J. Eur. Math. Soc. 20, 2209–2238

DOI 10.4171/JEMS/810

Thomas Dreyfus · Charlotte Hardouin · Julien Roques

Hypertranscendence of solutions of Mahler equations

Received July 9, 2015

Abstract. The last years have seen a growing interest from mathematicians in Mahler functions. This class of functions includes the generating series of the automatic sequences. The present paper is concerned with the following problem, which is rather frequently encountered in combinatorics: a set of Mahler functions $u_1, \dots, u_n$ being given, are $u_1, \dots, u_n$ and their successive derivatives algebraically independent? In this paper, we give general criteria ensuring an affirmative answer to this question. We apply our main results to the generating series attached to the so-called Baum–Sweet and Rudin–Shapiro automatic sequences. In particular, we show that these series are hyperalgebraically independent, i.e., these series and their successive derivatives are algebraically independent. Our approach relies on parametrized difference Galois theory (in this context, the algebro-differential relations between the solutions of a given Mahler equation are reflected by a linear differential algebraic group).

Keywords. Mahler functions, automatic sequences, difference Galois theory, parametrized difference Galois theory

Contents

T. Dreyfus: Institut de Recherche Mathématique Avancée, UMR 7501, Université de Strasbourg et CNRS, 7, rue René Descartes, 67084 Strasbourg, France; e-mail: dreyfus@math.unistra.fr

C. Hardouin: Université Paul Sabatier – Institut de Mathématiques de Toulouse, 118 route de Narbonne, 31062 Toulouse, France; e-mail: hardouin@math.univ-toulouse.fr

J. Roques: Université Grenoble Alpes, Institut Fourier, CNRS UMR 5582, 100 rue des Maths, BP 74, 38402 St Martin d'Hères, France; e-mail: Julien.Roques@ujf-grenoble.fr

Mathematics Subject Classification (2010): 39A06, 12H10

Introduction

This paper grew out of an attempt to understand the algebraic relations between classical Mahler functions and their successive derivatives. By *Mahler function*, we mean a function $f(z)$ such that

$$a_n(z)f(z^{p^n}) + a_{n-1}(z)f(z^{p^{n-1}}) + \cdots + a_0(z)f(z) = 0 \tag{1}$$

for some integers $p \geq 2$, $n \geq 1$, and some $a_0(z), \dots, a_n(z) \in \mathbb{C}(z)$ with $a_0(z)a_n(z) \neq 0$.

The study of this class of functions was originally motivated by the work of Mahler [Mah29, Mah30a, Mah30b] about the algebraic relations between special values at algebraic points of Mahler functions. This arithmetic aspect of the theory of the Mahler functions was developed further by several authors, e.g., Becker, Kubota, Loxton, van der Poorten, Masser, Nishioka, Töfer. We refer to Nishioka's book [Nis96] and Pellarin's paper [Pel09] for more information and references. We just mention that, quite recently, Philippon [Phi15] proved a refinement of Nishioka's analogue of the Siegel–Shidlovski theorem, in the spirit of Beukers' refinement of the Siegel–Shidlovski theorem [Beu06]. See also [AF16, AF17]. Roughly speaking, it says that the algebraic relations over $\overline{\mathbb{Q}}$ between the above-mentioned special values come from algebraic relations over $\overline{\mathbb{Q}}(z)$ between the functions themselves. These functional relations are at the heart of the present paper.

The renewed attractiveness of the theory of Mahler functions comes (to a large extent) from its close connection with automata theory: the generating series $f(z) = \sum_{k\geq 0} s_k z^k$ of any p-automatic sequence $(s_k)_{k\geq 0} \in \overline{\mathbb{Q}}^{\mathbb{N}}$ (and, actually, of any p-regular sequence) is a Mahler function: see Mendès France [MF80], Randé [Ran92], Dumas [Dum93], Becker [Bec94], Adamczewski and Bell [AB13], and the references therein. The famous examples are the generating series of the Thue–Morse, paper-folding, Baum–Sweet and Rudin–Shapiro sequences (see Allouche and Shallit's book [AS03]).

The Mahler functions also appear in many other circumstances, such as the combinatorics of partitions, the enumeration of words and the analysis of algorithms of divide and conquer type; see for instance [DF96] and the references therein.

It is a classical problem (in combinatorics in particular) to determine whether or not a given generating series is transcendental or even hypertranscendental over $\mathbb{C}(z)$.[1]

The hypertranscendence over $\mathbb{C}(z)$ of Mahler functions solutions of inhomogeneous Mahler equations of order one can be studied by using the work of Nishioka [Nis96]; see

[1] We say that a series $f(z) \in \mathbb{C}((z))$ is *hypertranscendental* over $\mathbb{C}(z)$ if $f(z)$ and all its derivatives are algebraically independent over $\mathbb{C}(z)$.

also the work of Nguyen [Ngu11, Ngu12] via difference Galois theory. This can be applied to the paper-folding generating series for instance. Actually, Randé [Ran92] already studied the functions $f(z)$ meromorphic on the unit disc $D(0,1) \subset \mathbb{C}$ which are solutions of some inhomogeneous Mahler equation of order one with coefficients in $\mathbb{C}(z)$; he proved that if $f(z)$ is hyperalgebraic over $\mathbb{C}(z)$, then $f(z) \in \mathbb{C}(z)$ (see [Ran92, Chapitre 5, Théorème 5.2]).

The present work started with the observation that, besides this case, very few things are known. For instance, the hypertranscendence of the Baum–Sweet or Rudin–Shapiro generating series was not known. The main objective of the present work is to develop an approach, as systematic as possible, to proving the hypertranscendence of such series.

To give an idea of the contents of this paper, we mention the following result (see Theorem 4.2), which is a consequence of one of our main hypertranscendence criteria. We consider the field $\mathbf{K} = \bigcup_{j \geq 1} \mathbb{C}(z^{1/j})$ endowed with the field automorphism ϕ given by $\phi(f(z)) = f(z^p)$. In this way we obtain a difference field with field of constants $\mathbf{K}^\phi = \mathbb{C}$, and we have at our disposal a difference Galois theory over $\mathbf{K}$ (see Section 1.1).

Theorem. *Assume that the difference Galois group over* $\mathbf{K}$ *of the Mahler equation* (1) *contains* $\mathrm{SL}_n(\mathbb{C})$ *and that* $a_n(z)/a_0(z)$ *is a monomial. Let* $f(z) \in \mathbb{C}((z))$ *be a nonzero solution of* (1)*. Then the series* $f(z), f(z^p), \ldots, f(z^{p^{n-1}})$ *and all their successive derivatives are algebraically independent over* $\mathbb{C}(z)$*. In particular,* $f(z)$ *is hypertranscendental over* $\mathbb{C}(z)$*.*

The hypothesis that $a_n(z)/a_0(z)$ is a monomial is satisfied in any of the above-mentioned cases. Moreover, for $n = 2$, there is an algorithm to determine whether or not the difference Galois group over $\mathbf{K}$ of (1) contains $\mathrm{SL}_2(\mathbb{C})$ [Roq15]. It turns out that the difference Galois groups involved in the Baum–Sweet and Rudin–Shapiro cases both contain $\mathrm{SL}_2(\mathbb{C})$ [Roq15, Section 9]. Therefore, we have the following consequences of the above theorem (see Theorems 4.3 and 4.4), with $f_{\mathrm{BS}}(z)$ and $f_{\mathrm{RS}}(z)$ denoting the generating series of the Baum–Sweet and Rudin–Shapiro sequences.

Corollary. *The series* $f_{\mathrm{BS}}(z)$, $f_{\mathrm{BS}}(z^2)$ *and all their successive derivatives are algebraically independent over* $\mathbb{C}(z)$*. In particular,* $f_{\mathrm{BS}}(z)$ *is hypertranscendental over* $\mathbb{C}(z)$*.*

Corollary. *The series* $f_{\mathrm{RS}}(z)$, $f_{\mathrm{RS}}(-z)$ *and all their successive derivatives are algebraically independent over* $\mathbb{C}(z)$*. In particular,* $f_{\mathrm{RS}}(z)$ *is hypertranscendental over* $\mathbb{C}(z)$*.*

Actually, our methods also allow one to study the relations between these series. We prove the following result (see Theorem 4.6).

Corollary. *The series* $f_{\mathrm{BS}}(z)$, $f_{\mathrm{BS}}(z^2)$, $f_{\mathrm{RS}}(z)$, $f_{\mathrm{RS}}(-z)$ *and all their successive derivatives are algebraically independent over* $\mathbb{C}(z)$*.*

We shall now say a few words about the proofs of these results. Our approach relies on the parametrized difference Galois theory developed by Hardouin and Singer [HS08]. Roughly speaking, to the difference equation (1), they attach a linear *differential* algebraic group over a differential closure $\widetilde{\mathbb{C}}$ of $\mathbb{C}$—called the parametrized difference Galois group—which reflects the algebro-differential relations between the solutions of the equation. The above theorem is actually a consequence of the following purely Galois theoretic statement (see Section 3.2 for more general results).

Theorem. *Assume that the difference Galois group over* $\mathbf{K}$ *of the Mahler equation* (1) *contains* $\mathrm{SL}_n(\mathbb{C})$ *and* $a_n(z)/a_0(z)$ *is a monomial. Then the parametrized difference Galois group of equation* (1) *is between* $\mathrm{SL}_n(\widetilde{\mathbb{C}})$ *and* $\mathbb{C}^\times \mathrm{SL}_n(\widetilde{\mathbb{C}})$.

Roughly speaking, the fact that the parametrized difference Galois group of equation (1) contains $\mathrm{SL}_n(\widetilde{\mathbb{C}})$ says that the algebro-differential relations between the elements of a basis $f_1, \dots, f_n$ of solutions (in a suitable sense) of (1) are generated by the relations satisfied by the determinant of the associated Wronskian matrix $(f_j(z^{p^{i-1}}))_{1\leq i,j\leq n}$. In particular, there is no nontrivial algebro-differential relation between the entries of a given column of this matrix, and this is exactly the conclusion of the first theorem stated in this introduction (with $f_1 = f$).

Note that, in order to use the parametrized difference Galois theory developed by Hardouin and Singer, one cannot work with the base field $\mathbf{K}$ endowed with the automorphism ϕ and the usual derivation d/dz because ϕ and d/dz do not commute. To solve this problem, Michael Singer uses, in an unpublished proof[2] of the fact that the Mahler function $\sum_{n\geq 0} z^{p^n}$ is hypertranscendental, the field $\mathbf{K}(\log(z))$ and the derivation $z\log(z)d/dz$. We follow this approach in the present paper. This idea also appears in Randé's [Ran92], but in a slightly different form. Indeed, Randé uses the change of variable $z = \exp(t)$ to transform the Mahler difference operator $z \mapsto z^p$ into the p-difference operator $t \mapsto pt$. Pulling back the usual Euler derivation td/dt to the z variable, we find the derivation $z\log(z)d/dz$. Note that Lemma 2.3 and Proposition 2.6 below are also due to Michael Singer and appear in the above mentioned unpublished manuscript.

This paper is organized as follows. Section 1 contains reminders and complements on difference Galois theory. Section 2 starts with reminders and complements on parametrized difference Galois theory. Then, we state and prove user-friendly hypertranscendence criteria for general difference equations of order one. We finish the section with complements on (projective) isomonodromy for general difference equations from a Galoisian point of view. In Section 3, we first study hypertranscendence of solutions of Mahler equations of order 1. We then come to higher order equations and give our main hypertranscendence criteria for Mahler equations. Section 4 provides user-friendly hypertranscendence criteria and is mainly devoted to applications of our main results to the generating series of classical automatic sequences.

General conventions. All rings considered are commutative with identity and contain the field of rational numbers. In particular, all fields are of characteristic zero.

1. Mahler equations and difference Galois theory

1.1. Difference Galois theory

For details on what follows, we refer to [vdPS97, Chapter 1].

[2] Letter from Michael Singer to the second author (February 25, 2010).

A ϕ*-ring* (R, ϕ) is a ring R together with a ring automorphism $\phi : R \to R$. An ideal of R stabilized by ϕ is called a ϕ*-ideal* of (R, ϕ). If R is a field, then (R, ϕ) is called a ϕ*-field*. To simplify the notation, most of the time we will write R instead of (R, ϕ).

The ring of constants of the ϕ-ring R is defined by

$$R^{\phi} := \{f \in R \mid \phi(f) = f\}.$$

If R^{ϕ} is a field, it is called the *field of constants*.

A ϕ*-morphism* (resp. ϕ*-isomorphism*) from the ϕ-ring (R, ϕ) to the $\widetilde{\phi}$-ring $(\widetilde{R}, \widetilde{\phi})$ is a ring morphism (resp. ring isomorphism) $\varphi : R \to \widetilde{R}$ such that $\varphi \circ \phi = \widetilde{\phi} \circ \varphi$.

Given a ϕ-ring (R, ϕ), a $\widetilde{\phi}$-ring $(\widetilde{R}, \widetilde{\phi})$ is an R*-*ϕ*-algebra* if $\widetilde{R}$ is a ring extension of R and $\widetilde{\phi}_{|R} = \phi$; in this case, we will often denote $\widetilde{\phi}$ by ϕ. Two R-ϕ-algebras $(\widetilde{R}_1, \widetilde{\phi}_1)$ and $(\widetilde{R}_2, \widetilde{\phi}_2)$ are isomorphic if there exists a ϕ-isomorphism φ from $(\widetilde{R}_1, \widetilde{\phi}_1)$ to $(\widetilde{R}_2, \widetilde{\phi}_2)$ such that $\varphi_{|R} = \mathrm{Id}_R$.

We fix a ϕ-field $\mathbf{K}$ such that $\mathbf{k} := \mathbf{K}^{\phi}$ is algebraically closed. We consider the linear difference system

$$\phi(Y) = AY \quad \text{with } A \in \mathrm{GL}_n(\mathbf{K}), n \in \mathbb{N}^*. \tag{2}$$

By [vdPS97, §1.1], there exists a $\mathbf{K}$-ϕ-algebra R such that

- there exists $U \in \mathrm{GL}_n(R)$ such that $\phi(U) = AU$ (such a U is called a *fundamental matrix* of solutions of (2));
- R is generated, as a $\mathbf{K}$-algebra, by the entries of U and $\det(U)^{-1}$;
- the only ϕ-ideals of R are $\{0\}$ and R.

Such an R is called a *Picard–Vessiot ring*, or PV ring for short, for (2) over $\mathbf{K}$. By [vdPS97, Lemma 1.8], we have $R^{\phi} = \mathbf{k}$. Two PV rings are isomorphic as $\mathbf{K}$-ϕ-algebras. A PV ring R is not always an integral domain. However, there exist idempotent elements $e_1, \ldots, e_s$ of R such that $R = R_1 \oplus \cdots \oplus R_s$ where the $R_i := Re_i$ are integral domains which are transitively permuted by ϕ. In particular, R has no nilpotent element and one can consider its *total ring of quotients* $\mathcal{Q}_R$, i.e., the localization of R with respect to the set of its nonzero divisors, which can be decomposed as the direct sum $\mathcal{Q}_R = K_1 \oplus \cdots \oplus K_s$ of the fields of fractions K_i of the R_i. The ring $\mathcal{Q}_R$ has a natural structure of R-ϕ-algebra and we have $\mathcal{Q}_R{}^{\phi} = \mathbf{k}$. Moreover, the K_i are transitively permuted by ϕ. We call the ϕ-ring $\mathcal{Q}_R$ a *total PV ring* for (2) over $\mathbf{K}$.

The following lemma gives a characterization of PV rings.

Lemma 1.1 ([HS08, Proposition 6.17]). *Let S be a $\mathbf{K}$-ϕ-algebra with no nilpotent element and let $\mathcal{Q}_S$ be its total ring of quotients. If*

(1) *there exists $V \in \mathrm{GL}_n(S)$ such that $\phi(V)V^{-1} = B \in \mathrm{GL}_n(\mathbf{K})$ and S is generated, as a $\mathbf{K}$-algebra, by the entries of V and by $\det(V)^{-1}$,*
(2) *$\mathcal{Q}_S{}^{\phi} = \mathbf{k}$,*

then S is a PV ring for the difference system $\phi(Y) = BY$ over $\mathbf{K}$.

As a corollary, we find

Lemma 1.2. *Let R be a PV ring over $\mathbf{K}$ and let S be a $\mathbf{K}$-ϕ-subalgebra of R. If there exists $V \in \mathrm{GL}_n(S)$ such that $\phi(V)V^{-1} = B \in \mathrm{GL}_n(\mathbf{K})$ and S is generated, as a $\mathbf{K}$-algebra, by the entries of V and by $\det(V)^{-1}$ then S is a PV ring for $\phi(Y) = BY$ over $\mathbf{K}$.*

Proof. Since R has no nilpotent element, S has no nilpotent element. By [HS08, Corollary 6.9], the total ring of quotients $\mathcal{Q}_S$ can be embedded into $\mathcal{Q}_R$. Since $\mathcal{Q}_R{}^{\phi} = \mathbf{k}$, we have $\mathcal{Q}_S{}^{\phi} = \mathbf{k}$. Lemma 1.1 yields the desired result. □

The *difference Galois group* $\mathrm{Gal}(\mathcal{Q}_R/\mathbf{K})$ of R over $\mathbf{K}$ is the group of $\mathbf{K}$-ϕ-automorphisms of $\mathcal{Q}_R$ commuting with ϕ:

$$\mathrm{Gal}(\mathcal{Q}_R/\mathbf{K}) := \{\sigma \in \mathrm{Aut}(\mathcal{Q}_R/\mathbf{K}) \mid \phi \circ \sigma = \sigma \circ \phi\}.$$

Abusing notation, we shall sometimes let $\mathrm{Gal}(\mathcal{Q}_R/F)$ denote the group $\{\sigma \in \mathrm{Aut}(\mathcal{Q}_R/F) \mid \phi \circ \sigma = \sigma \circ \phi\}$ for F a $\mathbf{K}$-ϕ-subalgebra of $\mathcal{Q}_R$.

An easy computation shows that, for any $\sigma \in \mathrm{Gal}(\mathcal{Q}_R/\mathbf{K})$, there exists a unique $C(\sigma) \in \mathrm{GL}_n(\mathbf{k})$ such that $\sigma(U) = UC(\sigma)$. By [vdPS97, Theorem 1.13], the faithful representation

$$\mathrm{Gal}(\mathcal{Q}_R/\mathbf{K}) \to \mathrm{GL}_n(\mathbf{k}), \quad \sigma \mapsto C(\sigma),$$

identifies $\mathrm{Gal}(\mathcal{Q}_R/\mathbf{K})$ with a linear algebraic subgroup of $\mathrm{GL}_n(\mathbf{k})$. If we choose another fundamental matrix of solutions U, we find a conjugate representation.

A fundamental theorem of difference Galois theory [vdPS97, Theorem 1.13] says that R is the coordinate ring of a G-torsor over $\mathbf{K}$. In particular, the dimension of $\mathrm{Gal}(\mathcal{Q}_R/\mathbf{K})$ as a linear algebraic group over $\mathbf{k}$ coincides with the transcendence degree of the K_i over $\mathbf{K}$. Thereby, the difference Galois group controls the algebraic relations satisfied by the solutions.

The following proposition gives a characterization of the normal algebraic subgroups of $\mathrm{Gal}(\mathcal{Q}_R/\mathbf{K})$.

Proposition 1.3. *An algebraic subgroup H of $\mathrm{Gal}(\mathcal{Q}_R/\mathbf{K})$ is normal if and only if the ϕ-ring $\mathcal{Q}_R{}^H := \{g \in \mathcal{Q}_R \mid \forall \sigma \in H, \sigma(g) = g\}$ is stable under the action of $\mathrm{Gal}(\mathcal{Q}_R/\mathbf{K})$. In this case, the $\mathbf{K}$-ϕ-algebra $\mathcal{Q}_R{}^H$ is a total PV ring over $\mathbf{K}$ and the following sequence of group morphisms is exact:*

$$0 \to H \xrightarrow{\iota} \mathrm{Gal}(\mathcal{Q}_R/\mathbf{K}) \xrightarrow{\pi} \mathrm{Gal}(\mathcal{Q}_R{}^H/\mathbf{K}) \to 0,$$

where ι is the inclusion of H in $\mathrm{Gal}(\mathcal{Q}_R/\mathbf{K})$ and π denotes the restriction of elements of $\mathrm{Gal}(\mathcal{Q}_R/\mathbf{K})$ to $\mathcal{Q}_R{}^H$.

Proof. Assume that H is normal in $\mathrm{Gal}(\mathcal{Q}_R/\mathbf{K})$. For all $\tau \in \mathrm{Gal}(\mathcal{Q}_R/\mathbf{K})$, $g \in \mathcal{Q}_R{}^H$, and $\sigma \in H$, we have

$$\sigma(\tau(g)) = \tau((\tau^{-1}\sigma\tau)(g)) = \tau(g).$$

This shows that $\mathcal{Q}_R{}^H$ is stable under the action of $\mathrm{Gal}(\mathcal{Q}_R/\mathbf{K})$. Conversely, assume that $\mathcal{Q}_R{}^H$ is stable under the action of $\mathrm{Gal}(\mathcal{Q}_R/\mathbf{K})$. Then we can consider the restriction morphism

$$\pi : \mathrm{Gal}(\mathcal{Q}_R/\mathbf{K}) \to \mathrm{Gal}(\mathcal{Q}_R{}^H/\mathbf{K}), \quad \sigma \mapsto \sigma|_{\mathcal{Q}_R{}^H}.$$

By Galois correspondence [HS08, Theorem 6.20], we have $\ker(\pi) = H$, and hence H is normal in $\mathrm{Gal}(\mathcal{Q}_R/\mathbf{K})$. The rest of the proof is [vdPS97, Corollary 1.30]. □

Corollary 1.4. *Let f be an invertible element of R such that $\phi(f) = af$ for some $a \in \mathbf{K}$. Let $\mathcal{Q}_f \subset \mathcal{Q}_R$ be the total ring of quotients of $\mathbf{K}[f, f^{-1}]$; this is a total PV ring for $\phi(y) = ay$ over $\mathbf{K}$. Then $\mathrm{Gal}(\mathcal{Q}_R/\mathcal{Q}_f)$ is a solvable algebraic group if and only if $\mathrm{Gal}(\mathcal{Q}_R/\mathbf{K})$ is a solvable algebraic group.*

Proof. We have ${\mathcal{Q}_R}^{\mathrm{Gal}(\mathcal{Q}_R/\mathcal{Q}_f)} = \mathcal{Q}_f$ (in virtue of the Galois correspondence [vdPS97, Theorem 1.29.3]) and $\mathcal{Q}_f$ is stable under the action of $\mathrm{Gal}(\mathcal{Q}_R/\mathbf{K})$ (because, for all $\sigma \in \mathrm{Gal}(\mathcal{Q}_R/\mathbf{K})$, we have $\sigma(f)f^{-1} \in {\mathcal{Q}_R}^{\phi} = \mathbf{k}$). By Proposition 1.3, $\mathrm{Gal}(\mathcal{Q}_R/\mathcal{Q}_f)$ is normal in $\mathrm{Gal}(\mathcal{Q}_R/\mathbf{K})$ and the sequence

$$0 \to \mathrm{Gal}(\mathcal{Q}_R/\mathcal{Q}_f) \xrightarrow{\iota} \mathrm{Gal}(\mathcal{Q}_R/\mathbf{K}) \xrightarrow{\pi} \mathrm{Gal}(\mathcal{Q}_f/\mathbf{K}) \to 0$$

is exact. Since $\mathrm{Gal}(\mathcal{Q}_f/\mathbf{K}) \subset \mathrm{GL}_1(\mathbf{k})$ is abelian, $\mathrm{Gal}(\mathcal{Q}_R/\mathbf{K})$ is solvable if and only if $\mathrm{Gal}(\mathcal{Q}_R/\mathcal{Q}_f)$ is. □

1.2. More specific results about Mahler equations

Now, we restrict ourselves to the Mahlerian context.

We let $p \geq 2$ be an integer. We consider the field

$$\mathbf{K} := \bigcup_{j \geq 1} \mathbb{C}(z^{1/j}).$$

The field automorphism

$$\phi : \mathbf{K} \to \mathbf{K}, \qquad f(z) \mapsto f(z^p),$$

gives a structure of ϕ-field on $\mathbf{K}$ such that $\mathbf{K}^{\phi} = \mathbb{C}$.

We also consider the field $\mathbf{K}' := \mathbf{K}(\log(z))$. The field automorphism

$$\phi : \mathbf{K}' \to \mathbf{K}', \qquad f(z, \log(z)) \mapsto f(z^p, p\log(z)),$$

gives a structure of ϕ-field on $\mathbf{K}'$ such that $\mathbf{K}'^{\phi} = \mathbb{C}$.

We shall consider Mahler equations over the ϕ-field $\mathbf{K}$ and also over its ϕ-field extension $\mathbf{K}'$. We now study the effect of the base extension from $\mathbf{K}$ to $\mathbf{K}'$ on the difference Galois groups.

We first state and prove a lemma.

Lemma 1.5. *Let L be a ϕ-subfield of $\mathbf{K}'$ that contains $\mathbf{K}$. Then there exists an integer $k \geq 0$ such that $L = \mathbf{K}(\log(z)^k)$.*

Proof. The case $L = \mathbf{K}$ is obvious (take $k = 0$). Now assume that $L \neq \mathbf{K}$. Lemma 1.1 ensures that $\mathbf{K}'$ is a total PV ring over $\mathbf{K}$ for the equation $\phi(y) = py$. The action of $\mathrm{Gal}(\mathbf{K}'/\mathbf{K})$ on $\log(z)$ allows us to see $\mathrm{Gal}(\mathbf{K}'/\mathbf{K})$ as an algebraic subgroup of $\mathbb{C}^{\times}$. Since $\log(z)$ is transcendental over $\mathbf{K}$, we have $\mathrm{Gal}(\mathbf{K}'/\mathbf{K}) = \mathbb{C}^{\times}$. Since $L \neq \mathbf{K}$, the group $\mathrm{Gal}(\mathbf{K}'/L)$ is a proper algebraic subgroup of $\mathbb{C}^{\times}$, and hence a group of roots of unity, so there exists an integer $k \geq 1$ such that $\mathrm{Gal}(\mathbf{K}'/L) = \mu_k := \{c \in \mathbb{C}^{\times} \mid c^k = 1\}$. Consequently, $\log(z)^k$ is fixed by $\mathrm{Gal}(\mathbf{K}'/L)$, and hence belongs to L by Galois correspondence. Since $\mathrm{Gal}(\mathbf{K}'/\mathbf{K}(\log(z)^k)) \subset \mu_k$, we get $L = \mathbf{K}(\log(z)^k)$. □

We consider the difference system

$$\phi(Y) = AY \tag{3}$$

with $A \in \mathrm{GL}_n(\mathbb{C}(z))$. Let R' be a PV ring for (3) over $\mathbf{K}'$; then $\mathcal{Q}_{R'}$ is a total PV ring for (3) over $\mathbf{K}'$. Let $U \in \mathrm{GL}_n(R')$ be a fundamental matrix of solutions of (3). Let R be the $\mathbf{K}$-subalgebra of R' generated by the entries of U and by $\det(U)^{-1}$. By [HS08, Corollary 6.9], we have $\mathcal{Q}_R \subset \mathcal{Q}_{R'}$. Since $\mathcal{Q}_{R'}{}^{\phi} = \mathbf{K}'^{\phi} = \mathbb{C}$, we have $\mathcal{Q}_R{}^{\phi} = \mathbb{C}$, and Lemma 1.1 shows that R is a PV ring for (3) over $\mathbf{K}$ and $\mathcal{Q}_R$ is a total PV ring for (3) over $\mathbf{K}$.

The restriction morphism

$$\iota : \mathrm{Gal}(\mathcal{Q}_{R'}/\mathbf{K}') \to \mathrm{Gal}(\mathcal{Q}_R/\mathbf{K})$$

is a closed immersion; we will freely identify $\mathrm{Gal}(\mathcal{Q}_{R'}/\mathbf{K}')$ with the subgroup $\iota(\mathrm{Gal}(\mathcal{Q}_{R'}/\mathbf{K}'))$ of $\mathrm{Gal}(\mathcal{Q}_R/\mathbf{K})$.

Proposition 1.6. *The difference Galois group* $\mathrm{Gal}(\mathcal{Q}_{R'}/\mathbf{K}')$ *is a normal subgroup of* $\mathrm{Gal}(\mathcal{Q}_R/\mathbf{K})$ *and the quotient* $\mathrm{Gal}(\mathcal{Q}_R/\mathbf{K})/\mathrm{Gal}(\mathcal{Q}_{R'}/\mathbf{K}')$ *is either trivial or isomorphic to* $\mathbb{C}^{\times}$.

Proof. We set $G' := \iota(\mathrm{Gal}(\mathcal{Q}_{R'}/\mathbf{K}'))$ and $G := \mathrm{Gal}(\mathcal{Q}_R/\mathbf{K})$. Let $F := (\mathcal{Q}_R)^{G'} = (\mathcal{Q}_{R'})^{G'} \cap \mathcal{Q}_R = \mathbf{K}' \cap \mathcal{Q}_R$. The Galois correspondence [HS08, Theorem 6.20] ensures that $G' = \mathrm{Gal}(\mathcal{Q}_R/F)$. Since $F/\mathbf{K}$ is a ϕ-subfield extension of $\mathbf{K}'/\mathbf{K}$, Lemma 1.5 yields an integer $k \geq 0$ such that $F = \mathbf{K}(\log(z)^k)$. Since $F^{\phi} = \mathbb{C}$, Lemma 1.1 shows that F is a total PV ring over $\mathbf{K}$ for $\phi(y) = p^k y$. Using Proposition 1.3, we see that G' is a normal subgroup of G and G/G' is isomorphic to the difference Galois group over $\mathbf{K}$ of $\phi(y) = p^k y$, which is trivial if $k = 0$ and equal to $\mathbb{C}^{\times}$ otherwise. □

Corollary 1.7. *If* $\mathrm{SL}_n(\mathbb{C}) \subset \mathrm{Gal}(\mathcal{Q}_R/\mathbf{K})$ *then* $\mathrm{SL}_n(\mathbb{C}) \subset \mathrm{Gal}(\mathcal{Q}_{R'}/\mathbf{K}')$.

2. Parametrized difference Galois theory

We will use standard notions and notation of difference and differential algebra which can be found in [Coh65] and [vdPS97].

2.1. Differential algebra

A *δ-ring* (R, δ) is a ring R endowed with a derivation $\delta : R \to R$ (this means that δ is additive and satisfies the Leibniz rule $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in R$). If R is a field, then (R, δ) is called a *δ-field*. To simplify the notation, most of the time we will write R instead of (R, δ).

We let R^{δ} denote the *ring of δ-constants* of the δ-ring R, i.e.,

$$R^{\delta} := \{c \in R \mid \delta(c) = 0\}.$$

If R^{δ} is a field, it is called the *field of δ-constants*.

Given a δ-ring (R, δ), a $\widetilde{\delta}$-ring $(\widetilde{R}, \widetilde{\delta})$ is an R-δ-*algebra* if $\widetilde{R}$ is a ring extension of R and $\widetilde{\delta}_{|R} = \delta$; in this case, we will often denote $\widetilde{\delta}$ by δ. Let $\mathbf{K}$ be a δ-field. If $\mathbf{L}$ is a $\mathbf{K}$-δ-algebra and a field, we say that $\mathbf{L}/\mathbf{K}$ is a δ-*field extension*. Let R be a $\mathbf{K}$-δ-algebra and let $a_1, \dots, a_n \in R$. We let $\mathbf{K}\{a_1, \dots, a_n\}$ denote the smallest $\mathbf{K}$-δ-subalgebra of R containing $a_1, \dots, a_n$. Let $\mathbf{L}/\mathbf{K}$ be a δ-field extension and let $a_1, \dots, a_n \in \mathbf{L}$. We let $\mathbf{K}\langle a_1, \dots, a_n\rangle$ denote the smallest $\mathbf{K}$-δ-subfield of $\mathbf{L}$ containing $a_1, \dots, a_n$.

The ring of δ-*polynomials* in the differential indeterminates $y_1, \dots, y_n$ and with coefficients in a differential field $(\mathbf{K}, \delta)$, denoted by $\mathbf{K}\{y_1, \dots, y_n\}$, is the ring of polynomials in the indeterminates $\{\delta^j y_i \mid j \in \mathbb{N}, 1 \le i \le n\}$ with coefficients in $\mathbf{K}$.

Let R be be a $\mathbf{K}$-δ-algebra and let $a_1, \dots, a_n \in R$. If there exists a nonzero δ-polynomial $P \in \mathbf{K}\{y_1, \dots, y_n\}$ such that $P(a_1, \dots, a_n) = 0$, then we say that $a_1, \dots, a_n$ are *hyperalgebraically dependent* over $\mathbf{K}$. Otherwise, we say that $a_1, \dots, a_n$ are *hyperalgebraically independent* over $\mathbf{K}$.

A δ-field $\mathbf{k}$ is called *differentially closed* if, for every (finite) set of δ-polynomials $\mathcal{F}$, if the system of differential equations $\mathcal{F} = 0$ has a solution with entries in some δ-field extension $\mathbf{L}$, then it has a solution with entries in $\mathbf{k}$. Note that the field of δ-constants $\mathbf{k}^\delta$ of any differentially closed δ-field $\mathbf{k}$ is algebraically closed. Any δ-field $\mathbf{k}$ has a *differential closure* $\widetilde{\mathbf{k}}$, i.e., a differentially closed δ-field extension, and we have $\widetilde{\mathbf{k}}^\delta = \mathbf{k}$.

From now on, we consider a differentially closed δ-field $\mathbf{k}$.

A subset $W \subset \mathbf{k}^n$ is *Kolchin-closed* (or δ-*closed*, for short) if there exists $S \subset \mathbf{k}\{y_1, \dots, y_n\}$ such that

$$W = \{a \in \mathbf{k}^n \mid \forall f \in S, f(a) = 0\}.$$

The Kolchin-closed subsets of $\mathbf{k}^n$ are the closed sets of a topology on $\mathbf{k}^n$, called the *Kolchin topology*. The *Kolchin-closure* of $W \subset \mathbf{k}^n$ is the closure of W in $\mathbf{k}^n$ for the Kolchin topology.

Following Cassidy [Cas72, Chapter II, Section 1, p. 905], we say that a subgroup $G \subset \mathrm{GL}_n(\mathbf{k}) \subset \mathbf{k}^{n\times n}$ is a *linear differential algebraic group* (LDAG) if G is the intersection of a Kolchin-closed subset of $\mathbf{k}^{n\times n}$ (identified with $\mathbf{k}^{n^2}$) with $\mathrm{GL}_n(\mathbf{k})$.

A δ-*closed subgroup*, or δ-*subgroup* for short, of an LDAG is a subgroup that is Kolchin-closed. The Zariski-closure of a LDAG $G \subset \mathrm{GL}_n(\mathbf{k})$ is denoted by $\overline{G}$ and is a linear algebraic group.

We will use the following fundamental result.

Proposition 2.1 ([Cas72, Proposition 42]). *Let* $\mathbf{k}$ *be a differentially closed field. Let* $\mathbf{C} := \mathbf{k}^\delta$. *A Zariski-dense* δ-*closed subgroup of* $\mathrm{SL}_n(\mathbf{k})$ *is either conjugate to* $\mathrm{SL}_n(\mathbf{C})$ *or equal to* $\mathrm{SL}_n(\mathbf{k})$.

We will also use the following result.

Lemma 2.2 ([MS13, Lemma 11]). *Let* $\mathbf{k}$ *be a differentially closed field. Let* $\mathbf{C} := \mathbf{k}^\delta$. *Then the normalizer of* $\mathrm{SL}_n(\mathbf{C})$ *in* $\mathrm{GL}_n(\mathbf{k})$ *is* $\mathbf{k}^\times \mathrm{SL}_n(\mathbf{C})$.

2.2. Difference-differential algebra

A (ϕ, δ)-*ring* (R, ϕ, δ) is a ring R endowed with a ring automorphism ϕ and a derivation $\delta : R \to R$ (in other words, (R, ϕ) is a ϕ-ring and (R, δ) is a δ-ring) such that ϕ commutes with δ. If R is a field, then (R, ϕ, δ) is called a (ϕ, δ)-*field*. If there is no possible confusion, we will write R instead of (R, ϕ, δ).

We have straightforward notions of (ϕ, δ)-ideals, (ϕ, δ)-morphisms, (ϕ, δ)-algebras, etc., similar to the notions recalled in Sections 1 and 2.1. We omit the details and refer for instance to [HS08, Section 6.2] and the references therein.

In order to use the parametrized difference Galois theory developed in [HS08], we will need to work with a base (ϕ, δ)-field $\mathbf{K}$ such that $\mathbf{k} := \mathbf{K}^{\phi}$ is differentially closed. Most of the common function fields do not satisfy this condition. The following result shows that any (ϕ, δ)-field with algebraically closed field of constants can be embedded into a (ϕ, δ)-field with differentially closed field of constants. The following lemma appears in an unpublished proof due to M. Singer (in a letter to the second author, February 25, 2010) of the fact that the Mahler function $\sum_{n\geq 0} z^{p^n}$ is hypertranscendental. It is close to [CHS08, Proposition 2.4] and [vdPS97, Lemma 1.11], but it is not completely similar.

Lemma 2.3. *Let F be a (ϕ, δ)-field with $\mathbf{k} := F^{\phi}$ algebraically closed. Let $\widetilde{\mathbf{k}}$ be a differentially closed field containing $\mathbf{k}$. Then the ring $\widetilde{\mathbf{k}} \otimes_{\mathbf{k}} F$ is an integral domain whose fraction field $\mathbf{K}$ is a (ϕ, δ)-field extension of F such that $\mathbf{K}^{\phi} = \widetilde{\mathbf{k}}$.*

Proof. The first assertion follows from the fact that since $\mathbf{k}$ is algebraically closed, the extension $\widetilde{\mathbf{k}}/\mathbf{k}$ is regular.

In what follows, we view F as embedded in $\widetilde{\mathbf{k}} \otimes_{\mathbf{k}} F$ and $\widetilde{\mathbf{k}}$ as embedded in $\widetilde{\mathbf{k}} \otimes_{\mathbf{k}} F$ via the maps

$$\begin{aligned} F &\to \widetilde{\mathbf{k}} \otimes_{\mathbf{k}} F, \\ f &\mapsto 1 \otimes f, \end{aligned} \quad \text{and} \quad \begin{aligned} \widetilde{\mathbf{k}} &\to \widetilde{\mathbf{k}} \otimes_{\mathbf{k}} F, \\ a &\mapsto a \otimes 1. \end{aligned}$$

The maps

$$\begin{aligned} \phi : \widetilde{\mathbf{k}} \otimes_{\mathbf{k}} F &\to \widetilde{\mathbf{k}} \otimes_{\mathbf{k}} F, \\ (a, b) &\mapsto a \otimes \phi(b), \end{aligned} \quad \text{and} \quad \begin{aligned} \delta : \widetilde{\mathbf{k}} \otimes_{\mathbf{k}} F &\to \widetilde{\mathbf{k}} \otimes_{\mathbf{k}} F, \\ (a, b) &\mapsto \delta(a) \otimes b + a \otimes \delta(b), \end{aligned}$$

are well-defined and endow $\widetilde{\mathbf{k}} \otimes_{\mathbf{k}} F$ with a structure of F-(ϕ, δ)-algebra.

To prove the second statement, we first show that any ϕ-ideal of $\widetilde{\mathbf{k}} \otimes_{\mathbf{k}} F$ is trivial. Let $(c_i)_{i\in I}$ be a $\mathbf{k}$-basis of $\widetilde{\mathbf{k}}$. Let $\mathfrak{I}$ be a nonzero ϕ-ideal of $\widetilde{\mathbf{k}} \otimes_{\mathbf{k}} F$ and let $w = \sum_{i=1}^{n} c_i \otimes f_i$ be a nonzero element of $\mathfrak{I}$ with $f_i \in F$ and n minimal. Without loss of generality, we can assume that $f_1 = 1$. Since $\phi(w) - w = \sum_{i=2}^{n} c_i \otimes (\phi(f_i) - f_i)$ is an element of $\mathfrak{I}$ with fewer terms than w has, it must be 0. This implies that, for all $i \in \{1, \dots, n\}$, $\phi(f_i) = f_i$, i.e., $f_i \in \mathbf{k}$. Then $w = (\sum_{i=1}^{n} c_i f_i) \otimes 1$ is invertible in $\widetilde{\mathbf{k}} \otimes_{\mathbf{k}} F$, and hence $\mathfrak{I} = \widetilde{\mathbf{k}} \otimes_{\mathbf{k}} F$.

Let $c \in \mathbf{K}^{\phi}$. Since $\mathfrak{I} := \{d \in \widetilde{\mathbf{k}} \otimes_{\mathbf{k}} F \mid dc \in \widetilde{\mathbf{k}} \otimes_{\mathbf{k}} F\}$ is a nonzero ϕ-ideal of $\widetilde{\mathbf{k}} \otimes_{\mathbf{k}} F$, we must have $\mathfrak{I} = \widetilde{\mathbf{k}} \otimes_{\mathbf{k}} F$. In particular, $1 \in \mathfrak{I}$, and hence $c \in \widetilde{\mathbf{k}} \otimes_{\mathbf{k}} F$. Writing $c = \sum_{i\in I} c_i \otimes f_i$, we see that $\phi(c) = c$ implies $\phi(f_i) = f_i$ for all $i \in \{1, \dots, n\}$. Therefore, the f_i are in $\mathbf{k}$, and hence c belongs to $\widetilde{\mathbf{k}}$. □

2.3. *Parametrized difference Galois theory*

For details on what follows, we refer to [HS08].

Let $\mathbf{K}$ be a (ϕ, δ)-field with $\mathbf{k} := \mathbf{K}^{\phi}$ differentially closed. We consider the linear difference system

$$\phi(Y) = AY \tag{4}$$

with $A \in \mathrm{GL}_n(\mathbf{K})$ for some integer $n \geq 1$.

By [HS08, § 6.2.1], there exists a $\mathbf{K}$-(ϕ, δ)-algebra S such that

- there exists $U \in \mathrm{GL}_n(S)$ such that $\phi(U) = AU$ (such a U is called a *fundamental matrix of solutions* of (4));
- S is generated, as a $\mathbf{K}$-δ-algebra, by the entries of U and by $\det(U)^{-1}$;
- the only (ϕ, δ)-ideals of S are $\{0\}$ and S.

Such an S is called a *parametrized Picard–Vessiot ring*, or PPV ring for short, for (4) over $\mathbf{K}$. It is unique up to isomorphism of $\mathbf{K}$-(ϕ, δ)-algebras. A PPV ring is not always an integral domain. However, there exist idempotent elements $e_1, \ldots, e_s$ of R such that $R = R_1 \oplus \cdots \oplus R_s$ where the $R_i := Re_i$ are integral domains stable by δ and transitively permuted by ϕ. In particular, S has no nilpotent element and one can consider its total ring of quotients $\mathcal{Q}_S$. It can be decomposed as the direct sum $\mathcal{Q}_S = K_1 \oplus \cdots \oplus K_s$ of the fields of fractions K_i of the R_i. The ring $\mathcal{Q}_S$ has a natural structure of R-(ϕ, δ)-algebra and we have ${\mathcal{Q}_S}^{\phi} = \mathbf{k}$. Moreover, the K_i are transitively permuted by ϕ. We call the (ϕ, δ)-ring $\mathcal{Q}_S$ a *total PPV ring* for (4) over $\mathbf{K}$.

The *parametrized difference Galois group* $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{K})$ of S over $(\mathbf{K}, \phi, \delta)$ is the group of $\mathbf{K}$-(ϕ, δ)-automorphisms of $\mathcal{Q}_S$:

$$\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{K}) := \{\sigma \in \mathrm{Aut}(\mathcal{Q}_S/\mathbf{K}) \mid \phi \circ \sigma = \sigma \circ \phi \text{ and } \delta \circ \sigma = \sigma \circ \delta\}.$$

Note that if $\delta = 0$, we recover the difference Galois groups considered in Section 1.1.

A straightforward computation shows that, for any $\sigma \in \mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{K})$, there exists a unique $C(\sigma) \in \mathrm{GL}_n(\mathbf{k})$ such that $\sigma(U) = UC(\sigma)$. By [HS08, Proposition 6.18], the faithful representation

$$\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{K}) \to \mathrm{GL}_n(\mathbf{k}), \qquad \sigma \mapsto C(\sigma),$$

identifies $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{K})$ with a linear differential algebraic subgroup of $\mathrm{GL}_n(\mathbf{k})$. If we choose another fundamental matrix of solutions U, we find a conjugate representation.

The parametrized difference Galois group $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{K})$ of (4) reflects the differential algebraic relations between the solutions of (4). In particular, the δ-dimension of $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{K})$ coincides with the δ-transcendence degree of the K_i over $\mathbf{K}$ (see [HS08, Proposition 6.26] for definitions and details).

A parametrized Galois correspondence holds between the δ-closed subgroups of $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{K})$ and the $\mathbf{K}$-(ϕ, δ)-subalgebras F of $\mathcal{Q}_S$ such that every nonzero divisor of F is a unit of F (see for instance [HS08, Theorem 6.20]). Abusing notation, we still let $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/F)$ denote the group of F-(ϕ, δ)-automorphisms of $\mathcal{Q}_S$. The following proposition is at the heart of the parametrized Galois correspondence.

Proposition 2.4 ([HS08, Theorem 6.20]). *Let S be a PPV ring over $\mathbf{K}$. Let F be a $\mathbf{K}$-(ϕ, δ)-subalgebra of $\mathcal{Q}_S$ such that every nonzero divisor of F is a unit of F. Let H be a δ-closed subgroup of $\mathrm{Gal}^\delta(\mathcal{Q}_S/\mathbf{K})$. Then*

- $\mathcal{Q}_S^{\mathrm{Gal}^\delta(\mathcal{Q}_S/F)} := \{f \in \mathcal{Q}_S \mid \forall \tau \in \mathrm{Gal}^\delta(\mathcal{Q}_S/F), \tau(f) = f\} = F$;
- $\mathrm{Gal}^\delta(\mathcal{Q}_S/\mathcal{Q}_S^H) = H$.

Let S be a PPV ring for (4) over $\mathbf{K}$ and let $U \in \mathrm{GL}_n(S)$ be a fundamental matrix of solutions. Then the $\mathbf{K}$-ϕ-algebra R generated by the entries of U and by $\det(U)^{-1}$ is a PV ring for (4) over $\mathbf{K}$ and we have $\mathcal{Q}_R \subset \mathcal{Q}_S$. One can identify $\mathrm{Gal}^\delta(\mathcal{Q}_S/\mathbf{K})$ with a subgroup of $\mathrm{Gal}(\mathcal{Q}_R/\mathbf{K})$ by restricting the elements of $\mathrm{Gal}^\delta(\mathcal{Q}_S/\mathbf{K})$ to $\mathcal{Q}_R$.

Proposition 2.5 ([HS08, Proposition 2.8]). $\mathrm{Gal}^\delta(\mathcal{Q}_S/\mathbf{K})$ *is a Zariski-dense subgroup of* $\mathrm{Gal}(\mathcal{Q}_R/\mathbf{K})$.

2.4. *Hypertranscendence criteria for equations of order one*

The hypertranscendence criteria contained in [HS08] are stated for (ϕ, δ)-fields $\mathbf{K}$ such that the δ-field $\mathbf{k} := \mathbf{K}^\phi$ is differentially closed. Recently some schematic versions (see for instance [Wib12] or [DVH12]) of [HS08] have been developed which allow one to work over (ϕ, δ)-fields with algebraically closed field of constants. One could use this schematic approach to show that the hypertranscendence criteria of [HS08] still hold over (ϕ, δ)-fields with algebraically closed field of constants (not necessarily differentially closed). However, for the sake of clarity and simplicity of exposition, we prefer to show that one can deduce these criteria directly from the ones contained in [HS08], via simple descent arguments. The following result is due to Michael Singer.

Proposition 2.6. *Let $\mathbf{K}$ be a (ϕ, δ)-field with $\mathbf{k} := \mathbf{K}^\phi$ algebraically closed and let $(a, b) \in \mathbf{K}^\times \times \mathbf{K}$. Let R be a $\mathbf{K}$-(ϕ, δ)-algebra and let $v \in R \setminus \{0\}$.*

- *If $\phi(v) - v = b$ and v is hyperalgebraic over $\mathbf{K}$, then there exist a nonzero linear homogeneous δ-polynomial $\mathcal{L}(y) \in \mathbf{k}\{y\}$ and an element $f \in \mathbf{K}$ such that*
$$\mathcal{L}(b) = \phi(f) - f.$$
- *Assume moreover that v is invertible in R. If $\phi(v) = av$ and if v is hyperalgebraic over $\mathbf{K}$, then there exist a nonzero linear homogeneous δ-polynomial $\mathcal{L}(y) \in \mathbf{k}\{y\}$ and an element $f \in \mathbf{K}$ such that*
$$\mathcal{L}\left(\frac{\delta(a)}{a}\right) = \phi(f) - f.$$

The converse of either statement is true if $R^\phi = \mathbf{k}$.

Proof. Let us prove the first statement. Let $\widetilde{\mathbf{k}}$ be a δ-closure of $\mathbf{k}$. Lemma 2.3 ensures that $\mathbf{L} := \mathrm{Frac}(\widetilde{\mathbf{k}} \otimes_{\mathbf{k}} \mathbf{K})$ is a (ϕ, δ)-field extension of $\mathbf{K}$ such that $\mathbf{L}^\phi = \tilde{\mathbf{k}}$. Let $\mathbf{L}\{y\}$ be the ring of δ-polynomials in one variable over $\mathbf{L}$ endowed with the structure of $\mathbf{L}$-(ϕ, δ)-algebra induced by setting $\phi(y) := y + b$. Without loss of generality, we can assume that $R = \mathbf{K}\{v\}$. We identify R with $\mathbf{K}\{y\}/\mathfrak{I}$ for some (ϕ, δ)-ideal $\mathfrak{I}$ of $\mathbf{K}\{y\}$. Since v is hyperalgebraic over $\mathbf{K}$, we have $\mathfrak{I} \neq \{0\}$. Moreover, $\mathfrak{I} \neq \mathbf{K}\{y\}$ because $R \neq \{0\}$. We claim that $(\mathfrak{I}) \cap \mathbf{K}\{y\} = \mathfrak{I}$ where $(\mathfrak{I})$ denotes the (ϕ, δ)-ideal generated by $\mathfrak{I}$ in $\mathbf{L}\{y\}$. Indeed, choose a $\mathbf{K}$-basis $(c_i)_{i\in I}$ of $\mathbf{L}$ with $c_{i_0} = 1$ for some $i_0 \in I$. Note that $(c_i)_{i\in I}$ is also a basis of the $\mathbf{K}\{y\}$-module $\mathbf{L}\{y\}$. Then $(\mathfrak{I})$ consists of the sums $\sum a_i c_i$ with $a_i \in \mathfrak{I}$. It follows easily that $(\mathfrak{I}) \cap \mathbf{K}\{y\} = \mathfrak{I}$, as claimed. In particular, $(\mathfrak{I})$ is a proper ideal of $\mathbf{L}\{y\}$, and hence is contained in some maximal (ϕ, δ)-ideal $\mathfrak{M}$ of $\mathbf{L}\{y\}$. The ring $S := \mathbf{L}\{y\}/\mathfrak{M}$ is a PPV ring for $\phi(y) = y + b$ over $\mathbf{L}$. The image u of y in S is hyperalgebraic over $\mathbf{L}$ (because $\mathfrak{M} \neq \{0\}$) and is a solution of $\phi(y) = y + b$. By [HS08, Proposition 3.1], there exist a nonzero linear homogeneous δ-polynomial $\mathcal{L}_0(y) \in \widetilde{\mathbf{k}}\{y\}$ and $g \in \mathbf{L}$ such that

$$\mathcal{L}_0(b) = \phi(g) - g. \tag{5}$$

Let $(h_i)_{i\in I}$ be a $\mathbf{k}$-basis of $\mathbf{K}$. Without loss of generality, we can assume that

$$\mathcal{L}_0(y) = \delta^{n+1}(y) + \sum_{i=0}^{n} c_i \delta^i(y) \quad \text{and} \quad g := \frac{\sum_{i=1}^{r} a_i \otimes h_i}{\sum_{i=1}^{s} b_i \otimes h_i}$$

where $a_i, b_i, c_i \in \widetilde{\mathbf{k}}$ and $b_1 = 1$. It is clear that (5) can be rewritten as

$$\sum_j P_j\big((a_i)_{i\in\{1,\dots,r\}}, (b_i)_{i\in\{2,\dots,s\}}, (c_i)_{i\in\{1,\dots,n\}}\big) \otimes h_j = 0$$

where the P_j are polynomials with coefficients in $\mathbf{k}$. Thus, for all j,

$$P_j\big((a_i)_{i\in\{1,\dots,r\}}, (b_i)_{i\in\{2,\dots,s\}}, (c_i)_{i\in\{1,\dots,n\}}\big) = 0.$$

Since $\mathbf{k}$ is algebraically closed, there exist $\alpha_i, \beta_i, \gamma_i \in \mathbf{k}$ such that, for all j,

$$P_j\big((\alpha_i)_{i\in\{1,\dots,r\}}, (\beta_i)_{i\in\{2,\dots,s\}}, (\gamma_i)_{i\in\{1,\dots,n\}}\big) = 0.$$

Set $\beta_1 := 1$. Then we see that

$$\mathcal{L}(y) := \delta^{n+1}(y) + \sum_{i=0}^{n} \gamma_i \delta^i(y) \quad \text{and} \quad f := \frac{\sum_i \alpha_i \otimes h_i}{\sum_i \beta_i \otimes h_i}$$

satisfy the conclusion of the first part of the proposition.

Conversely, if $R^\phi = \mathbf{k}$ and there exist a nonzero linear homogeneous δ-polynomial $\mathcal{L}(y) \in \mathbf{k}\{y\}$ and $f \in \mathbf{K}$ such that $\mathcal{L}(b) = \phi(f) - f$, then $\mathcal{L}(v) - f$ belongs to $R^\phi = \mathbf{k}$. Since $\mathcal{L}(y)$ is nonzero, v is differentially algebraic over $\mathbf{K}$.

The proof of the second statement is similar. It can also be deduced from the first by noticing that if $\phi(v) = av$ then $\phi(\frac{\delta v}{v}) = \frac{\delta v}{v} + \frac{\delta a}{a}$ and by using the fact that v is hyperalgebraic over $\mathbf{K}$ if and only if $\frac{\delta v}{v}$ is. □

Remark 2.7. In Proposition 2.6, we require that v is invertible in R. This is automatically satisfied if R is similar to a total PPV ring. More precisely, assume that $R = \bigoplus_{x\in\mathbb{Z}/s\mathbb{Z}} K_x$, where the K_x are δ-field extensions of $\mathbf{K}$ such that $\phi(K_x) = K_{x+\bar{1}}$. Then any nonzero solution $v \in R$ of $\phi(y) = ay$ for $a \in \mathbf{K}^\times$ is invertible. Indeed, $v = \sum_{x\in\mathbb{Z}/s\mathbb{Z}} v_x$ for some $v_x \in K_x$. Since $v \neq 0$, there exists $x_0 \in \mathbb{Z}/s\mathbb{Z}$ such that $v_{x_0} \neq 0$. From the equation $\phi(v) = av$, we get $\phi(v_{x_0-\bar{1}}) = av_{x_0}$. So, $v_{x_0-\bar{1}} \neq 0$. Iterating this argument, we see that $v_x \neq 0$ for all $x \in \mathbb{Z}/s\mathbb{Z}$. Hence, v is invertible in R.

2.5. Isomonodromy and projective isomonodromy

Let $\mathbf{K}$ be a (ϕ, δ)-field with $\mathbf{k} := \mathbf{K}^{\phi}$ algebraically closed. Let $\widetilde{\mathbf{k}}$ be a δ-closure of $\mathbf{k}$. Let $\mathbf{C} := \widetilde{\mathbf{k}}^{\delta}$ be the (algebraically closed) field of constants of $\widetilde{\mathbf{k}}$. Lemma 2.3 ensures that $\widetilde{\mathbf{k}} \otimes_{\mathbf{k}} \mathbf{K}$ is an integral domain and $\mathbf{L} := \mathrm{Frac}(\widetilde{\mathbf{k}} \otimes_{\mathbf{k}} \mathbf{K})$ is a (ϕ, δ)-field extension of $\mathbf{K}$ such that $\mathbf{L}^{\phi} = \widetilde{\mathbf{k}}$. We let $\mathcal{Q}_S$ be the total ring of quotients of a PPV ring S over $\mathbf{L}$ for the difference system $\phi(Y) = AY$ where $A \in \mathrm{GL}_n(\mathbf{K})$.

Proposition 2.8. *The following statements are equivalent:*

(i) $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{L})$ *is conjugate to a subgroup of* $\mathrm{GL}_n(\mathbf{C})$*;*
(ii) *there exists* $\widetilde{B} \in \mathbf{L}^{n\times n}$ *such that*

$$\phi(\widetilde{B}) = A\widetilde{B}A^{-1} + \delta(A)A^{-1}; \tag{6}$$

(iii) *there exists* $B \in \mathbf{K}^{n\times n}$ *such that*

$$\phi(B) = ABA^{-1} + \delta(A)A^{-1}.$$

Proof. The equivalence between (i) and (ii) is [HS08, Proposition 2.9]. To complete the proof, it remains to show that if (6) has a solution $\widetilde{B}$ in $\mathbf{L}^{n\times n}$, then it has a solution in $\mathbf{K}^{n\times n}$. This follows from an argument similar to the descent argument in the proof of Proposition 2.6. □

We now consider a "projective isomonodromic" situation, in the spirit of [MS13]. Let $U \in \mathrm{GL}_n(S)$ be a fundamental matrix of solutions of $\phi(Y) = AY$ and let $d := \det(U) \in\notin S^{\times}$.

Proposition 2.9. *Assume that the difference Galois group of* $\phi(Y) = AY$ *over the* ϕ*-field* $\mathbf{K}$ *contains* $\mathrm{SL}_n(\mathbf{k})$ *and that the parametrized difference Galois group of* $\phi(y) = \det(A)y$ *over the* (ϕ, δ)*-field* $\mathbf{L}$ *is included in* $\mathbf{C}^{\times}$*. Then either*

(i) $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{L})$ *is conjugate to a subgroup of* $\mathrm{GL}_n(\mathbf{C})$ *that contains* $\mathrm{SL}_n(\mathbf{C})$*, or*
(ii) $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{L})$ *is equal to a subgroup of* $\mathbf{C}^{\times}\mathrm{SL}_n(\widetilde{\mathbf{k}})$ *that contains* $\mathrm{SL}_n(\widetilde{\mathbf{k}})$.

Moreover, (i) *holds if and only if there exists* $B \in \mathbf{K}^{n\times n}$ *such that*

$$\phi(B) = ABA^{-1} + \delta(A)A^{-1}. \tag{7}$$

Proof. Let R be the $\mathbf{L}$-ϕ-algebra generated by the entries of U and by $\det(U)^{-1}$; this is a PV ring for $\phi(Y) = AY$ over the ϕ-field $\mathbf{L}$. Using [CHS08, Corollary 2.5], we see that the hypothesis that the difference Galois group of $\phi(Y) = AY$ over the ϕ-field $\mathbf{K}$ contains $\mathrm{SL}_n(\mathbf{k})$ implies that $\mathrm{Gal}(\mathcal{Q}_R/\mathbf{L})$ contains $\mathrm{SL}_n(\widetilde{\mathbf{k}})$. So, $\mathrm{Gal}(\mathcal{Q}_R/\mathbf{L})^{\mathrm{der}} = \mathrm{SL}_n(\widetilde{\mathbf{k}})$. Since $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{L})$ is Zariski-dense in $\mathrm{Gal}(\mathcal{Q}_R/\mathbf{L})$, we find that $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{L})^{\mathrm{der}_{\delta}}$ (the Kolchin-closure of the derived subgroup of $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{L})$; see Section 4.4.1) is Zariski-dense in $\mathrm{Gal}(\mathcal{Q}_R/\mathbf{L})^{\mathrm{der}} = \mathrm{SL}_n(\widetilde{\mathbf{k}})$. By Proposition 2.1, $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{L})^{\mathrm{der}_{\delta}}$ is either conjugate to $\mathrm{SL}_n(\mathbf{C})$ or equal to $\mathrm{SL}_n(\widetilde{\mathbf{k}})$. Since $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{L})^{\mathrm{der}_{\delta}}$ is a normal subgroup of $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{L})$, Lemma 2.2 ensures that $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{L})$ is either conjugate to a subgroup of $\widetilde{\mathbf{k}}^{\times}\mathrm{SL}_n(\mathbf{C})$ containing $\mathrm{SL}_n(\mathbf{C})$, or equal to a subgroup of $\mathrm{GL}_n(\widetilde{\mathbf{k}})$ containing $\mathrm{SL}_n(\widetilde{\mathbf{k}})$. But the parametrized difference Galois group of $\phi(y) = \det(A)y$ over $\mathbf{L}$, which can be identified with $\det(\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{L}))$, is contained in $\mathbf{C}^{\times}$. Therefore, $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{L})$ is contained in $\mathbf{C}^{\times}\mathrm{SL}_n(\mathbf{C}) = \mathrm{GL}_n(\mathbf{C})$ or in $\mathbf{C}^{\times}\mathrm{SL}_n(\widetilde{\mathbf{k}})$, which implies the first part of the proposition.

The "moreover" part follows from Proposition 2.8. □

Proposition 2.10. *Assume that the difference Galois group of $\phi(Y) = AY$ over the ϕ-field $\mathbf{K}$ contains $\mathrm{SL}_n(\mathbf{k})$ and that the parametrized difference Galois group of $\phi(y) = \det(A)y$ over the (ϕ, δ)-field $\mathbf{L}$ is $\widetilde{\mathbf{k}}^\times$. Then either*

(i) $\mathrm{Gal}^\delta(\mathcal{Q}_S/\mathbf{L})$ *is conjugate to* $\widetilde{\mathbf{k}}^\times \mathrm{SL}_n(\mathbf{C})$, *or*
(ii) $\mathrm{Gal}^\delta(\mathcal{Q}_S/\mathbf{L})$ *is equal to* $\mathrm{GL}_n(\widetilde{\mathbf{k}})$.

Moreover, (i) *holds if and only if there exists $B \in \mathbf{K}^{n\times n}$ such that*

$$\phi(B) = ABA^{-1} + \delta(A)A^{-1} - \frac{1}{n}\delta(\det(A))\det(A)^{-1}I_n. \tag{8}$$

Proof. Arguing as in the proof of Proposition 2.9, we see that $\mathrm{Gal}^\delta(\mathcal{Q}_S/\mathbf{L})$ is either conjugate to a subgroup of $\widetilde{\mathbf{k}}^\times \mathrm{SL}_n(\mathbf{C})$ containing $\mathrm{SL}_n(\mathbf{C})$, or equal to a subgroup of $\mathrm{GL}_n(\widetilde{\mathbf{k}})$ containing $\mathrm{SL}_n(\widetilde{\mathbf{k}})$. Now, the first part of the proposition follows from the fact that the parametrized difference Galois group of $\phi(y) = \det(A)y$ over $\mathbf{L}$, which can be identified with $\det(\mathrm{Gal}^\delta(\mathcal{Q}_S/\mathbf{L}))$, is equal to $\widetilde{\mathbf{k}}^\times$.

We shall now prove that (i) holds if and only if there exists $B \in \mathbf{L}^{n\times n}$ such that

$$\phi(B) = ABA^{-1} + \delta(A)A^{-1} - \frac{1}{n}\delta(\det(A))\det(A)^{-1}I_n. \tag{9}$$

First assume that $\mathrm{Gal}^\delta(\mathcal{Q}_S/\mathbf{L})$ is conjugate to $\widetilde{\mathbf{k}}^\times \mathrm{SL}_n(\mathbf{C})$. So, there exists a fundamental matrix of solutions $U \in \mathrm{GL}_n(S)$ of $\phi(Y) = AY$ such that, for all $\sigma \in \mathrm{Gal}^\delta(\mathcal{Q}_S/\mathbf{L})$, there exist $\rho_\sigma \in \widetilde{\mathbf{k}}^\times$ and $M_\sigma \in \mathrm{SL}_n(\mathbf{C})$ such that $\sigma(U) = U\rho_\sigma M_\sigma$. Note that $\sigma(d) = d\rho_\sigma^n$. Easy calculations show that the matrix

$$B := \delta(U)U^{-1} - \frac{1}{n}\delta(d)d^{-1}I_n \in S^{n\times n}$$

is left-invariant under the action of $\mathrm{Gal}^\delta(\mathcal{Q}_S/\mathbf{L})$, and hence belongs to $\mathbf{L}^{n\times n}$ in virtue of Proposition 2.4, and that B satisfies (9).

Conversely, assume that there exists $B \in \mathbf{L}^{n\times n}$ satisfying (9). Consider

$$B_1 = B + \frac{1}{n}\delta(d)d^{-1}I_n \in S^{n\times n}.$$

Note that

$$\phi(B_1) = AB_1A^{-1} + \delta(A)A^{-1}.$$

Let $U \in \mathrm{GL}_n(S)$ be a fundamental matrix of solutions of $\phi Y = AY$. As $\phi(\delta(U) - B_1U) = A(\delta(U) - B_1U)$, there exists $C \in \widetilde{\mathbf{k}}^{n\times n}$ such that $\delta(U) - B_1U = UC$. Since $\widetilde{\mathbf{k}}$ is differentially closed, we can find $D \in \mathrm{GL}_n(\widetilde{\mathbf{k}})$ such that $\delta(D) + CD = 0$. Then $V := UD$ is a fundamental matrix of solutions of $\phi Y = AY$ such that $\delta(V) = B_1V$. Consider $\sigma \in \mathrm{Gal}^\delta(\mathcal{Q}_S/\mathbf{L})$ and let $M_\sigma \in \mathrm{GL}_n(\widetilde{\mathbf{k}})$ be such that $\sigma(V) = VM_\sigma$; note that $\sigma(d) = d\rho_\sigma$ where $\rho_\sigma = \det(M_\sigma)$. On the one hand, $\sigma(\delta(V)) = \sigma(B_1V) = \left(B_1 + \frac{1}{n}\delta(\rho_\sigma)\rho_\sigma^{-1}I_n\right)VM_\sigma$. On the other hand, $\sigma(\delta(V)) = \delta(\sigma(V)) = \delta(VM_\sigma) = B_1VM_\sigma + V\delta(M_\sigma)$. So, $\frac{1}{n}\delta(\rho_\sigma)\rho_\sigma^{-1}M_\sigma = \delta(M_\sigma)$, i.e., the entries of $M_\sigma = (m_{i,j})_{1\le i,j\le n}$

are solutions of $\delta(y) = \frac{1}{n}\delta(\rho_\sigma)\rho_\sigma^{-1}y$. Let i_0, j_0 be such that $m_{i_0,j_0} \neq 0$. Then $M_\sigma = m_{i_0,j_0}M'$ with $M' = (1/m_{i_0,j_0})M_\sigma \in \mathrm{GL}_n(\widetilde{\mathbf{k}}^\delta) = \mathrm{GL}_n(\mathbf{C})$, whence the desired result.

To conclude the proof, we have to show that if (9) has a solution B in $\mathbf{L}^{n\times n}$ then it has a solution in $\mathbf{K}^{n\times n}$. This can be proved by using an argument similar to the descent argument in the proof of Proposition 2.6. □

3. Hypertranscendence of solutions of Mahler equations

Now, we focus our attention on Mahler equations.

We use the notation of Section 1.2: $p \geq 2$ is an integer, $\mathbf{K} := \bigcup_{j\geq 1}\mathbb{C}(z^{1/j})$ and $\mathbf{K}' := \mathbf{K}(\log(z))$. We endow $\mathbf{K}$ with the structure of ϕ-field given by $\phi(f(z)) := f(z^p)$. We endow $\mathbf{K}' := \mathbf{K}(\log(z))$ with the structure of ϕ-field given by $\phi(f(z, \log(z))) := f(z^p, p\log(z))$. We have $\mathbf{K}^\phi = \mathbf{K}'^\phi = \mathbb{C}$.

The derivation

$$\delta := z\log(z)\frac{d}{dz}$$

gives a structure of (ϕ, δ)-field over $\mathbf{K}'$ (so δ commutes with ϕ, and this is why we work with δ instead of a simplest derivation). We also set

$$\vartheta := z\frac{d}{dz}.$$

We let $\widetilde{\mathbb{C}}$ denote a differential closure of $(\mathbb{C}, \delta)$. We have $\widetilde{\mathbb{C}}^\delta = \mathbb{C}$. As in Lemma 2.3, we consider $\mathbf{L} = \mathrm{Frac}(\widetilde{\mathbb{C}} \otimes_{\mathbb{C}} \mathbf{K}') = \bigcup_{j\geq 1}\widetilde{\mathbb{C}}(z^{1/j})(\log(z))$, which is a (ϕ, δ)-field extension of $\mathbf{K}'$ such that $\mathbf{L}^\phi = \widetilde{\mathbb{C}}$.

3.1. Homogeneous Mahler equations of order one

In this section, we consider the difference equation of order one

$$\phi(y) = ay \tag{10}$$

where $a \in \mathbb{C}(z)^\times$. We let S be a PPV ring for (10) over $\mathbf{L}$.

Since S is an $\mathbf{L}$-(ϕ, δ)-algebra, it can be seen as a $\mathbb{C}(z)$-ϑ-algebra (i.e., over the differential field $(\mathbb{C}(z), \vartheta)$) by letting ϑ act as $\frac{1}{\log(z)}\delta$.

Proposition 3.1. *Let R be a $\mathbf{K}'$-(ϕ, δ)-algebra such that $R^\phi = \mathbb{C}$. Let u be an invertible element of R such that $\phi(u) = au$. The following statements are equivalent:*

(i) *u is hyperalgebraic over $(\mathbb{C}(z), \vartheta)$;*[3]
(ii) *$\mathrm{Gal}^\delta(\mathcal{Q}_S/\mathbf{L})$ is conjugate to a subgroup of $\mathbb{C}^\times$;*
(iii) *there exists $d \in \mathbb{C}(z)$ such that $\vartheta(a) = a(p\phi(d) - d)$;*
(iv) *there exist $c \in \mathbb{C}^\times$, $m \in \mathbb{Z}$ and $f \in \mathbb{C}(z)^\times$ such that $a = cz^m\frac{\phi(f)}{f}$.*

[3] Of course, u is hyperalgebraic over $(\mathbb{C}(z), \vartheta)$ if and only if it is hyperalgebraic over $(\mathbb{C}(z), d/dz)$.

Proof. We first prove the implication (iii)⇒(ii). Assume that there exists $d \in \mathbb{C}(z)$ such that $\vartheta(a) = a(p\phi(d) - d)$. Then $d_1 := d\log(z) \in \mathbf{K}'$ satisfies $\delta(a) = a(\phi(d_1) - d_1)$, and hence $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{L})$ is conjugate to a subgroup of $\mathbb{C}^{\times}$ in virtue of Proposition 2.8.

We now prove (ii)⇒(iii). We assume that $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{L})$ is conjugate to a subgroup of $\mathbb{C}^{\times}$. By Proposition 2.8, there exists $d_1 \in \mathbf{K}'$ such that $\delta(a) = a(\phi(d_1) - d_1)$. Therefore,

$$\vartheta(a) = a(p\phi(d_2) - d_2) \tag{11}$$

with $d_2 := d_1/\log(z) \in \mathbf{K}'$. We shall now prove that there exists $d_3 \in \mathbf{K}$ such that $\vartheta(a) = a(p\phi(d_3) - d_3)$. Indeed, let $u(X, Y) \in \mathbb{C}(X, Y)$, $k \geq 1$ and $v \in \mathbb{C}(X)$ be such that $d_2 = u(z^{1/k}, \log(z))$ and $\vartheta(a)/a = v(z)$. Equation (11) can be rewritten as

$$v(z) = pu(z^{p/k}, p\log(z)) - u(z^{1/k}, \log(z)).$$

Since $z^{1/k}$ and $\log(z)$ are algebraically independent over $\mathbb{C}$, we get

$$v(X^k) = pu(X^p, pY) - u(X, Y).$$

We view $u(X, Y)$ as an element of $\mathbb{C}(X)(Y) \subset \mathbb{C}(X)((Y))$ as follows: $u(X, Y) = \sum_{j \geq -N} u_j(X)Y^j$ for some $N \in \mathbb{Z}$. We have

$$v(X^k) = pu(X^p, pY) - u(X, Y) = \sum_{j \geq -N} (p^{j+1}u_j(X^p) - u_j(X))Y^j.$$

Equating the coefficients of Y^0, we obtain

$$pu_0(X^p) - u_0(X) = v(X^k).$$

Hence, $d_3 := u_0(z^{1/k})$ has the required property.

We claim that $d_3 \in \mathbb{C}(z)$. Indeed, suppose not. Let $k \geq 2$ be such that $d_3 \in \mathbb{C}(z^{1/k})$. We view d_3 as an element of $\mathbb{C}((z^{1/k}))$: $d_3 = \sum_{j \geq -N} d_{3,j} z^{j/k}$ for some $N \in \mathbb{Z}$. Let $j_0 \in \mathbb{Z}$ be such that $k \nmid j_0$ and $d_{3,j_0} \neq 0$, with $|j_0|$ minimal with this property. Then the coefficient of $z^{j_0/k}$ in $p\phi(d_3) - d_3$ is nonzero, contrary to $p\phi(d_3) - d_3 \in \mathbb{C}(z)$. This proves (iii).

We now prove (iii)⇒(i). Assume there exists $d \in \mathbb{C}(z)$ such that $\vartheta(a) = a(p\phi(d) - d)$. Then $d_1 := d\log(z) \in \mathbf{K}'$ satisfies $\delta(a) = a(\phi(d_1) - d_1)$, and hence Proposition 2.6 ensures that u is hyperalgebraic over $(\mathbf{K}', \delta)$. Therefore, u is hyperalgebraic over $(\mathbf{K}', \vartheta)$ and the conclusion follows, as $(\mathbf{K}', \vartheta)$ is hyperalgebraic over $(\mathbb{C}(z), \vartheta)$.

We now prove (i)⇒(iii). Proposition 2.6, applied to the difference equation $\phi(y) = ay$ over the (ϕ, δ)-field $\mathbf{K}'$, ensures that there exist $\mathcal{L}_1 := \sum_{i=1}^{\nu} \beta_i \delta^i$ with coefficients $\beta_1, \ldots, \beta_\nu = 1$ in $\mathbb{C}$ and $g_1 \in \mathbb{C}(z^{1/k}, \log(z))$ such that

$$\mathcal{L}_1\left(\frac{\delta(a)}{a}\right) = \phi(g_1) - g_1. \tag{12}$$

We shall now prove that there exists $g_2 \in \mathbb{C}(z^{1/k})$ such that

$$\vartheta^{\nu}\left(\frac{\vartheta(a)}{a}\right) = p^{\nu+1}\phi(g_2) - g_2.$$

Indeed, it is easily seen that there exists $v(X, Y) \in \mathbb{C}(X)[Y]$ such that $\mathcal{L}_1\left(\frac{\delta(a)}{a}\right) = v(z, \log(z))$. Using the fact that $\delta^i = \log(z)^i \vartheta^i$ + terms of lower degree in $\log(z)$, we see that

$$\mathcal{L}_1\left(\frac{\delta(a)}{a}\right) = \mathcal{L}_1\left(\log(z)\frac{\vartheta(a)}{a}\right) = \vartheta^{\nu}\left(\frac{\vartheta(a)}{a}\right)(\log(z))^{\nu+1} + \text{ terms of lower degree in } \log(z).$$

On the other hand, let $u(X, Y) \in \mathbb{C}(X, Y)$ and $k \geq 1$ be such that $g_1 = u(z^{1/k}, \log(z))$. Then (12) can be rewritten as

$$v(z, \log(z)) = u(z^{p/k}, p\log(z)) - u(z^{1/k}, \log(z)).$$

Since $z^{1/k}$ and $\log(z)$ are algebraically independent over $\mathbb{C}$, we get

$$v(X^k, Y) = u(X^p, pY) - u(X, Y).$$

We again view $u(X, Y)$ as an element of $\mathbb{C}(X)(Y) \subset \mathbb{C}(X)((Y))$ as follows: $u(X, Y) = \sum_{j\geq -N} u_j(X)Y^j$ for some $N \in \mathbb{Z}$. So,

$$v(X^k, Y) = u(X^p, pY) - u(X, Y) = \sum_{j\geq -N}(p^j u_j(X^p) - u_j(X))Y^j.$$

Equating the coefficients of $Y^{\nu+1}$, and letting $X = z^{1/k}$, we obtain

$$p^{\nu+1}u_{\nu+1}(z^{p/k}) - u_{\nu+1}(z^{1/k}) = \vartheta^{\nu}\left(\frac{\vartheta(a)}{a}\right).$$

Therefore, $g_2 = u_{\nu+1}(z^{1/k}) \in \mathbb{C}(z^{1/k})$ has the required property. One can show that $g_2 \in \mathbb{C}(z)$ by arguing as for $d_3 \in \mathbb{C}(z)$ in the proof of (ii)⇒(iii) above. We now claim that there exists $g_3 \in \mathbb{C}(z)$ such that

$$\frac{\vartheta(a)}{a} = p\phi(g_3) - g_3.$$

If $\nu = 0$, then $g_3 := g_2$ has the expected property. Assume that $\nu > 0$. Let $G_2 = \int \frac{g_2}{z}$ be some primitive of $\frac{g_2}{z}$ that we view as a function on some interval $(0, \epsilon)$, $\epsilon > 0$. We have

$$\vartheta(p^{\nu}\phi(G_2) - G_2) = p^{\nu+1}\phi(g_2) - g_2 = \vartheta^{\nu}\left(\frac{\vartheta(a)}{a}\right),$$

so there exists $C \in \mathbb{C}$ such that

$$p^{\nu}\phi(G_2) - G_2 = \vartheta^{\nu-1}\left(\frac{\vartheta(a)}{a}\right) + C.$$

Hence, $G_3 := G_2 - \frac{C}{p^{\nu}-1}$ satisfies

$$p^{\nu}\phi(G_3) - G_3 = \vartheta^{\nu-1}\left(\frac{\vartheta(a)}{a}\right).$$

But $G_3 = G_4 + \ell$ where $G_4 \in \mathbb{C}(z)$ and ℓ is a $\mathbb{C}$-linear combination of $\log(z)$ and of functions of the form $\log(1-z\xi)$[4] with $\xi \in \mathbb{C}^\times$. Using the $\mathbb{C}$-linear independence of any $\mathbb{C}$-linear combination of $\log(z)$ and of functions of the form $\log(1-z\xi)$ with $\xi \in \mathbb{C}^\times$ with any element of $\mathbb{C}(z)$, we see that the equality

$$p^\nu\phi(G_3) - G_3 = (p^\nu\phi(G_4) - G_4) + (p^\nu\phi(\ell) - \ell) = \vartheta^{\nu-1}\left(\frac{\vartheta(a)}{a}\right)$$

implies that

$$p^\nu\phi(G_4) - G_4 = \vartheta^{\nu-1}\left(\frac{\vartheta(a)}{a}\right).$$

Iterating this argument, we find $g_3 \in \mathbb{C}(z)$ with the expected property. This proves (iii).

We shall now prove (iii)⇒(iv). We assume that there exists $d \in \mathbb{C}(z)$ such that $\vartheta(a) = a(p\phi(d) - d)$. We write $a = cz^m l$ with $c \in \mathbb{C}^\times$, $m \in \mathbb{Z}$ and $l \in \mathbb{C}(z)$ without pole at 0 and such that $l(0) = 1$. Since $\frac{\vartheta(a)}{a} = \frac{\vartheta(c^{-1}a)}{c^{-1}a}$, we can assume that $c = 1$. A fundamental solution of $\phi(y) = ay$ is given by

$$f_0 = z^{m/(p-1)} \prod_{k\geq 0} \phi^k(l)^{-1} \in z^{m/(p-1)}\mathbb{C}[[z]] \subset \mathbb{C}((z^{1/(p-1)})).$$

We have $\delta(a)a^{-1} = \phi(\widetilde{d}) - \widetilde{d}$ with $\widetilde{d} = \log(z)d$. This is the integrability condition for the system of equations

$$\begin{cases} \phi(y) = ay, \\ \delta(y) = \widetilde{d}y, \text{ i.e., } \vartheta(y) = dy. \end{cases}$$

A straightforward calculation shows that $\delta(f_0) - \widetilde{d}f_0$ is a solution of $\phi(y) = ay$, so there exists $q \in \mathbb{C}$ such that $\delta(f_0) = (q + \widetilde{d})f_0$, i.e., $\log(z)\vartheta(f_0) = (q + \log(z)d)f_0$ (here, we work in the (ϕ,δ)-field $\mathbb{C}((z^{1/(p-1)}))(\log(z))$ and we have used the fact that the field of ϕ-constants of $\mathbb{C}((z^{1/(p-1)}))(\log(z))$ is $\mathbb{C}$, so that the solutions of $\phi(y) = ay$ in $\mathbb{C}((z^{1/(p-1)}))(\log(z))$ are of the form λf_0 for some $\lambda \in \mathbb{C}$). Therefore, $\vartheta(f_0) = df_0$. So, f_0 satisfies a nonzero linear differential equation with coefficients in **K**, and also a nonzero linear Mahler equation with coefficients in **K**. It follows from [Béz94, Theorem 1.3] that $f_0 \in \mathbb{C}(z^{1/(p-1)})$. Therefore, $f_0 = z^{m/(p-1)}h$ for some $h \in \mathbb{C}(z)$, and hence $a = \phi(f_0)f_0^{-1} = z^m\phi(h)h^{-1}$.

We shall now prove (iv)⇒(iii). We assume that there exist $c \in \mathbb{C}^\times$, $m \in \mathbb{Z}$ and $f \in \mathbb{C}(z)^\times$ such that $a = cz^m\frac{\phi(f)}{f}$. Then we have $\vartheta(a)/a = p\phi(d) - d$ with $d = m/(p-1) + \vartheta(f)/f \in \mathbb{C}(z)$, whence the desired result. □

Remark 3.2. The techniques employed above could also be used to recover a famous result of Nishioka [Nis84] about the hypertranscendence of solutions of inhomogeneous Mahler equations of order one. A Galoisian approach (but without parametrized Picard–Vessiot theory) to the work of Nishioka has been proposed by Nguyen [Ngu11].

[4] Here, $\log(z)$ is the principal determination of the logarithm, and $\log(1-z\xi)$ is such that $\log(1-0\xi) = 0$

3.2. Mahler equations of higher order with large classical difference Galois group

Consider the difference system

$$\phi(Y) = AY \tag{13}$$

with $A \in \mathrm{GL}_n(\mathbb{C}(z))$. We let S be a PPV ring for (13) over $\mathbf{L}$. The aim of the present section is to study the parametrized difference Galois group $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{L})$ of (13) over $\mathbf{L}$ under the following assumption.

Assumption 3.3. The difference Galois group of (13) over the ϕ-field $\mathbf{K}$ contains $\mathrm{SL}_n(\mathbb{C})$.

Note the following result.

Lemma 3.4. *Assume that Assumption 3.3 holds. Then the difference Galois group of (13) over the ϕ-field $\mathbf{L}$ contains $\mathrm{SL}_n(\widetilde{\mathbb{C}})$.*

Proof. Corollary 1.7 ensures that the difference Galois group of (13) over the ϕ-field $\mathbf{K}'$ contains $\mathrm{SL}_n(\mathbb{C})$. The conclusion is now a direct consequence of [CHS08, Corollary 2.5]. □

Let $U \in \mathrm{GL}_n(S)$ be a fundamental matrix of solutions of (13) and set

$$d := \det(U) \in S^{\times}.$$

Then d is a fundamental solution of $\phi(y) = \det(A)y$ in S. We split our study of $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{L})$ into two cases, depending on whether d is hyperalgebraic or hypertranscendental over $(\mathbf{L}, \delta)$. Note that Proposition 3.1 may be used to check whether d is hyperalgebraic or not.

3.2.1. Hyperalgebraic determinant. This section is devoted to the proof of the following result.

Theorem 3.5. *Assume that Assumption 3.3 holds and d is hyperalgebraic over $(\mathbb{C}(z), \vartheta)$ (or equivalently the parametrized difference Galois group of $\phi(y) = \det(A)y$ over $\mathbf{L}$ is included in $\mathbb{C}^{\times}$; see Proposition 3.1). Then the parametrized difference Galois group $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{L})$ is a subgroup of $\mathbb{C}^{\times}\mathrm{SL}_n(\widetilde{\mathbb{C}})$ containing $\mathrm{SL}_n(\widetilde{\mathbb{C}})$.*

Before proceeding to the proof of this theorem, we give some lemmas.

Lemma 3.6. *Assume that Assumption 3.3 holds and d is hyperalgebraic over $(\mathbb{C}(z), \vartheta)$ (or equivalently the parametrized difference Galois group of $\phi(y) = \det(A)y$ over $\mathbf{L}$ is included in $\mathbb{C}^{\times}$; see Proposition 3.1). Then either*

(i) *$\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{L})$ is conjugate to a subgroup of $\mathrm{GL}_n(\mathbb{C})$ containing $\mathrm{SL}_n(\mathbb{C})$, or*
(ii) *$\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{L})$ is equal to a subgroup of $\mathbb{C}^{\times}\mathrm{SL}_n(\widetilde{\mathbb{C}})$ containing $\mathrm{SL}_n(\widetilde{\mathbb{C}})$.*

Moreover, (i) *holds if and only if there exists $B \in \mathbf{K}^{n\times n}$ such that*

$$p\phi(B) = ABA^{-1} + \vartheta(A)A^{-1}. \tag{14}$$

Proof. Using Proposition 2.9, we are reduced to proving that the equation

$$\phi(B) = ABA^{-1} + \delta(A)A^{-1} \tag{15}$$

has a solution $B \in \mathbf{K}'^{n\times n}$ if and only if (14) has a solution $B \in \mathbf{K}^{n\times n}$.

Assume that (15) has a solution $B \in \mathbf{K}'^{n\times n}$. Let $u(X,Y) \in \mathbb{C}(X,Y)^{n\times n}$, $k \geq 1$, $v(X) \in \mathrm{GL}_n(\mathbb{C}(X))$ and $w(X) \in \mathbb{C}(X)^{n\times n}$ be such that

$$B = u(z^{1/k}, \log(z)), \qquad A = v(z), \qquad \delta(A)A^{-1} = \log(z)w(z).$$

Then (15) can be rewritten as

$$u(z^{p/k}, p\log(z)) = v(z)u(z^{1/k}, \log(z))v(z)^{-1} + \log(z)w(z).$$

Since $z^{1/k}$ and $\log(z)$ are algebraically independent over $\mathbb{C}$, we get

$$u(X^p, pY) = v(X^k)u(X,Y)v(X^k)^{-1} + Yw(X^k).$$

We view $u(X,Y)$ as an element of $\mathbb{C}(X)(Y)^{n\times n} \subset \mathbb{C}(X)((Y))^{n\times n}$ by writing $u(X,Y) = \sum_{j\geq -N} u_j(X)Y^j$ for some $N \in \mathbb{Z}$. We have

$$\sum_{j\geq -N} u_j(X^p)p^jY^j = \sum_{j\geq -N} v(X^k)u_j(X)v(X^k)^{-1}Y^j + Yw(X^k).$$

Equating the terms of degree 1 in Y, we get

$$pu_1(X^p) = v(X^k)u_1(X)v(X^k)^{-1} + w(X^k).$$

Therefore, $B_1 := u_1(z^{1/k}) \in \mathbf{K}$ is a solution of (14).

Conversely, assume that (14) has a solution $B \in \mathbf{K}^{n\times n}$. Then $B_1 := B\log(z) \in \mathbf{K}'^{n\times n}$ satisfies $\phi(B_1) = AB_1A^{-1} + \delta(A)A^{-1}$. □

Lemma 3.7. *Assume that the system $\phi(Y) = BY$, with $B \in \mathrm{GL}_n(\mathbf{K}')$, has a solution $u = (u_1, \dots, u_n)^t$ with coefficients in $\mathbb{C}((z^{1/k}))$ for some integer $k \geq 1$. Then there exists a PPV ring T for $\phi(Y) = BY$ over $\mathbf{L}$ that contains the $\mathbf{L}$-δ-algebra $\mathbf{L}\{u_1, \dots, u_n\}$.*

Proof. The result is obvious if $u = (0, \dots, 0)^t$. We shall now assume that $u \neq (0, \dots, 0)^t$. We consider the field $\widehat{\mathbf{K}'} := \bigcup_{j\geq 1} \mathbb{C}((z^{1/j}))(\log(z))$. We equip $\widehat{\mathbf{K}'}$ with the structure of (ϕ,δ)-field given by $\phi(f(z,\log(z))) = f(z^p, p\log(z))$ and $\delta = \log(z)z\frac{d}{dz}$. It is easily seen that $\widehat{\mathbf{K}'}^{\phi} = \mathbb{C}$. One can view $\mathbf{K}'$ as a (ϕ,δ)-subfield of $\widehat{\mathbf{K}'}$. We let $F = \mathbf{K}'\langle u_1, \dots, u_n\rangle$ be the δ-subfield of $\widehat{\mathbf{K}'}$ generated over $\mathbf{K}'$ by $u_1, \dots, u_n$; this is a (ϕ,δ)-subfield of $\widehat{\mathbf{K}'}$ such that $F^{\phi} = \mathbb{C}$. By Lemma 2.3, $\widetilde{\mathbb{C}} \otimes_{\mathbb{C}} F$ is an integral domain and its field of fractions $\mathbf{L}_1 = \mathbf{L}\langle u_1, \dots, u_n\rangle$ is a (ϕ,δ)-field such that $\mathbf{L}_1^{\phi} = \widetilde{\mathbb{C}}$. We consider a PPV ring S_1 for $\phi(Y) = BY$ over $\mathbf{L}_1$ and we let $U \in \mathrm{GL}_n(S_1)$ be a fundamental matrix of solutions of this difference system. We can assume that the first column of U is u. Then the $\mathbf{L}$-(ϕ,δ)-algebra T generated by the entries of U and by $\det(U)^{-1}$ contains $\mathbf{L}\{u_1, \dots, u_n\}$ and is a PPV ring for $\phi(Y) = BY$ over $\mathbf{L}$, whence the result. □

Lemma 3.8. *Let $u = (u_1, \ldots, u_n)^t$ be a vector with coefficients in $\widehat{\mathbf{K}} := \bigcup_{j\geq 1} \mathbb{C}((z^{1/j}))$ such that $\phi(u) = Bu$ for some $B \in \mathrm{GL}_n(\mathbf{K})$. Assume moreover that each u_i satisfies some nonzero linear differential equation with coefficients in $\bigcup_{j\geq 1} \widetilde{\mathbb{C}}(z^{1/j})$, with respect to the derivation ϑ. Then the u_i actually belong to $\mathbf{K}$.*

Proof. According to the cyclic vector lemma, there exists $P \in \mathrm{GL}_n(\mathbf{K})$ such that $Pu = (f, \phi(f), \ldots, \phi^{n-1}(f))^t$ for some $f \in \widehat{\mathbf{K}}$ which is a solution of a nonzero linear Mahler equation (i.e., a ϕ-difference equation) of order n with coefficients in $\mathbf{K}$. Moreover, f satisfies a nonzero linear differential equation with coefficients in $\bigcup_{j\geq 1} \widetilde{\mathbb{C}}(z^{1/j})$, with respect to the derivation ϑ, because it is a $\mathbf{K}$-linear combination of the u_i and the u_i themselves satisfy such equations. It follows from [Béz94, Theorem 1.3] that f belongs to $\mathbf{K}$. Hence, the entries of $u = P^{-1}(Pu) = P^{-1}(f, \phi(f), \ldots, \phi^{n-1}(f))^t$ actually belong to $\mathbf{K}$, as expected. □

Lemma 3.9. *There exists $c \in \mathbb{C}^\times$ such that the difference system $\phi(Y) = c^{-1}AY$ has a nonzero solution $u = (u_1, \ldots, u_n)^t$ with coefficients in $\widehat{\mathbf{K}} := \bigcup_{j\geq 1} \mathbb{C}((z^{1/j}))$.*

Proof. According to [Roq15, Section 4], the system $\phi(Y) = AY$ is triangularizable over $\widehat{\mathbf{K}}$, i.e., there exists $\widehat{P} \in \mathrm{GL}_n(\widehat{\mathbf{K}})$ such that $\phi(\widehat{P})^{-1}A\widehat{P} =: (v_{i,j})_{1\leq i,j\leq n}$ is upper-triangular. Let $c \in \mathbb{C}^\times$, $m \in \mathbb{Z}$ and $l \in 1 + z^{1/k}\mathbb{C}[[z^{1/k}]]$ be such that $v_{1,1} = cz^m l$. We consider $A_1 = c^{-1}A \in \mathrm{GL}_n(\mathbb{C}(z))$. Then the system $\phi(Y) = A_1Y$ has a nonzero solution with entries in $\widehat{\mathbf{K}}$, namely $u = (u_1, \ldots, u_n)^t := \widehat{P}(f, 0, \ldots, 0)^t$ with $f := z^{m/(p-1)} \prod_{j\geq 0} \phi^j(l)^{-1}$. □

Proof of Theorem 3.5. We let $c \in \mathbb{C}^\times$ and $u = (u_1, \ldots, u_n)^t$ be as in Lemma 3.9, and we set $A_1 := c^{-1}A \in \mathrm{GL}_n(\mathbb{C}(z))$. Thanks to Lemma 3.7, we can consider a PPV ring S_1 for $\phi(Y) = A_1Y$ over $\mathbf{L}$ that contains $\mathbf{L}\{u_1, \ldots, u_n\}$. We let $U_1 \in \mathrm{GL}_n(S_1)$ be a fundamental matrix of solutions of $\phi(Y) = A_1Y$ whose first column is u.

Let G denote the difference Galois group of $\phi(Y) = AY$ over the ϕ-field $\mathbf{K}$, and let G^δ be its parametrized difference Galois group over the (ϕ, δ)-field $\mathbf{L}$. Similarly, G_1 is the difference Galois group of $\phi(Y) = A_1Y$ over the ϕ-ring $\mathbf{K}$, and G_1^δ is its parametrized difference Galois group over the (ϕ, δ)-field $\mathbf{L}$.

We have $G_1^{\mathrm{der}} = G^{\mathrm{der}} = \mathrm{SL}_n(\mathbb{C})$, so G_1 contains $\mathrm{SL}_n(\mathbb{C})$. Moreover, the parametrized difference Galois group of $\phi(y) = \det(A_1)y = c^{-n}\det(A)y$ over $\mathbf{L}$ is a subgroup of $\mathbb{C}^\times$ (because the parametrized difference Galois group of $\phi(y) = \det(A)y$ over $\mathbf{L}$ is a subgroup of $\mathbb{C}^\times$ by hypothesis, and the parametrized difference Galois group of $\phi(y) = c^{-n}y$ over $\mathbf{L}$ has the same property).

We claim that G_1^δ is a subgroup of $\mathbb{C}^\times\mathrm{SL}_n(\widetilde{\mathbb{C}})$ that contains $\mathrm{SL}_n(\widetilde{\mathbb{C}})$. Indeed, according to Lemma 3.6, it is sufficient to prove that there is no $B \in \mathbf{K}^{n\times n}$ such that $\vartheta(A_1) = p\phi(B)A_1 - A_1B$. Suppose such a B exists. The equation $\vartheta(A_1) = p\phi(B)A_1 - A_1B$, which can be rewritten as $\delta(A_1) = \phi(\log(z)B)A_1 - A_1(\log(z)B)$, ensures the integrability of the system of equations

$$\begin{cases} \phi(Y) = A_1Y, \\ \delta(Y) = (\log(z)B)Y. \end{cases}$$

So, there exists $D \in \mathrm{GL}_n(\widetilde{\mathbb{C}})$ such that $V := U_1 D \in \mathrm{GL}_n(S_1)$ satisfies

$$\begin{cases} \phi(V) = A_1 V, \\ \delta(V) = (\log(z)B)V, \text{ i.e., } \vartheta(V) = BV. \end{cases}$$

Hence, we have the equalities $\vartheta(U_1)D + U_1\vartheta(D) = \vartheta(U_1 D) = \vartheta(V) = BV = BU_1 D$, so $\vartheta(U_1) = BU_1 - U_1\vartheta(D)D^{-1}$. This formula implies that the (finite-dimensional) $\bigcup_{j\geq 1} \widetilde{\mathbb{C}}(z^{1/j})$-vector space generated by the entries of U_1 is stable by ϑ. In particular, any u_i (recall that the u_i are the entries of the first column of U) satisfies a nonzero linear differential equation with coefficients in $\bigcup_{j\geq 1} \widetilde{\mathbb{C}}(z^{1/j})$, with respect to the derivation ϑ. It follows from Lemma 3.8 that the u_i belong to $\mathbf{K}$. Hence, the first column of U_1 is fixed by the Galois group G_1, and this contradicts the fact that G_1 contains $\mathrm{SL}_n(\mathbb{C})$.

Therefore, $(G^{\delta})^{\mathrm{der}} = (G_1^{\delta})^{\mathrm{der}}$ contains $\mathrm{SL}_n(\widetilde{\mathbb{C}})$. Now, the theorem follows from Lemma 3.6. □

3.2.2. Hypertranscendental determinant. In the case of a hypertranscendental determinant, we can reduce the computation of the parametrized difference Galois group to a question concerning the existence of a rational solution of a given Mahler equation as follows.

Lemma 3.10. *Assume that Assumption 3.3 holds and d is hypertranscendental over $(\mathbb{C}(z), \theta)$ (or equivalently the parametrized difference Galois group of $\phi(y) = \det(A)y$ over $\mathbf{L}$ is equal to $\widetilde{\mathbb{C}}^{\times}$). Then either*

(i) $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{L})$ *is conjugate to* $\widetilde{\mathbb{C}}^{\times}\mathrm{SL}_n(\mathbb{C})$, *or*
(ii) $\mathrm{Gal}^{\delta}(\mathcal{Q}_S/\mathbf{L})$ *is equal to* $\mathrm{GL}_n(\widetilde{\mathbb{C}})$.

Moreover, (i) *holds if and only if there exists $B \in \mathbf{K}^{n\times n}$ such that*

$$p\phi(B) = ABA^{-1} + \vartheta(A)A^{-1} - \frac{1}{n}\vartheta(\det(A))\det(A)^{-1}I_n. \tag{16}$$

Proof. Note that d is hypertranscendental over $(\mathbf{L}, \delta)$. Using Proposition 2.10, it remains to prove that the equation

$$\phi(B) = ABA^{-1} + \delta(A)A^{-1} - \frac{1}{n}\delta(\det(A))\det(A)^{-1}I_n \tag{17}$$

has a solution $B \in \mathbf{K}'^{n\times n}$ if and only if (16) has a solution $B \in \mathbf{K}^{n\times n}$. The proof of this fact is similar to the proof of Lemma 3.6. □

Unlike the situation in Section 3.2.1, it is not completely obvious that we can bypass the search of rational solutions of (16) to decide which of the two options of Lemma 3.10 is satisfied. However, we can still directly get some information on hypertranscendence of solutions in $\bigcup_{j\geq 1} \mathbb{C}(z^{1/j})$ as follows.

Theorem 3.11. *Assume that Assumption 3.3 holds and d is hypertranscendental over $(\mathbb{C}(z), \vartheta)$. Assume that the difference system $\phi(Y) = AY$ admits a nonzero solution $u = (u_1, \dots, u_n)^t$ with coefficients in $\mathbb{C}((z^{1/k}))$ for some integer $k \geq 1$. Then at least one of the u_i is hypertranscendental over $(\mathbb{C}(z), \vartheta)$.*

Note the following immediate corollary, which is particularly interesting when one works with difference equations rather than difference systems.

Corollary 3.12. *Assume that Assumption 3.3 holds and d is hypertranscendental over $(\mathbb{C}(z), \vartheta)$. Assume that the difference system $\phi(Y) = AY$ admits a nonzero solution $u = (f, \phi(f), \dots, \phi^{n-1}(f))^t$ for some $f \in \mathbb{C}((z^{1/k}))$ and some integer $k \geq 1$. Then f (and hence any $\phi^i(f)$) is hypertranscendental over $(\mathbb{C}(z), \vartheta)$.*

The arguments employed in the proof of Theorem 3.11 given below are very similar to the ones used in the hyperalgebraic case, but we need a new descent argument, contained in the following lemma.

Lemma 3.13. *Let L be a δ-field and let $L\langle a\rangle$ and $L\langle b_1, \dots, b_n\rangle$ be two δ-field extensions of L, both contained in the same δ-field extension of L. Assume that a is hypertranscendental over L and any b_i is hyperalgebraic over L. Then the field extensions $L\langle a\rangle$ and $L\langle b_1, \dots, b_n\rangle$ are linearly disjoint over L.*

Proof. If $L\langle a\rangle$ and $L\langle b_1, \dots, b_n\rangle$ are not linearly disjoint over L then a is hyperalgebraic over $L\langle b_1, \dots, b_n\rangle$. This implies that the differential transcendence degree of $L\langle a, b_1, \dots, b_n\rangle$ over $L\langle b_1, \dots, b_n\rangle$ is zero. Since the differential transcendence degree of $L\langle b_1, \dots, b_n\rangle$ over L is also zero by hypothesis, we find that the differential transcendence degree of $L\langle a, b_1, \dots, b_n\rangle$ over L is zero by the classical properties of transcendence degree. This implies that a is hyperalgebraic over L. □

Proof of Theorem 3.11. Thanks to Lemma 3.7, we can assume that the PPV ring S for $\phi(Y) = AY$ over $\mathbf{L}$ contains $\mathbf{L}\{u_1, \dots, u_n\}$. We can assume that the first column of the fundamental matrix of solutions $U \in \mathrm{GL}_n(S)$ of $\phi(Y) = AY$ is u.

We let G denote the difference Galois group of $\phi(Y) = AY$ over the ϕ-field $\mathbf{K}$, and G^δ its parametrized difference Galois group over the (ϕ, δ)-ring $\mathbf{L}$. Since d is hypertranscendental over $\mathbf{L}$, the parametrized difference Galois group of $\phi(y) = \det(A)y$ over $\mathbf{L}$ is $\widetilde{\mathbb{C}}^\times$.

Since Lemma 3.10 implies that G^δ is Kolchin-connected, S is an integral domain.

We claim that at least one of the u_i is hypertranscendental over $\mathbf{L}$. Suppose to the contrary that all of them are hyperalgebraic. In particular, G^δ is a strict subgroup of $\mathrm{GL}_n(\widetilde{\mathbb{C}})$. Lemma 3.10 ensures that there exists $B \in \mathbf{K}^{n\times n}$ such that

$$p\phi(B) = ABA^{-1} + \vartheta(A)A^{-1} - \frac{1}{n}\vartheta(\det(A))\det(A)^{-1}I_n. \tag{18}$$

This equation can be rewritten as

$$\phi(B_0) = AB_0A^{-1} + \delta(A)A^{-1} - \frac{1}{n}\delta(\det(A))\det(A)^{-1}I_n,$$

where $B_0 = \log(z)B$. Set $B_1 := B_0 + \frac{\delta(d)}{nd}$. Note that

$$\phi(B_1) = AB_1A^{-1} + \delta(A)A^{-1}.$$

This equation ensures the integrability of the system of equations

$$\begin{cases}\phi(Y) = AY,\\ \delta(Y) = B_1 Y.\end{cases}$$

So, there exists $D \in \mathrm{GL}_n(\widetilde{\mathbb{C}})$ such that $V := UD \in \mathrm{GL}_n(S)$ satisfies

$$\begin{cases}\phi(V) = AV,\\ \delta(V) = B_1 V, \text{ i.e., } \vartheta(V) = \left(B + \frac{\vartheta(d)}{nd}\right)V.\end{cases}$$

In particular, $\vartheta(U)D + U\vartheta(D) = \vartheta(U_1 D) = \vartheta(V) = \left(B + \frac{\vartheta(d)}{nd}\right)UD$, so

$$\vartheta(U) = \left(B + \frac{\vartheta(d)}{nd}\right)U - U\vartheta(D)D^{-1}.$$

If we set $F = \bigcup_{j\geq 1} \widetilde{\mathbb{C}}(z^{1/j})$, the previous formula implies that the $F\langle d\rangle$-vector subspace[5] of $\mathcal{Q}_S$ generated by the entries of U and all their successive ϑ-derivatives is of finite dimension. In particular, any u_i satisfies a nonzero linear differential equation $\mathcal{L}_i(y) = 0$ with coefficients in $F\langle d\rangle$, with respect to the derivation ϑ. We can assume that the coefficients of $\mathcal{L}_i(y)$ belong to $F\{d\}$. We write $\mathcal{L}_i(y) = \sum_\alpha L_{i,\alpha}(y)d_\alpha$ where $L_{i,\alpha}(y)$ is a linear differential operator with coefficients in F, with respect to the derivation ϑ, and d_α is a monomial in the $\vartheta^i(d)$'s. By Lemma 3.13, the ϑ-fields $F\langle d\rangle$ and $F\langle u_1, \dots, u_n\rangle$ are linearly disjoint over F. It follows easily that there exists a nonzero $L_{i,\alpha}(y)$ such that $L_{i,\alpha}(u_i) = 0$. Therefore, any u_i satisfies a nonzero linear differential equation with coefficients in F, with respect to the derivation ϑ. It follows from Lemma 3.8 that the u_i belong to $\mathbf{K}$. Hence, the first column of U is fixed by the difference Galois group G, which contradicts the fact that G contains $\mathrm{SL}_n(\mathbb{C})$. □

4. Applications

In this section, we will use the notation introduced at the beginning of Section 3.

4.1. *User-friendly hypertranscendence criteria*

Consider the Mahler system

$$\phi(Y) = AY \quad \text{with } A \in \mathrm{GL}_n(\mathbb{C}(z)). \tag{19}$$

Theorem 4.1. *Assume that the difference Galois group of the Mahler system (19) over the ϕ-field $\mathbf{K}$ contains $\mathrm{SL}_n(\mathbb{C})$ and that $\det A(z)$ is a monomial. Then the following properties hold:*

(i) *The parametrized difference Galois group of the Mahler system (19) over $\mathbf{L}$ is a subgroup of $\mathbb{C}^\times \mathrm{SL}_n(\widetilde{\mathbb{C}})$ containing $\mathrm{SL}_n(\widetilde{\mathbb{C}})$.*

(ii) *Let $u = (u_1, \dots, u_n)^t$ be a nonzero solution of (19) with entries in $\mathbb{C}((z))$. Then the series $u_1, \dots, u_n$ and all their successive derivatives are algebraically independent over $\mathbb{C}(z)$. In particular, any u_i is hypertranscendental over $\mathbb{C}(z)$.*

[5] Here, $F\langle d\rangle$ denotes the ϑ-field extension generated by d over F.

Proof. As $\det A(z)$ is a monomial, Proposition 3.1 ensures that the parametrized difference Galois group of $\phi(y) = \det(A)y$ is included in $\mathbb{C}^\times$, so Theorem 3.5 yields (i).

We claim that $u_1, \ldots, u_n$ are hyperalgebraically independent over $\mathbb{C}(z)$. Suppose to the contrary that they are hyperalgebraically dependent over $\mathbb{C}(z)$. Thanks to Lemma 3.7, there exists a PPV ring S for (19) over $\mathbf{L}$ containing $\mathbf{K}'\{u_1, \ldots, u_n\}$. Let $U \in \mathrm{GL}_n(S)$ be a fundamental matrix of solutions of (19) whose first column is u. Then $\det(U)$ is hyperalgebraic over $\mathbf{L}$ and the elements of the first column of U are hyperalgebraically dependent over $\mathbf{L}$. It follows easily that the δ-transcendence degree of S over $\mathbf{L}$ is at most n^2-2. This contradicts the fact that the δ-dimension of the parametrized difference Galois group of (19) over $\mathbf{L}$, namely $n^2 - 1$, is equal to the δ-transcendence degree of S over $\mathbf{L}$ [HS08, Proposition 6.26]. □

We shall now state a variant of the last theorem for Mahler equations. Consider the Mahler equation

$$a_n(z)y(z^{p^n}) + a_{n-1}(z)y(z^{p^{n-1}}) + \cdots + a_0(z)y(z) = 0 \tag{20}$$

for some integers $p \geq 2$, $n \geq 1$, and some $a_0(z), \ldots, a_n(z) \in \mathbb{C}(z)$ with $a_0(z)a_n(z) \neq 0$. In what follows, by "difference Galois group of (20)", we mean the difference Galois group of the associated system

$$\phi(Y) = AY \quad \text{with} \quad A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & \cdots & \cdots & -\frac{a_{n-1}}{a_n} \end{pmatrix} \in \mathrm{GL}_n(\mathbb{C}(z)). \tag{21}$$

Theorem 4.2. *Assume that the difference Galois group over the ϕ-field $\mathbf{K}$ of the Mahler equation (20) contains $\mathrm{SL}_n(\mathbb{C})$ and $a_n(z)/a_0(z)$ is a monomial. Then the following properties hold:*

(i) *The parametrized difference Galois group of (20) over $\mathbf{L}$ is a subgroup of $\mathbb{C}^\times\mathrm{SL}_n(\widetilde{\mathbb{C}})$ containing $\mathrm{SL}_n(\widetilde{\mathbb{C}})$.*

(ii) *Let $f(z) \in \mathbb{C}((z))$ be a nonzero solution of (20). Then the series $f(z), f(z^p), \ldots, f(z^{p^{n-1}})$ and all their successive derivatives are algebraically independent over $\mathbb{C}(z)$. In particular, $f(z)$ is hypertranscendental over $\mathbb{C}(z)$.*

Proof. Since the determinant of the matrix A given by (21) is a_0/a_n and if $f(z) \in \mathbb{C}((z))$ is a nonzero solution of (20), then $(f(z), f(z^p), \ldots, f(z^{p^{n-1}}))^t$ is a nonzero solution of (21) with entries in $\mathbb{C}((z))$, this theorem is a consequence of Theorem 4.1. □

4.2. The Baum–Sweet sequence

The *Baum–Sweet sequence* $(a_n)_{n\geq 0}$ is the automatic sequence defined by $a_n = 1$ if the binary representation of n contains no block of consecutive 0's of odd length, and $a_n = 0$ otherwise. It is characterized by the recursive equations

$$a_0 = 1, \quad a_{2n+1} = a_n, \quad a_{4n} = a_n, \quad a_{4n+2} = 0.$$

Let $f_{\mathrm{BS}}(z) = \sum_{n\geq 0} a_n z^n$ be the corresponding generating series. The recursive equations show that

$$Y(z) = \begin{pmatrix} f_{\mathrm{BS}}(z) \\ f_{\mathrm{BS}}(z^2) \end{pmatrix}$$

satisfies the Mahler system

$$\phi(Y) = AY \quad \text{where} \quad A = \begin{pmatrix} 0 & 1 \\ 1 & -z \end{pmatrix} \in \mathrm{GL}_2(\mathbf{K}). \tag{22}$$

Here $p=2$, $\mathbf{K}=\bigcup_{j\geq 1} \mathbb{C}(z^{1/j})$ and ϕ is the field automorphism of $\mathbf{K}$ such that $\phi(z)=z^2$.

Theorem 4.3. *The parametrized difference Galois group of* (22) *over* $\mathbf{L}$ *is* $\mu_4 \mathrm{SL}_2(\widetilde{\mathbb{C}})$, *where* $\mu_4 \subset \mathbb{C}^\times$ *is the group of* 4*th roots of unity. The series* $f_{\mathrm{BS}}(z)$, $f_{\mathrm{BS}}(z^2)$ *and all their successive derivatives are algebraically independent over* $\mathbb{C}(z)$.

Proof. According to [Roq15, Theorem 50], the difference Galois group of (22) over the ϕ-field $\mathbf{K}$ is $\mu_4\mathrm{SL}_2(\mathbb{C})$. Now, the result is a direct consequence of Theorem 4.2. □

4.3. *The Rudin–Shapiro sequence*

The *Rudin–Shapiro sequence* $(a_n)_{n\geq 0}$ is the automatic sequence defined by $a_n = (-1)^{b_n}$ where b_n is the number of pairs of consecutive 1's in the binary representation of n. It is characterized by the recurrence relations

$$a_0 = 1, \quad a_{2n} = a_n, \quad a_{2n+1} = (-1)^n a_n.$$

Let $f_{\mathrm{RS}}(z) = \sum_{n\geq 0} a_n z^n$ be the corresponding generating series. The recursive equations show that

$$Y(z) = \begin{pmatrix} f_{\mathrm{RS}}(z) \\ f_{\mathrm{RS}}(-z) \end{pmatrix}$$

satisfies the Mahler system

$$\phi(Y) = AY \quad \text{where} \quad A = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ \frac{1}{z} & -\frac{1}{z} \end{pmatrix} \in \mathrm{GL}_2(\mathbf{K}). \tag{23}$$

Here $p=2$, $\mathbf{K}=\bigcup_{j\geq 1} \mathbb{C}(z^{1/j})$ and ϕ is the field automorphism of $\mathbf{K}$ such that $\phi(z)=z^2$.

Theorem 4.4. *The parametrized difference Galois group of* (23) *over* $\mathbf{L}$ *is* $\mathrm{GL}_2(\widetilde{\mathbb{C}})$. *The series* $f_{\mathrm{RS}}(z)$, $f_{\mathrm{RS}}(-z)$ *and all their successive derivatives are algebraically independent over* $\mathbb{C}(z)$.

Proof. According to [Roq15, Theorem 54], the difference Galois group of (23) over the ϕ-field $\mathbf{K}$ is $\mathrm{GL}_2(\mathbb{C})$. Now, the result is a direct consequence of Theorem 4.1. □

4.4. *Direct sum of the Baum–Sweet and the Rudin–Shapiro equations*

The aim of this section is to illustrate how one can use the results of this paper to prove the hyperalgebraic independence of Mahler function solutions of distinct equations.

4.4.1. A differential-group-theoretic preliminary result. Let G° denote the neutral component of the linear algebraic group G, and G^{der} its derived subgroup. We recall that G° and G^{der} are Zariski-closed in G.

We let G°_δ} denote the neutral component of the differential algebraic group G (so here we consider Kolchin's topology), and let G^{der} be the derived subgroup of G. In general, G^{der} is not Kolchin-closed; let G^{der_δ} denote its Kolchin-closure in G.

Theorem 4.5. *Let* $\mathbf{k}$ *be a differentially closed* δ*-field. Let* $r \geq 2$ *be an integer and, for any* $i \in \{1, \ldots, r\}$*, let* G_i *be an algebraic subgroup of* $\mathrm{GL}_{n_i}(\mathbf{k})$*. Consider the linear algebraic group* $G = \prod_{i=1}^{r} G_i$*. Assume that, for any* $i \in \{1, \ldots, r\}$*,* $G_i^{\circ,\text{der}}$ *is quasi-simple and* $G^{\circ,\text{der}} = \prod_{i=1}^{r} G_i^{\circ,\text{der}}$*. Let* H *be a Zariski-dense differential algebraic subgroup of* G*. Let* H_i *be the projection of* H *in* $G_i \subset G$*. Then*

(i) *for all* $i \in \{1, \ldots, r\}$*,* $H_i^{\circ_\delta,\text{der}_\delta}$ *is Zariski-dense in* $G_i^{\circ,\text{der}}$*;*
(ii) *we have*

$$H^{\circ_\delta,\text{der}_\delta} = \prod_{i=1}^{r} H_i^{\circ_\delta,\text{der}_\delta} \subset \prod_{i=1}^{r} G_i^{\circ,\text{der}}.$$

Proof. By hypothesis, H is Zariski-dense in G, and hence H_i is Zariski-dense in G_i (because the projection $p_i : G \to G_i$ is continuous for the Zariski topology, and hence $G_i = p_i(G) = p_i(\overline{H}) \subset \overline{p_i(H)}$). Therefore, $H^{\circ_\delta,\text{der}_\delta}$ is Zariski-dense in $G^{\circ,\text{der}} = \prod_{i=1}^{r} G_i^{\circ,\text{der}}$ and $H_i^{\circ_\delta,\text{der}_\delta}$ is Zariski-dense in $G_i^{\circ,\text{der}}$. Recall that the $G_i^{\circ,\text{der}}$ are quasi-simple by hypothesis. It follows from [Cas89, Theorem 15] that

$$H^{\circ_\delta,\text{der}_\delta} = \prod_{i=1}^{r} K_i$$

for some δ-closed subgroups K_i of $G_i^{\circ,\text{der}}$. (In the terminology of [Cas89, Theorem 15], the simple component A_i of $G^{\circ,\text{der}}$ is $\{1\}^{i-1} \times G_i^{\circ,\text{der}} \times \{1\}^{r-i-1}$). We necessarily have $K_i = H_i^{\circ_\delta,\text{der}_\delta}$. □

4.4.2. Baum–Sweet and Rudin–Shapiro

Theorem 4.6. *The parametrized difference Galois group of the direct sum of the systems* (22) *and* (23) *is equal to the direct product of the parametrized difference Galois groups of* (22) *and* (23)*, namely* $\mu_4\mathrm{SL}_2(\widetilde{\mathbb{C}}) \times \mathrm{GL}_2(\widetilde{\mathbb{C}})$*. The series* $f_{\text{BS}}(z)$*,* $f_{\text{BS}}(z^2)$*,* $f_{\text{RS}}(z)$*,* $f_{\text{RS}}(-z)$ *and all their successive derivatives are algebraically independent over* $\mathbb{C}(z)$*.*

Proof. We let M_{BS} and M_{RS} denote the ϕ-modules associated to (22) and (23). It is proved in [Roq15, Section 9.3] that the difference Galois group over $\mathbf{K}$ of $M_{\text{BS}} \oplus M_{\text{RS}}$ is the direct product of the difference Galois groups, i.e., $\mu_4\mathrm{SL}(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$. It follows from Theorems 4.5, 4.3, and 4.4 that the parametrized difference Galois group of $M_{\text{BS}} \oplus M_{\text{RS}}$ contains $\mathrm{SL}_2(\widetilde{\mathbb{C}}) \times \mathrm{SL}_2(\widetilde{\mathbb{C}})$. The fact that the parametrized difference Galois group of $M_{\text{BS}} \oplus M_{\text{RS}}$ is $\mu_4\mathrm{SL}_2(\widetilde{\mathbb{C}}) \times \mathrm{GL}_2(\widetilde{\mathbb{C}})$ is now clear.

The proof of the last assertion is similar to the proof of the last statement of Theorem 4.2. □

Acknowledgments. We would like to thank Michael Singer for discussions and support of this work. We also thank the anonymous referee for interesting suggestions and references.

The first (resp. the second) author would like to thank the ANR-11-LABX-0040-CIMI within the program ANR-11-IDEX-0002-0 for its total (resp. partial) support. The second author's work is also supported by ANR Iso-Galois. This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme under the Grant Agreement No 648132.

References

[AB13] Adamczewski, B., Bell, J. P.: A problem about Mahler functions. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5) **17**, 1301–1355 (2017) Zbl 06836755 MR 3752528

[AF16] Adamczewski, B., Faverjon, C.: Méthode de Mahler, transcendance et relations linéaires : aspects effectifs. J. Théor. Nombres Bordeaux, to appear (2016)

[AF17] Adamczewski, B., Faverjon, C.: Méthode de Mahler : relations linéaires, transcendance et applications aux nombres automatiques. Proc. London Math. Soc. **115**, 55–90 (2017) Zbl 06774679 MR 3669933

[AS03] Allouche, J.-P., Shallit, J.: Automatic Sequences. Theory, Applications, Generalizations. Cambridge Univ. Press, Cambridge (2003) Zbl 1086.11015 MR 1997038

[Bec94] Becker, P.-G.: k-regular power series and Mahler-type functional equations. J. Number Theory **49**, 269–286 (1994) Zbl 0815.11037 MR 1315940

[Beu06] Beukers, F.: A refined version of the Siegel–Shidlovskii theorem. Ann. of Math. (2) **163**, 369–379 (2006) Zbl 1133.11044 MR 2195138

[Béz94] Bézivin, J.-P.: Sur une classe d'équations fonctionnelles non linéaires. Funkcial. Ekvac. **37**, 263–271 (1994) Zbl 0810.39006 MR 1299865

[Cas72] Cassidy, P. J.: Differential algebraic groups. Amer. J. Math. **94**, 891–954 (1972) Zbl 0389.12011 MR 0360611

[Cas89] Cassidy, P. J.: The classification of the semisimple differential algebraic groups and the linear semisimple differential algebraic Lie algebras. J. Algebra **121**, 169–238 (1989) Zbl 0678.14011 MR 0992323

[CHS08] Chatzidakis, Z., Hardouin, C., Singer, M. F.: On the definitions of difference Galois groups. In: Model Theory with Applications to Algebra and Analysis. Vol. 1, London Math. Soc. Lecture Note Ser. 349, Cambridge Univ. Press, Cambridge, 73–109 (2008) Zbl 1234.12005 MR 2441376

[Coh65] Cohn, R. M.: Difference Algebra. Interscience Publ. Wiley, New York (1965) Zbl 0127.26402 MR 0205987

[DVH12] Di Vizio, L., Hardouin, C.: Descent for differential Galois theory of difference equations: confluence and q-dependence. Pacific J. Math. **256**, 79–104 (2012) Zbl 1258.12004 MR 2928542

[Dum93] Dumas, P.: Récurrences mahlériennes, suites automatiques, études asymptotiques. Thèse de l'Université Bordeaux I, https://tel.archives-ouvertes.fr/tel-00614660 (1993) MR 1346304

[DF96] Dumas, P., Flajolet, P.: Asymptotique des récurrences mahlériennes : le cas cyclotomique. J. Théor. Nombres Bordeaux **8**, 1–30 (1996) Zbl 0869.11080 MR 1399944

[HS08] Hardouin, C., Singer, M. F.: Differential Galois theory of linear difference equations. Math. Ann. **342**, 333–377 (2008) Zbl 1163.12002 MR 2425146

[Mah29] Mahler, K.: Arithmetische Eigenschaften der Lösungen einer Klasse von Funktionalgleichungen. Math Ann. **101**, 342–366 (1929); Berichtigung, Math. Ann. **103**, 532 (1930) JFM 55.0115.01 JFM 56.0185.02 MR 151237 MR 1512635

[Mah30a] Mahler, K.: Arithmetische Eigenschaften einer Klasse transzendental-transzendente Funktionen. Math. Z. **32**, 545–585 (1930) JFM 56.0186.01 MR 1545184

[Mah30b] Mahler, K.: Über das Verschwinden von Potenzreihen mehrerer Veränderlichen in speziellen Punktfolgen. Math. Ann. **103**, 573–587 (1930) JFM 56.0185.03 MR 1512638

[MF80] Mendès France, M.: Nombres algébriques et théorie des automates. Enseign. Math. (2) **26**, 193–199 (1981) (1980) Zbl 0472.10037 MR 0610520

[MS13] Mitschi, C., Singer, M. F.: Projective isomonodromy and Galois groups. Proc. Amer. Math. Soc. **141**, 605–617 (2013) Zbl 1268.34187 MR 2996965

[Ngu11] Nguyen, P.: Hypertranscedance [sic] de fonctions de Mahler du premier ordre. C. R. Math. Acad. Sci. Paris **349**, 943–946 (2011) Zbl 1268.12004 MR 2838240

[Ngu12] Nguyen, P.: Équations de Mahler et hypertranscendance. Thèse de l'Institut de Mathématiques de Jussieu (2012)

[Nis84] Nishioka, K.: A note on differentially algebraic solutions of first order linear difference equations. Aequationes Math. **27**, 32–48 (1984) Zbl 0542.12012 MR 0758857

[Nis96] Nishioka, K.: Mahler Functions and Transcendence. Lecture Notes in Math. 1631, Springer, Berlin (1996) Zbl 0876.11034 MR 1439966

[Pel09] Pellarin, F.: An introduction to Mahler's method for transcendence and algebraic independence. In: G. Boeckle et al. (eds.), to appear in the EMS proceedings of the conference "Hodge Structures, Transcendence and Other Motivic Aspects" (2009)

[Phi15] Philippon, P.: Groupes de Galois et nombres automatiques. J. London Math. Soc. **92**, 596–614 (2015) Zbl 06522279 MR 3431652

[Ran92] Randé, B.: Équations fonctionnelles de Mahler et applications aux suites p-régulières. Thèse de l'Université Bordeaux I, https://tel.archives-ouvertes.fr/tel-01183330 (1992)

[Roq15] Roques, J.: Algebraic relations between Mahler functions. Trans. Amer. Math. Soc. **370**, 321–355 (2018) Zbl 1376.39002 MR 3717982

[vdPS97] van der Put, M., Singer, M. F.: Galois Theory of Difference Equations. Lecture Notes in Math. 1666, Springer, Berlin (1997) Zbl 0930.12006 MR 1480919

[Wib12] Wibmer, M.: Existence of ∂-parameterized Picard–Vessiot extensions over fields with algebraically closed constants. J. Algebra **361**, 163–171 (2012) Zbl 1280.12003 MR 2921616