (3.21). Since $\mathscr{A}$ holds for T, there exists a positive integer m_0 and an infinite set A of integers such that for all $m \in A$ there is $\tau \in \mathcal{C}_m(1/(2\alpha_m) - 2\mathbf{i})$ with

$$\operatorname{Im} \tau_j > T/\alpha_j \quad \text{for } m_0 + 1 \le j \le m - 1.$$

Note that by making T larger, m_0 becomes larger. In particular, we may assume that besides satisfying (3.17) and (3.21), T is so large that the corresponding m_0 is greater than m_0'.

Recall the set G_m defined in Section 3.4. By Lemmas 3.17 and 3.22, we obtain

$$\lim_{m \in A,\, m \to +\infty} \frac{|G_m|}{a_m + 1} = 1.$$

The Brjuno sum for α_{m_0}, that is,

$$\sum_{m=m_0+1}^{+\infty} \beta_{m_0}^{-1} \beta_{m-1} \log \alpha_m^{-1},$$

is finite. Also, by Lemma 3.4, $\lim_{n \to +\infty} B_{n,m_0}$ exists. Hence, there is $m_0'' > 0$ such that for all $m \ge m_0$,

$$\beta_{m_0}^{-1} \beta_{m-1} \log \alpha_m^{-1} < 2,$$

and

$$\left| \lim_{n \to +\infty} B_{n,m_0} - B_{m,m_0} \right| < 1.$$

Fix $m \in A$ such that $m \ge m_0''$, and let $w_m \in X_{\alpha_m}$ where X_{α_m} is defined in (3.7). Given integers l_i, $m \le i \le m_0 + 1$, with $0 \le l_i \le a_{i-1}$, consider the sequence of points

$$w_{i-1} = \chi_{i,l_i}(w_i) \quad \text{for } m \le i \le m_0 + 1.$$

By Proposition 3.18, we have

$$\begin{aligned} \operatorname{Im} w_{m_0} &\ge B_{m,m_0} - \tfrac{1}{2} \beta_{m_0}^{-1} \beta_{m-1} \log(1/\alpha_m) - 4 \\ &> \lim_{n \to \infty} B_{n,m_0} - 6 \quad (\text{since } m \ge m_0''). \end{aligned}$$

Hence,

$$\operatorname{Im} w_0 \ge \max_{w \in \partial \tilde{\Delta}(f_j)} \operatorname{Im} w - H - 6.$$

Now since $m > m_0'$ (the argument in the second paragraph), all further lifts of w_0 to the level 0 are in the δ'-neighbourhood of $\tilde{\Delta}(f_0)$.

By definition, for all $j \in G_m$, we have $f_m^{\circ j}(J_m) \subset \mathcal{P}_m$ and $\Phi_m \circ f_m^{\circ j}(J_m) \subset X_{\alpha_m}$. Therefore, by Proposition 2.13, and since $m \geq m_0'$, we conclude that

$$1 \geq \lim_{m\to\infty,\, m\in A} \frac{|H(m_i+1,\delta)|}{q_{m_i+1}} \geq \lim_{m\to\infty,\, m\in A} \frac{|G_m|}{a_m+1} = 1.$$

This finishes the proof of the proposition when $\mathscr{A}$ holds. □

As a corollary of the above proof we state the following property for future reference.

Corollary 3.24. *Assume $\alpha \in HT_N$ satisfies $\mathscr{A}$. Then for every $\delta' > 0$, there are infinitely many $m \in \mathbb{N}$ such that for all $\tau \in \mathcal{C}_m(1/(2\alpha_m) + 2\mathbf{i})$, we have*

$$\tau_0 \in B_{\delta'}(\tilde{\Delta}(f_0)) \setminus \tilde{\Delta}(f_0).$$

Proof. By the proof of Proposition 3.2 when $\mathscr{A}$ holds, we only need to show that $\tau_0 \notin \tilde{\Delta}(f_0)$ for all $m \in \mathbb{N}$. Indeed, since $\operatorname{Im} \tau_m = -2$, we have $\tau_m \notin \tilde{\Delta}(f_m)$, and since the changes of coordinates preserve the Siegel disks, τ_0 cannot belong to $\tilde{\Delta}(f_0)$. □

Proof of Proposition 3.2 when $\mathscr{B}$ holds. For $n \geq 0$, consider the sets

$$E_n = \bigcup_{i=0}^{a_n} \chi_{n+1,i}(\mathcal{D}_{n+1}).$$

By Lemma 2.1, for every $n \geq 0$, the δ_0-neighbourhood of E_n is contained in $\mathcal{D}_n$.

Fix $\delta > 0$. Choose $\delta' > 0$ such that if $w \in B_{\delta'}(\tilde{\Delta}(f_0)) \cap \mathcal{D}_0$ then $\Phi_0^{-1}(w) \in B_\delta(\Delta(f_0))$. As discussed in the proof of Proposition 3.2 when $\mathscr{A}$ holds, Lemma 2.1 implies that for every $H > 0$ there is $m_0' \geq 1$ with the following property. For every $m \geq m_0'$ and all integers l_j, $1 \leq j \leq m$, with $0 \leq l_j \leq a_{j-1}$, as well as all $w \in E_m$ with either the Euclidean distance $d(w, \partial\tilde{\Delta}(f_m))$ or the hyperbolic distance $d_{\text{hyp}}(w, \partial\tilde{\Delta}(f_m))$ at most H,

$$\chi_{1,l_1} \circ \chi_{2,l_2} \circ \cdots \circ \chi_{m,l_m}(w) \in B_{\delta'}(\tilde{\Delta}(f_0)) \cap \mathcal{D}_0.$$

Let T, C, and the sequence of integers $m_1 < m_2 < \cdots$ be the data obtained from $\mathscr{B}$. We break the proof into two cases.

Case I: $\limsup_{i\to\infty} \alpha_{m_i} > 0$. Let $\eta > 0$ and integers $n_1 < n_2 < \cdots$ be such that $\alpha_{n_i} > \eta$ for all $i \geq 1$. It follows from $\mathscr{B}$ that for all $i \geq 1$ and all $w \in E_{n_i}$ there is $w' \in \partial\tilde{\Delta}(f_{n_i})$ with $d(w, w') \leq T/\eta + C$. Therefore, by the above paragraph, there is n' in the sequence n_i such that for all integers l_j, $1 \leq j \leq n'$, with $0 \leq l_j \leq a_{j-1}$,

$$\chi_{1,l_1} \circ \chi_{2,l_2} \circ \cdots \circ \chi_{n',l_{n'}}(E_{n'}) \subseteq B_{\delta'}(\tilde{\Delta}(f_0)) \cap \mathcal{D}_0.$$

Here, we have used $\chi_{n,i}(\tilde{\Delta}(f_n)) \subset \tilde{\Delta}(f_{n-1})$ for all $n \geq 1$ and $0 \leq i \leq a_{i-1}$, which follows from Propositions 2.7 and 2.8. Thus, by Proposition 2.13, for all $m \geq n' + 1$,

$$|H(m,\delta)|/q_m = 1.$$

As δ was arbitrary, this finishes the proof in this case.

Case II: $\lim_{i\to\infty} \alpha_{m_i} = 0$. Recall the constant $\hat{\mathbf{k}}$ obtained in Lemma 1.9. Consider the sets

$$E'_n = \left\{ w \in E_n \;\middle|\; \frac{1}{4\alpha_n} - \hat{\mathbf{k}} \leq \operatorname{Re} w \leq \frac{3}{4\alpha_n} + \hat{\mathbf{k}} \right\}.$$

Since α_{m_i} tends to 0, there is $i_0 \geq 1$ such that $E'_{m_i} \subset E_{m_i}$ for all $i \geq i_0$.

By Proposition 3.6, for every $i \geq i_0$ the derivative of $\chi_{m_i,0}$ is comparable to $1/\alpha_{m_i}$ on E'_{m_i}, with constants independent of i. Therefore, there is a constant $H > 0$ such that for all $i \geq i_0$ and $0 \leq l_i \leq a_{m_i-1}$, the set $\chi_{m_i,l_i}(E'_{m_i})$ has Euclidean diameter bounded from above by H. Thus, by the second paragraph in this proof, there exists $i_1 \geq i_0$ such that for all $i \geq i_1$ and all integers l_j, $1 \leq j \leq m_i$, with $0 \leq l_j \leq a_{j-1}$,

$$\chi_{1,l_1} \circ \chi_{2,l_2} \circ \cdots \circ \chi_{m_i,l_{m_i}}(E_{m_i}) \subseteq B_{\delta'}(\tilde{\Delta}(f_0)) \cap \mathcal{D}_0.$$

Each set E'_n contains a vertical strip of width $1/(2\alpha_n) + 2\hat{\mathbf{k}}$. Thus, by Lemma 2.1, for all $i \geq i_0$, $\chi_{m_i+1,l}(\mathcal{D}_{m_i+1})$ is contained in E'_{m_i} for at least half of the integers l with $0 \leq l \leq a_{m_i} - 1$.

Fix $\varepsilon > 0$. Choose $i_2 \geq i_1$ such that $(1/2)^{i_2-i_1} < \varepsilon$. Now consider $m \geq m_{i_2} + 1$. Of the lifts $\chi_{m_{i_2},l_{i_2}} \circ \cdots \circ \chi_{m,l_m}(\mathcal{D}_m)$, at least half lie in $E_{m_{i_2}}$, and hence their further lifts lie in $B_{\delta'}(\tilde{\Delta}(f_0)) \cap \mathcal{D}_0$. Of all the remaining lifts up to level m_{i_2}, at least half of the lifts $\chi_{m_{i_2}-1,l_{i_2}-1} \circ \cdots \circ \chi_{m,l_m}(\mathcal{D}_m)$ lie in $E_{m_{i_2}-1}$, and hence their further lifts lie in $B_{\delta'}(\tilde{\Delta}(f_0)) \cap \mathcal{D}_0$, and so on. Altogether, Proposition 2.13 implies that for all $m \geq m_{i_2}+1$,

$$|H(m+1,\delta)|/q_{m+1} \geq 1 - \varepsilon.$$

This finishes the proof in this case. □

Proof of Theorem 0.3. For $m \geq 0$ let σ_m denote the non-zero fixed point of f_m that lies on the boundary of $\mathcal{P}_m$. Recall that by Lemma 2.4, $\Psi_m : \mathcal{P}_m \to \mathcal{P}_0$ conjugates f_m on $\mathcal{P}'_m \subset \mathcal{P}_m$ to $f_0^{\circ q_m}$ on $\Psi_m(\mathcal{P}'_m)$, and $\psi_m : \mathcal{P}_m \to \mathcal{P}_{m-1}$ conjugates f_m on $\mathcal{P}'_m$ to some iterate of f_{m-1} on $\psi_m(\mathcal{P}'_m)$. Since $\mathcal{P}_m$ is bounded by piecewise analytic curves, ψ_m and Ψ_m extend to some continuous maps on the closure of $\mathcal{P}_m$. In particular, Ψ_m maps σ_m to a periodic point of f_0 of period q_m. Similarly, ψ_m maps σ_m to a periodic point of f_{m-1} of period say b_{m-1}. We want to show that every neighbourhood of $\Delta(f_0)$ contains the cycle of infinitely many periodic points of type $\Psi_m(\sigma_m)$, for some $m \in \mathbb{N}$.

For $m \geq 1$, define

$$\mathcal{O}_{m-1} = \{\Phi_{m-1}(\psi_m(\sigma_m)) + j \mid j \in \mathbb{Z},\ 0 \leq j \leq b_{m-1} - 1\}.$$

By definition of ψ_m and $\mathbb{E}\mathrm{xp}$, the set $\mathcal{O}_{m-1}$ projects under $\mathbb{E}\mathrm{xp}$ onto the non-zero fixed point of $\mathcal{R}(f_{m-1})$, which is either σ_m or $s(\sigma_m)$. Lemma 3.9 implies that there is a constant $C'_1 > 0$, independent of m, such that

$$\left| \operatorname{Im} \mathcal{O}_{m-1} - \frac{1}{2\pi} \log \frac{1}{\alpha_m} \right| \leq C'_1, \tag{3.23}$$

and by Lemma 2.1, the δ_0-neighbourhood of $\mathcal{O}_{m-1}$ is contained in $\mathcal{D}_{m-1}$.

Fix $\delta > 0$, and choose $\delta' > 0$ such that for every $w \in \mathcal{D}_0 \cap B_{\delta'}(\tilde{\Delta}(f_0))$ we have $\Phi_0^{-1}(w) \in B_\delta(\Delta(f_0))$. We break the rest of the argument into two cases.

Case I: Assume that $\mathscr{A}$ holds for α. By Lemma 3.12, for every $\tau \in \mathcal{C}_m(1/(2\alpha_m) - 2\mathbf{i})$ we have

$$\left|\operatorname{Im} \tau_{m-1} - \frac{1}{2\pi} \log \frac{1}{\alpha_m}\right| \le C'.$$

Thus, by Lemma 2.1, for every element of $\mathcal{O}_{m-1}$ there is an integer l with $0 \le l \le a_{m-1}$ such that the hyperbolic distance between that element and $\chi_{m,l}(1/(2\alpha_m) - 2\mathbf{i})$ is uniformly bounded from above by a constant, say H, depending only on C', C_1', and δ_0. By the uniform contraction in Lemma 3.8, there is $m_0 \ge 1$ such that if $m \ge m_0$ then any two points at hyperbolic distance bounded by H are mapped to two points at Euclidean distance bounded from above by $\delta'/2$.

On the other hand, by Corollary 3.24 for $\delta'/2$, there are infinitely many m such that for all $\tau \in \mathcal{C}_m(1/(2\alpha_m) + 2\mathbf{i})$ we have $\tau_0 \in B_{\delta'/2}(\tilde{\Delta}(f_0))$. Combining this with the above paragraph, we conclude that for all such $m \ge m_0$ all lifts of elements of $\mathcal{O}_{m-1}$ to the level 0 are contained in $B_{\delta'}(\tilde{\Delta}(f_0))$. Therefore, by Proposition 2.13, the cycle of $\Psi_m(\sigma_m)$ is contained in $B_\delta(\Delta(f_0))$.

Case II: Assume that $\mathscr{B}$ holds for α. Let T, C, and $m_1 < m_2 < \cdots$ be provided by $\mathscr{B}$. By definition, for all $i \ge 1$, $\operatorname{Im} \tilde{\Delta}(f_{m_i}) \subset [-2, T/\alpha_{m_i} + C]$. In particular, there is $w_{m_i} \in \partial \tilde{\Delta}(f_{m_i})$ with

$$\operatorname{Re} w_{m_i} = 1/(2\alpha_{m_i}) \quad \text{and} \quad \operatorname{Im} w_{m_i} \in [-2, T/\alpha_{m_i} + C].$$

By Proposition 2.8, $\chi_{m_i,l}(w_{m_i}) \in \partial \tilde{\Delta}(f_{m_i-1})$ for all l with $0 \le l \le a_{m_i-1}$. Moreover, by Proposition 3.6, there is a constant $C_2 > 0$, independent of i and l, such that

$$\left|\operatorname{Im} \chi_{m_i,l}\left(\frac{1}{2\alpha_{m_i}} + 2\mathbf{i}\right) - \operatorname{Im} \chi_{m_i,l}(w_{m_i})\right| \le C_2.$$

Thus, by Lemma 3.12,

$$\left|\operatorname{Im} \chi_{m_i,l}(w_{m_i}) - \frac{1}{2\pi} \log \frac{1}{\alpha_{m_i}}\right| \le C_2 + C'.$$

Combining the above inequality with (3.23), and using Lemma 2.1, we conclude that every element of $\mathcal{O}_{m_i}$ lies within uniformly bounded (depending on $C_1' + C_2 + C'$ and δ_0) hyperbolic distance from $\partial \tilde{\Delta}(f_{m_i-1})$. Then, by the contraction in Lemma 3.8, for sufficiently large i, $\mathcal{O}_{m_i}$ lifts to a set of points in $B_{\delta'}(\tilde{\Delta}(f_0))$. Finally, Proposition 2.13 implies that the cycle of $\Psi_{m_i}(\sigma_{m_i})$ is contained in $B_\delta(\Delta(f_0))$. □

4. Unique ergodicity

We work with the following equivalent definition of unique ergodicity (see for instance [Mañ87, Theorem 9.2]). A continuous map $f : X \to X$, where X is a compact metric space, is *uniquely ergodic* if for every continuous function $\varphi : X \to \mathbb{R}$ and every $x \in X$

the limit of the Birkhoff averages

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\varphi(f^{\circ j}(x))$$

exists and is independent of x.

We have a simple candidate for the unique invariant measure when the map is not linearizable, that is, the Dirac measure at 0. Because of this, the proof of Theorem 0.1 becomes slightly simpler for non-linearizable maps. Also, it conveys the idea of proof for the linearizable ones where the invariant measure is more complicated. Hence, although both cases may be treated simultaneously, we present the proof for non-linearizable maps first.

4.1. Non-linearizable maps

Proof of Theorem 0.1 when α is non-Brjuno. We assume N is the integer determined in Section 1.4 and is subject to (1.8).

Let $\varphi\colon \mathcal{PC}(f_0)\to\mathbb{R}$ be a continuous function with

$$M=\max_{z\in\mathcal{PC}(f_0)}|\varphi(z)|<\infty.$$

Let $\varepsilon>0$. There is $\delta>0$ such that

$$\forall w\in B_\delta(0),\qquad |\varphi(w)-\varphi(0)|<\varepsilon.$$

Recall the set $G(n,\delta)$ defined in Section 3. By Proposition 3.1, there is an integer $n>0$ such that

$$|G(n,\delta)|/q_n\geq 1-\varepsilon.$$

If $z=0$, then clearly the Birkhoff average of φ along the orbit of 0 is the constant sequence with terms equal to $\varphi(0)$. We need to show that these averages along the orbit of every point in $\mathcal{PC}(f_0)$ are equal to $\varphi(0)$.

Recall that by Proposition 2.6, $\mathcal{PC}(f_0)\subset\Upsilon^n$. Given a non-zero $\zeta\in\mathcal{PC}(f_0)$, by Proposition 2.11, there is a one-to-one map

$$\tau\colon\{i\in\mathbb{Z}\mid 0\leq i\leq q_n-1\}\to\{i\in\mathbb{Z}\mid 0\leq i\leq q_n-1\},$$

depending on ζ, such that $f_0^{\circ i}(\zeta)\in f_0^{\circ\tau(i)}(I^n)$ for $0\leq i\leq q_n-1$. Then

$$\begin{aligned}
&\left|\frac{1}{q_n}\sum_{k=0}^{q_n-1}\varphi(f_0^{\circ k}(\zeta))-\varphi(0)\right|\\
&\quad\leq\frac{1}{q_n}\sum_{\substack{0\leq k\leq q_n-1\\ \tau(k)\in G(n,\delta)}}|\varphi(f_0^{\circ k}(\zeta))-\varphi(0)|+\frac{1}{q_n}\sum_{\substack{0\leq k\leq q_n-1\\ \tau(k)\notin G(n,\delta)}}|\varphi(f_0^{\circ k}(z))-\varphi(0)|\\
&\quad\leq\frac{1}{q_n}|G(n,\delta)|\cdot\varepsilon+\frac{1}{q_n}(q_n-|G(n,\delta)|)\cdot 2M\leq\varepsilon+\varepsilon 2M=\varepsilon(1+2M).
\end{aligned}\tag{4.1}$$

Fix z in $\mathcal{PC}(f_0)$. Let N be a positive integer. Dividing $N-1$ by q_n we obtain non-negative integers m and r such that $N-1=mq_n+r$ with $0\leq r\leq q_n-1$. In particular, there is $N_0 > 0$ such that for all $N \geq N_0$ we have

$$\frac{r}{N} 2M \leq \varepsilon.$$

Applying (4.1), to the points $f_0^{\circ i q_n}(z)$ for $i \geq 0$, we conclude that for every $N \geq N_0$,

$$\begin{aligned}
&\left| \frac{1}{N} \sum_{k=0}^{N-1} \varphi(f_0^{\circ k}(z)) - \varphi(0) \right| \\
&\quad \leq \frac{1}{N} \sum_{i=0}^{m-1} \sum_{k=iq_n}^{(i+1)q_n-1} |\varphi(f_0^{\circ k}(z)) - \varphi(0)| + \frac{1}{N} \sum_{k=mq_n}^{N-1} |\varphi(f_0^{\circ k}(z)) - \varphi(0)| \\
&\quad = \frac{mq_n}{N} \frac{1}{m} \sum_{i=0}^{m-1} \frac{1}{q_n} \sum_{k=iq_n}^{(i+1)q_n-1} |\varphi(f_0^{\circ k}(z)) - \varphi(0)| + \frac{1}{N} \sum_{k=mq_n}^{N-1} |\varphi(f_0^{\circ k}(z)) - \varphi(0)| \\
&\quad \leq 1 \cdot \frac{1}{m} \cdot m \cdot \varepsilon(1+2M) + \frac{1}{N} \cdot r \cdot 2M \leq \varepsilon(1+2M) + \varepsilon = \varepsilon(2+2M).
\end{aligned}$$

As ε was chosen arbitrarily, we conclude that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \varphi(f_0^{\circ k}(z)) = \varphi(0). \qquad \square$$

4.2. *Linearizable maps*

Proof of Theorem 0.1 when α is Brjuno. The integer N is determined in Section 1.4 and is subject to (1.8).

Let $\varphi\colon \mathcal{PC}(f_0) \to \mathbb{R}$ be a continuous map with

$$M = \max_{z \in \mathcal{PC}(f_0)} |\varphi(z)| < \infty.$$

Fix $z \in \mathcal{PC}(f_0)$ and $\varepsilon > 0$. There is $\delta' > 0$ such that for all w, w' in $\mathcal{PC}(f_0)$ with $|w' - w| < \delta'$ we have $|\varphi(w') - \varphi(w)| < \varepsilon$.

Recall that by Proposition 2.6, $\mathcal{PC}(f_0) \subset \Upsilon^n$ for all $n \geq 0$. Also, recall the definition of the sets $H(n, \delta)$ from Section 3.1. By Proposition 3.2, there are an integer n_0 and real $\delta_0 > 0$ such that

$$\forall n \geq n_0,\ \forall \delta \leq \delta_0,\ \forall k \in H(n, \delta), \qquad \operatorname{diam}(f_0^{\circ k}(I^n) \setminus \Delta(f_0)) < \delta'.$$

By the same proposition, there is an integer $n \geq n_0$ such that

$$|H(n, \delta)|/q_n \geq 1 - \varepsilon.$$

From Proposition 2.11, there is a permutation τ of $\{i \in \mathbb{Z} \mid 0 \leq i \leq q_n - 1\}$ such that

$$f_0^{\circ i}(z) \in f_0^{\circ \tau(i)}(I^n) \qquad \text{for } 0 \leq i \leq q_n - 1.$$

Given $w \in \mathcal{PC}(f_0)$, let ρ be a permutation of $\{i \in \mathbb{Z} \mid 0 \le i \le q_n - 1\}$, obtained from Proposition 2.11, such that

$$f_0^{\circ i}(w) \in f_0^{\circ \rho(i)}(I^n) \quad \text{for } 0 \le i \le q_n - 1.$$

Note that

$$f_0^{\circ \rho^{-1}(\tau(i))}(w) \in f_0^{\circ \tau(i)}(I^n) \quad \text{for } 0 \le i \le q_n - 1.$$

Using this, we observe that

$$\begin{aligned}
&\left| \frac{1}{q_n} \sum_{k=0}^{q_n-1} \varphi(f_0^{\circ k}(z)) - \frac{1}{q_n} \sum_{k=0}^{q_n-1} \varphi(f_0^{\circ k}(w)) \right| \\
&\qquad \le \frac{1}{q_n} \sum_{k=0}^{q_n-1} |\varphi(f_0^{\circ k}(z)) - \varphi(f_0^{\circ \rho^{-1}(\tau(k))}(w))| \\
&\qquad = \frac{1}{q_n} \sum_{\substack{0 \le k \le q_n-1 \\ k \in H(n,\delta)}} |\varphi(f_0^{\circ k}(z)) - \varphi(f_0^{\circ \rho^{-1}(\tau(k))}(w))| \\
&\qquad\quad + \frac{1}{q_n} \sum_{\substack{0 \le k \le q_n-1 \\ k \notin H(n,\delta)}} |\varphi(f_0^{\circ k}(z)) - \varphi(f_0^{\circ \rho^{-1}(\tau(k))}(w))| \\
&\qquad \le \frac{|H(n,\delta)|}{q_n} \cdot \varepsilon + \frac{1 - |H(n,\delta)|}{q_n} \cdot 2M \le \varepsilon + \varepsilon \cdot 2M. \qquad (4.2)
\end{aligned}$$

For every positive integer N there are non-negative integers m and r with

$$N - 1 = mq_n + r \quad \text{and} \quad 0 \le r \le q_n - 1.$$

Let N be large enough that

$$\frac{r}{N} 2M \le \varepsilon.$$

Applying the estimate in (4.2) to the points $w = f_0^{\circ i q_n}(z)$, $i \ge 0$, we get

$$\begin{aligned}
&\left| \frac{1}{N} \sum_{k=0}^{N-1} \varphi(f_0^{\circ k}(z)) - \frac{1}{q_n} \sum_{k=0}^{q_n-1} \varphi(f_0^{\circ k}(z)) \right| \\
&\quad \le \left| \frac{1}{N} \sum_{i=0}^{m-1} \sum_{k=iq_n}^{(i+1)q_n-1} \varphi(f_0^{\circ k}(z)) + \frac{1}{N} \sum_{k=mq_n}^{N-1} \varphi(f_0^{\circ k}(z)) - \frac{1}{q_n} \sum_{k=0}^{q_n-1} \varphi(f_0^{\circ k}(z)) \right| \\
&\quad \le \left| \frac{mq_n}{N} \frac{1}{m} \sum_{i=0}^{m-1} \frac{1}{q_n} \sum_{k=iq_n}^{(i+1)q_n-1} \varphi(f_0^{\circ k}(z)) - \frac{1}{q_n} \sum_{k=0}^{q_n-1} \varphi(f_0^{\circ k}(z)) \right| + \frac{1}{N} \sum_{k=mq_n}^{N-1} |\varphi(f_0^{\circ k}(z))| \\
&\quad \le 1 \cdot \frac{1}{m} \cdot m \cdot \varepsilon(1 + 2M) + \frac{1}{N} \cdot r \cdot 2M \le \varepsilon(1 + 2M) + \varepsilon = \varepsilon(2 + 2M).
\end{aligned}$$

As ε was chosen arbitrarily, the above bound implies that the sequence of Birkhoff averages along the orbit of z is a Cauchy sequence. In particular, the sequence of averages along the orbit of every $z \in \mathcal{PC}(f_0)$ is convergent.

On the other hand, by the above argument, for every z and w in $\mathcal{PC}(f_0)$ and every $\varepsilon > 0$ there is $n \geq 0$ such that (4.2) holds. This implies that the limits of the Birkhoff averages along the orbits of z and w are equal; that is, the limit of the sequence is independent of z. □

Proof of Corollary 0.2. As mentioned in the introduction, the limit set of Lebesgue almost every point in $J(f_0)$ is contained in $\mathcal{PC}(f_0)$. (Indeed, in [Che13] we show that for Q_α with $\alpha \in HT_N$ the limit set of the orbit of almost every point in the Julia set is equal to $\mathcal{PC}(f_0)$.) For any such point z, every convergent subsequence of the sequence of measures

$$\mu_n = \frac{1}{n}\sum_{k=0}^{n-1} \delta_{f_0^{\circ k}(z)}$$

is an f_0-invariant probability supported on $\mathcal{PC}(f_0)$. However, by Theorem 0.1 there is only one invariant probability measure supported on $\mathcal{PC}(f_0)$. Hence, the above sequence of measures is convergent, and converges either to the Dirac measure at 0 or the harmonic measure on the boundary of the Siegel disk, depending on the type of α. □

4.3. *Hedgehogs and the postcritical set*

Theorem 0.5 follows from the following proposition and Theorem 0.1.

Proposition 4.1. *Let $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $\alpha \in HT_N$ and let K be a Siegel compactum of f. Then, either*

- *∂K is an invariant analytic curve in the Siegel disk of f, or*
- *∂K is contained in the postcritical set of f.*

The above proposition is proved for the quadratic maps Q_α which are not linearizable at 0 in [Chi08], where only the second possibility may arise. We break the proof of the proposition into several lemmas.

Given $f \in \mathcal{IS}_\alpha$ with $\alpha \in HT_N$, let f_j, for $j \geq 0$, denote the sequence of maps defined in Section 1.4 with $f_j'(0) = e^{2\pi \mathbf{i}\alpha_j}$. Recall the sets $\mathcal{C}_n^{-k_n}$, $n \geq 0$, defined in Section 2.1.

In the next lemma,

$$b_n = (k_n + a_n - \mathbf{k} - 1)q_n + q_{n-1}.$$

Lemma 4.2. *For every $n \geq 0$ there is an integer l_n with $0 \leq l_n \leq k_n q_n + q_{n-1}$ such that $f_0^{\circ l_n}(\Psi_n(\mathcal{C}_n^{-k_n}))$ contains the critical point of f_0. Moreover,*

$$\lim_{n\to\infty} \operatorname{diam} f_0^{\circ l_n}(\Psi_n(\mathcal{C}_n^{-k_n})) = 0,$$
$$\lim_{n\to\infty} \sup\{|f_0^{\circ b_n}(z) - z| : z \in f_0^{\circ l_n}(\Psi_n(\mathcal{C}_n^{-k_n})) \cap \mathcal{PC}(f_0)\} = 0.$$

Proof. For every $n \geq 0$, the map $f_n^{\circ k_n}\colon \mathcal{C}_n^{-k_n} \to \mathcal{C}_n$ has a unique critical point. By Lemma 2.4, $\Psi_n \circ f_n^{\circ k_n} = f_0^{\circ k_n q_n + q_{n-1}} \circ \Psi_n$ on $\mathcal{C}_n^{-k_n}$. This implies that $f_0^{\circ k_n q_n + q_{n-1}}$ has a critical point in $\Psi_n(\mathcal{C}_n^{-k_n})$. Since f_0 has a unique critical point in its domain of definition, there must be an integer l_n with $0 \leq l_n \leq k_n q_n + q_{n-1}$ such that $f_0^{\circ l_n}(\Psi_n(\mathcal{C}_n^{-k_n}))$ contains the critical point of f_0.

By Proposition 1.7 there is $\delta > 0$ such that for all $n \geq 0$, the δ-neighbourhood of $\mathcal{C}_n^{-k_n}$ is contained in $\operatorname{Dom} f_n \setminus \{0\}$, and $\operatorname{diam} \mathcal{C}_n^{-k_n} \leq 2/\delta$. This implies that for every $n \geq 0$ there is a simply connected region $E_n \subset \operatorname{Dom} f_n \setminus \{0\}$ such that the conformal modulus of $E_n \setminus \mathcal{C}_n^{-k_n}$ is uniformly bounded from below. As $\Phi_{n-1}(S_{n-1}^0)$ projects onto $\operatorname{Dom} f_n \setminus \{0\}$ under $\mathbb{E}\mathrm{xp}$, or $s \circ \mathbb{E}\mathrm{xp}$, we conclude that $\chi_{n,0} \circ \Phi_n(\mathcal{C}_n^{-k_n})$ has uniformly bounded hyperbolic diameter in $\mathcal{D}_{n-1}$. On the other hand, the uniform bound in Proposition 1.6 implies that

$$\inf\{\operatorname{Re} \mathcal{C}_n^{-k_n}\} \geq a_n + k_n - \mathbf{k} - 2 - \mathbf{k}''.$$

By a similar argument, for every $z' \in \mathcal{C}_n^{-k_n}$ the hyperbolic distance between $\chi_{n,0} \circ \Phi_n(z')$ and $\chi_{n,0} \circ \Phi_n(f_n^{\circ a_n + k_n - \mathbf{k} - 1}(z'))$ in $\mathcal{D}_{n-1}$ is uniformly bounded from above by a constant independent of n and $z' \in \mathcal{C}_n^{-k_n}$.

Recall from Sections 2.3 and 2.4 that $\psi_n(\mathcal{C}_n^{-k_n}) \subset J_{n-1}$ and $\Psi_n(\mathcal{C}_n^{-k_n}) \subset I_n$. Proposition 2.13 implies that there are integers i_j with $0 \leq i_j \leq a_{j-1}$, for $1 \leq j \leq n-1$, such that

$$f_0^{\circ l_n}(\mathcal{C}_n^{-k_n}) = \Phi_0^{-1} \circ \chi_{1,i_1} \circ \chi_{2,i_2} \circ \chi_{n-1,i_{n-1}}(\Phi_{n-1}(\psi_n(\mathcal{C}_n^{-k_n}))).$$

Since the image of χ_{1,i_1} is well contained in $\mathcal{D}_0$ (see Lemma 2.1), the hyperbolic metric on $\mathcal{D}_0$ and the Euclidean metric on $\mathcal{D}_0$ are comparable on $\chi_{1,i_1}(\mathcal{D}_1)$. Now, the uniform contraction of the changes of coordinates χ_{j,i_j} with respect to the hyperbolic metrics in Lemma 3.8 implies that the Euclidean diameters must shrink to zero. $\square$

The second limit in the above proposition is a special case of a more general statement. It is proved in [Che10] that the maps $f_0^{\circ q_n}$ converge to the identity map on certain sets containing $\mathcal{PC}(f_0)$ (and $f_0^{\circ l_n}(\Psi_n(\mathcal{C}_n^{-k_n}))$). But we do not need this stronger statement here.

Recall that $S_n^0 = \mathcal{C}_n^{-k_n} \cup (\mathcal{C}_n^\sharp)^{-k_n}$. We break each set Υ^n into two sets as follows. Define

$$A_a^n = \bigcup_{i=0}^{k_n + a_n - \mathbf{k} - 2} f_0^{\circ(iq_n)}(\Psi_n((\mathcal{C}_n^\sharp)^{-k_n})), \quad A_b^n = f_0^{\circ q_{n-1}}(A_a^n), \quad A^n = A_a^n \cup A_b^n,$$
$$B_a^n = \bigcup_{i=0}^{k_n + a_n - \mathbf{k} - 2} f_0^{\circ(iq_n)}(\Psi_n(\mathcal{C}_n^{-k_n})), \quad B_b^n = f_0^{\circ q_{n-1}}(B_a^n), \quad B^n = B_a^n \cup B_b^n.$$

For $n \geq 1$, let

$$\mathcal{A}^n = \bigcup_{i=0}^{q_n - 1} f_0^{\circ i}(A^n) \cup \{0\}, \quad \mathcal{B}^n = \bigcup_{i=0}^{q_n - 1} f_0^{\circ i}(B^n).$$

For every $n \geq 1$, we have $\Upsilon^n = \mathcal{A}^n \cup \mathcal{B}^n$. The sets $\mathcal{A}^n$, $\mathcal{B}^n$, and Υ^n are bounded by piecewise smooth curves (see Lemma 2.5). The set $(\mathcal{C}_n^\sharp)^{-k_n}$ is bounded by three (closed) smooth curves, denoted by γ_n, ν_n, and η_n, such that

$$\begin{aligned}
\Phi_n(f_n^{\circ k_n}(\gamma_n(t))) &= 1/2 + (-2+t)\mathbf{i}, && \forall t \in [0,\infty),\\
\Phi_n(f_n^{\circ k_n}(\nu_n(t))) &= 3/2 + (-2+t)\mathbf{i}, && \forall t \in [0,\infty),\\
\Phi_n(f_n^{\circ k_n}(\eta_n(t))) &= 1/2 + t - 2\mathbf{i}, && \forall t \in [0,1].
\end{aligned}$$

Lemma 4.3. *For every $n \geq 0$,*

(a) *γ_n is contained in the interior of $\bigcup_{m=k_n}^{k_n+a_n-\mathbf{k}-2}(f_n^{\circ m}(S_n^0))$;*
(b) *$f_n^{\circ(k_n+a_n-\mathbf{k}-2)}(\nu_n)$ is contained in the interior of $\bigcup_{m=0}^{k_n-1}(f_n^{\circ m}(S_n^0))$.*

Proof. Recall from Section 1.1 that the ellipse E is contained in $B(0,2)$. By a simple calculation, $B(0,8/9) \subset U$. One can verify that the polynomial P is one-to-one on $B(0,1/3)$. By the 1/4-theorem, $\psi(U)$ contains $B(0,2/9)$, and $\psi(B(0,1/3))$ contains $B(0,1/12)$.

By Theorem 1.4 and the above paragraph, $\mathcal{R}(f_n)$ is univalent on $B(0,1/12)$. In particular, $B\big(0, \frac{4}{27}e^{-4\pi}\big) \subset B(0,1/12)$, and by the Koebe distortion theorem,

$$\mathcal{R}(f_n)^{-1}\big(B\big(0, \tfrac{4}{27}e^{-4\pi}\big)\big) \subset B\big(0, \tfrac{4}{27}e^{+4\pi}\big).$$

Therefore, by the definition of renormalization,

$$\mathbb{E}\mathrm{xp}(\Phi_n(\mathcal{C}_n^\sharp)^{-k_n}) \subset \mathbb{E}\mathrm{xp}(\Phi_n(f_n^{\circ k_n}(S_n))).$$

This implies (a).

On the other hand,

$$\operatorname{Im}\Phi_n(f_n^{\circ(k_n+a_n-\mathbf{k}-2)}(\nu_n)) \geq 2$$

and $\mathcal{R}(f_n)$ is defined on $B\big(0, \frac{4}{27}e^{-4\pi}\big)$. Since $\operatorname{Re}\Phi_n(f_n^{\circ(k_n+a_n-\mathbf{k}-2)}(\nu_n)) = a_n - \mathbf{k}_n - 1/2$, (b) follows. □

Lemma 4.4. *For every $n \geq 1$, the closure of $\mathcal{A}^n$ is contained in the interior of Υ^n. In particular, the boundary of Υ^n is contained in the closure of $\mathcal{B}^n$.*

Proof. Fix $n \geq 1$ and z in the closure of $\mathcal{A}^n$. As $z = 0$ belongs to the interior of Υ^n, below we assume that z is non-zero.

By the definition of $\mathcal{A}^n$, there is an integer l of the form $iq_n + j$ or $iq_n + q_{n-1} + j$, with $0 \leq i \leq k_n + a_n - \mathbf{k} - 2$ and $0 \leq j \leq q_n - 1$, such that $z \in f_0^{\circ l}(\Psi^n((\mathcal{C}_n^\sharp)^{-k_n}))$.

If z is in the interior of $f_0^{\circ l}(\Psi_n((\mathcal{C}_n^\sharp)^{-k_n}))$ then it is in the interior of Υ^n and we are done. If z lies on the curve $f_0^{\circ l}(\Psi_n(\eta_n))$ minus its end points, then it is in the interior of $f_0^{\circ l}(S_0^n)$, and hence in the interior of Υ^n. It remains to consider the situation where z lies on the boundary curves $f_0^{\circ l}(\Psi_n(\gamma_n \cup \nu_n))$.

First assume that $z \in f_0^{\circ l}(\Psi_n(\nu_n))$, and choose $z' \in \nu_n$ with $z = f_0^{\circ l}(\Psi_n(z'))$. We consider three cases:

(a) $i < a_n + k_n - \mathbf{k} - 2$;
(b) $i = a_n + k_n - \mathbf{k} - 2$ and $l = iq_n + q_{n-1} + j$;
(c) $i = a_n + k_n - \mathbf{k} - 2$ and $l = iq_n + j$.

Assume that (a) holds. There is $w \in A_a^n \cup A_b^n$ such that $z = f_0^{\circ j}(w)$ and $w = f_0^{\circ(l-j)}(\Psi_n(z'))$, where $l - j$ is either iq_n or $iq_n + q_{n-1}$. Note that z' is in the interior of $S_n^0 \cup f_n(S_n^0)$. Also, Ψ_n has a univalent extension onto $f_n(S_n)$ through compositions of the lifts $\chi_{\cdot,0}$. Then $f_0^{\circ l}(\Psi_n(f_n(S_n^0))) = f_0^{\circ l+q_n}(\Psi_n(S_n^0)) \subset \Upsilon^n$. By the open mapping property of holomorphic and anti-holomorphic maps, this implies that w must be in the interior of

$$f_0^{\circ l-j}(\Psi_n(S_n^0 \cup f_n(S_n^0))) \subset A_a^n \cup A_b^n.$$

Hence, $z = f_0^{\circ j}(w)$ is in the interior of Υ^n.

Assume that (b) holds. In this case, there is $w \in A_b^n$ such that $z = f_0^{\circ j}(w)$ and $w = f_0^{\circ(l-j)}(\Psi_n(z'))$. By Lemma 4.3(b), $f_n^{\circ k_n + a_n - \mathbf{k} - 2}(\nu_n)$ is contained in the interior of $\bigcup_{m=0}^{k_n-1}(S_n^0)$. Then, by Lemma 2.4, $f_0^{\circ(l-j)}(\Psi_n(\nu_n))$ is contained in the interior of

$$\bigcup_{m=0}^{k_n-1} f_0^{\circ(mq_n)}(\Psi_n(S_n^0)) \subset A_a^n.$$

By the open mapping property of $f_0^{\circ j}$, we conclude that z lies in the interior of Υ^n.

Assume that (c) holds. Choose $w \in f_0^{\circ iq_n}(\Psi_n(\nu_n))$ such that $z = f_0^{\circ j}(w)$. We have

$$f_0^{\circ(iq_n)}(\Psi_n(\nu_n)) = f_0^{\circ(q_n - q_{n-1})} \circ f_0^{\circ q_{n-1}} \circ f_0^{\circ((i-1)q_n}(\Psi_n(\nu_n)).$$

Let $w' \in f_0^{\circ q_{n-1}} \circ f_0^{\circ((i-1)q_n}(\Psi_n(\nu_n))$ be such that $f_0^{\circ(q_n - q_{n-1})}(w') = w$. By the argument in case (a), w' is in the interior of $A_a^n \cup A_b^n = A^n$. As $f_0^{\circ(q_n - q_{n-1})}$ maps open sets to open sets, w is in the interior of $f_0^{\circ(q_n - q_{n-1})}(A^n)$. In particular, if $j \le q_{n-1} - 1$, we conclude that z is in the interior of Υ^n.

On the other hand, if $j \ge q_{n-1}$, then by case (b) above, $f_0^{\circ q_{n-1}}(w)$ is in the interior of A_b^n. Therefore, for every j with $q_{n-1} \le j \le q_n - 1$, z is in the interior of $f_0^{\circ(j - q_{n-1})}(A_b^n) \subset \Upsilon^n$.

Now assume $z \in f_0^{\circ l}(\Psi_n(\gamma_n))$. Choose $z' \in \gamma_n$ with $z = f_0^{\circ l}(\Psi_n(z'))$ and consider the following three cases:

- $i \neq 0$;
- $i = 0$ and $l = iq_n + q_{n-1} + j$;
- $i = 0$ and $l = iq_n + j$.

The arguments in these cases are similar to the above ones, except that one uses (a) of Lemma 4.3 instead of (b). We leave further details to the reader. □

Assume that $W \ni 0$ is a Jordan domain such that $f \in \mathcal{IS}_\alpha$ and f^{-1} are defined and univalent on a neighbourhood of the closure of W. Let K denote the invariant Siegel compactum of f associated to W. Recall that if $\varepsilon_0 = +1$ then $f_0 = f$, and if $\varepsilon_0 = -1$ then $f_0 = s \circ f \circ s$. Define W' to be W if $\varepsilon_0 = +1$, and $W' = s(W)$ if $\varepsilon_0 = -1$. Then the Siegel compactum of f_0 in the closure of W', denoted by K', is either K or $s(K)$, depending on the sign of ε_0.

Lemma 4.5. *Let K' be a Siegel compactum of f_0. There is an integer $n_0 \geq 0$ such that for all $n \geq n_0$, the set K' does not intersect the closure of $\mathcal{B}^n$.*

Proof. Let cp_0 denote the critical point of f_0. As f_0 is univalent on a neighbourhood of K', there is $\delta > 0$ such that $B(\mathrm{cp}_0, \delta) \cap K' = \emptyset$. By Lemma 4.2, there is n_0 such that for all $n \geq n_0$, we have $\operatorname{diam} f_0^{\circ l_n}(\Psi_n(\mathcal{C}_n^{-k_n})) \leq \delta/3$ and $|f_0^{\circ b_n}(z) - z| \leq \delta/3$ for all $z \in \Psi_n(\mathcal{C}_n^{-k_n})$. In particular, for all $n \geq n_0$, K' does not intersect the closure of $f_0^{\circ l_n}(\Psi_n(\mathcal{C}_n^{-k_n}))$.

As $f_0(K') = K'$, for all $n \geq n_0$ and all i with $0 \leq i \leq l_n$, K' cannot intersect the closure of $f_0^{\circ i}(\Psi_n(\mathcal{C}_n^{-k_n}))$. However, we cannot immediately use the backward invariance of K' to conclude the same for other values of i, because K' is only fully invariant when f_0 is restricted to W', and there is no relation between W' and Υ^n.

Assume that there is $n \geq n_0$ and an integer i with $l_n \leq i \leq b_n - 1$ such that K' intersects the closure of $f_0^{\circ i}(\Psi_n(\mathcal{C}_n^{-k_n}))$. Let $z' \in K'$ be in the closure of $f_0^{\circ i}(\Psi_n(\mathcal{C}_n^{-k_n}))$, and choose z in the closure of $\Psi_n(\mathcal{C}_n^{-k_n})$ such that $f_0^{\circ i}(z) = z'$. Then, by the invariance of K', $f_0^{\circ (b_n - i)}(z') \in K'$, and

$$d(\mathrm{cp}_0, f_0^{\circ(b_n-i)}(z')) \leq d(\mathrm{cp}_0, z) + d(z, f_0^{\circ b_n}(z)) \leq \delta/3 + \delta/3.$$

That is, there is an element of K' within $2\delta/3$ of cp_0, contradicting the choice of δ. ◻

Proof of Proposition 4.1. Since f_0 is conjugate to f, it is enough to prove the proposition for f_0 and K'. As K' is connected and contains 0, the previous lemma yields $K' \subseteq \bigcap_{n \geq n_0} \Upsilon^n$. On the other hand, by Proposition 2.7, $\bigcap_{n\geq n_0} \Upsilon^n = \Delta(f_0) \cup \mathcal{PC}(f_0)$ and $\partial\Delta(f_0) \subseteq \mathcal{PC}(f_0)$. Thus, K' is a connected invariant region in $\Delta(f_0) \cup \mathcal{PC}(f_0)$. This implies that either K is equal to the region bounded by an analytic curve in $\Delta(f_0)$, or it contains $\Delta(f_0)$. ◻

As a corollary of the above lemmas we conclude the following result.

Theorem 4.6. *For every $f \in \mathcal{IS}_\alpha \cup \{Q_\alpha\}$ with $\alpha \in HT_N$, $f\colon \mathcal{PC}(f) \to \mathcal{PC}(f)$ is one-to-one.*

Proof. For every $n \geq 1$, if there are z and z' in Υ^n with $f_0(z) = f_0(z')$ then z and z' must be in $f_0^{\circ l_n}(\mathcal{C}_n^{-k_n})$ where l_n is as in Lemma 4.2. Since $\operatorname{diam} f_0^{\circ l_n}(\mathcal{C}_n^{-k_n}) \to 0$ as $n \to \infty$, we conclude that f_0 is one-to-one on $\bigcap_{n\geq 1} \Upsilon^n$. By Proposition 2.6, the postcritical set of f_0 is contained in this intersection. ◻

The original conjecture of Pérez-Marco on the unique ergodicity of hedgehog dynamics is stated in the full generality of the hedgehogs of all holomorphic maps with an irrationally indifferent fixed point. There are new examples of hedgehogs with surprisingly wild topological behaviour constructed by Chéritat [Ché11]. While our argument here proves that this conjecture holds for quadratic maps and elements of the Inou–Shishikura class, it is not clear if one can adapt the construction of Chéritat to give counterexamples to this unique ergodicity conjecture.

Acknowledgments. D.C. gratefully acknowledges funding from EPSRC, grant No EP/M01746X/1 (rigidity and small divisors in holomorphic dynamics).

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