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Philipp Hieronymi

A tame Cantor set

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Abstract. A Cantor set is a non-empty, compact subset of $\mathbb{R}$ that has neither interior nor isolated points. In this paper a Cantor set $K \subseteq \mathbb{R}$ is constructed such that every set definable in $(\mathbb{R}, <, +, \cdot, K)$ is Borel. In addition, we prove quantifier elimination and completeness results for $(\mathbb{R}, <, +, \cdot, K)$, making the set K the first example of a model-theoretically tame Cantor set. This answers questions raised by Friedman, Kurdyka, Miller and Speissegger. Our work depends crucially on results about automata on infinite words, in particular Büchi's celebrated theorem on the monadic second-order theory of one successor and McNaughton's theorem on Muller automata, which have never been used in the setting of expansions of the real field.

Keywords. Expansions of the real field, Cantor set, tame geometry, Borel sets, quantifier elimination, monadic second-order theory of one successor

1. Introduction

Let $\overline{\mathbb{R}} := (\mathbb{R}, <, +, \cdot)$ denote the real ordered field. The results in this paper contribute to the research program of understanding expansions of $\overline{\mathbb{R}}$ by constructible sets. A set is *constructible* if it is a finite boolean combination of open sets. The motivation behind this work is the following natural question which lies at the intersection of model theory and descriptive set theory:[1]

> *What can be said about sets definable in such an expansion in terms of the real projective hierarchy?*

As is well known, when expanding the real field by constructible sets, arbitrarily complicated projective sets can happen to be definable. Indeed, every projective subset of $\mathbb{R}^n$ is definable in $(\overline{\mathbb{R}}, \mathbb{N})$ (see for example Kechris [15, 37.6]). However, there are many examples of expansions of $\overline{\mathbb{R}}$ whose definable sets are all constructible; among these structures are all o-minimal expansions of $\overline{\mathbb{R}}$ and several non-o-minimal ones (see [21, 11, 22]). This paper aims to determine what kind of expansions lie between these two

P. Hieronymi: Department of Mathematics, University of Illinois at Urbana-Champaign,
1409 West Green Street, Urbana, IL 61801, USA; e-mail: phierony@illinois.edu

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[1] The question was first raised in [10, p. 1311].

extremes. Surprisingly little is known. The primary result in this direction is due to Friedman, Kurdyka, Miller and Speissegger [10]. They construct a constructible set $E \subseteq [0, 1]$ such that $(\overline{\mathbb{R}}, E)$ defines sets on every level of the projective hierarchy (that is, for each $N \in \mathbb{N}$ there is a definable set in $\mathbf{\Sigma}^1_{N+1} \setminus \mathbf{\Sigma}^1_N$), but does not define every projective set. At the end of [10] the question is discussed whether there is a constructible set K and $N \in \mathbb{N}$ such that $(\overline{\mathbb{R}}, K)$ defines non-constructible sets, yet every definable set is $\mathbf{\Sigma}^1_N$. In this paper, we will answer this question positively.

Theorem A. *There is a constructible set $K \subseteq \mathbb{R}$ such that $(\overline{\mathbb{R}}, K)$ defines non-constructible sets, yet every definable set in $(\overline{\mathbb{R}}, K)$ is Borel.*

This paper is also a contribution to the study of model-theoretic tameness in expansions of the real field. Both sets E from [10] and K from this paper are Cantor sets. For our purposes, a *Cantor set* is a non-empty, compact subset of $\mathbb{R}$ that has neither interior nor isolated points. By Fornasiero, Hieronymi and Miller [9] an expansion of $\overline{\mathbb{R}}$ does not define $\mathbb{N}$ (and hence not every projective set) if and only if every definable Cantor set of $\mathbb{R}$ is Minkowski null.[2] While prohibiting the existence of definable Cantor sets of positive Minkowski dimension in expansions that do not define $\mathbb{N}$, this result does not say much about definable sets in expansions that define Minkowski null Cantor sets. Again, the only result in this direction is the result from [10], because the set E is a Cantor set. The non-definability of $\mathbb{N}$ in $(\overline{\mathbb{R}}, E)$ is deduced from the property that every subset of $\mathbb{R}$ definable in this expansion either has interior or is nowhere dense. While this statement can be interpreted as a weak form of topological tameness of the definable sets, it surely cannot be considered as tameness in terms of model theory. In fact, the structure $(\overline{\mathbb{R}}, E)$ defines a Borel isomorph of $(\overline{\mathbb{R}}, \mathbb{N})$ and therefore does not satisfy any notion of what could reasonably be considered as model-theoretic tameness. Just to give an example: its theory is obviously undecidable and there is no bound on the quantifier complexity needed to define all definable sets in this structure. This observation made Friedman and Miller ask the following question in personal communication with the author:[3]

Is there a (model-theoretically) tame Cantor set?

Here we give a positive answer to their question using the Cantor set K from Theorem A. While it will follow easily from [10] that every bounded unary definable set in $(\overline{\mathbb{R}}, K)$ either has interior or is Minkowski null, we will say significantly more about the first-order theory of $(\overline{\mathbb{R}}, K)$ and definable sets in this structure. We will give a natural axiomatization of its theory (see Section 4 and Theorem 4.5) and prove a quantifier elimination result in a suitably extended language (see Theorem 7.1). Since the precise axiomatization and quantifier elimination results are technical, we postpone their statement.

In [10] it is already pointed out that new ideas seem to be necessary to say more about definable sets in expansions of $\overline{\mathbb{R}}$ by a Cantor set. This is indeed the case. In this

[2] A bounded set $A \subseteq \mathbb{R}^n$ is *Minkowski null* if $\lim_{r\to 0^+} r^{\varepsilon} N(A, r) = 0$ for all $\varepsilon > 0$, where $N(A, r)$ is the minimum number of balls of radius r needed to cover A.

[3] Already at the end of [10] the question is raised whether there is a Cantor set different from E such that more can be said about definable sets in the expansion by that set.

paper we use some of the techniques from [10], but we will have to develop several new tools to prove Theorem A and the existence of a tame Cantor set. Above all other we rely on a novel use of results about automata on infinite words. In particular, we recognize a deep connection between this research program and Büchi's famous theorem about the monadic second-order theory of one successor [2]. To the author's knowledge, this and related results have never been used for studying expansions of the real field. We regard this new relation between these research areas as one of the main contributions of this paper, and anticipate potential for further applications. We will outline some of these applications at the end of this introduction. First, we briefly describe how this connection arises.

Many of the results in and around Büchi's paper are stated in terms of second-order logic and in terms of automata on infinite words, but all of them can be restated in terms of first-order model theory. Let $\mathcal{B}$ be the two-sorted structure $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \in, s_{\mathbb{N}})$, where $s_{\mathbb{N}}$ is the successor function on $\mathbb{N}$ and $\in$ is the relation on $\mathbb{N} \times \mathcal{P}(\mathbb{N})$ such that $\in(t, X)$ iff $t \in X$. In [2] the decidability of the theory of $\mathcal{B}$ and a quantifier elimination result are established. The latter result, which is the most relevant to this paper, was later significantly strengthened by McNaughton [19].

Here we will show that when a Cantor set K is sufficiently regular, the expansion $(\overline{\mathbb{R}}, K)$ defines an isomorphic copy of $\mathcal{B}$. And not only is such an isomorphic copy definable: we will see that for well chosen K the complexity of the definable sets in $(\overline{\mathbb{R}}, K)$ is controlled by the complexity of the definable sets in $\mathcal{B}$. Hence the results bounding the complexity of definable sets in $\mathcal{B}$, such as the ones mentioned above, will bound the complexity of definable sets in $(\overline{\mathbb{R}}, K)$.

Theorem A and the existence of a tame Cantor set are proved not only for expansions of the real field, but for a larger class of o-minimal expansions of $\overline{\mathbb{R}}$. An expansion $\mathcal{R}$ of $\overline{\mathbb{R}}$ is *exponentially bounded* if for every function $f : \mathbb{R} \to \mathbb{R}$ definable in $\mathcal{R}$ there exists $m \in \mathbb{N}$ such that f is bounded at $+\infty$ by the m-th compositional iterate of exp. All known o-minimal expansions of the real field are exponentially bounded.

Theorem B. *There is a Cantor set $K \subseteq \mathbb{R}$ such that for every exponentially bounded o-minimal expansion $\mathcal{R}$ of $\overline{\mathbb{R}}$, every definable set in $(\mathcal{R}, K)$ is Borel.*

Here is a very rough outline of the proof. Following Cantor's classical construction we define a Cantor set K by inductively removing middle 'thirds' of a line segment. However, as in [10], instead of always removing exactly a third of the previous segment, we remove increasingly larger and larger portions of the segments. This construction results in a Cantor set that is homeomorphic to the classical Cantor ternary set, but Minkowski null. Indeed, it follows from results from [10] that every image of K^n under functions definable in $\mathcal{R}$ is Minkowski null. Let Q denote the set of lengths of complementary intervals of K. Note that Q is definable in $(\overline{\mathbb{R}}, K)$. We show that there is a set $\epsilon \subseteq Q \times K$ definable in $(\overline{\mathbb{R}}, K)$ such that the two-sorted structures (Q, K, ϵ, s_Q) and $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \in, s_{\mathbb{N}})$ are isomorphic, where s_Q denotes the successor function on $(Q, <)$. Then we use known results about the latter structure to control the complexity of definable sets in (Q, K, ϵ, s_Q) and hence in $(\overline{\mathbb{R}}, K)$. Because K is constructed to be very sparse, we are then able to show that the o-minimal structure does not induce new definable sets on Q and K other than the ones coming from (Q, K, ϵ, s_Q). This last step requires most of the technical work in this paper and involves a wide array of tools from o-minimality.

Throughout this paper we assume familiarity with basic definitions and results in model theory, o-minimality and descriptive set theory. We refer to Marker [18] for model theory, to van den Dries [6] for o-minimality, and to Kechris [15] for descriptive set theory. This paper aims to be self-contained with respect to ingredients from fractal geometry and from the theory of automata on infinite words. Nevertheless, a good reference on the former is Falconer [8] and on the latter is Khoussainov and Nerode [16].

Remarks

We conclude this introduction with a few remarks about the optimality of the results and the applicability of the methods of this paper to other open questions.

1. Because every Cantor set is interdefinable over $\overline{\mathbb{R}}$ with the set of midpoints of its complementary intervals, there is a discrete set $D \subseteq \mathbb{R}$ such that $(\overline{\mathbb{R}}, D)$ defines non-constructible sets, yet every definable set in $(\overline{\mathbb{R}}, D)$ is Borel. By [13, Theorem B] we can even take D to be closed and discrete.

2. In [10] it was suggested that there exists a Cantor set K such that every definable set in $(\overline{\mathbb{R}}, K)$ is not only Borel, but even a boolean combination of F_σ sets. We do not know whether or not the Cantor set constructed in this paper has this stronger property.

3. Another question from [10] asks whether there is a constructible set $E \subseteq \mathbb{R}$ and $N \in \mathbb{N}$ such that every definable set in $(\overline{\mathbb{R}}, E)$ is $\mathbf{\Sigma}^1_N$ and $(\overline{\mathbb{R}}, E)$ defines a non-Borel set. We imagine that the ideas presented in this paper can be used to give a positive answer to this question. However, since $\mathcal{B}$ does not define non-Borel sets, one has to replace the use of $\mathcal{B}$ by the use of more expressive structures. For example, structures based on Rabin's work on the monadic second-order theory of multiple successor [23] might prove useful here.

4. An open question related to the optimality of [9] is whether there is an expansion of $\overline{\mathbb{R}}$ that does not define $\mathbb{N}$, but defines both a Cantor set and a dense and codense set. The tools from [10] are known not to be enough to construct such an expansion. However, it seems reasonable to expect that the work in this paper can be adjusted to construct a Cantor set K and a dense and codense subset $X \subseteq \mathbb{R}$ such that $(\overline{\mathbb{R}}, K, X)$ not only does not define $\mathbb{N}$, but is model-theoretically well behaved. An amalgamation of the proofs from [5] and from this paper should yield this result.

5. A model-theorist might ask what happens when we look at expansions of the ordered real additive group by Cantor sets. Although to the author's knowledge this was never stated explicitly in the literature, strong results can be deduced easily from known theorems. Consider the famous Cantor ternary set C. It is not Minkowski null (its Minkowski dimension is $\log_3 2$). Therefore the theory of $(\overline{\mathbb{R}}, C)$ is undecidable by [9]. The situation is very different when we replace the real field by the ordered real additive group. For $r \in \mathbb{N}_{>2}$, consider the expansion $\mathcal{T}_r$ of $(\mathbb{R}, <, +, \mathbb{Z})$ by a ternary predicate $V_r(x, u, k)$

that holds if and only if u is a positive integer power of r, $k \in \{0, \dots, r-1\}$ and the digit of a base-r representation of x in the position corresponding to u is k. As shown by Boigelot, Rassart and Wolper [1], it follows from Büchi's work that the theory of $\mathcal{T}_r$ is decidable. Since C is precisely the set of real numbers in $[0, 1]$ in one of whose ternary expansions the digit 1 does not appear, C is $\emptyset$-definable in $\mathcal{T}_3$. Therefore the theory of $(\mathbb{R}, <, +, C)$ is decidable.

Notations

Throughout, definable means definable with parameters. If we need to be specific about the language $\mathcal{L}$ and the parameters X used to define a set, we say this set is $\mathcal{L}$-X-*definable*. If we say that φ is an $\mathcal{L}$-formula, we mean that there are no additional parameters appearing in φ. For an arbitrary language $\mathcal{L}$ and an arbitrary $\mathcal{L}$-theory T, we denote the type of a tuple z of elements of a model M of T over some subset X of the universe of M by $\mathrm{tp}_{\mathcal{L}}(z|X)$. Whenever there is a second model of N of T and an $\mathcal{L}$-embedding of X into N, we write $\beta\,\mathrm{tp}_{\mathcal{L}}(z|X)$ for the $\mathcal{L}$-type over $\beta(X)$ given by

$$\{\varphi(y, \beta(x_1), \dots, \beta(x_m)) : \varphi(y, x_1, \dots, x_n) \in \mathrm{tp}_{\mathcal{L}}(z|X)\}.$$

We will sometimes drop the subscript $\mathcal{L}$ when the language is clear from the context. The variables i, j, k, m, n always range over $\mathbb{N} = \{0, 1, 2, \dots\}$. Given sets $X, Y, Z \subseteq X \times Y$ and $x \in X$, we write Z_x for $\{y \in Y : (x, y) \in Z\}$. We will use $\pi : X \times Y \to X$ for the projection onto the first factor. Moreover, if X is ordered by $<$ and $x \in X$, we write $X_{\le x}$ for $\{z \in X : z \le x\}$. We write π for the projection of Z onto X. Moreover, if $(Y, \prec)$ is a linear order such that every element except the minimum and maximum of Y has a predecessor and successor, we denote the predecessor function on Y by p_Y and the successor function on Y by s_Y. If X is a subset of a topological space, we denote the closure of X by $\mathrm{cl}(X)$. If $x \in \mathbb{R}^n$ and $\varepsilon \in \mathbb{R}_{>0}$, we denote the ball of radius ε around x by $B_\varepsilon(x)$. Moreover, if X is linearly ordered by $<$, we will also write $<$ for the lexicographic ordering on X^n given by $<$. If M is a real closed field, $c = (c_1, \dots, c_n) \in M^n$ and $q = (q_1, \dots, q_n) \in \mathbb{Q}^n$, we write $q \cdot c$ for $(q_1c_1, \dots, q_nc_n)$.

2. Construction of K

Fix an o-minimal expansion $\mathcal{R}$ of the real field $\overline{\mathbb{R}}$. We denote the language of $\mathcal{R}$ by $\mathcal{L}$ and the $\mathcal{L}$-theory of $\mathcal{R}$ by T. Throughout, we assume that $\mathcal{R}$ is exponentially bounded. By combining Miller [20], Speissegger [24], and Lion, Miller and Speissegger [17], $(\mathcal{R}, \exp)$ is an exponentially bounded o-minimal expansion of $\overline{\mathbb{R}}$ if the same holds for $\mathcal{R}$. Since Theorem B holds for $\mathcal{R}$ if it holds for $(\mathcal{R}, \exp)$, we assume that $\mathcal{R}$ defines exp.

We denote the m-th compositional iterate of exp by $\exp_m$. Take an increasing sequence $(P_k)_{k\in\mathbb{N}}$ of finite subsets of $\mathbb{Q}$ such that $\bigcup_{k\in\mathbb{N}} P_k = \mathbb{Q}$. Set $q_0 = 1$. Now fix a sequence $(q_k)_{k\in\mathbb{N}_{>0}}$ of positive algebraically independent real numbers such that

(A) $q_{k+1} > 3q_k$ for $k \in \mathbb{N}$,
(B) $|\sum_{i=0}^{k} p_i q_i^{-1}| > q_{k+1}^{-1}$ for $k \in \mathbb{N}$ and any $p_0, \dots, p_k \in P_k$ not all zero,
(C) $\lim_{k\to\infty} \exp_m(q_k)/q_{k+1} = 0$ for $m \in \mathbb{N}$.

We denote the range of this sequence by Q. Set $K_0 := [0, 1]$, and for $i \geq 1$,

$$K_{i+1} := K_i \setminus \bigcup_c (c + q_{i+1}^{-1}, c + q_i^{-1} - q_{i+1}^{-1}),$$

where c ranges over the right endpoints of the complementary intervals of K_i. Set $K := \bigcap_i K_i$. We fix this Q and this K for the rest of the paper. The construction of Q and K was already given in [10, p. 1320].[4] As is pointed out there, one can easily check that K is a Cantor set and homeomorphic to the Cantor ternary set.

Monadic second-order theory of one successor

The work in this paper depends crucially on well known results about the monadic second-order theory of one successor. Because we expect a significant portion of the readers to be unfamiliar with many of the results, we will review them here. For details and proofs we refer to [16].

Let $\mathcal{B}$ be the two-sorted structure $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \in, s_{\mathbb{N}})$, where $s_{\mathbb{N}}$ is the successor function on $\mathbb{N}$ and $\in$ is the relation on $\mathbb{N} \times \mathcal{P}(\mathbb{N})$ such that $\in(t, X)$ iff $t \in X$. We denote the language of $\mathcal{B}$ by $\mathcal{L}_B$. The theory of $\mathcal{B}$ is called the *monadic second-order theory of one successor*. In a landmark paper Büchi [2] showed that the theory of $\mathcal{B}$ is decidable. He established the decidability by proving that a subset of $\mathcal{P}(\mathbb{N})^n$ is definable in $\mathcal{B}$ if and only if it is recognizable by what was later named a Büchi automaton.

In this paper we will use a substantial strengthening of Büchi's characterization of definability in $\mathcal{B}$ due to McNaughton [19]. This generalization states that a set is definable in $\mathcal{B}$ if and only if it is recognizable by a deterministic Muller automaton. For the purposes of this paper we are not so much interested in what exactly a deterministic Muller automaton is, but rather in what this characterization tells us about the Borel complexity of any given subset of $\mathcal{P}(\mathbb{N})^n$ in $\mathcal{B}$. Viewing $\mathcal{P}(\mathbb{N})$ as the product $\{0, 1\}^{\mathbb{N}}$, we can endow $\mathcal{P}(\mathbb{N})$ with the topology that corresponds to the usual product topology on $\{0, 1\}^{\mathbb{N}}$. Among other things, the Borel complexity with respect to this topology of subsets of $\mathcal{P}(\mathbb{N})$ definable in $\mathcal{B}$ was studied in Büchi and Landweber [3]. There the following result was deduced from McNaughton's Theorem.

Fact 2.1 ([3, Corollary 1]). *Every subset of $\mathcal{P}(\mathbb{N})^n$ definable in $\mathcal{B}$ is a boolean combination of sets of $\mathbf{\Pi}_2^0$ and hence in $\mathbf{\Delta}_3^0$.*

We will also use easy facts about definability in $\mathcal{B}$, such as the definability in $\mathcal{B}$ of the usual order on $\mathbb{N}$ and of the set of finite subsets of $\mathbb{N}$, which we denote by $\mathcal{P}_{\text{fin}}(\mathbb{N})$.

We now explain the connection with the topic of this paper. It is well known that the Cantor ternary set C and $\mathcal{P}(\mathbb{N})$ are homeomorphic (see for example [15, I.3.4] or [16, 6.9.1]). Since K is constructed in almost exactly the same way as C, one can easily see that the same construction gives a homeomorphism h between K and $\mathcal{P}(\mathbb{N})$. From this construction, it is clear that this homeomorphism can be extended to an isomorphism between the two-sorted structures $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \in, s_{\mathbb{N}})$ and (Q, K, ϵ, s_Q) for a certain $\epsilon \subseteq Q \times K$. We will now remind the reader what the precise definitions of h and ϵ are.

[4] In [10], Q is used to denote the set of reciprocals of our Q.

Recall that Q was defined to be the range of a sequence $(q_i)_{i\in\mathbb{N}}$ of real numbers and that we defined $K_0 := [0,1]$ and for $i \geq 1$, $K_{i+1} := K_i \setminus \bigcup_c (c + q_{i+1}^{-1}, c + q_i^{-1} - q_{i+1}^{-1})$, where c ranges over the right endpoints of the complementary intervals of K_i, and $K := \bigcap_i K_i$. Let $g : \mathbb{N} \to Q$ be the map taking n to q_n and let $h : \mathcal{P}(\mathbb{N}) \to K$ map $X \subseteq \mathbb{N}$ to $\sum_{n\in X}(q_{n-1}^{-1} - q_n^{-1})$. We will leave it to the reader to check that h is well defined.[5] Let R_n be the set of right endpoints of complementary intervals of K_n and let R be the set of right endpoints of complementary intervals of K. Define $e : Q \times K \to R$ to be the function that maps (q_n, c) to the largest $r \in R_n$ with $r \leq c$. From the construction, we immediately see that $0 \leq c - e(q,c) \leq q^{-1}$ for every $c \in K$ and $q \in Q$. Let $\epsilon \subseteq Q \times K$ be the set of all (q_n, c) such that $e(q_n, c) \in R_n \setminus R_{n-1}$.

Proposition 2.2. *The map $\beta = (g,h)$ is an isomorphism between the two-sorted structures $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \in, s_{\mathbb{N}})$ and (Q, K, ϵ, s_Q) and h is a homeomorphism.*

Proof. Checking that h is a homeomorphism and β is an isomorphism between the two structures is routine and we leave the details to the reader. □

We can deduce immediately from the definition of h that if $c = \sum_{n\in X}(q_{n-1}^{-1} - q_n^{-1})$ for some $c \in K$ and $X \subseteq \mathbb{N}$, then $e(q_m, c) = \sum_{n\in X,\, n\leq m}(q_{n-1}^{-1} - q_n^{-1})$.

It is now a good time to point out why we need to use these results about $\mathcal{B}$. Observe that $(\overline{\mathbb{R}}, K)$ defines the discrete set D of midpoints of complementary intervals of K and a map $f : D \to K$ that maps an element $d \in D$ to $\sup K \cap (-\infty, d]$ (see [9, proof of Theorem]). The image $f(D)$ is dense in K. The complexity of the definable sets in $(\overline{\mathbb{R}}, K)$ can be seen as a direct consequence of the definability of f (for evidence see [12, Theorem 1.1] and Hieronymi and Tychonievich [14, Theorem A]). Most of the technical work in this paper, and in particular the use of results about $\mathcal{B}$, will be towards controlling this map.

Definable sets in $(\overline{\mathbb{R}}, K)$

In this section we will study the Cantor set K and the discrete set Q in more detail. The goal is to show that Q and $\epsilon \subseteq Q \times K$ from the previous section are $\emptyset$-definable in $(\overline{\mathbb{R}}, K)$. It then follows from Proposition 2.2 that $(\overline{\mathbb{R}}, K)$ defines an isomorphic copy of $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \in, s_{\mathbb{N}})$.

First, we define $L \subseteq [0,1]$ to be the left endpoints of complementary intervals of K. It is easy to see that both L and R are $\emptyset$-definable in $(\overline{\mathbb{R}}, K)$. Note also that elements in L (or R) are left (resp. right) endpoints of complementary intervals of K_n for some n. We denote the set of left endpoints by L_n.

Definition 2.3. Let $v : (L \cup R) \setminus \{0,1\} \to \mathbb{R}$ map $d \in (L \cup R) \setminus \{0,1\}$ to the length of the complementary interval of K one of whose endpoints is d.

[5] This is really the same construction as for the Cantor ternary set C. In the case of C, the set Q is $3^{\mathbb{N}}$, and hence $q_n = 3^n$. Thus $q_{n-1}^{-1} - q_n^{-1} = 3^{-n-1} - 3^{-n} = 2 \cdot 3^{-n}$. Since C is the set of all numbers between 0 and 1 that have a ternary expansion consisting only of 0's and 2's, one can see directly that a function defined in the same way as h maps into C.

Note that v is $\emptyset$-definable in $(\overline{\mathbb{R}}, K)$. From the construction of K, in particular from $3q_n < q_{n+1}$, we directly get the following lemma.

Lemma 2.4. *Let $d \in R$. Then $v(d) = q_{n-1}^{-1} - 2q_n^{-1}$ if and only if $d \in R_n \setminus R_{n-1}$. The same holds with R replaced by L and R_n by L_n.*

Hence for every $n \in \mathbb{N}$, both L_n and R_n are $\emptyset$-definable in $(\overline{\mathbb{R}}, K)$.

Definition 2.5. Let $w : R \to L$ map $r \in R$ to the smallest $l \in L$ with $l > r$ and $v(l) \geq v(r)$ if such an l exists, and to 1 otherwise.

Since v is definable in $(\overline{\mathbb{R}}, K)$, so is w.

Corollary 2.6. *The set Q is equal to $\{w(r) - r : r \in R\}$ and hence is $\emptyset$-definable in $(\overline{\mathbb{R}}, K)$.*

Proof. Let $r \in R_n \setminus R_{n-1}$. By Lemma 2.4, $w(r)$ is the smallest left endpoint of the complementary interval of K_n, larger than r. It follows immediately from the construction of K_n that $w(r) - r = q_n^{-1}$. Hence $Q = \{w(r) - r : r \in R\}$ and Q is $\emptyset$-definable in $(\overline{\mathbb{R}}, K)$, since w is. □

Recall that the predecessor function on Q is denoted by p_Q. Since Q is $\emptyset$-definable in $(\overline{\mathbb{R}}, K)$, so is p_Q. For $q \in Q$, we set $R_q := \{r \in R : v(r) \geq p_Q(q)^{-1} - 2q^{-1}\}$. By Lemma 2.4, if $q = q_n$ for some n, then $R_q = R_n$. We immediately get $\emptyset$-definability of e and ϵ in $(\overline{\mathbb{R}}, K)$.

Corollary 2.7. *The function e and the set ϵ are $\emptyset$-definable in $(\overline{\mathbb{R}}, K)$.*

3. Preliminaries from o-minimality

O-minimal structures

Throughout this paper, the reader is assumed to know the basic results about o-minimal structures and theories, as can be found in [6]. The only o-minimal theory we will consider is the $\mathcal{L}$-theory T. Let $M \models T$. For a subset $X \subseteq M$, we denote the *definable closure* of X in M by $\mathrm{dcl}_{\mathcal{L}}(X)$. When it is clear from the context which language $\mathcal{L}$ is used, we simply write $\mathrm{dcl}(X)$. As is well known, in an o-minimal structure the definable closure operator is a pregeometry. We will make use of this fact without explicit mention.

Every complete o-minimal theory expanding the theory of real closed fields has definable Skolem functions. Hence by extending the language $\mathcal{L}$ and the theory T by definitions, we may assume that T has quantifier elimination and is universally axiomatizable, and that $\mathcal{L}$ has no relation symbol other than $<$.

Limit points of images of K^n under $\mathcal{L}$-definable functions

We now recall some definitions and results from [10]. For details and proofs, the interested reader should consult the original source.

Define $\psi : [0, \infty) \to \mathbb{R}$ by

$$\psi(t) := \begin{cases} 0, & t = 0, \\ e^{-1/t}, & 0 < t < 1, \\ t - 1 + e^{-1}, & t \geq 1. \end{cases}$$

Note that ψ is $\emptyset$-definable in $\mathcal{R}$. For $m \in \mathbb{Z}$, we denote the m-th compositional iterate of ψ by ψ_m. Note that $\lim_{k\to\infty} \psi_m(q_k^{-1})/q_{k+1} = 0$ for every $m \in \mathbb{N}$.

For $n \in \mathbb{N}$ and $l \in \mathbb{Z}$ define

$$S_{n,l} = \{x \in \mathbb{R}^n : 0 < x_n < \psi_l(x_{n-1}) < \cdots < \psi_{(n-1)l}(x_1)\}.$$

Again, note that every $S_{n,l}$ is $\emptyset$-definable in $\mathcal{R}$. Let $\mathcal{T}_n$ be the group of symmetries, regarded as linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$, of the polyhedron inscribed in the unit ball of $\mathbb{R}^n$ whose vertices are the intersection of the unit sphere in $\mathbb{R}^n$ with $\{tu : t > 0 \wedge u \in \{-1, 0, 1\}^n\}$.

Fact 3.1 (cp. [10, 1.8]). *Let $X \subseteq \mathbb{R}^n$ and $f : X \to [0, 1]$ be definable in $\mathcal{R}$. Then there is $j \in \mathbb{N}$ such that for all $y \in \mathrm{cl}(X)$ there is a $\delta \in \mathbb{R}_{>0}$ such that for every $m \leq n$ and every $T \in \mathcal{T}_n$ the restriction of f to $B_\delta(y) \cap X \cap (y + T(S_{m,j} \times \{0\}^{n-m}))$ is continuous and extends continuously to the closure.*

Note that j is chosen uniformly for all $y \in \mathrm{cl}(X)$. In [10, 1.8], j could a priori depend on y. However, as stated directly after [10, 1.8], the statement of [10, 1.8] holds in general exponentially bounded o-minimal structures, in particular in any elementary extensions of $\mathcal{R}$. Thus by compactness one can easily check that there is a $j \in \mathbb{N}$ such that the conclusion of [10, 1.8] holds for this j for every $y \in \mathrm{cl}(X)$.

Fact 3.2 (cp. [10, (ii) on p. 1320]). *For every $j > 0$ there is $\delta > 0$ such that for all $n > 0$,*

$$(K^n - K^n) \cap (-\delta, \delta)^n \subseteq \bigcup_{m\leq n} \bigcup_{T\in\mathcal{T}_n} T(S_{m,j} \times \{0\}^{n-m}).$$

Fact 3.2 was proved in [10] not for K, but for a $\mathbb{Q}$-linearly independent subset of K. One can check that the proof only needs minor adjustments to give Fact 3.2. The next result we want to state is also only shown in [10] for a $\mathbb{Q}$-linearly independent subset of K. However, given Fact 3.2 the same proof works for K.

Fact 3.3 (cp. [10, proof of Theorem B on p. 1319]). *Every bounded unary definable set in $(\mathcal{R}, K)$ either has interior or is Minkowski null.*

We will collect a few easy corollaries of Fact 3.2. Since these results were not stated in [10], not even for $\mathbb{Q}$-linearly independent subsets of K, we will give proofs here.

Lemma 3.4. *Let $X \subseteq \mathbb{R}^{l+n}$ and $f : X \to [0, 1]$ be $\mathcal{L}$-$\emptyset$-definable and continuous. Then there are $\mathcal{L}$-$\emptyset$-definable functions $g_1, \ldots, g_k : \mathbb{R}^{l+n} \to \mathbb{R}$ such that for every $x \in \pi(X)$, every $c \in K^n \cap \mathrm{cl}(X_x)$ and every $\varepsilon > 0$ there is $\delta > 0$ such that for $c' \in K^n \cap B_\delta(c) \cap X_x$,*

$$f(x, c') \in B_\varepsilon(g_1(x, c)) \cup \cdots \cup B_\varepsilon(g_k(x, c)).$$

Proof. Let $j > 0$ be as given by Fact 3.1. Define $g_{m,T} : \mathbb{R}^{l+n} \to \mathbb{R}$ to map (x, y) to $\lim_{z\to y} f(x, z)$, where z ranges over $y + \bigcup_{T\in\mathcal{T}_n} T(S_{m,j} \times \{0\}^{n-m}) \cap X_x$ if $y \in \operatorname{cl}(X_x)$ and $y + \bigcup_{T\in\mathcal{T}_n} T(S_{m,j} \times \{0\}^{n-m}) \cap X_x \neq \emptyset$, and to 0 otherwise. By our choice of j, the function $g_{m,T}$ is well defined and for every $x \in \mathbb{R}^l$ the function $g_{m,T}(x, -)$ extends $f(x, -)$ because of the continuity of f. Now take $x \in \mathbb{R}^l$, $c \in K^n$ and $\varepsilon > 0$. By Fact 3.2 we get $\delta > 0$ small enough that

$$(K^n - c) \cap (-\delta, \delta)^n \subseteq \bigcup_{m\le n} \bigcup_{T\in\mathcal{T}_n} T(S_{m,j} \times \{0\}^{n-m}).$$

By Fact 3.1 we can reduce δ so that for any m, T the function f is continuous on $B_\delta(c) \cap (c + \bigcup_{T\in\mathcal{T}_n} T(S_{m,j} \times \{0\}^{n-m}) \cap X_x)$ and extends continuously to the closure. Hence by further reducing δ, we conclude that for every $c' \in K^n \cap B_\delta(c) \cap \operatorname{cl}(X_x)$,

$$f(x, c') \in \bigcup_{m\le n} \bigcup_{T\in\mathcal{T}_n} B_\varepsilon(g_{m,T}(x, c)).$$

□

Corollary 3.5. *Let $X \subseteq \mathbb{R}^{l+n}$ and $f : X \to [0, 1]$ be $\mathcal{L}$-$\emptyset$-definable and continuous. Let $g_1, \ldots, g_k$ be as given in Lemma 3.4. Then*

$$\operatorname{cl}(f(x, X_x \cap K^n)) \subseteq \bigcup_{i=1}^{k} g_i(x, \operatorname{cl}(X_x) \cap K^n).$$

For every $x \in \pi(X)$ and every $z \in \operatorname{cl}(f(x, X_x \cap K^n))$, the set

$$\{c \in K^n \cap \operatorname{cl}(X_x) : z = g_i(x, c) \text{ for some } i = 1, \ldots, k\}$$

is closed.

Proof. Let $z \in \operatorname{cl}(f(x, X_x \cap K^n))$. Let $(c_j)_{j\in\mathbb{N}}$ be a sequence of elements in $K^n \cap X_x$ such that $\lim_{j\to\infty} f(x, c_j) = z$. Since K is bounded, we can assume $(c_j)_{j\in\mathbb{N}}$ converges. Since $\operatorname{cl}(X_x) \cap K^n$ is closed, there is $c \in \operatorname{cl}(X_x) \cap K^n$ such that $\lim_{j\to\infty} c_j = c$. By Lemma 3.4, $\lim_{j\to\infty} f(x, c_j) = g_i(x, c)$ for some $i \in \{1, \ldots, k\}$.

For the second statement, let $x \in \pi(X)$ and $z \in \operatorname{cl}(f(x, X_x \cap K^n))$, and suppose there is $c \in K^n \cap \operatorname{cl}(X_x)$ such that $z \neq g_i(x, c)$ for all i. Let $\varepsilon > 0$ be such that $2\varepsilon < \min_{i=1,\ldots,k} |g_i(x, c) - z|$. By Lemma 3.4 there is $\delta > 0$ such that $|f(x, c') - g_i(x, c)| < \varepsilon$ for all $c' \in K^n \cap X_x \cap B_\delta(c)$ and all i. Let $d \in K^n \cap \operatorname{cl}(X_x) \cap B_{\delta/2}(c)$. By Lemma 3.4 there is $\gamma > 0$ such that $\gamma < \delta/2$ and $|f(x, c') - g_i(x, d)| < \varepsilon$ for all $c' \in K^n \cap B_\gamma(c) \cap X_x$ and all i. Let $c' \in K^n \cap X_x \cap B_\gamma(d)$. Since $\gamma < \delta/2$, we have $c' \in B_\delta(c)$. Then

$$|g_i(x, d) - g_i(x, c)| \le |g_i(x, d) - f(x, c')| + |g_i(x, c) - f(x, c')| < 2\varepsilon = |g_i(x, c) - z|.$$

Thus $g_i(x, d) \neq z$ for all i and all $d \in K^n \cap \operatorname{cl}(X_x) \cap B_{\delta/2}(c)$. Therefore the set in the second statement of the corollary is closed. □

T-levels and T-convexity

We will now recall some less well known results about o-minimal theories. We start by a review of the notion of T-levels as introduced by Tyne [25]. For more details and proofs we refer to [25] and [26]. For this section, fix a model M of T.

Definition 3.6. Let $x \in M$. We write $0 \ll x$ if x is greater than every element of $\mathrm{dcl}(\emptyset)$. For $0 \ll x \in M$, the T*-level* of x, denoted by $[x]$, is the convex hull in M of the set of all values $f(x)$, with f ranging over all $\mathcal{L}$-$\emptyset$-definable functions $f : M \to M$ that are strictly increasing and unbounded from above.

Fact 3.7 ([25, Corollary 3.11]). *Let $X \subseteq M$ and $a \in M$ be such that $[a] \cap X = \emptyset$. Then*

$$\{x \in \mathrm{dcl}(X \cup \{a\}) : 0 \ll x\} = \bigcup_{0 \ll x \in X} [x] \cup [a].$$

We will need the following generalization which can easily be deduced by induction on the size of A.

Fact 3.8. *Let $X \subseteq M$ and let $A \subseteq M$ be finite such that for all $a, b \in A$ we have $0 \ll a$, $a \notin \bigcup_{0 \ll x \in X} [x]$, and $[a] \neq [b]$ whenever $a \neq b$. Then*

$$\{x \in \mathrm{dcl}(X \cup A) : 0 \ll x\} = \bigcup_{0 \ll x \in X} [x] \cup \bigcup_{a \in A} [a].$$

Throughout this paper we will use the fact that given an elementary substructure of M, the $\mathcal{L}$-type of a tuple of elements of M over X whose pairwise disjoint T-levels do not intersect X is determined just by the order of the elements in the tuple. The next lemma makes this statement precise.

Lemma 3.9. *Let $X \preceq M$ and let $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in M^n$ be such that*

(i) $0 \ll a_1 < \cdots < a_n$ *and* $0 \ll b_1 < \cdots < b_n$,
(ii) $a_i \notin \bigcup_{0 \ll x \in X} [x]$ *and* $b_i \notin \bigcup_{0 \ll x \in X} [x]$ *for* $i = 1, \ldots, n$,
(iii) $[a_i] \neq [a_j]$ *and* $[b_i] \neq [b_j]$,
(iv) $\mathrm{tp}_{\mathcal{L}}(a_i|X) = \mathrm{tp}_{\mathcal{L}}(b_i|X)$.

Then $\mathrm{tp}_{\mathcal{L}}(a|X) = \mathrm{tp}_{\mathcal{L}}(b|X)$.

Proof. We use induction on n. For $n = 1$, the statement follows immediately from (iv). Now suppose it holds for $n - 1$. Hence

$$\mathrm{tp}_{\mathcal{L}}(a_1, \ldots, a_{n-1}|X) = \mathrm{tp}_{\mathcal{L}}(b_1, \ldots, b_{n-1}|X). \tag{3.1}$$

Let $f : M^{n-1} \to M$ be an $\mathcal{L}$-X-definable function. We will write a' for $(a_1, \ldots, a_{n-1})$ and b' for $(b_1, \ldots, b_{n-1})$. In order to show that $\mathrm{tp}_{\mathcal{L}}(a_n|X, a') = \mathrm{tp}_{\mathcal{L}}(b_n|X, b')$, it is enough to show that $f(a') \neq a_n$, $f(b') \neq b_n$, and $f(a') < a_n$ iff $f(b') < b_n$. By (3.1), $0 \ll f(a')$ iff $0 \ll f(b')$. Since $0 \ll a_n$ and $0 \ll b_n$ by (ii), we can assume that $0 \ll f(a')$ and $0 \ll f(b')$. Then by Fact 3.8, $f(a') \in \bigcup_{i=1}^{n-1} [a_j] \cup \bigcup_{0 \ll x \in X} [x]$ and $f(b') \in \bigcup_{i=1}^{n-1} [b_j] \cup \bigcup_{0 \ll x \in X} [x]$. Since $a_n \notin \bigcup_{i=1}^{n-1} [a_j] \cup \bigcup_{0 \ll x \in X} [x]$ and $b_n \notin \bigcup_{i=1}^{n-1} [b_j] \cup \bigcup_{0 \ll x \in X} [x]$ by (ii) and (iii), we see that $f(a') \neq a_n$ and $f(b') \neq b_n$. It remains to establish that $f(a') < a_n$ iff $f(b') < b_n$. First suppose there is $j \in \{1, \ldots, n\}$ such that $[f(a')] = [a_j]$. By (3.1), $[f(b')] = [b_j]$. Hence by (iii), $f(a') < a_n$ iff $a_j < a_n$ iff $b_j < b_n$ iff $f(b') < b_n$. Now suppose there is $x \in X$ such that $[f(a')] = [x]$. By (3.1), $[f(b')] = [x]$. By (ii) and (iv), $f(a') < a_n$ iff $x < a_n$ iff $x < b_n$ iff $f(b') < b_n$. □

We now turn our attention to the notion of a T-convex subring, which was introduced by van den Dries and Lewenberg [7]. For more details and proofs we refer the reader to [7] and its companion paper [4].

Definition 3.10. A convex subring V of M is called T*-convex* if $f(V) \subseteq V$ for all $\mathcal{L}$-$\emptyset$-definable functions $f : M \to M$.

Lemma 3.11. *Let $a \in M$ and let $U_a := \mathrm{dcl}(\emptyset) \cup \bigcup_{x \in M, [x]<[a]} [x]$. Then $V_a = U_a \cup -U_a$ is a T-convex subring. Its maximal ideal $\mathfrak{m}_a$ is $\{x \in V_a : [|x|^{-1}] \geq [a]\}$.*

Proof. The fact that V_a is T-convex follows immediately from the definition of V_a and [25, Lemma 10.2]. Since $\mathfrak{m}_a$ is exactly the set of non-units of V_a, the description of $\mathfrak{m}_a$ in the lemma holds. □

For the rest of this section fix $a \in A$ with $0 \ll a$. Throughout, for $x \in M$, we denote the residue class of x mod $\mathfrak{m}_a$ by $\overline{x}^a$. For a subset $X \subseteq M$, we write $\overline{X}^a$ for $\{\overline{x}^a : x \in X\}$. By [7, Remark 2.16], $\overline{V_a}^a$ expands naturally into a model of T that by [7, Remark 2.11] is isomorphic to a tame[6] substructure of $\mathcal{R}$. We will always consider $\overline{V_a}^a$ as a model T in this way. Let $y \in V_a$ and $X \subseteq V_a$. When saying $\overline{y}^a$ is dcl-dependent over $\overline{X}^a$, we mean dcl-dependent with respect to this T-model on $\overline{V_a}^a$. The following lemma can easily be derived from [4, Proposition 1.7].

Lemma 3.12. *Let $x \in V_a^m$ and $y \in V_a$. Then the following are equivalent:*

(i) *there is a continuous $\mathcal{L}$-$\emptyset$-definable $f : U \subseteq M^m \to M$ such that $x \in U$, U is open and $[|f(x) - y|^{-1}] \geq [a]$,*
(ii) *$\overline{y}^a$ is* dcl*-dependent over $\overline{x}^a$.*

4. The theory and its consequences

In this section, we begin the study of the first-order theory of $(\mathcal{R}, K, Q, \epsilon)$. We will define a theory $\tilde{T}$ in the language of the structure. In addition to deriving first consequences of this theory, we will prove that this structure is a model of $\tilde{T}$. The completeness of $\tilde{T}$ will be established later. We assume that the language $\mathcal{L}$ of the underlying o-minimal structure $\mathcal{R}$ already contains constant symbols for each element of Q and for each element of C. From the construction of Q, we know that for every $n \in \mathbb{N}$ there is $m \in \mathbb{N}$ such that $s_Q(p) > \exp_n(p)$ for all $p \in Q$ with $p \geq q_m$. We denote the minimal such m by $\omega(n)$. For $p \in \mathbb{Q}^n$ we denote by $\xi(p)$ the minimal $k \in \mathbb{N}$ such that

$$\left\{ \frac{s \cdot p}{2(t \cdot p) + 1} : s, t \in \{-1, 0, 1\}^n,\ 2(t \cdot p) \neq -1 \right\} \subseteq P_{k-1}.$$

[6] Tame here means tame in the sense of [7, p. 76].

Notations and conventions

We consider structures

$$\mathcal{M} := (M, C, A, E),$$

where M is an $\mathcal{L}$-structure, $C, A \subseteq M$ and $E \subseteq M^2$. Let $\mathcal{L}_C$ be the language of this structure.

Let $\mathcal{M}$ be such an $\mathcal{L}_C$-structure. With the usual abuse of notation, we will write Q for the set of interpretations of the constant symbols corresponding to elements in Q, and we will write K for the set of interpretations of the constant symbols corresponding to elements in K. Similarly for $q \in Q$ and $c \in K$, we will use q for the interpretation of the constant corresponding to q in M, and we will use c for the interpretation of the constant corresponding to c in M.

In the following we want to restrict ourselves to $\mathcal{L}_C$-structures that satisfy a certain $\mathcal{L}_C$-theory. Before we can describe this theory, we have to introduce some further notations.

Definition 4.1. For every $c \in C$, let $S(c)$ be the set $\{a \in A : E(a, c)\}$. Let $e : A \times C \to C$ map (a, c) to the unique $d \in C$ such that $S(d) = S(c)_{\leq a}$ if such a d exists, and to 0 otherwise.

One of the sentences in the $\mathcal{L}_C$-theory we are going to define will guarantee that the unique d in the definition of e will indeed always exist. If $c = (c_1, \dots, c_n) \in C^n$ and $a \in A$, we write $e(a, c) := (e(a, c_1), \dots, e(a, c_n))$.

Definition 4.2. For $a \in A$ and $c \in C$, we set

$$\delta_{a,c} := \begin{cases} 1 & \text{if } E(a, c), \\ 0 & \text{otherwise.} \end{cases}$$

For $q = (q_1, \dots, q_n) \in \mathbb{Q}^n$, let $\mu_q : C^n \to A$ map $c = (c_1, \dots, c_n)$ to the minimum $a \in A$ such that $\sum_{i=1}^n q_i \delta_{a,c_i} \neq 0$ if such an a exists, and to 0 otherwise.

We recall that given an ordered set $(Y, \prec)$, we denote the predecessor function on Y by p_Y and the successor function on Y by s_Y if such functions are well defined. So whenever A is a closed and discrete subset of M, p_A and s_A will denote the predecessor function and the successor function on A with respect to $<$.

The theory

We are now ready to define the desired $\mathcal{L}_C$-theory.

Definition 4.3. Let $\tilde{T}$ be the $\mathcal{L}_C$-theory consisting of the first-order $\mathcal{L}_C$-sentences expressing the following statements:

- T1. $M \models T$.
- T2. $C \subseteq [0, 1]$ is closed, has no isolated points and empty interior, and $K \subseteq C$.
- T3. A is an infinite, unbounded, closed and discrete subset of $M_{\geq 1}$ with initial segment $Q \subseteq A$.
- T4. For all $n \in \mathbb{N}$ and all $a \in A$, if $a > \omega(n)$, then $s_A(a) > \exp_n(a)$.

T5. $(A, C, E, s_A) \equiv (\mathbb{N}, \mathcal{P}(\mathbb{N}), \in, s_{\mathbb{N}})$.

T6. For all $c_1, \ldots, c_n \in K$ and all $\mathcal{L}_B$-formulas φ,

$$(Q, K, \epsilon, s_Q) \models \varphi(c_1, \ldots, c_n) \quad \text{if and only if} \quad (A, C, E, s_A) \models \varphi(c_1, \ldots, c_n).$$

T7. For all $c \in C$ and $a \in A$, $0 \le c - e(a, c) \le a^{-1}$.

T8. For all $c, d \in C$ and $a \in A$,

$$\text{if} \quad 0 \le d - e(a, c) \le a^{-1}, \quad \text{then} \quad e(a, c) = e(a, d).$$

T9. For all $c \in C$ and $a \in A$, $c - e(a, c) \in C$ and

$$S(c - e(a, c)) = S(c)_{>a}.$$

T10. For all $c \in C$ and $a, b \in A$, if $E(d, c)$ for all $d \in A \cap (a, b]$, then

$$e(b, c) = e(a, c) + a^{-1} - b^{-1}$$

T11. For all $p \in \mathbb{Q}^n$, all $c \in C^n$ and all $a \in A$ with $a \ge \xi(p)$,

(i) $p \cdot c = 0$ if and only if $\mu_p(c) = 0$,

(ii) if $0 < |p \cdot c| < a^{-1}$, then $\mu_p(c) \ge a$.

T12. For every $X \subseteq M^{l+n}$ and $f : X \to [0, 1]$ $\mathcal{L}$-$\emptyset$-definable continuous, and $g_1, \ldots, g_k : M^{l+n} \to M$ as given in Lemma 3.4, and for every $x \in \pi(X)$ with $X_x \cap C^n \neq \emptyset$ and every $z \in \mathrm{cl}(f(x, X_x \cap C^n))$, there exists a lexicographically minimal $c \in C^n$ such that there is $i \in \{1, \ldots, k\}$ with $g_i(x, c) = z$.

One has to check that there is such a first-order $\mathcal{L}_C$-theory $\tilde{T}$. In most cases it is routine to show that the above axioms can be expressed by first-order statements. Of course, statements like Axiom T4, Axiom T5 or Axiom T6 have to be expressed by axiom schemes rather than by a single $\mathcal{L}_C$-sentence.

Note that by Axiom T5 the unique d in the definition of e in Definition 4.1 indeed always exists. By Axiom T5 we also see that the function S is injective. Moreover Axiom T3 guarantees that the predecessor function p_A and the successor s_A on A are well defined. So in particular our use of s_A in Axiom T4 is unproblematic.

Proposition 4.4. $(\mathcal{R}, K, Q, \epsilon) \models \tilde{T}$.

Proof. It follows immediately from the definitions of $\mathcal{R}$, K and Q that Axioms T1–T4 hold. One can easily deduce Axioms T5–T10 from Proposition 2.2. Axiom T12 follows from Corollary 3.5. Axiom T11 requires a bit more explanation. We will just show (ii), because (i) can be established similarly. Let $p \in \mathbb{Q}^n$, $c \in K^n$ and $k \in \mathbb{N}$ be such that $0 < |p \cdot c| \le q_k^{-1}$ and $q_k \ge \xi(p)$. Towards a contradiction, suppose $\mu_p(c) < q_k$. Note that there are $s_0, \ldots, s_k \in \{-1, 0, 1\}^n$ such that

$$p \cdot e(q_k, c) = \sum_{i=0}^{k} (s_i \cdot p) q_i^{-1}. \tag{4.1}$$

Since $\mu_p(c) < q_k$, there is at least one $i < k$ such that $s_i \cdot p \neq 0$. By the algebraic independence of elements of Q, we get $\sum_{i=0}^{k-1}(s_i \cdot p)q_i^{-1} \neq 0$. Since $q_k \geq \xi(p)$,

$$\left|\sum_{i=0}^{k-1}(s_i \cdot p)q_i^{-1}\right| \geq \Big(2 \max_{t\in\{-1,0,1\}^n}(t \cdot p) + 1\Big)q_k^{-1}. \tag{4.2}$$

Note that $|p \cdot (c - e(q_k, c))| < \max_{t\in\{-1,0,1\}^n}(t \cdot p) \cdot q_k^{-1}$ (see Axiom T7). From this statement, (4.1) and (4.2), the reader can now easily deduce $|p \cdot c| > q_k^{-1}$. □

One of the main results of this paper is the following. It will be proved towards the end of the paper.

Theorem 4.5. *The theory $\tilde{T}$ is complete.*

Consequences of $\tilde{T}$

In this subsection we establish first consequences of $\tilde{T}$. So throughout this subsection, let $\mathcal{M} \models \tilde{T}$. We start by collecting some results about the function e, in particular how e interacts with the arithmetic operations on M.

Lemma 4.6. *The unique element $c \in C$ with $S(c) = \emptyset$ is* 0.

Proof. By Axiom T6, $S(0) = \emptyset$. Uniqueness follows from the injectivity of S. □

Lemma 4.7. *Let $c \in C$ and $a \in A$ be such that $E(b, c)$ for all $b \in A$ with $b > a$. Then $c = e(a, c) + a^{-1}$.*

Proof. Let $b \in A$ with $b > a$. By Axiom T10, $e(b, c) = e(a, c) + a^{-1} - b^{-1}$. Thus

$$|c - (e(a, c) + a^{-1})| \leq |c - e(b, c)| + |e(b, c) - (e(a, c) + a^{-1})| \leq 2b^{-1}.$$

Since A is unbounded in M by Axiom T3, $c = e(a, c) + a^{-1}$. □

Corollary 4.8. *Let $a, b \in A$. Then a^{-1} is the unique element c in C with $S(c) = A_{>a}$, and*

$$a^{-1} - e(b, a^{-1}) = \begin{cases} a^{-1} & \text{if } b \leq a, \\ b^{-1} & \text{otherwise.} \end{cases}$$

Proof. Let c be the unique element in C such that $S(c) = A_{>a}$. Note that $e(b, c) = 0$ by Lemma 4.6 for all $b \leq a$. By Lemma 4.7, $c = e(a, c) + a^{-1} = a^{-1}$. Now suppose $b > a$. Then $a^{-1} - e(b, a^{-1})$ is in C by Axiom T9 and

$$S(a^{-1} - e(b, a^{-1})) = S(a^{-1})_{>b} = S(b^{-1}).$$

By injectivity of S, we get $a^{-1} - e(b, a^{-1}) = b^{-1}$. □

Lemma 4.9. *Let $c \in C$ and $a, b \in A$ with $a < b$. Then*

$$c - e(a, c) - e(b, c - e(a, c)) = c - e(b, c).$$

Proof. By Axiom T9, $c - e(a,c) \in C$. Thus $e(b, c - e(a,c)) \in C$. Again by Axiom T9, $S(c - e(b,c)) = S(c)_{>b}$ and $S(c - e(a,c)) = S(c)_{>a}$. By Axiom T9 once more, $c - e(a,c) - e(b, c - e(a,c)) = S(c - e(a,c))_{>b} = S(c)_{>b}$. By Axiom T5, $c - e(a,c) - e(b, c - e(a,c)) = c - e(b,c)$. □

Lemma 4.10. *Let $c \in C$ and $a, b \in A$. If $\neg E(d,c)$ for all $d \in A \cap (a,b]$, then $e(b,c) = e(a,c)$.*

Proof. From the assumptions on a and b we can directly conclude that $S(e(a,c)) = S(e(b,c))$. By Axiom T5, $e(a,c) = e(b,c)$. □

Corollary 4.11. *Let $c \in C$ and $a \in A$. Then*

$$e(s_A(a), c) = e(a,c) + \delta_{s_A(a),c}(a^{-1} - s_A(a)^{-1}).$$

Proof. The statement follows immediately from Lemma 4.10 when $\neg E(s_A(a), c)$, and from Axiom T10 when $E(s_A(a), c)$. □

We get the following lemma directly from Lemma 4.10 and Corollary 4.11.

Lemma 4.12. *Let $c \in C$ and let $a, b \in A$ be such that $a < b$ and b is the minimal element in A with $b > a$ and $E(b,c)$. Then $e(b,c) = e(a,c) + p_A(b)^{-1} - b^{-1}$.*

We now collect a few easy corollaries of Axiom T11.

Lemma 4.13. *Let $c \in C^n$ and $q \in \mathbb{Q}^n$, and let $a \in A$ be such that $a < \mu_q(c)$. Then $q \cdot e(a,c) = 0$.*

Proof. Since $a < \mu_q(c)$, we have $\mu_q(e(a,c)) = 0$. By Axiom T11, $q \cdot e(a,c) = 0$. □

Corollary 4.14. *Let $q = (q_1, \dots, q_n) \in \mathbb{Q}^n$ and $c = (c_1, \dots, c_n) \in C^n$ be such that $\mu_q(c) > 0$. Then*

$$q \cdot c = \sum_{i=1}^{n} q_i \delta_{\mu_q(c), c_i}(p_A(\mu_q(c))^{-1} - \mu_q(c)^{-1}) + q \cdot (c - e(\mu_q(c), c)).$$

Proof. The statement is a direct consequence of Corollary 4.11 and Lemma 4.13. □

Lemma 4.15. *Let $a \in A$, $c = (c_1, \dots, c_n) \in C^n$, $d \in C$ and $p \in \mathbb{Q}^n$ be such that $d = p \cdot c$. Then $e(a,d) = p \cdot e(a,c)$.*

Proof. By Axiom T11, $\mu_{(-1,p)}(d,c) = 0$. By the definition of μ, $\delta_{b,d} = \sum_{i=1}^{n} p_i \delta_{b,c_i}$ for all $b \in A$. Note that for every $c' \in C$ and $b \in A$, we have $\delta_{b,c'} = \delta_{b,e(a,c')}$ if $b \le a$, and $\delta_{b,e(a,c')} = 0$ otherwise. Therefore $0 = -\delta_{b,e(a,d)} + \sum_{i=1}^{n} p_i \delta_{b,e(b,c_i)}$ for every $b \in A$. Hence $\mu_{(-1,p)}(e(a,d), e(a,c_1), \dots, e(a,c_n)) = 0$. Axiom T11 yields the conclusion. □

Because $e(b, e(a,c)) = e(b,c)$ for all $c \in C$ and $a, b \in A$ with $b \le a$, the following corollary can be deduced directly from Lemma 4.15.

Corollary 4.16. *Let $a, b \in A$ with $b \le a$, $c \in C^n$, $d \in C$ and $p \in \mathbb{Q}^n$ be such that $e(a,d) = p \cdot e(a,c)$. Then $e(b,d) = p \cdot e(b,c)$.*

Complementary intervals

Another set of important and still rather easy consequences of $\tilde{T}$ concerns the set of complementary intervals of C. By a *complementary interval* of C, we mean an interval (x, y) between two elements $x, y \in M$ such that $(x, y) \cap C = \emptyset$ and $x, y \in C$. Given $a \in A$ and $c \in C$, we will first consider complementary intervals strictly between $e(a, c)$ and $e(a, c) + a^{-1}$. Afterwards we will consider complementary intervals that have one of these two points as an endpoint.

Lemma 4.17. *Let $a \in A$ and $c \in C$. Then*

$$\left(e(a, c) + s_A(a)^{-1}, e(a, c) + a^{-1} - s_A(a)^{-1}\right)$$

is a complementary interval of C.

Proof. It follows immediately from Corollary 4.11, Axiom T5 and Lemma 4.7 that the two endpoints of the interval are in C. Suppose there is $d \in C$ in the interval. By Axiom T8, $e(a, d) = e(a, c)$. By Corollary 4.11, $e(s_A(a), d) \in \{e(a, c), e(a, c) + a^{-1} - s_A(a)^{-1}\}$. Hence by Axiom T7, either $d \in [e(a, c), e(a, c) + s_A(a)^{-1}]$ or $d \in [e(a, c) + a^{-1} - s_A(a)^{-1}, e(a, c) + a^{-1}]$. This contradicts our assumption on d. □

We will now use Lemma 4.17 to show that $e(a, c) + a^{-1}$ is the left endpoint of a complementary interval. Its right endpoint will depend on whether or not $E(a, c)$ holds.

Corollary 4.18. *Let $a, b \in A$ and $c \in C$ be such that $b \in A$ is maximal in A with $b \le a$ and $\neg E(b, c)$. Then $(e(a, c) + a^{-1}, e(a, c) + a^{-1} + p_A(b)^{-1} - 2b^{-1})$ is a complementary interval of C.*

Proof. Since $\neg E(b, c)$, we have $e(p_A(b), c) = e(b, c)$ by Lemma 4.10. Therefore by Lemma 4.17, $(e(b, c) + b^{-1}, e(b, c) + p_A(b)^{-1} - b^{-1})$ is a complementary interval of C. Since $e(a, c) + a^{-1} = e(b, c) + b^{-1}$ by Axiom T10, the statement follows. □

Note that if a in the previous corollary satisfies $\neg E(a, c)$, the right endpoint of the interval is $e(a, c) + p_A(a)^{-1} - a^{-1}$. Finally, we will now show that $e(a, c)$ is the right endpoint of a complementary interval of C.

Lemma 4.19. *Let $a, b \in A$ and $c \in C$ be such that b is maximal in A with $b \le a$ and $E(b, c)$. Then $(e(a, c) - p_A(b)^{-1} + 2b^{-1}, e(a, c))$ is a complementary interval of C.*

Proof. By Lemma 4.17, $(e(p_A(b), c)+b^{-1}, e(p_A(b), c)+p(b)^{-1}-b^{-1})$ is a complementary interval of C. By Corollary 4.11, $e(a, c) = e(b, c) = e(p_A(b), c) + p_A(b)^{-1} - b^{-1}$. The statement follows. □

5. The closest element in A and C

As before, let $\mathcal{M} \models \tilde{T}$. In this section, we will study two important definable functions λ and ν and their interaction with the $\mathcal{L}$-structure on M.

Definition 5.1. Let $\lambda : M \to A$ map $x \in M$ to $\max A \cap (-\infty, x]$ if this maximum exists, and to 0 otherwise. Let $\nu : M \to C$ map $x \in M$ to $\max C \cap (-\infty, x]$ if this maximum exists, and to 0 otherwise.

Since A and C are closed, the maximum in the definition of λ exists for all $x \geq 1$, and the maximum in the definition of ν exists for all $x \geq 0$. The two main results of this section are as follows. The first is that two distinct elements of A have to lie in different T-levels. This will allow us to show that elements of C that are $\mathbb{Q}$-linearly independent over K are also dcl-independent over K. The second main result states that if $a \in A$, $x, y \in M$ and $x - y \in \mathfrak{m}_a$, then $e(a, \nu(y))$ can be expressed in terms of $e(a, \nu(x))$.

T-levels and A

In this subsection we will study consequences of Axiom T4, in particular on the T-levels of M and on the function λ. Because M is exponentially bounded, from $Q \subseteq \mathrm{dcl}(\emptyset)$ and Axiom T4 we directly get the following lemma.

Lemma 5.2. *Let $a \in A$. Then $0 \ll a$ iff $a \notin Q$.*

Lemma 5.2 will be used routinely throughout this paper. We now show that if $a, b \in A \setminus Q$ and $a \neq b$, then a and b have to lie in different T-levels.

Lemma 5.3. *Let $a, b \in A$ be such that $a \neq b$, $0 \ll a$ and $0 \ll b$. Then $[a] \cap [b] = \emptyset$.*

Proof. Without loss of generality, assume that $a < b$. Suppose that the conclusion fails. Then there are $\mathcal{L}$-$\emptyset$-definable strictly increasing functions $f, g : M \to M$ such that $f(a) < b < g(a)$. Since g is strictly increasing, it is invertible. Since $0 \ll a$ and $0 \ll b$, we get $s_A(a) > g(a)$ and $p_A(a) < f(a)$ by Axiom T4 and exponential boundedness of M. Hence $p_A(a) < b < s_A(a)$. This contradicts $a \neq b$. □

We immediately get the following corollary.

Corollary 5.4. *Let $a_1, \ldots, a_n \in A$ such that $0 \ll a_1 < \cdots < a_n$. Then*

$$[a_j] \cap \bigcup_{i \neq j} [a_i] = \emptyset.$$

Throughout the paper we will have to compare not only elements of A, but also their inverses. The following lemma states that any $\mathbb{Q}$-linear combination of inverses of elements of A is always dominated by the inverse of the smallest element.

Corollary 5.5. *Let $a_1, \ldots, a_n \in A$ be such that $0 \ll a_1 < \cdots < a_n$, and let $(q_1, \ldots, q_n) \in \mathbb{Q}^n$ with $q_1 \neq 0$. Then for every $\varepsilon \in \mathbb{Q}_{>0}$ there are $u_1, u_2 \in \mathbb{Q}$ such that $u_1 a_1^{-1} < \sum_{i=1}^n q_i a_i^{-1} < u_2 a_1^{-1}$ and $|q_1 - u_j| < \varepsilon$ for $j = 1, 2$.*

Proof. By Corollary 5.4, $[a_1] < \cdots < [a_n]$. Thus for any $r, s \in \mathbb{Q}_{>0}$ and $i > 1$, we have $ra_1 < sa_i$ and $sa_i^{-1} < ra_1^{-1}$. The statement follows easily. □

Lemma 5.6. *Let $q = (q_1, \ldots, q_n) \in \mathbb{Q}^n$ and $c = (c_1, \ldots, c_n) \in C^n$ be such that $0 \ll \mu_q(c)$. Then $[|q \cdot c|^{-1}] = [p_A(\mu_q(c))]$.*

Proof. For ease of notation, we set $b := \mu_q(c)$. By Corollary 4.14,

$$\begin{aligned} q \cdot c &= \sum_{i=1}^{n} q_i \delta_{b,c_i} (p_A(b)^{-1} - b^{-1}) + q \cdot (c - e(b,c)) \\ &= \sum_{i=1}^{n} q_i \delta_{b,c_i} p_A(b)^{-1} - \sum_{i=1}^{n} q_i \delta_{b,c_i} b^{-1} + q \cdot (c - e(b,c)). \end{aligned}$$

Set $u := \sum_{i=1}^{n} q_i \delta_{b,c_i}$. By Axiom T7 we have $0 \le c_i - e(b, c_i) \le b^{-1}$ for each $i = 1, \dots, n$. Hence there are $v_1, v_2 \in \mathbb{Q}$ such that $up_A(b)^{-1} + v_1 b^{-1} \le q \cdot c \le up_A(b)^{-1} + v_2 b^{-1}$. By Corollary 5.5 there are $u_1, u_2 \in \mathbb{Q}$ such that $u_1 p_A(b)^{-1} \le q \cdot c \le u_2 p_A(b)^{-1}$. Thus $[|q \cdot c|^{-1}] = [p_A(b)]$. □

Corollary 5.7. *Let $a \in A$, $c \in C^n$ and $q \in \mathbb{Q}^n$ be such that $0 \ll a$ and $q \cdot c \in \mathfrak{m}_a$. Then $q \cdot e(a, c) = 0$.*

Proof. If $\mu_q(c)$ is zero, then so is $\mu_q(e(a, c))$. By Axiom T11, $q \cdot e(a, c) = 0$. Now assume that $\mu_q(c) > 0$. Since $0 \ll a$, we get $0 < |q \cdot c| < p_A(a)^{-1}$. By Axiom T11, $\mu_q(c) \ge p_A(a)$. In particular, $0 \ll \mu_q(c)$. By Lemma 5.6, $[|q \cdot c|^{-1}] = [p_A(\mu_q(c))]$. As $q \cdot c \in \mathfrak{m}_\mathfrak{a}$, we have $a \le p_A(\mu_q(c))$. By Lemma 4.13 we finally get $q \cdot e(a, c) = 0$. □

The following corollary of Lemma 5.3 essentially shows that it is enough to understand λ for a single member of each T-level of M.

Corollary 5.8. *Let $x, y \in M$ be such that $0 \ll x$, $0 \ll y$ and $[x] = [y]$. Then*

$$\lambda(y) = \begin{cases} s_A(\lambda(x)), & y \ge s_A(\lambda(x)), \\ p_A(\lambda(x)), & y < \lambda(x), \\ \lambda(x), & \textit{otherwise}. \end{cases} \tag{5.1}$$

Proof. First consider the case $[x] = [\lambda(x)]$. By Lemma 5.3, $\lambda(x)$ is the only element of A with that property. Therefore $p_A(\lambda(x)) < z < s_A(\lambda(x))$ for every $z \in [x]$. Hence (5.1) holds. Now suppose $\lambda(x) < z$ for all $z \in [x]$. By Lemma 5.3, $s_A(\lambda(x))$ is the only element of A that can possibly lie in $[x]$. Again, it follows easily that (5.1) holds. □

Let $Z \subseteq C$. In the following, we say that $c \in C^n$ is $\mathbb{Q}$*-linearly independent over* Z if there are no $d \in Z^m$, $p \in \mathbb{Q}^m$ and non-zero $q \in \mathbb{Q}^n$ such that $(p, q) \cdot (d, c) = 0$.

Lemma 5.9. *Let $c = (c_1, \dots, c_n) \in C^n$ be $\mathbb{Q}$-linearly independent over K. Then there are*

- $d = (d_1, \dots, d_n) \in K^n$,
- *tuples* $r_1, \dots, r_n, q_1, \dots, q_n \in \mathbb{Q}^n$,
- *pairwise distinct* $a_1, \dots, a_n \in A$

such that for every $i \le n$,

- $q_{i,i} \ne 0$ *and* $q_{i,j} = 0$ *if* $j > i$,
- $0 \ll \mu_{(r_i, q_i)}(d, c) = a_i$.

Proof. For $i = 1, \ldots, n$ let d_i be the unique element in K such that for all $q \in Q$, we have $E(q, d_i)$ if and only if $E(q, c_i)$. The existence of such d_i's follows from Axiom T6. We directly get from Axiom T11 and $\mathbb{Q}$-linear independence of c_i over K that $0 \ll \mu_{(-1,1)}(d_i, c_i)$. We now use induction on n.

For $n = 1$, the conclusion holds with $r_{1,1} = -1$ and $q_{1,1} = 1$.

Now suppose that it holds for $n - 1$. Then there are pairwise distinct $a_1, \ldots, a_{n-1} \in A$ and tuples $r_1, \ldots, r_{n-1}, q_1, \ldots, q_{n-1} \in \mathbb{Q}^n$ such that the conclusion holds for all $i \leq n - 1$. We can assume that $a_1 < \cdots < a_{n-1}$. Let $l \in \{0, 1, \ldots, n-1\}$ be maximal such that there are $p = (p_1, \ldots, p_n), s = (s_1, \ldots, s_n) \in \mathbb{Q}^n$ with $p_n \neq 0$ and $\mu_{(s,p)}(d, c) = a_l$, if such an l exists, and $l = 0$ otherwise.

If $l = 0$, then $\mu_{(0,\ldots,0,-1),(0,\ldots,0,1)}(d, c) \notin \{a_1, \ldots, a_{n-1}\}$. Hence the statement holds with $q_n = (0, \ldots, 0, 1)$ and $r_n = (0, \ldots, 0, -1)$.

Now consider the case $l > 0$. Set

$$t := \frac{\sum_{j=1}^n s_j \delta_{a_l,d_j} + \sum_{j=1}^n p_j \delta_{a_l,c_j}}{\sum_{j=1}^n r_{l,j} \delta_{a_l,d_j} + \sum_{j=1}^n q_{l,j} \delta_{a_l,c_j}}.$$

The denominator is non-zero, because $\mu_{(r_l,q_l)}(d, c) = a_l$. Let $v = (v_1, \ldots, v_n)$, $w = (w_1, \ldots, w_n) \in \mathbb{Q}^n$ be such that

$$v := tr_l - s \quad \text{and} \quad w := tq_l - p.$$

We will now show that $\mu_{(v,w)}(d, c) > a_l$. Since $c_1, \ldots, c_n$ are $\mathbb{Q}$-linearly independent over K, we have $\mu_{(v,w)}(d, c) \neq 0$ by Axiom T11. Now let $b < a_l$. Since $\mu_{(r_l,q_l)}(d, c) = \mu_{(s,p)}(c) = a_l$, we get

$$\begin{aligned}\sum_{j=1}^n v_j \delta_{b,d_j} + \sum_{j=1}^n w_j \delta_{b,c_j} &= \sum_{j=1}^n tr_{l,j} \delta_{b,d_j} - \sum_{j=1}^n s_j \delta_{b,d_j} + \sum_{j=1}^n tq_{l,j} \delta_{b,c_j} - \sum_{j=1}^n p_j \delta_{b,c_j} \\ &= t\Big(\sum_{j=1}^n r_{l,j} \delta_{b,d_j} + \sum_{j=1}^n q_{l,j} \delta_{b,c_j}\Big) - \Big(\sum_{j=1}^n s_j \delta_{b,d_j} + \sum_{j=1}^n p_j \delta_{b,c_j}\Big) \\ &= t0 - 0 = 0.\end{aligned}$$

By our choice of t, we also get

$$\begin{aligned}\sum_{j=1}^n v_j \delta_{a_l,d_j} + \sum_{j=1}^n w_j \delta_{a_l,c_j} &= \sum_{j=1}^n tr_{l,j} \delta_{a_l,d_j} - \sum_{j=1}^n s_j \delta_{a_l,d_j} + \sum_{j=1}^n tq_{l,j} \delta_{a_l,c_j} - \sum_{j=1}^n p_j \delta_{a_l,c_j} \\ &= t\Big(\sum_{j=1}^n r_{l,j} \delta_{a_l,d_j} + \sum_{j=1}^n q_{l,j} \delta_{a_l,c_j}\Big) - \Big(\sum_{j=1}^n s_j \delta_{a_l,d_j} + \sum_{j=1}^n p_j \delta_{a_l,c_j}\Big) = 0.\end{aligned}$$

Thus $\mu_{(v,w)}(d, c) > a_l$. Since l was chosen to be maximal, $\mu_{(v,w)}(c) \notin \{a_1, \ldots, a_n\}$. □

By inspection of the proof of Lemma 5.9, the reader can see that $d \in K^n$ is only used to make sure that $0 \ll a_i$. If this statement is dropped from the conclusion, the above proof still gives the following lemma. Because we weaken the conclusion, we are also able to weaken the assumption of $\mathbb{Q}$-linear independence over K to just $\mathbb{Q}$-linear independence.

Lemma 5.10. *Let $c = (c_1, \ldots, c_n) \in C^n$ be $\mathbb{Q}$-linearly independent. Then there are $r_1, \ldots, r_n \in \mathbb{Q}^n$ and pairwise distinct $a_1, \ldots, a_n \in A$ such that for every $i \leq n$,*

- $q_{i,i} \neq 0$ *and* $q_{i,j} = 0$ *if* $j > i$,
- $\mu_{q_i}(c) = a_i$.

Lemma 5.9 is a crucial result and will play an important role later on. The reason is that it allows us to transform any given $\mathbb{Q}$-linearly independent tuple of elements of C via linear operations into a tuple of elements of C whose inverses are in different T-levels. Among other things, the resulting elements of C will not only be $\mathbb{Q}$-linearly independent, but also dcl-independent.

Proposition 5.11. *Let $c = (c_1, \ldots, c_n) \in C^n$ and let $f : M^n \to M$ be $\mathcal{L}$-$\emptyset$-definable. Then there are $r \in \mathbb{Q}^n$, $q \in \mathbb{Q}^m$ and $d \in K^m$ such that for every $a \in A$ with $0 \ll a$,*

$$f(c) \in \mathfrak{m}_a \setminus \{0\} \;\Rightarrow\; (r, q) \cdot (d, c) \in \mathfrak{m}_a.$$

Proof. We can directly reduce the proof to the case where $c_1, \ldots, c_n$ are $\mathbb{Q}$-linearly independent over K. By Lemma 5.9 there are $d \in K^n$, $q_1, \ldots, q_n, r_1, \ldots, r_n \in \mathbb{Q}^n$ and pairwise distinct $a_1, \ldots, a_n \in A$ such that for $i = 1, \ldots, n$, $q_{i,i} \neq 0$, $q_{i,j} = 0$ for $j > i$ and $0 \ll \mu_{r_i,q_i}(d, c) = a_i$. By Lemma 5.6 we get $[|(r_i, q_i) \cdot (d, c)|^{-1}] = [p_A(a_i)]$. For $i = 1, \ldots, n$, set $v_i := |(r_i, q_i) \cdot (d, c)|^{-1}$. Since $a_1, \ldots, a_n$ are pairwise distinct and $0 \ll a_i$, we get $[v_i] \cap [v_j] = \emptyset$ for $i \neq j$ by Corollary 5.4. Since $q_{i,i} \neq 0$ for each i, and $d \in K^n$, there is an $\mathcal{L}$-$\emptyset$-definable function $g : M^n \to M$ such that $g(v_1, \ldots, v_n) = f(c)^{-1}$. Let $a \in A$ be such that $0 \ll a$ and suppose that $f(c) \in \mathfrak{m}_a$. If $[v_i] \geq [a]$ for some i, then $(r_i, q_i) \cdot (d, c) \in \mathfrak{m}_a$. Therefore the conclusion holds with $(r, q) = (r_i, q_i)$. It remains to consider the case where $[v_i] < [a]$ for all $i \in \{1, \ldots, n\}$. Since $[v_i] \cap [v_j] = \emptyset$ for $i \neq j$, using Fact 3.8 we conclude that $[g(v_1, \ldots, v_n)] < [a]$. Hence $[f(c)^{-1}] < [a]$. This contradicts $f(c) \in \mathfrak{m}_a$. ◻

Corollary 5.12. *Let $c = (c_1, \ldots, c_{n-1}) \in C^{n-1}$ and let $f : M^{n-1} \to M$ be $\mathcal{L}$-$\emptyset$-definable with $f(c) \in C$. Then there are $r \in \mathbb{Q}^m$, $q \in \mathbb{Q}^{n-1}$ and $d \in K^m$ such that*

$$f(c) = (r, q) \cdot (d, c).$$

Proof. Suppose not. We can easily reduce the proof to the case where $c_1, \ldots, c_{n-1}, f(c)$ are $\mathbb{Q}$-linearly independent over K. By Lemma 5.9 there are $d \in K^n$, $q_1, \ldots, q_n$, $r_1, \ldots, r_n \in \mathbb{Q}^n$ and pairwise distinct $a_1, \ldots, a_n \in A$ such that for $i = 1, \ldots, n$, $q_{i,i} \neq 0$, $q_{i,j} = 0$ for $j > i$ and $0 \ll \mu_{(r_i,q_i)}(d, c, f(c)) = a_i$. By Lemma 5.6 we find that $[|(r_i, q_i) \cdot (d, c, f(c))|^{-1}] = [p_A(a_i)]$. For $i = 1, \ldots, n$, set $v_i := |(r_i, q_i) \cdot (d, c, f(c))|^{-1}$. Since $a_1, \ldots, a_n$ are pairwise distinct and $0 \ll a_i$, we get $[v_i] \cap [v_j] = \emptyset$ for $i \neq j$ by Corollary 5.4. Since $d \in K^n$, there is an $\mathcal{L}$-$\emptyset$-definable function $g : M^n \to M$ such that $g(v_1, \ldots, v_{n-1}) = v_n$. This contradicts Fact 3.8. ◻

Understanding ν

We now turn our attention to ν. By Lemma 4.17, if $x \in [0, 1] \setminus C$ and $a \in A$ are such that

$$x \in \left[e(p_A(a), \nu(x)) + a^{-1}, e(p_A(a), \nu(x)) + p_A(a)^{-1} - a^{-1}\right),$$

then $\nu(x) = e(p_A(a), \nu(x)) + a^{-1}$. We will show that for every $x \in [0, 1] \setminus C$ there is such an $a \in A$, and it is $\mathcal{L}_B$-definable from $\nu(x)$.

We first establish the following lemma.

Lemma 5.13. *Let $x \in (0, 1) \setminus C$. Then there is $a \in A$ such that $\nu(x) = e(a, \nu(x)) + a^{-1}$.*

Proof. Let $a = \lambda((x - \nu(x))^{-1})$. Because $\nu(x) \neq x$, we see that $a \neq 0$. Indeed, a is the unique element in A such that $s_A(a)^{-1} < x - \nu(x) < a^{-1}$. By Axiom T7,

$$x < \nu(x) + a^{-1} \leq e(a, \nu(x)) + a^{-1} + a^{-1} = e(a, \nu(x)) + 2a^{-1}.$$

First consider the case $x < e(a, \nu(x)) + a^{-1}$. Note that

$$e(a, \nu(x)) + s_A(a)^{-1} \leq \nu(x) + s_A(a)^{-1} < x.$$

We will now show that $x < e(a, \nu(x)) + a^{-1} - s_A(a)^{-1}$. Suppose not. Then we have $x \geq e(a, \nu(x)) + a^{-1} - s_A(a)^{-1}$. Since $e(a, \nu(x)) + a^{-1} - s_A(a)^{-1} \in C$, we get $\nu(x) \geq e(a, \nu(x)) + a^{-1} - s_A(a)^{-1}$. Then

$$\begin{aligned} x - \nu(x) &\leq x - (e(a, \nu(x)) + a^{-1} - s_A(a)^{-1}) \\ &< e(a, \nu(x)) + a^{-1} - (e(a, \nu(x)) + a^{-1} - s_A(a)^{-1}) = s_A(a)^{-1}. \end{aligned}$$

This contradicts our choice of a. Hence

$$x \in \big(e(a, \nu(x)) + s_A(a^{-1}), e(a, \nu(x)) + a^{-1} - s_A(a)^{-1}\big).$$

Thus $\nu(x) = e(a, \nu(x)) + s_A(a^{-1})$ by Lemma 4.17. Since $e(s_A(a), e(a, \nu(x))) = e(a, \nu(x))$ by the definition of e, and $0 < \nu(x) - e(a, \nu(x)) \leq s_A(a)^{-1}$, we see from Axiom T8 that $e(s_A(a), \nu(x)) = e(s_A(a), e(a, \nu(x))) = e(a, \nu(x))$. So $\nu(x) = e(s_A(a), \nu(x)) + s_A(a^{-1})$.

Now suppose $x \geq e(a, \nu(x)) + a^{-1}$. Then $x \in (e(a, \nu(x)) + a^{-1}, e(a, \nu(x)) + 2a^{-1})$. By Corollary 4.18, $\nu(x) = e(a, \nu(x)) + a^{-1}$. □

We immediately get the following corollary, which implies that the interpretation of ν in $(\mathcal{R}, K)$ is Borel.

Corollary 5.14. *Let $x \in [0, 1]$ and $c \in C$. Then $\nu(x) = c$ iff either*

- *$x \in C$ and $c = x$, or*
- *there exist $a \in A$ and $d \in C$ with $d = e(a, d)$ and $c = d + a^{-1}$.*

Corollary 5.15. *Let $x \in [0, 1] \setminus C$ and $d \in A$ be such that d is maximal in A with $\neg E(d, \nu(x))$. Then $\nu(x) = e(p_A(d), \nu(x)) + d^{-1}$.*

Proof. By Lemma 5.13 there is $a \in A$ such that $\nu(x) = e(a, \nu(x)) + a^{-1}$. By Lemma 4.7 we see that $E(b, \nu(x))$ for all $b \in A$ with $b > a$. Hence there is a maximal $d \in A$ such that $\neg E(d, \nu(x))$. Then by Lemma 4.7, $\nu(x) = e(p_A(d), \nu(x)) + d^{-1}$. □

We now show that we can express the image of a $\mathbb{Q}$-linear combination of elements of C under ν as a $\mathbb{Q}$-linear combination of images of the elements under e.

Lemma 5.16. *Let $c = (c_1, \dots, c_n) \in C^n$, $q = (q_1, \dots, q_n) \in \mathbb{Q}^n$ with $0 < q \cdot c < 1$, and let a be the minimal element in A such that $\sum_{i=1}^n q_i \delta_{s(a),c_i} \notin \{0, 1\}$. If $0 \ll a$, then*

$$\nu(q \cdot c) = \begin{cases} s(a)^{-1} + q \cdot e(a, c) & \text{if } 0 < \sum_{i=1}^n q_i \delta_{s(a),c_i} < 1, \\ a^{-1} + q \cdot e(a, c) & \text{if } 1 < \sum_{i=1}^n q_i \delta_{s(a),c_i}, \\ b^{-1} + q \cdot e(b, c) & \text{otherwise,} \end{cases}$$

where b is the largest element in A with $b \le a$ such that $\sum_{i=1}^n q_i \delta_{b,c_i} = 1$.

Proof. Let $d \in C$ be such that for all $a' \in A$, the following statement is true: if $a' \le a$, then $E(a', d)$ iff $\sum_{i=1}^n q_i \delta_{a,c_i} = 1$, and if $a' > a$, then $\neg E(a', d)$. By Axiom T5 such a d exists. Hence $\delta_{a',d} = \sum_{i=1}^n q_i \delta_{a',c_i}$ for all $a' \le a$, and thus $\mu_{(q,-1)}(c, d) = s_A(a)$ by our choice of a. Since $e(a, d) = d$, we get $d = q \cdot e(a, c)$ by Axiom T11. By Corollary 4.14,

$$q \cdot c - d = \sum_{i=1}^n q_i \delta_{s_A(a),c_i} (a^{-1} - s_A(a)^{-1}) + q \cdot (c - e(s_A(a), c)).$$

Set $u := \sum_{i=1}^n q_i \delta_{s_A(a),c_i}$. By Axiom T7, $c_i - e(s_A(a), c_i) < s_A(a)^{-1}$ for each $i = 1, \dots, n$.

First consider the case $0 < u < 1$. By Corollary 5.5 there are $u_1, u_2 \in \mathbb{Q}$ with $0 < u_1 < u_2 < 1$ such that $u_1 a^{-1} < q \cdot c - d < u_2 a^{-1}$. Since $s_A(a) > ra$ for every $r \in \mathbb{Q}_{>0}$, we get

$$d < d + s_A(a)^{-1} < d + u_1 a^{-1} < q \cdot c < d + u_2 a^{-1} < d + a^{-1} - s_A(a)^{-1}.$$

By Lemma 4.17 we get $\nu(q \cdot c) = d + s_A(a)^{-1}$.

Now suppose that $u > 1$. By Corollary 5.5 there are $u_1, u_2 \in \mathbb{Q}$ with $1 < u_1 < u_2$ such that $u_1 a^{-1} < q \cdot c - d < u_2 a^{-1}$. Suppose $E(a', d)$ holds for all $a' \le a$. Then $d = 1 - a^{-1}$ and $q \cdot c > 1$. Hence we can assume there is $a' \in A$ maximal such that $a' \le a$ and $\neg E(a', d)$. Then $e(a, d) + a^{-1} = e(a', d) + a'^{-1}$ by Axiom T10. Since $\neg E(a', d)$, we have $e(p_A(a'), d) = e(a', d)$. Thus

$$\begin{aligned} e(p_A(a'), d) + a'^{-1} &= d + a^{-1} < d + u_1 a^{-1} < q \cdot c \\ &< d + u_2 a^{-1} < e(p_A(a'), d) + p_A(a')^{-1} - a'^{-1}. \end{aligned}$$

Hence by Lemma 4.17, $\nu(q \cdot c) = d + a^{-1}$.

Suppose finally that $u < 0$. There are $u_1, u_2 \in \mathbb{Q}$ with $u_1 < u_2 < 0$ such that $u_1 a^{-1} < q \cdot c - d < u_2 a^{-1}$. Let $b \in A$ be the largest element in A with $b \le a$ such that $E(b, d)$. Because $q \cdot c > 0$, such a b exists. By Lemma 4.10 and Corollary 4.11, $d = e(b, d) = e(p_A(b), d) + p_A(b)^{-1} - b^{-1}$. By Axiom T4, $b^{-1} < p_A(b)^{-1} - b^{-1} + u_1 a^{-1}$. Therefore

$$\begin{aligned} e(p_A(b), d) + b^{-1} &< e(p_A(b), d) + p_A(b)^{-1} - b^{-1} + u_1 a^{-1} < d + u_1 a^{-1} \\ &< q \cdot c < d + u_2 a^{-1} < d = e(p_A(b), d) + p_A(b)^{-1} - b^{-1}. \end{aligned}$$

Hence $\nu(q \cdot c) = d + b^{-1}$ by Lemma 4.17. □

Closed elements in images under $\mathcal{L}$-definable functions

Before finishing this section, we need to mention two further classes of $\mathcal{L}_C$-definable functions. While ν maps an element z of $(0, 1)$ to the left endpoint of the complementary interval of C in whose closure z lies, we also have to understand functions that map z to left and right endpoints of complementary intervals of the closure of the image of C^n under an $\mathcal{L}$-definable function. Here Axiom T12 is the key.

Definition 5.17. Let $f : X \subseteq M^{l+n} \to [0, 1]$ be $\mathcal{L}$-$\emptyset$-definable and continuous. Let $g_1, \dots, g_k : M^{l+n} \to M$ be as in Axiom T12. Define $\nu_f : M^{l+1} \to C^n$ to map $(x, y) \in M^{l+1}$ to the lexicographically minimal $c \in C^n \cap \mathrm{cl}(X_x)$ such that there is $i \in \{1, \dots, k\}$ with

$$g_i(x, c) = \sup_{d \in C^n \cap X_x,\ f(x,d) \leq y} f(x, d)$$

when $C^n \cap X_x \neq \emptyset$, and to 0 otherwise. Let $\tau_f : M^{l+1} \to C^n$ map $(x, y) \in M^{l+1}$ to the lexicographically minimal $c \in C^n \cap \mathrm{cl}(X_x)$ such that there is $i \in \{1, \dots, k\}$ with

$$g_i(x, c) = \inf_{d \in C^n \cap X_x,\ f(x,d) \geq y} f(x, d)$$

when $C^n \cap X_x \neq \emptyset$, and to 0 otherwise.

The existence of the lexicographically minimal elements of C^n in Definition 5.17 follows immediately from Axiom T12 when $x \in \pi(X)$ and y is bounded above and below by an element of $f(x, C^n \cap X_x)$.

One can easily deduce the following lemma from Definition 5.17.

Lemma 5.18. *Let $f : X \subseteq M^{l+n} \to [0, 1]$ be $\mathcal{L}$-$\emptyset$-definable and continuous, and let $x \in \pi(X)$ and $y \in M$. Let $g_1, \dots, g_k : M^{l+n} \to M$ be as in Axiom* T12. *If there are $c, d \in C^n \cap X_x$ such that $f(x, c) < y < f(x, d)$, then there are $i, j \in \{1, \dots, k\}$ with*

$$\big(g_i(x, \nu_f(x, y)), g_j(x, \tau_f(x, y))\big) \cap f(C^n \cap X_x) = \emptyset.$$

Hence, for every y in the convex closure of $f(x, C^n \cap X_x)$ but not in $f(x, C^n \cap X_x)$, there are i, j such that $g_i(x, \nu_f(x, y))$ and $g_j(x, \tau_f(x, y))$ are the endpoints of the complementary interval of $f(x, C^n \cap X_x)$ whose closure contains y. Thus $\nu_f(x, -)$ and $\tau_f(x, -)$ are constant on complementary intervals of $f(x, C^n \cap X_x)$.

Corollary 5.19. *Let $f : X \subseteq M^{l+n} \to [0, 1]$ be $\mathcal{L}$-$\emptyset$-definable and continuous, let $x \in \pi(X)$, and let $g_1, \dots, g_k : M^{l+n} \to M$ be as in Axiom* T12. *Then*

(i) *$\nu_f(x, -)$ and $\tau_f(x, -)$ are continuous on $M \setminus \mathrm{cl}(f(x, X_x \cap K^n))$,*
(ii) *if $y \in \mathrm{cl}(f(x, X_x \cap K^n))$, then $y = g_i(x, \nu_f(x, y))$ or $y = g_i(x, \tau_f(x, y))$ for some i.*

6. Büchi expressibility

Recall that the language of $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \in, s_{\mathbb{N}})$ was denoted by $\mathcal{L}_B$. Let $\mathcal{M} \models \tilde{T}$. We will write $\mathcal{B}(M)$ for the $\mathcal{L}_B$-structure (A, C, E, s_A). In this section, we will study this structure in detail. In particular, we will show that certain algebraic conditions on elements of C are equivalent to statements expressible in $\mathcal{B}(M)$.

Lemma 6.1. *Let $a \in A$ and $c \in C$. Then*

(i) *a^{-1} is $\mathcal{L}_B$-definable from a,*
(ii) *$c - e(a, c)$ is $\mathcal{L}_B$-definable from a and c.*

Proof. For (i), by Corollary 4.8, a^{-1} is the unique element d in C such that $E(b, d)$ holds iff $b > a$ for all b. Since this property is $\mathcal{L}_B$-definable over a, a^{-1} is $\mathcal{L}_B$-definable from a. For (ii), note that by Axiom T8, $c - e(a, c)$ is the unique element $d \in C$ with $S(d) = S(c)_{>a}$. Because this property is $\mathcal{L}_B$-definable over a and c, it follows that $c - e(a, c)$ is $\mathcal{L}_B$-definable from a and c. □

Proposition 6.2. *Let $q = (q_1, \ldots, q_n) \in \mathbb{Q}^n$. There are $\mathcal{L}_B$-formulas $\chi_1(x, y)$, $\chi_2(x, y)$, $\varphi(x, y)$, $\psi(x, y)$, $\theta(x)$, $\omega(x)$ such that for every $a \in A$ and every $c = (c_1, \ldots, c_n) \in C^n$,*

(i) *$\sum_{i=1}^{n} q_i \delta_{a,c_i} = 0$ iff $\mathcal{B}(M) \models \chi_1(c, a)$,*
(ii) *$\sum_{i=1}^{n} q_i \delta_{a,c_i} > 0$ iff $\mathcal{B}(M) \models \chi_2(c, a)$,*
(iii) *$\mu_q(c) \geq a$ iff $\mathcal{B}(M) \models \varphi(c, a)$,*
(iv) *$\mu_q(c) = a$ iff $\mathcal{B}(M) \models \psi(c, a)$,*
(v) *$q \cdot c = 0$ iff $\mathcal{B}(M) \models \theta(c)$,*
(vi) *$q \cdot c \in C$ iff $\mathcal{B}(M) \models \omega(c)$,*

Proof. It is easy to see that there are $\mathcal{L}_B$-formulas $\chi_1(x, y)$, $\chi_2(x, y)$ satisfying (i) and (ii). For (iii), let $\varphi(x, y)$ be the formula $\forall z \in A\ (z < y) \rightarrow \chi_1(x, z)$. It follows immediately from the definition of μ that (iii) holds with this choice of φ. For (iv), let $\psi(x, y)$ be $\neg\chi_1(x, y) \wedge \varphi(x, y)$. Again it follows immediately from the definition of μ that (iv) holds for this ψ. For (v), let $\theta(x)$ be the formula $\forall y \in A\ \chi_1(x, y)$. First, suppose that $\mathcal{B}(M) \models \theta(c)$. By the definitions of μ and χ_1 we have $\mu_q(c) = 0$. By Axiom T11, $q \cdot c = 0$. Conversely, suppose that $q \cdot c = 0$. Then by Axiom T11, $\mu_q(c) = 0$. Thus $\mathcal{B}(M) \models \theta(c)$. For (vi), note that by (v) there is an $\mathcal{L}_B$-formula $\theta'(x, y)$ such that $\mathcal{B}(M) \models \theta'(c, d)$ iff $d = q \cdot c$. Now set $\omega(x)$ to be the $\mathcal{L}_B$-formula $\exists y \in C\ \theta'(x, y)$. Statement (vi) follows. □

The $\mathcal{L}_B$-formulas in Proposition 6.2 dependent on the given tuple q.

7. Towards quantifier elimination

In this section the first steps toward a quantifier elimination statement for $\tilde{T}$ are made. We will not show that the theory $\tilde{T}$ has quantifier elimination in the language $\mathcal{L}_C$. Rather, we extend this language and theory by definitions to a language $\mathcal{L}_C^+$ and a theory $\tilde{T}^+$ and show quantifier elimination for this expansion.

For this section, we fix a model $\mathcal{M} = (M, C, A, E) \models \tilde{T}$. In the following we will consider substructures of M. Whenever X is an $\mathcal{L}_C$-substructure of $\mathcal{M}$, we denote by $C(X)$ and $A(X)$ the interpretations of the symbols for C and A in X. Since X is a substructure, $C(X) = X \cap C$ and $A(X) = X \cap A$. Whenever $A(X)$ is closed under s_A, we will write $\mathcal{B}(X)$ for the two-sorted structure $(C(X), A(X), E|_{A(X)\times C(X)}, s_{A(X)})$. We write that $a \in A$ is $\mathcal{L}_B$-definable from X whenever a is $\mathcal{L}_B$-definable from $A(X) \cup C(X)$ in $\mathcal{B}(M)$.

Languages and theories

We will now introduce the expansions of $\mathcal{L}_C$ and $\tilde{T}$. Since (A, C, E, s_A) is definable in $\mathcal{M}$, it is easy to see that for every $\mathcal{L}_B$-formula $\varphi(x_1, \ldots, x_m, y_1, \ldots, y_n)$, where the variables $x_1, \ldots, x_m$ are of the first sort and the variables $y_1, \ldots, y_n$ are of the second sort, there is an $\mathcal{L}_C$-formula φ^C such that

$$\mathcal{M} \models \varphi^C(a, c) \quad \text{iff} \quad a \in A^m, c \in C^n \text{ and } \mathcal{B}(M) \models \varphi(a, c).$$

Whenever $X, Y \subseteq M$, we simply write $\text{tp}_{\mathcal{L}_B}(Y|X)$ for $\text{tp}_{\mathcal{L}_B}(\mathcal{B}(Y)|\mathcal{B}(X))$.

Let $\mathcal{L}_C^B$ be the language $\mathcal{L}_C$ augmented by n-ary predicate symbols P_φ for each $\mathcal{L}_B$-formula φ in n free variables. Let $\tilde{T}^B$ be the $\mathcal{L}_C^B$-theory extending $\tilde{T}$ by axioms

$$\forall x \; (P_\varphi(x) \leftrightarrow \varphi^C(x))$$

for every $\mathcal{L}_B$-formula φ.

Let $\mathcal{L}_C^*$ be the language $\mathcal{L}_C^B$ augmented by function symbols for λ, ν, e and for every $\mathcal{L}_B$-$\emptyset$-definable $g : A^m \times C^n \to A$. Let $\mathcal{L}_C^+$ be the language $\mathcal{L}_C^*$ augmented by function symbols for ν_f and τ_f for every $\mathcal{L}$-$\emptyset$-definable function f that satisfies the assumption of Definition 5.17. For each of the function symbols f added, let $\varphi_f(x, y)$ be the $\mathcal{L}_C$-formula defining the $\mathcal{L}_C$-definable function corresponding to f. Let $\tilde{T}^+$ be the $\mathcal{L}_C^*$-theory extending $\tilde{T}^B$ by axioms

$$\forall x \; (\varphi_f(x, y) \leftrightarrow f(x) = y)$$

for each new function symbol in $\mathcal{L}_C^+$.

Note that every model of $\tilde{T}$ naturally extends to a model of $\tilde{T}^+$. From now on we will regard each $\tilde{T}$-model also as a $\tilde{T}^+$-model whenever needed. If $X \subseteq M$ is an $\mathcal{L}_C^*$-substructure of M, we write $X \trianglelefteq^* M$. If $X \subseteq M$ is even an $\mathcal{L}_C^+$-substructure of M, we write $X \trianglelefteq^+ M$. Our main quantifier elimination results can now be stated as follows.

Theorem 7.1. *$\tilde{T}^+$ has quantifier elimination.*

$\mathcal{L}_C^+$-substructures

Towards proving Theorem 7.1, substructures of $\mathcal{M}$ in the extended languages $\mathcal{L}_C^*$ and $\mathcal{L}_C^+$ will be studied in this subsection. We are in particular interested in the question how to extend an $\mathcal{L}_C^+$-substructure to an $\mathcal{L}_C^*$-substructure that contains a given subset of C. Before making this statement precise, we will prove several lemmas. We recall that dcl denotes the definable closure operator in the o-minimal reduct M.

Lemma 7.2. *Let $X \trianglelefteq^+ \mathcal{M}$. Then there is $Z \subseteq X$ such that Z is* dcl-*independent over C and $X = \text{dcl}(Z \cup C(X))$.*

Proof. Let $Z \subseteq X$ be maximal such that Z is dcl-independent over C. It suffices to show that $X = \text{dcl}(Z \cup C(X))$. Let $x \in X$. Without loss of generality, we can assume that $x \in [0, 1]$. By maximality of Z, there are $z \in Z^m$ and $c \in C^n$ such that $x \in \text{dcl}(z, c)$.

By o-minimality of T we can assume that there is an $\emptyset$-definable open cell U and an $\mathcal{L}$-$\emptyset$-definable continuous function $f : U \to [0, 1]$ such that $(z, c) \in U$ and $f(z, c) = x$. Let $g_1, \dots, g_k$ be as in Definition 5.17. Since $f(z, c) = x$, there is $i \in \{1, \dots, k\}$ such that $g_i(z, \tau_f(z, x)) = x$. Because $x, z \in X$ and $X \trianglelefteq^+ \mathcal{M}$ we have $\tau_f(z, x) \in C(X)^n$. Therefore $x \in \operatorname{dcl}(Z \cup C(X))$. □

Lemma 7.3. *Let $X \trianglelefteq^+ M$, $z \in (X \cap [0, 1])^m$, $y \in X$ and $a \in A$ be such that $0 \ll a$ and $\overline{y}^a$ is* dcl-*dependent over $\overline{z}^a$ and $\overline{C}^a$. Then $\overline{y}^a$ is* dcl-*dependent over $\overline{z}^a$ and $\overline{C(X)}^a$.*

Proof. We can assume $y \in [0, 1]$. By Lemma 3.12 there is an open $\mathcal{L}$-$\emptyset$-cell U, a continuous $\mathcal{L}$-$\emptyset$-definable function $f : U \to M$ and $c \in C^n$ such that $[|f(z, c) - y|^{-1}] \geq [a]$. Let $g_1, \dots, g_k$ be as in Definition 5.17. Then there is $i \in \{1, \dots, k\}$ such that $[|g_i(z, \nu_f(z, y)) - y|^{-1}] \geq [a]$ or $[|g_i(z, \tau_f(z, y)) - y|^{-1}] \geq [a]$. Since $X \trianglelefteq^+ M$, we have $\nu_g(x, y), \tau_g(x, y) \in C(X)^n$. It can easily be deduced from Lemma 3.12 that $\overline{y}^a$ is dcl-dependent over $\overline{z}^a$ and $\overline{C(X)}^a$. □

Lemma 7.4. *Let $X \trianglelefteq^+ \mathcal{M}$, and let $c \in C^n$ and $a \in A$ be such that $0 \ll a$ and $\overline{c}^a$ is* dcl-*dependent over $\overline{X}^a$. Then $\overline{c}^a$ is* dcl-*dependent over $\overline{C(X)}^a$.*

Proof. Take $z \in (X \cap [0, 1])^l$ and $d \in C(X)^m$ such that $\overline{c}^a$ is dcl-dependent over $\overline{(z, d)}^a$ and $\overline{z}^a$ is dcl-independent over $\overline{C(X)}^a$. By Lemma 7.3, $\overline{z}^a$ is dcl-independent over $\overline{C}^a$. Since $(d, c) \in C^{m+n}$ and $\overline{c}^a$ is dcl-dependent over $\overline{(z, d)}^a$, we conclude that $\overline{c}^a$ is dcl-dependent over $\overline{d}^a$. □

Lemma 7.5. *Let $X \trianglelefteq^+ \mathcal{M}$, $c \in C^n$, and let $f : M^n \to M$ be $\mathcal{L}$-X-definable such that $f(c) \in C$. Then there are $x \in C(X)^m$, $p \in \mathbb{Q}^m$ and $q \in \mathbb{Q}^n$ such that*

$$f(c) = (p, q) \cdot (x, c).$$

Proof. Because $(c, f(c)) \in C^{n+1}$, it follows from Lemma 7.2 that there are $x \in C(X)^m$ and an $\mathcal{L}$-$\emptyset$-definable function $g : M^{m+n} \to M$ such that $g(x, c) = f(c)$. By Corollary 5.12 we can extend x by elements of K such that there are $p \in \mathbb{Q}^m$ and $q \in \mathbb{Q}^n$ with $f(c) = (p, q) \cdot (x, c)$. □

Hence any $\mathcal{L}$-dependence among elements of C over an $\mathcal{L}_C^+$-substructure is just a $\mathbb{Q}$-linear dependence. We immediately get the following corollary.

Corollary 7.6. *Let $X \trianglelefteq^+ \mathcal{M}$, $D \subseteq C$ and $Y := \operatorname{dcl}(X \cup D)$. Then every element in $C(Y)$ is a $\mathbb{Q}$-linear combination of elements in $C(X) \cup D$.*

Lemma 7.7. *Let $X \trianglelefteq^+ \mathcal{M}$, let $c \in C^n$, $d \in C$, $a \in A$ with $0 \ll a$, and let $f : M^n \to M$ be $\mathcal{L}$-X-definable such that $f(c) - d \in \mathfrak{m}_a$. Then there are $x \in C(X)^m$, $p \in \mathbb{Q}^m$ and $q \in \mathbb{Q}^n$ such that $e(a, d) = (p, q) \cdot (e(a, x), e(a, c))$.*

Proof. By Lemmas 7.4 and 3.12 there are $x \in C(X)^m$ and an $\mathcal{L}$-$\emptyset$-definable function $g : M^{m+n+1} \to M$ such that $g(x, c, d) \in \mathfrak{m}_a$. Then by Proposition 5.11 we can extend x by elements from K such that there are $p \in \mathbb{Q}^m$ and $q \in \mathbb{Q}^n$ with $d - (p \cdot x + q \cdot c) \in \mathfrak{m}_a$. By Corollary 5.7, $e(a, d) = (p, q) \cdot (e(a, x), e(a, c))$. □

Extensions by elements of C

In the following we will need to consider a special kind of $\mathcal{L}_C^*$-substructures of $\mathcal{M}$, given by extending $\mathcal{L}_C^+$-substructures by elements of C.

Definition 7.8. A subset $X \subseteq M$ is a *special $\mathcal{L}_C^*$-substructure* if there are $D \subseteq C$ and $Z \subseteq M$ such that $\operatorname{dcl}(Z \cup D) \trianglelefteq^* M$ and either Z is $\emptyset$ or $Z \trianglelefteq^+ M$. In this case, we write $X \trianglelefteq M$.

While Lemmas 7.5 and 7.7 do not generalize to arbitrary $\mathcal{L}_C^*$-substructures, both statements hold for special substructures, as can easily be checked.

Lemma 7.9. *Let $X \trianglelefteq \mathcal{M}$, $c \in C^n$, and let $f : M^n \to M$ be $\mathcal{L}$-X-definable such that $f(c) \in C$. Then there are $x \in C(X)^m$, $p \in \mathbb{Q}^m$ and $q \in \mathbb{Q}^n$ such that*

$$f(c) = (p, q) \cdot (x, c).$$

Lemma 7.10. *Let $X \trianglelefteq \mathcal{M}$, $c \in C^n$, $d \in C$, $a \in A$ with $0 \ll a$, and let $f : M^n \to M$ be $\mathcal{L}$-X-definable such that $f(c) - d \in \mathfrak{m}_a$. Then there are $x \in C(X)^m$, $p \in \mathbb{Q}^m$ and $q \in \mathbb{Q}^n$ such that $e(a, d) = (p, q) \cdot (e(a, x), e(a, c))$.*

Definition 7.11. A set $Z \subseteq M$ is *A-closed* if Z is closed under λ and contains all $a \in A$ that are $\mathcal{L}_B$-definable from Z.

We are now ready to state the main result of this subsection.

Proposition 7.12. *Let $X \trianglelefteq \mathcal{M}$, $Y \subseteq \mathcal{M}$ and $D \subseteq C$ be such that*

(i) *$Y = \operatorname{dcl}(X \cup D)$,*
(ii) *D is closed under $e(a, -)$ for each $a \in A(Y)$,*
(iii) *Y is A-closed.*

Then $Y \trianglelefteq \mathcal{M}$.

Proof. Because $D \subseteq C$, it suffices to show that Y is closed under ν and $e(a, -)$ for every $a \in A(Y)$. We start by showing the latter statement. Let $y \in C(Y)$ and $a \in A(Y)$. By (i) there is an $\mathcal{L}$-X-definable function $f : M^n \to M$ such that $y = f(c)$ for some $c = (c_1, \ldots, c_n) \in D^n$. By Lemma 7.9 there are $x \in C(X)^m$, $p \in \mathbb{Q}^m$ and $q \in \mathbb{Q}^n$ such that $y = (p, q) \cdot (x, c)$. By Lemma 4.15, $e(a, y) = (p, q) \cdot (e(a, x), e(a, c))$. By (ii), $e(a, c) \in D^n$. By (i), $e(a, y) \in Y$.

We will now check that Y is closed under ν. Let $y \in Y \cap (0, 1)$. We immediately reduce the argument to the case $\nu(y) \neq y$. By (i) there is an $\mathcal{L}$-X-definable function $f : M^n \to M$ such that $y = f(c)$ for some $c \in D^n$. By Corollary 5.15 there is $b \in A$ maximal such that $\neg E(b, \nu(f(c)))$ and $\nu(f(c)) = e(p_A(b), \nu(f(c))) + b^{-1}$. We can assume that $0 \ll b$. Hence

$$\begin{aligned} e(p_A(b), \nu(f(c))) &< e(p_A(b), \nu(f(c))) + b^{-1} = \nu(f(c)) < f(c) \\ &< e(p_A(b), \nu(f(c))) + p_A(b)^{-1} - b^{-1} < e(p_A(b), \nu(f(c))) + p_A(b)^{-1}. \end{aligned}$$

We first observe that $\nu(f(c)) + p_A(b)^{-1} - 2b^{-1} = e(p_A(b), \nu(f(c))) + p_A(b)^{-1} - b^{-1}$. Set

$$a_1 := \lambda(f(c) - \nu(f(c))^{-1}),$$
$$a_2 := \lambda\big((e(p_A(b), \nu(f(c))) + p_A(b)^{-1} - b^{-1} - f(c))^{-1}\big).$$

Note that $a_1, a_2 \geq p_A(b)$. Set $a := \max\{a_1, a_2\}$. Thus either

$$\nu(f(c)) - f(c) \in \mathfrak{m}_a \quad \text{or} \quad \nu(f(c)) + p_A(b)^{-1} - 2b^{-1} - f(c) \in \mathfrak{m}_a.$$

By Lemma 7.10 there are $x \in C(X)^m$, $p \in \mathbb{Q}^m$ and $q \in \mathbb{Q}^n$ such that either

$$\nu(f(c)) - (p, q) \cdot (x, c) \in \mathfrak{m}_a \quad \text{or} \quad \nu(f(c)) + p_A(b)^{-1} - 2b^{-1} - (p, q) \cdot (x, c) \in \mathfrak{m}_a.$$

Set $z := (p, q)(x, c)$. By Lemma 5.16 and assumption (iii), we have $\nu(z) \in Y$.

We will now consider several different cases. In each case we will conclude that $\nu(f(c)) \in Y$. The arguments that follow are not complicated, but the details are tiresome. Because the arguments itself are not crucial for the rest of the paper, the reader might prefer to skip them or just read the first two cases which already contain the main ideas.

First, consider the case $e(p_A(b), \nu(f(c))) + b^{-1} \leq z < e(p_A(b), \nu(f(c))) + p_A(b)^{-1} - b^{-1}$. By Lemma 4.17, $\nu(z) = e(p_A(b), \nu(f(c))) + b^{-1}$. Hence $\nu(f(c)) = \nu(z) \in Y$.

Now suppose that $z < e(p_A(b), \nu(f(c)))$. Let $d \in A$ be maximal such that $d \leq p_A(b)$ and $E(d, \nu(f(c)))$. By Lemma 4.19,

$$C \cap \big(e(p_A(b), \nu(f(c))) - p_A(d)^{-1} + 2d^{-1}, e(p_A(b), \nu(f(c)))\big) = \emptyset. \tag{7.1}$$

Because $0 < e(p_A(b), \nu(f(c))) - z < \nu(f(c)) - z \in \mathfrak{m}_a$ and $a \geq p_A(b) > p_A(d)$, we get $\nu(z) = e(p_A(b), \nu(f(c))) - p_A(d)^{-1} + 2d^{-1}$. Since $z < e(p_A(b), \nu(f(c)))$, we have $\nu(f(c)) - z > b^{-1}$. Since $\nu(f(c)) - z \in \mathfrak{m}_{\mathfrak{a}}$, $f(c) - \nu(f(c)) < p_A(b)^{-1} + 2b^{-1}$ and $a \geq p_A(b)$, we deduce that $b^{-1} < f(c) - z < p_A(p_A(b))^{-1}$. Hence $\lambda((f(c) - z)^{-1})$ has to be either $p_A(b)$ or $p_A(p_A(b))$. Therefore $b \in Y$, since $\lambda((f(c) - z)^{-1}) \in Y$ and Y is closed under p_A and s_A. Let $d' \in A$ be the largest element in A such that $\neg E(d', \nu(z))$. Because $\nu(z) \in Y$ and Y is A-closed, $d' \in Y$. By Corollary 5.15, $\nu(z) = e(p_A(d'), \nu(z)) + d'^{-1}$ and $e(p_A(b), \nu(f(c))) = e(p_A(d'), \nu(z)) + p_A(d')^{-1} - d'^{-1}$ by (7.1). Thus $e(p_A(b), \nu(f(c))) \in Y$. Since $b \in Y$ and $\nu(f(c)) = e(p_A(b), \nu(f(c))) + b^{-1}$, we get $\nu(f(c)) \in Y$.

Consider the case $z > e(p_A(b), \nu(f(c))) + p_A(b)^{-1}$. Note that

$$z - \big(e(p_A(b), \nu(f(c))) + p_A(b)^{-1}\big) < z - (\nu(f(c)) + p_A(b)^{-1} - 2b^{-1}) \in \mathfrak{m}_a.$$

Since $a \geq p_A(b)$, Corollary 4.18 yields $\nu(z) = e(p_A(b), \nu(f(c))) + p_A(b)^{-1}$. Because $f(c) - (\nu(f(c)) + p_A(b)^{-1} - 2b^{-1}) \in \mathfrak{m}_a$, $z - (\nu(f(c)) + p_A(b)^{-1} - 2b^{-1}) \in \mathfrak{m}_a$ and $\nu(z) - (\nu(f(c)) + p_A(b)^{-1} - 2b^{-1}) = b^{-1}$, we again deduce that $b^{-1} \leq z - f(c) < p_A(p_A(b))^{-1}$. Hence $\lambda((z - f(c))^{-1})$ has to be either $p_A(b)$ or $p_A(p_A(b))$. Consequently, $p_A(b), b \in Y$ as argued above. Because $\nu(z) \in Y$ and $e(p_A(b), \nu(f(c))) + p_A(b)^{-1} \in Y$, we get $\nu(f(c)) \in Y$.

Now suppose that $z \in [e(p_A(b), \nu(f(c))), \nu(f(c)))$ and $a_1 = p_A(b)$. It follows immediately that $a = p_A(b)$ and $\nu(f(c)) - z \in \mathfrak{m}_{p_A(b)}$. Since $\lambda(f(c) - \nu(f(c)))^{-1} = p_A(b)$, $\lambda((f(c) - z)^{-1})$ is either $p_A(b)$ or $p_A(p_A(b))$. Thus b and $p_A(b)$ are in Y. Because $e(p_A(b), \nu(f(c))) \leq z < e(p_A(b), \nu(f(c))) + p_A(b)^{-1}$, we conclude that $e(p_A(b), \nu(f(c))) = e(p_A(b), \nu(z)) \in Y$. Hence $\nu(f(c)) \in Y$.

Suppose that $z \in [e(p_A(b), \nu(f(c))), \nu(f(c)))$ and $a_1 > p_A(b)$. Again it follows easily that $a = a_1$ and $\nu(f(c)) - z \in \mathfrak{m}_a$. Since $\lambda((f(c) - \nu(f(c)))^{-1}) = a$, we deduce that $s_A(a)^{-1} \leq f(c) - z < p_A(a)^{-1}$. Hence $\lambda((f(c) - z)^{-1})$ is either a or $p_A(a)$. Thus $a \in Y$. Because $\nu(f(c)) - z \in \mathfrak{m}_a$, we get $\nu(f(c)) - p_A(a)^{-1} < z < \nu(f(c))$. Since $\nu(f(c)) = e(p_A(b), \nu(f(c))) + b^{-1}$, we have $\nu(f(c)) - p_A(a)^{-1} = e(p_A(b), \nu(f(c)) + b^{-1} - p_A(a)^{-1}$. Because of $a \geq p_A(b)$ we deduce from Axiom T10 and Lemma 4.7 that

$$e(p_A(a), \nu(f(c))) = e(p_A(b), \nu(f(c))) + b^{-1} - p_A(a)^{-1} = \nu(f(c)) - p_A(a)^{-1}.$$

Since $\nu(f(c)) - p_A(a)^{-1} < z < \nu(f(c))$, we have $e(p_A(a), \nu(z)) = \nu(f(c)) - p_A(a)^{-1}$ by Axiom T8. From $p_A(a) \in Y$ we conclude that $e(p_A(a), \nu(z)) \in Y$. Hence $\nu(f(c)) \in Y$.

The case $z \in [e(p_A(b), \nu(f(c))) + p_A(b)^{-1} - b^{-1}, \nu(f(c)) + p_A(b)^{-1})$ can be handled similarly to the last two cases. We leave the details to the reader. □

Interaction between elements of A and substructures

We finish this section with two lemmas on the interplay of elements of A and $\mathcal{L}_C^*$-substructures. These results will be used in the next section.

Lemma 7.13. *Let $X \trianglelefteq \mathcal{M}$ and $a \in A$. If $[a] = [x]$ for some $x \in X$, then $a \in A(X)$.*

Proof. By Lemma 5.2 we can assume that $0 \ll a$. Let $x \in X$ be such that $[a_i] = [x]$. Since $X \trianglelefteq \mathcal{M}$, we get $a_i \in A(X)$ by Corollary 5.8. □

Lemma 7.14. *Let $X \trianglelefteq \mathcal{M}$. Let $c = (c_1, \dots, c_n) \in C(X)^n$, $(a_1, \dots, a_n) \in A$ and $p, q \in \mathbb{Q}^n$ be such that*

$$\mu_{(p,q)}\big(a_1^{-1}, \dots, a_n^{-1}, c_1 - e(a_1, c_1), \dots, c_n - e(a_n, c_n)\big) \in A(X).$$

Then $(a_1^{-1}, \dots, a_n^{-1}, c_1 - e(a_1, c_1), \dots, c_n - e(a_n, c_n))$ is $\mathbb{Q}$-linearly dependent over $C(X)$.

Proof. Set $b = \mu_{(p,q)}(a_1^{-1}, \dots, a_n^{-1}, c_1 - e(a_1, c_1), \dots, c_n - e(a_n, c_n))$. Because $b \in A(X)$ and $X \trianglelefteq \mathcal{M}$, we have $p_A(b) \in A(X)$ and $c_i - e(p_A(b), c_i) \in X$. By Corollary 4.8 we see that for all $a \in A$,

$$a^{-1} - e(p_A(b), a^{-1}) = \begin{cases} a^{-1} & \text{if } p_A(b) \leq a, \\ p_A(b)^{-1} & \text{otherwise.} \end{cases}$$

By Lemmas 4.13 and 4.9,

$$\begin{aligned}
&\sum_{i=1}^{n} p_i a_i^{-1} + \sum_{i=1}^{n} q_i(c_i - e(a_i, c_i)) \\
&\quad = \sum_{i=1}^{n} p_i(a_i^{-1} - e(p_A(b), a_i^{-1})) + \sum_{i=1}^{n} q_i\big(c_i - e(a_i, c_i) - e(p_A(b), c_i - e(a_i, c_i))\big) \\
&\quad = \sum_{p_A(b)\le a_i} p_i a_i^{-1} + \sum_{p_A(b)>a_i} p_i p_A(b)^{-1} + \sum_{i=1}^{n} q_i(c_i - e(p_A(b), c_i)),
\end{aligned}$$

and the statement follows. □

8. Quantifier elimination

In this section we prove Theorems 4.5 and 7.1. The actual proof will use several embedding lemmas that we will establish first. Let $\kappa = |\mathcal{L}_C^+|$ and let $\mathcal{M}, \mathcal{N} \models \tilde{T}^+$ be such that $|\mathcal{M}| \le \kappa$ and $\mathcal{N}$ is κ^+-saturated. Let $X \trianglelefteq \mathcal{M}$ and suppose that $\beta : X \to \mathcal{N}$ is an $\mathcal{L}_C^*$-embedding.

Types of elements of A

We first consider types of elements of $A(M)$ over X.

Lemma 8.1. *Let $a = (a_1, \dots, a_n) \in A(M)^n$ and $b = (b_1, \dots, b_n) \in A(N)^n$. Then*

$$\beta\, \mathrm{tp}_{\mathcal{L}_B}(a|X) = \mathrm{tp}_{\mathcal{L}_B}(b|\beta(X)) \;\Rightarrow\; \beta\, \mathrm{tp}_{\mathcal{L}}(a|X) = \mathrm{tp}_{\mathcal{L}}(b|\beta(X)).$$

Proof. By Lemma 5.2 we can assume that $0 \ll a_i$ and $0 \ll b_i$ for $i = 1, \dots, n$. By Corollary 5.4 and Lemma 3.9 it is enough show that the statement holds for $n = 1$. Let $a \in A(M)$ and $b \in A(N)$ be such that $\beta\, \mathrm{tp}_{\mathcal{L}_B}(a|X) = \mathrm{tp}_{\mathcal{L}_B}(b|\beta(X))$. We can immediately reduce the proof to the case where $a \notin X$ and $b \notin \beta(X)$. It suffices to show that b lies in the image of the cut of a over X under β. Suppose there are $x, y \in X$ such that $x < a < y$. Because X is closed under λ, p_A and s_A, we have $s_A(\lambda(x)) \in X$ and $\lambda(y) \in X$. Since $a \notin X$, we get $s_A(\lambda(x)) < a < \lambda(y)$. Because b satisfies $\beta\, \mathrm{tp}_{\mathcal{L}_B}(a/X)$ and β is an $\mathcal{L}_C^*$-embedding, we conclude that $s_A(\lambda(\beta(x))) < b < \lambda(\beta(y))$. Consequently, $\beta(x) < b < \beta(y)$. □

Corollary 8.2. *Let $c \in M$, $d \in N$, $a \in A(M) \setminus A(X)$ and $b \in A(N) \setminus A(\beta(X))$ be such that $[c] = [a]$ and $[d] = [b]$. Then*

$$\beta\, \mathrm{tp}_{\mathcal{L}_B}(a|X) = \mathrm{tp}_{\mathcal{L}_B}(b|\beta(X)) \;\Rightarrow\; \beta\, \mathrm{tp}_{\mathcal{L}}(c|X) = \mathrm{tp}_{\mathcal{L}}(d|\beta(X)).$$

Proof. Since $a \notin A(X)$ and $[a] = [c]$, there is no $x \in X$ such that $[x] = [c]$ by Lemma 7.13. Therefore $x < a < y$ iff $x < c < y$ for all $x, y \in X$. Similarly one shows that $\beta(x) < b < \beta(y)$ iff $\beta(x) < d < \beta(y)$ for all $x, y \in X$. By Lemma 8.1, $\beta\, \mathrm{tp}_{\mathcal{L}}(a|X) = \mathrm{tp}_{\mathcal{L}}(b|\beta(X))$. Thus $\beta\, \mathrm{tp}_{\mathcal{L}}(c|X) = \mathrm{tp}_{\mathcal{L}}(d|\beta(X))$. □

Later we will have to extend X not only by elements of $A(M)$, but also by their images under $e(-,z)$ for certain $z \in C(M)$. The next lemma shows that for $a \in A^n$ the $\mathcal{L}_B$-type of a over X does not only determine the $\mathcal{L}$-type of a, but also the $\mathcal{L}$-type of $e(a,c)$ and a over X for every $c \in C(X)$.

Proposition 8.3. *Let $c = (c_1, \dots, c_n) \in C(X)^n$, $a = (a_1, \dots, a_n) \in A(M)^n$ and $b = (b_1, \dots, b_n) \in A(N)^n$. If $\beta\, \mathrm{tp}_{\mathcal{L}_B}(a|X) = \mathrm{tp}_{\mathcal{L}_B}(b|\beta(X))$, then*

$$\beta\, \mathrm{tp}_{\mathcal{L}}(a, e(a_1,c_1), \dots, e(a_n,c_n)|X) = \mathrm{tp}_{\mathcal{L}}(b, e(b_1,\beta(c_1)), \dots, e(b_n,\beta(c_n))|\beta(X)).$$

Proof. By Lemma 5.2 we can assume that $0 \ll a_i$ and $0 \ll b_i$ for $i = 1, \dots, n$. We can easily reduce the proof to the case where $a_i \neq a_j$ and $b_i \neq b_j$ for $i, j \in \{1, \dots, n\}$ and $i \neq j$. By Corollary 5.4, $[a_i] \neq [a_j]$ and $[b_i] \neq [b_j]$ for $i \neq j$. Suppose there is $x \in X$ such that $[a_i] = [x]$. Since $X \trianglelefteq \mathcal{M}$, we get $a_i \in A(X)$ by Corollary 5.8. Thus $b_i \in A(\beta(X))$, because b_i satisfies $\beta\, \mathrm{tp}_{\mathcal{L}_B}(a_i|X)$. Therefore we can assume that there is no $x \in X$ with $[a_i] = [x]$. Similarly we can assume that no $x \in X$ satisfies $[b_i] = [\beta(x)]$.

After reordering $c_1, \dots, c_n$, we can assume that there is $l \in \mathbb{N}$ such that $a_1^{-1}, \dots, a_n^{-1}, c_1 - e(a_1,c_1), \dots, c_l - e(a_l,c_l)$ are $\mathbb{Q}$-linearly independent over $C(X)$, and for all $k > l$, $c_k - e(a_k,c_k)$ is $\mathbb{Q}$-linearly dependent over $C(X)$ and $a_1^{-1}, \dots a_n^{-1}, c_1 - e(a_1,c_1), \dots, c_l - e(a_l,c_l)$. Because a_i^{-1} and $c_i - e(a_i,c_i)$ are $\mathcal{L}_B$-definable over $X \cup \{a_i\}$ for each i and $\beta\, \mathrm{tp}_{\mathcal{L}_B}(a|X) = \mathrm{tp}_{\mathcal{L}_B}(b|\beta(X))$, we deduce from Proposition 6.2(v) that $c_k - e(b_k,c_k)$ satisfies the same $\mathbb{Q}$-linear dependency over $C(\beta(X))$ and $b_1^{-1}, \dots, b_n^{-1}, \beta(c_1) - e(b_1,\beta(c_1)), \dots, \beta(c_l) - e(b_l,\beta(c_l))$. It remains to show that

$$\beta\, \mathrm{tp}_{\mathcal{L}}(a, e(a_1,c_1), \dots, e(a_l,c_l)|X) = \mathrm{tp}_{\mathcal{L}}(b, e(b_1,\beta(c_1)), \dots, e(b_l,\beta(c_l))|\beta(X)).$$

We will use the abbreviations

$$u := (a_1^{-1}, \dots, a_n^{-1}, c_1 - e(a_1,c_1), \dots, c_l - e(a_l,c_l)),$$
$$v := (b_1^{-1}, \dots, b_n^{-1}, \beta(c_1) - e(b_1,\beta(c_1)), \dots, \beta(c_l) - e(b_l,\beta(c_l))).$$

Because $c_i \in X$ for each i, it is enough to show that $\beta\, \mathrm{tp}_{\mathcal{L}}(u|X) = \mathrm{tp}_{\mathcal{L}}(v|\beta(X))$. Since $\beta\, \mathrm{tp}_{\mathcal{L}_B}(a|X) = \mathrm{tp}_{\mathcal{L}_B}(b|\beta(X))$, we have $\beta\, \mathrm{tp}_{\mathcal{L}_B}(u|X) = \mathrm{tp}_{\mathcal{L}_B}(v|\beta(X))$ by Lemma 6.1. By $\mathbb{Q}$-linear independence of u and Lemma 5.10 there are tuples of rational numbers $r_1, \dots, r_{n+l} \in \mathbb{Q}^{n+l}$ with $r_{i,i} \neq 0$ and $r_{i,j} = 0$ for $j > i$ such that $0 \neq \mu_{r_i}(u) \neq \mu_{r_j}(u)$ for $i \neq j$. Because μ_r is an $\mathcal{L}_B$-definable function for every $r \in \mathbb{Q}^m$ and $\beta\, \mathrm{tp}_{\mathcal{L}_B}(u|X) = \mathrm{tp}_{\mathcal{L}_B}(v|\beta(X))$, we see that $0 \neq \mu_{r_i}(v) \neq \mu_{r_j}(v)$ for $i \neq j$. Moreover, from Lemma 7.14 and u being $\mathbb{Q}$-linearly independent over $C(X)$ we conclude that $\mu_{r_i}(u) \notin X$ for each i. Since $\beta\, \mathrm{tp}_{\mathcal{L}_B}(u|X) = \mathrm{tp}_{\mathcal{L}_B}(v|\beta(X))$, we deduce that $\mu_{r_i}(v) \notin \beta(X)$ for each i. This implies that $0 \ll \mu_{r_i}(u)$ and $0 \ll \mu_{r_i}(v)$. By Lemma 5.6, $[|r_i \cdot u|^{-1}] = [p_A(\mu_{r_i}(u))]$ and $[|r_i \cdot v|^{-1}] = [p_A(\mu_{r_i}(v))]$. Since $\beta\, \mathrm{tp}_{\mathcal{L}_B}(u|X) = \mathrm{tp}_{\mathcal{L}_B}(v|\beta(X))$, we get $\beta\, \mathrm{tp}_{\mathcal{L}_B}(p_A(\mu_{r_i}(u))|X) = \mathrm{tp}_{\mathcal{L}_B}(p_A(\mu_{r_i}(v))|\beta(X))$. By Corollary 8.2, $\beta\, \mathrm{tp}_{\mathcal{L}}(r_i \cdot u|X) = \mathrm{tp}_{\mathcal{L}}(r_i \cdot v|\beta(X))$. Hence by Lemma 3.9,

$$\beta\, \mathrm{tp}_{\mathcal{L}}(r_1 \cdot u, \dots, r_n \cdot u|X) = \mathrm{tp}_{\mathcal{L}}(r_1 \cdot v, \dots, r_n \cdot v|\beta(X)).$$

As $r_{i,i} \neq 0$ for each i and $r_{i,j} = 0$ for $j > i$, we get $\beta\, \mathrm{tp}_{\mathcal{L}}(u|X) = \mathrm{tp}_{\mathcal{L}}(v|\beta(X))$. □

Types of elements of C

Now consider types of elements of $C(M)$ over X. In contrast to the results about types of tuples of elements of A, we will only consider types of a single element of $C(M)$.

Lemma 8.4. *Let $c \in C(M)$ and $d \in C(N)$. Then*

$$\beta\,\mathrm{tp}_{\mathcal{L}_B}(c|X) = \mathrm{tp}_{\mathcal{L}_B}(d|\beta(X)) \;\Rightarrow\; \beta\,\mathrm{tp}_{\mathcal{L}}(c|X) = \mathrm{tp}_{\mathcal{L}}(d|\beta(X)).$$

Proof. Suppose $\beta\,\mathrm{tp}_{\mathcal{L}_B}(c|X) = \mathrm{tp}_{\mathcal{L}_B}(d|\beta(X))$. We can easily reduce the proof to the case where $c \notin X$ and $d \notin \beta(X)$. Since $X \trianglelefteq \mathcal{M}$ and β is an $\mathcal{L}_C^*$-embedding, both X and $\beta(X)$ are closed under ν. Consequently, for all $x \in X$ with $\nu(x) < c$, we have $x < c$, and for all $y \in X$ with $c < y$, we have $c < \nu(y)$. Similarly for all $x \in X$ with $\nu(\beta(x)) < d$, we have $\beta(x) < d$, and for all $y \in X$ with $d < \beta(y)$, we have $d < \nu(\beta(y))$. Let $x, y \in X$. Since $\beta\,\mathrm{tp}_{\mathcal{L}_B}(c|X) = \mathrm{tp}_{\mathcal{L}_B}(d|\beta(X))$, it follows that $x < c < y$ iff $\nu(x) < c < \nu(y)$ iff $\nu(\beta(x)) < d < \nu(\beta(y))$ iff $\beta(x) < d < \beta(y)$. Thus d lies in the image of the cut of c over X under β. Therefore $\beta\,\mathrm{tp}_{\mathcal{L}}(c|X) = \mathrm{tp}_{\mathcal{L}}(d|\beta(X))$. ◻

Corollary 8.5. *Let $\varphi(x)$ be an $\mathcal{L}$-X-formula and $p(x)$ a complete $\mathcal{L}_B$-type over X. Then there is an $\mathcal{L}_B$-X-formula $\psi(x) \in p(x)$ such that either $\mathcal{M} \models \forall x \in C\ \psi(x) \to \varphi(x)$ or $\mathcal{M} \models \forall x \in C\ \psi(x) \to \neg\varphi(x)$.*

Proof. By replacing $\mathcal{M}$ by a κ^+-saturated elementary extension, we can assume that $\mathcal{M} = \mathcal{N}$. Suppose the conclusion of the corollary fails. By saturation of $\mathcal{M}$ there are $c, d \in C(M)$ such that $p = \mathrm{tp}_{\mathcal{L}_B}(c|X) = \mathrm{tp}_{\mathcal{L}_B}(d|X)$ and $\varphi(c) \wedge \neg\varphi(d)$. By Lemma 8.4, $\mathrm{tp}_{\mathcal{L}}(c|X) = \mathrm{tp}_{\mathcal{L}}(d|X)$. This contradicts $\varphi(c) \wedge \neg\varphi(d)$. ◻

Proposition 8.6. *Let $c \in C(M), d \in C(N)$ and $a_1, \ldots, a_n \in A(X)$. If $\beta\,\mathrm{tp}_{\mathcal{L}_B}(c|X) = \mathrm{tp}_{\mathcal{L}_B}(d|\beta(X))$, then*

$$\beta\,\mathrm{tp}_{\mathcal{L}}(c, e(a_1, c), \ldots, e(a_n, c)|X) = \mathrm{tp}_{\mathcal{L}}(d, e(\beta(a_1), d), \ldots, e(\beta(a_n), d)|\beta(X)). \tag{8.1}$$

Proof. Let $c \in C(M)$ and $d \in C(N)$ be such that $\beta\,\mathrm{tp}_{\mathcal{L}_B}(c|X) = \mathrm{tp}_{\mathcal{L}_B}(d|\beta(X))$. Let $a_1, \ldots, a_n \in A(X)$ with $a_1 < \cdots < a_n$. Because $e(x, y)$ is $\mathcal{L}_B$-definable from x, y and $\beta\,\mathrm{tp}_{\mathcal{L}_B}(c|X) = \mathrm{tp}_{\mathcal{L}_B}(d|\beta(X))$,

$$\beta\,\mathrm{tp}_{\mathcal{L}_B}(c, e(a_1, c), \ldots, e(a_n, c)|X) = \mathrm{tp}_{\mathcal{L}_B}(d, e(\beta(a_1), d), \ldots, e(\beta(a_n), d)|\beta(X)).$$

We will now argue by induction on n. By Proposition 6.2(v) we can directly reduce the proof to the case where $e(a_1, c), \ldots, e(a_n, c)$ are $\mathbb{Q}$-linearly independent over $C(X)$ and $e(\beta(a_1), d), \ldots, e(\beta(a_n), d)$ are $\mathbb{Q}$-linearly independent over $C(\beta(X))$. For $n = 0$, note that $\mathrm{tp}_{\mathcal{L}}(c|X) = \mathrm{tp}_{\mathcal{L}}(d|\beta(X))$ by Lemma 8.4. For the induction step, suppose

$$\beta\,\mathrm{tp}_{\mathcal{L}}(c, e(a_1, c), \ldots, e(a_{n-1}, c)|X) = \mathrm{tp}_{\mathcal{L}}(d, e(\beta(a_1), d), \ldots, e(\beta(a_{n-1}), d)|\beta(X)). \tag{8.2}$$

We will use the abbreviations

$$u := (e(a_1, c), \ldots, e(a_{n-1}, c)), \qquad v := (e(\beta(a_1), d), \ldots, e(\beta(a_{n-1}), d)).$$

Suppose that $\overline{u}^{a_n}, \overline{c}^{a_n}$ are dcl-dependent over $\overline{X}^{a_n}$. By Lemma 7.10 and since $e(a_n, e(a_j, c)) = e(a_j, c)$ for $j \leq n$, there are $x \in C(X)^m$, $p \in \mathbb{Q}^m$ and $q \in \mathbb{Q}^n$ such that $e(a_n, c) = (p, q)(x, u)$. This contradicts our assumption that $e(a_1, c), \ldots, e(a_n, c)$ are $\mathbb{Q}$-linearly independent over $C(X)$. Similarly we can rule out $\overline{v}, \overline{d}^{a_n}$ being dcl-dependent over $\overline{\beta(X)}^{a_n}$.

Now suppose that $\overline{u}^{a_n}, \overline{c}^{a_n}$ are dcl-independent over $\overline{X}^{a_n}$. From Lemma 6.1 and the fact that $\beta\, \mathrm{tp}_{\mathcal{L}_B}(c|X) = \mathrm{tp}_{\mathcal{L}_B}(d|\beta(X))$, we deduce that $\beta\, \mathrm{tp}_{\mathcal{L}_B}(c - e(a_n, c)|X) = \mathrm{tp}_{\mathcal{L}_B}(c - e(a_n, d)|\beta(X))$. Thus by Lemma 8.4,

$$\beta\, \mathrm{tp}_{\mathcal{L}}(c - e(a_n, c)|X) = \mathrm{tp}_{\mathcal{L}}(d - e(a_n, d)|\beta(X)). \tag{8.3}$$

Let $Z \subseteq M^{m+n}$ be open and let $f, g : Z \to M$ be $\mathcal{L}$-$\emptyset$-definable continuous functions. Let $x = (x_1, \ldots, x_m) \in X^m$. By o-minimality of T it is enough to show that

$$f(x, c, u) < c - e(a_n, c) < g(x, c, u) \quad \text{iff} \quad f(\beta(x), d, v) < d - e(\beta(a_n), d) < g(\beta(x), d, v). \tag{8.4}$$

By regular cell decomposition in o-minimal structures and (8.3), we can reduce the proof to the case where f, g are not constant in all the last n coordinates. By our assumptions on c and d we have $f(x, c, u), g(x, c, u) \notin \mathfrak{m}_{a_n}$, but $c - e(a_n, c) \in \mathfrak{m}_{a_n}$ by Axiom T7. Therefore

$$f(x, c, u) < c - e(a_n, c) < g(x, c, u) \quad \text{iff} \quad f(x, c, u) < 0 < g(x, c, u).$$

A similar argument shows that

$$f(\beta(x), d, v) < d - e(\beta(a_n), d) < g(\beta(x), d, v) \text{ iff } f(\beta(x), d, v) < 0 < g(\beta(x), d, v).$$

Hence (8.4) follows from (8.2). □

Lemma 8.7. *Let $U \subseteq C(M)$, $V \subseteq C(N)$ and $\gamma : \mathrm{dcl}(X \cup U) \to \mathrm{dcl}(\beta(X) \cup V)$ be such that*

(i) $\mathrm{dcl}(X \cup U) \trianglelefteq \mathcal{M}$,
(ii) *γ is an $\mathcal{L}$-isomorphism extending β with $\gamma(U) = V$,*
(iii) $\beta\, \mathrm{tp}_{\mathcal{L}_B}(U|X) = \mathrm{tp}_{\mathcal{L}_B}(V|\beta(X))$.

Then $\mathrm{dcl}(\beta(X) \cup V) \trianglelefteq \mathcal{N}$ and γ is an $\mathcal{L}^*_C$-isomorphism.

Proof. For ease of notation, set $X' := \mathrm{dcl}(X \cup U)$ and $Y' := \mathrm{dcl}(\beta(X) \cup V)$. We will establish the conclusion by proving a sequence of claims.

Claim 1. *Let $c \in C(X')$. Then c is $\mathcal{L}_B$-definable over $X \cup U$, $\gamma(c) \in C(Y')$ and $\gamma\, \mathrm{tp}_{\mathcal{L}_B}(c|X \cup U) = \mathrm{tp}_{\mathcal{L}_B}(\gamma(c)|\beta(X) \cup V)$.*

Proof of Claim 1. Since $X \trianglelefteq \mathcal{M}$, there are $p \in \mathbb{Q}^m$, $q \in \mathbb{Q}^n$, $x \in X^m$ and $u \in U^n$ such that $c = p \cdot x + q \cdot u$ by Lemma 7.9. By (ii), $\gamma(c) = p \cdot \beta(x) + q \cdot \gamma(u)$ and $\gamma(u) \in V^n$. By (iii) and Proposition 6.2(vi), $p \cdot x + q \cdot u \in C(M)$ iff $p \cdot \gamma(x) + q \cdot \beta(u) \in C(N)$. Hence $\gamma(c) = p \cdot \gamma(z) + q \cdot \gamma(u) \in C(Y')$. □

Because every element in $C(X')$ is $\mathcal{L}_B$-definable over $X \cup U$ by Claim 1, we conclude that $\gamma\, \mathrm{tp}_{\mathcal{L}_B}(C(X')|X \cup U) = \mathrm{tp}_{\mathcal{L}_B}(\gamma(C(X'))|\beta(X) \cup V)$.

Claim 2. *Let $a \in A(X')$. Then a is $\mathcal{L}_B$-definable over $X \cup U$, $\gamma(a) \in A(Y')$ and $\gamma\,\mathrm{tp}_{\mathcal{L}_B}(a|X \cup U) = \gamma\,\mathrm{tp}_{\mathcal{L}_B}(\gamma(a)|\beta(X) \cup V)$.*

Proof. Since $a \in A(X')$, we have $a^{-1} \in C(X')$. Thus $\gamma(a^{-1}) = \gamma(a)^{-1}$ by (ii) and $\gamma(a)^{-1} \in C(Y')$ by Claim 1. Since $a \in A(X')$, there is a unique $b \in A(M)$ such that $E(b, a^{-1})$. By Claim 1, $\gamma\,\mathrm{tp}_{\mathcal{L}_B}(a^{-1}|X \cup U) = \mathrm{tp}_{\mathcal{L}_B}(\gamma(a)^{-1}|\beta(X) \cup V)$. Therefore there is also a unique $b \in A(N)$ such that $E(b, \gamma(a)^{-1})$. The existence of b implies that $\gamma(a) \in A(Y')$. By Lemma 6.1(i), $\gamma\,\mathrm{tp}_{\mathcal{L}_B}(a|X \cup U) = \mathrm{tp}_{\mathcal{L}_B}(\gamma(a)|\beta(X) \cup V)$. □

Claim 3. *$A(Y') = \gamma(A(X'))$ and $\lambda(\gamma(x)) = \gamma(\lambda(x))$ for all $x \in X'$.*

Proof. We first prove the second statement. Let $y \in Y'$ and $x \in X'$ be such that $\gamma(x) = y$. By Claim 2 we have $\gamma(\lambda(x)) \in A(Y')$, $\gamma(s_A(\lambda(x))) \in A(Y')$ and $s_A(\gamma(\lambda(x))) = \gamma(s_A(\lambda(x)))$. Hence $\gamma(\lambda(x)) \le \gamma(x) = y < s_A(\gamma(\lambda(x)))$. Consequently, $\lambda(y) = \gamma(\lambda(x)) \in A(Y')$. For the proof of the first statement let $a \in A(Y')$ and $b \in X'$ be such that $\gamma(b) = a$. Then $a = \lambda(a) = \lambda(\gamma(b)) = \gamma(\lambda(b))$. Thus $\lambda(b) = b$ and $b \in A(X')$. □

It follows immediately from Claim 3 that Y' is closed under λ.

Claim 4. *$C(Y') = \gamma(C(X'))$ and $\nu(\gamma(x)) = \gamma(\nu(x))$ for all $x \in X'$.*

Proof. Again we prove the second statement first. Let $y \in Y'$ and $x \in X'$ be such that $\gamma(x) = y$. We can assume that $y \in (0, 1)$, $\nu(y) \neq 0$ and $\nu(y) \neq y$. Let $d \in A(M)$ be maximal such that $\neg E(d, \nu(x))$. By Corollary 5.15,

$$x \in \big[e(p_A(d), \nu(x)) + d^{-1}, e(p_A(d), \nu(x)) + p_A(d)^{-1} - d^{-1}\big).$$

Because d is $\mathcal{L}_B$-definable from $\nu(x)$ and $X' \trianglelefteq \mathcal{M}$, d is in $A(X')$ and so is $p_A(d)$. Since X' is closed under e, $e(p_A(d), \nu(x)) \in X'$. By Claim 2, $\gamma(d) \in A(Y')$. By Claim 1, $\gamma(\nu(x)), \gamma(e(p_A(d), \nu(x))) \in C(Y')$ and $\gamma(e(p_A(d), \nu(x))) = e(p_A(\gamma(d)), \gamma(\nu(x)))$. Since γ is an $\mathcal{L}$-isomorphism, we have

$$y \in \big[e(p_A(\gamma(d)), \gamma(\nu(x))) + \gamma(d)^{-1}, e(p_A(\gamma(d)), \gamma(\nu(x))) + p_A(\gamma(d))^{-1} - \gamma(d)^{-1}\big).$$

By Lemma 4.17, $\nu(y) = e(p_A(\gamma(d)), \gamma(\nu(x))) + \gamma(d)^{-1} \in C(Y')$. Hence $\nu(y) = \gamma(\nu(x))$.

For the first statement, let $y \in C(Y')$ and $x \in X'$ be such that $\gamma(x) = y$. Then $\gamma(\nu(x)) = \nu(y) = y$. Since γ is bijective, $\nu(x) = x$. Therefore $x \in C(X')$. □

Claim 4 shows that Y' is closed under ν. Combining this claim with the statement after Claim 1 we see that $\beta\,\mathrm{tp}_{\mathcal{L}_B}(C(X')|X) = \mathrm{tp}_{\mathcal{L}_B}(C(Y')|\beta(X))$. Hence γ is an $\mathcal{L}_C^B$-isomorphism. Since $X' \trianglelefteq \mathcal{M}$, it follows easily that Y' is closed under e and under all $\mathcal{L}_C$-definable functions into A, and that these functions commute with γ. Since we already know that Y' is closed under λ and ν, and that γ commutes with ν and λ, we conclude that $Y' \trianglelefteq \mathcal{N}$ and γ is an $\mathcal{L}_C^*$-isomorphism. □

Embedding lemmas

We will now prove the necessary embedding lemmas for our quantifier elimination result. We still assume that $\kappa = |\mathcal{L}_C^+|$ and $\mathcal{M}, \mathcal{N} \models \tilde{T}^+$ are such that $|\mathcal{M}| \leq \kappa$ and $\mathcal{N}$ is κ^+-saturated.

Definition 8.8. Let $Z \subseteq M$. We define $\mathcal{D}(Z)$ as the union of $\lambda(Z)$ and the set of all elements in $A(M)$ that are $\mathcal{L}_B$-definable from Z.

Proposition 8.9. *Let $X \trianglelefteq \mathcal{M}$, $\beta : X \to \mathcal{N}$ an $\mathcal{L}_C^*$-embedding and $c \in C(M)$. Then there is an $\mathcal{L}_C^*$-embedding γ into $\mathcal{N}$ extending β such that $c \in \mathrm{dom}(\gamma)$ and $\mathrm{dom}(\gamma) \trianglelefteq \mathcal{M}$.*

Proof. Set $U_0 := A(X)$ and $V_0 := \emptyset$, and recursively define

$$U_{i+1} := \mathcal{D}(X \cup \{c\} \cup U_i \cup V_i), \qquad V_{i+1} := e(U_i, C(X) \cup \{c\}).$$

Set $U := \bigcup_{i \in \mathbb{N}} U_i$ and $V := \bigcup_{i \in \mathbb{N}} V_i$. Since β is an $\mathcal{L}_C^B$-isomorphism and $\mathcal{N}$ is saturated, there is $W \subseteq A(N)$ such that

$$\beta\, \mathrm{tp}_{\mathcal{L}_B}(U|X) = \mathrm{tp}_{\mathcal{L}_B}(W|\beta(X)). \tag{8.5}$$

By Proposition 8.3 we have

$$\beta\, \mathrm{tp}_{\mathcal{L}}(U \cup e(U, C(X))|X) = \mathrm{tp}_{\mathcal{L}}(W \cup e(W, C(\beta(X)))|\beta(X)).$$

Therefore β extends to an $\mathcal{L}$-isomorphism β' between $X' := \mathrm{dcl}(X \cup U \cup e(U, C(X)))$ and $Y' := \mathrm{dcl}(Y \cup W \cup e(W, C(\beta(X))))$ such that $\beta'(U) = W$ and $e(u, c) = e(\beta'(u), \beta(c))$ for all $u \in U$ and $c \in C(X)$. By (8.5), it follows that $\beta\, \mathrm{tp}_{\mathcal{L}_B}(U \cup e(U, C(X))|X) = \mathrm{tp}_{\mathcal{L}_B}(W \cup e(W, C(\beta(X))|\beta(X)))$. By Lemma 8.7 it remains to show that $X' \trianglelefteq \mathcal{M}$.

We will prove that X' satisfies the assumptions of Proposition 7.12. We first establish that X' is A-closed. Note that by construction of U and V, for every $x \in X'$ there is $i \in \mathbb{N}$ such that $x \in \mathrm{dcl}(X \cup U_i \cup V_i)$. Hence $\lambda(y) \in U_{i+1}$. Thus $A(X') = U$. If $a \in A(M)$ is $\mathcal{L}_B$-definable from X', there is $i \in \mathbb{N}$ such that a is $\mathcal{L}_B$-definable from $X \cup U_i \cup V_i$. Hence $a \in U_{i+1}$ and X' is A-closed. Since $U \cup e(U, C(X))$ is closed under $e(a, -)$ for each $a \in U$ and $A(X') = U$, we get $X' \trianglelefteq \mathcal{M}$ by Proposition 7.12. Therefore β' is an $\mathcal{L}_C^*$-embedding.

We now extend β' to an $\mathcal{L}_C^*$-embedding γ whose domain contains c. Because β' is an $\mathcal{L}_C^B$-embedding and $\mathcal{N}$ is saturated, there is $d \in C(N)$ such that

$$\beta'\, \mathrm{tp}_{\mathcal{L}_B}(c|X') = \mathrm{tp}_{\mathcal{L}_B}(d|Y'). \tag{8.6}$$

By Proposition 8.6, $\beta'\, \mathrm{tp}_{\mathcal{L}}(c \cup e(A(X'), c)|X') = \mathrm{tp}_{\mathcal{L}}(d \cup e(A(Y'), d)|Y')$. Consequently, β' extends to an $\mathcal{L}$-isomorphism γ between $X'' := \mathrm{dcl}(X' \cup \{c\} \cup e(A(X'), c))$ and $Y'' := \mathrm{dcl}(Y' \cup \{d\} \cup e(A(Y'), d))$ mapping c to d and $e(a, c)$ to $e(\beta'(a), d)$ for every $a \in A(X')$. By (8.6), $\beta'\, \mathrm{tp}_{\mathcal{L}_B}(c \cup e(A(X'), c)|X') = \mathrm{tp}_{\mathcal{L}_B}(d \cup e(A(Y'), d)|Y')$. In order to show that γ is an $\mathcal{L}_C^*$-embedding, it is again enough to show that $X'' \trianglelefteq \mathcal{M}$.

We will establish that the conditions of Proposition 7.12 are satisfied. We first prove that X'' is A-closed. By construction of U and V, for every $x \in X''$ there is $i \in \mathbb{N}$

such that $x \in \mathrm{dcl}(X \cup \{c\} \cup U_i \cup V_i)$. Thus for $x \in X''$ we can find $i \in \mathbb{N}$ such that $x \in \mathrm{dcl}(X \cup \{c\} \cup U_i \cup V_i)$. Hence $\lambda(x) \in U_{i+1}$. In particular, $A(X'') = U$. If $a \in A(M)$ is $\mathcal{L}_B$-definable from X'', there is $i \in \mathbb{N}$ such that a is $\mathcal{L}_B$-definable from $X \cup \{c\} \cup U_i \cup V_i$. Hence $a \in U_{i+1}$ and X'' is A-closed. Since $A(X'') = U = A(X')$, we also see that $\{c\} \cup e(A(X'), c)$ is closed under $e(a, -)$ for each $a \in A(X'')$. By Proposition 7.12, $X'' \trianglelefteq \mathcal{M}$. Thus γ is an $\mathcal{L}_C^*$-embedding. □

Corollary 8.10. *Let $X \trianglelefteq \mathcal{M}$, $\beta : X \to \mathcal{N}$ be an $\mathcal{L}_C^*$-embedding and $Z \subseteq C(M)$ be such that $|Z| \le \kappa$. Then there is an $\mathcal{L}_C^*$-embedding γ extending β such that $Z \subseteq \mathrm{dom}(\gamma)$ and $\mathrm{dom}(\gamma) \trianglelefteq \mathcal{M}$.*

So far we have only extended β to another $\mathcal{L}_C^*$-embedding. In order to extend β to an $\mathcal{L}_C^+$-embedding, we need to better understand the interaction of β with ν_f and τ_f.

Lemma 8.11. *Let $X \trianglelefteq \mathcal{M}$ and $\beta : X \to \mathcal{N}$ be an $\mathcal{L}_C^*$-embedding. Let $x \in X^l$ and $y \in M$, and let $f : U \subseteq M^{l+n} \to [0,1]$ be $\mathcal{L}$-$\emptyset$-definable and continuous. Let γ be an $\mathcal{L}$-embedding extending β with $y \in \mathrm{dom}(\gamma)$. If $\nu_f(x,y), \tau_f(x,y) \in X$, then $\beta(\nu_f(x,y)) = \nu_f(\beta(x), \gamma(y))$ and $\beta(\tau_f(x,y)) = \tau_f(\beta(x), \gamma(y))$.*

Proof. Let $g_1, \dots, g_k : M^{l+n} \to M$ be as in Axiom T12 for f. Suppose $\nu_f(x,y), \tau_f(x,y) \in X$. Let $i, j \le k$ be such that

$$g_i(x, \nu_f(x,y)) = \sup_{d \in C(M)^n \cap U_x,\, f(x,d) \le y} f(x,d),$$
$$g_j(x, \tau_f(x,y)) = \inf_{d \in C(M)^n \cap U_x,\, f(x,d) \ge y} f(x,d).$$

Let $\varphi(z, x, u, v)$ be the $\mathcal{L}$-formula stating that $(x,z) \in U$ and one of the following three statements holds:

(i) $f(x,z) = g_i(x,u)$ and z is lexicographically smaller than u,
(ii) $f(x,z) = g_j(x,v)$ and z is lexicographically smaller than v,
(iii) $g_i(x,u) < f(x,z) < g_j(x,v)$.

By definition of ν_f and τ_f and our choice of i, j, there is no $c \in C(M)$ such that $\varphi(c, x, \nu_f(x,y), \tau_f(x,y))$. We now show that there is no $d \in C(N)$ with $\varphi(d, \beta(x), \beta(\nu_f(x,y)), \beta(\nu_f(x,y)))$. Suppose there is such a d. Let $p(z)$ be the $\mathcal{L}_B$-type $\beta^{-1}\, \mathrm{tp}_{\mathcal{L}_B}(d|\beta(X))$. By Corollary 8.5 there is an $\mathcal{L}_B$-formula $\psi(z, x') \in p(z)$, where $x' \in X^n$ is such that

$$\mathcal{M} \models \forall z \in C(M)\ \psi(z, x') \to \varphi(c, x, \nu_f(x,y), \tau_f(x,y)). \tag{8.7}$$

Since $\mathcal{N} \models \psi(d, \beta(x'))$, we have $\mathcal{N} \models \exists z \in C(N)\ \psi(z, \beta(x'))$. Because β is a partial $\mathcal{L}_C^*$-isomorphism and ψ is an $\mathcal{L}_B$-formula, $\mathcal{M} \models \exists z \in C(M)\ \psi(z, x')$. Hence there is $d' \in C(M)$ such that $\psi(d', x)$. By (8.7) we get $\varphi(d', x, \nu_f(x,y), \tau_f(x,y))$, a contradiction.

Since γ is an $\mathcal{L}$-embedding,

$$g_i(\beta(x), \beta(\nu_f(x,y)) \le \gamma(y) \le g_j(\beta(x), \beta(\tau_f(x,y)).$$

Because there is no $d \in C(N)$ with $\varphi(d, \beta(x), \beta(\nu_f(x,y)), \beta(\nu_f(x,y))$, we conclude that $\beta(\nu_f(x,y)) = \nu_f(\beta(x), \gamma(y))$ and $\beta(\tau_f(x,y)) = \tau_f(\beta(x), \gamma(y))$. □

Proposition 8.12. *Let $X \trianglelefteq^+ \mathcal{M}$, $\beta : X \to \mathcal{N}$ be an $\mathcal{L}_C^+$-embedding and $c \in C(M)$. Then there is an $\mathcal{L}_C^+$-embedding γ into $\mathcal{N}$ extending β such that $c \in \mathrm{dom}(\gamma)$.*

Proof. By Proposition 8.9 we can extend β to an $\mathcal{L}_C^*$-embedding $\gamma : X' \to Y'$ such that $c \in X'$ and $X' \trianglelefteq \mathcal{M}$. For $Z \subseteq M$, we define $\mathcal{E}(Z)$ to be the union of all $\nu_f(Z^l, Z) \cup \tau_f(Z^l, Z)$ for all continuous $\mathcal{L}$-$\emptyset$-definable $f : U \subseteq M^{l+n} \to [0, 1]$. Since $\kappa > |\mathcal{L}|$, we see that $|\mathcal{E}(Z)| \le \kappa$ if $|Z| \le \kappa$. Hence by Corollary 8.10 we can extend γ to an $\mathcal{L}_C^*$-embedding $\gamma_0 : X_0 \to Y_0$ such that $\mathcal{E}(X') \subseteq X_0$ and $X_0 \trianglelefteq \mathcal{M}$. For $n \in \mathbb{N}$, define $\gamma_{n+1} : X_{n+1} \to Y_{n+1}$ to be an $\mathcal{L}_C^*$-embedding extending γ_n such that $\mathcal{E}(X_n) \subseteq X_{n+1}$ and $X_{n+1} \trianglelefteq \mathcal{M}$. Let $\gamma_\infty = \bigcup_{n=0}^\infty \gamma_n : X_\infty \to Y_\infty$. Because each γ_n is an $\mathcal{L}_C^*$-embedding, so is γ_∞. Moreover, since $X_n \trianglelefteq \mathcal{M}$ for each n, we have $X_\infty \trianglelefteq \mathcal{M}$. It is easy to see that by the construction of γ_∞, X_∞ is closed under all ν_f's and τ_f's. Hence $X_\infty \trianglelefteq^+ \mathcal{M}$. By Lemma 8.11, γ_∞ is an $\mathcal{L}_C^+$-embedding. □

We are now ready to prove two of the main results of the paper: quantifier elimination for $\tilde{T}^+$ and completeness of $\tilde{T}$.

Proof of Theorem 7.1. Let $\kappa = |\mathcal{L}_C^+|$ and let $\mathcal{M}, \mathcal{N} \models \tilde{T}^+$ be such that $|\mathcal{M}| \le \kappa$ and $\mathcal{N}$ is κ^+-saturated. Let $X \trianglelefteq^+ \mathcal{M}$ and suppose that $\beta : X \to \mathcal{N}$ is an $\mathcal{L}_C^+$-embedding. It is enough to show that β can be extended. By Proposition 8.12 we can assume $C(M) \subseteq X$. Let $u \in \mathcal{M}$. We can assume that $u \in [0, 1]$. We will extend β to an $\mathcal{L}_C^+$-embedding γ such that $u \in \mathrm{dom}(\gamma)$. Let $v \in \mathcal{N}$ be such that $\mathrm{tp}_{\mathcal{L}}(v|\beta(X)) = \beta\,\mathrm{tp}_{\mathcal{L}}(u|X)$. Because β is an $\mathcal{L}$-embedding, such a v exists. Then β extends to an $\mathcal{L}$-embedding γ between $\mathrm{dcl}(X \cup \{u\})$ and $\mathrm{dcl}(\beta(X) \cup \{v\})$ with $\gamma(u) = v$. Since $C(M) \subseteq X$, it is easy to check that $\mathrm{dcl}(X \cup \{u\}) \trianglelefteq^+ \mathcal{M}$. It remains to show that $\mathrm{dcl}(\beta(X) \cup \{v\}) \trianglelefteq^+ \mathcal{N}$ and that γ is an $\mathcal{L}_C^+$-embedding. Because β is also an $\mathcal{L}_C^+$-embedding, it is enough to prove that

$$\mathrm{dcl}(\beta(X) \cup \{v\}) \cap C(N) \subseteq \beta(X).$$

Suppose there is $d \in C(N)$ such that $d \in \mathrm{dcl}(\beta(X) \cup \{v\}) \setminus \beta(X)$. By o-minimality of T there is a continuous $\mathcal{L}$-$\emptyset$-definable function $f : U \subseteq M^{l+1} \to [0, 1]$ and $x \in X^l$ such that $f(\beta(x), d) = v$. Let $g_1, \ldots, g_k : M^{l+n} \to M$ be as in Axiom T12 for f. Since $u \notin X$ and $C(M) \subseteq X$, we have $g_i(x, \nu_f(x, u)) < u < g_i(x, \tau_f(x, u))$ for $i = 1, \ldots, k$. By Lemma 8.11, $\gamma(\nu_f(x, u)) = \nu_f(\beta(x), v)$ and $\gamma(\tau_f(x, u)) = \tau_f(\beta(x), v)$. Since $f(\beta(x), d) = v$, there is $i \in \{1, \ldots, k\}$ such that either $v = g_i(\beta(x), \nu_f(\beta(x), v))$ or $v = g_i(\beta(x), \tau_f(\beta(x), v))$. But then for this i, $u = g_i(x, \nu_f(x, u))$ or $u = g_i(x, \tau_f(x, u))$, contradicting our assumption on u.

Because β is an $\mathcal{L}_C^+$-embedding, it now follows easily that γ is an $\mathcal{L}_C^+$-embedding and $\mathrm{dcl}(\beta(X) \cup \{v\}) \trianglelefteq^+ \mathcal{N}$. □

Proof of Theorem 4.5. Let $\kappa = |\mathcal{L}_C^+|$ and let $\mathcal{M}, \mathcal{N} \models \tilde{T}^+$ be such that $\mathcal{M}$ and $\mathcal{N}$ are κ^+-saturated. Reccall that by K we denote the interpretation of the $\mathcal{L}$-constant symbols c_k, where $k \in K$. By Axiom T2, $K \subseteq C(M)$ and $K \subseteq C(N)$. By Axiom T1 there is an $\mathcal{L}$-isomorphism β between $\mathrm{dcl}(\emptyset) \subseteq M$ and $\mathrm{dcl}(\emptyset) \subseteq N$ with $\beta(K) = K$. By Axioms T5 and T6, $\beta\,\mathrm{tp}_{\mathcal{L}_B}(K) = \mathrm{tp}_{\mathcal{L}_B}(K)$. Note that $Q \subseteq \mathrm{dcl}(\emptyset)$ and $\mathrm{dcl}(\emptyset)$ is archimedean. Therefore $A(M) \cap \mathrm{dcl}(\emptyset) = Q$ and $\mathrm{dcl}(\emptyset)$ is A-closed. By Axiom T6, $\mathrm{dcl}(\emptyset)$ is closed under $e(a, -)$ for every $a \in Q$. By Proposition 7.12, $\operatorname{dcl}(\emptyset) \trianglelefteq \mathcal{M}$. By the same argument, $\operatorname{dcl}(\emptyset) \trianglelefteq \mathcal{N}$. By Lemma 8.7, β is an $\mathcal{L}_C^*$-isomorphism. As in the proof of Proposition 8.12, we can extend this $\mathcal{L}_C^*$-embedding to a partial $\mathcal{L}_C^+$-isomorphism $\gamma : X \to Y$ such that $X \trianglelefteq^+ \mathcal{M}$ and $Y \trianglelefteq^+ \mathcal{N}$. Because $\tilde{T}^+$ has quantifier elimination, we have $X \equiv \mathcal{M}$ and $Y \equiv \mathcal{N}$. Since X and Y are isomorphic, $X \equiv Y$. Hence $\mathcal{M} \equiv \mathcal{N}$. Thus $\tilde{T}^+$ is complete and so is $\tilde{T}$. ◻

9. Definable sets are Borel

In this section it will be shown that every set definable in $(\mathcal{R}, K)$ is Borel. The main ingredient of the proof is the quantifier elimination result established in the previous section. Using Fact 2.1 we will establish that the interpretation of every $\mathcal{L}_C^+$-relation symbol and $\mathcal{L}_C^+$-function symbol in $(\mathcal{R}, K)$ is Borel. It then follows easily from Theorem 7.1 using elementary results from descriptive set theory that every definable set is Borel.

We first introduce some new notation. We write $\mathcal{R}_K$ for the $\mathcal{L}_C$-structure $(\mathcal{R}, K, Q, \epsilon)$. As usual we will consider it is an $\mathcal{L}_C^+$-structure. For $q \in Q$, set $K_q := \{c \in K : e(q, c) = c\}$, and set $K_{\text{fin}} := \bigcup_{q \in Q} K_q$. Note that K_q is $\mathcal{L}_C$-definable over q, and K_{fin} is $\mathcal{L}_C$-$\emptyset$-definable. Also note that K_q is finite for each $q \in Q$, and K_{fin} is countable. Let $f : Q \to \mathbb{R}$ be an $\mathcal{L}_C$-definable function. We define $\lim_{q \in Q} f(q)$ as the element $x \in [0, 1]$ such that for all $\varepsilon > 0$ there is $b \in Q$ such that $|f(b') - x| < \varepsilon$ for all $b' \in Q_{>b}$. Obviously, if such an x exists, it is unique. We set

$$\liminf_{q \in Q} f(q) := \lim_{q \in Q} \inf\{f(b) : b \in Q_{\geq q}\}.$$

For $c = (c_1, \ldots, c_n) \in K^n$ and $q \in Q$, set $U_{c,q} := [c_1, c_1 + q^{-1}] \times \cdots \times [c_n, c_n + q^{-1}]$.

Now fix a continuous $\mathcal{L}$-$\emptyset$-definable function $f : X \subseteq \mathbb{R}^{l+n} \to [0, 1]$ such that X is open. Let $g_1, \ldots, g_k$ be as in Axiom T12. For $(x, y) \in \mathbb{R}^{l+1}$, let

$$l(x, y) := \sup_{d \in K_{\text{fin}} \cap X_x,\, f(x,d) \leq y} f(x, d), \qquad r(x, y) := \inf_{d \in K_{\text{fin}} \cap X_x,\, f(x,d) \geq y} f(x, d).$$

Because X is open and K_{fin} is dense in K, we directly get

$$l(x, y) = \sup_{d \in K \cap X_x,\, f(x,d) \leq y} f(x, d), \qquad r(x, y) = \inf_{d \in K \cap X_x,\, f(x,d) \geq y} f(x, d).$$

Since K_{fin} is countable, it follows easily that the graphs of l and r are Borel.

Definition 9.1. For $q \in Q$ and $(x, y) \in \mathbb{R}^{l+1}$, let $D_{q,x,y} \subseteq K_q$ be the set of all $d \in K_q$ such that for all $b \in Q$ there exists $d' \in K_{\text{fin}} \cap X_x$ such that $d' \in U_{d,q}$ and $0 \leq l(x, y) - f(x, d') < b^{-1}$.

Since Q and K_{fin} are countable and the graph of l is Borel, the set $\{(q, x, y, c) : c \in D_{q,x,y}\}$ is Borel as well.

Lemma 9.2. *Let $g : Q \to K_{\text{fin}}$ be such that $g(q) \in D_{q,x,y}$. Then there is $i \in \{1, \ldots, k\}$ such that $l(x, y) = g_i(x, \liminf_{q \in Q} g(q))$.*

Proof. Let $c := \liminf_{q \in Q} g(q)$. Since K is closed, $c \in K$. Suppose towards a contradiction that $l(x, y) \neq g_i(x, c)$ for each $i = 1, \dots, k$. Then by Lemma 3.4 there are $a, q \in Q$ such that $|l(x, y) - f(x, d)| > a^{-1}$ for all $d \in B_{q^{-1}}(c) \cap X_x \cap K_{\text{fin}}$. Let $b \in Q$ be such that $b > \max\{2nq, a\}$ and $|c - g(b)| < q^{-1}/2$. Since $g(b) \in D_{b,x,y}$, there is $d \in U_{g(b),b}$ such that $|l(x, y) - f(x, d)| < b^{-1}$. Because $d \in U_{g(b),b}$ and $b > 2nq$,

$$|d - c| < |d - g(b)| + |c - g(b)| < nb^{-1} + q^{-1}/2 < q^{-1},$$

contradicting our assumption on q. □

For $m, n \in \mathbb{N}$ with $m < n$, $c \in K^n$ and $q \in Q$ we set

$$V_{c,q,m} := \{d = (d_1, \dots, d_n) \in K_q^n : e(q, c_i) = d_i \wedge e(q, c_{m+1}) > d_{m+1}\}.$$

Proposition 9.3. *Let $c = (c_1, \dots, c_n) \in K^n$. The following are equivalent:*

(i) $c = \nu_f(x, y)$,
(ii) $l(x, y) = g_i(x, c)$ *for some* $i \in \{1, \dots, k\}$, *and for* $m = 0, \dots, n - 1$,

$$\forall q \in Q\ \forall d \in D_{q,x,y} \cap V_{c,q,m}\ \exists b \in Q_{\geq q} \quad U_{d,q} \cap D_{b,x,y} \cap V_{c,b,m} = \emptyset.$$

Proof. Suppose $c = \nu_f(x, y)$ and (ii) fails. Then there are $q \in Q$ and $d = (d_1, \dots, d_n) \in D_{q,x,y} \cap V_{c,q,m}$ such that $U_{d,q} \cap D_{b,x,y} \cap V_{c,b,m} \neq \emptyset$ for all $b \in Q_{\geq q}$. Let $h : Q \to \mathbb{R}$ be the $\mathcal{L}_C$-definable function that maps b to the lexicographically smallest $d' \in U_{d,q} \cap D_{b,x,y} \cap V_{c,b,m}$ if $b \geq q$, and to 0 otherwise; the lexicographic minimum exists because $D_{b,x,y}$ is finite. Set $c' = (c'_1, \dots, c'_n) := \liminf_{b \in Q} h(b)$. Because $h(b) \in V_{c,b,m}$ for $b \in \mathbb{Q}_{\geq q}$, we get $c_i = c'_i$ for $i = 1, \dots, m$ and $c_{m+1} \geq c'_{m+1}$. However, $h(b)$ is in $U_{q,d}$ for each $b \geq q$, and hence so is c'. Since $d_{m+1} < e(q, c_{m+1})$, we have $d_{m+1} + q^{-1} < e(q, c_{m+1})$. Hence $c_{m+1} > c'_{m+1}$. Thus c' is lexicographically smaller than c. This contradicts $c = \nu_f(x, y)$, because by Lemma 9.2 there is $i \in \{1, \dots, k\}$ such that $l(x, y) = g_i(x, \liminf_{q \in Q} g(q)) = g_i(x, c')$.

Now suppose that c satisfies (ii). By definition of ν_f it suffices show that c is lexicographically minimal such that there is $i \in \{1, \dots, k\}$ with $l(x, y) = g_i(x, c)$. Suppose not. Let $c' = (c'_1, \dots, c'_n) \in K^n$ be lexicographically smaller than c and such that there is $i \in \{1, \dots, k\}$ with $l(x, y) = g_i(x, c')$. Let $m < n$ be such that $c_i = c'_i$ for $i = 1, \dots, m$ and $c_{m+1} > c'_{m+1}$. Then $e(b, c_i) = e(b, c'_i)$ for all $i = 1, \dots, m$ and all $b \in Q$, and there is $q \in Q$ such that $e(b, c_{m+1}) > e(b, c'_{m+1})$ for all $b \in Q$ with $b \geq q$. Thus $e(b, c') \in V_{c,b,m}$ for every $b \geq q$. Since $l(x, y) = g_i(x, c')$ for some i, we have $e(b, c') \in D_{b,x,y}$ for every $b \in Q$. Hence $e(b, c') \in U_{e(q,c'),q} \cap D_{b,x,y} \cap V_{c,b,m}$ for all $b \geq q$. This contradicts (ii). □

Corollary 9.4. *The graph of ν_f is Borel.*

Similarly it can be shown that the graph of τ_f is Borel. We leave the details to the reader. We are now ready to finish the proof of Theorem B.

Proof of Theorem B. First note that the interpretation of A in $\mathcal{R}_K$ is Q, and hence countable and Borel. The same is true for K_{fin}. The interpretation of C in $\mathcal{R}_K$ is K and closed, in particular Borel.

We will first show that the interpretation of each of the function symbols from $\mathcal{L}_C^+$ is a Borel function. It is enough to check that the graph of each function is Borel. Since $\mathcal{R}$ is o-minimal, the graph of every $\mathcal{L}$-definable function is Borel. It follows immediately from its definition that the graph of λ is Borel. By Corollary 5.14 the graph of ν is Borel. By Corollary 9.4 the graphs of ν_f and τ_f are Borel. Since sets definable from $\mathcal{L}_B$-formulas are Borel by Fact 2.1, every $\mathcal{L}_B$-definable function is Borel. Hence all interpretations of function symbols from $\mathcal{L}_C^+$ are Borel.

Again, because sets definable from an $\mathcal{L}_B$-formula or an $\mathcal{L}$-formula are Borel, the interpretation of any predicate symbol from $\mathcal{L}_C^+$ is Borel. Since Borel sets are closed under preimages under Borel functions and under boolean combinations, every set definable by a quantifier-free $\mathcal{L}_C^+$-formula is Borel. By Theorem 7.1 every set definable in $\mathcal{R}_K$ is Borel. □

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