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Mean curvature flow with surgery of mean convex surfaces in three-manifolds

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Abstract. In a previous paper, we introduced a notion of mean curvature flow with surgery for embedded, mean convex surfaces in $\mathbb{R}^3$. In this paper, we extend this construction to embedded, mean convex surfaces in a Riemannian three-manifold. Moreover, by combining our results with earlier work of Brian White, we are able to give a precise description of the longtime behavior of the surgically modified flow.

Keywords. Mean curvature flow, singularities

1. Introduction

This is a sequel to our earlier paper [7], where we introduced a notion of mean curvature flow with surgery for embedded, mean convex surfaces in $\mathbb{R}^3$. This construction extends earlier work of Huisken and Sinestrari in the higher-dimensional case. An alternative approach was later described by Haslhofer and Kleiner [11]. Both approaches are inspired by the spectacular work of Hamilton [8], [9] and Perelman [17]–[19] on the formation of singularities in the Ricci flow.

In this paper, we extend the surgery construction of [7] to the more general setting of embedded, mean convex surfaces in Riemannian three-manifolds. It is well known that embeddedness and mean convexity are preserved under mean curvature flow in any ambient Riemannian manifold. The fact that mean convexity is preserved follows from the maximum principle together with the evolution equation for the mean curvature:

$$\frac{\partial}{\partial t} H = \Delta H + (|A|^2 + \mathrm{Ric}(\nu, \nu)) H.$$

Our first main result asserts that there exists a surgically modified flow on any given bounded time interval.

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Theorem 1.1. *Let X be a compact Riemannian manifold of dimension 3, and let $M_0 = \partial\Omega_0$ be a closed, embedded, mean convex surface in X. Finally, let $T > 0$ be a given number. Then there exists a mean curvature flow with surgery starting from M_0 which is defined on the time interval $[0, T]$. We allow for the possibility that the flow may become extinct before time T.*

Theorem 1.1 relies on the insight that finite-time singularities of embedded, mean convex surfaces are either spherical or contain a cylindrical neck. This is not true for flows of immersed surfaces: in that case, the flow may form singularities modeled after the translating *grim reaper* solution crossed with a line. Without the assumption of mean convexity, many more singularity models are possible.

We briefly recall the surgery algorithm from [7] and [15]. We specify three curvature thresholds $H_3 = 10H_2 \gg H_1$. We start from the given initial surface M_0, and evolve it by the smooth mean curvature flow until the maximum of the mean curvature reaches the threshold H_3 for the first time. We then perform surgery on suitably chosen necks. On each neck, the mean curvature is comparable to H_1. After surgery, the maximum of the mean curvature drops below H_2. We then evolve the surgically modified surface by smooth mean curvature flow until the maximum of the mean curvature reaches the threshold H_3 for the second time. Again, by performing surgery on suitably chosen necks, we are able to push the maximum of the mean curvature below H_2. We then run the smooth flow again. This process can be repeated until we reach time T or the flow becomes extinct.

One of the main ingredients in the proof of Theorem 1.1 is a sharp estimate, established in [5], for the inscribed radius along mean curvature flow (see [3] for a survey). Given a hypersurface M in a Riemannian manifold X, we can define the *inscribed radius* at x as

$$\left[\sup_{y \in M,\, 0 < d(x,y) \leq \frac{1}{2}\operatorname{inj}(X)} \left(-\frac{2\langle \exp_x^{-1}(y), \nu(x)\rangle}{d(x,y)^2}\right)\right]^{-1}.$$

Similarly, the *outer radius* can be defined as

$$\left[\max\left\{\sup_{y \in M,\, 0 < d(x,y) \leq \frac{1}{2}\operatorname{inj}(X)} \frac{2\langle \exp_x^{-1}(y), \nu(x)\rangle}{d(x,y)^2}, 0\right\}\right]^{-1}.$$

Here, exp and d denote the exponential map and Riemannian distance in the ambient manifold X. In the Euclidean setting, the inscribed radius can be interpreted as the radius of the largest ball which lies inside M and touches M at x. This quantity was first considered by Sheng and Wang [20].

For embedded, mean-convex solutions of mean curvature flow in Euclidean space, the inscribed radius is bounded from below by a constant multiple of $1/H$. This was first proved by Brian White [21], [22]. Alternative proofs were given by Sheng–Wang [20], and by Andrews [1]. In [4] and [5], this was improved to a sharp estimate, which also holds for flows in Riemannian manifolds. More precisely, if M_t is a family of closed, embedded, mean convex hypersurfaces evolving under mean curvature flow, then the inscribed radius is bounded from below by $\frac{1}{(1+\delta)H}$ at points where the curvature is large (see [5]). This estimate is particularly useful for flows of two-dimensional surfaces in three-manifolds where it serves as a substitute for the cylindrical estimate of [15].

In the second step, we want to extend the flow to the interval $[0, \infty)$, and analyze its asymptotic behavior for $t \to \infty$. Here, we encounter several new phenomena. For example, while mean curvature flow in $\mathbb{R}^3$ becomes extinct in finite time, the mean curvature flow in a Riemannian manifold may exist for all time. For the level-set flow, Brian White [21] has shown that the flow will either become extinct in finite time, or will converge to a finite collection of stable minimal surfaces as $t \to \infty$. It turns out that there is an analogous picture for mean curvature flow with surgery.

Theorem 1.2. *Let $M_0 = \partial\Omega_0$ be a closed, embedded, mean convex surface in a Riemannian three-manifold X. Then there exists a mean curvature flow with surgery starting from M_0 which is defined for all $t \in [0, \infty)$. Furthermore, the solution either becomes extinct in finite time, or else the flow is smooth for t sufficiently large, and the surfaces converge smoothly to a union of finitely many embedded stable minimal surfaces.*

We briefly sketch the main ideas involved in the proof of Theorem 1.2. By work of Brian White [21], we can fix a time T such that the level-set flow is smooth for $t \in (T - 2, \infty)$. We then consider the surgically modified flow on the interval $[0, T]$. By a suitable choice of the surgery parameters, we can ensure that the surgically modified flow is close to the level-set flow in the sense of geometric measure theory (see [16]). A result from [6] then implies that the surgically modified flow is, in fact, free of surgeries for $t \in (T - 1, T)$. We then restart the flow at time T. This flow turns out to have a smooth solution which is defined on the time interval $[T, \infty)$ and has the desired asymptotic behavior as $t \to \infty$.

We note that Theorem 1.2 can be applied to mean curvature flows starting from large coordinate spheres in asymptotically flat three-manifolds. In this case, the flow will converge to an outermost minimal surface in the limit.

2. Auxiliary results needed for the proof of Theorem 1.1

In order to define the surgery algorithm, we need a number of auxiliary results. These results were established in [7] in the special case when the ambient manifold is $\mathbb{R}^3$. In the following, we list the auxiliary results needed, and indicate the necessary adaptations in the Riemannian setting.

We first describe how to perform a single surgery on a mean convex surface M in a Riemannian three-manifold. We first recall a definition from [7]:

Definition 2.1. Let M be an embedded, mean convex surface in $\mathbb{R}^3$, and let N be a region in M. As usual, we denote by ν the outward pointing unit normal vector field. We say that N is an $(\hat{\alpha}, \hat{\delta}, \varepsilon, L)$-*neck* of size r if (in a suitable coordinate system in $\mathbb{R}^3$) the following holds:

- There is a simple closed, convex curve $\Gamma \subset \mathbb{R}^2$ with $\mathrm{dist}_{C^{20}}(r^{-1}N, \Gamma \times [-L, L]) \leq \varepsilon$.
- At each point on Γ, the inscribed radius is at least $\frac{1}{(1+\hat{\delta})\kappa}$, where κ denotes the geodesic curvature of Γ.

- $\sum_{l=1}^{18} |\nabla^l \kappa| \leq 1/100$ at each point on Γ.
- There exists a point on Γ where the geodesic curvature κ is equal to 1.
- The region $\{x + a\nu(x) : x \in N,\ a \in (0, 2\hat{\alpha}r)\}$ is disjoint from M.

Here, it is understood that ε is much smaller than $\hat{\delta}$.

We can extend this definition to Riemannian manifolds as follows:

Definition 2.2. Let M be a closed surface in a Riemannian three-manifold X, and let $o \in M$. We say that o *lies at the center of an* $(\hat{\alpha}, \hat{\delta}, \varepsilon, L)$*-neck* of size r if 0 lies at the center of an $(\hat{\alpha}, \hat{\delta}, \varepsilon, L)$-neck of size r in $\exp_o^{-1}(M \cap B_{\frac{1}{4}\mathrm{inj}(X)}(o)) \subset T_o X$ in the sense of [7, Definition 2.4]. Here, it is understood that ε is much smaller than $\hat{\delta}$, and the product Lr is much smaller than the injectivity radius of the ambient three-manifold X.

There is now an obvious way to perform surgery on a neck in Riemannian three-manifold X. Namely, if o lies at the center of an $(\hat{\alpha}, \hat{\delta}, \varepsilon, L)$-neck N in X, then $\exp_o^{-1}(N)$ is an $(\hat{\alpha}, \hat{\delta}, \varepsilon, L)$-neck in Euclidean space. Hence, we can perform the surgery procedure described in [7] on $\exp_o^{-1}(N)$. This gives a surgically modified surface in Euclidean space, which we can paste back into X using the exponential map $\exp_o$. This surgery procedure depends on a parameter Λ, and enjoys good properties as long as Λ is large and Lr is small:

Theorem 2.3 (Properties of surgery). *Given any number $\hat{\alpha} > \alpha$, there exists a real number δ_0 with the following properties. Suppose that we are given a pair of real numbers δ and $\hat{\delta}$ such that $\hat{\delta} < \delta < \delta_0$. Then we can find numbers $\bar{\varepsilon}$, Λ, and $\bar{r}$, depending only on δ and $\hat{\delta}$, such that the following holds. Suppose that N is an $(\hat{\alpha}, \hat{\delta}, \varepsilon, L)$-neck of size r which is contained in an embedded, mean convex surface in X, where $\varepsilon \leq \bar{\varepsilon}$, $L/1000 \geq \Lambda$, and $Lr \leq \bar{r}$. If we perform a Λ-surgery on N, then the resulting surface $\tilde{N}$ will be $\frac{1}{1+\delta}$-noncollapsed. Furthermore, the outer radius is at least α/H at each point on $\tilde{N}$. Finally, if $\tilde{p} \in \tilde{N} \setminus N$ is a point in the surgically modified region, then either $\lambda_1(\tilde{p}) \geq 0$, or else there exists a point $p \in N$ such that $\lambda_1(\tilde{p}) \geq \lambda_1(p)$ and $H(\tilde{p}) \geq H(p)$.*

Proof. We sketch the modifications needed to prove Theorem 2.3. By [7, Theorem 2.5], we can choose the surgery parameters $\bar{\varepsilon}$ and Λ so that performing surgery on an $(\hat{\alpha}, \hat{\delta}, \varepsilon, L)$-neck in $\mathbb{R}^3$ will produce a surface which is $\frac{2}{2+\delta+\hat{\delta}}$-noncollapsed with respect to the Euclidean metric. Suppose now that o lies at the center of an $(\hat{\alpha}, \hat{\delta}, \varepsilon, L)$-neck N in X. By definition, $\exp_o^{-1}(N)$ is an $(\hat{\alpha}, \hat{\delta}, \varepsilon, L)$-neck with respect to the Euclidean metric. Hence, if we perform surgery on $\exp_o^{-1}(N)$, then the resulting surface is $\frac{2}{2+\delta+\hat{\delta}}$-noncollapsed with respect to the Euclidean metric. We now paste this surface back into X using the exponential map. If Lr is sufficiently small, then we obtain a surface $\tilde{N}$ in X which is $\frac{1}{1+\delta}$-noncollapsed. An analogous argument works for the outer radius.

From now on, we will fix a Riemannian three-manifold X, and an initial surface $M_0 = \partial\Omega_0$ in X. We assume that M_0 is closed, embedded, and mean convex. In view of the evolution equation for the mean curvature, the mean curvature H has a positive lower bound which may deteriorate exponentially in t. The following proposition tells us that the noncollapsing constant can only deteriorate in a controlled way.

Proposition 2.4 (Coarse noncollapsing estimate). *We can find positive functions* $\Xi(t)$ *and* $\alpha(t)$ *with the following properties:*

- *The initial surface* M_0 *satisfies* $|A| + 1 \leq \Xi(0)H$. *Moreover, the inscribed radius and the outer radius of* M_0 *are bounded from below by* $\alpha(0)/H$.
- *Suppose that, for some time* t, *the surface* M_t *satisfies* $|A| + 1 \leq \Xi(t)H$, *and the inscribed radius and the outer radius of* M_t *are bounded from below by* $\alpha(t)/H$. *Then this remains so for all later times.*

Proposition 2.4 follows directly from results in [5]. Note that the functions $\Xi(t)$ and $\alpha(t)$ are uniformly bounded from below on any finite time interval, but may deteriorate as $t \to \infty$. *From now on, we will fix the time interval* $[0, T]$ *and the functions* $\Xi(t)$ *and* $\alpha(t)$ *throughout the proof of Theorem* 1.1. For abbreviation, we define $\alpha_{\min} := \min_{t \in [0,T]} \alpha(t)$.

Assumption 2.5. In the following, we will assume that M_t is a solution of the mean curvature flow with surgery which starts from M_0 and is defined on a subinterval of $[0, T]$. We will assume that this flow has the following properties:

- Each surgery procedure on M_t involves performing a Λ-surgery on an $(\hat{\alpha}(t), \hat{\delta}, \varepsilon, L)$-neck of size $r \in [1/(2H_1), 2/H_1]$, where $\hat{\alpha}(t) > \alpha(t)$, $\hat{\delta} \leq 1/10$, $L/1000 \geq \Lambda$, and H_1 is sufficiently large.
- The region Ω_t enclosed by M_t shrinks as t increases.
- For each t, the surface M_t is outward-minimizing within the region Ω_0.
- For each t, the inscribed radius and the outer radius of M_t are at least $\alpha(t)/H$.

The precise values of the function $\hat{\alpha}(t)$ and the surgery parameters $\hat{\delta}$, Λ, ε, L, and H_1 will be specified later.

In the first step, we want to apply the Pseudolocality Theorem to obtain derivative bounds shortly after a surgery. We begin by showing that surgeries are separated in space:

Proposition 2.6 (Separation of surgery regions). *Let* M_t *be a mean curvature flow with surgery satisfying Assumption* 2.5. *Suppose that* $t_0 < t_1$ *are two surgery times, and* $x_0 \in M_{t_0+}$ *and* $x_1 \in M_{t_1+}$ *are two points in the surgically modified regions. Then* $x_1 \notin B_{\frac{\alpha_{\min}}{1000} H_1^{-1}}(x_0)$.

Proof. The proof of [7, Proposition 2.7] carries over directly to the Riemannian setting.

Thus, if t_0 is a surgery time and x_0 is a point in the surgically modified region at time t_0+, then the flow $M_t \cap B_{\frac{\alpha_{\min}}{1000} H_1^{-1}}(x_0)$, $t > t_0$, is free of surgeries. Using the Pseudolocality Theorem of [7], we can draw the following conclusion:

Proposition 2.7. *There exist positive constants* $\beta_* \in (0, \alpha_{\min}/1000)$ *and* C_* *with the following property. Let* M_t *be a mean curvature flow with surgery satisfying Assumption* 2.5. *Suppose that* $t_0 \in [0, T]$ *is a surgery time and* x_0 *is a point in the surgically modified region at time* t_0+. *Then*

$$H_1^{-1}|A| + H_1^{-2}|\nabla A| + H_1^{-3}|\nabla^2 A| \leq C_*$$

for all times $t \in (t_0, t_0 + \beta_* H_1^{-2}]$ *and all points* $x \in M_t \cap B_{\beta_* H_1^{-1}}(x_0)$. *The constants* β_* *and* C_* *may depend on the noncollapsing constant* $\alpha_{\min}$, *but they are independent of the surgery parameters* $\hat{\alpha}(t)$, $\hat{\delta}$, Λ, ε, L, *and* H_1.

The exact values of the surgery parameters will depend on the value of the constant in the derivative estimate, which in turn depends on β_* and C_*. It is therefore critically important that the constants β_* and C_* do not depend on the exact choice of the surgery parameters $\hat{\alpha}(t)$, $\hat{\delta}$, Λ, ε, L, and H_1.

We next recall the following gradient estimate due to Haslhofer and Kleiner [10]. This estimate is stated for solutions of mean curvature flow in Euclidean space, but their arguments easily carry over to the Riemannian setting.

Theorem 2.8 (cf. Haslhofer–Kleiner [10, Theorem 1.8′]). *Let M_t be a mean curvature flow with surgery satisfying Assumption 2.5. Let p be a point on M_{t_0} for some $t_0 \in [0, T]$, and let $r \leq 1/H(p, t_0)$. Assume that the surfaces $M_t \cap B_{4r}(p)$, $t \in [t_0 - r^2, t_0]$, are free of surgeries. Moreover, assume that the surface $M_t \cap B_{4r}(p)$ is outward-minimizing within the ball $B_{4r}(p)$ for each $t \in [t_0 - r^2, t_0]$. Then $r^2|\nabla A| \leq C(\alpha_{\min})$ and $r^3|\nabla^2 A| \leq C(\alpha_{\min})$ at the point p.*

Combining Proposition 2.7 with the interior gradient estimate of Haslhofer and Kleiner [10], we obtain pointwise bounds for the first and second derivatives of the second fundamental form which hold even in the presence of surgeries.

Proposition 2.9 (Pointwise derivative estimate). *There exists a constant $C_\#$ with the following properties. Suppose that M_t is a mean curvature flow with surgery satisfying Assumption 2.5. Then $|\nabla A| \leq C_\#(H + H_1)^2$ and $|\nabla^2 A| \leq C_\#(H + H_1)^3$ for $0 \leq t \leq T$. The constant $C_\#$ may depend on the noncollapsing constant $\alpha_{\min}$, but is independent of the surgery parameters $\hat{\alpha}(t)$, $\hat{\delta}$, Λ, ε, L, and H_1.*

Proof. Let us consider an arbitrary time t_1 and an arbitrary point $x_1 \in M_{t_1}$ for which we want to verify the estimate. Without any loss of generality, we may assume that t_1 is not a surgery time. There are two cases:

Case 1: There exists a surgery time t_0 and a point x_0 such that $|x_1 - x_0| \leq \beta_* H_1^{-1}$, $0 < t_1 - t_0 \leq \beta_* H_1^{-2}$, and x_0 lies in the surgically modified region at time t_0+. Applying Proposition 2.7, we conclude that

$$H_1^{-1}|A| + H_1^{-2}|\nabla A| + H_1^{-3}|\nabla^2 A| \leq C_*$$

at the point (x_1, t_1). Hence, $|\nabla A| \leq C_*(H + H_1)^2$ and $|\nabla^2 A| \leq C_*(H + H_1)^3$ at (x_1, t_1).

Case 2: There does not exist a surgery time t_0 and a point x_0 as in Case 1. Then the flow $M_t \cap B_{\beta_* H_1^{-1}}(x_1)$, $t \in (t_1 - \beta_* H_1^{-2}, t_1]$, is free of surgeries. Moreover, the ball $B_{\beta_* H_1^{-1}}(x_1)$ is contained in the region Ω_0, so the surfaces M_t are outward-minimizing within that ball. Hence, Theorem 2.8 implies that $|\nabla A| \leq B(H + H_1)^2$ and $|\nabla^2 A| \leq B(H + H_1)^3$ at the point (x_1, t_1). Here, B is a positive constant that depends only on β_* and the noncollapsing constant $\alpha_{\min}$. This completes the proof.

Having fixed the constant $C_\#$ in the derivative estimate, we next define $\Theta = 400/\alpha_{\min}$, $\theta_0 = 10^{-6} \min\{\alpha_{\min}, 1/(C_\# \Theta^3)\}$, and $\hat{\alpha}(t) = \alpha(t)/(1 - \theta_0/8)$. Hence, if we start at a point (p_0, t_0) with $H(p_0, t_0) \geq H_1/\Theta$ and follow this point back in time, then the mean curvature at the resulting point will be between $\frac{1}{2}H(p_0, t_0)$ and $2H(p_0, t_0)$ provided that $t \in (t_0 - 2\theta_0 H(p_0, t_0)^{-2}, t_0]$.

We next recall two auxiliary results from [7] concerning curves in $\mathbb{R}^2$. As in [7], we will apply these results to a blow-up limit that splits off a line. This will be used to show that the noncollapsing constants of a neck improve prior to surgery; this improvement offsets the deterioration of the noncollapsing constants under surgery (see Theorem 2.3 above).

Proposition 2.10. *We can find a real number $\delta > 0$ such that the following holds:*

- *Suppose that Γ is a (possibly nonclosed) embedded curve in the plane with $\kappa > 0$, $\left|\frac{d\kappa}{ds}\right| \leq C_\#(\kappa + 2\Theta)^2$, and $\left|\frac{d^2\kappa}{ds^2}\right| \leq C_\#(\kappa + 2\Theta)^3$. Moreover, suppose that the inscribed radius is at least $\frac{1}{(1+\delta)\kappa}$ at each point on Γ, and the outer radius is at least $\alpha_{\min}/\kappa$ at each point on Γ. Finally, assume that $\kappa(p) = 1$ for some point $p \in \Gamma$. Then $L(\Gamma) \leq 3\pi$ and $\sup_\Gamma |\kappa - 1| \leq 1/100$.*
- *Suppose that Γ_t, $t \in (-2\theta_0, 0]$, is a family of simple, closed, convex curves in the plane which evolve by curve shortening flow. Assume that, for each $t \in (-2\theta_0, 0]$, the curve Γ_t satisfies the derivative estimates $\left|\frac{d\kappa}{ds}\right| \leq C_\#(\kappa + 2\Theta)^2$ and $\left|\frac{d^2\kappa}{ds^2}\right| \leq C_\#(\kappa + 2\Theta)^3$. Moreover, assume that at each point on Γ_t, the inscribed radius is at least $\frac{1}{(1+\delta)\kappa}$ and the outer radius is at least $\alpha_{\min}/\kappa$. Finally, assume that the geodesic curvature of Γ_0 is equal to 1 somewhere. Then the curve Γ_0 satisfies $\sum_{l=1}^{18} |\nabla^l \kappa| \leq 1/1000$. Moreover, $\sup_{\Gamma_{-\theta_0}} \kappa \leq 1 - \theta_0/4$.*

We assume that δ is chosen so small that $\delta < \delta_0$, where δ_0 is the constant in Theorem 2.3. In the next step, we choose $\hat{\delta}$ such that the following holds:

Proposition 2.11. *Given $\theta_0, \delta > 0$, we can find a real number $\hat{\delta} \in (0, \delta)$ with the following property. Consider a simple, closed, convex solution Γ_t, $t \in (-2\theta_0, 0]$, of the curve shortening flow in the plane which satisfies the derivative estimates $\left|\frac{d\kappa}{ds}\right| \leq C_\#(\kappa + 2\Theta)^2$ and $\left|\frac{d^2\kappa}{ds^2}\right| \leq C_\#(\kappa + 2\Theta)^3$. Moreover, assume that at each point on Γ_t, the inscribed radius is at least $\frac{1}{(1+\delta)\kappa}$ and the outer radius is at least $\alpha_{\min}/\kappa$. Finally, assume that the geodesic curvature of Γ_0 is 1 somewhere. Then Γ_0 is $\frac{1}{1+\hat{\delta}}$-noncollapsed.*

Having fixed the values of $\alpha(t), \hat{\alpha}(t), \delta, \hat{\delta}$, we will choose $\bar{\varepsilon}$ and Λ such that the conclusion of Theorem 2.3 holds.

We next observe that the convexity estimates of Huisken and Sinestrari [13], [14] still hold for mean curvature flow with surgery.

Proposition 2.12 (Huisken–Sinestrari [15, Section 4]). *Suppose that $\bar{\varepsilon}$ and Λ are chosen so that the conclusion of Theorem 2.3 holds. Moreover, let M_t be a mean curvature flow with surgery satisfying Assumption 2.5, where $\varepsilon \leq \bar{\varepsilon}$ and $L \geq 1000\Lambda$. Given any $\eta > 0$, there exists a constant $C_1(\eta)$ such that $\lambda_1 \geq -\eta H - C_1(\eta)$ for $0 \leq t \leq T$. The constant $C_1(\eta)$ depends only on η, T, the initial surface M_0, and the ambient manifold X, but is independent of the remaining surgery parameters ε, L, and H_1.*

Theorem 2.3 implies that performing Λ-surgery on an $(\hat{\alpha}(t), \hat{\delta}, \varepsilon, L)$-neck will produce a surface which is $\frac{1}{1+\delta}$-noncollapsed provided that $\varepsilon \leq \bar{\varepsilon}$ and $L \geq 1000\Lambda$. This allows us to show that the cylindrical estimate from [5] holds in the presence of surgeries:

Proposition 2.13 (Cylindrical estimate). *Let δ and $\hat{\delta}$ be chosen as above. Moreover, suppose that $\bar{\varepsilon}$ and Λ are chosen so that the conclusion of Theorem 2.3 holds. Finally, let M_t be a mean curvature flow with surgery satisfying Assumption 2.5, where $\varepsilon \leq \bar{\varepsilon}$ and $L \geq 1000\Lambda$. Then $\mu \leq (1+\delta)H + CH^{1-\sigma}$ for $0 \leq t \leq T$, where μ denotes the reciprocal of the inscribed radius. Here, σ and C may depend on δ, T, M_0, and X, but they are independent of the exact choice of ε, L, and H_1.*

Proof. In view of Proposition 2.12, we can find a large constant

$$K_0 \geq 8\,\mathrm{inj}(X)^{-1}\Big(\inf_{t\in[0,T]} \inf_{M_t} \min\{H, 1\}\Big)^{-1}$$

such that

$$(n-1)\lambda_1 \geq -\frac{\delta}{2}H - K_0 \min\{H, 1\}.$$

Note that K_0 is a constant that depends only on δ, T, the initial surface M_0, and the ambient manifold X. We next define

$$f_{\delta,\sigma} = H^{\sigma-1}(\mu - (1+\delta)H) - K_0,$$

where μ denotes the reciprocal of the inscribed radius. By results in [5], we can find a constant c_0, depending only on δ, T, M_0, and X, with the following property: if $p \geq 1/c_0$ and $\sigma \leq c_0 p^{-1/2}$, then

$$\frac{d}{dt}\Big(\int_{M_t} f_{\delta,\sigma,+}^p\Big) \leq C\sigma p \int_{M_t} f_{\delta,\sigma,+}^p + \sigma p K_0^p \int_{M_t} |A|^2 + (Cp)^p |M_t|$$

in between surgery times. Here, C depends only on δ, T, the initial surface M_0, and the ambient manifold X, but not on σ or p.

In the next step, we show that the integral $\int_{M_t} f_{\delta,\sigma,+}^p$ does not increase under surgery. To see this, we consider a surgery time t_0. By assumption, each surgery is being performed on an $(\hat{\alpha}(t_0), \hat{\delta}, \varepsilon, L)$-neck with $\varepsilon \leq \bar{\varepsilon}$ and $L \geq 1000\Lambda$. Hence, Theorem 2.3 implies that the inscribed radius of M_{t_0+} is at least $\frac{1}{(1+\delta)H}$ in the surgically modified region. In other words, $f_{\delta,\sigma} \leq 0$ in the surgically modified region of M_{t_0+}. Consequently, $\int_{M_{t_0+}} f_{\delta,\sigma,+}^p \leq \int_{M_{t_0-}} f_{\delta,\sigma,+}^p$.

Arguing as in [5], we conclude that there exists a constant c_0, depending only on δ, T, M_0, and X, with the following property: if $p \geq 1/c_0$ and $\sigma \leq c_0 p^{-1/2}$, then

$$\int_{M_t} f_{\delta,\sigma,+}^p \leq C$$

for all t. Here, C is a constant that depends on p, σ, δ, T, M_0, and X. Having established this, we can now use Stampacchia iteration to show that $f_{\delta,\sigma} \leq C$, where σ and C depend only on δ, T, M_0, and X. This completes the proof of Proposition 2.13.

Using the convexity estimate and the cylindrical estimate, we can now prove the Neck Detection Lemma. The proof is the same as in [7], and will be omitted. As in [15], we need two slightly different versions.

Theorem 2.14 (Neck Detection Lemma, Version A). *Let δ and $\hat{\delta}$ be chosen as above, and let $\bar{\varepsilon}$ and Λ be chosen so that the conclusion of Theorem 2.3 holds. Let M_t be a mean curvature flow with surgery satisfying Assumption 2.5, where $\varepsilon \leq \bar{\varepsilon}$ and $L \geq 1000\Lambda$. Then, given $\varepsilon_0 > 0$ and $L_0 \geq 100$, we can find $\eta_0 > 0$ and K_0 with the following properties. Suppose that $t_0 \in [0, T]$ and $p_0 \in M_{t_0}$ satisfy*

- *$H(p_0, t_0) \geq \max\{K_0, H_1/\Theta\}$ and $\lambda_1(p_0, t_0)/H(p_0, t_0) \leq \eta_0$,*
- *the parabolic neighborhood $\hat{\mathcal{P}}(p_0, t_0, L_0 + 4, 2\theta_0)$ does not contain surgeries.*[1]

Then (p_0, t_0) lies at the center of an $(\hat{\alpha}(t_0), \hat{\delta}, \varepsilon_0, L_0)$-neck of size $H(p_0, t_0)^{-1}$. Finally, the constants η_0 and K_0 may depend on ε_0, L_0, δ, $\hat{\delta}$, T, M_0, and X, but they are independent of the remaining surgery parameters ε, L, and H_1.

Theorem 2.15 (Neck Detection Lemma, Version B). *Let δ and $\hat{\delta}$ be chosen as above, and let $\bar{\varepsilon}$ and Λ be chosen so that the conclusion of Theorem 2.3 holds. Let M_t be a mean curvature flow with surgery satisfying Assumption 2.5, where $\varepsilon \leq \bar{\varepsilon}$ and $L \geq 1000\Lambda$. Then, given $\theta, \varepsilon_0 > 0$ and $L_0 \geq 100$, we can find positive numbers η_0 and K_0 with the following properties. Suppose that $t_0 \in [0, T]$ and $p_0 \in M_{t_0}$ satisfy*

- *$H(p_0, t_0) \geq \max\{K_0, H_1/\Theta\}$ and $\lambda_1(p_0, t_0)/H(p_0, t_0) \leq \eta_0$,*
- *the parabolic neighborhood $\hat{\mathcal{P}}(p_0, t_0, L_0 + 4, \theta)$ does not contain surgeries.*

Let us dilate the surface $\{x \in M_{t_0} : d_{g(t_0)}(p_0, x) \leq L_0 H(p_0, t_0)^{-1}\}$ by the factor $H(p_0, t_0)$. Then the resulting surface is ε_0-close to a product $\Gamma \times [-L_0, L_0]$ in the C^3-norm. Here, Γ is a closed, convex curve satisfying $L(\Gamma) \leq 3\pi$ and $\sup_\Gamma |\kappa - 1| \leq 1/100$. The constant K_0 may depend on θ, ε_0, L_0, δ, $\hat{\delta}$, T, M_0, and X, but it is independent of the remaining surgery parameters ε, L, and H_1.

The proof of the Neck Continuation Theorem in Section 3 will require both versions of the Neck Detection Lemma. The main difference between the two versions is that Version A requires the assumption that $\hat{\mathcal{P}}(p_0, t_0, L_0 + 4, 2\theta_0)$ does not contain surgeries, whereas Version B only requires that the parabolic neighborhood $\hat{\mathcal{P}}(p_0, t_0, L_0 + 4, \theta)$ is free of surgeries. (Note that θ can be much smaller than θ_0.)

The following result serves as a replacement for [15, Lemma 7.12]:

Proposition 2.16 (Replacement for [15, Lemma 7.12]). *Let M_t be a mean curvature flow with surgery satisfying Assumption 2.5. Suppose that (p_1, t_1) is a point in spacetime such that $H(p_1, t_1) \geq H_1$ and the parabolic neighborhood $\hat{\mathcal{P}}(p_1, t_1, \tilde{L} + 4, 2\theta_0)$ contains at least one point belonging to a surgery region. Then there exists a point $q_1 \in M_{t_1}$ and an open set $V \subset M_{t_1}$ such that $d_{g(t_1)}(p_1, q_1) \leq (\tilde{L} + 4) H(p_1, t_1)^{-1}$, $\{x \in M_{t_1} : d_{g(t_1)}(q_1, x) \leq 500 H_1^{-1}\} \subset V$, and V is diffeomorphic to a disk. Moreover, the mean curvature is at most $40 H_1$ at each point in V.*

[1] See [15, pp. 189–190] for the definition of $\hat{\mathcal{P}}(p_0, t_0, L_0 + 4, 2\theta_0)$.

Proof. We will consider a surgical cap that was inserted shortly before time t_1, and follow this cap forward in time. More precisely, in view of our assumption, the parabolic neighborhood $\hat{\mathcal{P}}(p_1, t_1, \tilde{L} + 4, 2\theta_0)$ contains a point which belongs to a surgery region. Consequently, we can find a surgery time $t_0 \in [t_1 - 2\theta_0 H(p_1, t_1)^{-2}, t_1)$ and a point $q_1 \in M_{t_1}$ such that the following holds:

- $d_{g(t_1)}(p_1, q_1) \leq (\tilde{L} + 4) H(p_1, t_1)^{-1}$.
- If we follow the point $q_1 \in M_{t_1}$ back in time, then the corresponding point $q_0 \in M_{t_0+}$ lies in the region modified by surgery at time t_0.

Let us consider the region modified by surgery at time t_0, and let U_0 denote the connected component of this set that contains the point q_0. In other words, $U_0 \subset M_{t_0+}$ is a cap that was inserted at time t_0. We next define $V_0 = \{x \in M_{t_0+} : \text{dist}_{g(t_0+)}(U_0, x) \leq 1000 H_1^{-1}\}$. Clearly, V_0 is diffeomorphic to a disk. Let

$$D = \bigcup_{x \in V_0} B_{\frac{\alpha_{\min}}{1000} H_1^{-1}}(x).$$

Arguing as in Proposition 2.6 above, we can show that, for every surgery time $t > t_0$, the set D is disjoint from the region modified by surgery at time t. Consequently, the surfaces $M_t \cap D$ form a regular mean curvature flow for $t > t_0$. In other words, the surfaces $M_t \cap D$ evolve smoothly for $t > t_0$, but we allow the possibility that some components of $M_t \cap D$ may disappear as a result of surgeries in other regions.

At each point on $V_0 \subset M_{t_0+}$, the mean curvature is at most $20 H_1$. We now follow the surface $V_0 \subset M_{t_0+}$ forward in time. This gives a one-parameter family of surfaces which are all diffeomorphic to a disk. It follows from Proposition 2.9 that, for $t \in (t_0, t_0 + 2\theta_0 H_1^{-2}]$, the resulting surfaces remain inside the region D and have mean curvature at most $40 H_1$. Moreover, since $q_1 \in M_{t_1}$, the resulting surfaces cannot disappear before time t_1.

Let $V_1 \subset M_{t_1}$ denote the region in M_{t_1} which is obtained by following the region $V_0 \subset M_{t_0+}$ forward in time. Clearly, V_1 is diffeomorphic to a disk, and the mean curvature is at most $40 H_1$ at each point in V_1. Since $q_0 \in V_0$, we have $q_1 \in V_1$. Furthermore, since $\text{dist}_{g(t_0+)}(q_0, \partial V_0) \geq 1000 H_1^{-1}$, we obtain $\text{dist}_{g(t_1)}(q_1, \partial V_1) \geq 500 H_1^{-1}$. From this, we deduce that

$$\{x \in M_{t_1} : d_{g(t_1)}(q_1, x) \leq 500 H_1^{-1}\} \subset V_1.$$

Hence, if we set $V := V_1$, then V has the required properties.

Since we have a pointwise estimate for the derivatives of the second fundamental form, we are able to prove an analogue of [15, Theorem 7.14].

Proposition 2.17. *Consider a closed surface M in X which satisfies the estimate $|\nabla A| \leq C_\#(H + H_1)^2$ for suitable constants $C_\#$ and H_1, and let $d(\cdot, \cdot)$ denote the intrinsic distance on M. Given any $\eta > 0$, we can find large numbers ρ and γ_0 (depending only on $C_\#$ and η) with the following properties. Suppose that p is a point on M with $\lambda_1(p) > \eta H(p)$ and $H(p) \geq \gamma_0 H_1$, where H_1 is sufficiently large. Then one of the following statements holds:*

- $\lambda_1(q') > \eta H(q')$ *and* $d(p, q') \leq \rho/H(p)$ *for each point* $q' \in M$.
- *There exists a point* $q \in M$ *such that* $\lambda_1(q) \leq \eta H(q)$ *and* $d(p, q) \leq \rho/H(p)$.

Furthermore, $H(q) \geq H(p)/\gamma_0 \geq H_1$ *for each point* $q' \in M$ *satisfying* $d(p, q') \leq \rho/H(p)$.

Proof. Let $\rho = (1/C_\#)e^{20\pi C_\#/\eta}$ and $\gamma_0 = 1 + 10C_\#\rho$. We claim that ρ and γ_0 have the desired property. To see this, suppose that p is a point in M satisfying $\lambda_1(p) > \eta H(p)$ and $H(p) \geq \gamma_0 H_1$. Using the curvature derivative estimate, we obtain

$$H(q) \geq \frac{H(p)}{1 + 10C_\# d(p, q)H(p)} \geq \frac{H(p)}{\gamma_0} \geq H_1$$

for all points $q \in M$ satisfying $d(p, q) \leq \rho/H(p)$.

If there exists a point $q \in M$ such that $\lambda_1(q) \leq \eta H(q)$ and $d(p, q) \leq \rho/H(p)$, then we are done. Hence, it remains to consider the case that $\lambda_1(q) > \eta H(q)$ for all $q \in M$ satisfying $d(p, q) \leq \rho/H(p)$. In this case, we will show that $d(p, q) \leq \rho/H(p)$ for all $q \in M$. To prove this, we choose a local height function $u : B_\sigma(p) \to \mathbb{R}$ in ambient space such that $|\nabla u| = 1$ at each point on $B_\sigma(p)$, $u(p) = 0$, and $\nabla u(p) = -\nu(p)$. Here, σ is a positive constant which depends only on the ambient manifold X. In the following, we assume that H_1 is chosen so large that $\rho/H(p) \leq \rho/(\gamma_0 H_1) \leq \sigma/2$. We next consider the flow on M generated by the vector field $\omega^T/|\omega^T|^2$, where $\omega = \nabla u$ and ω^T denotes the projection of ω to the tangent space of M. For $y > 0$ small, we can find a closed curve Γ_y around p which is contained in the level set $\{u = y\}$. By following the trajectories of the ODE, we can extend this to a maximal foliation Γ_y, $y \in (0, y_{\max})$.

We claim that $\sup_{q \in \Gamma_y} d(p, q) \leq \rho/H(p)$ for each $y \in (0, y_{\max})$ provided that H_1 is sufficiently large. Suppose this is false. Let $\bar{y} \in (0, y_{\max})$ be the smallest number with the property that $\sup_{q \in \Gamma_{\bar{y}}} d(p, q) \geq \rho/H(p)$. Along a trajectory of the ODE, we have

$$\frac{d}{dy}\langle \nu, \omega \rangle = \frac{\langle \bar{D}_{\omega^T}\nu, \omega \rangle + \langle \nu, \bar{D}_{\omega^T}\omega \rangle}{|\omega^T|^2} = \frac{h(\omega^T, \omega^T) + \langle \nu - \langle \nu, \omega \rangle \omega, \bar{D}_{\omega^T}\omega \rangle}{|\omega^T|^2},$$

where in the last step we have used the fact that ω has unit length. Using the identity $|\nu - \langle \nu, \omega \rangle \omega| = |\omega - \langle \omega, \nu \rangle \nu| = |\omega^T|$, we conclude that

$$\frac{d}{dy}\langle \nu, \omega \rangle \geq \lambda_1 - |\bar{D}\omega|.$$

Hence, if H_1 is sufficiently large, then $\frac{d}{dy}\langle \nu, \omega \rangle \geq \eta H - |\bar{D}\omega| \geq \frac{\eta}{2} H$ for all points $q \in \bigcup_{y \in (0, \bar{y})} \Gamma_y$. Consequently, along a trajectory of the ODE, we have

$$\frac{d}{dy}\bigg(\frac{\eta}{20C_\#}\log(1 + 10C_\# d(p, q)H(p)) - \arcsin\langle \nu, \omega \rangle\bigg)$$
$$\leq \frac{\eta}{2}\bigg(\frac{H(p)}{1 + 10C_\# d(p, q)H(p)} - H(q)\bigg)\frac{1}{\sqrt{1 - \langle \nu, \omega \rangle^2}} \leq 0$$

for all $q \in \bigcup_{y \in (0,\bar{y})} \Gamma_y$. This implies

$$\frac{\eta}{20C_\#} \log(1 + 10C_\# d(p,q) H(p)) - \arcsin\langle \nu, \omega \rangle \leq \frac{\pi}{2},$$

hence

$$d(p,q) \leq \frac{1}{10C_\# H(p)} \big(e^{20\pi C_\#/\eta} - 1\big) \leq \frac{\rho}{10H(p)}$$

for all $q \in \bigcup_{y \in (0,\bar{y})} \Gamma_y$. This contradicts the fact that $\sup_{q \in \Gamma_{\bar{y}}} d(p,q) \geq \rho/H(p)$.

Thus, $\sup_{q \in \Gamma_y} d(p,q) \leq \rho/H(p)$ for all $y \in (0, y_{\max})$. In particular, $\lambda_1 \geq \eta H \geq \eta H_1$ at each point in $\bigcup_{y \in (0,y_{\max})} \Gamma_y$. Using this uniform convexity property together with elementary Morse theory, we conclude that $M \setminus (\{p\} \cup \bigcup_{y \in (0,y_{\max})} \Gamma_y)$ consists of a single point, and furthermore $\nu = \omega$ at that point. Consequently, $d(p,q) \leq \rho/H(p)$ for all $q \in M$. This completes the proof of Proposition 2.17.

3. The Neck Continuation Theorem and the proof of Theorem 1.1

In this section, we use the auxiliary results from Section 2 to establish an analogue of the Neck Continuation Theorem of Huisken and Sinestrari [15].

We begin by finalizing our choice of the surgery parameters. This step is similar to the discussion in [15, pp. 208–209]. Recall that the parameters δ, $\hat{\delta}$, $\hat{\alpha}(t)$ and the constants $C_\#$, θ_0, Θ have already been chosen at this stage. Moreover, $\bar{\varepsilon}$ and Λ have been chosen so that the conclusion of Theorem 2.3 holds.

In the next step, we choose numbers ε_0 and L_0 so that $\varepsilon_0 < \bar{\varepsilon}$ and $L_0 > 1000\Lambda$. In addition, we require that the mean curvature on an $(\hat{\alpha}(t), \hat{\delta}, \varepsilon_0, L_0)$-neck varies by at most a factor of $1 + L_0^{-1}$. (This can always be achieved by choosing ε_0 very small.) We then choose real numbers $\eta_0 > 0$ and K_0 so that the conclusion of Version A of the Neck Detection Lemma can be applied for each $\tilde{L} \in [100, L_0]$. In other words, if (p_0, t_0) satisfies $H(p_0, t_0) \geq \max\{K_0, H_1/\Theta\}$ and $\lambda_1(p_0, t_0) \leq \eta_0 H(p_0, t_0)$, and if the parabolic neighborhood $\hat{\mathcal{P}}(p_0, t_0, \tilde{L} + 4, 2\theta_0)$ is free of surgeries for some $\tilde{L} \in [100, L_0]$, then (p_0, t_0) lies at the center of an $(\hat{\alpha}(t_0), \hat{\delta}, \varepsilon_0, \tilde{L})$-neck in M_{t_0}.

In the next step, we set $\varepsilon_1 = \eta_0/10$. By Version A of the Neck Detection Lemma, we can find constants $\eta_1 < \eta_0$ and $K_1 > K_0$ such that the following holds: if (p_0, t_0) satisfies $H(p_0, t_0) \geq \max\{K_1, H_1/\Theta\}$, $\lambda_1(p_0, t_0) \leq \eta_1 H(p_0, t_0)$, and if the parabolic neighborhood $\hat{\mathcal{P}}(p_0, t_0, 104, 2\theta_0)$ is free of surgeries, then (p_0, t_0) lies at the center of an $(\hat{\alpha}(t_0), \hat{\delta}, \varepsilon_1, 100)$-neck in M_{t_0}.

Having chosen η_1, we next choose γ_0 and ρ so that the conclusion of Proposition 2.17 holds with $\eta = \eta_1$.

By Version B of the Neck Detection Lemma, we can find a number $K_2 > K_1$ such that the following holds. Suppose that (p_0, t_0) satisfies $H(p_0, t_0) \geq \max\{K_2, H_1/\Theta\}$ and $\lambda_1(p_0, t_0) \leq \eta_2 H(p_0, t_0)$, and the parabolic neighborhood $\hat{\mathcal{P}}(p_0, t_0, 104, 10^{-6}\Theta^{-2}\gamma_0^{-2})$ does not contain surgeries. Then, if we dilate the surface $\{x \in M_{t_0} : d_{g(t_0)}(p_0, x) \leq 100H(p_0, t_0)^{-1}\}$ by the factor $H(p_0, t_0)$, the resulting surface is $\varepsilon_1/10$-close to a product

$\Gamma \times [-100, 100]$ in the C^3-norm. Here, Γ is a closed, convex curve satisfying $L(\Gamma) \leq 3\pi$ and $\sup_\Gamma |\kappa - 1| \leq 1/100$.

Finally, we choose $H_1 \geq 1000\Theta K_2$, and define $H_2 = 1000\gamma_0 H_1$ and $H_3 = 10H_2$. Note that we may choose H_1 arbitrarily large (cf. [15, remark at the bottom of p. 209]).

We note that the distance between consecutive surgery times is bounded from below by

$$\frac{1}{3}(H_2^{-2} - H_3^{-2}) > \frac{1}{4}(1000\gamma_0 H_1)^{-2} = 10^{-6}\Theta^{-2}\gamma_0^{-2}\left(\frac{2H_1}{\Theta}\right)^{-2}$$

(cf. [15, p. 210]). In particular, if $H(p_0, t_0) \geq 2H_1/\Theta$ and $\lambda_1(p_0, t_0) \leq \eta_2 H(p_0, t_0)$, then the parabolic neighborhood $\hat{\mathcal{P}}(p_0, t_0, 104, 10^{-6}\Theta^{-2}\gamma_0^{-2})$ does not contain surgeries, so we may apply Version B of the Neck Detection Lemma.

As in [7], the Neck Detection Lemma requires that a certain parabolic neighborhood be free of surgeries, but this assumption can be removed when the curvature is at least $1000H_1$:

Proposition 3.1. *Suppose that M_t is a mean curvature flow with surgeries satisfying Assumption 2.5, where $\varepsilon \leq \bar{\varepsilon}$ and $L \geq 1000\Lambda$. Moreover, suppose that (p_0, t_0) satisfies $H(p_0, t_0) \geq 1000H_1$ and $\lambda_1(p_0, t_0) \leq \eta_0 H(p_0, t_0)$, where η_0 and H_1 are defined as above. Then p_0 lies at the center of an $(\hat{\alpha}(t_0), \hat{\delta}, \varepsilon_0, L_0)$-neck.*

Proof. We distinguish two cases:

Case 1: Suppose first that the parabolic neighborhood $\hat{\mathcal{P}}(p_0, t_0, 104, 2\theta_0)$ contains a point modified by surgery. By Proposition 2.16, we can find a point $q \in M_{t_0}$ and an open set $V \subset \{x \in M_{t_0} : H(x, t_0) \leq 40H_1\}$ such that $d_{g(t_0)}(p_0, q) \leq 104H(p_0, t_0)^{-1}$ and $\{x \in M_{t_0} : d_{g(t_0)}(q, x) \leq 500H_1^{-1}\} \subset V$. Clearly, $p_0 \in V$. Consequently, $H(p_0, t_0) \leq 40H_1$, contrary to our assumption.

Case 2: We now assume that the parabolic neighborhood $\hat{\mathcal{P}}(p_0, t_0, 104, 2\theta_0)$ is free of surgeries. Let $\tilde{L} \in [100, L_0]$ be the largest number such that $\hat{\mathcal{P}}(p_0, t_0, \tilde{L} + 4, 2\theta_0)$ is free of surgeries. By Version A of the Neck Detection Lemma, the point (p_0, t_0) lies at the center of an $(\hat{\alpha}, \hat{\delta}, \varepsilon_0, \tilde{L})$-neck N. If $\tilde{L} = L_0$, we are done. Hence, it remains to consider the case when $\tilde{L} < L_0$. In this case, the parabolic neighborhood $\hat{\mathcal{P}}(p_0, t_0, \tilde{L} + 5, 2\theta_0)$ must contain a point modified by surgery. By Proposition 2.16, we can find a point $q \in M_{t_0}$ and an open set $V \subset \{x \in M_{t_0} : H(x, t_0) \leq 40H_1\}$ such that $d_{g(t_0)}(p_0, q) \leq (\tilde{L} + 5)H(p_0, t_0)^{-1}$ and $\{x \in M_{t_0} : d_{g(t_0)}(q, x) \leq 500H_1^{-1}\} \subset V$. Since the set $\{x \in M_{t_0} : d_{g(t_0)}(p_0, x) \leq (\tilde{L} - 1)H(p_0, t_0)^{-1}\}$ is contained in N, we conclude that $\mathrm{dist}_{g(t_0)}(q, N) \leq 6H(p_0, t_0)^{-1} \leq 6H_1^{-1}$. Consequently, $N \cap V \neq \emptyset$. On the other hand, $H \geq \frac{1}{2}H(p_0, t_0) \geq 500H_1$ at each point of N, and $H \leq 40H_1$ at each point of V. This contradiction completes the proof of Proposition 3.1.

Theorem 3.2 (Neck Continuation Theorem). *Suppose that M_t is a mean curvature flow with surgery satisfying Assumption 2.5, where $\varepsilon \leq \bar{\varepsilon}$ and $L \geq 1000\Lambda$. Suppose that (p_0, t_0) satisfies $H(p_0, t_0) \geq 1000H_1$ and $\lambda_1(p_0, t_0) \leq \eta_1 H(p_0, t_0)$, where η_1 is defined as above and H_1 is sufficiently large. Then there exists a finite collection of points $p_1, \ldots, p_l$ with the following properties:*

- *For each $i = 0, 1, \dots, l$, the point p_i lies at the center of an $(\hat{\alpha}(t_0), \hat{\delta}, \varepsilon_0, L_0)$-neck $N^{(i)} \subset M_{t_0}$, and $H(p_i, t_0) \geq H_1$.*
- *For each $i = 1, \dots, l-1$, the point p_{i+1} lies on the neck $N^{(i)}$, and*

$$\mathrm{dist}_{g(t_0)}(p_{i+1}, \partial N^{(i)} \setminus N^{(i-1)}) \in [(L_0 - 100) H(p_i, t_0)^{-1}, (L_0 - 50) H(p_i, t_0)^{-1}].$$

- *Finally, at least one of the following four statements holds: either the union $\mathcal{N} = \bigcup_{i=1}^{l} N^{(i)}$ covers the entire surface; or $H(p_l, t_0) \in [H_1, 2H_1]$; or there exists a closed curve in $\mathcal{N} \cap \{x \in M_{t_0} : H(x, t_0) \leq 40H_1\}$ which is homotopically nontrivial in $\mathcal{N}$ and bounds a disk in $\{x \in M_{t_0} : H(x, t_0) \leq 40H_1\}$; or the outer boundary $\partial N^{(l)} \setminus N^{(l-1)}$ bounds a convex cap.*

Proof. By Proposition 3.1, the point p_0 lies at the center of an $(\hat{\alpha}(t_0), \hat{\delta}, \varepsilon_0, L_0)$-neck $N^{(0)} \subset M_{t_0}$. The construction of the points $p_1, p_2, \dots$ is by induction. Suppose that we have constructed points $p_1, \dots, p_k$ and necks $N^{(1)}, \dots, N^{(k)}$ with the following properties:

- For each $i = 0, 1, \dots, k$, the point p_i lies at the center of an $(\hat{\alpha}(t_0), \hat{\delta}, \varepsilon_0, L_0)$-neck $N^{(i)} \subset M_{t_0}$, and $H(p_i, t_0) \geq H_1$.
- For each $i = 1, \dots, k-1$, the point p_{i+1} lies on the neck $N^{(i)}$, and $\mathrm{dist}_{g(t_0)}(p_{i+1}, \partial N^{(i)} \setminus N^{(i-1)}) \in [(L_0 - 100) H(p_i, t_0)^{-1}, (L_0 - 50) H(p_i, t_0)^{-1}]$.

If $H(p_k, t_0) \in [H_1, 2H_1]$, then we are done. Hence, for the remainder of the proof, we will assume that $H(p_k, t_0) \geq 2H_1$. We break the discussion into several cases:

Case 1: Suppose that there exists a point $p \in N^{(k)}$ such that

$$\mathrm{dist}_{g(t_0)}(p, \partial N^{(k)} \setminus N^{(k-1)}) \in [(L_0 - 100) H(p_k, t_0)^{-1}, (L_0 - 50) H(p_k, t_0)^{-1}], \quad (\dagger)$$

and the parabolic neighborhood $\hat{\mathcal{P}}(p, t_0, L_0 + 4, 2\theta_0)$ contains a point modified by surgery. In this case, Proposition 2.16 implies that there exists a point $q \in M_{t_0}$ and an open set $V \subset \{x \in M_{t_0} : H(x, t_0) \leq 40H_1\}$ such that $d_{g(t_0)}(p, q) \leq (L_0 + 4) H(p, t_0)^{-1}$, $\{x \in M_{t_0} : d_{g(t_0)}(q, x) \leq 500 H_1^{-1}\} \subset V$, and V is diffeomorphic to a disk.

By our choice of ε_0 and L_0, the mean curvature on $N^{(k)}$ varies at most by a factor of $1 + L_0^{-1}$. Hence, $H(p_k, t_0) \leq (1 + L_0^{-1}) H(p, t_0)$. Since the set $\{x \in M_{t_0} : d_{g(t_0)}(p, x) \leq (L_0 - 100) H(p_k, t_0)^{-1}\}$ is contained in $N^{(k)}$, we conclude that

$$\begin{aligned} \mathrm{dist}_{g(t_0)}(q, N^{(k)}) &\leq (L_0 + 4) H(p, t_0)^{-1} - (L_0 - 100) H(p_k, t_0)^{-1} \\ &\leq (L_0 + 4)(1 + L_0^{-1}) H(p_k, t_0)^{-1} - (L_0 - 100) H(p_k, t_0)^{-1} \\ &\leq 200 H(p_k, t_0)^{-1} \leq 100 H_1^{-1}. \end{aligned}$$

Consequently, there exists a closed curve which is contained in $N^{(k)} \cap V$ and is homotopically nontrivial in $N^{(k)}$. Since V is diffeomorphic to a disk, this curve bounds a disk in V, and we are done.

Case 2: We now assume that the parabolic neighborhood $\hat{\mathcal{P}}(p, t_0, L_0 + 4, 2\theta_0)$ is free of surgeries for all points $p \in N^{(k)}$ satisfying $(\dagger)$. There are two possibilities:

Subcase 2.1: Suppose that there exists a point $p \in N^{(k)}$ satisfying (†) and $\lambda_1(p, t_0) \leq \eta_0 H(p, t_0)$. By Version A of the Neck Detection Lemma, the point p lies at the center of an $(\hat{\alpha}(t_0), \hat{\delta}, \varepsilon_0, L_0)$-neck N. Moreover, since $p \in N^{(k)}$ and $H(p_k, t_0) \geq 2H_1$, we have $H(p, t_0) \geq H_1$. Hence, we can set $p^{(k+1)} := p$ and $N^{(k+1)} := N$ and continue the process.

Subcase 2.2: Suppose that $\lambda_1(p, t_0) > \eta_0 H(p, t_0)$ for all points $p \in N^{(k)}$ satisfying (†). Let $\mathcal{N} = \bigcup_{i=0}^{k} N^{(i)}$, and let $\mathcal{A}$ be the set of all points $x \in \mathcal{N}$ satisfying $\text{dist}_{g(t_0)}(p, \partial N^{(k)} \setminus N^{(k-1)}) \geq (L_0 - 50) H(p_k, t_0)^{-1}$ and $\lambda_1(x, t_0) \leq \eta_1 H(x, t_0)$. The assumptions of Theorem 3.2 imply that the initial point p_0 belongs to $\mathcal{A}$, so $\mathcal{A}$ is nonempty. Let us consider a point p^* which has maximal intrinsic distance from p_0 among all points in $\mathcal{A}$.

Subcase 2.2.1: Suppose that the parabolic neighborhood $\hat{\mathcal{P}}(p^*, t_0, 104, 2\theta_0)$ contains a point modified by surgery. In this case, Proposition 2.16 implies that there exists a point $q \in M_{t_0}$ and an open set $V \subset \{x \in M_{t_0} : H(x, t_0) \leq 40H_1\}$ such that $d_{g(t_0)}(p^*, q) \leq 104H(p^*, t_0)^{-1}$, $\{x \in M_{t_0} : d_{g(t_0)}(q, x) \leq 500H_1^{-1}\} \subset V$, and V is diffeomorphic to a disk. Since $H(p^*, t_0) \geq H_1/2$, this implies

$$\begin{aligned}\{x \in M_{t_0} : d_{g(t_0)}(p^*, x) \leq 100H(p^*, t_0)^{-1}\} &\subset \{x \in M_{t_0} : d_{g(t_0)}(q, x) \leq 204H(p^*, t_0)^{-1}\} \\ &\subset \{x \in M_{t_0} : d_{g(t_0)}(q, x) \leq 500H_1^{-1}\} \subset V.\end{aligned}$$

Consequently, there exists a closed curve in $\mathcal{N} \cap V$ which is homotopically nontrivial in $\mathcal{N}$. This curve bounds a disk which is contained in V. Hence, we can again terminate the process.

Subcase 2.2.2: Suppose, finally, that the parabolic neighborhood $\hat{\mathcal{P}}(p^*, t_0, 104, 2\theta_0)$ is free of surgeries. In this case, Version A of the Neck Detection Lemma implies that the point p^* lies at the center of an $(\hat{\alpha}(t_0), \hat{\delta}, \varepsilon_1, 100)$-neck N^*. Clearly, $\lambda_1 \leq \varepsilon_1 H$ at each point on N^*. Consequently, the set N^* is disjoint from the set $\{p \in N^{(k)} :$ (†) holds$\}$. Furthermore, since p^* has maximal distance from p_0 among all points in $\mathcal{A}$, we conclude that $\lambda_1 \geq \eta_1 H \geq \eta_1 H_1/2$ on the part of $\mathcal{N}$ that lies between the neck N^* and the set $\{p \in N^{(k)} :$ (†) holds$\}$. In particular, the part of $\mathcal{N}$ that lies between N^* and the latter set has diameter $O(H_1^{-1})$.

Let $u : B_\sigma(p^*) \to \mathbb{R}$ be a local height function in ambient space such that $|\nabla u| = 1$ at each point on $B_\sigma(p^*)$ and ∇u agrees with the first eigenvector of the second fundamental form at p^*. Here, σ is a positive constant which depends only on the ambient manifold X. As above, we consider the flow on M_{t_0} generated by the vector field $\omega^T/|\omega^T|^2$, where $\omega = \nabla u$ and ω^T denotes the projection of ω to the tangent space of M_{t_0}. It was shown in the proof of Proposition 2.17 that $\frac{d}{dy}\langle \nu, \omega \rangle \geq \lambda_1 - |\bar{D}\omega|$ along each trajectory of this ODE. In particular, if H_1 is large enough, then the function $\langle \nu, \omega \rangle$ is increasing on the part of $\mathcal{N}$ that lies between the neck N^* and the set $\{p \in N^{(k)} :$ (†) holds$\}$. From this, we deduce that $\langle \nu, \omega \rangle \geq -\varepsilon_1$ for all points p in the latter set. Moreover, $\lambda_1(p, t_0) > \eta_0 H(p, t_0)$ for all such p. Putting these facts together (and using the fact that $\eta_0 \geq 10\varepsilon_1$), we conclude that $\langle \nu, \omega \rangle \geq 4\varepsilon_1$ for all $p \in N^{(k)}$ satisfying $\text{dist}_{g(t_0)}(p, \partial N^{(k)} \setminus N^{(k-1)}) \in [(L_0 - 100)H(p_k, t_0)^{-1}, (L_0 - 75)H(p_k, t_0)^{-1}]$.

We claim that the boundary curve $\partial N^{(k)} \setminus N^{(k-1)}$ bounds a convex cap. To prove this, we follow the argument of [15, pp. 215–216]. Choose a curve $\Gamma_0 \subset \{p \in N^{(k)} : \text{dist}_{g(t_0)}(p, \partial N^{(k)} \setminus N^{(k-1)}) \in [(L_0-100)H(p_k,t_0)^{-1}, (L_0-75)H(p_k,t_0)^{-1}]\}$ contained in a level set of u. For each point on Γ_0, we continue to follow the integral curves of the vector field $\omega^T/|\omega^T|^2$. This gives a family of curves $\Gamma_y \subset M_{t_0}$, each of which is contained in a level set of u. The curves Γ_y are well-defined for $y \in [0, y_{\max})$. Moreover, there exists a point $p \in \Gamma_0$ such that $\nu(\gamma(y,p)) \to \omega$ as $y \to y_{\max}$.

Following the arguments in [15, p. 215], we can show that

$$\langle \nu, \omega \rangle < 1, \qquad \lambda_1 \geq \eta_2 H, \qquad H > 2H_1/\Theta, \qquad \langle \nu, \omega \rangle > \varepsilon_1 \tag{$\star$}$$

for all $y \in [0, y_{\max})$. Indeed, the inequalities in ($\star$) are clearly satisfied for $y = 0$. If one of the inequalities in ($\star$) fails for some $y > 0$, we consider the smallest value of y for which that happens. The first inequality in ($\star$) cannot fail first by definition of $y_{\max}$. If the second inequality in ($\star$) is the first one to fail, then $\lambda_1 \leq \eta_2 H$. Since $H \geq 2H_1/\Theta$, we may apply Version B of the Neck Detection Lemma to conclude that we are $\varepsilon_1/10$-close to a Cartesian product, but this is ruled out by the fourth inequality in ($\star$). If the third inequality in ($\star$) is the first one to fail, we obtain a contradiction with [7, Proposition 2.18]. Finally, as long as ($\star$) holds, we have $\frac{d}{dy}\langle \nu, \omega \rangle \geq \lambda_1 - |\bar{D}\omega| \geq \eta_2 H - |\bar{D}\omega| \geq 2\eta_2 H_1/\Theta - |\bar{D}\omega|$. Note that η_2 and Θ have already been fixed at this stage. Hence, if we choose the curvature threshold H_1 sufficiently large, then $\langle \nu, \omega \rangle$ is increasing along each trajectory of the ODE. This implies that the fourth inequality in ($\star$) cannot fail first. Thus, the inequalities in ($\star$) hold for all $y \in [0, y_{\max})$. Consequently, the union of the curves Γ_y is a convex cap, and we can terminate the process. This completes the construction of the sequence $p_1, p_2, \ldots$.

If the sequence $p_1, p_2, \ldots$ terminates after finitely many steps, then the theorem is proved. Otherwise, the necks $N^{(1)}, N^{(2)}, \ldots$ will eventually cover the entire surface. This completes the proof of Theorem 3.2.

Having completed the proof of the Neck Continuation Theorem, we are now ready to implement the surgery algorithm of Huisken and Sinestrari [15]. Starting from the given initial surface M_0, we run the mean curvature flow until the maximum of the mean curvature reaches the threshold H_3 for the first time. Let us denote this time by T_1. By Proposition 2.4, the inscribed radius and the outer radius are bounded from below by $\alpha(t)/H$ for $0 \leq t \leq T_1$. Moreover, it is easy to see that the surfaces M_t are outward-minimizing for $0 \leq t \leq T_1$. Therefore, Assumption 2.5 is satisfied for $0 \leq t \leq T_1$. Consequently, we may apply the Neck Detection Lemma and the Neck Continuation Theorem for $0 \leq t \leq T_1$. At time T_1, we perform surgeries on suitably chosen $(\hat{\alpha}(T_1), \hat{\delta}, \varepsilon_0, L_0)$-necks. This allows us to remove all regions where the mean curvature is between H_2 and H_3. Immediately after surgery, the maximum of the mean curvature drops to a level below H_2. We then run the flow again until the maximum of the mean curvature reaches H_3 for the second time. Let us denote this time by T_2. We claim that, for $0 \leq t \leq T_2$, the flow satisfies Assumption 2.5 with $\varepsilon = \varepsilon_0$ and $L = L_0$. Indeed, Theorem 2.3 implies that the inscribed radius and the outer radius of the surface M_{T_1+} are bounded from below by $\alpha(t)/H$, and this property continues to hold for all $T_1 < t \leq T_2$ by Proposition 2.4. Furthermore, the outward-minimizing property follows from work of Head [12, Lemma 5.2]. Therefore, Assumption 2.5 is satisfied for $0 \le t \le T_2$ with $\varepsilon = \varepsilon_0$ and $L = L_0$. Hence, we can again apply the Neck Detection Lemma and the Neck Continuation Theorem for $0 \le t \le T_2$. By performing surgery on suitably chosen $(\hat{\alpha}(T_2), \hat{\delta}, \varepsilon_0, L_0)$-necks, we can push the maximum of the mean curvature below H_2. We then restart the flow again. This process can now be repeated until we reach time T or the solution becomes extinct.

4. Longtime behavior of the flow

In this final section, we prove Theorem 1.2. Let $\bar{M}_t = \partial \bar{\Omega}_t$ denote the level-set solution of mean curvature flow with initial surface M_0. If the level-set solution becomes extinct in finite time, then the solution of mean curvature flow with surgery also becomes extinct in finite time. Hence, it is enough to consider the case that the level-set flow does not become extinct in finite time. We recall the following fundamental theorem due to Brian White:

Theorem 4.1 (Brian White [21, Theorem 11.1]). *Each connected component of $\bar{\Omega}_\infty = \bigcap_{t \ge 0} \bar{\Omega}_t$ is either an embedded stable minimal surface or a compact domain bounded by one or more embedded stable minimal surfaces. Moreover, the level-set flow $\bar{M}_t$ is smooth for t sufficiently large. Finally, $\bar{M}_t$ converges smoothly to $\bar{\Omega}_\infty$. Near a point $\bar{p} \in \bar{\Omega}_\infty$, the convergence is locally one-sheeted if the connected component of $\bar{\Omega}_\infty$ containing $\bar{p}$ has nonempty interior; otherwise the convergence is locally two-sheeted.*

Let us fix a positive real number T such that the level-set flow $\bar{M}_t$ is smooth for $t \in (T-2, \infty)$. By Theorem 1.1, there exists a solution of mean curvature flow with surgery on the interval $[0, T]$. Let $\mathcal{M}^{(j)}$ be a sequence of mean curvature flows with surgery starting from the initial surface M_0 with curvature thresholds $H_1^{(j)} \to \infty$. We assume that the surgery parameters δ, $\hat{\delta}$, Λ, ε, and L are independent of j and are chosen so that the monotonicity formula in [6] holds. It follows from work of Lauer [16] that the flows $\mathcal{M}^{(j)}$ converge to the level-set flow $\bar{\mathcal{M}}$ in the Hausdorff sense. Using the outward-minimizing property, we conclude that the flows $\mathcal{M}^{(j)}$ converge to $\bar{\mathcal{M}}$ in the sense of geometric measure theory for each $t \in (T-2, T]$ (see also [12]). Since the flow $\bar{\mathcal{M}}$ is smooth for all $t \in (T-2, T]$, the results in [6] imply that the flow $\mathcal{M}^{(j)}$ is free of surgeries for $t \in (T-1, T]$ provided that j is sufficiently large. Moreover, as $j \to \infty$, the flows $\mathcal{M}^{(j)}$ converge smoothly to $\bar{\mathcal{M}}$ for all $t \in (T-1, T]$. Furthermore, by Lauer's result, $\bar{\Omega}_{T-\tau_j} \subset \Omega_T^{(j)} \subset \bar{\Omega}_T^{(j)}$ for some sequence $\tau_j \to 0$.

For each j, we consider the unique maximal solution of the smooth mean curvature flow with initial surface $M_T^{(j)}$. Let us denote this solution by $\{M_t^{(j)} : t \in [T, T_j)\}$. Clearly, $\bar{\Omega}_{t-\tau_j} \subset \Omega_t^{(j)} \subset \bar{\Omega}_t^{(j)}$ for all $t \in [T, T_j)$. We next show that these flows are defined for all time and have uniformly bounded curvature.

Proposition 4.2. *We have*

$$\limsup_{j \to \infty} \sup_{t \in [T, T_j)} \sup_{M_t^{(j)}} |A| < \infty.$$

In particular, $T_j = \infty$ if j is sufficiently large.

Proof. Suppose that the upper limit is infinite. Then there exists a sequence of points $p_j \in M^{(j)}_{t_j}$ such that $t_j \in [T, T_j)$ and $|A(p_j, t_j)| \to \infty$. Clearly, $t_j \to \infty$. After passing to a subsequence, we may assume that the points p_j converge to a point $\bar{p}$ in ambient space. Note that $\bar{p} \in \partial\bar{\Omega}_\infty$. We next consider a small geodesic ball $B_{2\sigma}(\bar{p})$ in ambient space. After passing to a subsequence, we may assume that the points p_j all lie in one connected component of $B_{2\sigma}(\bar{p}) \setminus \bar{\Omega}_\infty$. Let us denote this connected component by U. Note that U is close to a half-ball if $\sigma > 0$ is sufficiently small. It follows from Brian White's theorem that the flows $\bar{M}_{s+t_j} \cap U$, $s \in [-1, 0]$, converge smoothly to $\partial\bar{\Omega}_\infty \cap B_{2\sigma}(\bar{p})$, and the convergence is one-sheeted. Using the inclusion $\bar{\Omega}_{s+t_j-\tau_j} \subset \Omega^{(j)}_{s+t_j} \subset \bar{\Omega}^{(j)}_{s+t_j}$ together with the outward-minimizing property of the set $\Omega^{(j)}_{s+t_j}$, we conclude that the flows $M^{(j)}_{s+t_j} \cap U$, $s \in [-1, 0]$, converge to $\partial\bar{\Omega}_\infty \cap B_{2\sigma}(\bar{p})$ in the sense of geometric measure theory, and the limit has multiplicity 1. By Brakke's local regularity theorem, the flows $M^{(j)}_{s+t_j} \cap U \cap B_\sigma(\bar{p})$, $s \in [-1, 0]$, converge smoothly to $\partial\bar{\Omega}_\infty \cap B_\sigma(\bar{p})$. This contradicts the fact that $|A(p_j, t_j)| \to \infty$.

Thus, we conclude that the upper limit in the statement is finite. This immediately implies that $T_j = \infty$ if j is sufficiently large.

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