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Combinatorial positivity of translation-invariant valuations and a discrete Hadwiger theorem

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Abstract. We introduce the notion of combinatorial positivity of translation-invariant valuations on convex polytopes that extends the nonnegativity of Ehrhart h^*-vectors. We give a surprisingly simple characterization of combinatorially positive valuations that implies Stanley's nonnegativity and monotonicity of h^*-vectors and generalizes work of Beck et al. (2010) from solid-angle polynomials to all translation-invariant simple valuations. For general polytopes, this yields a new characterization of the volume as the unique combinatorially positive valuation up to scaling. For lattice polytopes our results extend work of Betke–Kneser (1985) and give a discrete Hadwiger theorem: There is essentially a unique combinatorially-positive basis for the space of lattice-invariant valuations. As byproducts, we prove a multivariate Ehrhart–Macdonald reciprocity and we show universality of weight valuations studied in Beck et al. (2010).

Keywords. Ehrhart polynomials, h^*-vectors, combinatorial positivity, translation-invariant valuations, discrete Hadwiger theorem, multivariate reciprocity

1. Introduction

A celebrated result of Ehrhart [15] states that for a convex lattice polytope $P = \operatorname{conv}(V)$, $V \subset \mathbb{Z}^d$, the function $\mathrm{E}_P(n) := |nP \cap \mathbb{Z}^d|$ agrees with a polynomial—the *Ehrhart polynomial* of P. More precisely, there are unique $h_0^*, h_1^*, \dots, h_r^* \in \mathbb{Z}$ with $r = \dim P$ such that

$$\mathrm{E}_P(n) = h_0^* \binom{n+r}{r} + h_1^* \binom{n+r-1}{r} + \cdots + h_r^* \binom{n}{r} \tag{1}$$

for all $n \in \mathbb{Z}_{\geq 0}$. In the language of generating functions this states

$$\sum_{n \geq 0} \mathrm{E}_P(n) z^n = \frac{h_0^* + h_1^* z + \cdots + h_r^* z^r}{(1-z)^{r+1}}.$$

Ehrhart polynomials miraculously occur in many areas such as combinatorics [5, 11, 28], commutative algebra and algebraic geometry [25], and representation theory [6, 12]. The

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question which polynomials can occur as Ehrhart polynomials is well-studied [2, 9, 18, 31] but wide open. Groundbreaking contributions to that question are two theorems of Stanley [29, 30]. Define the *h^*-vector*[1] of P as $h^*(P) := (h_0^*, h_1^*, \ldots, h_d^*)$ where we set $h_i^* = 0$ for $i > \dim P$. Stanley showed that h^*-vectors of lattice polytopes satisfy a nonnegativity and monotonicity property: If $P \subseteq Q$ are lattice polytopes, then

$$0 \leq h_i^*(P) \leq h_i^*(Q)$$

for all $i = 0, \ldots, d$.

McMullen [22] generalized Ehrhart's result to translation-invariant valuations. For now, let $\Lambda \in \{\mathbb{Z}^d, \mathbb{R}^d\}$ and $\mathcal{P}(\Lambda)$ be the collection of polytopes with vertices in Λ. A map $\varphi : \mathcal{P}(\Lambda) \to \mathbb{R}$ is a *translation-invariant valuation* if $\varphi(\emptyset) = 0$ and

$$\varphi(P \cup Q) + \varphi(P \cap Q) = \varphi(P) + \varphi(Q)$$

whenever $P, Q, P \cup Q, P \cap Q \in \mathcal{P}(\Lambda)$, and $\varphi(t + P) = \varphi(P)$ for all $t \in \Lambda$. Valuations are a cornerstone of modern discrete and convex geometry. The study of valuations invariant under the action of a group of transformations is an area of active research with beautiful connections to algebra and combinatorics [20, 23]. For example, for $\Lambda = \mathbb{Z}^d$, the *discrete volume* $\mathrm{E}(P) := |P \cap \Lambda|$ is clearly a translation-invariant valuation.

McMullen showed that for every r-dimensional polytope $P \in \mathcal{P}(\Lambda)$, there are unique $h_0^\varphi, h_1^\varphi, \ldots, h_r^\varphi$ such that

$$\varphi_P(n) := \varphi(nP) = h_0^\varphi \binom{n+r}{r} + h_1^\varphi \binom{n+r-1}{r} + \cdots + h_r^\varphi \binom{n}{r} \tag{2}$$

for all $n \in \mathbb{Z}_{\geq 0}$. Hence, every translation-invariant valuation φ comes with the notion of an h^*-vector $h^\varphi(P) := (h_0^\varphi, h_1^\varphi, \ldots, h_d^\varphi)$ with $h_i^\varphi = 0$ for $i > \dim P$. We call a valuation φ *combinatorially positive* if $h_i^\varphi(P) \geq 0$ and *combinatorially monotone* if $h_i^\varphi(P) \leq h_i^\varphi(Q)$ whenever $P \subseteq Q$. The natural question that motivated the research presented in this paper was

Which valuations are combinatorially positive/monotone?

The Euler characteristic shows that not every translation-invariant valuation is combinatorially positive. Beck, Robins, and Sam [4] showed that *solid-angle* polynomials are combinatorially positive/monotone, and they gave a sufficient condition for combinatorial positivity/monotonicity of general *weight valuations*. Unfortunately, this condition is not correct; see the discussion after Corollary 3.9. We will revisit the construction of weight valuations in Section 2 and show that they are universal for $\Lambda = \mathbb{Z}^d$. Our main result is the following simple complete characterization.

Theorem. *For a translation-invariant valuation $\varphi : \mathcal{P}(\Lambda) \to \mathbb{R}$, the following are equivalent:*

(i) *φ is combinatorially monotone;*
(ii) *φ is combinatorially positive;*

[1] Also called the δ-vector or Ehrhart h-vector.

(iii) *for every simplex* $\Delta \in \mathcal{P}(\Lambda)$,

$$\varphi(\operatorname{relint} \Delta) := \sum_{F} (-1)^{\dim \Delta - \dim F} \varphi(F) \geq 0,$$

where the sum is over all faces $F \subseteq \Delta$.

The combinatorial positivity/monotonicity for the discrete volume (Corollary 3.7) and solid angles (Corollary 3.9) are simple consequences, and we show that Steiner polynomials are not combinatorially positive (Example 3.10). In Section 5, we investigate the relation of combinatorial positivity/monotonicity to the more common notion of nonnegativity and monotonicity of a valuation. In particular, we show that combinatorially positive valuations are necessarily monotone and hence nonnegative. All implications are strict.

Condition (iii) above is linear in φ. Hence, the combinatorially positive valuations constitute a pointed convex cone in the vector space of translation-invariant valuations. In Section 6, we investigate the nested cones of combinatorially positive, monotone, and nonnegative valuations. For $\Lambda = \mathbb{R}^d$, this gives a new characterization of the volume as the unique, up to scaling, combinatorially positive valuation. For $\Lambda = \mathbb{Z}^d$, these cones are more intricate. By results of Betke and Kneser [8], the vector space of valuations on $\mathcal{P}(\mathbb{Z}^d)$ that are invariant under lattice transformations is of dimension $d + 1$. We show that the cone of lattice-invariant valuations that are combinatorially positive is full-dimensional and simplicial.

Hadwiger's characterization theorem [17] states that the coefficients of the Steiner polynomial give a basis for the continuous rigid-motion invariant valuations on convex bodies that can be characterized in terms of homogeneity, nonnegativity, and monotonicity, respectively. Betke and Kneser [8] proved a discrete analog: a homogeneous basis for the vector space of lattice-invariant valuations on $\mathcal{P}(\mathbb{Z}^d)$ is given by the coefficients of the Ehrhart polynomial in the monomial basis. Unfortunately, nonnegativity and monotonicity are genuinely lost. In Section 7 we prove a discrete characterization theorem: Up to scaling there is a unique combinatorially positive basis for lattice-invariant valuations. We close with an explicit descriptions of the three cones of combinatorially positive, monotone, and nonnegative lattice-invariant valuations for $d = 2$.

While Stanley's approach made use of the strong ties between Ehrhart polynomials and commutative algebra, our main tool are *half-open* decompositions introduced by Köppe and Verdolaage [21]. We give a general introduction to translation-invariant valuations in Section 2 and we use half-open decompositions to give a simple proof of McMullen's result (2) in Section 3. As a byproduct, we recover and extend the famous Ehrhart–Macdonald reciprocity to multivariate Ehrhart polynomials in Section 4.

2. Translation-invariant valuations

Let $\Lambda \subset \mathbb{R}^d$ be a lattice (i.e. discrete subgroup) or a finite-dimensional vector subspace over a subfield of $\mathbb{R}$. Following [22], a convex polytope $P \subset \mathbb{R}^d$ with vertices in Λ is called a Λ-*polytope* and we denote all Λ-polytopes by $\mathcal{P}(\Lambda)$. A map $\varphi : \mathcal{P}(\Lambda) \to G$ into some abelian group G is a *valuation* if $\varphi(\emptyset) = 0$ and φ satisfies the valuation property

$$\varphi(P_1 \cup P_2) = \varphi(P_1) + \varphi(P_2) - \varphi(P_1 \cap P_2)$$

for all $P_1, P_2 \in \mathcal{P}(\Lambda)$ with $P_1 \cup P_2, P_1 \cap P_2 \in \mathcal{P}(\Lambda)$. It can be shown that valuations satisfy the more general *inclusion-exclusion property*: For any $P_1, \dots, P_k \in \mathcal{P}(\Lambda)$ such that $P = P_1 \cup \cdots \cup P_k \in \mathcal{P}(\Lambda)$ and $P_I := \bigcap_{i \in I} P_i \in \mathcal{P}(\Lambda)$ for all $I \subseteq [k]$,

$$\varphi(P) = \sum_{\emptyset \neq I \subseteq [k]} (-1)^{|I|-1} \varphi(P_I). \tag{3}$$

For Λ a vector subspace this was first shown by Volland [32]; for the case that Λ is a lattice this is due to Betke (unpublished) in the case of real-valued valuations and by McMullen [24] in general. A valuation $\varphi \colon \mathcal{P}(\Lambda) \to G$ is *translation-invariant* with respect to Λ and called a Λ-*valuation* if $\varphi(t + P) = \varphi(P)$ for all $P \in \mathcal{P}(\Lambda)$ and $t \in \Lambda$. We write $\mathcal{V}(\Lambda, G)$ for the family of Λ-valuations into G.

Many well-known valuations can be obtained as integrals over polytopes such as the d-dimensional *volume* $V(P) = \int_P dx$. The volume is an example of a *homogeneous* valuation, that is, $V(nP) = n^d V(P)$ for all $n \geq 0$. An important valuation that cannot be represented as an integral is the *Euler characteristic* χ defined by $\chi(P) = 1$ for all nonempty polytopes P. The volume and the Euler characteristic are Λ-valuations with respect to any Λ. If Λ is discrete, the *discrete volume* $\mathrm{E}(P) := |P \cap \Lambda|$ is a Λ-valuation.

We mention two particular techniques to manufacture new valuations from old ones. If Λ is a vector space over a subfield of $\mathbb{R}$, then $P \cap Q \in \mathcal{P}(\Lambda)$ whenever $P, Q \in \mathcal{P}(\Lambda)$, i.e. $\mathcal{P}(\Lambda)$ is an intersectional family. For a fixed valuation φ and a polytope $Q \in \mathcal{P}(\Lambda)$, the map

$$\varphi^{\cap Q}(P) := \varphi(P \cap Q)$$

is a valuation. Observe that $\varphi^{\cap Q}$ is not translation-invariant unless $Q = \emptyset$.

The *Minkowski sum* of two $P, Q \in \mathcal{P}(\Lambda)$ is the Λ-polytope

$$P + Q = \{p + q : p \in P, q \in Q\}.$$

For a fixed Λ-polytope Q and valuation φ, we define

$$\varphi^{+Q}(P) := \varphi(P + Q)$$

for $P \in \mathcal{P}(\Lambda)$. That this defines a valuation follows from the fact that

$$\begin{aligned}(K_1 \cup K_2) + K_3 &= (K_1 + K_3) \cup (K_2 + K_3),\\ (K_1 \cap K_2) + K_3 &= (K_1 + K_3) \cap (K_2 + K_3)\end{aligned}$$

for any convex bodies $K_1, K_2, K_3 \subset \mathbb{R}^d$ [27, Section 3.1]. Observe that φ^{+Q} is translation-invariant whenever φ is.

A result that we alluded to in the introduction regards the behavior of Λ-valuations with respect to dilations. It was first shown for the discrete volume by Ehrhart [15] and then for all Λ-valuations by McMullen [22].

Theorem 2.1. *Let $\varphi : \mathcal{P}(\Lambda) \to G$ be a Λ-valuation. Then for every r-dimensional Λ-polytope $P \subset \mathbb{R}^d$ there are unique $h_0^\varphi, h_1^\varphi, \dots, h_r^\varphi \in G$ such that*

$$\varphi_P(n) := \varphi(nP) = h_0^\varphi \binom{n+r}{r} + h_1^\varphi \binom{n+r-1}{r} + \cdots + h_r^\varphi \binom{n}{r}.$$

That is, $\varphi_P(n)$ agrees with a polynomial for all $n \geq 0$. We define the h^**-vector* of φ and P as the vector of coefficients $h^\varphi(P) := (h_0^\varphi, \dots, h_d^\varphi)$ with $h_i^\varphi = 0$ for $i > \dim P$. We will give a simple proof of this result in Section 3 whose inner workings we will need for our main results.

We define the *Steiner valuation* of a polytope $P \subset \mathbb{R}^d$ as

$$S(P) := V^{+B_d}(P) = V(P + B_d).$$

Using Theorem 2.1, we obtain the *Steiner polynomial*

$$S_P(n) := V(nP + B_d) = \sum_{i=0}^{d} \binom{d}{i} W_{d-i}(P) n^i. \tag{4}$$

The coefficient $W_i(P)$, called the i-th *quermassintegral*, is a homogeneous valuation of degree $d - i$ [16, Sect. 6.2]. The Steiner valuation is invariant under rigid motions and so are the quermassintegrals. Hadwiger's characterization theorem [17] states that for any real-valued valuation φ on convex bodies in $\mathbb{R}^d$ that is continuous and invariant under rigid motions, there are unique $\alpha_0, \dots, \alpha_d \in \mathbb{R}$ such that

$$\varphi = \alpha_0 W_0 + \cdots + \alpha_d W_d.$$

Let Λ be a lattice. A less well-known Λ-valuation is the solid-angle valuation. The solid angle of a polytope P at the origin is defined as

$$\omega(P) := \lim_{\varepsilon \to 0} \frac{V(\varepsilon B_d \cap P)}{V(\varepsilon B_d)},$$

where B_d is the unit ball centered at the origin. It is easy to see that ω is a valuation. The *solid-angle valuation* of $P \in \mathcal{P}(\Lambda)$ is defined as

$$A(P) := \sum_{p \in \Lambda} \omega(-p + P)$$

By construction, this is a Λ-valuation and an example of a *simple* valuation: $A(P) = 0$ whenever $\dim P < d$.

Beck, Robins, and Sam [4] considered a class of Λ-valuations that generalize the idea underlying the solid-angle valuation. Slightly rectifying the definitions in [4], a *system of weights* $\boldsymbol{\nu} = (\nu_p)$ is a choice of a valuation $\nu_p : \mathcal{P}(\Lambda) \to G$ for every lattice point $p \in \Lambda$ such that

$$N_{\boldsymbol{\nu}}(P) := \sum_{p \in \Lambda} \nu_p(P)$$

is defined for all $P \in \mathcal{P}(\Lambda)$. Certainly a sufficient condition for this is that ν_p has *bounded support*, i.e. $\nu_p(P) = 0$ whenever $P \cap (R \cdot B_d - p) = \emptyset$ for some $R = R(\nu_p) > 0$. We call $N_{\boldsymbol{\nu}}$ a *weight valuation*. If we choose $\nu_p(P) := \varphi(-p + P)$ for some fixed valuation φ, then $N_{\boldsymbol{\nu}}$ is a Λ-valuation. This generalizes the solid-angle valuation for $\nu_p(P) = \omega(-p + P)$ as well as the discrete volume for $\nu_p(P) = 1$ if and only if $p \in P$. For other valuations it is in general not clear if they can be represented by weight valuations.

Example 2.2 (Euler characteristic). Let $t \in \mathbb{R}^d$ be an irrational vector. For a nonempty lattice polytope $P \in \mathcal{P}(\mathbb{Z}^d)$ there is always a unique vertex $v_t \in Q$ such that $\langle t, x\rangle \le \langle t, v_t\rangle$ for all $x \in Q$. Let ν_p be the function defined by $\nu_p(P) = 1$ if $v_t = p$ and zero otherwise. In particular, $\nu_p(\emptyset) = 0$. It is easy to check that this is a valuation and that N_ν is the Euler characteristic.

Before we ponder the general case, let us consider one more example.

Example 2.3 (Volume). We write $C_d = [0, 1]^d \subset \mathbb{R}^d$ for the standard cube and we define $\nu := V^{\cap C_d}$. The induced weights are then

$$\nu_p(P) = V(P \cap (p + C_d))$$

for $p \in \mathbb{Z}^d$. Since V is a simple valuation, we get

$$N_{\boldsymbol{\nu}}(P) = V\Big(\bigcup\{P \cap (p + C_d) : p \in \mathbb{Z}^d\}\Big) = V(P).$$

The example already hints at the fact that general valuations on rational polytopes can be expressed as weight valuations. The following result is phrased in terms of the standard lattice $\Lambda = \mathbb{Z}^d$, but of course can be adapted to any lattice Λ.

Proposition 2.4. *Let $\varphi : \mathcal{P}(\mathbb{Q}^d) \to G$ be a valuation on rational polytopes. Then there is a system of weights $\boldsymbol{\nu}$ such that $\varphi|_{\mathcal{P}(\mathbb{Z}^d)} = N_{\boldsymbol{\nu}}$.*

Proof. Let $C_d = [0, 1]^d$ be the standard cube and define $F_i := C_d \cap \{x_i = 0\}$ for $i = 1, \ldots, d$. The set $H_d := C_d \setminus (F_1 \cup \cdots \cup F_d) = (0, 1]^d$ is the *half-open* standard cube. It is clear that $\{p + H_d\}_{p\in\mathbb{Z}^d}$ is a partition of $\mathbb{R}^d$. Let us define the valuation

$$\varphi^{\cap H_d} = \sum_{I\subseteq[d]} (-1)^{|I|} \varphi^{\cap F_I},$$

where $F_I := \bigcap\{F_i : i \in I\}$ and $F_\emptyset := C_d$. Then

$$\sum_{p\in\mathbb{Z}^d} \varphi(P \cap (p + H_d)) = \varphi\Big(P \cap \biguplus\{p + H_d : p \in \mathbb{Z}^d\}\Big) = \varphi(P),$$

which proves the claim with $\nu_p(P) = \varphi(P \cap (p + H_d))$. □

Note that this result does not require φ to be invariant with respect to translations. The main result of this section is a representation theorem for $\mathbb{Z}^d$-valuations in terms of weight valuations.

Theorem 2.5. *Let $\varphi : \mathcal{P}(\mathbb{Z}^d) \to G$ be a $\mathbb{Z}^d$-valuation taking values in a divisible abelian group G. Then $\varphi = N_{\boldsymbol{v}}$ for some system of weights $\boldsymbol{v}$.*

This result is a direct consequence of Proposition 2.4 and the following lemma which is of interest in its own right.

Lemma 2.6. *Let $\varphi : \mathcal{P}(\mathbb{Z}^d) \to G$ be a $\mathbb{Z}^d$-valuation taking values in a divisible abelian group. Then there is a valuation $\bar{\varphi} : \mathcal{P}(\mathbb{Q}^d) \to G$ that is invariant under translations by $\mathbb{Z}^d$, and $\bar{\varphi}(P) = \varphi(P)$ for all lattice polytopes $P \in \mathcal{P}(\mathbb{Z}^d)$.*

Proof. Since G is divisible, we can rewrite Theorem 2.1 as

$$\varphi_P(n) = \varphi_d(P)n^d + \cdots + \varphi_0(P)$$

for all $P \in \mathcal{P}(\mathbb{Z}^d)$. The coefficients $\varphi_i(P)$ are $\mathbb{Z}^d$-valuations homogeneous of degree i. It is sufficient to show that we can extend φ_i to rational polytopes.

For $Q \in \mathcal{P}(\mathbb{Q}^d)$, let $\ell \in \mathbb{Z}_{>0}$ be such that $\ell Q \in \mathcal{P}(\mathbb{Z}^d)$. We define

$$\bar{\varphi}_i(Q) := \ell^{-i}\varphi_i(\ell Q).$$

To see that $\bar{\varphi}_i$ is well-defined, observe that $\ell Q \in \mathcal{P}(\mathbb{Z}^d)$ if and only if $\ell = k\ell_0$ where ℓ_0 is the least common multiple of the denominators of the vertex coordinates of Q, and $k \in \mathbb{Z}_{\geq 1}$. Hence, by homogeneity

$$\varphi_i(\ell Q) = k^i\varphi_i(\ell_0 Q).$$

It remains to show that $\bar{\varphi}_i$ satisfies the valuation property. Let Q, Q' be rational polytopes such that $Q \cup Q' \in \mathcal{P}(\mathbb{Q}^d)$. Choose $\ell > 0$ such that ℓQ, $\ell Q'$, $\ell(Q \cup Q')$ and $\ell(Q \cap Q')$ are lattice polytopes. Then

$$\begin{aligned}\ell^i\bar{\varphi}_i(Q \cup Q') &= \varphi_i(\ell(Q \cup Q')) = \varphi_i(\ell Q) + \varphi_i(\ell Q') - \varphi_i(\ell(Q \cap Q'))\\ &= \ell^i\bar{\varphi}_i(Q) + \ell^i\bar{\varphi}_i(Q') - \ell^i\bar{\varphi}_i(Q \cap Q'),\end{aligned}$$

which finishes the proof. □

Note that Lemma 2.6 not necessarily yields the extension one would expect: The discrete volume E clearly extends to rational polytopes. However, the following example shows that this is not the extension furnished by Lemma 2.6.

Example 2.7. Consider the discrete volume E in dimension $d = 1$. For lattice polytopes $P \subset \mathbb{R}$, the polynomial expansion is given by

$$\mathrm{E}_P(n) = V(P)n + \chi(P),$$

where V is the 1-dimensional volume. By Lemma 2.6, there is an extension of E to rational segments and we compute

$$\bar{\mathrm{E}}\left(\left[0, \tfrac{1}{3}\right]\right) = \tfrac{1}{3}V\left(3\left[0, \tfrac{1}{3}\right]\right) + \chi\left(3\left[0, \tfrac{1}{3}\right]\right) = \tfrac{1}{3} + 1 \neq |Q \cap \mathbb{Z}|.$$

Since every abelian group G can be embedded into a divisible group $\overline{G}$, Theorem 2.5 can be extended to abelian groups if we allow the weights to take values in $\overline{G}$. However, the assumption that φ is translation-invariant with respect to $\mathbb{Z}^d$ is necessary for our proof.

Question 1. *Can Lemma 2.6 be extended to general valuations $\varphi : \mathcal{P}(\mathbb{Z}^d) \to G$?*

3. Half-open decompositions and h^*-vectors

For a polytope $P \in \mathcal{P}(\Lambda)$ and a valuation, we defined in the introduction

$$\varphi(\operatorname{relint} P) := \sum_F (-1)^{\dim P - \dim F} \varphi(F), \tag{5}$$

where the sum is over all faces F of P. Using Möbius inversion, this definition is consistent with

$$\varphi(P) = \sum_F \varphi(\operatorname{relint} F). \tag{6}$$

In this section we will extend φ to half-open polytopes, which allows us to use half-open decompositions of polytopes for a proof of Theorem 2.1 that avoids inclusion-exclusion of any sort.

Let $P \subset \mathbb{R}^d$ be a full-dimensional polytope with facets $F_1, \ldots, F_m$. A point $q \in \mathbb{R}^d$ is *general* with respect to P if q is not contained in any facet-defining hyperplane. The point q is *beneath* or *beyond* the facet F_i if q and P are on the same side or, respectively, on different sides of the facet hyperplane $\operatorname{aff}(F_i)$. We write $I_q(P) \subset [m]$ for the set indexing the facets for which q is beyond. Since we assume P to be full-dimensional, we always have $I_q(P) \neq [m]$. A *half-open* polytope is a set of the form

$$\mathrm{H}_q P := P \setminus \bigcup \{F_i : i \in I_q(P)\}.$$

We will write $P^\bullet$ for a half-open polytope $\mathrm{H}_q P$ obtained from P with respect to some general point q.

Our interest in half-open polytopes stems from the following result of Köppe and Verdoolaege [21] that is already implicit in the works of Stanley and Ehrhart (see [28]). A *dissection* of a polytope P is a presentation $P = P_1 \cup \cdots \cup P_k$, where each P_i is a polytope of dimension $\dim P$ and $\dim(P_i \cap P_j) < d$ for all $i \neq j$.

Lemma 3.1 ([21, Thm. 3]). *Let $P = P_1 \cup \cdots \cup P_k$ be a dissection. If q is a point that is general with respect to P_i for all $i = 1, \ldots, k$, then*

$$\mathrm{H}_q P = \mathrm{H}_q P_1 \uplus \cdots \uplus \mathrm{H}_q P_k.$$

For the sake of completeness we include a short proof of this result.

Proof. We only need to show that for every $p \in \mathrm{H}_q P$ there is a unique P_i with $p \in \mathrm{H}_q P_i$. There is a P_i such that for every $\varepsilon > 0$ sufficiently small, the point $p' := p + \varepsilon(q - p)$ is in the interior and p possibly on the boundary. In particular, the segment $[q, p]$ meets P_i in the interior of P_i, which shows that $p \in \mathrm{H}_q P_i$. If $p \in P_j$ for some $j \neq i$, then there is a facet-hyperplane H of P_j through p that separates P_j from p'. This, however, shows that q and P_j are on different sides of H and hence $p \notin \mathrm{H}_q P_j$. ◻

For a valuation φ we define

$$\varphi(\mathrm{H}_q P) := \varphi(P) - \sum_{\emptyset \neq J \subseteq I_q(P)} (-1)^{|J|} \varphi(F_J),$$

where we set $F_J := \bigcap_{i \in J} F_i$. Lemma 3.1 now implies the following.

Corollary 3.2. *Let $P = P_1 \cup \cdots \cup P_k$ be a dissection with $P_1, \ldots, P_k \in \mathcal{P}(\Lambda)$. If φ is a valuation on $\mathcal{P}(\Lambda)$, then for a general $q \in \operatorname{relint} P$,*

$$\varphi(P) = \varphi(\mathrm{H}_q P_1) + \cdots + \varphi(\mathrm{H}_q P_k).$$

It is well-known (see for example [13]) that every (lattice) polytope P can be dissected into (lattice) simplices. Thus, Theorem 2.1 follows from Corollary 3.2 and the following proposition.

Proposition 3.3. *Let $S^\bullet$ be a full-dimensional, half-open Λ-simplex and φ a Λ-valuation. Then the function $\varphi_{S^\bullet}(n) = \varphi(nS^\bullet)$ is a polynomial in n of degree at most d.*

Proof. Let S be the Λ-simplex such that $S^\bullet = \mathrm{H}_q S$ for some general q and set $I = I_q(S)$. Now, S has vertices $v_1, \ldots, v_{d+1}$ and facets $F_1, \ldots, F_{d+1}$ labeled in such a way that $v_i \notin F_i$ for $i \in [d+1]$. An intrinsic description of $S^\bullet$ is given by

$$S^\bullet = \Big\{ \sum_i \lambda_i v_i : \sum_i \lambda_i = 1,\ \lambda_i \geq 0 \text{ for } i \notin I,\ \lambda_i > 0 \text{ for } i \in I \Big\}.$$

Define $\bar{v}_i = (v_i, 1) \in \mathbb{R}^{d+1}$ and consider the half-open polyhedral cone

$$C := \{\mu_1 \bar{v}_1 + \cdots + \mu_{d+1}\bar{v}_{d+1} : \mu_1, \ldots, \mu_{d+1} \geq 0,\ \mu_i > 0 \text{ for } i \in I\}.$$

For $n \geq 0$, the hyperplane $H_n = \{x \in \mathbb{R}^{d+1} : x_{d+1} = n\}$ can be naturally identified with $\mathbb{R}^d$ such that $H_n \cap C = nS^\bullet$, where $0S^\bullet := \emptyset$ unless $I = \emptyset$. Define the (half-open) parallelepiped

$$\Pi := \big\{\mu_1 \bar{v}_1 + \cdots + \mu_{d+1}\bar{v}_{d+1} : 0 \leq \mu_i < 1 \text{ for } i \notin I,\ 0 < \mu_i \leq 1 \text{ for } i \in I\big\}.$$

Then for every $p \in C$ there are unique $\mu_i \in \mathbb{Z}_{\geq 0}$ and $r \in \Pi$ such that $p = \sum_i \mu_i \bar{v}_i + r$. Let us write

$$\Pi_i := \Pi \cap H_i \qquad \text{for } 0 \leq i \leq d. \tag{7}$$

In general, the Π_j are *not* half-open polytopes but *partly open*: they are Λ-polytopes with certain relatively open faces removed. It follows that

$$nS^\bullet = C \cap H_n = \biguplus_{\substack{k,r \geq 0 \\ k+r=n}} \{\bar{v}_{i_1} + \cdots + \bar{v}_{i_k} + \Pi_r : 1 \leq i_1 \leq \cdots \leq i_k \leq d+1\}.$$

This is a partition of $nS^\bullet$ into partly open polytopes. Using the translation-invariance of φ yields

$$\varphi_{S^\bullet}(n) = \varphi(\Pi_0)\binom{n+d}{d} + \varphi(\Pi_1)\binom{n+d-1}{d} + \cdots + \varphi(\Pi_d)\binom{n}{d}, \tag{8}$$

where we have used (6) to compute $\varphi(\Pi_j)$. □

A notion developed in the proof that will be of importance later is the following. For a (half-open) simplex S, we define the j-th (*partly open*) *hypersimplex* $\Pi_j(S)$ through (7). Proposition 3.3 prompts the definition of an h^*-vector for half-open polytopes. The proof of Proposition 3.3 then yields

Corollary 3.4. *If $S^\bullet \subset \mathbb{R}^d$ is a half-open Λ-simplex and φ a Λ-valuation, then*

$$h_j^\varphi(S^\bullet) = \varphi(\Pi_j(S^\bullet)) \quad \text{for all } 0 \le j \le d.$$

The following is an immediate consequence of Corollary 3.2 and Proposition 3.3.

Corollary 3.5. *Let $P \in \mathcal{P}(\Lambda)$ be a polytope and φ a Λ-valuation. Let $P = P_1 \cup \cdots \cup P_k$ be a dissection into Λ-simplices and $q \in \operatorname{relint} P$ a point general with respect to P_i for all $i = 1, \ldots, k$. Then*

$$h^\varphi(P) = h^\varphi(\mathrm{H}_q P_1) + \cdots + h^\varphi(\mathrm{H}_q P_k).$$

3.1. Combinatorial positivity and monotonicity

We now assume that G is an abelian group together with a partial order $\preceq$ compatible with the group structure, that is, $(G, \preceq)$ is a poset such that for all $a, b, c \in G$,

$$a \preceq b \ \Rightarrow\ a + c \preceq b + c.$$

A Λ-valuation $\varphi : \mathcal{P}(\Lambda) \to G$ is called *combinatorially positive* or *h^*-nonnegative* if

$$h_i^\varphi(P) \succeq 0 \quad \text{for all } P \in \mathcal{P}(\Lambda) \text{ and } 0 \le i \le d,$$

and *combinatorially monotone* or *h^*-monotone* if

$$h_i^\varphi(P) \preceq h_i^\varphi(Q) \quad \text{for } P \subseteq Q \text{ in } \mathcal{P}(\Lambda) \text{ and } 0 \le i \le d.$$

Our main theorem from the introduction is a special case of the following.

Theorem 3.6. *For a Λ-valuation $\varphi : \mathcal{P}(\Lambda) \to G$ into a partially ordered abelian group G, the following are equivalent:*

(i) *φ is combinatorially monotone;*
(ii) *φ is combinatorially positive;*
(iii) *$\varphi(\operatorname{relint} \Delta) \succeq 0$ for every Λ-simplex Δ.*

Proof. The implication (i)$\Rightarrow$(ii) simply follows from the fact that $\emptyset$ is trivially a Λ-polytope. Hence, $h_i^\varphi(P) \succeq h_i^\varphi(\emptyset) = 0$ for every $P \in \mathcal{P}(\Lambda)$ and all i.

For (ii)$\Rightarrow$(iii), let Δ be a Λ-simplex of dimension r. Note that the $(r-1)$-th partly open hypersimplex Π_{r-1} of Δ is a translate of $\operatorname{relint}(-\Delta)$. Combinatorial positivity implies that $0 \preceq h_{r-1}^\varphi(-\Delta) = \varphi(\Pi_{r-1}(-\Delta)) = \varphi(\operatorname{relint} \Delta)$.

(iii)$\Rightarrow$(i): Let $P \subseteq Q$ be Λ-polytopes. If $r = \dim P = \dim Q$, let $Q = T_1 \cup \cdots \cup T_N$ be a dissection of Q into r-dimensional Λ-simplices such that $P = T_{M+1} \cup T_{M+2} \cup \cdots \cup T_N$ for some $M < N$. Such a dissection can be constructed by using, for example, the *Beneath-Beyond* algorithm [14, Section 8.4]. For a point $q \in \operatorname{relint} P$ general with respect to all T_i, it follows from Corollary 3.5 that

$$h_i^\varphi(Q) - h_i^\varphi(P) = h_i^\varphi(\mathrm{H}_q T_1) + \cdots + h_i^\varphi(\mathrm{H}_q T_M).$$

Hence, it is sufficient to show

$$h_i^\varphi(S^\bullet) \succeq 0$$

for any *half-open* Λ-simplex $S^\bullet$. For $0 \le i \le \dim S^\bullet$, let $\Pi_i = \Pi_i(S^\bullet)$ be the corresponding i-th hypersimplex and let $\overline{\Pi}_i$ be its closure. Pick a triangulation $\mathcal{T}$ of $\overline{\Pi}_i$ into Λ-simplices. Then $\mathcal{T}' = \{\sigma \in \mathcal{T} : \operatorname{relint}\sigma \subset \Pi_i\}$ is a triangulation of the partly open hypersimplex. From Corollary 3.4 and inclusion-exclusion, we obtain

$$h_i^\varphi(S^\bullet) = \varphi(\Pi_i) = \sum_{\sigma \in \mathcal{T}'} \varphi(\operatorname{relint}\sigma) \succeq 0,$$

which completes the proof for the case $\dim P = \dim Q$.

Let $r := \dim Q - \dim P > 0$. Set $P^0 := P$ and $P^i := \operatorname{conv}(P^{i-1} \cup q_i)$ for $i = 1, \dots, r-1$, where $q_i \in (Q \cap \Lambda) \setminus \operatorname{aff}(P^{i-1})$. This yields a chain of Λ-polytopes

$$P = P^0 \subset P^1 \subset \cdots \subset P^r \subseteq Q$$

with $\dim P^i = \dim P^{i-1} + 1$ for $1 \le i \le r$. So, it remains to prove that $h^\varphi(P) \preceq h^\varphi(Q)$ when Q is a pyramid with base P and apex a. Let $P = P_1 \cup \cdots \cup P_k$ be a dissection of P into Λ-simplices. This induces a dissection of Q with pieces $Q_i = \operatorname{conv}(P_i \cup a)$. A point $q \in \operatorname{relint} Q$ general with respect to all Q_i gives half-open simplices $Q_i^\bullet = \mathrm{H}_q Q_i$ with half-open facets $P_i^\bullet = Q_i^\bullet \cap P_i$. For $0 \le j \le d$, it is easy to see that $\Pi_j(P_i^\bullet) \subseteq \Pi_j(Q_i^\bullet)$ is a (partly open) face. For fixed j we compute, from a triangulation $\mathcal{T}$ of $\Pi_j(Q_i^\bullet)$,

$$h_j^\varphi(Q_i^\bullet) - h_j^\varphi(P_i^\bullet) = \sum \{\varphi(\operatorname{relint}\sigma) : \sigma \in \mathcal{T}, \operatorname{relint}\sigma \subsetneq \Pi_j(P_i^\bullet)\} \succeq 0,$$

and hence

$$h_j^\varphi(Q) - h_j^\varphi(P) = \sum_i (h_j^\varphi(Q_i^\bullet) - h_j^\varphi(P_i^\bullet)) \succeq 0. \qquad \square$$

As a direct consequence we recover Stanley's results regarding the h^*-vector for the discrete volume.

Corollary 3.7. *Let Λ be a lattice. Then the discrete volume $\mathrm{E}(P) = |P \cap \mathbb{Z}^d|$ is an h^*-nonnegative and h^*-monotone valuation.*

Proof. By Theorem 3.6, it suffices to prove that $\mathrm{E}(\operatorname{relint} P) \ge 0$ for all $P \in \mathcal{P}(\mathbb{Z}^d)$. From the definition of $\mathrm{E}(\operatorname{relint} P)$ it follows that $\mathrm{E}(\operatorname{relint} P) = |\mathbb{Z}^d \cap \operatorname{relint} P| \ge 0$. □

Another simple application gives the following.

Corollary 3.8. *A simple Λ-valuation $\varphi : \mathcal{P}(\Lambda) \to G$ is combinatorially positive if and only if $\varphi(P) \succeq 0$ for all $P \in \mathcal{P}(\Lambda)$.*

Proof. For a simple valuation, we observe that

$$\varphi(\operatorname{relint} P) = \sum_F (-1)^{\dim P - \dim F} \varphi(F) = \varphi(P).$$

Theorem 3.6 yields the claim. □

Since the solid-angle valuation is simple, this implies the main results of Beck, Robins, and Sam [4].

Corollary 3.9. *The solid-angle valuation $A(P)$ is h^*-nonnegative and h^*-monotone.*

Beck, Robins, and Sam also give a sufficient condition for the h^*-nonnegativity/-monotonicity of general weight valuations. Theorems 3 and 4 of [4] state that $N_{\boldsymbol{\nu}}$ is h^*-nonnegative and h^*-monotone if and only if $\nu_p(P) \geq 0$ for all $P \in \mathcal{P}(\mathbb{Z}^d)$ and all $p \in \mathbb{Z}^d$. Unfortunately, this condition is not correct, as Example 2.2 shows.

The Steiner valuation S also turns out not to be combinatorially positive/monotone.

Example 3.10. Let $P = [0, \alpha e_1] \subset \mathbb{R}^d$ be a segment of length $\alpha > 0$ in dimension $d > 1$. Then

$$S(\operatorname{relint} P) = V(P + B_d) - V(0 + B_d) - V(\alpha e_1 + B_d) = \alpha V_{d-1}(B_{d-1}) - V_d(B_d) < 0$$

for α sufficiently small.

4. Reciprocity and a multivariate Ehrhart–Macdonald Theorem

A fascinating result in Ehrhart theory and an important tool in geometric and enumerative combinatorics is the reciprocity theorem of Ehrhart and Macdonald.

Theorem 4.1. *Let $P \subset \mathbb{R}^d$ be a lattice polytope and $\mathrm{E}_P(n)$ its Ehrhart polynomial. Then*

$$(-1)^{\dim P}\mathrm{E}(-n) = \mathrm{E}(\operatorname{relint}(nP)) = |\operatorname{relint}(nP) \cap \mathbb{Z}^d|.$$

McMullen [22] generalized this result to all Λ-valuations as follows.

Theorem 4.2. *Let $\varphi : \mathcal{P}(\Lambda) \to G$ be a Λ-valuation and $P \in \mathcal{P}(\Lambda)$. Then*

$$(-1)^{\dim P}\varphi_P(-n) = \varphi(\operatorname{relint}(-nP)).$$

In this section we succumb to the temptation to give a simple proof of Theorem 4.2 using the machinery of half-open decompositions developed in Section 3. As a corollary we obtain McMullen's multivariate version of Theorem 2.1 for Minkowski sums $\varphi(n_1P_1 + \cdots + n_kP_k)$ and, from the perspective of weight valuations, we give a multivariate Ehrhart–Macdonald reciprocity (Theorem 4.8). This section is not necessary for the remainder of the paper and can, if necessary, be skipped.

We start with a generalization of Lemma 3.1. Let $P \subset \mathbb{R}^d$ be a full-dimensional polytope with facets $F_1, \ldots, F_m$. For a general point $q \in \mathbb{R}^d$, we have defined $I_q(P) = \{i \in [m] : q \text{ beyond } F_i\}$, which led us to the definition of half-open polytopes. We now define

$$\mathrm{H}^q P := P \setminus \bigcup\{F_i : i \notin I_q(P)\} = P \setminus \overline{\partial \mathrm{H}_q P}.$$

In a more general setting the relation between $\mathrm{H}_q P$ and $\mathrm{H}^q P$ was studied in [1].

Lemma 4.3. *Let $P = P_1 \cup \cdots \cup P_k$ be a dissection and q general with respect to all P_i. Then*

$$\mathrm{H}^q P = \mathrm{H}^q P_1 \uplus \cdots \uplus \mathrm{H}^q P_k.$$

Proof. For a polytope $P \subset \mathbb{R}^d$, define the homogenization $\widehat{P} := \{(x,t) : t \geq 0, x \in tP\}$. This is a polyhedral cone and P can be identified with $\rho(\widehat{P}) := \{(x,1) \in \widehat{P}\}$. Let $\widehat{q} = \binom{q}{1}$. Then $\mathrm{H}^q P_i = \rho(\mathrm{H}_{-\widehat{q}} \widehat{P}_i)$. Applying Lemma 3.1 with $-\widehat{q}$ to $\widehat{P} = \widehat{P}_1 \cup \cdots \cup \widehat{P}_k$ then proves the claim. □

The following reciprocity is a simple extension of Stanley's result for reciprocal domains [28]. Observe that for $q \in \operatorname{relint} P$ we get $\mathrm{H}^q P = \operatorname{relint} P$, and hence the following theorem subsumes Theorem 4.2.

Theorem 4.4. *Let P be a Λ-polytope and φ be a Λ-valuation. Then*

$$(-1)^{\dim P} \varphi_{\mathrm{H}_q P}(-n) = \varphi(-n\mathrm{H}^q P).$$

Proof. Since $\varphi_{\mathrm{H}_q P}(n) = \varphi_{n\mathrm{H}_q P}(1)$, we only have to prove that $(-1)^{\dim P} \varphi_{\mathrm{H}_q P}(-1) = \varphi(-\mathrm{H}^q P)$. Let us first assume that P is a simplex of dimension d. With the notation from the proof of Proposition 3.3 and (8) we obtain

$$(-1)^{\dim P} \varphi_{\mathrm{H}_q P}(-n) = \varphi(\Pi_0)\binom{n-1}{d} + \varphi(\Pi_1)\binom{n}{d} + \cdots + \varphi(\Pi_d)\binom{n+d-1}{d}, \tag{9}$$

where $\Pi_i = \Pi_i(\mathrm{H}_q P)$ and we have used the identity $(-1)^b\binom{-a+b}{b} = \binom{a-1}{b}$. Thus,

$$(-1)^{\dim P} \varphi_{\mathrm{H}_q P}(-1) = \varphi(\Pi_d) = \varphi(-\mathrm{H}^q P),$$

since Π_d is a translate of $-\mathrm{H}^q P$. Now, let P be an arbitrary Λ-polytope, and let $P = T_1 \cup \cdots \cup T_k$ be a dissection into Λ-simplices. Then

$$\begin{aligned}(-1)^{\dim P} \varphi_{\mathrm{H}_q P}(-1) &= (-1)^{\dim P}\left(\varphi_{\mathrm{H}_q T_1}(-n) + \cdots + \varphi_{\mathrm{H}_q T_k}(-n)\right) \\ &= \varphi(-\mathrm{H}^q T_1) + \cdots + \varphi(-\mathrm{H}^q T_k) = \varphi(-\mathrm{H}^q P)\end{aligned}$$

by Lemma 4.3. □

Corollary 4.5. *Let $N_{\boldsymbol{\nu}}$ be a translation-invariant weight valuation and P be a lattice polytope. Then also $P \mapsto (-1)^{\dim P} N_{\boldsymbol{\nu}}(-\operatorname{relint} P)$ is a weight valuation, and*

$$(-1)^{\dim P} (N_{\boldsymbol{\nu}})_P(-n) = \sum_{p \in \mathbb{Z}^d} \nu_p(\operatorname{relint}(-nP)).$$

4.1. Multivariate Ehrhart–Macdonald reciprocity

A multivariate version of Theorem 2.1 was given by Bernstein [7] for the discrete volume and by McMullen [22] for general Λ-valuations.

Theorem 4.6 ([22, Theorem 6]). *Let $P_1, \ldots, P_k \in \mathcal{P}(\Lambda)$ and let $\varphi : \mathcal{P}(\Lambda) \to G$ be a Λ-valuation. Then the function*

$$\varphi_{P_1,\ldots,P_k}(n_1,\ldots,n_k) = \varphi(n_1P_1 + \cdots + n_kP_k)$$

agrees with a polynomial of total degree at most $\dim P_1 + \cdots + P_k$ *for all* $n_1, \ldots, n_k \geq 0$.

Proof. For $k = 1$, this is just Theorem 2.1. For $k > 1$, consider for fixed P_k the Λ-valuation φ^{+P_k}. By induction, $\varphi_{P_1,\dots,P_{k-1}}(P_k; n_1, \dots, n_{k-1}) := \varphi^{+P_k}(n_1 P_1 + \cdots + n_{k-1}P_{k-1})$ is a polynomial in $n_1, \dots, n_{k-1}$. In particular, the map

$$P_k \mapsto \varphi_{P_1,\dots,P_{k-1}}(P_k) := \varphi_{P_1,\dots,P_{k-1}}(P_k; n_1, \dots, n_{k-1}) \in G[n_1, \dots, n_{k-1}]$$

is a Λ-valuation. Hence, again by Theorem 2.1,

$$(\varphi_{P_1,\dots,P_{k-1}})_{P_k}(n_k) = \varphi(n_1 P_1 + \cdots + n_{k-1}P_{k-1} + n_k P_k) \in G[n_1, \dots, n_{k-1}][n_k]$$

is a multivariate polynomial. The total degree of $\varphi_{P_1,\dots,P_k}(n_1, \dots, n_k)$ is equal to the degree of $\varphi_{P_1,\dots,P_k}(n, n, \dots, n) = \varphi(n(P_1 + \cdots + P_k))$ in n, which, by Theorem 2.1, is $\leq \dim(P_1 + \cdots + P_k)$. □

Specializing Theorem 4.6 to the discrete volume, we find that, for any lattice polytopes $P_1, \dots, P_k \subset \mathbb{R}^d$,

$$\mathrm{E}_{P_1,\dots,P_k}(n_1, \dots, n_k) = |(n_1 P_1 + \cdots + n_k P_k) \cap \mathbb{Z}^d|$$

agrees with a polynomial for all $n_1, \dots, n_k \geq 0$. Using Ehrhart–Macdonald reciprocity (Theorem 4.1), we can interpret $(-1)^r \mathrm{E}_{P_1,\dots,P_k}(-n_1, \dots, -n_k)$ for $n_1, \dots, n_k \geq 0$ as the number of lattice points in the relative interior of $P = n_1 P_1 + \cdots + n_k P_k$ where $r = \dim P$. This raises the natural question if there is a combinatorial interpretation for the evaluation

$$\mathrm{E}_{P_1,\dots,P_k}(-n_1, \dots, -n_l, n_{l+1}, \dots, n_k) \tag{10}$$

for $n_1, \dots, n_k \geq 0$ and $1 < l < k$. The following example shows that there cannot be a straightforward generalization of Theorem 4.1.

Example 4.7. Let $P = [0, 1]^2$ and $Q = [(0, 0), (1, 1)]$. Then

$$\mathrm{E}_{P,Q}(n, m) = (n + 1)^2 + 2nm + m.$$

Therefore

$$\begin{aligned} \mathrm{E}_{P,Q}(-n, m) &< 0 \qquad \text{for } 0 < n \ll m, \\ \mathrm{E}_{P,Q}(-n, m) &> 0 \qquad \text{for } 0 < m \ll n. \end{aligned}$$

However, from the perspective of weight valuations, we can give an interpretation of (10) in terms of the topology of certain polyhedral complexes. We first note that for (10),

$$\mathrm{E}_{P_1,\dots,P_k}(-n_1, \dots, -n_l, n_{l+1}, \dots, n_k) = \mathrm{E}_{P,Q}(-1, 1) = \mathrm{E}_P^{+Q}(-1)$$

where $P := n_1 P_1 + \cdots + n_l P_l$ and $Q = n_{l+1}P_{l+1} + \cdots + n_k P_k$. Hence, it is sufficient to find an interpretation for $\mathrm{E}_{P,Q}(-1, 1)$ for general lattice polytopes P, Q.

For two polytopes $P, Q \subset \mathbb{R}^d$, the *$Q$-complement* is the polyhedral complex

$$\mathcal{C}_Q(P) := \{F \subseteq P \text{ face} : F \cap Q = \emptyset\}.$$

Recall that the *reduced Euler characteristic* of a polyhedral complex $\mathcal{K}$ is defined as $\tilde{\chi}(\mathcal{K}) := \sum\{(-1)^{\dim F} : F \in \mathcal{K}\}$. Here is our generalization of Ehrhart–Macdonald reciprocity to Minkowski sums of lattice polytopes.

Theorem 4.8. *Let $P, Q \subset \mathbb{R}^d$ be nonempty lattice polytopes. Then $P \mapsto \tilde{\chi}(\mathcal{C}_Q(P))$ defines a valuation on $\mathcal{P}(\mathbb{Z}^d)$ and*

$$\mathrm{E}_{P,Q}(-1,1) = -\sum_{p \in \mathbb{Z}^d} \tilde{\chi}(\mathcal{C}_Q(P+p)).$$

Proof. Consider $\varphi := \chi^{\cap(-Q)}$ and define a system of weights $\boldsymbol{\nu}$ by $\nu_p(P) := \varphi(-p+P)$. We have $\nu_p(P) = 1$ if and only if $(-p+P) \cap (-Q) \neq \emptyset$ if and only if $p \in P+Q$. Hence,

$$\mathrm{E}^{+Q}(P) = \sum_{p \in \mathbb{Z}^d} \nu_p(P) = N_{\boldsymbol{\nu}}(P).$$

By Corollary 4.5, we obtain

$$\begin{aligned} \mathrm{E}_P^{+Q}(-1) &= \sum_{p \in \mathbb{Z}^d} (-1)^{\dim(P)} \chi^{\cap(-Q)}(-(p + \operatorname{relint} P)) \\ &= \sum_{p \in \mathbb{Z}^d} \sum \{(-1)^{\dim F} : F \subseteq P \text{ face},\ (F+p) \cap Q \neq \emptyset\} \\ &= -\sum_{p \in \mathbb{Z}^d} \tilde{\chi}(\mathcal{C}_Q(P+p)) \end{aligned}$$

where the last equation follows from the fact that the complex of faces of P has reduced Euler characteristic zero. $\square$

For $Q = \{0\}$, we recover Ehrhart–Macdonald reciprocity: For $p \in \mathbb{Z}^d$, set

$$\mathcal{C}_p := \mathcal{C}_Q(-p+P) = \{F \subseteq P \text{ face} : p \notin F\}.$$

For $p \in \operatorname{relint} P$, $\mathcal{C}_p$ is a sphere of dimension $\dim P - 1$. For $p \notin P$ and $p \in \partial P$, the complex $\mathcal{C}_p$ is a ball and so $\tilde{\chi}(\mathcal{C}_p) = 0$. Hence, Theorem 4.8 yields

$$\mathrm{E}_P(-1) = \sum_{p \in \mathbb{Z}^d \cap \operatorname{relint} P} (-1)^{\dim P} = (-1)^{\dim P} \mathrm{E}(\operatorname{relint} P).$$

One could hope that the Q-complements are combinatorially well-behaved (e.g. shellable, Cohen–Macaulay, Gorenstein, etc.), but it turns out that Q-complements are universal.

Proposition 4.9. *Let $\mathcal{C}$ be a simplicial complex. Then there are lattice polytopes P and Q such that*

$$\mathcal{C} \cong \mathcal{C}_Q(P).$$

Proof. Let $\mathcal{C}$ be a simplicial complex on the vertex set $[m]$. Let $P = \operatorname{conv}(e_1, \ldots, e_m)$ be a lattice $(m-1)$-simplex in $\mathbb{R}^m$. For $I \subseteq [m]$ let

$$w_I := \frac{1}{|I|} \sum_{i \in I} e_i$$

be the barycenter of the face $F_I := \operatorname{conv}(e_i : i \in I) \subseteq P$. Let $Q = \operatorname{conv}(w_I : I \notin \mathcal{C})$. Then $F_I \cap Q = \emptyset$ if and only if $I \notin \mathcal{C}$. Hence, $\mathcal{C}_Q(P)$ is a geometric realization of $\mathcal{C}$. Observing that $m!Q \subseteq m!P$ are lattice polytopes finishes the proof. $\square$

In particular, the weights appearing in Theorem 4.8 can be arbitrary. This, however, does not exclude the possibility that there are combinatorial interpretations of $\mathrm{E}_{P,Q}(m, n)$ for certain regimes $\mathcal{R} \subset \mathbb{Z}^2$, and it would certainly be interesting to find such interpretations.

5. Weak h^*-nonnegativity, monotonicity, and nonnegativity

The Euler characteristic is a simple example of Λ-valuation that is not combinatorially positive. Indeed, for an r-polytope $P \neq \emptyset$ we have

$$h_i^{\chi}(P) = (-1)^i \binom{r}{i}.$$

In this section we consider a weaker notion than h^*-nonnegativity that clarifies the relation of combinatorial positivity/monotonicity to the usual nonnegativity and monotonicity of valuations. A Λ-valuation $\varphi \in \mathcal{V}(\Lambda, G)$ is *weakly combinatorially monotone* or *weakly h^*-monotone* if $\varphi(\{0\}) \succeq 0$ and

$$h_i^{\varphi}(P) \preceq h_i^{\varphi}(Q)$$

for all Λ-polytopes $P \subseteq Q$ such that $\dim P = \dim Q$. Clearly, every combinatorially monotone valuation is also weakly combinatorially monotone. Moreover, the Euler characteristic is weakly h^*-monotone, which also shows that weakly h^*-monotone does not imply h^*-monotone. The main result of this section exactly characterizes the weakly h^*-monotone valuations.

Theorem 5.1. *For a Λ-valuation $\varphi : \mathcal{P}(\Lambda) \to G$ into a partially ordered abelian group G, the following are equivalent:*

(i) *φ is weakly h^*-monotone;*
(ii) *$\varphi(\operatorname{relint} \Delta) + \varphi(\operatorname{relint} F) \succeq 0$ for every Λ-simplex Δ and every facet F of Δ;*
(iii) *$\varphi(S^\bullet) \succeq 0$ for every half-open Λ-simplex $S^\bullet$.*

Proof. (i)$\Rightarrow$(ii): Let $\Delta = \operatorname{conv}(v_0, \ldots, v_r)$ be a Λ-simplex of dimension r. We can assume that $v_0 = 0$. If $r = 0$, then $\varphi(\operatorname{relint} \Delta) \succeq 0$ by definition. For $r > 0$, the truncated pyramid $T = \overline{2\Delta \setminus \Delta}$ is contained in 2Δ and is of dimension r. Since φ is weakly h^*-monotone, we obtain

$$0 \preceq h_r^{\varphi}(-2\Delta) - h_r^{\varphi}(-T) = \varphi(\operatorname{relint}(2\Delta)) - \varphi(\operatorname{relint} T) = \varphi(\operatorname{relint} \Delta) + \varphi(\operatorname{relint} F),$$

where F denotes the facet opposite to $v_0 = 0$.

(ii)$\Rightarrow$(iii): Let $S^\bullet$ be a half-open simplex of dimension r and let $f = f(S^\bullet)$ be the number of facets present in $S^\bullet$. If $f = 1$ or $r = 0$, then $\varphi(S^\bullet) = \varphi(\operatorname{relint} S) + \varphi(\operatorname{relint} F) \succeq 0$ by (ii). For $f > 1$, let $F \subset S^\bullet$ be a half-open facet. Then $T = S^\bullet \setminus F$ is a half-open simplex with $f(T) < f$, and by induction on f and r we get

$$\varphi(S^\bullet) = \varphi(T) + \varphi(F) \succeq 0.$$

(iii)⇒(i): Let $P \subseteq Q$ be two Λ-polytopes with $r - 1 = \dim P = \dim Q$. As in the proof of Theorem 3.6, we can choose a dissection $Q = T_1 \cup \cdots \cup T_N$ of Q into $(r-1)$-dimensional Λ-simplices such that $P = T_{M+1} \cup T_{M+2} \cup \cdots \cup T_N$ for some $M < N$. For a point $q \in \operatorname{relint} P$ general with respect to all T_i, it follows from Corollary 3.5 that

$$h_i^{\varphi}(Q) - h_i^{\varphi}(P) = h_i^{\varphi}(\mathrm{H}_q T_1) + \cdots + h_i^{\varphi}(\mathrm{H}_q T_M).$$

It is thus sufficient to show

$$h_i^{\varphi}(S^{\bullet}) \succeq 0$$

for any *proper half-open* Λ-simplex $S^{\bullet}$, that is, $S^{\bullet} = \mathrm{H}_q S$ for some general $q \notin S$. We will show that the corresponding partly open hypersimplex $\Pi_i = \Pi_i(S^{\bullet})$ can be dissected into half-open simplices. By a change of coordinates, we can assume $S = \{x \in V : x \geq 0\}$, where $V = \{x \in \mathbb{R}^r : x_1 + \cdots + x_r = 1\}$, and

$$S^{\bullet} = \{x \in S : x_j > 0 \text{ for } j \in I\}$$

with $I = I_q(S) \neq \emptyset$. We can also assume that the general point $q \in V$ satisfies $q_j > 1$ for $j \notin I$. The corresponding i-th partly open hypersimplex is

$$\Pi_i = \{x \in i \cdot V : x_j > 0 \text{ for } j \in I, x_j < 1 \text{ for } j \notin I\} = \mathrm{H}_{q'} \overline{\Pi}_i,$$

where $q' = i \cdot q$. Hence, Π_i is a half-open polytope and after choosing a dissection $\overline{\Pi}_i = S_1 \cup \cdots \cup S_l$ into simplices, we obtain from Lemma 3.1

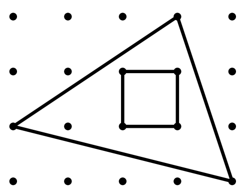
$$\Pi_i = \mathrm{H}_{q'} S_1 \cup \cdots \cup \mathrm{H}_{q'} S_l,$$

and thus

$$\varphi(\Pi_i) = \sum_{l=1}^{k} \varphi(H_{q'}(S_k)) \succeq 0. \qquad \square$$

A Λ-valuation is *monotone* if $\varphi(P) \preceq \varphi(Q)$ for all Λ-polytopes $P \subseteq Q$ and *nonnegative* if $\varphi(P) \succeq 0$ for all $P \in \mathcal{P}(\Lambda)$. Clearly, every monotone valuation is nonnegative but the converse is in general not true as the following example shows.

Example 5.2. For $\Lambda = \mathbb{Z}^2$, define the $\mathbb{Z}^2$-valuation $b(P) := \mathrm{E}(P) - V_2(P) - \chi(P)$. If $\dim P \leq 1$, then $b(P) = V_1(P)$. For $\dim P = 2$, $2b(P) = |\partial P \cap \mathbb{Z}^2|$. This is clearly a nonnegative valuation. But the following figure shows that b is not monotone.



We call a Λ-valuation *weakly monotone* if $\varphi(\{0\}) \succeq 0$ and $\varphi(P) \preceq \varphi(Q)$ for all Λ-polytopes $P \subseteq Q$ with $\dim P = \dim Q$. It turns out that monotonicity and weak monotonicity are in fact equivalent.

Proposition 5.3. *Let φ be a Λ-valuation. Then φ is monotone if and only if φ is weakly monotone.*

Proof. For Λ-polytopes $P \subseteq Q$ we construct a chain of Λ-polytopes

$$P = P_0 \subseteq P_1 \subseteq \cdots \subseteq P_r \subseteq Q,$$

where $P_{i+1} = \operatorname{conv}(P_i \cup q_i)$ for some $q_i \in (Q \cap \Lambda) \setminus \operatorname{aff}(P_i)$ for all $0 \leq i \leq r-1$, and $\dim P_r = \dim Q$. Hence, it suffices to prove that $\varphi(P) \preceq \varphi(Q)$ when Q is a pyramid over P with apex $a = 0$. If $P = \emptyset$, then $Q = \{0\}$ and $\varphi(Q) \succeq 0$ by definition. If $\dim P \geq 0$, then the truncated pyramid $T := 2Q \setminus (Q \setminus P)$ is contained in $2Q$ and is of equal dimension. Therefore

$$0 \preceq \varphi(2Q) - \varphi(T) = \varphi(Q) - \varphi(P). \qquad \square$$

The next result gives us the relation to monotone valuations.

Proposition 5.4. *Let φ be a weakly h^*-monotone Λ-valuation. Then φ is monotone.*

Proof. We have to show that $\varphi(P) \leq \varphi(Q)$ for Λ-polytopes $P \subseteq Q$. By Proposition 5.3 we may assume that $\dim P = \dim Q$. Let $Q = T_1 \cup \cdots \cup T_N$ be a dissection of Q into Λ-simplices such that $P = T_{M+1} \cup T_{M+2} \cup \cdots \cup T_N$ for some $M < N$. For a point $q \in \operatorname{relint} P$ general with respect to all T_i we obtain

$$\varphi(Q) - \varphi(P) = \sum_{i=1}^{M} \varphi(H_q T_i) \succeq 0$$

by Theorem 5.1. $\square$

The converse, however, is not true.

Example 5.5. Let R be the lattice triangle with vertices $a = \binom{0}{0}, b = \binom{2}{0}, c = \binom{2}{1}$. Consider the valuation E^{+Q} where $Q = [(0,0),(1,1)]$. It is easy to see that E^{+Q} is monotone. To see that E^{+Q} is not weakly h^*-monotone, we appeal to Theorem 5.1 and compute, for the facet $F = \operatorname{conv}(b, c)$,

$$\mathrm{E}^{+Q}(\operatorname{relint} R) + \mathrm{E}^{+Q}(\operatorname{relint} F) = (-1) + 0 < 0.$$

We close this section by summarizing the various relationships in the following diagram:

$$\begin{array}{c} h^*\text{-nonnegative} \\ \Updownarrow \\ h^*\text{-monotone} \end{array} \Longrightarrow \begin{array}{c} \text{weakly} \\ h^*\text{-monotone} \end{array} \Longrightarrow \begin{array}{c} \text{monotone} \\ \Updownarrow \\ \text{weakly monotone} \end{array} \Longrightarrow \text{nonnegative}$$

6. Cones of combinatorially positive valuations

Let us assume that G is a finite-dimensional $\mathbb{R}$-vector space. Then

$$\mathcal{V}(\Lambda, G) = \{\varphi : \mathcal{P}(\Lambda) \to G \text{ a } \Lambda\text{-valuation}\}$$

inherits the structure of a real vector space. Let $C \subset G$ be a closed and pointed convex cone. Then we can define a partial order on G by

$$x \preceq_C y \;:\Leftrightarrow\; y - x \in C.$$

This partial order is compatible with the group structure on G and $C = \{x \in G : x \succeq 0\}$. Throughout this section, G will be partially ordered by some C.

We will write $\mathcal{V}_{\mathrm{CP}}(\Lambda, G)$ for the collection of combinatorially positive Λ-valuations $\varphi : \mathcal{P}(\Lambda) \to G$. Observing that condition (iii) in Theorem 3.6 is linear in φ shows that $\mathcal{V}_{\mathrm{CP}}(\Lambda, G)$ has typically a nice structure.

Proposition 6.1. *The set $\mathcal{V}_{\mathrm{CP}}(\Lambda, G)$ is a convex cone.*

In the following sections we will study the geometry of this cone for $\Lambda = \mathbb{R}^d$ and $\Lambda = \mathbb{Z}^d$.

6.1. $\mathbb{R}^d$-valuations

Our main result for $\Lambda = \mathbb{R}^d$ gives a precise description of $\mathcal{V}_{\mathrm{CP}}(\mathbb{R}^d, G)$.

Theorem 6.2. *Let G be a finite-dimensional real vector space partially ordered by a closed and pointed convex cone C. Then*

$$\mathcal{V}_{\mathrm{CP}}(\mathbb{R}^d, G) \cong C.$$

The isomorphism takes c to cV_d.

If $\dim G = 1$ and hence up to isomorphism $G = \mathbb{R}$ with the usual order, we obtain a new characterization of the volume.

Corollary 6.3. *The volume is, up to scaling, the unique real-valued combinatorially positive $\mathbb{R}^d$-valuation.*

As a first step towards the proof of Theorem 6.2, we recall the following result of McMullen.

Theorem 6.4 ([22, Theorem 8]). *Every monotone $\mathbb{R}^d$-valuation $\varphi : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ is continuous with respect to the Hausdorff metric.*

Since every combinatorially positive valuation is monotone (Proposition 5.4), we conclude that the cone $\mathcal{V}_{\mathrm{CP}}(\mathbb{R}^d, G)$ is indeed a closed convex cone. We recall the following well-known result (see, for example, Gruber [16, Chapter 16]).

Lemma 6.5. *If $\varphi : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ is a simple, monotone $\mathbb{R}^d$-valuation, then $\varphi = \lambda V_d$ for some $\lambda \geq 0$.*

Proof of Theorem 6.2. Let φ be a combinatorially positive valuation. We will show that for every linear form $\ell : G \to \mathbb{R}$ that is nonnegative on C, the real-valued $\mathbb{R}^d$-valuation $\ell \circ \varphi$ is a nonnegative multiple of the volume. Since C is pointed, this then proves $\varphi = cV_d$ for $c = \varphi([0, 1]^d) \in C$.

Since $\ell \geq 0$ on C, $\ell \circ \varphi$ is monotone and by Theorem 6.4 continuous in the Hausdorff metric. In light of Lemma 6.5 it thus suffices to prove that φ is simple.

For every polytope $P \in \mathcal{P}(\mathbb{R}^d)$ let $g(P) = (g_0(P), g_1(P), \dots, g_d(P)) \in G^{d+1}$ be such that

$$\sum_{n \geq 0} \varphi(nP)t^n = \frac{g_0(P) + g_1(P)t + \cdots + g_d(P)t^d}{(1-t)^{d+1}}.$$

We denote the numerator polynomial by $g_P(t)$. For all $0 \leq i \leq d$, every g_i is a continuous $\mathbb{R}^d$-valuation. If $\dim P = r$, then

$$g_P(t) = (1-t)^{d-r} \sum_{i=0}^{r} h_i^{\varphi}(P)t^i.$$

In particular, if $\dim P = d$, then $g_i(P) = h_i^{\varphi}(P) \in C$ for all $0 \leq i \leq d$.

Now let Q be of dimension $r < d$. Consider the sequence of polytopes $Q_n = Q + \frac{1}{n}[0,1]^d$. Then $\dim Q_n = d$ for all $n \geq 1$ and $h_i^{\varphi}(Q_n) = g_i(Q_n) \to g_i(Q)$ as $n \to \infty$. Since C is closed, we have $g_i(Q) \in C$ for all i. On the other hand, $(1-t) \mid g_Q(t)$ and therefore $\sum_{i=0}^{d} g_i(Q) = 0$. Since C is pointed, we conclude that $g_i(Q) = 0$ for all i and thus $\varphi(Q) = 0$. □

Using similar techniques, we can describe the cone

$$\mathcal{V}_{\mathrm{WCP}}(\mathbb{R}^d, G) := \{\varphi : \mathcal{P}(\mathbb{R}^d) \to G \text{ weakly } h^*\text{-monotone}\}.$$

Theorem 6.6. *Let G be a finite-dimensional real vector space partially ordered by a closed and pointed convex cone C. Then*

$$\mathcal{V}_{\mathrm{WCP}}(\mathbb{R}^d, G) \cong C \times C.$$

The isomorphism takes (c_1, c_2) to $c_1\chi + c_2 V_d$.

Proof. Proposition 5.4 shows that weakly h^*-monotone implies monotone. It follows that for $c_1 := \varphi(\{0\}) \in C$,

$$\psi := \varphi - c_1\chi$$

is still a weakly h^*-monotone $\mathbb{R}^d$-valuation and, in particular, monotone. Analogous to the proof of Theorem 6.2, we show that ψ is simple and conclude that $\psi = c_2 V_d$ for some $c_2 \in C$.

Let $P \subseteq Q$ be two polytopes of dimension $r < d$. Consider the d-polytopes $P_n := P + \frac{1}{n}[0,1]^d$ and $Q_n := Q + \frac{1}{n}[0,1]^d$. Then $\dim P_n = \dim Q_n = d$ and $P_n \subseteq Q_n$ for all $n \geq 1$. Following the proof of Theorem 6.2, we infer that $g_{Q_n}(t) - g_{P_n}(t)$ has all coefficients in C and

$$g_{Q_n}(t) - g_{P_n}(t) \xrightarrow{n \to \infty} g_Q(t) - g_P(t).$$

However, since $\dim P = \dim Q < d$, we have $g_P(1) - g_Q(1) = 0$. Since C is pointed, this implies that $g_P(t) = g_Q(t)$ and $\psi(P) = \psi(Q)$.

Let us assume that $0 \in P$. Then $P \subseteq nP$ for all $n \geq 1$ and hence $\psi(nP) = c$ for all $n \geq 1$. In particular $\psi(0P) = \psi(\{0\}) = c$, which implies that $\psi(P) = 0$. □

Corollary 6.7. *The Steiner valuation $S(P) = V_d(P + B_d)$ is not weakly h^*-monotone for $d > 1$.*

Proof. The quermassintegrals are linearly independent $\mathbb{R}^d$-valuations with W_0 being the volume and W_d proportional to the Euler characteristic. Hence the representation (4) shows that for $d > 1$, S is not in the cone spanned by χ and V_d. □

It is known (see [16]) that the quermassintegrals are nonnegative and monotone with respect to inclusion. Hence, by Hadwiger's characterization result, the cone of nonnegative and the cone of monotone rigid-motion invariant continuous valuations on convex bodies in $\mathbb{R}^d$ coincide and are isomorphic to $\mathbb{R}^{d+1}_{\geq 0}$. Meanwhile, the corresponding cones of rigid-motion invariant (weakly) h^*-monotone valuations are still given by Theorems 6.2 and 6.6.

6.2. *Lattice-invariant valuations*

Let Λ be a lattice of rank d that, without loss of generality, we can assume to be $\mathbb{Z}^d$. A valuation $\varphi : \mathcal{P}(\mathbb{Z}^d) \to G$ is *lattice-invariant* if $\varphi(T(P)) = \varphi(P)$ for all $P \in \mathcal{P}(\mathbb{Z}^d)$ and every affine map T with $T(\mathbb{Z}^d) = \mathbb{Z}^d$. A fundamental result on the structure of lattice-invariant valuations was obtained by Betke and Kneser [8]. For $0 \leq i \leq d$, we define the *i-th standard simplex* as $\Delta_i := \operatorname{conv}\{0, e_1, \ldots, e_i\}$, where $\{e_1, \ldots, e_d\}$ is a fixed basis for Λ.

Theorem 6.8 (Betke–Kneser [8]). *For every $a_0, a_1, \ldots, a_d \in G$ there is a unique lattice-invariant valuation $\varphi : \mathcal{P}(\mathbb{Z}^d) \to G$ such that*

$$\varphi(\Delta_i) = a_i \quad \textit{for all } 0 \leq i \leq d.$$

In particular, there are lattice-invariant valuations $\varphi_0, \ldots, \varphi_d : \mathcal{P}(\mathbb{Z}^d) \to \mathbb{Z}$ such that $\varphi_j(\Delta_i) = \delta_{ij}$ and every valuation $\varphi : \mathcal{P}(\mathbb{Z}^d) \to G$ admits a unique presentation as

$$\varphi = \varphi(\Delta_0)\varphi_0 + \cdots + \varphi(\Delta_d)\varphi_d. \tag{11}$$

This implies that

$$\overline{\mathcal{V}}(\mathbb{Z}^d, G) := \{\varphi : \mathcal{P}(\mathbb{Z}^d) \to G : \varphi \text{ lattice-invariant}\} \cong G^{d+1}.$$

We assume that G is a real vector space of finite dimension, partially ordered by a closed and pointed convex cone C. In this section we study the cone of combinatorially positive, lattice-invariant valuations,

$$\overline{\mathcal{V}}_{\mathrm{CP}}(\mathbb{Z}^d, G) := \mathcal{V}_{\mathrm{CP}}(\mathbb{Z}^d, G) \cap \overline{\mathcal{V}}(\mathbb{Z}^d, G).$$

In contrast to the case of (rigid-motion invariant) $\mathbb{R}^d$-valuations, this is a proper convex cone.

Proposition 6.9. *The cone $\overline{\mathcal{V}}_{\mathrm{CP}}(\mathbb{Z}^d, G)$ is of full dimension $(d+1) \cdot \dim C$.*

Proof. For $\ell = 1, \ldots, d+1$, define the valuation $\mathrm{E}^\ell(P) := \mathrm{E}(\ell \cdot P)$. Then E^ℓ is lattice-invariant, and

$$\mathrm{E}^\ell(\operatorname{relint} P) = \mathrm{E}(\operatorname{relint}(\ell \cdot P)) \geq 0$$

shows that E^ℓ is combinatorially positive. Moreover, $\mathrm{E}^1, \ldots, \mathrm{E}^{d+1}$ are linearly independent. Indeed, assume that $\alpha_1\mathrm{E}^1 + \cdots + \alpha_{d+1}\mathrm{E}^{d+1} = 0$. We have $\mathrm{E}^\ell(n[0,1]^d) = (\ell n + 1)^d$, and

$$\alpha_1(n+1)^d + \alpha_2(2n+1)^d + \cdots + \alpha_{d+1}((d+1)n+1)^d = 0$$

for all n implies $\alpha_i = 0$ for all i.

Now, let $m = \dim C$ and let $c_1, \ldots, c_m \in C$ be linearly independent. The lattice-invariant valuations $\{c_i E^\ell : 1 \leq i \leq m, 1 \leq \ell \leq d+1\}$ are linearly independent and combinatorially positive, which proves the claim. $\square$

We will give a detailed description of $\overline{\mathcal{V}}_{\mathrm{CP}}(\mathbb{Z}^d, G)$ that complements the Betke–Kneser theorem.

Theorem 6.10. *A lattice-invariant valuation $\varphi : \mathcal{P}(\mathbb{Z}^d) \to G$ is combinatorially positive if and only if $\varphi(\operatorname{relint} \Delta_i) \succeq 0$ for all standard simplices Δ_i, $i = 0, \ldots, d$. In particular,*

$$\overline{\mathcal{V}}_{\mathrm{CP}}(\mathbb{Z}^d, G) \cong C^{d+1}.$$

The theorem is equivalent to

$$\overline{\mathcal{V}}_{\mathrm{CP}}(\mathbb{Z}^d, G) = \{\varphi \in \overline{\mathcal{V}}(\mathbb{Z}^d, G) : \varphi(\operatorname{relint} \Delta_i) \succeq 0 \text{ for all } i = 0, \ldots, d\}. \tag{12}$$

The inclusion '$\subseteq$' follows from Theorem 3.6(iii). To prove the reverse inclusion it is sufficient to show that every lattice-invariant valuation φ is combinatorially positive if $\varphi(\operatorname{relint} \Delta_i) \succeq 0$ for all $i = 0, \ldots, d$. In dimensions $d \leq 2$, this is true since every lattice polytope can be triangulated into unimodular simplices. In dimension $d = 3$, a direct approach uses the classification of empty lattice simplices due to Reznick [26, Corollary 2.7] and induction on the lattice volume similar to Betke–Kneser [8].

Our proof of Theorem 6.10 pursues a different strategy: Since the right-hand side of (12) is a polyhedral cone, it is sufficient to verify it is generated by a set of combinatorially positive valuations. For the case $(G, C) = (\mathbb{R}, \mathbb{R}_{\geq 0})$, such generators will be given in the next section.

7. A discrete Hadwiger theorem

Hadwiger's characterization theorem [17] states that every continuous rigid-motion invariant valuation φ on convex bodies in $\mathbb{R}^d$ is uniquely determined by the evaluations $(\varphi(S_i))_{i=0,\ldots,d}$ where $S_0, \ldots, S_d \subset \mathbb{R}^d$ are arbitrary but fixed convex bodies with $\dim S_r = r$. From this it is easy to deduce that the quermassintegrals W_i, i.e. the coefficients of the Steiner polynomial

$$V(tK + B_d) = \sum_{i=0}^{d} \binom{d}{i} W_{d-i}(K) n^i,$$

are linearly independent and hence span the space of continuous rigid-motion invariant valuations. The quermassintegral W_i is homogeneous of degree $d-i$ and hence up to scaling $W_0, \dots, W_d$ is the unique homogeneous basis for this space.

The Betke–Kneser result (Theorem 6.8) is a natural discrete counterpart: Every lattice-invariant valuation $\varphi : \mathcal{P}(\mathbb{Z}^d) \to G$ is uniquely determined by its values on $d+1$ lattice simplices of different dimensions. A homogeneous basis for the space of lattice-invariant valuations is given by the coefficients of the Ehrhart polynomial

$$\mathrm{E}_P(n) = e_d(P)n^d + \cdots + e_0(P).$$

However, there are many desirable properties of quermassintegrals that the valuations e_i lack. As they are special mixed volumes, the quermassintegrals are nonnegative and monotone. These properties distinguish them from all other bases for the space of rigid-motion invariant valuations: The cones of nonnegative and, equivalently, monotone rigid-motion invariant valuations are spanned by the quermassintegrals. Unfortunately, the valuations e_i are neither monotone nor nonnegative [3, Chapter 3]. This was Stanley's original motivation for the h^*-monotonicity result [30] given in Corollary 3.7. In this section we study a basis for $\overline{\mathcal{V}}(\mathbb{Z}^d, \mathbb{Z})$ that is combinatorially positive and hence by the results of Section 5 also nonnegative and monotone. This yields a discrete Hadwiger Theorem.

In a different binomial basis Ehrhart's result (1) states that

$$\mathrm{E}_P(n) = f_0^*(P)\binom{n-1}{0} + f_1^*(P)\binom{n-1}{1} + \cdots + f_d^*(P)\binom{n-1}{d}. \tag{13}$$

for some $f_i^*(P) \in \mathbb{Z}$. These coefficients take the role of the quermassintegrals for combinatorial positivity.

Theorem 7.1. *Let $\varphi : \mathcal{P}(\mathbb{Z}^d) \to \mathbb{R}$ be a lattice-invariant valuation. Then φ is combinatorially positive if and only if*

$$\varphi = \alpha_0 f_0^* + \alpha_1 f_1^* + \cdots + \alpha_d f_d^* \quad \text{for some } \alpha_0, \dots, \alpha_d \geq 0.$$

Since $\binom{n-1}{0}, \dots, \binom{n-1}{d}$ is a basis for univariate polynomials of degree $\leq d$, the valuations $f_0^*, \dots, f_d^*$ are a basis for $\overline{\mathcal{V}}(\mathbb{Z}^d, \mathbb{R})$. The following lemma gives an explicit expression of φ in terms of this basis.

Lemma 7.2. *For all $i, j = 0, 1, \dots, d$,*

$$f_j^*(\operatorname{relint} \Delta_i) = \delta_{ij}.$$

In particular, for every lattice invariant valuation $\varphi \in \overline{\mathcal{V}}(\mathbb{Z}^d, G)$,

$$\varphi = \varphi(\operatorname{relint} \Delta_0) f_0^* + \varphi(\operatorname{relint} \Delta_1) f_1^* + \cdots + \varphi(\operatorname{relint} \Delta_d) f_d^*.$$

Proof. For the first claim, we simply note that $\mathrm{E}_{\operatorname{relint} \Delta_i}(n) = \binom{n-1}{i}$. For the second claim, observe that if $\varphi(\operatorname{relint} \Delta_i) = a_i$ for all $i = 0, \dots, d$, then (6) together with the fact that every r-face of Δ_i is lattice isomorphic to Δ_r yields

$$\varphi(\Delta_i) = \sum_{r=0}^{i} \binom{i+1}{r+1} a_r.$$

By Theorem 6.8, there is a unique valuation taking these values on standard simplices, and (5) finishes the proof. □

Thus, if φ is combinatorially positive, then $\alpha_i = \varphi(\operatorname{relint} \Delta_i) \geq 0$, which proves necessity in Theorem 7.1. For sufficiency, we need to show that f_j^* is combinatorially positive for all j, that is, $f_j^*(\operatorname{relint} \Delta) \geq 0$ for all lattice simplices Δ.

For a lattice polytope $P \in \mathcal{P}(\mathbb{Z}^d)$, $f^*(P) = (f_0^*(P), \dots, f_d^*(P))$ is called the *f^*-vector*. The f^*-vector was introduced and studied by Breuer [10]. He showed that $f_j^*(\operatorname{relint} P) \geq 0$ and gave an enumerative interpretation for lattice simplices. We deduce the nonnegativity result from more general considerations. For a translation-invariant valuation $\varphi : \mathcal{P}(\Lambda) \to G$, where Λ is not restricted to lattices, we define its *f^*-vector* $f^\varphi = (f_0^\varphi, \dots, f_d^\varphi)$ to be such that for every $P \in \mathcal{P}(\Lambda)$,

$$\varphi_P(n) = \sum_{i=0}^{d} f_i^\varphi(P) \binom{n-1}{i}$$

for all $n \geq 0$. Equivalently,

$$f_i^\varphi(P) := \sum_{k=0}^{i} \binom{i}{k} (-1)^{i-k} \varphi((k+1)P).$$

Notice the f_i^φ are translation-invariant Λ-valuations.

Theorem 7.3. *Let $\Lambda \subset \mathbb{R}^d$ be a lattice or a finite-dimensional vector subspace over a subfield of $\mathbb{R}$, and G a partially ordered abelian group. For a Λ-valuation $\varphi : \mathcal{P}(\Lambda) \to G$ the following are equivalent:*

(i) *φ is combinatorially positive;*
(ii) *f_i^φ is combinatorially positive for all $i = 0, \dots d$.*

Proof. For the implication (ii)⇒(i) simply observe that

$$\varphi(\operatorname{relint} P) = \varphi_{\operatorname{relint} \Delta}(1) = f_0^\varphi(\operatorname{relint} P) \geq 0$$

for all $P \in \mathcal{P}(\Lambda)$. The claim now follows from Theorem 3.6.

For (i)⇒(ii), we claim that

$$f_{r-k}^\varphi(\operatorname{relint}(-P)) = \sum_{i=k}^{r} h_i^\varphi(P) \binom{i}{k}$$

for any r-dimensional Λ-polytope P. Assuming that φ is h^*-nonnegative then shows combinatorial positivity of f_i^φ. To prove the claim, we use Theorem 4.2 together with the identity $(-1)^r\binom{-n+r-k}{r} = \binom{n-1+k}{r}$ to get

$$\varphi_{\mathrm{relint}(-P)}(n)=(-1)^r\varphi_P(-n)=h_0^\varphi(P)\binom{n-1}{r}+h_1^\varphi(P)\binom{n}{r}+\cdots+h_r^\varphi(P)\binom{n-1+r}{r},$$

and collecting terms completes the proof. □

To complete the proof of Theorem 7.1, we use Stanley's nonnegativity of the h^*-vector (Corollary 3.7) together with Theorem 7.3. The same reasoning also yields a proof of Theorem 6.10.

Proof of Theorem 6.10. The map $\Psi : \overline{\mathcal{V}}(\mathbb{Z}^d, G) \to G^{d+1}$ given by

$$\varphi \mapsto (\varphi(\mathrm{relint}\,\Delta_i))_{i=0,\ldots,d}$$

is an isomorphism by Lemma 7.2. In particular Ψ takes $\overline{\mathcal{V}}_{\mathrm{CP}}(\mathbb{Z}^d, G)$ into C^{d+1}. To show that Ψ is a surjection, we use Theorem 7.3 to infer that for every $a = (a_0, \ldots, a_d) \in C^{d+1}$ the valuation $\varphi = a_0 f_0^* + \cdots + a_d f_d^*$ is combinatorially positive with $\Psi(\varphi) = a$. □

It turns out that there is also a Hadwiger-type result for weakly h^*-monotone valuations. For this consider the Ehrhart polynomial in the basis

$$E_P(n) = \widetilde{f}_0^*(P)\binom{n}{0} + \cdots + \widetilde{f}_0^*(P)\binom{n}{d}. \tag{14}$$

Theorem 7.4. *A lattice-invariant valuation $\varphi : \mathcal{P}(\mathbb{Z}^d) \to \mathbb{R}$ is weakly h^*-monotone if and only if*

$$\varphi = \alpha_0\widetilde{f}_0^* + \alpha_1\widetilde{f}_1^* + \cdots + \alpha_d\widetilde{f}_d^* \quad \textit{for some } \alpha_0, \ldots, \alpha_d \geq 0.$$

As for the proof of Theorem 7.1, the crucial observation is that φ is weakly h^*-monotone if and only if an analogous extension $\widetilde{f}_i^\varphi$ is weakly h^*-monotone for all i. Necessity follows from the proof of Theorem 5.1 where it is shown that if φ is weakly h^*-monotone then $h_i^*(S^\bullet) \succeq 0$ for all proper half-open simplices $S^\bullet$.

7.1. Dimension $d = 2$

In this section we study in detail the cone $\overline{\mathcal{V}}_{\mathrm{CP}}(\mathbb{Z}^2, \mathbb{R})$ in relation to the cones

$$\overline{\mathcal{V}}_{\mathrm{M}}(\mathbb{Z}^2, \mathbb{R}) := \{\varphi \in \overline{\mathcal{V}}(\mathbb{Z}^2, \mathbb{R}) : \varphi(P) \geq \varphi(Q) \text{ for lattice polytopes } Q \subseteq P\},$$
$$\overline{\mathcal{V}}_{+}(\mathbb{Z}^2, \mathbb{R}) := \{\varphi \in \overline{\mathcal{V}}(\mathbb{Z}^2, \mathbb{R}) : \varphi(P) \geq 0 \text{ for } P \in \mathcal{P}(\mathbb{Z}^2)\}.$$

The results of Section 5 imply

$$\overline{\mathcal{V}}_{\mathrm{CP}}(\mathbb{Z}^2, \mathbb{R}) \subsetneq \overline{\mathcal{V}}_{\mathrm{M}}(\mathbb{Z}^2, \mathbb{R}) \subsetneq \overline{\mathcal{V}}_{+}(\mathbb{Z}^2, \mathbb{R}).$$

We study these cones in the usual monomial basis. From Pick's theorem [3, Theorem 2.8] the Ehrhart polynomial of a lattice polytope can be expressed as

$$E_P(n) = V_2(P)n^2 + b(P)n + \chi(P),$$

where $b(P)$ was introduced in Example 5.2. In particular, the coefficients V_2, b, χ are lattice-invariant, nonnegative and homogeneous of degrees 2, 1, 0, respectively.

Proposition 7.5. *The cone $\overline{\mathcal{V}}_+$ is the simplicial cone generated by V_2, b and χ.*

From Theorem 6.10 we know that $\overline{\mathcal{V}}_{\mathrm{CP}}$ is simplicial and generated by

$$\mathrm{E} = V_2 + b + \chi, \quad V_2, \quad 3V_2 + b.$$

Determining the cone of monotone valuations is harder since b, as opposed to V_2 and χ, is *not* monotone (see Example 5.2).

Theorem 7.6. *The cone $\overline{\mathcal{V}}_{\mathrm{M}}$ is simplicial and generated by*

$$\chi, \quad b + V_2, \quad V_2.$$

Proof. First, since $b + V_2 = \mathrm{E} - \chi$, the given valuations are indeed monotone.

Now, let $\varphi = \alpha V_2 + \beta b + \gamma\chi$ be a monotone translation-invariant valuation. Since φ is monotone, we have $\alpha, \beta, \gamma \geq 0$. We can assume that $\gamma = 0$ as $\varphi - \varphi(0)$ is still monotone. Let $Q_n = [0, n]^2$ be the n-th dilated unit square and set $P_n = \mathrm{conv}(Q_n \cup \{(-1, -1)\})$. Then

$$\varphi(Q_n) = \alpha n^2 + 2\beta n, \qquad \varphi(P_n) = \alpha(n^2 + n) + \beta(n + 1).$$

By monotonicity, we obtain

$$0 \leq \varphi(P_n) - \varphi(Q_n) = (\alpha - \beta)n + \beta$$

for all $n \geq 0$, and thus $\alpha \geq \beta$. The cone generated by the inequalities $\alpha \geq 0$, $\gamma \geq 0$ and $\alpha \geq \beta$ is generated by the rays V_2, $V_2 + b$, and χ. □

In the space $\overline{\mathcal{V}}(\mathbb{Z}^2, \mathbb{R}) = \{\alpha V_2 + \beta b + \gamma\chi : \alpha, \beta, \gamma \in \mathbb{R}\}$, a cross-section of the cones with $\{\alpha + \beta + \gamma = 1\}$ is given in Figure 1.

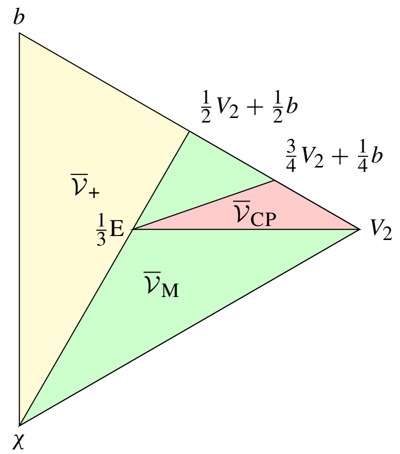


Fig. 1. Cross-section of the nested cones $\overline{\mathcal{V}}_{\mathrm{CP}} \subset \overline{\mathcal{V}}_{\mathrm{M}} \subset \overline{\mathcal{V}}_+$ for $\Lambda = \mathbb{Z}^2$.

It would be very interesting to see if a Hadwiger-type result can be given for monotone or nonnegative valuations. In the language of cones, we conjecture the following.

Conjecture 1. *The cones of lattice-invariant valuations $\varphi : \mathcal{P}(\mathbb{Z}^d) \to \mathbb{R}$ that are monotone or respectively nonnegative are simplicial.*

In dimension $d = 2$, it can also be observed that the cone of lattice-invariant monotone valuations coincides with the cone of weakly h^*-monotone valuations. Example 5.5 shows that this is not true without the restriction to *lattice-invariant* valuations. We do not believe that these cones coincide in general. However, we currently do not have a counterexample.

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