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Convergence of marked point processes of excesses for dynamical systems

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Abstract. We consider stochastic processes arising from dynamical systems simply by evaluating an observable function along the orbits of the system and study marked point processes associated to extremal observations of such time series corresponding to exceedances of high thresholds. Each exceedance is marked by a quantity intended to measure the severity of the exceedance. In particular, we consider marked point processes measuring the aggregate damage by adding all the excesses over the threshold that mark each exceedance (AOT) or simply by adding the largest excesses in a cluster of exceedances (POT). We provide conditions for the convergence of such marked point processes to a compound Poisson process, for whose multiplicity distribution we give an explicit formula. These conditions are shown to follow from a strong form of decay of correlations of the system. Moreover, we prove that the convergence of the marked point processes for a 'nice' first return induced map can be carried over to the original system. The systems considered include non-uniformly expanding maps (in one or higher dimensions) and maps with intermittent fixed points or non-recurrent critical points. For a general class of examples, the compound Poisson limit process is computed explicitly, and in particular in the POT case we obtain a generalised Pareto multiplicity distribution.

Keywords. Extreme value theory, return time statistics, stationary stochastic processes, random measures, extremal index

1. Introduction

In the past few years the study of extremal behaviour of dynamical systems has drawn much attention (see for example [8, 9, 11, 19, 28, 7, 27]). The occurrence of extreme or rare events is often seen as the entrance of an orbit in some small (hence rare) target set in the phase space. These target sets are usually taken either as cylinders or shrinking balls around some determined point ζ in the phase space and we want to study the time elapsed before hitting such targets. This is obviously related to the recurrence properties

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of the system and can also be associated to the occurrence of extreme observations (or exceedances of high thresholds) for a given potential, so that entrances in the target set mean that the respective observations of the potential achieve very high values. This relationship between hitting times and extreme values was formally established in [9, 10]. In this paper, we will use an extreme value approach rather than a hitting time approach, but bear in mind that these are two sides of the same coin as can be fully appreciated in the aforementioned papers.

One of the motivations for studying such properties is that extreme events are associated with risk assessment and understanding their likelihood is of crucial importance. One way of keeping track of extreme events is through the study of point processes, which keep record of the number of exceedances (entrances in the target sets) observed in a certain normalised time frame. In [9, 12, 2] these processes were studied in a dynamical setting and called Rare Events Point Processes (REPP). REPP could be described in a simplified way as follows (see the precise definition in Section 2). Let $X_0, X_1, \ldots$ be a stationary stochastic process arising from a dynamical system by observing a given potential along its orbits. Let u be a high threshold, consider the time interval $[0, t)$ and a normalising scale factor v_u that depends on u and which will be made precise below. Let

$$N_u(t) = \sum_{j=0}^{\lfloor v_u t \rfloor} \mathbf{1}_{X_j > u},$$

where $\mathbf{1}_A$ is the indicator function of the set A. Note that $N_u(t)$ gives the number of exceedances during the normalised time interval $[0, v_u t)$.

The convergence of REPP is affected significantly by the presence or absence of clustering of exceedances. As shown in [9], in the absence of clustering, REPP converges to a Poisson process of intensity 1, meaning that, in particular, $N_u(t)$ converges in distribution to a Poisson random variable of average t. In [12], under the presence of clustering, REPP was shown to converge to a compound Poisson process of intensity $0 < \theta < 1$ and geometric multiplicity distribution with mean θ^{-1}, which can be interpreted as the average cluster size. This parameter θ is called the Extremal Index (EI). In particular, this means that $N_u(t)$ converges in distribution to a Pólya–Aeppli distribution. One can think of the compound Poisson process as having two independent components: the Poisson events on the time axis ruled by exponential interarrival times, and the multiplicity of each event (or weight associated to each event) that in the latter case is determined by a geometric distribution.

The convergence of REPP can be used to obtain relevant information such as the expected time between the occurrence of catastrophic events, the intensity of clustering, the distribution of higher order statistics of a finite size sample, which ultimately are crucial for assessing risk. However, in many circumstances such as in actuarial science, or in structural safety, not only the frequency of rare undesirable events is relevant for the evaluation of risk associated to certain phenomena. In fact, insurance companies and safety regulation agencies are also very much interested in, on the one hand, the severity of high impacts, and on the other hand, in the effects of aggregate damage. This motivates studying other point processes that are not limited to count the number of exceedances but rather quantify somehow the amount of damage by adding the excesses over a certain high threshold:

$$A_u(t) = \sum_{j=0}^{\lfloor v_u t \rfloor} (X_j - u)_+,$$

where $(x)_+ = \max\{0, x\}$.

When there is no clustering, $A_u(t)$ gives rise to an Excesses Over Threshold (EOT) marked point process. If there is clustering then one has (at least) two natural possibilities to handle the excesses within a cluster: either we are interested in the aggregate damage and in that case we sum all the excesses within a cluster to obtain an Area Over Threshold (AOT) marked point process; or we are interested in the record impact of the highest exceedance and in that case we take the maximum excess within a cluster to obtain a Peak Over Threshold (POT) marked point point process (in this case we need to adjust the definition of $A_u(t)$, but we postpone it to Section 2).

Interest in AOT arises for example in situations where immediate large observations have an accumulated detrimental effect on a certain structure or a company's financial situation, which ultimately results in a system failure/collapse or bankruptcy. On the other hand, interest in POT may appear when there is some sort of recovery mechanism that softens the effect of small exceedances but one is mostly worried with the sensitivity to singular very high impacts.

Several difficulties arise in studying convergence of such point processes. The most obvious is that instead of expecting a discrete multiplicity distribution (geometric distribution), as in [12], here we expect continuous multiplicity distributions of Pareto type. This means that we have to build on the work of [12], adapting the mixing conditions considered there in order to study joint Laplace transforms associated with these processes and ultimately prove their convergence for systems with good mixing rates.

In the classical setting of stationary stochastic processes the limit of REPP was proved to be a compound Poisson process in [20, 25] under assumption $\Delta(u_n)$ (very similar to Leadbetter's $D(u_n)$ introduced in [22]) and assuming the existence of an EI. In the dynamical setting, in [16], the authors prove the convergence of $N_u(t)$ to a Pólya–Aeppli distribution for cylinder target sets. In [12], which builds on [11], some conditions were devised to prove the existence of an EI when the target sets are balls around repelling periodic points; the authors proved the convergence of REPP to a compound Poisson process with geometric multiplicity distribution. The conditions proposed in [12] can be checked for systems with sufficient decay of correlations (in contrast to $D(u_n)$ or $\Delta(u_n)$) and allow one to prove the existence of an EI and compute its value from the expansion rate at the repelling periodic point.

In the classical paper [23], Leadbetter shows the convergence of EOT, for independent and identically distributed (i.i.d.) random variables, and of the POT marked point process to a compound Poisson process with multiplicity distribution given by a generalised Pareto distribution, whose type is determined by the tail of the distribution of X_0. The convergence of the latter is obtained for stationary stochastic processes under condition $\Delta(u_n)$ that cannot be verified in a dynamical setting. The result is obtained under the assumption of existence of an EI. In [24], convergence of AOT under $\Delta(u_n)$

is also addressed but assuming the existence of an unknown limit for the multiplicity distribution.

In the dynamical setting the appearance of clustering was linked to periodicity of the point ζ playing the role of base of the target sets in [18, 16, 11, 12]. In fact, as proved in [2], when the target sets are balls around ζ then we have a dichotomy regarding the convergence of REPP for systems with a strong form of decay of correlations known as decay of correlations against L^1 (see Definition 2.10 below): either ζ is periodic and converges to a compound Poisson process of intensity θ and geometric multiplicity distribution, or ζ is not periodic and we have no clustering and convergence to a standard Poisson process. In a very recent paper [3], the authors use multiple maxima $\zeta_1, \ldots, \zeta_k$ correlated by belonging to the same orbit to create a fake periodic effect that ultimately creates clustering, in this case, with possibly different multiplicity distributions.

In this paper we give conditions (long range and short range conditions on the dependence structure of the stochastic processes) to guarantee the convergence of the EOT, AOT, POT marked point processes, which can also be used to prove the convergence of REPP. In fact, the result (Theorem 2.A) is quite general and can be used to prove the convergence of other marked point processes associated to exceedances by using other possible marks over each exceedance. The result applies both in the presence and absence of clustering. The conditions are devised to be applied in the dynamical setting (in contrast to $\Delta(u_n)$) and to simplify the proof of the existence of an EI. Moreover, from these new conditions we provide a new formula to compute the multiplicity distribution of the limiting compound Poisson process. Furthermore, the conditions can be used in a wide range of scenarios including target sets around multiple maxima as in [3] or discontinuity points as in [2] or even other more geometrically intricate sets.

Then in Theorem 2.B we show that such conditions can be easily verified if the system has for example decay of correlations against L^1 observables, which allows us to apply the theory to uniformly expanding maps of the interval (such as Rychlik maps) or higher dimensional uniformly expanding systems like the ones studied by Saussol [30].

Motivated by an idea introduced in [5] and extended in the recent paper [17], we prove Theorem 2.C, which states that if a system admits a 'nice' first return time induced map for which we can prove the convergence of marked point processes associated to the exceedances (such as EOT, AOT or POT) then the original system shares the same property. This allows the application of our results to maps with intermittent fixed points, like the Manneville–Pommeau maps or Liverani–Saussol–Vaienti maps, or maps with critical points such as Misiurewicz quadratic maps.

In order to exemplify the application of the main results to proving the convergence of marked point processes and actually computing the limit distributions (using the formula we provide to compute its multiplicity distribution), we consider the case where the targets are balls around a single maximum at ζ with some natural regularity conditions, to obtain a result (Theorem 3.A) stating that for a fairly large scope of examples the EOT and POT marked point processes converge to a compound Poisson process with intensity θ (for which we provide a precise formula) and with multiplicity distribution corresponding to a generalised Pareto. Then in Theorem 3.B we address the more difficult AOT case, for which, under some more restrictive assumptions on the system, we also compute the multiplicity of the limiting compound Poisson distribution.

2. The setting and statement of results

Take a system $(\mathcal{X}, \mathcal{B}, \mathbb{P}, f)$, where $\mathcal{X}$ is a Riemannian manifold, $\mathcal{B}$ is the Borel σ-algebra, $f : \mathcal{X} \to \mathcal{X}$ is a measurable map and $\mathbb{P}$ an f-invariant probability measure. Suppose that the time series $X_0, X_1, \ldots$ arises from such a system simply by evaluating a given observable $\varphi : \mathcal{X} \to \mathbb{R} \cup \{\pm\infty\}$ along the orbits of the system, or in other words, the time evolution given by successive iterations by f:

$$X_n = \varphi \circ f^n \quad \text{for each } n \in \mathbb{N}. \tag{2.1}$$

Clearly, $X_0, X_1, \ldots$ defined in this way is not necessarily an independent sequence. However, f-invariance of $\mathbb{P}$ guarantees that this stochastic process is stationary.

The simplest point processes keep record of the exceedances of the high thresholds u_n by counting the number of such exceedances on a rescaled time interval. The sequence $(u_n)_{n\in\mathbb{N}}$ of thresholds is chosen such that

$$n\mathbb{P}(X_0 > u_n) \to \tau \quad \text{for some } \tau > 0 \text{ as } n \to \infty, \tag{2.2}$$

so that the number of exceedances among the first n observations is kept, approximately, at constant rate $\tau > 0$. These counting processes were called Rare Events Point Processes (REPP) and were studied in [9, 12, 15, 6]. Here, we consider even more sophisticated cases like when each exceedance is marked by the respective excess over the threshold u_n. In fact, the marked point processes will be defined by keeping record of the occurrence of clusters of exceedances and each such occurrence will be marked by the number of exceedances in the cluster (which allows us to recover REPP), the sum of the excesses of all exceedances in a cluster, the maximum excess in the cluster or any other measure weighing the intensity of each cluster.

In order to provide a proper framework of the problem we next introduce the necessary formalism to state the results regarding the convergence of point processes and random measures. We recommend the book of Kallenberg [21] for further details on these topics.

2.1. *Random measures and weak convergence*

First we introduce the notions of *random measures* and in particular *point processes* and *marked point processes* on the positive real line. Consider the interval $[0, \infty)$ and its Borel σ-algebra $\mathcal{B}_{[0,\infty)}$. A positive measure ν on $\mathcal{B}_{[0,\infty)}$ is said to be a *Radon measure* if $\nu(A) < \infty$ for every bounded set $A \in \mathcal{B}_{[0,\infty)}$. Let $\mathcal{M} := \mathcal{M}([0,\infty))$ denote the space of all Radon measures defined on $([0,\infty), \mathcal{B}_{[0,\infty)})$. We equip this space with the vague topology, i.e., $\nu_n \to \nu$ in $\mathcal{M}([0,\infty))$ whenever $\nu_n(\psi) \to \nu(\psi)$ for any continuous function $\psi : [0,\infty) \to \mathbb{R}$ with compact support. Consider the subsets of $\mathcal{M}$ defined by $\mathcal{M}_p := \{\nu \in \mathcal{M} : \nu(A) \in \mathbb{N} \text{ for all } A \in \mathcal{B}_{[0,\infty)}\}$ and $\mathcal{M}_a := \{\nu \in \mathcal{M} : \nu \text{ is an atomic measure}\}$. A *random measure* M on $[0,\infty)$ is a random element of $\mathcal{M}$, i.e., if $(\mathcal{X}, \mathcal{B}_{\mathcal{X}}, \mathbb{P})$ is a probability space, then any measurable $M : \mathcal{X} \to \mathcal{M}$ is a random measure on $[0, \infty)$. A *point process* N and *marked point process* A are defined similarly as random elements of $\mathcal{M}_p$ and $\mathcal{M}_a$, respectively.

The elements ν of $\mathcal{M}_p$ can be interpreted as counting measures, i.e., $\nu = \sum_{i=1}^{\infty} \delta_{x_i}$, where $x_1, x_2, \dots$ is a collection of not necessarily distinct points in $[0, \infty)$ and δ_{x_i} is the Dirac measure at x_i, i.e., for every $A \in \mathcal{B}_{[0,\infty)}$, we have $\delta_{x_i}(A) = 1$ if $x_i \in A$, and $\delta_{x_i}(A) = 0$ otherwise. The elements ν of $\mathcal{M}_a$ can be written as $\nu = \sum_{i=1}^{\infty} d_i \delta_{x_i}$, where $x_1, x_2, \dots \in [0, \infty)$ and $d_1, d_2, \dots \in [0, \infty)$.

To give a concrete example of a marked point process, which in particular will appear as the limit of marked point processes, we consider

Definition 2.1. Let $T_1, T_2, \dots$ be an i.i.d. sequence of r.v.'s with common exponential distribution of mean $1/\theta$. Let $D_1, D_2, \dots$ be another i.i.d. sequence of r.v.'s, independent of the previous one, and with d.f. π. Given these sequences, for $J \in \mathcal{B}_{[0,\infty)}$, set

$$A(J) = \int \mathbf{1}_J \, d\Big(\sum_{i=1}^{\infty} D_i \delta_{T_1+\dots+T_i}\Big).$$

Let $\mathcal{X}$ denote the space of all possible realisations of $T_1, T_2, \dots$ and $D_1, D_2, \dots$, equipped with the product σ-algebra and measure. Then $A : \mathcal{X} \to \mathcal{M}_a([0, \infty))$ is a marked point process which we call a *compound Poisson process* of *intensity* θ and *multiplicity d.f.* π.

Remark 2.2. When $D_1, D_2, \dots$ are integer valued positive random variables, π is completely defined by the values $\pi_k = \mathbb{P}(D_1 = k)$ for every $k \in \mathbb{N}_0$, and A is actually a point process. Note that if $\pi_1 = 1$ and $\theta = 1$, then A is the standard Poisson process and, for every $t > 0$, the random variable $A([0, t))$ has a Poisson distribution of mean t.

Now, we define what we mean by convergence of random measures (see [21] for more details).

Definition 2.3. Let $(M_n)_{n\in\mathbb{N}} : \mathcal{X} \to \mathcal{M}$ be a sequence of random measures defined on a probability space $(\mathcal{X}, \mathcal{B}_{\mathcal{X}}, \mu)$ and let $M : Y \to \mathcal{M}$ be another random measure defined on a possibly distinct probability space $(Y, \mathcal{B}_Y, \nu)$. We say that M_n *converges in distribution* to M if $\mu \circ M_n^{-1}$ converges weakly to $\nu \circ M^{-1}$, i.e., for every bounded continuous function φ defined on $\mathcal{M}$, we have $\lim_{n\to\infty} \int \varphi \, d\mu \circ M_n^{-1} = \int \varphi \, d\nu \circ M^{-1}$. We write $M_n \stackrel{\mu}{\Longrightarrow} M$.

In order to check convergence of random measures it is useful to translate it into convergence in distribution of more tractable random variables or express it in terms of Laplace transforms. To this end, we let $\mathcal{S}$ denote the semiring of subsets of $\mathbb{R}_0^+$ whose elements are intervals of the type $[a, b)$ for $a, b \in \mathbb{R}_0^+$. Let $\mathcal{R}$ denote the ring generated by $\mathcal{S}$. Recall that for every $J \in \mathcal{R}$ there are $\varsigma \in \mathbb{N}$ and ς disjoint intervals $I_1, \dots, I_\varsigma \in \mathcal{S}$ such that $J = \dot\bigcup_{i=1}^{\varsigma} I_j$. In order to fix notation, let $a_j, b_j \in \mathbb{R}_0^+$ be such that $I_j = [a_j, b_j) \in \mathcal{S}$.

Definition 2.4. Let Z be a non-negative random variable with distribution F. For every $y \in \mathbb{R}_0^+$, the *Laplace transform* $\phi(y)$ of the distribution F is given by

$$\phi(y) := \mathbb{E}(\mathrm{e}^{-yZ}) = \int \mathrm{e}^{-yZ}\, d\mu_F,$$

where μ_F is the Lebesgue–Stieltjes probability measure associated to the distribution function F.

Definition 2.5. For a random measure M on $\mathbb{R}_0^+$ and ς disjoint intervals $I_1, \ldots, I_\varsigma \in \mathcal{S}$ and non-negative $y_1, \ldots, y_\varsigma$, we define the *joint Laplace transform* $\psi(y_1, \ldots, y_\varsigma)$ by

$$\psi_M(y_1, \ldots, y_\varsigma) = \mathbb{E}\big(\mathrm{e}^{-\sum_{\ell=1}^{\varsigma} y_\ell M(I_\ell)}\big).$$

If M is a compound Poisson point process with intensity λ and multiplicity distribution π, then given ς disjoint intervals $I_1, \ldots, I_\varsigma \in \mathcal{S}$ and non-negative $y_1, \ldots, y_\varsigma$ we have

$$\psi_M(y_1, \ldots, y_\varsigma) = \mathrm{e}^{-\lambda \sum_{\ell=1}^{\varsigma}(1-\phi(y_\ell))|I_\ell|},$$

where $\phi(y)$ is the Laplace transform of the multiplicity distribution π.

Remark 2.6. By [21, Theorem 4.2], a sequence $(M_n)_{n\in\mathbb{N}}$ of random measures converges in distribution to a random measure M iff the sequence of vector r.v.'s $(M_n(J_1), \ldots, M_n(J_\varsigma))$ converges in distribution to $(M(J_1), \ldots, M(J_\varsigma))$ for every $\varsigma \in \mathbb{N}$ and all disjoint $J_1, \ldots, J_\varsigma \in \mathcal{S}$ such that $M(\partial J_\ell) = 0$ a.s. for $\ell = 1, \ldots, \varsigma$, which will be the case if the respective joint Laplace transforms $\psi_{M_n}(y_1, \ldots, y_\varsigma)$ converge to the joint Laplace transform $\psi_M(y_1, \ldots, y_\varsigma)$ for all $y_1, \ldots, y_\varsigma \in [0, \infty)$.

2.2. *Marked point processes of rare events*

We start by defining some concepts and events that will be used in the definition of marked point processes and of dependence conditions needed to ensure their convergence.

Let $A \in \mathcal{B}$. We define a function $r_A : \mathcal{X} \to \mathbb{N} \cup \{\infty\}$, which we refer to as the *first hitting time function* to A, by

$$r_A(x) = \min\{j \in \mathbb{N} \cup \{\infty\} : f^j(x) \in A\}. \tag{2.3}$$

The restriction of r_A to A is called the *first return time function* to A. We define the *first return time* to A, denoted by $R(A)$, as

$$R(A) = \operatorname*{ess\,inf}_{x\in A} r_A(x). \tag{2.4}$$

We define, for each $j > 1$, the j-th *waiting* (or *inter-hitting*) *time* as

$$w_A^j(x) := r_A\big(f^{\sum_{i=1}^{j-1} w_A^i(x)}(x)\big), \tag{2.5}$$

where $w_A^1(x) := r_A(x)$, and the *j-th hitting time* as

$$r_A^j(x) := \sum_{i=1}^{j} w_A^i(x). \tag{2.6}$$

For $u \in \mathbb{R}$, $p, i, \kappa, s \in \mathbb{N}_0$ and $\ell \in \mathbb{N}$, we set $U_p^{(0)}(u) = U(u) = \{X_0 > u\}$ and define the following events:

$$U_p^{(\kappa)}(u) := U(u) \cap \bigcap_{i=1}^{\kappa} \{w_{U(u)}^i \leq p\},$$
$$U_p^{(\infty)}(u) := U(u) \cap \bigcap_{i=1}^{\infty} \{w_{U(u)}^i \leq p\} = \bigcap_{\kappa=0}^{\infty} U_p^{(\kappa)}(u),$$
$$Q_{p,i}^{\kappa}(u) := f^{-i}(U_p^{(\kappa)}(u) \setminus U_p^{(\kappa+1)}(u)) = f^{-i}(U_p^{(\kappa)}(u) \cap \{w_{U(u)}^{\kappa+1} > p\}).$$

If $p = 0$ then $U_0^{(\kappa)}(u) = \emptyset$ for all $\kappa \geq 1$ and $Q_{0,0}^0(u) = U(u) = \{X_0 > u\}$. One of the main ideas of [11], further developed in [12], is that the events $Q_{p,0}^0(u) = \{X_0 > u, X_1 \leq u, \ldots, X_p \leq u\}$ (when $p > 0$) play a key role in determining and identifying the clusters. In fact, every cluster ends with an entrance in $Q_{p,0}^0(u)$, meaning that the inter-cluster exceedances must be separated at most by p units of time. Hence, given an interval $I \in \mathcal{S}$, $x \in \mathcal{X}$ and $u \in \mathbb{R}$, we define

$$N(I)(x, u) = \sum_{j \in I \cap \mathbb{N}_0} \mathbf{1}_{f^{-j}(Q_{p,0}^0(u))}(x).$$

Let $i_1(x, u) < i_2(x, u) < \cdots < i_{N(I)(x,u)}(x, u)$ denote the times at which the orbit of x enters $Q_{p,0}^0(u)$, i.e., $f^{i_k(x,u)}(x) \in Q_{p,0}^0(u)$ for all $k = 1, \ldots, N(I)(x, u)$. We now define the cluster periods: for every $j = 1, \ldots, N(I)(x, u) - 1$ let $I_j(x, u) = (i_j(x, u), i_{j+1}(x, u)]$ and set $I_0(x, u) = [\min I, i_1(x, u)]$ and $I_{N(I)(x,u)}(x, u) = (i_{N(I)(x,u)}(x, u), \sup(I))$. In order to define the marks for each cluster we consider the following mark functions that depend on the level u and on the random variables in a certain time frame $I^* \in \mathcal{S}$:

$$m_u(\{X_i\}_{i \in I^* \cap \mathbb{N}_0}) := \begin{cases} \sum_{i \in I^* \cap \mathbb{N}_0} (X_i - u)_+, & \text{AOT case,} \\ \max_{i \in I^* \cap \mathbb{N}_0} \{(X_i - u)_+\}, & \text{POT case,} \\ \sum_{i \in I^* \cap \mathbb{N}_0} \mathbf{1}_{X_i > u}, & \text{REPP case,} \end{cases} \tag{2.7}$$

where $(y)_+ = \max\{y, 0\}$ and when $I^* \neq \emptyset$. Also set $m_u(\emptyset) := 0$.

We now define the *cluster marks* for each $j = 0, 1, \ldots, N(I)(x, u)$ by

$$D_j(x, u) := m_u(\{X_i\}_{i \in I_j(x,u) \cap \mathbb{N}_0}).$$

Finally, we set

$$\mathscr{A}_u(I)(x) := \sum_{j=0}^{N(I)(x,u)} D_j(x, u). \tag{2.8}$$

In order to define the marked point processes in such a way that they admit a non-degenerate limit, we introduce a link between the number of observations and the thresholds by considering the sequence $(u_n)_{n \in \mathbb{N}}$ of levels satisfying (2.2) and by rescaling time by the factor

$$v_n := 1/\mathbb{P}(X_0 > u_n)$$

given by Kac's Theorem so that the expected number of exceedances of the level u_n in each time frame considered is kept 'constant' as $n \to \infty$. Hence, we introduce the following notation. For $I = [a, b) \in \mathcal{S}$ and $\alpha \in \mathbb{R}$, we denote $\alpha I := [\alpha a, \alpha b)$ and $I + \alpha := [a + \alpha, b + \alpha)$. Similarly, for $J \in \mathcal{R}$ such that $J = J_1 \dot\cup \cdots \dot\cup J_k$, define $\alpha J := \alpha J_1 \dot\cup \cdots \dot\cup \alpha J_k$ and $J + \alpha := (J_1 + \alpha) \dot\cup \cdots \dot\cup (J_k + \alpha)$.

Definition 2.7. We define the *marked rare events point process* (MREPP) by setting, for every $J \in \mathcal{R}$ with $J = J_1 \dot\cup \cdots \dot\cup J_k$ where $J_i \in \mathcal{S}$ for all $i = 1, \dots, k$,

$$A_n(J) := \sum_{i=1}^{k} \mathscr{A}_{u_n}(v_n J_i). \tag{2.9}$$

When m_u given in (2.7) is as in the AOT case, then the MREPP A_n computes the sum of all excesses over the threshold u_n, and in that case we will refer to A_n as being an *area over threshold* or AOT MREPP. Observe that in this case we may write

$$A_n(J) = \sum_{j \in v_n J \cap \mathbb{N}_0} (X_j - u_n)_+.$$

When m_u given in (2.7) is as in the POT case, then the MREPP A_n computes the sum of the largest excesses (peaks) over the threshold u_n within each cluster; in that case, we will refer to A_n as being a *peaks over threshold* or POT MREPP.

When m_u given in (2.7) is as in the REPP case, then the MREPP A_n is actually a point process that counts the number of exceedances of u_n; in that case we will refer to A_n as being a *rare events point process*, or REPP, as in [12]. Observe that in this case

$$A_n(J) = \sum_{j \in v_n J \cap \mathbb{N}_0} \mathbf{1}_{X_j > u_n}.$$

If $p = 0$, then $Q^0_{p,0}(u_n) = U(u_n) = \{X_0 > u_n\}$, and in this case the AOT MREPP and the POT MREPP coincide and both compute the sum of all excesses over the threshold u_n. In that situation we say that A_n is an *excesses over threshold* (EOT) MREPP.

Now, we introduce the dependence conditions $Д_p(u_n)^*$ and $Д'_p(u_n)^*$, with the same flavour as $Д_p(u_n)$ and $Д'_p(u_n)$ considered in [13] but designed to establish the convergence of MREPP (whether they are of the type AOT, POT or simpler REPP), which allows us to state the main result of this paper. The mixing type condition $Д_p(u_n)^*$ also follows easily from sufficiently fast decay of correlations, which makes it particularly useful to apply to stochastic processes arising from dynamical systems, in contrast to condition $\Delta(u_n)$ used by Leadbetter [23] or any other similar such condition available in the literature.

For $u \in \mathbb{R}$, $x \geq 0$, $p, i, \kappa, s \in \mathbb{N}_0$ and $\ell \in \mathbb{N}$, we define the following events:

$$U^\kappa_{p,i}(u, x) := f^{-i}\big(Q^\kappa_{p,0}(u) \cap \{m_u(\{X_j\}_{0 \leq j \leq r^\kappa_{U(u)}}) > x\}\big),$$

$$U_{p,i}(u, x) := f^{-i}\Big(\bigcup_{\kappa=0}^{\infty} U^\kappa_{p,0}(u, x) \cup U^{(\infty)}_p(u)\Big),$$

$$R_{p,i}(u,x) := f^{-i}(U_{p,0}(u,x) \cap \{r_{U_{p,0}(u,x)} > p\}),$$
$$\mathscr{I}_{p,s,\ell}(u,x) := \bigcap_{i=s}^{s+\ell-1} (U_{p,i}(u,x))^c, \quad \mathcal{R}_{p,s,\ell}(u,x) := \bigcap_{i=s}^{s+\ell-1} (R_{p,i}(u,x))^c.$$

In particular, for $x = 0$ we have

$$U^{\kappa}_{p,i}(u,0) = Q^{\kappa}_{p,i}(u), \quad U_{p,i}(u,0) = f^{-i}\Big(\bigcup_{\kappa=0}^{\infty} Q^{\kappa}_{p,0}(u) \cup U^{(\infty)}_p(u)\Big) = \{X_i > u\},$$
$$R_{p,i}(u,0) = \{X_i > u,\ X_{i+1} \le u, \dots, X_{i+p} \le u\} = Q^0_{p,i}(u),$$

and for $p = 0$ we have

$$\begin{aligned} &U^{(\kappa)}_0(u) = \emptyset \text{ for } \kappa > 0, \quad Q^0_{0,i}(u) = \{X_i > u\} \text{ and } Q^{\kappa}_{0,i}(u) = \emptyset \text{ for } \kappa > 0,\\ &U^0_{0,i}(u,x) = \{X_i > u, m_u(\{X_i\}) > x\} \text{ and } U^{\kappa}_{0,i}(u,x) = \emptyset \text{ for } \kappa > 0,\\ &R_{0,i}(u,x) = U_{0,i}(u,x). \end{aligned} \tag{2.10}$$

Condition $Д_p(u_n^*)$. We say that $Д_p(u_n)^*$ holds for the sequence $X_0, X_1, X_2, \dots$ if for $t, n \in \mathbb{N}$, $x_1, \dots, x_\varsigma \ge 0$ and any $J = \bigcup_{i=2}^{\varsigma} I_j \in \mathcal{R}$ with $\inf\{x : x \in J\} \ge t$,

$$\left|\mathbb{P}\Big(R_{p,0}(u_n,x_1) \cap \bigcap_{j=2}^{\varsigma}\{\mathscr{A}_{u_n}(I_j) \le x_j\}\Big) - \mathbb{P}(R_{p,0}(u_n,x_1))\mathbb{P}\Big(\bigcap_{j=2}^{\varsigma}\{\mathscr{A}_{u_n}(I_j) \le x_j\}\Big)\right| \le \gamma(n,t),$$

where for each n, $\gamma(n,t)$ is nonincreasing in t, and $n\gamma(n,t_n) \to 0$ as $n \to \infty$ for some sequence $t_n = o(n)$; here $\mathscr{A}_{u_n}$ is given by (2.8).

As mentioned before, this mixing condition is easy to check for stochastic processes arising from dynamical systems with sufficiently fast decay of correlations, as can be seen in Theorem 2.B (see also Remark 3.1). This is the main advantage of this condition when compared with Leadbetter's $\Delta(u_n)$ and others of the same kind.

For some fixed $p \in \mathbb{N}_0$, consider the sequence $(t_n)_{n\in\mathbb{N}}$ given by $Д_p(u_n)^*$ and let $(k_n)_{n\in\mathbb{N}}$ be such that

$$k_n \to \infty \quad \text{and} \quad k_n t_n = o(n). \tag{2.11}$$

Condition $Д'_p(u_n^*)$. We say that $Д'_p(u_n)^*$ holds for the sequence $X_0, X_1, X_2, \dots$ if there exists a sequence $(k_n)_{n\in\mathbb{N}}$ satisfying (2.11) such that

$$\lim_{n\to\infty} n \sum_{j=p+1}^{\lfloor n/k_n \rfloor - 1} \mathbb{P}(Q^0_{p,0}(u_n) \cap \{X_j > u_n\}) = 0.$$

In this approach, it is rather important to observe the prominent role played by condition $Д'_p(u_n)^*$. In particular, note that if $Д'_p(u_n)^*$ holds for some particular $p = p_0 \in \mathbb{N}_0$, then it holds for all $p \ge p_0$. This suggests that when trying to prove the existence of Extreme

Value Law (EVL), one should check the values $p = p_0$ until one finds the smallest one that makes $Д'_p(u_n)$ hold. Assume that there exists $p \in \mathbb{N}_0$ such that

$$p := \min\Big\{ j \in \mathbb{N}_0 : \lim_{n\to\infty} R(Q^0_{j,0}(u_n)) = \infty \Big\}, \tag{2.12}$$

where R is as in (2.4). This p is a natural candidate to check for the validity of $Д'_p(u_n)$ and then define

$$\theta_n := \frac{\mathbb{P}(Q^0_{p,0}(u_n))}{\mathbb{P}(U(u_n))}. \tag{2.13}$$

If $Д'_{p_0}(u_n)^*$ holds and if the limit of θ_n in (2.13) exists for such $p = p_0$, it will also exist for all $p \geq p_0$ and will take always the same value. In this case, $\theta = \lim_{n\to\infty} \theta_n$ is called the Extremal Index (EI). When $p = 0$, observe that $Д'_p(u_n)^*$ is condition $D'(u_n)$ from Leadbetter [22], which prevents clustering of exceedances. In particular, in this case $\theta_n = 1$ for all $n \in \mathbb{N}$, and we get an EI equal to 1.

When $p > 0$, we have clustering of exceedances, i.e., the exceedances have a tendency to appear aggregated in groups (called clusters), whose mean size is typically given by the inverse of the value of the EI θ.

We will also assume:

Multiplicity limit condition. There exists a normalising sequence $(a_n)_{n\in\mathbb{N}}$ and a probability distribution π such that

$$\lim_{n\to\infty} \frac{\mathbb{P}(R_{p,0}(u_n, x/a_n))}{\mathbb{P}(U(u_n))} = \theta(1 - \pi(x)), \qquad \forall x \geq 0. \tag{2.14}$$

We will see that (2.14) provides a nice formula to compute the multiplicity distribution of the limiting compound Poisson process, which will be used in Sections 3.4 and 3.5.

Finally, we give a technical condition which imposes a sufficiently fast decay of the probability of having very long clusters. We will call it $ULC_p(u_n)$, which stands for 'Unlikely Long Clusters'. Of course this condition is trivially satisfied when there is no clustering. Moreover, it can be easily checked (see Proposition 2.13 below) when ζ is a repelling periodic point.

Condition $ULC_p(u_n)$. We say that condition $ULC_p(u_n)$ holds if for all $y > 0$,

$$\limsup_{n\to\infty} n \int_0^\infty y e^{-yx} \delta_{p,\lfloor n/k_n\rfloor,u_n}(x/a_n)\,dx < \infty$$

where a_n is as in (2.14), $\delta_{0,s,u}(x) := 0$, and for $p > 0$,

$$\begin{aligned}\delta_{p,s,u}(x) &:= \sum_{\kappa=1}^{\lfloor s/p\rfloor} \kappa p \mathbb{P}(U^\kappa_{p,0}(u,x)) + s \sum_{\kappa=\lfloor s/p\rfloor+1}^{\infty} \mathbb{P}(U^\kappa_{p,0}(u,x)) + p\mathbb{P}(U_{p,0}(u,x)) \\ &= p \sum_{\kappa=0}^{\lfloor s/p\rfloor} (\kappa+1)\mathbb{P}(U^\kappa_{p,0}(u,x)) + (s+p) \sum_{\kappa=\lfloor s/p\rfloor+1}^{\infty} \mathbb{P}(U^\kappa_{p,0}(u,x))\end{aligned} \tag{2.15}$$

is an integrable function on $\mathbb{R}^+$ for u sufficiently close to $u_F = \varphi(\zeta)$.

Note that, by definition, condition $ULC_0(u_n)$ always holds.

We emphasise that this is indeed a technical condition that hardly imposes any restriction on applications to dynamical systems. In fact, although we do not address such examples here, it can also be checked in situations when ζ is a discontinuity point as in [2] or when we have multiple correlated or uncorrelated maximal points $\zeta_1, \ldots, \zeta_k$ as in [3].

We are now ready to state the main convergence result:

Theorem 2.A. *Let $X_0, X_1, \ldots$ be given by (2.1) and $(u_n)_{n\in\mathbb{N}}$ be a sequence satisfying (2.2). Assume that $Д_p(u_n)^*$, $Д'_p(u_n)^*$ and $ULC_p(u_n)^*$ hold, for some $p \in \mathbb{N}_0$. Assume that $\lim_{n\to\infty} \theta_n = \theta \in (0, 1]$ and assume the existence of a normalising sequence $(a_n)_{n\in\mathbb{N}}$ and a probability distribution π such that (2.14) holds. Then the MREPP $a_n A_n$, where A_n is given by Definition 2.7 for any of the three mark functions considered in (2.7), converges in distribution to a compound Poisson process A with intensity θ and multiplicity d.f. π.*

Remark 2.8. In the proof of this theorem, what is essential about the mark function m_u considered in (2.7) to define the respective MREPP is that it satisfies the following assumptions:

(1) $m_u(\{X_i\}_{i\in I^*\cap\mathbb{N}_0}) \geq 0$ and $m_u(\emptyset) = 0$,
(2) $m_u(\{X_i\}_{i\in I^*\cap\mathbb{N}_0}) \leq m_u(\{X_i\}_{i\in J^*\cap\mathbb{N}_0})$ if $I^* \subset J^*$,
(3) $m_u(\{X_i\}_{i\in I^*\cap\mathbb{N}_0}) = m_u(\{X_i\}_{i\in J^*\cap\mathbb{N}_0})$ if $X_i \leq u$, for all $i \in (I^* \setminus J^*) \cap \mathbb{N}_0$.

Note that in particular we must have $m_u(\{X_i\}_{i\in I^*\cap\mathbb{N}_0}) = 0$ if $X_i \leq u$ for all $i \in I^* \cap \mathbb{N}_0$.

As long as the above assumptions hold, the conclusion of Theorem 2.A holds for the MREPP defined from a mark function m_u satisfying the three assumptions above.

Remark 2.9. The main purpose of this paper is to develop a methodology to prove the convergence of marked rare events point processes for stochastic processes arising from chaotic dynamics. For that reason we assume a priori that the processes are generated as in (2.1). However, Theorem 2.A holds for general stationary stochastic processes, which can be seen by realising that every stationary stochastic process can be modelled by (2.1). In fact, if $X_0, X_1, \ldots$ is a stationary stochastic process, then taking $\mathcal{X}$ to be the space of each possible realisation of the stochastic process, f to be the shift map on that space and φ to be the projection on the first coordinate, we can write any stationary stochastic process in the form (2.1).

In order to have an idea of the scope of applications to specific dynamical systems we consider the type of properties that a system must have in order to satisfy the abstract conditions of Theorem 2.A.

First we start by explaining what exceeding a high threshold means in terms of dynamics. To that end, we suppose that the r.v. $\varphi : \mathcal{X} \to \mathbb{R} \cup \{\pm\infty\}$ achieves its maximum value at a finite number of points $\zeta_1, \ldots, \zeta_N \in \mathcal{X}$ (we allow $\varphi(\zeta_i) = +\infty$).

We assume that φ and $\mathbb{P}$ are sufficiently regular, so that

(R1) for u sufficiently close to $u_F := \varphi(\zeta_i)$ $(i \in \{1, \ldots, N\})$,

$$U(u) := \{x \in \mathcal{X} : \varphi(x) > u\} = \{X_0 > u\}$$

corresponds to a disjoint union of balls centred at the points ζ_i, i.e., $U(u) = \bigcup_{i=1}^{N} B_{\varepsilon_i}(\zeta_i)$ with $\varepsilon_i = \varepsilon_i(u)$. Moreover, the quantity $\mathbb{P}(U(u))$, as a function of u, varies continuously on a neighbourhood of u_F.

$Д_p(u_n)^*$ and $Д'_p(u_n)^*$ are conditions on the long range and short range dependence structure of the processes, respectively, and they can be easily checked if the system has some strong form of decay of correlations such as decay of correlations against L^1 observables, which we define next.

Definition 2.10 (Decay of correlations). Let $\mathcal{C}_1, \mathcal{C}_2$ denote Banach spaces of real valued measurable functions defined on $\mathcal{X}$. We denote the *correlation* of non-zero functions $\phi \in \mathcal{C}_1$ and $\psi \in \mathcal{C}_2$ with respect to a measure $\mathbb{P}$ as

$$\mathrm{Cor}_{\mathbb{P}}(\phi, \psi, n) := \frac{1}{\|\phi\|_{\mathcal{C}_1} \|\psi\|_{\mathcal{C}_2}} \left| \int \phi\,(\psi \circ f^n)\,d\mathbb{P} - \int \phi\,d\mathbb{P} \int \psi\,d\mathbb{P} \right|.$$

We say that we have *decay of correlations*, with respect to the measure $\mathbb{P}$, for observables in $\mathcal{C}_1$ *against* observables in $\mathcal{C}_2$ if, for all $\phi \in \mathcal{C}_1$ and $\psi \in \mathcal{C}_2$,

$$\mathrm{Cor}_{\mathbb{P}}(\phi, \psi, n) \to 0 \quad \text{as } n \to \infty.$$

We say that we have *decay of correlations against L^1 observables* whenever this holds for $\mathcal{C}_2 = L^1(\mathbb{P})$ and $\|\psi\|_{\mathcal{C}_2} = \|\psi\|_1 = \int |\psi|\,d\mathbb{P}$.

Examples of systems with the latter property include

- uniformly expanding maps on the circle/interval (see [4]);
- Markov maps (see [4]);
- piecewise expanding maps of the interval with countably many branches, like Rychlik maps (see [29]);
- higher dimensional piecewise expanding maps studied by Saussol [30].

Remark 2.11. In the first three examples above, the Banach space $\mathcal{C}_1$ for the decay of correlations can be taken to be the space of functions of bounded variation. In the fourth example, $\mathcal{C}_1$ is the space of functions with finite quasi-Hölder norm studied in [30]. We refer the readers to [4, 30] or [2] for precise definitions but mention that if $I \subset \mathbb{R}$ is an interval then $\mathbf{1}_I$ is of bounded variation and its BV-norm is 2, $\|\mathbf{1}_I\|_{\mathrm{BV}} = 2$, and if A denotes a ball or an annulus then $\mathbf{1}_A$ has a finite quasi-Hölder norm.

Theorem 2.B. *Let $f : \mathcal{X} \to \mathcal{X}$ be a system with summable decay of correlations against L^1 observables, i.e., for all $\phi \in \mathcal{C}_1$ and $\psi \in L^1$, $\mathrm{Cor}(\phi, \psi, n) \leq \rho_n$ with $\sum_{n \geq 1} \rho_n < \infty$. Assume that there exists $p \in \mathbb{N}_0$ such that (2.12) holds and there exists $C > 0$ such that for all $n \in \mathbb{N}$ and $x \in \mathbb{R}_0^+$ we have $\mathbf{1}_{R_{p,0}(u_n,x)} \in \mathcal{C}_1$ and $\|\mathbf{1}_{R_{p,0}(u_n,x)}\|_{\mathcal{C}_1} \leq C$. Then conditions $Д_p(u_n)^*$ and $Д'_p(u_n)^*$ hold.*

Remark 2.12. Although we have assumed for simplicity that $\mathbf{1}_{R_{p,0}(u_n,x)} \in \mathcal{C}_1$ in the last theorem to simplify the proof of $Д_p(u_n)^*$, which can easily be verified when $\mathcal{C}_1$ is the space of functions of bounded variation or quasi-Hölder, one can still check condition $Д_p(u_n)^*$ when $\mathcal{C}_1$ is the space of Hölder functions, for example, in which case $\mathbf{1}_{R_{p,0}(u_n,x)} \notin \mathcal{C}_1$. This can be proved by minor adjustments to [12, Proposition 3.1].

As shown in [11], the appearance of clustering of exceedances in a dynamical setting is associated to periodic behaviour. This was seen in [16, 11, 12] when the maximum value of φ is attained at a single point ζ that happens to be a repelling periodic point[1] but, as in [3], it can also appear due to fake periodicity created by taking multiple maximal points which are related by belonging to the orbit of the same point ξ. To show that condition $ULC_p(u_n)$ is very easily checked, we will prove that it holds whenever we have a single maximum ζ, which is a repelling periodic point of prime period p. Assume that φ and $\mathbb{P}$ are sufficiently regular at ζ so that

(R2) the periodicity of ζ implies that for all large u, $\{X_0 > u\} \cap f^{-p}(\{X_0 > u\}) \neq \emptyset$, and the fact that the prime period is p implies $\{X_0 > u\} \cap f^{-j}(\{X_0 > u\}) = \emptyset$ for all $j = 1, \ldots, p-1$. The fact that ζ is repelling means that we have backward contraction, implying that $U_p^{(\infty)}(u) = \{\zeta\}$ and that there exists $0 < \theta < 1$ such that $\bigcap_{j=0}^{\kappa} f^{-jp}(X_0 > u)$ is a ball around ζ with

$$\mathbb{P}\Big(\bigcap_{j=0}^{\kappa} f^{-jp}(\{X_0 > u\})\Big) \sim (1-\theta)^{\kappa}\mathbb{P}(X_0 > u).$$

Proposition 2.13. *Let $f : \mathcal{X} \to \mathcal{X}$ be a system and let $\varphi : \mathcal{X} \to \mathbb{R}$ have a global maximum at ζ, which is a repelling periodic point of prime period p for which* (R2) *holds. Then condition $ULC_p(u_n)$ is satisfied.*

Remark 2.14. We remark that for examples considered in [3], condition $ULC_p(u_n)$ can also be checked with the same amount of effort necessary to prove the last proposition. For its proof see the end of Section 3.2.

The assumption on decay of correlations against L^1 observables is quite strong. In fact, as shown in [1], summable decay of correlations against L^1 implies exponential decay of correlations of Hölder observables against L^∞ ones. From the examples listed above, one perceives that it holds essentially (to the best of our knowledge) in the uniformly expanding realm.

One way of expanding the scope of applications is to consider systems which admit nice first return induced maps, for which we can prove the existence of limits for MREPP, and then pass that information to the original system. In [5], the authors showed that the original system and the first return induced system shared the same Hitting Times Statistics for ball targets shrinking to ζ (which plays the role of the single maximum of φ). Their statement held for a.e. ζ and the standard exponential law. Then in [12], the authors showed that the same limit for REPP applies to the original system and the first return induced system when ζ is a repelling periodic point. In [17], the result of [5] was generalised to all points ζ, and in [14] the latter was generalised to the convergence of REPP. However, the statement of [14, Theorem 3] holds only for point processes and its proof relies on [31, Corollary 6], which was only proved for point processes. Hence, in order to be able to extend our results here to systems admitting a nice first return induced

[1] We say that ζ is a periodic point of *prime period* p if $f^p(\zeta) = \zeta$ and $f^j(\zeta) \neq \zeta$ for all $j = 1, \ldots, p-1$. A periodic point is said to be *repelling* if Df^p is defined at ζ and $\|(Df^p(\zeta))^{-1}\| < 1$, where $\|\cdot\|$ is the norm on the tangent space to $\mathcal{X}$ at ζ given by the Riemannian structure.

map, we need to prove a generalisation of [14, Theorem 3] to atomic random measures, for which we cannot use [31, Corollary 6]. Hence, we will prove Theorem 2.C below.

Let $f : \mathcal{X} \to \mathcal{X}$ be a system with an ergodic f-invariant probability measure $\mathbb{P}$, choose a subset $B \subset \mathcal{X}$ and let $F_B : B \to B$ be the first return map f^{r_B} to B (note that F_B may be undefined at a zero Lebesgue measure set of points which do not return to B, but most of these points are not important, so we will abuse notation here). Let $\mathbb{P}_B(\cdot) = \mathbb{P}(\cdot \cap B)/\mathbb{P}(B)$ be the conditional measure on B. By Kac's Theorem $\mathbb{P}_B$ is F_B-invariant.

Setting $v_n^B = 1/\mathbb{P}_B(X_0 > u_n)$, for the induced process $X_i^B = \varphi \circ F_B^i$ we define, for every $J \in \mathcal{R}$ with $J = J_1 \dot{\cup} \cdots \dot{\cup} J_k$ and $J_i \in \mathcal{S}$ for all $i = 1, \ldots, k$,

$$A_n^B(J) := \sum_{i=1}^{k} \mathscr{A}_{u_n}^B(v_n^B J_i),$$

where, for every interval $I \in \mathcal{S}$,

$$\mathscr{A}_u^B(I)(x) := \sum_{j=0}^{N(I)(x,u)} m_u(\{X_i^B\}_{i \in I_j(x,u) \cap \mathbb{N}_0}).$$

For all $I \in \mathcal{S}$ and $\varepsilon < |I|$ we define

$$I^{\varepsilon+} = (I + \varepsilon) \cup (I - \varepsilon) \in \mathcal{S}, \qquad I^{\varepsilon-} = (I + \varepsilon) \cap (I - \varepsilon) \in \mathcal{S}.$$

If $J \in \mathcal{R}$ we define $J^{\varepsilon\pm}$ accordingly.

Theorem 2.C. *For $\varepsilon > 0$, assume that the limit marked point process $A(I^{\varepsilon\pm})$ is continuous in ε for all small ε. Also assume that $U(u_n) \subset B \in \mathcal{B}$ for n sufficiently large. Then*

$$A_n^B \overset{\mathbb{P}_B}{\Longrightarrow} A \text{ as } n \to \infty \quad \text{implies} \quad A_n \overset{\mathbb{P}}{\Longrightarrow} A \text{ as } n \to \infty.$$

As consequence, if a system $f : \mathcal{X} \to \mathcal{X}$ admits first return time induced systems $F_B : B \to B$ such that F_B has decay of correlations against L^1 so that we can apply Theorem 2.B to prove the convergence of MREPP, then we may use Theorem 2.C to prove the convergence of the corresponding MREPP for the original system f.

Two examples of systems that admit such 'nice' first return induced maps are:

- *Manneville–Pomeau* (MP) map equipped with an absolutely continuous invariant probability measure. These maps, given in [26, 5], are defined, for $\alpha \in (0, 1)$, by

$$f = f_\alpha(x) = \begin{cases} x(1 + 2^\alpha x^\alpha) & \text{for } x \in [0, 1/2), \\ 2x - 1 & \text{for } x \in [1/2, 1]. \end{cases}$$

  They are often referred to as Liverani–Saussol–Vaienti maps since their actual equation was first introduced in [26]. Let $\mathcal{P}$ be the *renewal partition*, that is, the partition defined inductively by $\mathrm{Z} \in \mathcal{P}$ if $\mathrm{Z} = [1/2, 1)$ or $f(\mathrm{Z}) \in \mathcal{P}$. Now let $Y \in \mathcal{P}$ and let F_Y be the first return map to Y and μ_Y be the conditional measure on Y. It is well-known that (Y, F_Y, μ_Y) is a Bernoulli map, and hence in particular a Rychlik system (see [29] or [2, Section 3.2.1] for essential information about such systems).

- We consider a class of C^3 unimodal interval maps $f : I \to I$ with an invariant probability measure absolutely continuous with respect to Lebesgue measure. Let c be the critical point. Such a map is called *S-unimodal* if it has negative *Schwarzian derivative*, i.e., $D^3 f(x)/Df(x) - \frac{3}{2}(D^2 f(x)/Df(x))^2 < 0$ for any $x \in I \setminus \{c\}$. We say that c is *non-flat* if there exists $\ell \in (1, \infty)$ such that $\lim_{x\to c} |f(x) - f(c)|/|x - c|^\ell$ exists and is positive. Here ℓ is called the *order* of the critical point.
  As in [5], if the critical point has an orbit which is not dense in I (e.g. in the Misiurewicz case), it is possible to construct a first return map which gives a Rychlik system.

In Section 3 we will address the issue of convergence in (2.14) which is related to the shape of the observable φ and its behaviour near its maximum value, as well as to the regularity of $\mathbb{P}$. In particular, for certain examples of dynamical systems we will show the convergence of AOT, POT MREPP and compute the limit multiplicity distributions.

3. Applications to dynamical systems

3.1. Conditions on the dependence structure

We begin by proving Theorem 2.B which allows us to automatically verify conditions $Д_p(u_n)^*$ and $Д'_p(u_n)^*$ from decay of correlations against L^1 observables. The proof follows the same lines as the verification of earlier conditions of the same type (like $Д_p(u_n)$ and $Д'_p(u_n)$) in [13] or similar conditions in [11, 12, 2], under the same assumption. However, for completeness and because it is short, we give it here.

Proof of Theorem 2.B. Recall that by assumption $\mathrm{Cor}_{\mathbb{P}}(\phi, \psi, n) \le \rho_n$ with $\sum_{n\ge1} \rho_n < \infty$. As mentioned earlier, condition $Д_p(u_n)^*$, as its predecessors, is designed to follow easily from decay of correlations (and it does not need to be against L^1). Take $\phi = \mathbf{1}_{R_{p,0}(u_n,x_1)}$ and $\psi = \mathbf{1}_{\bigcap_{j=2}^{\varsigma}\{\mathscr{A}_{u_n}(I_j - t)\le x_j\}}$. By assumption, there exists $C' > 0$ such that $\|\mathbf{1}_{R_{p,0}(u_n,x_1)}\|_{\mathcal{C}_1} \le C'$ for all $n \in \mathbb{N}$ and $x_1 \in \mathbb{R}_0^+$. Hence, condition $Д_p(u_n)^*$ holds with $\gamma(n, t) = \gamma(t) := C'\rho_t$ and with a sequence $(t_n)_{n\in\mathbb{N}}$ such that $t_n = o(n)$ and $\lim_{n\to\infty} n\rho_{t_n} = 0$.

We now turn to condition $Д'_p(u_n)^*$. Notice that $Q^0_{p,0}(u_n) = R_{p,0}(u_n, 0)$, so taking $\phi = \mathbf{1}_{Q^0_{p,0}(u_n)}$ and $\psi = \mathbf{1}_{X_0>u_n}$ we easily get

$$
\begin{aligned}
\mathbb{P}(Q^0_{p,0}(u_n) \cap f^{-j}(\{X_0 > u_n\})) &\le \mathbb{P}(Q^0_{p,0}(u_n))\mathbb{P}(X_0 > u_n) + \|\mathbf{1}_{Q^0_{p,0}(u_n)}\|_{\mathcal{C}_1}\mathbb{P}(X_0 > u_n)\rho_j \\
&\le \mathbb{P}(X_0 > u_n)(\mathbb{P}(Q^0_{p,0}(u_n)) + C'\rho_j). \qquad (3.1)
\end{aligned}
$$

Recalling that $n\mathbb{P}(X_0 > u_n) \to \tau \ge 0$ and p is such that (2.12) holds, we find that

$$
\begin{aligned}
n \sum_{j=p+1}^{\lfloor n/k_n \rfloor - 1} \mathbb{P}(Q^0_{p,0}(u_n) \cap f^{-j}(\{X_0 > u_n\})) &= n \sum_{j=R(Q^0_{p,0}(u_n))}^{\lfloor n/k_n \rfloor - 1} \mathbb{P}(Q^0_{p,0}(u_n) \cap f^{-j}(\{X_0 > u_n\})) \\
\le \frac{n^2}{k_n}\mathbb{P}(X_0 > u_n)\mathbb{P}(Q^0_{p,0}(u_n)) + n\mathbb{P}(X_0 > u_n)C' &\sum_{j=R(Q^0_{p,0}(u_n))}^{\infty} \rho_j \to 0 \quad \text{as } n \to \infty.
\end{aligned}
$$

□

Remark 3.1. Note that in the proof of Theorem 2.B, the fact that the decay of correlations holds against all L^1 observables was only used in the proof of $Д'_p(u_n)^*$. In fact, as mentioned earlier, by adapting the proof of [12, Proposition 3.1], one can easily show that $Д_p(u_n)^*$ follows from decay of correlations of Hölder observables against L^∞ ones.

In the proof of Theorem 2.B we use the fact that we can find p such that (2.12) holds, and consequently $R(Q^0_{p,0}(u_n)) \to \infty$ as $n \to \infty$. If we take $q = \max\{n \in \mathbb{N} : f^n(\zeta_i) = \zeta_j, i, j = 1, \dots, k\}$ then under mild assumptions on the system we have $R(Q^0_{q,0}(u_n)) \to \infty$ as $n \to \infty$. For example, if the system is continuous along the orbits of ζ_i, $i = 1, \dots, k$, then using a continuity argument and the Hartman–Grobman theorem (when a ζ_i is periodic) one can show the above convergence (see [3, Lemmas 4.1 and 5.1]). One can prove it even when the orbit of some ζ_i hits a discontinuity point of f as in [2, Section 3.3].

3.2. *Scenarios of possible application*

As in [3], having multiple maximal points creates a large range of possibilities since the local behaviour of φ and of the measure $\mathbb{P}$ at each point raises an enormous number of cases. Our goal here is to illustrate our convergence theorem and compute the limit marked point process for some examples. Since it would be extremely difficult to cover all the possibilities in a systematic way, we make some assumptions from this point until the end of this section intended to simplify the presentation but maintain, as much as possible, the key aspects of potential application.

Assumption 1: Single global maximum. There exists a single point $\zeta \in \mathcal{X}$ where φ achieves its global maximum value. We allow $\varphi(\zeta) = \infty$.

Assumption 2: Shape of the observable. The observable $\varphi : \mathcal{X} \to \mathbb{R} \cup \{\infty\}$ is of the form

$$\varphi(x) = g(\mathrm{dist}(x, \zeta)), \tag{3.2}$$

where $g : V \to W$ is a strictly decreasing homeomorphism in a neighbourhood V of 0 and has one of the following three types of behaviour:

Type 1: there exists a strictly positive function $q : W \to \mathbb{R}$ such that for all $y \in \mathbb{R}$,

$$\lim_{s \to g(0)} \frac{g^{-1}(s + yq(s))}{g^{-1}(s)} = \mathrm{e}^{-y}. \tag{3.3}$$

Type 2: $g(0) = \infty$ and there exists $\beta > 0$ such that for all $y > 0$,

$$\lim_{s \to \infty} \frac{g^{-1}(sy)}{g^{-1}(s)} = y^{-\beta}. \tag{3.4}$$

Type 3: $g(0) = D < \infty$ and there exists $\gamma > 0$ such that for all $y > 0$,

$$\lim_{s \to 0} \frac{g^{-1}(D - sy)}{g^{-1}(D - s)} = y^{\gamma}. \tag{3.5}$$

Examples of each of the three types are, respectively, as follows: $g(x) = -\log x$ (in this case (3.3) is easily verified with $q \equiv 1$); $g(x) = x^{-1/\alpha}$ for some $\alpha > 0$ (condition (3.4) holds with $\beta = \alpha$); and $g(x) = D - x^{1/\alpha}$ for some $D \in \mathbb{R}$ and $\alpha > 0$ (condition (3.5) holds with $\gamma = \alpha$).

Assumption 3: Regularity of $\mathbb{P}$. Now assume $\mathbb{P}$ is absolutely continuous with respect to Lebesgue measure and its Radon–Nikodym derivative is sufficiently regular so that for all $x \in \mathcal{X}$,

$$\lim_{\varepsilon\to 0} \frac{\mathbb{P}(B_\varepsilon(x))}{\mathrm{Leb}(B_\varepsilon(x))} = \frac{d\mathbb{P}}{d\mathrm{Leb}}(x). \tag{3.6}$$

Remark 3.2. Note that if f is a one-dimensional smooth map modelled by the full shift as in [13, Section 7.1] and the derivative is sufficiently regular then, as seen in [13, Section 7.3], the invariant density is fairly smooth and formula (3.6) holds for all $x \in \mathcal{X}$.

Remark 3.3. The different types of g imply that the distribution of X_0 falls in the domain of attraction for maxima of the Gumbel, Fréchet and Weibull distributions, respectively.

We recall that as shown in [2], under decay of correlations against L^1 and the previous assumptions, either we have clustering when ζ is a repelling periodic point, or at any other non-periodic point ζ we have no clustering of exceedances and an EI equal to 1. Moreover, under the previous assumptions condition (R1) is always satisfied, and if ζ is a repelling periodic point of prime period p, then (R2) is also satisfied with

$$\theta = 1 - \frac{1}{\det Df^p(\zeta)}. \tag{3.7}$$

In particular the limit of θ_n defined in (2.13) exists and equals θ.

Remark 3.4. If $\mathbb{P}$ is not absolutely continuous with respect to Lebesgue measure, we can use instead observables of the form $\varphi(x) = g(\mu_\phi(B_{\mathrm{dist}(x,\zeta)}(\zeta)))$, as introduced in [10], and the analysis we will carry out could be easily adjusted in order to obtain essentially the same results. In particular, when $\mathbb{P}$ is the more general equilibrium state associated to a potential ψ then condition (R2) can be verified as in [11, Lemma 3.1] and the EI is given by the formula $\theta = 1 - \mathrm{e}^{\psi(\zeta)+\cdots+\psi(f^{p-1}(\zeta))}$.

As mentioned above, if ζ is not periodic then condition $ULC_0(u_n)$ is trivially satisfied. If ζ is a periodic point of prime period p, then since the above assumptions guarantee that (R2) is satisfied, condition $ULC_p(u_n)$ can also be easily verified, as follows.

Proof of Proposition 2.13. Since by (2.15),

$$\begin{aligned}\delta_{p,s,u}(x) &= p\Big(\sum_{\kappa=0}^{\lfloor s/p\rfloor}(\kappa+1)\mathbb{P}(U^\kappa_{p,0}(u,x)) + \sum_{\kappa=\lfloor s/p\rfloor+1}^{\infty}(s/p+1)\mathbb{P}(U^\kappa_{p,0}(u,x))\Big)\\ &\le p\sum_{\kappa=0}^{\infty}(\kappa+1)\mathbb{P}(U^\kappa_{p,0}(u,x)) \le p\sum_{\kappa=0}^{\infty}(\kappa+1)\mathbb{P}(Q^\kappa_{p,0}(u))\end{aligned}$$

for all $x \in \mathbb{R}_0^+$ and $y \in \mathbb{R}^+$, we have

$$\int_0^\infty y\mathrm{e}^{-yx}\delta_{p,\lfloor n/k_n\rfloor,u_n}(x)\,dx \leq p\sum_{\kappa=0}^{\infty}(\kappa+1)\mathbb{P}(Q_{p,0}^{\kappa}(u_n))\int_0^\infty y\mathrm{e}^{-yx}\,dx$$
$$= p\sum_{\kappa=0}^{\infty}(\kappa+1)\mathbb{P}(Q_{p,0}^{\kappa}(u_n)).$$

So, it is sufficient to check if

$$\limsup_{n\to\infty} n\sum_{\kappa=0}^{\infty}(\kappa+1)\mathbb{P}(Q_{p,0}^{\kappa}(u_n)) < \infty.$$

By (R2), there exists $0<\theta<1$ such that $\bigcap_{j=0}^{i} f^{-jp}(\{X_0>u\})$ is a ball around ζ with

$$\mathbb{P}\Big(\bigcap_{j=0}^{\kappa} f^{-jp}(\{X_0>u\})\Big) \sim (1-\theta)^{\kappa}\mathbb{P}(X_0>u)$$

for all u sufficiently large. So, we have

$$\mathbb{P}(U_p^{(\kappa)}(u_n)) \sim (1-\theta)^{\kappa}\mathbb{P}(U(u_n)),$$
$$\mathbb{P}(Q_{p,0}^{\kappa}(u_n)) = \mathbb{P}(U_p^{(\kappa)}(u_n)) - \mathbb{P}(U_p^{(\kappa+1)}(u_n)) \sim \theta(1-\theta)^{\kappa}\mathbb{P}(U(u_n)),$$
$$\sum_{\kappa=0}^{\infty}(\kappa+1)\mathbb{P}(Q_{p,0}^{\kappa}(u_n)) \sim \sum_{\kappa=0}^{\infty}(\kappa+1)\theta(1-\theta)^{\kappa}\mathbb{P}(U(u_n)) = \frac{1}{\theta}\mathbb{P}(U(u_n)).$$

Since by (2.2) we have $\lim_{n\to\infty} n\mathbb{P}(U(u_n)) = \tau$, we conclude that condition $ULC_p(u_n)$ is always satisfied when $\zeta \in \mathcal{X}$ is a repelling periodic point of prime period $p \in \mathbb{N}$ satisfying (R2). □

3.3. Convergence of REPP

When the mark function m_u defined in (2.7) counts the number of exceedances, then our atomic random measure A_n is actually a REPP as the one considered in [12], namely, $A_n(J) = \sum_{j\in v_n J\cap\mathbb{N}_0} \mathbf{1}_{X_j>u}$. We realise here that if we have a system that admits a first return induced map on a base B with decay of correlations against L^1, and $\zeta \in B$ is the only global maximum of φ, which is a periodic point satisfying (R2), which is the case if Assumptions 1–3 hold, then we recover the main result in [12], which states that A_n converges in distribution to a compound Poisson process of intensity θ and geometric multiplicity distribution.

To see this, we note that

$$U_p^{(\kappa)}(u) = U(u) \cap \bigcap_{i=1}^{\kappa}\{w_{U(u)}^{i} = p\} = \{X_0>u,\, X_p>u,\, \ldots,\, X_{\kappa p}>u\},$$
$$Q_{p,i}^{\kappa} = \{X_i>u,\, X_{i+p}>u,\ldots,X_{i+\kappa p}>u,\, X_{i+(\kappa+1)p}\leq u\},$$

$$m_u(\{X_j\}_{0\le j\le r^\kappa_{U(u)}}) > x \Leftrightarrow \kappa \ge \lfloor x\rfloor, \qquad U^\kappa_{p,0}(u,x) = \begin{cases} Q^\kappa_{p,0}(u) & \text{if } \kappa \ge \lfloor x\rfloor, \\ \emptyset & \text{if } \kappa < \lfloor x\rfloor, \end{cases}$$

$$U_{p,0}(u,x) = \bigcup_{\kappa=\lfloor x\rfloor}^{\infty} Q^\kappa_{p,0}(u) \cup \{\zeta\} = U_p^{(\lfloor x\rfloor)}(u),$$

$$R_{p,0}(u,x) = U_p^{(\lfloor x\rfloor)}(u) \cap \{r_{U_p^{(\lfloor x\rfloor)}(u)} > p\} = Q^{\lfloor x\rfloor}_{p,0}(u).$$

Moreover, $\mathbb{P}(U_p^{(\kappa)}(u_n)) \sim (1-\theta)^\kappa \mathbb{P}(U(u_n))$ and $\mathbb{P}(Q^\kappa_{p,0}(u_n)) \sim \theta(1-\theta)^\kappa \mathbb{P}(U(u))$. The result now follows from observing that

$$\lim_{n\to\infty} \frac{\mathbb{P}(R_{p,0}(u_n,x))}{\mathbb{P}(U(u_n))} = \lim_{n\to\infty} \frac{\mathbb{P}(Q^{\lfloor x\rfloor}_{p,0}(u_n))}{\mathbb{P}(U(u_n))} = \lim_{n\to\infty} \frac{\theta(1-\theta)^{\lfloor x\rfloor}\mathbb{P}(U(u_n))}{\mathbb{P}(U(u_n))}$$
$$= \theta(1-\theta)^{\lfloor x\rfloor} = \theta(1-\pi(x))$$

where $\pi(x) = 1-(1-\theta)^{\lfloor x\rfloor}$ is the cumulative distribution function of a geometric distribution of parameter θ, that is, $\pi(x) = \sum_{\kappa\le x,\kappa\in\mathbb{N}} \theta(1-\theta)^{\kappa-1}$.

Remark 3.5. If the point ζ is not periodic and a dichotomy holds, as in [2], for the first return induced system (which we are assuming to have decay of correlations against L^1), then condition $Д'_0(u_n)^*$ holds and REPP is easily seen to converge to a standard Poisson process (with intensity 1).

3.4. Computation of the limit of EOT and POT MREPP

When the mark function m_u defined in (2.7) weighs the maximum excess within a cluster, then our atomic random measure A_n is a POT MREPP. When there is no clustering then A_n is an EOT MREPP and, as observed above, the POT and AOT MREPP coincide and provide information about the sum of all observed excesses.

The result below states that for uniformly expanding and certain non-uniformly expanding dynamical systems the POT MREPP, in the presence of clustering, and the EOT MREPP, in its absence, both converge to a compound Poisson process with intensity given by the EI and whose multiplicity distribution is a Generalised Pareto Distribution (GPD), of type depending on the type of g chosen in Assumption 2.

Theorem 3.A. *Let $f : \mathcal{X} \to \mathcal{X}$ be a system admitting a first return induced map $F_B : B \to B$ with $B \subset \mathcal{X}$ and such that F_B has summable decay of correlations against L^1 observables, i.e., for all $\phi \in \mathcal{C}_1$ and $\psi \in L^1$,* $\mathrm{Cor}(\phi,\psi,n) \le \rho_n$ *with $\sum_{n\ge1}\rho_n < \infty$. Assume that for every ζ, for all balls $B_\varepsilon(\zeta)$ and annuli $B_{\varepsilon_1}(\zeta)\setminus B_{\varepsilon_2}(\zeta)$ with $\varepsilon > 0$, $0 < \varepsilon_1 < \varepsilon_2$, we have $\mathbf{1}_{B_\varepsilon(\zeta)} \in \mathcal{C}_1$, $\mathbf{1}_{B_{\varepsilon_1}(\zeta)\setminus B_{\varepsilon_2}(\zeta)} \in \mathcal{C}_1$ and their norms are uniformly bounded above.*

Let $X_0, X_1, \ldots$ be given by (2.1) *and $(u_n)_{n\in\mathbb{N}}$ be a sequence satisfying* (2.2)*. Assume that φ and $\mathbb{P}$ are such that Assumptions* 1–3 *hold, where $\zeta \in B$. Then*

- *if ζ is a periodic repelling point of prime period p, the POT MREPP a_nA_n converges in distribution to a compound Poisson process with intensity θ given by formula* (3.7) *and with multiplicity distribution*

$$\pi(x) = \begin{cases} 1 - \mathrm{e}^{-x} & \text{when } g \text{ is of type 1 and } a_n = (q(u_n))^{-1}, \\ 1 - (1+x)^{-\beta} & \text{when } g \text{ is of type 2 and } a_n = u_n^{-1}, \\ 1 - (1-x)^{\gamma} & \text{when } g \text{ is of type 3 and } a_n = (D - u_n)^{-1}; \end{cases} \tag{3.8}$$

- *if ζ is not periodic and f is continuous at the points of its orbit then the EOT MREPP $a_n A_n$ converges in distribution to a compound Poisson process with intensity 1 and multiplicity distribution given by (3.8).*

Proof. By Theorem 2.C we only need to prove the result for F_B since then it follows for f. First we consider the case of ζ not periodic ($p = 0$). By Assumptions 1 and 2, $R_{0,0}(u_n, x) = U^0_{0,0}(u_n, x) = B_\varepsilon(\zeta)$ for some $\varepsilon > 0$, and consequently $\mathbf{1}_{R_{0,0}(u_n,x)} \in \mathcal{C}_1$ and $\|\mathbf{1}_{R_{0,0}(u_n,x)}\|_{\mathcal{C}_1} \leq C$ for every $x > 0$ and $n \in \mathbb{N}$. Recalling that in this case $Q^0_{0,0}(u_n) = U(u_n)$, as in [2, Lemma 3.1], it follows using a continuity argument that $\lim_{n\to\infty} R(U(u_n)) = \infty$. Then all hypotheses of Theorem 2.B are satisfied, and so conditions $Д_0(u_n)^*$ and $Д'_0(u_n)^*$ hold. Moreover, as observed earlier, condition $ULC_0(u_n)$ is trivially satisfied.

Now we consider the case where ζ is a repelling periodic point of prime period p. By Assumptions 1 and 2, $R_{p,0}(u_n, x) = B_{\varepsilon_1}(\zeta) \setminus B_{\varepsilon_2}(\zeta)$ for some $\varepsilon_1, \varepsilon_2 > 0$, and consequently $\mathbf{1}_{R_{p,0}(u_n,x)} \in \mathcal{C}_1$ and $\|\mathbf{1}_{R_{p,0}(u_n,x)}\|_{\mathcal{C}_1} \leq C$ for every $x > 0$ and $n \in \mathbb{N}$. Moreover, as in [12, proof of Theorem 2], using the Hartman–Grobman theorem one can easily check that $\lim_{n\to\infty} R(Q^0_{p,0}(u_n)) = \infty$. Then all hypotheses of Theorem 2.B are satisfied, and so $Д_p(u_n)^*$ and $Д'_p(u_n)^*$ hold. By Assumptions 1–3 and the fact that ζ is a repelling periodic point (R2) holds and by Proposition 2.13 so does $ULC_p(u_n)$. Hence, the statements of the theorem follow as soon as we show that (2.14) holds with $\pi(x)$ as in (3.8). For u sufficiently close to $g(0)$, we have

$$U_{p,0}(u, x) = \{X_0 > u + x\} = B_{\frac{1}{2}g^{-1}(u+x)}(\zeta),$$

$$R_{p,0}(u, x) = U_{p,0}(u, x) \setminus U_{p,p}(u, x) = \{X_0 > u + x\} \setminus \bigcap_{j=0}^{1} f^{-jp}(\{X_0 > u + x\}).$$

By (R2), $\{X_0 > u + x\}$ and $\bigcap_{j=0}^{1} f^{-jp}(\{X_0 > u + x\})$ are both intervals, and

$$\mathbb{P}(R_{p,0}(u, x)) = \mathbb{P}(X_0 > u + x) - (1 - \theta)\mathbb{P}(X_0 > u + x) = \theta\mathbb{P}(X_0 > u + x).$$

Let $(u_n)_n$ be a normalizing sequence of levels satisfying (2.2) and $\lim_{n\to\infty} u_n = g(0)$. Given the assumptions (3.6) and (R1), of regularity of $\mathbb{P}$ and $U(u_n) = \{X_0 > u_n\}$ being a ball centred at ζ, respectively, we have

$$\mathbb{P}(X_0 > u_n) \sim \mathrm{Leb}(X_0 > u_n)\frac{d\mathbb{P}}{d\mathrm{Leb}}(\zeta) = g^{-1}(u_n)\frac{d\mathbb{P}}{d\mathrm{Leb}}(\zeta),$$

$$\begin{aligned}\mathbb{P}(R_{p,0}(u_n, x)) &= \theta\mathbb{P}(X_0 > u_n + x) \\ &\sim \theta\mathrm{Leb}(X_0 > u_n + x)\frac{d\mathbb{P}}{d\mathrm{Leb}}(\zeta) = \theta g^{-1}(u_n + x)\frac{d\mathbb{P}}{d\mathrm{Leb}}(\zeta).\end{aligned}$$

If there exists a strictly positive function $q : W \to \mathbb{R}$ and a strictly monotone homeomorphism h such that

$$\lim_{u \to g(0)} \frac{g(g^{-1}(u)h(x)) - u}{q(u)} = x$$

then, for $a_n = 1/q(u_n)$,

$$\begin{aligned}\lim_{n\to\infty} \frac{\mathbb{P}(R_{p,0}(u_n, x/a_n))}{\mathbb{P}(X_0 > u_n)} &= \theta \lim_{n\to\infty} \frac{g^{-1}(u_n + q(u_n)x)}{g^{-1}(u_n)} \\ &= \theta \lim_{n\to\infty} \lim_{u\to g(0)} \frac{g^{-1}\big(u_n + q(u_n)\frac{g(g^{-1}(u)h(x))-u}{q(u)}\big)}{g^{-1}(u_n)}.\end{aligned}$$

In particular, for $u = u_n$ we get

$$\lim_{n\to\infty} \frac{\mathbb{P}(R_{p,0}(u_n, x/a_n))}{\mathbb{P}(X_0 > u_n)} = \theta \lim_{n\to\infty} \frac{g^{-1}\big(u_n + q(u_n)\frac{g(g^{-1}(u_n)h(x))-u_n}{q(u_n)}\big)}{g^{-1}(u_n)} = \theta h(x)$$

and the probability distribution is $\pi(x) = 1 - h(x)$. We will analyse each type of behaviour separately.

Type 1: there exists a strictly positive function $p : W \to \mathbb{R}$ such that for all $y \in \mathbb{R}$,

$$\lim_{s\to g(0)} \frac{g^{-1}(s + yq(s))}{g^{-1}(s)} = \mathrm{e}^{-y}.$$

Then

$$\begin{aligned}&\lim_{u\to g(0)} \frac{g^{-1}(u - \log(x)q(u))}{g^{-1}(u)} = x, \\ &\lim_{u\to g(0)} \frac{g(g^{-1}(u)x) - u}{q(u)} = \lim_{u\to g(0)} \frac{g\big(g^{-1}(u)\frac{g^{-1}(u-\log(x)q(u))}{g^{-1}(u)}\big) - u}{q(u)} = -\log(x).\end{aligned}$$

Let $h(x) = \mathrm{e}^{-x}$, so that $h^{-1}(x) = -\log(x)$. Then $a_n A_n$ converges in distribution to a compound Poisson process A with intensity θ and multiplicity d.f. $\pi(x) = 1 - \mathrm{e}^{-x}$.

Type 2: $g(0) = \infty$ and there exists $\beta > 0$ such that for all $y > 0$,

$$\lim_{s\to\infty} \frac{g^{-1}(sy)}{g^{-1}(s)} = y^{-\beta}.$$

Then for $q(u) = u$ we have

$$\begin{aligned}&\lim_{u\to\infty} \frac{g^{-1}(ux^{-1/\beta})}{g^{-1}(u)} = x, \\ &\lim_{u\to\infty} \frac{g(g^{-1}(u)x) - u}{q(u)} = \lim_{u\to\infty} \frac{g\big(g^{-1}(u)\frac{g^{-1}(ux^{-1/\beta})}{g^{-1}(u)}\big)}{u} - 1 = x^{-1/\beta} - 1.\end{aligned}$$

Let $h(x) = (1+x)^{-\beta}$, so that $h^{-1}(x) = x^{-1/\beta} - 1$. Then $a_n A_n$ converges in distribution to a compound Poisson process A with intensity θ and multiplicity d.f. $\pi = 1-(1+x)^{-\beta}$.

Type 3: $g(0) = D < \infty$ and there exists $\gamma > 0$ such that for all $y > 0$,

$$\lim_{s\to 0} \frac{g^{-1}(D-sy)}{g^{-1}(D-s)} = y^\gamma.$$

Then for $q(u) = D - u$ we have

$$\lim_{u\to D} \frac{g^{-1}(D-(D-u)x^{1/\gamma})}{g^{-1}(u)} = x,$$

$$\lim_{u\to D} \frac{g(g^{-1}(u)x) - u}{q(u)} = \lim_{u\to D} \frac{g\big(g^{-1}(u)\frac{g^{-1}(D-(D-u)x^{1/\gamma})}{g^{-1}(u)}\big) - u}{D-u} = 1 - x^{1/\gamma}.$$

Let $h(x) = (1-x)^\gamma$, so that $h^{-1}(x) = 1 - x^{1/\gamma}$. Then $a_n A_n$ converges in distribution to a compound Poisson process A with intensity θ and multiplicity d.f. $\pi = 1 - (1-x)^\gamma$. □

3.5. Computation of the limit of AOT MREPP for specific systems

In the case of AOT MREPP it is technically much harder to compute the multiplicity distribution of the limiting compound Poisson process. In order to write an explicit formula for it we need to assume a specific backward contraction in a neighbourhood of the repelling periodic point, rather than an approximate rate as in the previous cases.

Theorem 3.B. *Let $f : \mathcal{X} \to \mathcal{X}$ be a system admitting a first return induced map $F_B : B \to B$ with $B \subset \mathcal{X}$ and such that F_B has summable decay of correlations against L^1 observables, i.e., for all $\phi \in \mathcal{C}_1$ and $\psi \in L^1$, $\mathrm{Cor}(\phi, \psi, n) \le \rho_n$ with $\sum_{n\ge 1} \rho_n < \infty$. Assume that for every ζ, for all balls $B_\varepsilon(\zeta)$ and annuli $B_{\varepsilon_1}(\zeta) \setminus B_{\varepsilon_2}(\zeta)$ with $\varepsilon > 0$, $0 < \varepsilon_1 < \varepsilon_2$ we have $\mathbf{1}_{B_\varepsilon(\zeta)} \in \mathcal{C}_1$, $\mathbf{1}_{B_{\varepsilon_1}(\zeta)\setminus B_{\varepsilon_2}(\zeta)} \in \mathcal{C}_1$ and their norms are uniformly bounded above.*

Let $X_0, X_1, \dots$ be given by (2.1) and $(u_n)_{n\in\mathbb{N}}$ be a sequence satisfying (2.2). Assume that φ and $\mathbb{P}$ are such that Assumptions 1–3 hold, where $\zeta \in B$. Additionally, suppose that

- *ζ is a periodic repelling point of period p;*
- *for some $M > 1$,*
$$\mathrm{dist}(f^p(x), \zeta) = M\,\mathrm{dist}(x, \zeta)$$
for all x in a neighbourhood of ζ (an example of such a dynamical system is $f : t \mapsto mt \bmod 1$ with $m \in \{2, 3, \dots\}$; in this case $M = m^p$);
- *there exists a strictly positive function $q : W \to \mathbb{R}$ and a strictly monotone homeomorphism h_k such that*

$$\lim_{u\to g(0)} \frac{g_{\kappa,u}(g^{-1}(u)h_\kappa(x))}{q(u)} = x, \qquad \forall \kappa \in \mathbb{N}_0, \tag{3.9}$$

where $g_{\kappa,u}(x) := \sum_{i=0}^{\kappa}(g(M^i x) - u)$ (as we will see, this holds when g has the form given in Assumption 2).

Then, for $a_n = q(u_n)^{-1}$ the AOT MREPP $a_n A_n$ converges in distribution to a compound Poisson process with intensity $\theta = 1 - 1/M$ and multiplicity d.f. π given by

$$\pi(x) = 1 - \lim_{n\to\infty} h_{\kappa(u_n,q(u_n)x)}(x)$$

where $\kappa = \kappa(u,x)$ is the only integer such that $x \in \left[g_{\kappa,u}\left(\frac{g^{-1}(u)}{M^\kappa}\right), g_{\kappa,u}\left(\frac{g^{-1}(u)}{M^{\kappa+1}}\right)\right)$.

Remark 3.6. In particular, when g is one of the three examples given above, the multiplicity d.f. can be computed as shown in the following table:

Examples of $g(x)$	Respective distribution $\pi(x)$
$-\log(x)$	$1-(\sqrt{M})^{-\lfloor\frac{\sqrt{1+8x/\log M}-1}{2}\rfloor}\exp\left(-\frac{x}{\lfloor\frac{\sqrt{1+8x/\log M}-1}{2}\rfloor+1}\right)$
$x^{-1/\alpha}, \alpha>0$	$1-\left(\frac{1-M^{-1/\alpha}}{1-M^{-(\kappa(x)+1)/\alpha}}\right)^{-\alpha}(\kappa(x)+1+x)^{-\alpha}$ where $\kappa=\kappa(x)$ is the only integer such that $\frac{M^{\kappa/\alpha}-M^{-1/\alpha}}{1-M^{-1/\alpha}} \le \kappa+1+x < \frac{M^{(\kappa+1)/\alpha}-1}{1-M^{-1/\alpha}}$
$D-x^{1/\alpha}, D\in\mathbb{R}, \alpha>0$	$1-\left(\frac{1-M^{1/\alpha}}{1-M^{(\kappa(x)+1)/\alpha}}\right)^{\alpha}(\kappa(x)+1-x)^{\alpha}$ where $\kappa=\kappa(x)$ is the only integer such that $\frac{1-M^{-(\kappa+1)/\alpha}}{M^{1/\alpha}-1} < \kappa+1-x \le \frac{M^{1/\alpha}-M^{-\kappa/\alpha}}{M^{1/\alpha}-1}$

Proof of Theorem 3.B. By Theorem 2.C we only need to prove the result for F_B since then it follows for f. If $R_{p,0}(u_n,x) = B_{\varepsilon_1}(\zeta)\setminus B_{\varepsilon_2}(\zeta)$ for some $\varepsilon_1,\varepsilon_2>0$ then $\mathbf{1}_{R_{p,0}(u_n,x)} \in \mathcal{C}_1$ and $\|\mathbf{1}_{R_{p,0}(u_n,x)}\|_{\mathcal{C}_1} \le C$ for every $x>0$ and $n\in\mathbb{N}$. Moreover, as in [12, proof of Theorem 2], using the Hartman–Grobman theorem, one can easily check that $\lim_{n\to\infty} R(Q^0_{p,0}(u_n)) = \infty$. Then all hypotheses of Theorem 2.B are satisfied, and so conditions $Д_p(u_n)^*$ and $Д'_p(u_n)^*$ hold. By Assumptions 1–3 and the fact that ζ is a repelling periodic point we find that (R2) holds and by Proposition 2.13 so does $ULC_p(u_n)$.

Hence, the statements of the theorem follow as soon as we show that $R_{p,0}(u_n,x) = B_{\varepsilon_1}(\zeta)\setminus B_{\varepsilon_2}(\zeta)$ for some $\varepsilon_1,\varepsilon_2>0$ and (2.14) holds with $\pi(x)$ as in (3.8).

For $u\in(0,g(0))$, let

$$g_{\kappa,u}(x) := \sum_{i=0}^{\kappa}(g(M^i x)-u), \qquad b_\kappa(u) := g_{\kappa,u}\left(\frac{g^{-1}(u)}{M^\kappa}\right).$$

For $j,\kappa\in\mathbb{N}_0$, t sufficiently close to ζ and u sufficiently close to $g(0)$, we have

$$X_{jp}(t) > u \iff g(2\,\mathrm{dist}(f^{jp}(t),\zeta)) > u \iff g(2M^j\,\mathrm{dist}(t,\zeta)) > u \iff t \in B_{\frac{g^{-1}(u)}{2M^j}}(\zeta),$$
$$m_u(\{X_0, X_p,\dots,X_{\kappa p}\})(t) > x \iff g_{\kappa,u}(2\,\mathrm{dist}(t,\zeta)) > x \iff t\in B_{\frac{1}{2}g^{-1}_{\kappa,u}(x)}(\zeta).$$

Notice that $(b_\kappa(u))_{\kappa\in\mathbb{N}_0}$ is an increasing sequence for any $u\in[0,g(0))$ since $g\left(\frac{g^{-1}(u)}{M^i}\right) > u$ for $i>0$. Moreover, $b_0(u)=0$ and $b_{\kappa+1}(u) = g_{\kappa+1,u}\left(\frac{g^{-1}(u)}{M^{\kappa+1}}\right) = g_{\kappa,u}\left(\frac{g^{-1}(u)}{M^{\kappa+1}}\right)$.

Then,

$$x \geq b_{\kappa+1}(u) \;\Leftrightarrow\; g_{\kappa,u}^{-1}(x) \leq \frac{g^{-1}(u)}{M^{\kappa+1}} \;\Leftrightarrow\; U_{p,0}^{\kappa}(u,x) = \emptyset,$$

$$x \leq b_{\kappa}(u) \;\Leftrightarrow\; g_{\kappa,u}^{-1}(x) \geq \frac{g^{-1}(u)}{M^{\kappa}} \;\Leftrightarrow\; U_{p,0}^{\kappa}(u,x) = B_{\frac{g^{-1}(u)}{2M^{\kappa}}}(\zeta) \setminus B_{\frac{g^{-1}(u)}{2M^{\kappa+1}}}(\zeta),$$

$$\begin{aligned} b_{\kappa}(u) \leq x \leq b_{\kappa+1}(u) \;&\Leftrightarrow\; \frac{g^{-1}(u)}{M^{\kappa+1}} \leq g_{\kappa,u}^{-1}(x) \leq \frac{g^{-1}(u)}{M^{\kappa}} \\ &\Leftrightarrow\; U_{p,0}^{\kappa}(u,x) = B_{\frac{1}{2}g_{\kappa,u}^{-1}(x)}(\zeta) \setminus B_{\frac{g^{-1}(u)}{2M^{\kappa+1}}}(\zeta). \end{aligned}$$

Since $(b_{\kappa}(u))_{\kappa}$ is an increasing sequence, there is at most one $\kappa = \kappa(u,x)$ such that $x \in [b_{\kappa}(u), b_{\kappa+1}(u))$. Notice that

$$\begin{aligned} b_{\kappa}(u) \leq x < b_{\kappa+1}(u) \;&\Leftrightarrow\; \frac{g^{-1}(u)}{M^{\kappa+1}} < g_{\kappa,u}^{-1}(x) \leq \frac{g^{-1}(u)}{M^{\kappa}} \;\Leftrightarrow\; M^{-\kappa+1} < \frac{g_{\kappa,u}^{-1}(x)}{g^{-1}(u)} \leq M^{-\kappa} \\ &\Leftrightarrow\; \kappa \leq -\log_M \frac{g_{\kappa,u}^{-1}(x)}{g^{-1}(u)} < \kappa+1 \;\Leftrightarrow\; \kappa = \left\lfloor \frac{\log(g^{-1}(u)) - \log(g_{\kappa,u}^{-1}(x))}{\log M} \right\rfloor. \end{aligned}$$

Hence,

$$\begin{aligned} U_{p,0}(u,x) &:= \bigcup_{\kappa=0}^{\infty} U_{p,0}^{\kappa}(u,x) \cup \{\zeta\} \\ &= \bigcup_{\kappa<\kappa(u,x)} U_{p,0}^{\kappa}(u,x) \cup U_{p,0}^{\kappa(u,x)}(u,x) \cup \bigcup_{\kappa>\kappa(u,x)} U_{p,0}^{\kappa}(u,x) \cup \{\zeta\} \\ &= \big(B_{\frac{1}{2}G_u(x)}(\zeta) \setminus B_{\frac{g^{-1}(u)}{2M^{\kappa(u,x)+1}}}(\zeta)\big) \cup \bigcup_{\kappa=\kappa(u,x)+1}^{\infty} \big(B_{\frac{g^{-1}(u)}{2M^{\kappa}}}(\zeta) \setminus B_{\frac{g^{-1}(u)}{2M^{\kappa+1}}}(\zeta)\big) \cup \{\zeta\} \\ &= B_{\frac{1}{2}G_u(x)}(\zeta) \end{aligned}$$

where $G_u(x) = g_{\kappa(u,x),u}^{-1}(x)$. Now, we note that

$$U_{p,0}(u,x) \cap U_{p,p}(u,x) = \bigcup_{\kappa=0}^{\infty} U_{p,0}^{\kappa}(u,x) \cap \bigcup_{\kappa=0}^{\infty} U_{p,p}^{\kappa}(u,x) \cup \{\zeta\}$$

and, for u sufficiently close to $g(0)$, $U_{p,0}^{i}(u,x) \cap U_{p,p}^{j}(u,x) \neq \emptyset$ only when $i = j+1$, so

$$\begin{aligned} U_{p,0}(u,x) \cap U_{p,p}(u,x) &= \bigcup_{\kappa=0}^{\infty} (U_{p,0}^{\kappa+1}(u,x) \cap U_{p,p}^{\kappa}(u,x)) \cup \{\zeta\} \\ &= \bigcup_{\kappa=0}^{\infty} \big\{X_0 > u,\ X_p > u, \ldots, X_{(\kappa+1)p} > u, \\ &\qquad X_{(\kappa+2)p} \leq u,\ m_u(\{X_p, \ldots, X_{(\kappa+1)p}\}) > x\big\} \cup \{\zeta\}. \end{aligned}$$

Then, for $\kappa \in \mathbb{N}_0$ and t sufficiently close to ζ, we have

$$m_u(\{X_p, \dots, X_{(\kappa+1)p}\})(t) > x \;\Leftrightarrow\; g_{\kappa,u}(2M\,\mathrm{dist}(t,\zeta)) > x \;\Leftrightarrow\; t \in B_{\frac{g_{\kappa,u}^{-1}(x)}{2M}}(\zeta),$$

$$\Big(B_{\frac{g^{-1}(u)}{2M^{\kappa+1}}}(\zeta) \setminus B_{\frac{g^{-1}(u)}{2M^{\kappa+2}}}(\zeta)\Big) \cap B_{\frac{g_{\kappa,u}^{-1}(x)}{2M}}(\zeta) = \begin{cases} \emptyset & \text{if } \kappa < \kappa(u,x), \\ B_{\frac{G_u(x)}{2M}}(\zeta) \setminus B_{\frac{g^{-1}(u)}{2M^{\kappa(u,x)+2}}}(\zeta) & \text{if } = \kappa(u,x), \\ B_{\frac{g^{-1}(u)}{2M^{\kappa+1}}}(\zeta) \setminus B_{\frac{g^{-1}(u)}{2M^{\kappa+2}}}(\zeta) & \text{if } \kappa > \kappa(u,x). \end{cases}$$

Hence, for u sufficiently close to $g(0)$,

$$\begin{aligned} U_{p,0}(u,x) \cap U_{p,p}(u,x) &= \bigcup_{\kappa=0}^{\infty} \Big(\Big(B_{\frac{g^{-1}(u)}{2M^{\kappa+1}}}(\zeta) \setminus B_{\frac{g^{-1}(u)}{2M^{\kappa+2}}}(\zeta)\Big) \cap B_{\frac{g_{\kappa,u}^{-1}(x)}{2M}}(\zeta)\Big) \cup \{\zeta\} \\ &= \Big(B_{\frac{G_u(x)}{2M}}(\zeta) \setminus B_{\frac{g^{-1}(u)}{2M^{\kappa(u,x)+2}}}(\zeta)\Big) \cup \bigcup_{\kappa=\kappa(u,x)+1}^{\infty} \Big(B_{\frac{g^{-1}(u)}{2M^{\kappa+1}}}(\zeta) \setminus B_{\frac{g^{-1}(u)}{2M^{\kappa+2}}}(\zeta)\Big) \cup \{\zeta\} \\ &= B_{\frac{G_u(x)}{2M}}(\zeta) \end{aligned}$$

and

$$R_{p,0}(u,x) = U_{p,0}(u,x) \setminus (U_{p,0}(u,x) \cap U_{p,p}(u,x)) = B_{\frac{1}{2}G_u(x)}(\zeta) \setminus B_{\frac{G_u(x)}{2M}}(\zeta).$$

Let $(u_n)_n$ be a normalizing sequence of levels satisfying (2.2) such that $\lim_{n\to\infty} u_n = g(0)$. Given the assumptions (3.6) and (R1), of regularity of $\mathbb{P}$ and $U(u_n) = \{X_0 > u_n\}$ being a ball centred at ζ, respectively, we have

$$\begin{aligned} \mathbb{P}(X_0 > u_n) &\sim \mathrm{Leb}(X_0 > u_n)\frac{d\mathbb{P}}{d\mathrm{Leb}}(\zeta) = g^{-1}(u_n)\frac{d\mathbb{P}}{d\mathrm{Leb}}(\zeta), \\ \mathbb{P}(R_{p,0}(u_n,x)) &= \mathbb{P}\big(B_{\frac{1}{2}G_{u_n}(x)}(\zeta)\big) - \mathbb{P}\big(B_{\frac{G_{u_n}(x)}{2M}}(\zeta)\big) \\ &\sim \mathrm{Leb}\big(B_{\frac{1}{2}G_{u_n}(x)}(\zeta)\big)\frac{d\mathbb{P}}{d\mathrm{Leb}}(\zeta) - \mathrm{Leb}\big(B_{\frac{G_{u_n}(x)}{2M}}(\zeta)\big)\frac{d\mathbb{P}}{d\mathrm{Leb}}(\zeta) \\ &= \left(G_{u_n}(x) - \frac{G_{u_n}(x)}{M}\right)\frac{d\mathbb{P}}{d\mathrm{Leb}}(\zeta) = \theta G_{u_n}(x)\frac{d\mathbb{P}}{d\mathrm{Leb}}(\zeta) \end{aligned}$$

where $\theta = 1 - 1/M$.

If there exists a strictly positive function $q : W \to \mathbb{R}$ and a strictly monotone homeomorphism h_k such that

$$\lim_{u\to g(0)} \frac{g_{\kappa,u}(g^{-1}(u)h_\kappa(x))}{q(u)} = x, \qquad \forall \kappa \in \mathbb{N}_0,$$

then, for $a_n = 1/q(u_n)$,

$$\begin{aligned} \lim_{n\to\infty} \frac{\mathbb{P}(R_{p,0}(u_n, x/a_n))}{\mathbb{P}(X_0 > u_n)} &= \lim_{n\to\infty} \frac{\theta G_{u_n}(q(u_n)x)}{g^{-1}(u_n)} = \theta \lim_{n\to\infty} \frac{g^{-1}_{\kappa(u_n,q(u_n)x),u_n}(q(u_n)x)}{g^{-1}(u_n)} \\ &= \theta \lim_{n\to\infty} \lim_{u\to g(0)} \frac{g^{-1}_{\kappa(u_n,q(u_n)x),u_n}\Big(q(u_n)\frac{g_{\kappa,u}(g^{-1}(u)h_\kappa(x))}{q(u)}\Big)}{g^{-1}(u_n)}, \qquad \forall \kappa \in \mathbb{N}_0. \end{aligned}$$

In particular, for $u = u_n$ and $\kappa = \kappa(u_n, q(u_n)x)$, we get

$$\lim_{n\to\infty} \frac{\mathbb{P}(R_{p,0}(u_n, x/a_n))}{\mathbb{P}(X_0 > u_n)}$$
$$= \theta \lim_{n\to\infty} \frac{g^{-1}_{\kappa(u_n,q(u_n)x),u_n}\left(q(u_n)\frac{g_{\kappa(u_n,q(u_n)x),u_n}(g^{-1}(u_n)h_{\kappa(u_n,q(u_n)x)}(x))}{q(u_n)}\right)}{g^{-1}(u_n)}$$
$$= \theta \lim_{n\to\infty} h_{\kappa(u_n,q(u_n)x)}(x)$$

and the probability distribution is given by

$$\pi(x) = 1 - \lim_{n\to\infty} h_{\kappa(u_n,q(u_n)x)}(x).$$

Now, we will analyse each type of behaviour separately.

Type 1: there exists some strictly positive function $q : W \to \mathbb{R}$ such that for all $y \in \mathbb{R}$,

$$\lim_{s\to g(0)} \frac{g^{-1}(s + yq(s))}{g^{-1}(s)} = \mathrm{e}^{-y}.$$

Then

$$\lim_{u\to g(0)} \frac{g^{-1}(u - \log(x)q(u))}{g^{-1}(u)} = x, \quad \text{so} \quad \lim_{u\to g(0)} \frac{g(g^{-1}(u)x) - u}{q(u)} = -\log(x),$$

$$\lim_{u\to g(0)} \frac{g_{\kappa,u}(g^{-1}(u)x)}{q(u)} = \lim_{u\to g(0)} \frac{\sum_{i=0}^{\kappa}(g(M^i g^{-1}(u)x) - u)}{q(u)} = -\sum_{i=0}^{\kappa} \log(xM^i)$$
$$= -\sum_{i=0}^{\kappa}(\log(x) + i\log M) = -(\kappa+1)\log(x) - \frac{\kappa(\kappa+1)}{2}\log M$$
$$= -(\kappa+1)(\log(x) + \kappa\log\sqrt{M}).$$

Let $h_\kappa(x) = \mathrm{e}^{-\frac{x}{\kappa+1} - \kappa\log\sqrt{M}}$, so that $h_\kappa^{-1}(x) = -(\kappa+1)(\log(x) + \kappa\log\sqrt{M})$. Then $a_n A_n$ converges in distribution to a compound Poisson process A with intensity $\theta = 1 - 1/M$ and multiplicity d.f. π given by

$$\pi(x) = 1 - \lim_{n\to\infty} h_{\kappa(u_n,q(u_n)x)}(x) = 1 - \lim_{n\to\infty} \mathrm{e}^{-\frac{x}{\kappa(u_n,q(u_n)x)+1} - \kappa(u_n,q(u_n)x)\log\sqrt{M}}.$$

If $g(x) = -\log(x)$, then

$$g_{\kappa,u}(x) = \sum_{i=0}^{\kappa}(-\log(M^i x) - u) = -(\kappa+1)(\log(x) + u + \kappa\log\sqrt{M}),$$
$$b_\kappa(u) = g_{\kappa,u}\left(\frac{\mathrm{e}^{-u}}{M^\kappa}\right) = \frac{\kappa(\kappa+1)}{2}\log M,$$
$$b_{\kappa+1}(u) = g_{\kappa,u}\left(\frac{\mathrm{e}^{-u}}{M^{\kappa+1}}\right) = \frac{(\kappa+1)(\kappa+2)}{2}\log M.$$

Let $\kappa = \kappa(u, x)$ be the only integer such that $x \in [b_\kappa(u), b_{\kappa+1}(u))$, or equivalently

$$\frac{\kappa(\kappa+1)}{2} \log M \le x < \frac{(\kappa+1)(\kappa+2)}{2} \log M.$$

Then $\kappa(u, x) = \lfloor \frac{\sqrt{1+8x/\log M}-1}{2} \rfloor$ and

$$\begin{aligned}\pi(x) &= 1 - \mathrm{e}^{-\frac{x}{\kappa(u,x)+1} - \kappa(u,x) \log \sqrt{M}} \\ &= 1 - (\sqrt{M})^{-\lfloor \frac{\sqrt{1+8x/\log M}-1}{2} \rfloor} \exp\left(-\frac{x}{\lfloor \frac{\sqrt{1+8x/\log M}-1}{2} \rfloor + 1}\right).\end{aligned}$$

Type 2: $g(0) = \infty$ and there exists $\alpha > 1$ such that for all $y > 0$,

$$\lim_{s\to\infty} \frac{g^{-1}(sy)}{g^{-1}(s)} = y^{-\alpha}.$$

Then for $q(u) = u$ we have

$$\lim_{u\to\infty} \frac{g^{-1}(ux^{-1/\alpha})}{g^{-1}(u)} = x, \quad \text{so} \quad \lim_{u\to\infty} \frac{g(g^{-1}(u)x) - u}{q(u)} = x^{-1/\alpha} - 1,$$

$$\begin{aligned}\lim_{u\to\infty} \frac{g_{\kappa,u}(g^{-1}(u)x)}{q(u)} &= \lim_{u\to\infty} \frac{\sum_{i=0}^{\kappa}(g(M^i g^{-1}(u)x) - u)}{q(u)} = \sum_{i=0}^{\kappa}((M^{-1/\alpha})^i x^{-1/\alpha} - 1) \\ &= \frac{1 - M^{-(\kappa+1)/\alpha}}{1 - M^{-1/\alpha}} x^{-1/\alpha} - (\kappa+1).\end{aligned}$$

Let $h_\kappa(x) = \left(\frac{1-M^{-1/\alpha}}{1-M^{-(\kappa+1)/\alpha}}\right)^{-\alpha}(\kappa+1+x)^{-\alpha}$, so that $h_\kappa^{-1}(x) = \frac{1-M^{-(\kappa+1)/\alpha}}{1-M^{-1/\alpha}} x^{-1/\alpha} - (\kappa+1)$. Then $a_n A_n$ converges in distribution to a compound Poisson process A with intensity $\theta = 1 - 1/M$ and multiplicity d.f. π given by

$$\begin{aligned}\pi(x) &= 1 - \lim_{n\to\infty} h_{\kappa(u_n, q(u_n)x)}(x) \\ &= 1 - \lim_{n\to\infty}\left(\frac{1 - M^{-1/\alpha}}{1 - M^{-(\kappa(u_n,u_n x)+1)/\alpha}}\right)^{-\alpha} (\kappa(u_n, u_n x) + 1 + x)^{-\alpha}.\end{aligned}$$

If $g(x) = x^{-1/\alpha}$ for some $\alpha > 0$, then

$$\begin{aligned}g_{\kappa,u}(x) &= \sum_{i=0}^{\kappa}((M^i x)^{-1/\alpha} - u) = \frac{1 - M^{-(\kappa+1)/\alpha}}{1 - M^{-1/\alpha}} x^{-1/\alpha} - (\kappa+1)u, \\ b_\kappa(u) &= g_{\kappa,u}\left(\frac{u^{-\alpha}}{M^\kappa}\right) = \left(\frac{M^{\kappa/\alpha} - M^{-1/\alpha}}{1 - M^{-1/\alpha}} - (\kappa+1)\right)u, \\ b_{\kappa+1}(u) &= g_{\kappa,u}\left(\frac{u^{-\alpha}}{M^{\kappa+1}}\right) = \left(\frac{M^{(\kappa+1)/\alpha} - 1}{1 - M^{-1/\alpha}} - (\kappa+1)\right)u.\end{aligned}$$

Let $\kappa = \kappa(u, ux)$ be the only integer such that $ux \in [b_\kappa(u), b_{\kappa+1}(u))$, or equivalently

$$\frac{M^{\kappa/\alpha} - M^{-1/\alpha}}{1 - M^{-1/\alpha}} \le \kappa + 1 + x < \frac{M^{(\kappa+1)/\alpha} - 1}{1 - M^{-1/\alpha}}.$$

Notice that $\kappa(u, ux)$ does not depend on u; hence,

$$\pi(x) = 1 - \left(\frac{1 - M^{-1/\alpha}}{1 - M^{-(\kappa(x)+1)/\alpha}}\right)^{-\alpha} (\kappa(x) + 1 + x)^{-\alpha}$$

where $\kappa(x) = \kappa(u, ux)$.

Type 3: $g(0) = D < \infty$ and there exists $\alpha > 0$ such that for all $y > 0$,

$$\lim_{s\to 0} \frac{g^{-1}(D - sy)}{g^{-1}(D - s)} = y^\alpha.$$

Then for $q(u) = D - u$ we have

$$\lim_{u\to D} \frac{g^{-1}(D - (D - u)x^{1/\alpha})}{g^{-1}(u)} = x, \quad \text{so} \quad \lim_{u\to D} \frac{g(g^{-1}(u)x) - u}{q(u)} = 1 - x^{1/\alpha},$$

$$\lim_{u\to D} \frac{g_{\kappa,u}(g^{-1}(u)x)}{q(u)} = \lim_{u\to D} \frac{\sum_{i=0}^{\kappa}(g(M^i g^{-1}(u)x) - u)}{q(u)} = \sum_{i=0}^{\kappa}(1 - (M^{1/\alpha})^i x^{1/\alpha})$$

$$= \kappa + 1 - \frac{1 - M^{(\kappa+1)/\alpha}}{1 - M^{1/\alpha}} x^{1/\alpha}.$$

Let $h_\kappa(x) = \left(\frac{1-M^{1/\alpha}}{1-M^{(\kappa+1)/\alpha}}\right)^\alpha (\kappa + 1 - x)^\alpha$, so that $h_\kappa^{-1}(x) = \kappa + 1 - \frac{1-M^{(\kappa+1)/\alpha}}{1-M^{1/\alpha}} x^{1/\alpha}$. Then $a_n A_n$ converges in distribution to a compound Poisson process A with intensity $\theta = 1 - 1/M$ and multiplicity d.f. π given by

$$\pi(x) = 1 - \lim_{n\to\infty} h_{\kappa(u_n, q(u_n)x)}(x)$$

$$= 1 - \lim_{n\to\infty} \left(\frac{1 - M^{1/\alpha}}{1 - M^{(\kappa(u_n,(D-u_n)x)+1)/\alpha}}\right)^\alpha \big(\kappa(u_n, (D - u_n)x) + 1 - x\big)^\alpha.$$

If $g(x) = D - x^{1/\alpha}$ for some $D \in \mathbb{R}$ and $\alpha > 0$, then

$$g_{\kappa,u}(x) = \sum_{i=0}^{\kappa}(D - (M^i x)^{1/\alpha} - u) = (\kappa + 1)(D - u) - \frac{1 - M^{(\kappa+1)/\alpha}}{1 - M^{1/\alpha}} x^{1/\alpha},$$

$$b_\kappa(u) = g_{\kappa,u}\left(\frac{(D - u)^\alpha}{M^\kappa}\right) = \left(\kappa + 1 - \frac{M^{1/\alpha} - M^{-\kappa/\alpha}}{M^{1/\alpha} - 1}\right)(D - u),$$

$$b_{\kappa+1}(u) = g_{\kappa,u}\left(\frac{(D - u)^\alpha}{M^{\kappa+1}}\right) = \left(\kappa + 1 - \frac{1 - M^{-(\kappa+1)/\alpha}}{M^{1/\alpha} - 1}\right)(D - u).$$

Let $\kappa = \kappa(u, (D - u)x)$ be the only integer such that $(D - u)x \in [b_\kappa(u), b_{\kappa+1}(u))$, or equivalently

$$\frac{1 - M^{-(\kappa+1)/\alpha}}{M^{1/\alpha} - 1} < \kappa + 1 - x \le \frac{M^{1/\alpha} - M^{-\kappa/\alpha}}{M^{1/\alpha} - 1}.$$

Notice that $\kappa(u, (D - u)x)$ does not depend on u; hence,

$$\pi(x) = 1 - \left(\frac{1 - M^{1/\alpha}}{1 - M^{(\kappa(x)+1)/\alpha}}\right)^\alpha (\kappa(x) + 1 - x)^\alpha$$

where $\kappa(x) = \kappa(u, (D - u)x)$. □

4. Convergence of marked rare events point processes

This section is dedicated to the proof of Theroem 2.A. The argument follows the same thread as in [12, proof of Theorem 1] but it is much more involved due to the sophistication associated to MREPP and the degree of generalisation and cases addressed (like allowing multiple maxima and the absence of clustering). One of the highlights of the proof below is the way we handle the gap created by considering general distributions for the marking of clusters, when compared to the distributions defined on the integers in [12], which significantly simplified the proof of [12, Theorem 1]. The major step to overcome this difficulty is made with Proposition 4.B, which is of independent interest since it provides a formula to compute the Laplace transform of multiple random variables with general distributions, possibly diffuse with respect to Lebesgue measure.

We start with a lemma which says that the probability of not entering $U_{p,0}(u,x)$ can be approximated by the probability of not entering $R_{p,0}(u,x)$ during the same period of time.

Lemma 4.1. *For any $p \in \mathbb{N}_0$, $s \in \mathbb{N}$, $x \geq 0$ and $u > 0$ we have*

$$|\mathbb{P}(\mathscr{I}_{p,0,s}(u,x)) - \mathbb{P}(\mathcal{R}_{p,0,s}(u,x))| \leq p\mathbb{P}(U_{p,0}(u,x)).$$

Proof. For $p = 0$ this is trivial since $U_{0,i}(u,x) = R_{0,i}(u,x)$. For $p > 0$, first observe that since $R_{p,i}(u,x) \subset U_{p,i}(u,x)$, we have $\mathscr{I}_{p,0,s}(u,x) \subset \mathcal{R}_{p,0,s}(u,x)$. Next, observe that if $\mathcal{R}_{p,0,s}(u,x) \setminus \mathscr{I}_{p,0,s}(u,x)$ occurs, then we may choose $j \in \{0,1,\ldots,s-1\}$ such that $X_j \in U_{p,0}(u,x)$. But since $\mathcal{R}_{p,0,s}(u,x)$ does occur, we must have $X_{j+j_1} \in U_{p,0}(u,x)$ for some $1 \leq j_1 \leq p$, otherwise $R_{p,j}(u,x)$ would occur. Similarly, if $j + j_1 < s$ then $X_{j+j_1+j_2} \in U_{p,0}(u,x)$ for some $1 \leq j_2 \leq p$ and so on. We conclude that $X_i \in U_{p,0}(u,x)$ for some $i \in \{s-p,\ldots,s-1\}$, and this means that

$$\mathcal{R}_{p,0,s}(u,x) \setminus \mathscr{I}_{p,0,s}(u,x) \subset \bigcup_{i=s-p}^{s-1} U_{p,i}(u,x).$$

Hence, by stationarity,

$$\begin{aligned}|\mathbb{P}(\mathscr{I}_{p,0,s}(u,x)) - \mathbb{P}(\mathcal{R}_{p,0,s}(u,x))| &= \mathbb{P}(\mathcal{R}_{p,0,s}(u,x) \setminus \mathscr{I}_{p,0,s}(u,x)) \\ &\leq p\mathbb{P}(U_{p,0}(u,x)).\end{aligned}$$

□

Next we give an approximation for the probability of not entering $R_{p,0}(u,x)$ during a certain period of time.

Lemma 4.2. *For any $p \in \mathbb{N}_0$, $s \in \mathbb{N}$, $x \geq 0$ and $u > 0$ we have*

$$|\mathbb{P}(\mathcal{R}_{p,0,s}(u,x)) - (1 - s\mathbb{P}(R_{p,0}(u,x)))| \leq s \sum_{j=p+1}^{s-1} \mathbb{P}(Q^0_{p,0}(u) \cap U_{p,j}(u,x)).$$

Proof. Since $(\mathcal{R}_{p,0,s}(u,x))^c = \bigcup_{i=0}^{s-1} R_{p,i}(u,x)$, it is clear that

$$|1 - \mathbb{P}(\mathcal{R}_{p,0,s}(u,x)) - s\mathbb{P}(R_{p,0}(u,x))| \leq \sum_{i=0}^{s-1} \sum_{j=i+p+1}^{s-1} \mathbb{P}(R_{p,i}(u,x) \cap R_{p,j}(u,x)).$$

If $p > 0$, the result now follows by stationarity plus the following two facts: $R_{p,j}(u, x) \subset U_{p,j}(u, x)$, and between two entrances to $R_{p,0}(u, x)$, at times i and j, there must have existed an escape, i.e., an entrance in $Q^0_{p,0}(u)$ (otherwise, an entrance to $R_{p,0}(u, x)$ and therefore to $U_{p,0}(u, x)$ at time j would imply an entrance to $U_{p,0}(u, x)$ at some earlier time i_1 for $i + 1 \leq i_1 \leq i + p$, which would contradict the entrance to $R_{p,0}(u, x)$ at time i).

If $p = 0$, the result follows by stationarity plus the following two facts: $R_{0,j}(u, x) = U_{0,j}(u, x)$ and $R_{0,i}(u, x) \subset \{X_i > u\} = Q^0_{0,i}(u)$. □

The next lemma gives an error bound for the approximation of the probability of the process $\mathscr{A}_u([0, s))$ not exceeding x by the probability of not entering $R_{p,0}(u, x)$ during the period $[0, s)$. In what follows, we use the notation

$$\mathscr{A}^b_{u,a} := \mathscr{A}_u([a, b)), \qquad \mathscr{A}_u \text{ as in (2.8).} \tag{4.1}$$

Lemma 4.3. *For any $s \in \mathbb{N}$, $x \geq 0$ and $u > 0$ we have*

$$\begin{aligned}|\mathbb{P}(\mathscr{A}^s_{u,0} \leq x) - \mathbb{P}(\mathscr{I}_{p,0,s}(u, x))| &\leq (s - p) \sum_{j=p+1}^{s-1} \mathbb{P}(Q^0_{p,0}(u) \cap \{X_j > u\}) \\ &\quad + \sum_{\kappa=1}^{\lfloor s/p \rfloor} \kappa p \mathbb{P}(U^\kappa_{p,0}(u, x)) + s \sum_{\kappa=\lfloor s/p\rfloor+1}^{\infty} \mathbb{P}(U^\kappa_{p,0}(u, x))\end{aligned}$$

if $p > 0$, and in case $p = 0$ we have

$$\begin{aligned}|\mathbb{P}(\mathscr{A}^s_{u,0} \leq x) - \mathbb{P}(\mathscr{I}_{0,0,s}(u, x))| &\leq s \sum_{j=1}^{s-1} \mathbb{P}(X_0 > u, X_j > u\}) \\ &= s \sum_{j=1}^{s-1} \mathbb{P}(Q^0_{0,0}(u) \cap \{X_j > u\}).\end{aligned}$$

Proof. If $p > 0$, we start by observing that

$$\begin{aligned}A_{0,s}(u, x) &:= \{\mathscr{A}^s_{u,0} \leq x\} \cap (\mathscr{I}_{p,0,s}(u, x))^c \\ &\subset \bigcup_{i=s-p}^{s-1} U^1_{p,i}(u, x) \cup \bigcup_{i=s-2p}^{s-1} U^2_{p,i}(u, x) \\ &\quad \cup \cdots \cup \bigcup_{i=s-\lfloor s/p\rfloor p}^{s-1} U^{\lfloor s/p \rfloor}_{p,i}(u, x) \cup \bigcup_{i=0}^{s-1} \bigcup_{\kappa > \lfloor s/p \rfloor} U^\kappa_{p,i}(u, x)\end{aligned}$$

since $\bigcup_{i=0}^{s-\kappa p-1} U^\kappa_{p,i}(u, x) \subset \{\mathscr{A}^s_{u,0} > x\}$ for any $\kappa \leq \lfloor s/p \rfloor$. So, by stationarity,

$$\mathbb{P}(A_{0,s}(u, x)) \leq \sum_{\kappa=1}^{\lfloor s/p \rfloor} \kappa p \mathbb{P}(U^\kappa_{p,0}(u, x)) + s \sum_{\kappa=\lfloor s/p\rfloor+1}^{\infty} \mathbb{P}(U^\kappa_{p,0}(u, x)).$$

Now, we note that

$$B_{0,s}(u, x) := \{\mathscr{A}^s_{u,0} > x\} \cap \mathscr{I}_{p,0,s}(u, x) \subset \bigcup_{i=0}^{s-p-1} \bigcup_{j>i+p}^{s-1} Q^0_{p,i}(u) \cap \{X_j > u\}.$$

This is because no entrance in $U_{p,0}(u,x)$ during the time period $0,\dots,s-1$ implies that there must be at least two distinct clusters during that time. Since each cluster ends with an escape, i.e., an entrance in $Q^0_{p,0}(u)$, this must have happened at some time $i \in \{0,\dots,s-p-1\}$ which was then followed by another exceedance at some subsequent instant $j > i$ where a new cluster has begun. Consequently, by stationarity,

$$\mathbb{P}(B_{0,s}(u,x)) \le (s-p)\sum_{j=p+1}^{s-1}\mathbb{P}(Q^0_{p,0}(u)\cap\{X_j>u\}).$$

If $p=0$, we start by observing that $\{\mathscr{A}^s_{u,0}\le x\}\subset\mathscr{I}_{0,0,s}(u,x)$. Then, we note that

$$\{\mathscr{A}^s_{u,0}>x\}\cap\mathscr{I}_{0,0,s}(u,x)\subset\bigcup_{i=0}^{s-1}\bigcup_{j=i+1}^{s-1}\{X_i>u\}\cap\{X_j>u\}$$

because no entrance in $U_{0,0}(u,x)$ during the time period $0,\dots,s-1$ implies that there must be at least two exceedances during that time. Consequently, by stationarity,

$$\begin{aligned}|\mathbb{P}(\mathscr{A}^s_{u,0}\le x)-\mathbb{P}(\mathscr{I}_{0,0,s}(u,x))| &= \mathbb{P}(\{\mathscr{A}^s_{u,0}>x\}\cap\mathscr{I}_{0,0,s}(u,x))\\ &\le s\sum_{j=1}^{s-1}\mathbb{P}(X_0>u,X_j>u).\end{aligned}$$

□

As a consequence we obtain an approximation for the Laplace transform of $\mathscr{A}^s_{u,o}$.

Corollary 4.A. *For any $p\in\mathbb{N}_0$, $s\in\mathbb{N}$, $y\ge0$, $a>0$ and $u>0$ sufficiently close to $u_F=\sup\varphi$ we have*

$$\begin{aligned}&\left|\mathbb{E}\left(\mathrm{e}^{-ya\mathscr{A}^s_{u,0}}\right)-\left(1-s\int_0^\infty y\mathrm{e}^{-yx}\mathbb{P}(R_{p,0}(u,x/a))\,dx\right)\right|\\ &\qquad\le 2s\sum_{j=p+1}^{s-1}\mathbb{P}(Q^0_{p,0}(u)\cap\{X_j>u\})+\int_0^\infty y\mathrm{e}^{-yx}\delta_{p,s,u}(x/a)\,dx,\end{aligned}$$

where $\delta_{p,s,u}(x/a)$ is as in (2.15).

Proof. Using Lemmas 4.1–4.3, for every $x>0$ and $p>0$ we have

$$\begin{aligned}&|\mathbb{P}(\mathscr{A}^s_{u,0}\le x)-(1-s\mathbb{P}(R_{p,0}(u,x)))| \le |\mathbb{P}(\mathscr{A}^s_{u,0}\le x)-\mathbb{P}(\mathscr{I}_{p,0,s}(u,x))|\\ &\quad+|\mathbb{P}(\mathscr{I}_{p,0,s}(u,x))-\mathbb{P}(\mathcal{R}_{p,0,s}(u,x))|+|\mathbb{P}(\mathcal{R}_{p,0,s}(u,x))-(1-s\mathbb{P}(R_{p,0}(u,x)))|\\ &\le (s-p)\sum_{j=p+1}^{s-1}\mathbb{P}(Q^0_{p,0}(u)\cap\{X_j>u\})+\sum_{\kappa=1}^{\lfloor s/p\rfloor}\kappa p\mathbb{P}(U^\kappa_{p,0}(u,x))\\ &\quad+s\sum_{\kappa=\lfloor s/p\rfloor+1}^{\infty}\mathbb{P}(U^\kappa_{p,0}(u,x))+p\mathbb{P}(U_{p,0}(u,x))+s\sum_{j=p+1}^{s-1}\mathbb{P}(Q^0_{p,0}(u)\cap U_{p,j}(u,x))\\ &\le 2s\sum_{j=p+1}^{s-1}\mathbb{P}(Q^0_{p,0}(u)\cap\{X_j>u\})+\delta_{p,s,u}(x).\end{aligned}$$

When $p = 0$, we have

$$
\begin{aligned}
|\mathbb{P}(\mathscr{A}_{u,0}^s \le x) - (1 - s\mathbb{P}(R_{0,0}(u,x)))| &\le |\mathbb{P}(\mathscr{A}_{u,0}^s \le x) - \mathbb{P}(\mathscr{I}_{0,0,s}(u,x))| \\
&\quad + |\mathbb{P}(\mathscr{I}_{0,0,s}(u,x)) - \mathbb{P}(\mathcal{R}_{0,0,s}(u,x))| + |\mathbb{P}(\mathcal{R}_{0,0,s}(u,x)) - (1 - s\mathbb{P}(R_{0,0}(u,x)))| \\
&\le s\sum_{j=1}^{s-1}\mathbb{P}(Q_{0,0}^0(u) \cap \{X_j > u\}) + s\sum_{j=1}^{s-1}\mathbb{P}(Q_{0,0}^0(u) \cap U_{0,j}(u,x)) \\
&\le 2s\sum_{j=1}^{s-1}\mathbb{P}(Q_{0,0}^0(u) \cap \{X_j > u\}) + \delta_{0,s,u}(x).
\end{aligned}
$$

Since $\mathbb{P}(\mathscr{A}_{u,0}^s < 0) = 0$, using integration by parts we have

$$
\begin{aligned}
\mathbb{E}(\mathrm{e}^{-ya\mathscr{A}_{u,0}^s}) &= \mathrm{e}^{-y.0}\mathbb{P}(\mathscr{A}_{u,0}^s = 0) + \int_0^\infty \mathrm{e}^{-yx}\, d\mathbb{P}(\mathscr{A}_{u,0}^s \le x/a) \\
&= \mathbb{P}(\mathscr{A}_{u,0}^s = 0) + \lim_{x\to\infty}[\mathrm{e}^{-yx}\mathbb{P}(\mathscr{A}_{u,0}^s \le x/a) - \mathrm{e}^{-y.0}\mathbb{P}(\mathscr{A}_{u,0}^s \le 0)] \\
&\quad - \int_0^\infty \mathbb{P}(\mathscr{A}_{u,0}^s \le x/a)\, d\mathrm{e}^{-yx} \\
&= \mathbb{P}(\mathscr{A}_{u,0}^s = 0) - \mathbb{P}(\mathscr{A}_{u,0}^s \le 0) - \int_0^\infty (-y\mathrm{e}^{-yx})\mathbb{P}(\mathscr{A}_{u,0}^s \le x/a)\, dx \\
&= \int_0^\infty y\mathrm{e}^{-yx}\mathbb{P}(\mathscr{A}_{u,0}^s \le x/a)\, dx.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\left|\mathbb{E}\big(\mathrm{e}^{-ya\mathscr{A}_{u,0}^s}\big) - \left(1 - s\int_0^\infty y\mathrm{e}^{-yx}\mathbb{P}(R_{p,0}(u,x/a))\,dx\right)\right| \\
&\quad = \left|\int_0^\infty y\mathrm{e}^{-yx}\mathbb{P}(\mathscr{A}_{u,0}^s \le x/a)\,dx - \int_0^\infty y\mathrm{e}^{-yx}(1 - s\mathbb{P}(R_{p,0}(u,x/a)))\,dx\right| \\
&\quad \le \int_0^\infty y\mathrm{e}^{-yx}\left[2s\sum_{j=p+1}^{s-1}\mathbb{P}(Q_{p,0}^0(u) \cap \{X_j > u\}) + \delta_{p,s,u}(x/a)\right]dx \\
&\quad = 2s\sum_{j=p+1}^{s-1}\mathbb{P}(Q_{p,0}^0(u) \cap \{X_j > u\}) + \int_0^\infty y\mathrm{e}^{-yx}\delta_{p,s,u}(x/a)\,dx. \qquad \square
\end{aligned}
$$

The next result gives the main induction tool for the proof of Theorem 2.A.

Lemma 4.4. *Let $p \in \mathbb{N}_0$, $s, t, \varsigma \in \mathbb{N}$ and consider $x_1 \in \mathbb{R}_0^+$ and $\underline{x} = (x_2, \ldots, x_\varsigma) \in (\mathbb{R}_0^+)^{\varsigma-1}$, $s + t - 1 < a_2 < b_2 < a_3 < \cdots < b_\varsigma \in \mathbb{N}_0$. For $u > 0$ sufficiently close to $u_F = \varphi(\zeta)$ we have*

$$
\begin{aligned}
|\mathbb{P}(\mathscr{A}_{u,0}^s \le x_1, \mathscr{A}_{u,a_2}^{b_2} \le x_2, \ldots, \mathscr{A}_{u,a_\varsigma}^{b_\varsigma} \le x_\varsigma) \\
- \mathbb{P}(\mathscr{A}_{u,0}^s \le x_1)\mathbb{P}(\mathscr{A}_{u,a_2}^{b_2} \le x_2, \ldots, \mathscr{A}_{u,a_\varsigma}^{b_\varsigma} \le x_\varsigma)| \\
\le s\iota(u,t) + 4s\sum_{j=p+1}^{s-1}\mathbb{P}(Q_{p,0}^0(u) \cap \{X_j > u\}) + 2\delta_{p,s,u}(x_1)
\end{aligned}
$$

where $\delta_{p,s,u}$ *is as in* (2.15) *and*

$$\iota(u,t)=\sup_{s\in\mathbb{N}}\max_{i=0,\dots,s-1}\Big\{\Big|\mathbb{P}(R_{p,i}(u,x_1))\mathbb{P}\Big(\bigcap_{j=2}^{\varsigma}\{\mathscr{A}_{u,a_j}^{b_j}\le x_j\}\Big) - \mathbb{P}\Big(\bigcap_{j=2}^{\varsigma}\{\mathscr{A}_{u,a_j}^{b_j}\le x_j\}\cap R_{p,i}(u,x_1)\Big)\Big|\Big\}. \tag{4.2}$$

Proof. Let

$$A_{x_1,\underline{x}}:=\{\mathscr{A}_{u,0}^{s}\le x_1,\mathscr{A}_{u,a_2}^{b_2}\le x_2,\dots,\mathscr{A}_{u,a_\varsigma}^{b_\varsigma}\le x_\varsigma\},\qquad B_{x_1}:=\{\mathscr{A}_{u,0}^{s}\le x_1\},$$
$$\tilde{A}_{x_1,\underline{x}}:=\mathcal{R}_{p,0,s}(u,x_1)\cap\{\mathscr{A}_{u,a_2}^{b_2}\le x_2,\dots,\mathscr{A}_{u,a_\varsigma}^{b_\varsigma}\le x_\varsigma\},\quad \tilde{B}_{x_1}:=\mathcal{R}_{p,0,s}(u,x_1),$$
$$D^{\underline{x}}:=\{\mathscr{A}_{u,a_2}^{b_2}\le x_2,\dots,\mathscr{A}_{u,a_\varsigma}^{b_\varsigma}\le x_\varsigma\}.$$

If $x_1>0$, by Lemmas 4.1 and 4.3 we have

$$\begin{aligned}
&|\mathbb{P}(B_{x_1})-\mathbb{P}(\tilde{B}_{x_1})|\\
&\quad\le|\mathbb{P}(\mathscr{A}_{u,0}^{s}\le x_1)-\mathbb{P}(\mathscr{I}_{p,0,s}(u,x_1))|+|\mathbb{P}(\mathscr{I}_{p,0,s}(u,x_1))-\mathbb{P}(\mathcal{R}_{p,0,s}(u,x_1))|\\
&\quad\le|\mathbb{P}(\{\mathscr{A}_{u,0}^{s}\le x_1\}\bigtriangleup\mathscr{I}_{p,0,s}(u,x_1))|+|\mathbb{P}(\mathcal{R}_{p,0,s}(u,x_1)\setminus\mathscr{I}_{p,0,s}(u,x_1))|\\
&\quad\le(s-p)\sum_{j=p+1}^{s-1}\mathbb{P}(Q_{p,0}^{0}(u)\cap\{X_j>u\})+\sum_{\kappa=1}^{\lfloor s/p\rfloor}\kappa p\mathbb{P}(U_{p,0}^{\kappa}(u,x_1))\\
&\qquad+s\sum_{\kappa=\lfloor s/p\rfloor+1}^{\infty}\mathbb{P}(U_{p,0}^{\kappa}(u,x_1))+p\mathbb{P}(U_{p,0}(u,x_1))\\
&\quad\le s\sum_{j=p+1}^{s-1}\mathbb{P}(Q_{p,0}^{0}(u)\cap\{X_j>u\})+\delta_{p,s,u}(x_1)
\end{aligned}\tag{4.3}$$

and also

$$\begin{aligned}
&|\mathbb{P}(A_{x_1})-\mathbb{P}(\tilde{A}_{x_1})|\\
&\le|\mathbb{P}(\{\mathscr{A}_{u,0}^{s}\le x_1\}\cap D^{\underline{x}})-\mathbb{P}(\mathscr{I}_{p,0,s}(u,x_1)\cap D^{\underline{x}})|\\
&\qquad+|\mathbb{P}((\mathcal{R}_{p,0,s}(u,x_1)\setminus\mathscr{I}_{p,0,s}(u,x_1))\cap D^{\underline{x}}))|\\
&\le|\mathbb{P}((\{\mathscr{A}_{u,0}^{s}\le x_1\}\bigtriangleup\mathscr{I}_{p,0,s}(u,x_1))\cap D^{\underline{x}})|+|\mathbb{P}((\mathcal{R}_{p,0,s}(u,x_1)\setminus\mathscr{I}_{p,0,s}(u,x_1))\cap D^{\underline{x}}))|\\
&\le|\mathbb{P}(\{\mathscr{A}_{u,0}^{s}\le x_1\}\bigtriangleup\mathscr{I}_{p,0,s}(u,x_1))|+|\mathbb{P}(\mathcal{R}_{p,0,s}(u,x_1)\setminus\mathscr{I}_{p,0,s}(u,x_1))|\\
&\le s\sum_{j=p+1}^{s-1}\mathbb{P}(Q_{p,0}^{0}(u)\cap\{X_j>u\})+\delta_{p,s,u}(x_1).
\end{aligned}\tag{4.4}$$

If $x_1=0$, then $\{\mathscr{A}_{u,0}^{s}\le x_1\}=\{\mathscr{A}_{u,0}^{s}=0\}=\{X_0\le u,\dots,X_{s-1}\le u\}=\mathscr{I}_{p,0,s}(u,0)$, so estimates (4.3) and (4.4) are still valid by Lemma 4.1.

Using stationarity and adapting the proof of Lemma 4.2, we find that

$$|\mathbb{P}(\tilde{A}_{x_1,\underline{x}})-(1-s\mathbb{P}(R_{p,0}(u,x_1)))\mathbb{P}(D^{\underline{x}_1})|\le\mathrm{Err},$$

where

$$\mathrm{Err} = \left| s\mathbb{P}(R_{p,0}(u,x_1))\mathbb{P}(D^{\underline{x}}) - \sum_{i=0}^{s-1} \mathbb{P}(R_{p,i}(u,x_1) \cap D^{\underline{x}}) \right| + s \sum_{j=p+1}^{s-1} \mathbb{P}(Q^0_{p,0}(u) \cap U_{p,j}(u,x_1)).$$

Now, since, by definition of $\iota(u,t)$,

$$\left| s\mathbb{P}(R_{p,0}(u,x_1))\mathbb{P}(D^{\underline{x}}) - \sum_{i=0}^{s-1} \mathbb{P}(R_{p,i}(u,x_1) \cap D^{\underline{x}}) \right| \leq \sum_{i=0}^{s-1} |\mathbb{P}(R_{p,i}(u,x_1))\mathbb{P}(D^{\underline{x}}) - \mathbb{P}(R_{p,i}(u,x_1) \cap D^{\underline{x}})| \leq s\iota(u,t),$$

we conclude that

$$|\mathbb{P}(\tilde{A}_{x_1,\underline{x}}) - (1 - s\mathbb{P}(R_{p,0}(u,x_1)))\mathbb{P}(D^{\underline{x}})| \leq s\iota(u,t) + s \sum_{j=p+1}^{s-1} \mathbb{P}(Q^0_{p,0}(u) \cap U_{p,j}(u,x_1)). \tag{4.5}$$

Also, by Lemma 4.2 we have

$$|\mathbb{P}(\tilde{B}_{x_1})\mathbb{P}(D^{\underline{x}}) - (1 - s\mathbb{P}(R_{p,0}(u,x_1)))\mathbb{P}(D^{\underline{x}})| \leq s \sum_{j=p+1}^{s-1} \mathbb{P}(Q^0_{p,0}(u) \cap U_{p,j}(u,x_1)). \tag{4.6}$$

Combining (4.3)–(4.6) we get

$$\begin{aligned} &|\mathbb{P}(A_{x_1,\underline{x}}) - \mathbb{P}(B_{x_1})\mathbb{P}(D^{\underline{x}})| \\ &\quad \leq |\mathbb{P}(A_{x_1,\underline{x}}) - \mathbb{P}(\tilde{A}_{x_1,\underline{x}})| + |\mathbb{P}(\tilde{A}_{x_1,\underline{x}}) - (1 - s\mathbb{P}(R_{p,0}(u,x_1)))\mathbb{P}(D^{\underline{x}})| \\ &\qquad + |\mathbb{P}(\tilde{B}_{x_1})\mathbb{P}(D^{\underline{x}}) - (1 - s\mathbb{P}(R_{p,0}(u,x_1)))\mathbb{P}(D^{\underline{x}})| + |\mathbb{P}(B_{x_1}) - \mathbb{P}(\tilde{B}_{x_1})|\mathbb{P}(D^{\underline{x}}) \\ &\quad \leq s\iota(u,t) + 4s \sum_{j=p+1}^{s-1} \mathbb{P}(Q^0_{p,0}(u) \cap \{X_j > u\}) + 2\delta_{p,s,u}(x_1). \qquad \square \end{aligned}$$

Let $F : (\mathbb{R}^+_0)^n \to \mathbb{R}$ be right continuous in each variable separately and such that for each $R = (a_1, b_1] \times \cdots \times (a_n, b_n] \subset (\mathbb{R}^+_0)^n$ we have

$$\mu_F(R) := \sum_{c_i \in \{a_i, b_i\}} (-1)^{\#\{i \in \{1,\ldots,n\}:\, c_i = a_i\}} F(c_1, \ldots, c_n) \geq 0.$$

Such an F is called an *n-dimensional Stieltjes measure function*; then μ_F has a unique extension to the Borel σ-algebra in $(\mathbb{R}^+_0)^n$, which is called the *Lebesgue–Stieltjes measure* associated to F.

For each $I \subset \{1, \ldots, n\}$, let $F_I(\underline{x}) := F(\delta_1 x_1, \ldots, \delta_n x_n)$, where

$$\delta_i = \begin{cases} 1 & \text{if } i \in I, \\ 0 & \text{if } i \notin I. \end{cases}$$

If F is an n-dimensional Stieltjes measure function, it is easy to see that F_I is also an n-dimensional Stieltjes measure function, which has an associated Lebesgue–Stieltjes measure μ_{F_I}. We have the following proposition:

Proposition 4.B. *Given $n \in \mathbb{N}$, $I \subset \{1, \dots, n\}$ and two functions $F, G : (\mathbb{R}_0^+)^n \to \mathbb{R}$ such that F is a bounded n-dimensional Stieltjes measure function, let*

$$\int G(\underline{x})\, dF_I(\underline{x}) := \begin{cases} G(0,\dots,0)F(0,\dots,0) & \text{for } I = \emptyset, \\ \int G(\underline{x})\, d\mu_{F_I} & \text{for } I \neq \emptyset, \end{cases}$$

where μ_{F_I} is the Lebesgue–Stieltjes measure associated to F_I. Then

$$\int_0^\infty \!\!\cdots\! \int_0^\infty e^{-y_1x_1-\cdots-y_nx_n} F(\underline{x})\, dx_1 \dots dx_n = \frac{1}{y_1 \cdots y_n} \sum_{I \subset \{1,\dots,n\}} \int e^{-\sum_{i\in I} y_i x_i}\, dF_I(\underline{x}).$$

Proof. We use induction on n. For $n = 1$, using integration by parts,

$$\begin{aligned} \int_0^\infty e^{-y_1x_1} F(x_1)\, dx_1 &= \lim_{a\to\infty} \left[-\frac{e^{-y_1.a}}{y_1} F(a) + \frac{e^{-y_1.0}}{y_1} F(0) + \frac{1}{y_1} \int_0^a e^{-y_1x_1}\, dF(x_1) \right] \\ &= \frac{1}{y_1} \left(F(0) + \int_0^\infty e^{-y_1x_1}\, dF(x_1) \right) = \frac{1}{y_1} \sum_{I\subset\{1\}} \int e^{-\sum_{i\in I} y_ix_i}\, dF_I(x_1). \end{aligned}$$

For $n > 1$, using integration by parts again,

$$\begin{aligned} &\int_0^\infty \!\!\cdots\! \int_0^\infty e^{-y_1x_1-\cdots-y_nx_n} F(\underline{x})\, dx_1 \dots dx_n \\ &\quad = \lim_{a\to\infty} \int_0^a e^{-y_nx_n} \left(\int_0^\infty \!\!\cdots\! \int_0^\infty e^{-y_1x_1-\cdots-y_{n-1}x_{n-1}} F(\underline{x})\, dx_1 \dots dx_{n-1} \right) dx_n \\ &\quad = \lim_{a\to\infty} -\frac{e^{-y_na}}{y_n} \int_0^\infty \!\!\cdots\! \int_0^\infty e^{-y_1x_1-\cdots-y_{n-1}x_{n-1}} F(x_1,\dots,x_{n-1},a)\, dx_1 \dots dx_{n-1} \\ &\qquad + \frac{1}{y_n} \int_0^\infty \!\!\cdots\! \int_0^\infty e^{-y_1x_1-\cdots-y_{n-1}x_{n-1}} F(x_1,\dots,x_{n-1},0)\, dx_1 \dots dx_{n-1} \\ &\qquad + \frac{1}{y_n} \int_0^\infty e^{-y_nx_n}\, d\left(\int_0^\infty \!\!\cdots\! \int_0^\infty e^{-y_1x_1-\cdots-y_{n-1}x_{n-1}} F(\underline{x})\, dx_1 \dots dx_{n-1} \right). \end{aligned}$$

Since F is bounded, we have

$$\lim_{a\to\infty} -\frac{e^{-y_na}}{y_n} \int_0^\infty \!\!\cdots\! \int_0^\infty e^{-y_1x_1-\cdots-y_{n-1}x_{n-1}} F(x_1,\dots,x_{n-1},a)\, dx_1 \dots dx_{n-1} = 0.$$

Assuming that the result is valid for the $n-1$-dimensional functions $f_{x_n}(x_1,\dots,x_{n-1}) = F(x_1,\dots,x_{n-1},x_n)$ for every $x_n \geq 0$, we have

$$\begin{aligned}
\int_0^\infty\!\!\cdots\!\int_0^\infty & \mathrm{e}^{-y_1x_1-\cdots-y_{n-1}x_{n-1}} F(x_1,\dots,x_{n-1},0)\,dx_1\dots dx_{n-1} \\
&= \int_0^\infty\!\!\cdots\!\int_0^\infty \mathrm{e}^{-y_1x_1-\cdots-y_{n-1}x_{n-1}} f_0(x_1,\dots,x_{n-1})\,dx_1\dots dx_{n-1} \\
&= \frac{1}{y_1\dots y_{n-1}} \sum_{I\subset\{1,\dots,n-1\}} \int \mathrm{e}^{-\sum_{i\in I} y_ix_i}\, d(f_0)_I(x_1,\dots,x_{n-1}) \\
&= \frac{1}{y_1\dots y_{n-1}} \sum_{I\subset\{1,\dots,n-1\}} \int \mathrm{e}^{-\sum_{i\in I} y_ix_i}\, dF_I(\underline{x})
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{y_n}\int_0^\infty \mathrm{e}^{-y_nx_n} d\left(\int_0^\infty\!\!\cdots\!\int_0^\infty \mathrm{e}^{-y_1x_1-\cdots-y_{n-1}x_{n-1}} F(\underline{x})\,dx_1\dots dx_{n-1}\right) \\
&\quad= \frac{1}{y_n}\int_0^\infty \mathrm{e}^{-y_nx_n} d\left(\int_0^\infty\!\!\cdots\!\int_0^\infty \mathrm{e}^{-y_1x_1-\cdots-y_{n-1}x_{n-1}} f_{x_n}(x_1,\dots,x_{n-1})\,dx_1\dots dx_{n-1}\right) \\
&\quad= \frac{1}{y_n}\int_0^\infty \mathrm{e}^{-y_nx_n} d\left(\frac{1}{y_1\dots y_{n-1}} \sum_{J\subset\{1,\dots,n-1\}} \int \mathrm{e}^{-\sum_{i\in J} y_ix_i} d(f_{x_n})_J(x_1,\dots,x_{n-1})\right) \\
&\quad= \frac{1}{y_1\dots y_n} \sum_{J\subset\{1,\dots,n-1\}} \int_0^\infty \mathrm{e}^{-y_nx_n} d\left(\int \mathrm{e}^{-\sum_{i\in J} y_ix_i}\, dF_{J\cup\{n\}}(\underline{x})\right) \\
&\quad= \frac{1}{y_1\dots y_n} \sum_{I\subset\{1,\dots,n\},\, n\in I} \int \mathrm{e}^{-\sum_{i\in I} y_ix_i}\, dF_I(\underline{x}).
\end{aligned}$$

So,

$$\int_0^\infty\!\!\cdots\!\int_0^\infty \mathrm{e}^{-y_1x_1-\cdots-y_nx_n} F(\underline{x})\,dx_1\dots dx_n = \frac{1}{y_1\dots y_n} \sum_{I\subset\{1,\dots,n\}} \int \mathrm{e}^{-\sum_{i\in I} y_ix_i}\, dF_I(\underline{x}).$$

□

Corollary 4.C. *Let $p\in\mathbb{N}_0$, $s,t,\varsigma\in\mathbb{N}$ and consider $y_1,\dots,y_\varsigma\in\mathbb{R}_0^+$, $a>0$, and $s+t-1<a_2<b_2<a_3<\cdots<b_\varsigma\in\mathbb{N}_0$. For u sufficiently close to $u_F=\varphi(\zeta)$,*

$$\mathbb{E}\big(\mathrm{e}^{-y_1a\mathscr{A}_{u,0}^s - y_2a\mathscr{A}_{u,a_2}^{b_2}-\cdots-y_\varsigma a\mathscr{A}_{u,a_\varsigma}^{b_\varsigma}}\big) = \mathbb{E}\big(\mathrm{e}^{-y_1a\mathscr{A}_{u,0}^s}\big)\mathbb{E}\big(\mathrm{e}^{-y_2a\mathscr{A}_{u,a_2}^{b_2}-\cdots-y_\varsigma a\mathscr{A}_{u,a_\varsigma}^{b_\varsigma}}\big) + \mathrm{Err}$$

where $|\mathrm{Err}| \leq s\iota(u,t) + 4s\sum_{j=p+1}^{s-1}\mathbb{P}(Q_{p,0}^0(u)\cap\{X_j>u\}) + 2\int_0^\infty y_1\mathrm{e}^{-y_1x}\delta_{p,s,u}(x/a)\,dx$, *$\iota(u,t)$ is given by (4.2) and $\delta_{p,s,u}$ as in (2.15).*

Proof. Using the same notation as in the proof of Lemma 4.4, let $F^{(A)}(x_1,\dots,x_\varsigma) = \mathbb{P}(A_{x_1,\underline{x}})$, $F^{(B)}(x_1) = \mathbb{P}(B_{x_1})$ and $F^{(D)}(x_2,\dots,x_\varsigma) = \mathbb{P}(D^{\underline{x}})$. Then $F^{(A)}$, $F^{(B)}$ and $F^{(D)}$ are each bounded Stieltjes measure functions, with

$$\mu_{F^{(A)}}(U_1) = \mathbb{P}((a\mathscr{A}^s_{u,0}, a\mathscr{A}^{b_2}_{u,a_2}, \dots, a\mathscr{A}^{b_\varsigma}_{u,a_\varsigma}) \in U_1),$$
$$\mu_{F^{(B)}}(U_2) = \mathbb{P}(a\mathscr{A}^s_{u,0} \in U_2), \qquad \mu_{F^{(D)}}(U_3) = \mathbb{P}((a\mathscr{A}^{b_2}_{u,a_2}, \dots, a\mathscr{A}^{b_\varsigma}_{u,a_\varsigma}) \in U_3),$$

where U_1, U_2 and U_3 are Borel sets in $(\mathbb{R}^+_0)^\varsigma$, $\mathbb{R}^+_0$ and $(\mathbb{R}^+_0)^{\varsigma-1}$, respectively.

Therefore, we can apply the previous proposition and we obtain

$$\begin{aligned}
&\mathbb{E}\big(\mathrm{e}^{-y_1 a\mathscr{A}^s_{u,0} - y_2 a\mathscr{A}^{b_2}_{u,a_2} - \dots - y_\varsigma a\mathscr{A}^{b_\varsigma}_{u,a_\varsigma}}\big) - \mathbb{E}\big(\mathrm{e}^{-y_1 a\mathscr{A}^s_{u,0}}\big)\mathbb{E}\big(\mathrm{e}^{-y_2 a\mathscr{A}^{b_2}_{u,a_2} - \dots - y_\varsigma a\mathscr{A}^{b_\varsigma}_{u,a_\varsigma}}\big)\\
&= \sum_{I\subset\{1,\dots,\varsigma\}} \int \mathrm{e}^{-\sum_{i\in I} y_i a x_i}\, d(F^{(A)})_I(x_1,\dots,x_\varsigma)\\
&\quad - \sum_{I\subset\{1\}} \int \mathrm{e}^{-\sum_{i\in I} y_i a x_i}\, d(F^{(B)})_I(x_1) \sum_{I\subset\{2,\dots,\varsigma\}} \int \mathrm{e}^{-\sum_{i\in I} y_i a x_i}\, d(F^{(D)})_I(x_2,\dots,x_\varsigma)\\
&= y_1\dots y_\varsigma a^\varsigma \int_0^\infty\!\!\dots\!\int_0^\infty \mathrm{e}^{-y_1 a x_1 - \dots - y_\varsigma a x_\varsigma} F^{(A)}(x_1,\dots,x_\varsigma)\,dx_1\dots dx_\varsigma\\
&\quad - \left(y_1 a \int_0^\infty \mathrm{e}^{-y_1 a x_1} F^{(B)}(x_1)\,dx_1\right)\\
&\qquad\quad \times \left(y_2\dots y_\varsigma a^{\varsigma-1} \int_0^\infty\!\!\dots\!\int_0^\infty \mathrm{e}^{-y_2 a x_2 - \dots - y_\varsigma a x_\varsigma} F^{(D)}(x_2,\dots,x_\varsigma)\,dx_2\dots dx_\varsigma\right)\\
&= y_1\dots y_\varsigma a^\varsigma \int_0^\infty\!\!\dots\!\int_0^\infty \mathrm{e}^{-y_1 a x_1 - \dots - y_\varsigma a x_\varsigma} (F^{(A)} - F^{(B)}F^{(D)})(x_1,\dots,x_\varsigma)\,dx_1\dots dx_\varsigma.
\end{aligned}$$

Hence, by Lemma 4.4 and the change of variables $x = ax_1$,

$$\begin{aligned}
&\big|\mathbb{E}\big(\mathrm{e}^{-y_1 a\mathscr{A}^s_{u,0} - y_2 a\mathscr{A}^{b_2}_{u,a_2} - \dots - y_\varsigma a\mathscr{A}^{b_\varsigma}_{u,a_\varsigma}}\big) - \mathbb{E}\big(\mathrm{e}^{-y_1 a\mathscr{A}^s_{u,0}}\big)\mathbb{E}\big(\mathrm{e}^{-y_2 a\mathscr{A}^{b_2}_{u,a_2} - \dots - y_\varsigma a\mathscr{A}^{b_\varsigma}_{u,a_\varsigma}}\big)\big|\\
&\quad \le y_1\dots y_\varsigma a^\varsigma \int_0^\infty\!\!\dots\!\int_0^\infty \mathrm{e}^{-y_1 a x_1 - \dots - y_\varsigma a x_\varsigma} |\mathbb{P}(A_{x_1,\underline{x}}) - \mathbb{P}(B_{x_1})\mathbb{P}(D^{\underline{x}})|\,dx_1\dots dx_\varsigma\\
&\quad \le s\iota(u,t) + 4s\sum_{j=p+1}^{s-1} \mathbb{P}(Q^0_{p,0}(u)\cap\{X_j > u\}) + 2\int_0^\infty y_1 \mathrm{e}^{-y_1 x}\delta_{p,s,u}(x/a)\,dx. \qquad \square
\end{aligned}$$

Proposition 4.D. *Let $X_0, X_1, \dots$ be given by (2.1), where φ achieves a global maximum at ζ. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence satisfying (2.2) and $(a_n)_{n\in\mathbb{N}}$ a normalising sequence. Assume that conditions $Д_p(u_n)^*$, $Д'_p(u_n)^*$ and $ULC_p(u_n)$ hold for some $p \in \mathbb{N}_0$. Let $J = \bigcup_{\ell=1}^{\varsigma} I_\ell \in \mathcal{R}$ where $I_j = [a_j, b_j) \in \mathcal{S}$, $j = 1, \dots, \varsigma$ and $a_1 < b_1 < a_2 < \dots < b_{\varsigma-1} < a_\varsigma < b_\varsigma$. Then, for all $y_1, \dots, y_\varsigma \in \mathbb{R}^+_0$, we have*

$$\mathbb{E}\big(\mathrm{e}^{-\sum_{\ell=1}^{\varsigma} y_\ell a_n \mathscr{A}_{u_n}(nI_\ell)}\big) - \prod_{\ell=1}^{\varsigma} \mathbb{E}^{k_n|I_\ell|}\big(\mathrm{e}^{-y_\ell a_n \mathscr{A}^{\lfloor n/k_n\rfloor}_{u_n,0}}\big) \xrightarrow[n\to\infty]{} 0.$$

Proof. Without loss of generality, we can assume that $y_1, \dots, y_\varsigma \in \mathbb{R}^+$, because if $y_j = 0$ for some $j = 1, \dots, \varsigma$ then we could consider $J = \bigcup_{\ell=1}^{j-1} I_\ell \cup \bigcup_{\ell=j+1}^{\varsigma} I_\ell$ instead. Let

$h := \inf_{j\in\{1,\dots,\varsigma\}}\{b_j - a_j\}$, $H := \lceil \sup\{x : x \in J\}\rceil = \lceil b_\varsigma \rceil$, $\hat{y} := \inf\{y_j : j = 1,\dots,\varsigma\} > 0$ and $\hat{Y} := \sup\{y_j : j = 1,\dots,\varsigma\}$. Let n be sufficiently large so that in particular $k_n > 2/h$, and set $\varrho_n := \lfloor n/k_n \rfloor$. We consider the following partition of $n[0,H]\cap\mathbb{Z}$ into blocks of length ϱ_n: $J_1 = [0,\varrho_n)$, $J_2 = [\varrho_n, 2\varrho_n),\dots, J_{Hk_n} = [(Hk_n-1)\varrho_n, Hk_n\varrho_n)$, $J_{Hk_n+1} = [Hk_n\varrho_n, Hn)$. We further cut each J_i into two blocks:

$$J_i^* := [(i-1)\varrho_n, i\varrho_n - t_n) \quad \text{and} \quad J_i' := J_i \setminus J_i^*.$$

Note that $|J_i^*| = \varrho_n - t_n$ and $|J_i'| = t_n$.

Let $\mathscr{S}_\ell = \mathscr{S}_\ell(k)$ be the number of blocks J_j contained in nI_ℓ, that is,

$$\mathscr{S}_\ell := \#\{j \in \{1,\dots,Hk_n\} : J_j \subset nI_\ell\}.$$

By the relation between k_n and h, we have $\mathscr{S}_\ell > 1$ for every $\ell \in \{1,\dots,\varsigma\}$. For each such ℓ, we also define $i_\ell := \min\{j \in \{1,\dots,k\} : J_j \subset nI_\ell\}$. It follows that $J_{i_\ell}, J_{i_\ell+1},\dots, J_{i_\ell+\mathscr{S}_\ell-1} \subset nI_\ell$. Moreover, by choice of the size of each block,

$$\mathscr{S}_\ell \sim k_n|I_\ell|. \tag{4.7}$$

First of all, recall that for any $0 \le x_i, z_i \le 1$, we have

$$\left|\prod x_i - \prod z_i\right| \le \sum |x_i - z_i|. \tag{4.8}$$

We start by making the following approximation, in which we use (4.8) and stationarity:

$$\begin{aligned}
&\left|\mathbb{E}\left(e^{-\sum_{\ell=1}^{\varsigma} y_\ell a_n \mathscr{A}_{u_n}(nI_\ell)}\right) - \mathbb{E}\left(e^{-\sum_{\ell=1}^{\varsigma} y_\ell \sum_{j=i_\ell}^{i_\ell+\mathscr{S}_\ell-1} a_n\mathscr{A}_{u_n}(J_j)}\right)\right| \\
&\quad \le \mathbb{E}\left(1 - e^{-\sum_{\ell=1}^{\varsigma} y_\ell a_n \mathscr{A}_{u_n}(nI_\ell\setminus\bigcup_{j=i_\ell}^{i_\ell+\mathscr{S}_\ell-1} J_j)}\right) \\
&\quad \le \mathbb{E}\left(1 - e^{-2\sum_{\ell=1}^{\varsigma} y_\ell a_n\mathscr{A}_{u_n}(J_1)}\right) \le 2\varsigma K\mathbb{E}\left(1 - e^{-a_n\mathscr{A}_{u_n}(J_1)}\right),
\end{aligned}$$

where $\max\{y_1,\dots,y_\varsigma\} \le K \in \mathbb{N}$. In order to show that we are allowed to use the above approximation we just need to check that $\mathbb{E}(1 - e^{-a_n\mathscr{A}_{u_n}(J_1)}) \to 0$ as $n\to\infty$. By Corollary 4.A we have

$$\mathbb{E}\left(e^{-a_n\mathscr{A}_{u_n}(J_1)}\right) = 1 - \varrho_n\int_0^\infty e^{-x}\mathbb{P}(R_{p,0}(u_n, x/a_n))\,dx + \mathrm{Err}, \tag{4.9}$$

where

$$|\mathrm{Err}| \le 2\varrho_n\sum_{j=p+1}^{\varrho_n-1}\mathbb{P}(Q_{p,0}^0(u_n)\cap\{X_j > u_n\}) + \int_0^\infty e^{-x}\delta_{p,\varrho_n,u_n}(x/a_n)\,dx \to 0$$

as $n\to\infty$ by conditions $Д'_p(u_n)^*$ and $ULC_p(u_n)$. Since $\int_0^\infty e^{-x}\mathbb{P}(R_{p,0}(u_n, x/a_n))\,dx \le \int_0^\infty e^{-x}\mathbb{P}(U(u_n))\,dx = \mathbb{P}(U(u_n))$, we get $\mathbb{E}(e^{-a_n\mathscr{A}_{u_n}(J_1)}) \to 1$ as $n\to\infty$ by (2.2).

Now, we proceed with another approximation which consists of replacing J_j by J_j^*. Using (4.8), stationarity and (4.7), we have

$$\begin{aligned}\Big|\mathbb{E}\big(\mathrm{e}^{-\sum_{\ell=1}^{\varsigma} y_\ell \sum_{j=i_\ell}^{i_\ell+\mathscr{S}_\ell-1} a_n\mathscr{A}_{u_n}(J_j)}\big) - \mathbb{E}\big(\mathrm{e}^{-\sum_{\ell=1}^{\varsigma} y_\ell \sum_{j=i_\ell}^{i_\ell+\mathscr{S}_\ell-1} a_n\mathscr{A}_{u_n}(J_j^*)}\big)\Big| \\ \le \mathbb{E}\big(1-\mathrm{e}^{-\sum_{\ell=1}^{\varsigma} y_\ell \mathscr{S}_\ell a_n\mathscr{A}_{u_n}(J_1')}\big) \\ \le K\sum_{\ell=1}^{\varsigma}\mathscr{S}_\ell\mathbb{E}\big(1-\mathrm{e}^{-a_n\mathscr{A}_{u_n}(J_1')}\big) \lesssim KHk_n\mathbb{E}\big(1-\mathrm{e}^{-a_n\mathscr{A}_{u_n}(J_1')}\big),\end{aligned}$$

and we must show that $k_n\mathbb{E}(1-\mathrm{e}^{-a_n\mathscr{A}_{u_n}(J_1')})\to 0$, as $n\to\infty$, in order for the approximation to make sense. By Corollary 4.A we have

$$\mathbb{E}\big(\mathrm{e}^{-a_n\mathscr{A}_{u_n}(J_1')}\big) = 1 - t_n\int_0^\infty \mathrm{e}^{-x}\mathbb{P}(R_{p,0}(u_n, x/a_n))\,dx + \mathrm{Err}, \tag{4.10}$$

where

$$k_n|\mathrm{Err}| \le 2k_nt_n\sum_{j=p+1}^{t_n-1}\mathbb{P}(Q_{p,0}^0(u_n)\cap\{X_j>u_n\}) + k_n\int_0^\infty \mathrm{e}^{-x}\delta_{p,t_n,u_n}(x/a_n)\,dx \to 0$$

as $n\to\infty$ by $\mathcal{Д}'_p(u_n)^*$ and $ULC_p(u_n)$. By (2.2) as well,

$$k_n\mathbb{E}\big(1-\mathrm{e}^{-a_n\mathscr{A}_{u_n}(J_1')}\big) \sim k_nt_n\int_0^\infty \mathrm{e}^{-x}\mathbb{P}(R_{p,0}(u_n, x/a_n))\,dx \xrightarrow[n\to\infty]{} 0. \tag{4.11}$$

Let us fix now some $\hat\ell\in\{1,\dots,\varsigma\}$ and $i\in\{i_{\hat\ell},\dots,i_{\hat\ell}+\mathscr{S}_{\hat\ell}-1\}$. Let $M_i = y_{\hat\ell}\sum_{j=i}^{i_{\hat\ell}+\mathscr{S}_{\hat\ell}-1} a_n\mathscr{A}_{u_n}(J_j^*)$ and $L_{\hat\ell} = \sum_{\ell=\hat\ell+1}^{\varsigma} y_\ell\sum_{j=i_\ell}^{i_\ell+\mathscr{S}_\ell-1} a_n\mathscr{A}_{u_n}(J_j^*)$. Using stationarity and Corollary 4.C along with the facts that $\iota(u_n,t)\le\gamma(n,t)$ and $y_{\hat\ell}\mathrm{e}^{-y_{\hat\ell}x}\le\hat Y\mathrm{e}^{-\hat y x}$, we obtain

$$\Big|\mathbb{E}\big(\mathrm{e}^{-y_{\hat\ell}a_n\mathscr{A}_{u_n}(J_{i_{\hat\ell}}^*)-M_{i_{\hat\ell}+1}-L_{\hat\ell}}\big) - \mathbb{E}\big(\mathrm{e}^{-y_{\hat\ell}a_n\mathscr{A}_{u_n}(J_1^*)}\big)\mathbb{E}\big(\mathrm{e}^{-M_{i_{\hat\ell}+1}-L_{\hat\ell}}\big)\Big| \le \Upsilon_n,$$

where

$$\Upsilon_n = \varrho_n\gamma(n,t_n)+4\varrho_n\sum_{j=p+1}^{\varrho_n-1}\mathbb{P}(Q_{p,0}^0(u_n)\cap\{X_j>u_n\})+2\hat Y\int_0^\infty \mathrm{e}^{-\hat y x}\delta_{p,\varrho_n,u_n}(x/a_n)\,dx.$$

Since $\mathbb{E}(\mathrm{e}^{-y_{\hat\ell}a_n\mathscr{A}_{u_n}(J_1^*)})\le 1$, it follows by the same argument that

$$\begin{aligned}\Big|\mathbb{E}\big(\mathrm{e}^{-M_{i_{\hat\ell}}-L_{\hat\ell}}\big)-\mathbb{E}^2\big(\mathrm{e}^{-y_{\hat\ell}a_n\mathscr{A}_{u_n}(J_1^*)}\big)\mathbb{E}\big(\mathrm{e}^{-M_{i_{\hat\ell}+2}-L_{\hat\ell}}\big)\Big| \\ \le \Big|\mathbb{E}\big(\mathrm{e}^{-M_{i_{\hat\ell}}-L_{\hat\ell}}\big)-\mathbb{E}\big(\mathrm{e}^{-y_{\hat\ell}a_n\mathscr{A}_{u_n}(J_1^*)}\big)\mathbb{E}\big(\mathrm{e}^{-M_{i_{\hat\ell}+1}-L_{\hat\ell}}\big)\Big| \\ +\mathbb{E}\big(\mathrm{e}^{-y_{\hat\ell}a_n\mathscr{A}_{u_n}(J_1^*)}\big)\Big|\mathbb{E}\big(\mathrm{e}^{-M_{i_{\hat\ell}+1}-L_{\hat\ell}}\big)-\mathbb{E}\big(\mathrm{e}^{-y_{\hat\ell}a_n\mathscr{A}_{u_n}(J_1^*)}\big)\mathbb{E}\big(\mathrm{e}^{-M_{i_{\hat\ell}+2}-L_{\hat\ell}}\big)\Big| \\ \le 2\Upsilon_n.\end{aligned}$$

Hence, proceeding inductively with respect to $i \in \{i_{\hat\ell}, \dots, i_{\hat\ell} + \mathscr{S}_{\hat\ell} - 1\}$, we obtain

$$\left|\mathbb{E}\big(\mathrm{e}^{-M_{i_{\hat\ell}} - L_{\hat\ell}}\big) - \mathbb{E}^{\mathscr{S}_{\hat\ell}}\big(\mathrm{e}^{-y_{\hat\ell} a_n \mathscr{A}_{u_n}(J_1^*)}\big)\mathbb{E}\big(\mathrm{e}^{-L_{\hat\ell}}\big)\right| \le \mathscr{S}_{\hat\ell}\Upsilon_n.$$

In the same way, if we proceed inductively with respect to $\hat\ell \in \{1, \dots, \varsigma\}$, we get

$$\left|\mathbb{E}\big(\mathrm{e}^{-\sum_{\ell=1}^{\varsigma} y_\ell \sum_{j=i_\ell}^{i_\ell + \mathscr{S}_\ell - 1} a_n \mathscr{A}_{u_n}(J_j^*)}\big) - \prod_{\ell=1}^{\varsigma} \mathbb{E}^{\mathscr{S}_\ell}\big(\mathrm{e}^{-y_\ell a_n \mathscr{A}_{u_n}(J_1^*)}\big)\right| \le \sum_{\ell=1}^{\varsigma} \mathscr{S}_\ell \Upsilon_n.$$

By (4.7), we have $\sum_{\ell=1}^{\varsigma} \mathscr{S}_\ell \Upsilon_n \lesssim H k_n \Upsilon_n$ and

$$\begin{aligned} k_n\Upsilon_n &= k_n \varrho_n \gamma(n, t_n) + 4k_n\varrho_n \sum_{j=p+1}^{\varrho_n - 1} \mathbb{P}(Q_{p,0}^0(u_n) \cap \{X_j > u_n\}) \\ &\quad + 2k_n \hat{Y} \int_0^\infty \mathrm{e}^{-\hat{y}x} \delta_{p,\varrho_n,u_n}(x/a_n)\,dx \\ &\sim n\gamma(n, t_n) + 4n \sum_{j=p+1}^{\varrho_n - 1} \mathbb{P}(Q_{p,0}^0(u_n) \cap \{X_j > u_n\}) \\ &\quad + \frac{2\hat{Y}}{\hat{y}} k_n \int_0^\infty \hat{y}\mathrm{e}^{-\hat{y}x} \delta_{p,\varrho_n,u_n}(x/a_n)\,dx \\ &\to 0 \quad \text{as } n \to \infty, \end{aligned}$$

by $Д_p(u_n)^*$, $Д'_p(u_n)^*$ and $ULC_p(u_n)$.

Using (4.8) and stationarity, again, we have the final approximation

$$\left|\prod_{\ell=1}^{\varsigma} \mathbb{E}^{\mathscr{S}_\ell}\big(\mathrm{e}^{-y_\ell a_n \mathscr{A}_{u_n}(J_1)}\big) - \prod_{\ell=1}^{\varsigma} \mathbb{E}^{\mathscr{S}_\ell}\big(\mathrm{e}^{-y_\ell a_n \mathscr{A}_{u_n}(J_1^*)}\big)\right| \lesssim KHk_n\mathbb{E}\big(1 - \mathrm{e}^{-a_n\mathscr{A}_{u_n}(J_1')}\big).$$

Since in (4.11) we have already proved that $k_n\mathbb{E}(1 - \mathrm{e}^{-a_n\mathscr{A}_{u_n}(J_1')}) \to 0$ as $n \to \infty$, we only need to gather all the approximations and recall (4.7) to get the stated result. □

Proof of Theorem 2.A. In order to prove convergence of a_nA_n to a process A, it is sufficient to show that for any ς disjoint intervals $I_1, \dots, I_\varsigma \in \mathcal{S}$, the joint distribution of a_nA_n over these intervals converges to the joint distribution of A over the same intervals, i.e.,

$$(a_nA_n(I_1), \dots, a_nA_n(I_\varsigma)) \xrightarrow[n\to\infty]{} (A(I_1), \dots, A(I_\varsigma)),$$

which will be the case if the corresponding joint Laplace transforms converge. Hence, we only need to show that

$$\psi_{a_nA_n}(y_1, \dots, y_\varsigma) \to \psi_A(y_1, \dots, y_\varsigma) = \mathbb{E}\big(\mathrm{e}^{-\sum_{\ell=1}^{\varsigma} y_\ell A(I_\ell)}\big) \quad \text{as } n \to \infty,$$

for any non-negative values $y_1, \dots, y_\varsigma$, any disjoint intervals $I_1, \dots, I_\varsigma \in \mathcal{S}$ and each $\varsigma \in \mathbb{N}$. Note that $\psi_{a_n A_n}(y_1, \dots, y_\varsigma) = \mathbb{E}(\mathrm{e}^{-\sum_{\ell=1}^{\varsigma} y_\ell a_n A_n(I_\ell)}) = \mathbb{E}(\mathrm{e}^{-\sum_{\ell=1}^{\varsigma} y_\ell a_n \mathscr{A}_{u_n}(v_n I_\ell)})$ and

$$\begin{aligned}\Big|\mathbb{E}\big(\mathrm{e}^{-\sum_{\ell=1}^{\varsigma} y_\ell a_n \mathscr{A}_{u_n}(v_n I_\ell)}\big) - \mathbb{E}\big(\mathrm{e}^{-\sum_{\ell=1}^{\varsigma} y_\ell A(I_\ell)}\big)\Big| &\\ \le \Big|\mathbb{E}\big(\mathrm{e}^{-\sum_{\ell=1}^{\varsigma} y_\ell a_n \mathscr{A}_{u_n}(v_n I_\ell)}\big) - \prod_{\ell=1}^{\varsigma} \mathbb{E}^{k_n \frac{v_n}{n}|I_\ell|}\big(\mathrm{e}^{-y_\ell a_n \mathscr{A}_{u_n,0}^{\lfloor n/k_n\rfloor}}\big)\Big| &\\ + \Big|\prod_{\ell=1}^{\varsigma} \mathbb{E}^{k_n \frac{v_n}{n}|I_\ell|}\big(\mathrm{e}^{-y_\ell a_n \mathscr{A}_{u_n,0}^{\lfloor n/k_n\rfloor}}\big) - \mathbb{E}\big(\mathrm{e}^{-\sum_{\ell=1}^{\varsigma} y_\ell A(I_\ell)}\big)\Big|.&\end{aligned}$$

By Proposition 4.D, the first term on the right goes to 0 as $n \to \infty$. Also, by Corollary 4.A,

$$\mathbb{E}\big(\mathrm{e}^{-y a_n \mathscr{A}_{u_n,0}^{\lfloor n/k_n\rfloor}}\big) = 1 - \lfloor n/k_n\rfloor \int_0^\infty y\mathrm{e}^{-yx}\mathbb{P}(R_{p,0}(u_n, x/a_n))\,dx + \mathrm{Err},$$

where

$$|\mathrm{Err}| \le \frac{2n}{k_n}\sum_{j=p+1}^{\lfloor n/k_n\rfloor-1} \mathbb{P}(Q_{p,0}^0(u_n) \cap \{X_j > u_n\}) + \int_0^\infty y\mathrm{e}^{-yx}\delta_{p,\lfloor n/k_n\rfloor,u_n}(x/a_n)\,dx.$$

Since, by $Д'_p(u_n)^*$ and $ULC_p(u_n)$, we have $k_n|\mathrm{Err}| \to 0$ as $n \to \infty$, it follows that

$$\begin{aligned}\mathbb{E}^{k_n}\big(\mathrm{e}^{-y a_n \mathscr{A}_{u_n,0}^{\lfloor n/k_n\rfloor}}\big) &\sim \left(1 - \frac{n}{k_n}\int_0^\infty y\mathrm{e}^{-yx}\mathbb{P}(R_{p,0}(u_n, x/a_n))\,dx\right)^{k_n}\\ &\sim \mathrm{e}^{-n\int_0^\infty y\mathrm{e}^{-yx}\mathbb{P}(R_{p,0}(u_n,x/a_n))\,dx}\end{aligned}$$

as $n \to \infty$. Hence,

$$\begin{aligned}\prod_{\ell=1}^{\varsigma} \mathbb{E}^{k_n \frac{v_n}{n}|I_\ell|}\big(\mathrm{e}^{-y_\ell a_n \mathscr{A}_{u_n,0}^{\lfloor n/k_n\rfloor}}\big) &\sim \prod_{\ell=1}^{\varsigma}\big(\mathrm{e}^{-n\int_0^\infty y_\ell \mathrm{e}^{-y_\ell x}\mathbb{P}(R_{p,0}(u_n,x/a_n))\,dx}\big)^{\frac{v_n}{n}|I_\ell|}\\ &= \mathrm{e}^{-v_n\sum_{\ell=1}^{\varsigma}|I_\ell|\int_0^\infty y_\ell \mathrm{e}^{-y_\ell x}\mathbb{P}(R_{p,0}(u_n,x/a_n))\,dx} = \mathrm{e}^{-\sum_{\ell=1}^{\varsigma}|I_\ell|\int_0^\infty y_\ell \mathrm{e}^{-y_\ell x}\frac{\mathbb{P}(R_{p,0}(u_n,x/a_n))}{P(U(u_n))}\,dx},\end{aligned}$$

$$\begin{aligned}\lim_{n\to\infty}\int_0^\infty y\mathrm{e}^{-yx}\frac{\mathbb{P}(R_{p,0}(u_n, x/a_n))}{P(U(u_n))}\,dx &= \int_0^\infty y\mathrm{e}^{-yx}\theta(1-\pi(x))\,dx\\ &= \theta(1-\pi(0) - \int_0^\infty \mathrm{e}^{-yx}\,d\pi(x)) = \theta(1-\phi(y)),\end{aligned}$$

where ϕ is the Laplace transform of π, and

$$\lim_{n\to\infty} \mathrm{e}^{-\sum_{\ell=1}^{\varsigma}|I_\ell|\int_0^\infty y_\ell \mathrm{e}^{-y_\ell x}\frac{\mathbb{P}(R_{p,0}(u_n,x/a_n))}{P(U(u_n))}dx} = \mathrm{e}^{-\theta\sum_{\ell=1}^{\varsigma}|I_\ell|(1-\phi(y_l))} = \mathbb{E}\big(\mathrm{e}^{-\sum_{\ell=1}^{\varsigma} y_\ell A(I_\ell)}\big),$$

where A is a compound Poisson process of intensity θ and multiplicity d.f. π. □

5. Convergence of random measures for induced and original systems

In this section we prove Theorem 2.C. We start by settling notation. For all $A, B \in \mathcal{B}$ and $j \in \mathbb{N}_0$ we define $r^j_{A,B}$ as r^j_A simply by replacing iterations by f by iterations by F_B. To ease the notation we let $U_n := U(u_n)$. We will assume throughout that n is so large that $U_n \subset B$.

We start with the following simple observation.

Lemma 5.1. *If $x \in \{r_B > j\}$ then $r^i_{U_n}(f^j(x)) = r^i_{U_n}(x) - j$ for all $i \in \mathbb{N}$.*

Proof. We will use induction. Note that since $U_n \subset B$, $r_B(x) > j$ implies $r_{U_n}(x) > j$ and then it is clear that $r_{U_n}(f^j(x)) = r_{U_n}(x) - j$. Moreover, $F_{U_n}(f^j(x)) = F_{U_n}(x)$ since $F_{U_n}(f^j(x)) = f^{r_{U_n}(f^j(x))}(f^j(x)) = f^{r_{U_n}(x)-j}(f^j(x)) = f^{r_{U_n}(x)}(x) = F_{U_n}(x)$.

Assume now that the statement holds for i and $F^i_{U_n}(f^j(x)) = F^i_{U_n}(x)$. Then $r^{i+1}_{U_n}(f^j(x)) = r_{U_n}(F^i_{U_n}(f^j(x))) + r^i_{U_n}(f^j(x)) = r_{U_n}(F^i_{U_n}(x)) + r^i_{U_n}(x) - j = r^{i+1}_{U_n}(x) - j$. Moreover, $F^{i+1}_{U_n}(f^j(x)) = f^{r_{U_n}(F^i_{U_n}(f^j(x)))}(F^i_{U_n}(f^j(x))) = f^{r_{U_n}(F^i_{U_n}(x))}(F^i_{U_n}(x)) = F^{i+1}_{U_n}(x)$. □

The next two lemmas show that we can replace $\mathbb{P}$ by $\mathbb{P}_B$ to study the distribution of A_n. Let $J = \bigcup_{l=1}^k I_j \in \mathcal{R}$, where $I_j = [a_j, b_j) \in \mathcal{S}$ are disjoint intervals. Let $\mathbf{x} = (x_1, \dots, x_k) \in \mathbb{R}^k$ and define the event

$$\mathbb{A}(J, \mathbf{x}, n) = \{A_n(I_1) > x_1, \dots, A_n(I_k) > x_k\}. \tag{5.1}$$

We begin by proving that $\mathbb{P}(\mathbb{A}(J, \mathbf{x}, n))$ can be approximated by $\int_B r_B \mathbf{1}_{\mathbb{A}(J,\mathbf{x},n)}\, d\mathbb{P}$. First we recall two useful formulas that are standard for induced maps:

$$\int_B r_B \, d\mathbb{P}_B = \sum_{j=0}^{\infty} \mathbb{P}_B(r_B > j), \tag{5.2}$$

$$\mathbb{P}(A) = \sum_{j=0}^{\infty} \mathbb{P}(B \cap \{r_B > j\} \cap f^{-j}(A)). \tag{5.3}$$

Lemma 5.2. *For any small $\varepsilon_0, \varepsilon_1 > 0$ and n sufficiently large we have*

$$\int_B r_B \mathbf{1}_{\mathbb{A}(J^{\varepsilon_1 -},\mathbf{x},n)}\, d\mathbb{P} - \varepsilon_0 \le \mathbb{P}(\mathbb{A}(J, \mathbf{x}, n)) \le \int_B r_B \mathbf{1}_{\mathbb{A}(J^{\varepsilon_1 +},\mathbf{x},n)}\, d\mathbb{P} + \varepsilon_0.$$

Proof. By Lemma 5.1,

$$r_{U_n} \circ f^j = r_{U_n} - j \quad \text{ in } \{r_B > j\} \subset \{r_{U_n} > j\}.$$

Let $\varepsilon_0, \varepsilon_1 > 0$. We start by choosing N^* such that

$$\sum_{j > N^*} \mathbb{P}(B \cap \{r_B > j\}) < \varepsilon_0,$$

which is possible since $\int_B r_B \, d\mathbb{P} < \infty$.

Let N_1 be so large that $N^* \mathbb{P}(U_n) = N^* v_n^{-1} < \varepsilon_1$ for all $n > N_1$.

We first prove the second inequality of the lemma. We have

$$\mathbb{P}(\mathbb{A}(J,\mathbf{x},n)) < \sum_{j=0}^{N^*}\mathbb{P}(B\cap\{r_B > j\}\cap f^{-j}(\mathbb{A}(J,\mathbf{x},n)))+\varepsilon_0 \tag{5.4}$$

$$< \sum_{j=0}^{N^*}\mathbb{P}(B\cap\{r_B > j\}\cap\{\mathbb{A}(J^{\varepsilon_1+},\mathbf{x},n)\})+\varepsilon_0 < \int_B r_B\mathbf{1}_{\mathbb{A}(J^{\varepsilon_1+},\mathbf{x},n)}\,d\mathbb{P}+\varepsilon_0. \tag{5.5}$$

Inequality (5.4) follows from (5.3). The first inequality in (5.5) holds because if $x \in B\cap\{r_B > j\}$, then $x \in B\cap\{r_{U_n} > j\}$, which implies that $r^i_{U_n}\circ f^j(x) = r^i_{U_n}(x) - j$. Thus, if $r^i_{U_n}\circ f^j(x) \in v_n J$ then $r^i_{U_n}(x) \in v_n J^{\varepsilon_1+}$, because $v_n\varepsilon_1 > N^* \geq j$ and so $r^i_{U_n}(x) = r^i_{U_n}\circ f^j(x) + j \in v_n J^{\varepsilon_1+}$. The second inequality in (5.5) follows from (5.2). Thus, the second inequality of Lemma 5.2 holds.

Now we turn to the first inequality. We have

$$\mathbb{P}(\mathbb{A}(J,\mathbf{x},n)) > \sum_{j=0}^{N^*}\mathbb{P}(B\cap\{r_B > j\}\cap f^{-j}(\mathbb{A}(J,\mathbf{x},n)))$$

$$\geq \sum_{j=0}^{N^*}\mathbb{P}(B\cap\{r_B > j\}\cap\{\mathbb{A}(J^{\varepsilon_1-},\mathbf{x},n)\}) \tag{5.6}$$

$$\geq \sum_{j=0}^{\infty}\mathbb{P}(B\cap\{r_B > j\}\cap\{\mathbb{A}(J^{\varepsilon_1-},\mathbf{x},n)\})-\varepsilon_0 = \int_B r_B\mathbf{1}_{\mathbb{A}(J^{\varepsilon_1-},\mathbf{x},n)}\,d\mathbb{P}-\varepsilon_0. \tag{5.7}$$

The inequality in (5.6) holds because if $x \in \{r_B > j\} \subset \{r_{U_n} > j\}$, then, by Lemma 5.1, $r^i_{U_n}(f^j(x)) = r^i_{U_n}(x) - j$. Thus, if $r^i_{U_n}(x) \in v_n J^{\varepsilon_1-}$ then $r^i_{U_n}(f^j(x)) \in v_n J$, because $v_n\varepsilon_1 > N^* \geq j$. The inequality in (5.7) follows from (5.3). □

The next lemma shows that $\int_B r_B\mathbf{1}_{\mathbb{A}(J,\mathbf{x},n)}\,d\mathbb{P}$ can be approximated by $\mathbb{P}_B(\mathbb{A}(J,\mathbf{x},n))$.

Lemma 5.3. *For any small $\varepsilon_0, \varepsilon_1 > 0$ and n sufficiently large we have*

$$\mathbb{P}_B(\mathbb{A}(J^{\varepsilon_1-},\mathbf{x},n)) - \varepsilon_0 \leq \int_B r_B\mathbf{1}_{\mathbb{A}(J,\mathbf{x},n)}\,d\mathbb{P} \leq \mathbb{P}_B(\mathbb{A}(J^{\varepsilon_1+},\mathbf{x},n)) + \varepsilon_0.$$

Proof. We start by noting that since F_B is $\mathbb{P}$-invariant in B,

$$\int_B r_B\mathbf{1}_{\mathbb{A}(J,\mathbf{x},n)}\,d\mathbb{P} = \frac{1}{M}\sum_{j=0}^{M-1}\int_B r_B\circ F_B^j\mathbf{1}_{\mathbb{A}(J,\mathbf{x},n)}\circ F_B^j\,d\mathbb{P}.$$

Let $\varepsilon_0, \varepsilon_1 > 0$. We will see that for n sufficiently large,

$$\int_B r_B\circ F_B^j\mathbf{1}_{\mathbb{A}(J^{\varepsilon_1-},\mathbf{x},n)}\,d\mathbb{P} - \varepsilon_0/2 \leq \int_B r_B\circ F_B^j\mathbf{1}_{\mathbb{A}(J,\mathbf{x},n)}\circ F_B^j\,d\mathbb{P}$$

$$\leq \int_B r_B\circ F_B^j\mathbf{1}_{\mathbb{A}(J^{\varepsilon_1+},\mathbf{x},n)}\,d\mathbb{P} + \varepsilon_0/2.$$

As in Lemma 5.1, we have

$$r^i_{U_n}\circ F^j_B = r^i_{U_n} - r^j_B \quad \text{in } B\cap\{r_{U_n} > r^j_B\} = B\cap\{r_{U_n,B} > j\}.$$

Now, let ε_2 be such that if $\mathbb{P}(D) < \varepsilon_2$ for some $D\subset B$ then

$$\int_D r_B\circ F^j_B\, d\mathbb{P} < \varepsilon_0/2, \qquad \forall j\in\mathbb{N}_0. \tag{5.8}$$

Let M^* be so large that $\mathbb{P}(\{r^M_B > M^*\}\cap B) < \varepsilon_2/2$. Let N_2 be such that for all $n > N_2$, $\mathbb{P}(B\cap\{r_{U_n,B}\le M\}) < \varepsilon_2/2$. We also assume that there exists $N_3\in\mathbb{N}$ such that $M^*\mathbb{P}(U_n) = M^* v_n^{-1} < \varepsilon_1$ for all $n > N_3$. Let $G_n = B\cap\{r^M_B\le M^*\}\cap\{r_{U_n,B} > M\}$.

By construction, $\mathbb{P}(B\setminus G_n) \le \mathbb{P}(B\cap\{r_{U_n,B}\le M\}) + \mathbb{P}(B\cap\{r^M_B > M^*\}) < \varepsilon_2/2+\varepsilon_2/2 = \varepsilon_2$ for $n > N_2$.

Since r_B is integrable in B and $\mathbb{P}|_B$ is F_B-invariant, the sequence $\{r_B\circ F^j_B\}_{j\in\mathbb{N}}$ of functions is uniformly integrable in B, i.e., $\int_B r_B\circ F^j_B\, d\mathbb{P} = \int_B r_B\circ F^{j'}_B\, d\mathbb{P}$ for all $j, j'\in\mathbb{N}$.

Observe now that in G_n and for $n > \max\{N_2, N_3\}$ we have

$$r^i_{U_n}\circ F^j_B = r^i_{U_n} - r^i_B,$$

because if $x\in G_n$ then $x\in B\cap\{r_{U_n,B} > M\}$, which implies $x\in B\cap\{r_{U_n,B} > j\}$. If $x\in G_n$ then $r^j_B(x)\le r^M_B(x)\le M^*$, and since $n > N_3$, we have $r^j_B(x)v_n^{-1} < \varepsilon_1$, and

$$r^i_{U_n}\circ F^j_B(x)\in v_n J,\ \text{so}\ r^i_{U_n}(x)\in v_n J^{\varepsilon_1+} \quad \text{and} \quad r^i_{U_n}(x)\in v_n J^{\varepsilon_1-},\ \text{so}\ r^i_{U_n}\circ F^j_B(x)\in v_n J.$$

In this way, we deduce that, for $n > N_3$,

$$\mathbb{A}(J^{\varepsilon_1-},\mathbf{x},n)\cap G_n \subset F_B^{-j}(\mathbb{A}(J,\mathbf{x},n))\cap G_n, \tag{5.9}$$

$$F_B^{-j}(\mathbb{A}(J,\mathbf{x},n))\cap G_n \subset \mathbb{A}(J^{\varepsilon_1+},\mathbf{x},n)\cap G_n. \tag{5.10}$$

We may then write

$$\int_B r_B\circ F^j_B \mathbf{1}_{\mathbb{A}(J,\mathbf{x},n)}\circ F^j_B\, d\mathbb{P} = \int_{G_n} r_B\circ F^j_B \mathbf{1}_{\mathbb{A}(J,\mathbf{x},n)}\circ F^j_B\, d\mathbb{P} + \int_{B\setminus G_n} r_B\circ F^j_B \mathbf{1}_{\mathbb{A}(J,\mathbf{x},n)}\circ F^j_B\, d\mathbb{P}.$$

By the choice of N_2 and M^*,

$$0\le \int_{B\setminus G_n} r_B\circ F^j_B \mathbf{1}_{\mathbb{A}(J,\mathbf{x},n)}\circ F^j_B\, d\mathbb{P} \le \int_{B\setminus G_n} r_B\circ F^j_B\, d\mathbb{P} < \varepsilon_0/2.$$

The last inequality follows from (5.8) since $\mathbb{P}(B\setminus G_n) < \varepsilon_2$. Thus,

$$\int_B r_B\circ F^j_B \mathbf{1}_{\mathbb{A}(J,\mathbf{x},n)}\circ F^j_B\, d\mathbb{P} \le \int_{G_n} r_B\circ F^j_B \mathbf{1}_{\mathbb{A}(J,\mathbf{x},n)}\circ F^j_B\, d\mathbb{P} + \varepsilon_0/2 \le \int_B r_B\circ F^j_B \mathbf{1}_{\mathbb{A}(J^{\varepsilon_1+},\mathbf{x},n)}\, d\mathbb{P} + \varepsilon_0/2.$$

The first inequality follows from (5.9).

On the other hand,

$$\int_B r_B \circ F_B^j \mathbf{1}_{\mathbb{A}(J,\mathbf{x},n)} \circ F_B^j \, d\mathbb{P} \geq \int_{G_n} r_B \circ F_B^j \mathbf{1}_{\mathbb{A}(J,\mathbf{x},n)} \circ F_B^j \, d\mathbb{P}$$
$$\geq \int_B r_B \circ F_B^j \mathbf{1}_{\mathbb{A}(J^{\varepsilon_1 -},\mathbf{x},n)} \, d\mathbb{P} - \int_{B\setminus G_n} r_B \circ F_B^j \mathbf{1}_{\mathbb{A}(J^{\varepsilon_1 -},\mathbf{x},n)} \, d\mathbb{P}$$
$$\geq \int_B r_B \circ F_B^j \mathbf{1}_{\mathbb{A}(J^{\varepsilon_1 -},\mathbf{x},n)} \, d\mathbb{P} - \varepsilon_0/2.$$

The second inequality above follows from (5.10) and the last one follows from (5.8). Hence,

$$\int_B r_B \circ F_B^j \mathbf{1}_{\mathbb{A}(J^{\varepsilon_1 -},\mathbf{x},n)} \, d\mathbb{P} - \varepsilon_0/2 \leq \int_B r_B \circ F_B^j \mathbf{1}_{\mathbb{A}(J,\mathbf{x},n)} \circ F_B^j \, d\mathbb{P}$$
$$\leq \int_B r_B \circ F_B^j \mathbf{1}_{\mathbb{A}(J^{\varepsilon_1 +},\mathbf{x},n)} \, d\mathbb{P} + \varepsilon_0/2.$$

By ergodicity of F_B, the Ergodic Theorem and Kac's Theorem we conclude that if M^* is sufficiently large, then

$$\int_B \left| \frac{1}{M} \sum_{j=0}^{M-1} r_B \circ F_B^j - \mathbb{P}(B)^{-1} \right| d\mathbb{P} < \varepsilon_0/2.$$

Consequently,

$$\int_B \mathbb{P}(B)^{-1} \mathbf{1}_{\mathbb{A}(J^{\varepsilon_1 -},\mathbf{x},n)} \, d\mathbb{P} - \varepsilon_0 \leq \frac{1}{M} \sum_{j=0}^{M-1} \int_B r_B \circ F_B^j \mathbf{1}_{\mathbb{A}(J,\mathbf{x},n)} \circ F_B^j \, d\mathbb{P}$$
$$\leq \int_B \mathbb{P}(B)^{-1} \mathbf{1}_{\mathbb{A}(J^{\varepsilon_1 +},\mathbf{x},n)} \, d\mathbb{P} + \varepsilon_0,$$

and the result follows. □

Finally, the last lemma allows us to approximate $\mathbb{P}_B(\mathbb{A}(J,\mathbf{x},n))$ by $\mathbb{P}_B(\mathbb{A}^B(J,\mathbf{x},n))$, where $\mathbb{A}^B(J,\mathbf{x},n)$ is defined as $\mathbb{A}(J,\mathbf{x},n)$ with A_n replaced by A_n^B.

Lemma 5.4. *For any small $\varepsilon_0, \varepsilon_1, \varepsilon_1' > 0$ and n sufficiently large we have*

$$\mathbb{P}_B(\mathbb{A}^B(J^{\varepsilon_1 -},\mathbf{x},n)) - \varepsilon_0 \leq \mathbb{P}_B(\mathbb{A}(J,\mathbf{x},n)) \leq \mathbb{P}_B(\mathbb{A}^B(J^{\varepsilon_1' +},\mathbf{x},n)) + \varepsilon_0.$$

Proof. We recall that

$$A_n(I_l)(x) = \sum_{j=0}^{N(I_l)(x,u)} m_u(\{X_i\}_{i \in v_n(I_l)_j(x,u) \cap \mathbb{N}_0})$$

and

$$A_n^B(I_l)(x) = \sum_{j=0}^{N(I_l)(x,u)} m_u(\{X_i^B\}_{i \in v_n^B(I_l)_j(x,u) \cap \mathbb{N}_0})$$

where $X_j^B = \varphi \circ F_B^j$ and $v_n^B = \frac{1}{\mathbb{P}_B(U_n)} = \frac{\mathbb{P}(B)}{\mathbb{P}(U_n)}$. Note that $r_B^j(x) = \sum_{i=0}^{j-1} r_B \circ F_B^i(x)$.

By the Ergodic Theorem and Kac's Theorem, $\left|\frac{1}{j}\sum_{i=0}^{j-1} r_B\circ F_B^i(x)-\frac{1}{\mathbb{P}(B)}\right|\to 0$ $\mathbb{P}_B$-a.e. because F_B is ergodic with respect to $\mathbb{P}_B$ and $\int_B r_B\,d\mathbb{P}_B=\frac{1}{\mathbb{P}(B)}$.

Observe that

$$\left|\frac{1}{j}\sum_{i=0}^{j-1} r_B\circ F_B^i(x)-\frac{1}{\mathbb{P}(B)}\right|<\delta \;\Leftrightarrow\; \left(\frac{1}{\mathbb{P}(B)}-\delta\right)j<r_B^j(x)<\left(\frac{1}{\mathbb{P}(B)}+\delta\right)j.$$

Define

$$E_M^{\varepsilon_3}:=\left\{x\in B:\left(\frac{1}{\mathbb{P}(B)}-\varepsilon_3\right)j\le\sum_{i=0}^{j-1} r_B\circ F_B^i(x)\le\left(\frac{1}{\mathbb{P}(B)}+\varepsilon_3\right)j,\ \forall j\ge M\right\}.$$

Note that $\mathbb{P}_B(B\setminus E_M^{\varepsilon_3})\to 0$ as $M\to\infty$. Let $F_M=\{r_{U_n,B}\ge M\}$. We have $B\setminus F_M=B\cap(F_B^{-1}U_n\cup\cdots\cup F_B^{-(M-1)}U_n)$, and so $\mathbb{P}_B(B\setminus F_M)\le M\mathbb{P}_B(U_n)\to 0$ as $n\to\infty$.

Let M be so large that $\mathbb{P}_B(B\setminus E_M^{\varepsilon_3})<\varepsilon_0/2$, and N_4 so large that $\mathbb{P}_B(B\setminus F_M)<\varepsilon_0/2$ for all $n>N_4$.

We have $F_M\subset\{r_{U_n,B}^i\ge M\}$ for all $i\in\mathbb{N}$, since $r_{U_n,B}^i\ge r_{U_n,B}$. Moreover, if $x\in E_M^{\varepsilon_3}\cap F_M$, then

$$\begin{aligned}\left(\frac{1}{\mathbb{P}(B)}-\varepsilon_3\right)r_{U_n,B}^i(x)\le r_{U_n}^i(x)&=\sum_{i=0}^{r_{U_n,B}^i(x)-1} r_B\circ F_B^i(x)=r_B^{r_{U_n,B}^i(x)}(x)\\&\le\left(\frac{1}{\mathbb{P}(B)}+\varepsilon_3\right)r_{U_n,B}^i(x).\end{aligned}$$

So, we may write

$$r_{U_n}^i(x)=(1+\alpha)\mathbb{P}(B)^{-1}r_{U_n,B}^i(x), \tag{5.11}$$

where $|\alpha|<\varepsilon_3\mathbb{P}(B)$. Consequently,

$$\begin{aligned}r_{U_n}^i(x)\in v_nJ&\;\Leftrightarrow\; v_n^{-1}r_{U_n}^i(x)\in J\;\Leftrightarrow\;(1+\alpha)(v_n^B)^{-1}r_{U_n,B}^i(x)\in J\\&\;\Leftrightarrow\;(v_n^B)^{-1}r_{U_n,B}^i(x)\in(1+\alpha)^{-1}J\Rightarrow(v_n^B)^{-1}r_{U_n,B}^i(x)\in J^{\frac{\varepsilon_3}{1-\varepsilon_3}J_{\sup}+},\end{aligned}$$

where $J_{\sup}=\sup J$.

On the other hand, using again (5.11), we get

$$\begin{aligned}r_{U_n,B}^i(x)\in v_n^BJ^{\varepsilon_3J_{\sup}-}&\;\Leftrightarrow\;\mathbb{P}(B)^{-1}r_{U_n,B}^i(x)v_n^{-1}\in J^{\varepsilon_3J_{\sup}-}\\&\;\Leftrightarrow\; r_{U_n}^i(x)v_n^{-1}(1+\alpha)^{-1}\in J^{\varepsilon_3J_{\sup}-}\\&\;\Leftrightarrow\; r_{U_n}^i(x)\in v_n(1+\alpha)J^{\varepsilon_3J_{\sup}-}\Rightarrow r_{U_n}^i(x)\in v_nJ.\end{aligned}$$

Thus

$$\mathbb{A}(J,\mathbf{x},n)\cap F_M\cap E_M^{\varepsilon_3}\subset\mathbb{A}^B(J^{\frac{\varepsilon_3}{1-\varepsilon_3}J_{\sup}+},\mathbf{x},n)\cap F_M\cap E_M^{\varepsilon_3}, \tag{5.12}$$

$$\mathbb{A}^B(J^{\varepsilon_3J_{\sup}-},\mathbf{x},n)\cap F_M\cap E_M^{\varepsilon_3}\subset\mathbb{A}(J,\mathbf{x},n)\cap F_M\cap E_M^{\varepsilon_3}. \tag{5.13}$$

By (5.12),

$$\begin{aligned}\mathbb{P}_B(\mathbb{A}(J,\mathbf{x},n)) &\le \mathbb{P}_B(\mathbb{A}(J,\mathbf{x},n)\cap F_M\cap E_M^{\varepsilon_3}) + \mathbb{P}_B(B\setminus F_M) + \mathbb{P}_B(B\setminus E_M^{\varepsilon_3})\\ &\le \mathbb{P}_B(\mathbb{A}(J,\mathbf{x},n)\cap F_M\cap E_M^{\varepsilon_3}) + \varepsilon_0 \le \mathbb{P}_B(\mathbb{A}^B(J^{\varepsilon_1'+},\mathbf{x},n)) + \varepsilon_0,\end{aligned}$$

where $\varepsilon_1' = \frac{\varepsilon_3}{1-\varepsilon_3} J_{\sup}$. By (5.13),

$$\begin{aligned}\mathbb{P}_B(\mathbb{A}(J,\mathbf{x},n)) &\ge \mathbb{P}_B(\{\mathbb{A}(J,\mathbf{x},n)\}\cap F_M\cap E_M^{\varepsilon_3})\\ &\ge \mathbb{P}_B(\mathbb{A}^B(J^{\varepsilon_3 J_{\sup}-},\mathbf{x},n)) - \mathbb{P}_B(B\setminus F_M) - \mathbb{P}_B(B\setminus E_M^{\varepsilon_3})\\ &\ge \mathbb{P}_B(\mathbb{A}^B(J^{\varepsilon_1 -},\mathbf{x},n)) - \varepsilon_0,\end{aligned}$$

where $\varepsilon_1 = \varepsilon_3 J_{\sup}$, concluding the proof. □

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