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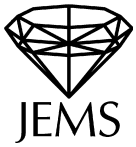

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The calculus of thermodynamical formalism

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Abstract. Given an onto map T acting on a metric space Ω and an appropriate Banach space of functions $\mathcal{X}(\Omega)$, one classically constructs for each potential $A \in \mathcal{X}$ a transfer operator $\mathscr{L}_A$ acting on $\mathcal{X}(\Omega)$. Under suitable hypotheses, it is well-known that $\mathscr{L}_A$ has a maximal eigenvalue λ_A, has a spectral gap and defines a unique Gibbs measure μ_A. Moreover there is a unique normalized potential of the form $B = A + f - f \circ T + c$ acting as a representative of the class of all potentials defining the same Gibbs measure.

The goal of the present article is to study the geometry of the set $\mathcal{N}$ of normalized potentials, of the normalization map $A \mapsto B$, and of the Gibbs map $A \mapsto \mu_A$. We give an easy proof of the fact that $\mathcal{N}$ is an analytic submanifold of $\mathcal{X}$ and that the normalization map is analytic; we compute the derivative of the Gibbs map; and we endow $\mathcal{N}$ with a natural weak Riemannian metric (derived from the asymptotic variance) with respect to which we compute the gradient flow induced by the pressure with respect to a given potential, e.g. the metric entropy functional. We also apply these ideas to recover in a wide setting existence and uniqueness of equilibrium states, possibly under constraints.

Keywords. Transfer operators, equilibrium states, entropy, regularity, Wasserstein space

1. Introduction

The goal of this article is to propose a differential-geometric approach to the thermodynamical formalism for maps whose transfer operators satisfy the conclusion of the Ruelle–Perron–Frobenius theorem (for example, expanding maps).

Many of our results stated below are already well-known for classes of dynamical systems; let us stress what we believe are our main contributions:

- we propose a point of view, based on differential geometry in the space of potentials, which provides new and efficient[1] proofs of strong results (e.g. Fréchet dérivatives are computed instead of mere directional derivatives) valid in a fairly general framework,

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[1] See for example Corollary 3.6.

- we give a formula for the derivative of $\int \varphi \, \mathrm{d}\mu_A$ with respect to A (Theorem C), from which the links between (among others) the convexity of pressure and the variance are formalized through a natural Riemannian metric (Theorem D),
- we use this metric to define the gradient of natural functionals, which leads to a gradient flow modeling a system out of equilibrium (Section 7.3.2),
- we show that the map sending a potential to its Gibbs measure is very far from being smooth in the sense of optimal transportation (Theorem E),
- we improve a result of Kucherenko and Wolf, identifying precisely the equilibrium state of a potential under a finite set of linear constraints (Theorem G, see also Example H).

1.1. Transfer operator, Gibbs measures and normalization

Let Ω be a metric space and $T : \Omega \to \Omega$ be an onto map, defining a discrete-time dynamical system. The model cases we have in mind are uniformly expanding maps such as $x \mapsto dx \bmod 1$ on the circle or the shift over right-infinite words on a finite alphabet, but we shall consider a very general setting by mostly requiring[2] that for each potential A in an open set $\mathcal{U}$ of a suitable function space $\mathcal{X}(\Omega)$, the Ruelle–Perron–Frobenius theorem holds for the transfer operator $\mathscr{L}_A : \mathcal{X}(\Omega) \to \mathcal{X}(\Omega)$ defined by

$$\mathscr{L}_A f(x) = \sum_{T(y)=x} e^{A(y)} f(y),$$

i.e. $\mathscr{L}_A$ has a positive, simple leading eigenvalue λ_A associated with a positive eigenfunction h_A; its dual operator $\mathscr{L}_A^*$ acting on measures has a unique eigenprobability ν_A; and $\mathscr{L}_A$ has a spectral gap below λ_A. Then the measure $\mu_A = h_A \nu_A$ (where the multiplicative constant in h_A is chosen so as to make μ_A a probability measure) is an invariant measure for T, which we will call here the *Gibbs measure* associated to the potential A.

The open set $\mathcal{U}$ can (and will) be chosen to be invariant under translations by constants and coboundaries, and two different potentials A, B which differ by a constant and a coboundary define the same Gibbs measure. One can thus parametrize the set of Gibbs measures by the quotient space $\widehat{\mathcal{U}} = \mathcal{U}/\mathcal{C}$ where

$$\mathcal{C} = \{c + g - g \circ T \mid c \in \mathbb{R},\ g \in \mathcal{X}(\Omega)\}.$$

The subset $\mathcal{N} \subset \mathcal{U}$ of normalized potentials (i.e. such that $\lambda_A = 1$ and $h_A = \mathbf{1}$) contains exactly one representative of each class modulo $\mathcal{C}$, making $\mathcal{N}$ another natural parameter space for Gibbs measures.

Our main object of study is the first-order variation of μ_A with respect to A, which means we consider $\mu_{A+\zeta}$ for small $\zeta \in \mathcal{X}(\Omega)$; of course, adding to ζ a constant and a coboundary will have no effect. In the literature, one often requires ζ to satisfy the normalizing condition $\int \zeta \, \mathrm{d}\mu_A = 0$ to get rid of the constant, and then considers ζ up to coboundaries. We argue that instead, it makes things simpler and clearer to go fully with one of these points of view: either consider both A and ζ modulo $\mathcal{C}$, or restrict the

[2] See Section 2 for the precise hypotheses.

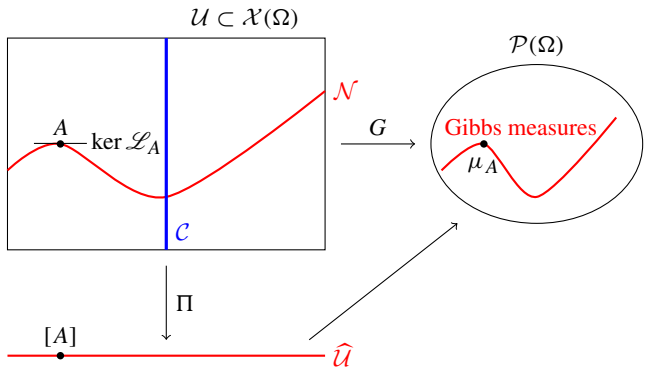

Fig. 1. Potentials and Gibbs measures.

considerations to normalized A and constrain ζ to be tangent to $\mathcal{N}$. Our first result gives a solid ground to this principle (Theorem 3.4, see Figure 1):

Theorem A. *The set $\mathcal{N}$ of normalized potentials is an analytic submanifold of $\mathcal{U}$ and its tangent space at A is* $\ker \mathscr{L}_A$*, which is a topological complement to $\mathcal{C}$.*

From this we will easily deduce the analyticity and derivative of several important maps. Consider:

- the *normalization map* $N : \mathcal{U} \to \mathcal{N}$ which sends A to the unique normalized potential in its class modulo $\mathcal{C}$,
- the *leading eigenvalue map* $\Lambda : A \mapsto \lambda_A$,
- the *leading eigenfunction map* $H : A \mapsto h_A$ (suitably normalized),
- the *Gibbs map* $G : A \mapsto \mu_A$ taking its values in $\mathcal{X}(\Omega)^*$ with the convention that $\mu_A(\varphi) = \int \varphi \, \mathrm{d}\mu_A$.

Then we get (Theorem 3.5 and Corollaries 3.6 and 3.7)

Corollary B. *The maps N, Λ, H, G are analytic and for all $A \in \mathcal{U}$:*

- *the differential DN_A of N at A is the linear projection on* $\ker \mathscr{L}_{N(A)}$ *along $\mathcal{C}$,*
- *$D(\log \Lambda)_A = \mu_A$ as a linear form on $\mathcal{X}(\Omega)$, i.e.*

$$\frac{\mathrm{d}}{\mathrm{d}t} \log \lambda_{A+t\zeta}\Big|_{t=0} = \int \zeta \, \mathrm{d}\mu_A \quad \forall A, \zeta \in \mathcal{X}(\Omega).$$

The analyticity of these maps and the derivative of $\log \Lambda$ are well-known for many dynamical systems,[3] but our framework is quite general, based mainly on the spectral gap, and our method pretty elementary: we only use basic differential calculus, not complex analysis or Kato's theory of regularity of eigendata for operators (as done, for example, in

[3] For a historical account of the problem, see the introduction of [BCV12] and references therein (among others [PP90], [Ma90], [BS01]).

[PP90], [dS+14]). We apply this idea only to variation of the potential A, not to variations of the map T. This latter problem is very important, appearing in the abundant literature on "linear response", and we believe that our point of view could apply there too, as long as the transfer operator varies smoothly (or analytically) with the dynamics T (which is not a mild hypothesis).

1.2. From integral differentiation to a Riemannian metric

Both derivatives above are really easy to obtain, but the derivative of G is slightly more complicated; it will be convenient to use the notation $\varphi_A = \varphi - \int \varphi \, \mathrm{d}\mu_A$ (and similarly ζ_A, η_A) for the translate of a function φ (or ζ, η) which has vanishing average with respect to the Gibbs measure of the potential A. We obtain the following (Theorem 4.1):

Theorem C. *For all $A \in \mathcal{U}$ and $\varphi, \zeta \in \mathcal{X}(\Omega)$ we have*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \varphi \, \mathrm{d}\mu_{A+t\zeta} \bigg|_{t=0} = \int (I - \mathscr{L}_{N(A)})^{-1}(\varphi_A) \cdot DN_A(\zeta) \, \mathrm{d}\mu_A .$$

(Note that of course the left-hand side is $DG_A(\zeta) \in \mathcal{X}(\Omega)^*$ applied at φ.)

This derivative can then be expressed in various forms using standard computations (see Sections 4 and 5), and some interesting connections appear clearly.

Theorem D. *All the expressions*

$$\begin{aligned}
\langle \zeta, \eta \rangle_A &= D^2(\log \Lambda)_A(\zeta, \eta), \\
\langle \zeta, \eta \rangle_A &= \frac{\mathrm{d}}{\mathrm{d}t} \int \eta \, \mathrm{d}\mu_{A+t\zeta} \bigg|_{t=0}, \\
\langle \zeta, \zeta \rangle_A &= \mathrm{Var}(\zeta_A, \mu_A) := \lim_{n\to\infty} \frac{1}{n} \int \Big(\sum_{i=0}^{n-1} \zeta_A \circ T^i \Big)^2 \mathrm{d}\mu_A, \\
\langle \zeta, \eta \rangle_A &= \int \zeta \eta \, \mathrm{d}\mu_A \quad \text{whenever } A \in \mathcal{N},\ \zeta, \eta \in T_A\mathcal{N}.
\end{aligned}$$

define the same analytic map $A \mapsto \langle \cdot, \cdot \rangle_A$ from $\mathcal{U}$ to the Banach space of symmetric linear 2-forms on $\mathcal{X}(\Omega)$ such that $\langle \cdot, \cdot \rangle_A$ is positive-semi-definite with kernel $\mathcal{C}$ for all A. This map induces by restriction a weak Riemannian metric on $\mathcal{N}$, and then by projection a weak Riemannian metric on $\widehat{\mathcal{U}} = \mathcal{U}/\mathcal{C}$.

The metric $\langle \cdot, \cdot \rangle_A$ is thus a close cousin to McMullen's variance metric introduced in the context of Teichmüller space [McM08][4] (up to a conformal rescaling by entropy), contains the derivative of the Gibbs map, measures the convexity of $\log \Lambda$ and extends the $L^2(\mu_A)$ metric on $\mathcal{N}$ at the same time. We show that the Central Limit Theorem holds in our general setting (Theorem 5.6), justifying the name "variance" for $\langle \zeta, \zeta \rangle_A$.

[4] See also [BCS15] by Bridgeman, Canary and Sambarino and references therein, and [PS14] by Pollicott and Sharp for an analogous metric of Weil–Petersson type on spaces of metric graphs.

In the closing Section 8, we show a concrete example of this metric approach. When the dynamics is just the shift on the Bernoulli space $\{1, 2\}^{\mathbb{N}}$ and the potential depends only on two coordinates, we exhibit the metric explicitly and we compute the curvature (Proposition 8.1), which is positive. In analogy with the work of McMullen, one could conjecture that when our metric is rescaled by the entropy, the curvature is negative, but we show that this is not the case.

1.3. Optimal transportation approach to differentiability of measure-valued maps

Above we took a very common point of view, considering the Gibbs map $G : A \mapsto \mu_A$ as taking values in (an affine subspace of) the Banach space $\mathcal{X}(\Omega)^*$, yielding an obvious differential structure in which each $\varphi \in \mathcal{X}(\Omega)$ defines a "coordinate function" by $\mu_A \mapsto \mu_A(\varphi) = \int \varphi \, \mathrm{d}\mu_A$. We call this the "affine differential structure".

However, this is not the only way to study the regularity of such a map, and in Section 6 we study the "Wasserstein differential structure" aspect of the question. One can see G as taking values in the subset $\mathcal{P}_T(\Omega)$ of T-invariant measures in the set $\mathcal{P}(\Omega)$ of all probability measures, and use the differential framework based on the 2-Wasserstein distance W_2 from optimal transportation which has been developed in the last fifteen years.[5] This point of view proved useful in the study of the action of expanding circle maps near the absolutely continuous invariant measure by one of the present authors (see [Kl13, Kl15a]); here, we show that with the 2-Wasserstein metric the Gibbs map $A \mapsto \mu_A$ is far from being differentiable even in the simplest smoothest case.

Theorem E. *Assume $T = x \mapsto dx \bmod 1$ is the standard d-self-covering map of the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ and $\mathcal{X}(\Omega)$ is the space of α-Hölder functions for some $\alpha \in (0, 1]$. Then given any smooth path (A_t) in $\mathcal{X}(\Omega)$, the path (μ_{A_t}) of Gibbs measures is not absolutely continuous in $(\mathcal{P}(\Omega), W_2)$ unless it is constant.*

Recall that a path in a metric space defined on an interval I is said to be *absolutely continuous* when it has a metric speed in $L^1(I)$: this is a very weak regularity, so that Theorem E can be interpreted as meaning that a small perturbation of the potential induces a brutal reallocation of mass in the sense of W_2. This contrasts with a Lipschitz regularity result obtained for the 1-Wasserstein metric in [KLS14] (but note that W_1 does not yield a differential structure).

1.4. Applications to equilibrium states

We end this introduction by presenting some applications and illustrations of our differential calculus setting and the metric obtained above. In these applications, because we rely on convexity of some functionals, we assume the spectral gap holds on the whole space (i.e. $\mathcal{U} = \mathcal{X}(\Omega)$).

[5] The full story does not fit in a footnote, but let us mention the important cornerstones which are the works of Otto [O01], Benamou and Brenier [BB00], and Ambrosio, Gigli and Savaré [AGS05]; see also [Vi03], [Vi09] and [Gi11].

First we show that our approach makes it easy to derive the existence and uniqueness of equilibrium states. Consider the following optimization problems and induced functionals:

$$\mathrm{h}_{\mathcal{X}}(\mu) := \inf_{A\in\mathcal{X}(\Omega)} \left(\log \lambda_A - \int A \,\mathrm{d}\mu \right) \qquad \text{for } \mu \in \mathcal{P}_T(\Omega),$$

$$\mathrm{Pr}(B) := \sup_{\mu\in\mathcal{P}_T(\Omega)} \left(\mathrm{h}_{\mathcal{X}}(\mu) + \int B \,\mathrm{d}\mu \right) \qquad \text{for } B \in \mathcal{X}(\Omega)$$

(recall that $\mathcal{X}(\Omega)$ is a suitable space of potentials $\Omega \to \mathbb{R}$ which we can choose with some freedom).

Theorem F. *For all $B \in \mathcal{X}(\Omega)$, the supremum in the definition of* $\mathrm{Pr}(B)$ *is uniquely realized by μ_B, and* $\mathrm{Pr}(B) = \log \lambda_B$.

We show in Remark 7.5 that for the case of the classical thermodynamical formalism in the sense of [PP90] (the shift acting on the Bernoulli space) and for any invariant probability μ we have equality between $\mathrm{h}_{\mathcal{X}}(\mu)$ and the metric entropy of μ. In this case the pressure Pr defined above also coincides with the usual topological pressure. We consider however more general hypotheses in our reasoning. We will refer to $\mathrm{h}_{\mathcal{X}}$ and Pr as "entropy" and "pressure" from now on.

We then observe that the metric $\langle\cdot,\cdot\rangle_A$ enables us to define the gradient of various natural dynamical quantities, including entropy and pressure (see Proposition 7.10). This gives a meaning to the gradient flow of the functional

$$A \mapsto \mathrm{h}_{\mathcal{X}}(\mu_A) + \int B \,\mathrm{d}\mu_A$$

obtained by composing G with the functional defining the pressure. This gradient flow has a *linear* form when expressed in the quotient space $\widehat{\mathcal{X}}$ and can serve as a model for non-equilibrium dynamics, according to which a system out of equilibrium behaves just like a system at equilibrium with a varying potential (Section 7.3.2). In case of a mere change in the temperature of the system's environment, this model predicts the physically sound property that the system evolves only in its temperature (Remark 7.11).

As a consequence of Theorems D and F we obtain several results related to work of Kucherenko and Wolf. The first result, obtained in [KW14] under somewhat different hypotheses, is a prescription result. Given a tuple $\Phi = (\varphi_1, \ldots, \varphi_K)$ of test functions in $\mathcal{X}(\Omega)$, the "rotation vector"

$$\mathrm{rv}(\mu) = \left(\int \varphi_1 \,\mathrm{d}\mu, \ldots, \int \varphi_K \,\mathrm{d}\mu \right)$$

of a T-invariant measure runs over some convex set $\mathrm{Rot}(\Phi) \subset \mathbb{R}^K$. The result is then that for all base potentials B, every interior value of $\mathrm{Rot}(\Phi)$ can be realized uniquely as the rotation vector of the Gibbs measure of a potential of the form $B + a_1\varphi_1 + \cdots + a_K\varphi_K$ (Theorem 7.13).

The second result states existence and uniqueness of equilibrium states under linear constraints; it is very close to Theorem B of [KW15], but even disregarding the difference in the hypotheses we obtain a more precise description of the equilibrium state: the parameter s of [KW15] is always equal to 1. In other words:

Theorem G. *Let* $\Phi = (\varphi_1, \dots, \varphi_K) \in \mathcal{X}(\Omega)$ *be such that* $0 \in \operatorname{int}\operatorname{Rot}(\Phi)$. *Given any* $B \in \mathcal{X}(\Omega)$, *the restriction of*

$$P_B : \mu \mapsto \mathrm{h}_{\mathcal{X}}(\mu) + \int B \, \mathrm{d}\mu$$

to the set $\mathcal{P}_T[\Phi]$ *of invariant measures realizing* $\int \varphi_k \, \mathrm{d}\mu = 0$ *for all* k *is uniquely maximized at the unique Gibbs measure in* $\mathcal{P}_T[\Phi]$ *that is defined by a potential of the form* $B + a_1\varphi_1 + \cdots + a_K\varphi_K$.

We also recover Theorem B of [KW14] (the supremum of entropy of measures realizing a given vector in the interior of $\operatorname{Rot}(\Phi)$ depends analytically on the vector, Corollary 7.16), and by nature our method could be applied to more general constraints (e.g. requiring $\operatorname{rv}(\mu)$ to belong to some submanifold of $\operatorname{Rot}(\Phi)$).

Theorem G notably shows that when T is the shift map and the test functions and the potential B all depend only on n coordinates, so does the potential of the constrained equilibrium state, which is thus an $(n-1)$-Markovian measure (Remark 7.17, which also follows from the results of [KW15] but is not stated there).

This result is precise enough to yield explicit solutions to some concrete maximizing questions, which as far as we know would be difficult to solve without it. Let us give a toy example which turns out to have a non-trivial answer.

Example H. *Assume* T *is the shift map on* $\Omega = \{0, 1\}^{\mathbb{N}}$. *Among shift-invariant measures* μ *such that* $\mu(01*) = 2\mu(11*)$, *the Markov measure associated to the transition probabilities*

$$\begin{aligned} \mathbb{P}(0 \to 0) &= 1 - a, \quad & \mathbb{P}(0 \to 1) &= a, \\ \mathbb{P}(1 \to 0) &= 2/3, & \mathbb{P}(1 \to 1) &= 1/3, \end{aligned}$$

where a *is the only real solution to*

$$(1-a)^5 = \tfrac{4}{27}a^2 \qquad (a \simeq 0.487803),$$

uniquely maximizes entropy.

As a final remark, we mention that optimization problems such as the one we solve in Theorem G appear naturally in multifractal analysis: see [BS01], [BSS02], [Cli14]. Our approach might lead to explicit computations in that field.

2. Notation and preliminaries

We shall consider the thermodynamical formalism associated with a discrete-time, continuous-space dynamical system. The phase space will be denoted by Ω, and will be assumed to be a metric space with metric d. The time evolution is then described by a map $T : \Omega \to \Omega$ which will be assumed to be an onto map (it will often be finite-to-one). We will denote by $\mathcal{P}(\Omega)$ the set of probability measures on Ω, and by $\mathcal{P}_T(\Omega)$ the subset of T-invariant measures.

Typical examples are the shift on $\mathcal{A}^{\mathbb{N}}$ where $\mathcal{A}$ is a finite alphabet, and the maps $x \mapsto dx \bmod 1$ acting on the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. Our results also apply, for example, to one-sided subshifts of finite type with Hölder potentials (see [PP90]) and to piecewise expanding maps of the interval with bounded variation potentials. The end of the section is dedicated to a few examples.

Remark 2.1. We also want to consider cases such as the tent map

$$x \mapsto \begin{cases} 2x & \text{if } x \le 1/2, \\ 2 - 2x & \text{if } x \ge 1/2, \end{cases}$$

on the interval $[0, 1]$. This map has a particularity shared with other finite-to-one maps: one point, $1/2$, has only one inverse image while the neighboring points have two. This will make it necessary to adjust some of the definitions below. Let us formalize a property of the tent map which we will refer to when we explain these modifications: the tent map *has local inverse branches* in the sense that for all $x \in \Omega$ there is an integer $d \ge 2$ (to be implicitly taken minimal), a neighborhood V of x and continuous maps $y_k : V \to V_k \subset \Omega$ (where $k \in \{1, \dots, d\}$) such that for all $x' \in V$ we have

$$T^{-1}(x') = \{y_1(x'), \dots, y_d(x')\}.$$

Remark 2.2. The case of T invertible does not directly fit into our framework: indeed, the sum in the transfer operator then only has one term, and the only possible normalized potential is the constant 0. Then $\mathcal{N}$ is reduced to a point and studying its differential geometry makes little sense. It might be the case, however, that through coding and the correspondence between one-sided and two-sided subshifts the study of some invertible dynamical systems could benefit from our work.

2.1. Working hypotheses

2.1.1. Space of potentials. The first set of assumptions we make concerns the regularity of potentials; in designing the hypotheses below we have tried to keep them general enough not to rule out discontinuous potentials; e.g. in some settings bounded variation functions are meaningful (in particular when T is only piecewise continuous).

Throughout the article we fix a space $\mathcal{X}(\Omega)$ of functions, endowed with a norm $\|\cdot\|$, satisfying the following.

(H1) $\mathcal{X}(\Omega)$ is a Banach space of Borel-measurable, bounded functions $\Omega \to \mathbb{R}$, which includes all constant functions; for all $f, g \in \mathcal{X}(\Omega)$ we have

$$\|fg\| \le \|f\| \, \|g\|;$$

for every $f \in \mathcal{X}(\Omega)$ that is positive and bounded away from 0, the function $\log f$ also lies in $\mathcal{X}(\Omega)$; and for some constant C,

$$\|f\| \ge C \sup_{x \in \Omega} |f(x)|.$$

In particular for each probability measure μ on Ω, the linear form defined by $f \mapsto \int f \, \mathrm{d}\mu$ is continuous: in other words, every probability measure can be seen as an element of $\mathcal{X}(\Omega)^*$ (equipped with its operator norm).

Note that since we assume $\mathcal{X}(\Omega)$ to be a Banach *algebra*, whenever $f \in \mathcal{X}(\Omega)$ we have $e^f \in \mathcal{X}(\Omega)$. In particular, if $f \in \mathcal{X}(\Omega)$ is positive and bounded away from 0, then $1/f = e^{-\log f}$ is also in $\mathcal{X}(\Omega)$.

Remark 2.3. In some circumstances, one works with a norm satisfying only the weak multiplicativity condition $\|fg\| \leq C\|f\|\,\|g\|$ for some positive constant C. Then one can define a new, equivalent norm $\|\cdot\|' = C\|\cdot\|$ which is multiplicative.

Remark 2.4. We work here with a *real* Banach algebra, so that we do not have to enter into subtleties involving branches of the logarithm. It would certainly be possible to adjust most of our work to the case of complex potentials, but we felt it was not worth the extra pages and cases to consider. Note that we prove the usual Central Limit Theorem below without having to resort to complex potentials.

Example 2.5. The space $\mathrm{Hol}_\alpha(\Omega)$ of α-Hölder functions (for some $\alpha \in (0, 1]$) with its usual norm

$$\|f\|_\alpha = \sup_{x\in\Omega} |f(x)| + \sup_{x\neq y\in\Omega} \frac{|f(x) - f(y)|}{d(x, y)^\alpha}$$

satisfies (H1). When $\alpha = 1$, we get the space $\mathrm{Lip}(\Omega)$ of Lipschitz-continuous functions. Note that d^α is a distance on Ω, and $\mathrm{Hol}_\alpha(\Omega)$ coincides with $\mathrm{Lip}(\Omega, d^\alpha)$.

Next, we need a compatibility hypothesis between T and $\mathcal{X}(\Omega)$.

(H2) *T preserves $\mathcal{X}(\Omega)$ forward and backward*, i.e. the composition operator $f \mapsto f \circ T$ is well-defined and continuous from $\mathcal{X}(\Omega)$ to itself, and the map sending $f \in \mathcal{X}(\Omega)$ to

$$g : x \mapsto \sum_{T(y)=x} f(y) \in \mathcal{X}$$

is a linear, continuous operator with values in $\mathcal{X}(\Omega)$ (in particular $\|g\| \leq C\|f\|$ for some constant C).

This hypothesis in particular means that even if T is not finite-to-one, the sum $\sum_{T(y)=x} f(y)$ is required to converge in norm for all $x \in \Omega$ and $f \in \mathcal{X}(\Omega)$.

Example 2.6. When $\mathcal{X}(\Omega) = \mathrm{Hol}_\alpha(\Omega)$, it is sufficient to require the map T to be finite-to-one and a local bi-Lipschitz homeomorphism to obtain (H2).

Remark 2.7. The tent map does not strictly speaking satisfy this compatibility when for example $\mathcal{X}(\Omega) = \mathrm{Hol}_\alpha(\Omega)$, because $1/2$ only has one inverse image and g is usually not even continuous. One can fix such cases by introducing a suitable weight in all sums $\sum_{T(y)=x} f(y)$, i.e. $\sum_{T(y)=1/2} f(y)$ should be interpreted as $f(1) + f(1)$ to ensure continuity in x of $\sum_{T(y)=x} f(y)$. In other words, if needed $\sum_{T(y)=x} f(y)$ can be replaced everywhere by $\sum_k f(y_k(x))$ where y_k are the local inverse branches of T.

2.1.2. Transfer operator. The composition operator arising from T is the natural functional counterpart to our dynamical system; in fact, most properties of T of ergodic flavor are naturally formulated in terms of the composition operator on a certain class of functions. However, it is useful to investigate its "inverses", the transfer operators. Given a "potential" $A \in \mathcal{X}(\Omega)$, one defines the *transfer operator* (also called the *Ruelle operator*) by

$$\mathscr{L}_A(f)(x) = \sum_{T(y)=x} e^{A(y)} f(y);$$

note that since $\mathcal{X}(\Omega)$ is a Banach algebra, e^A lies in $\mathcal{X}(\Omega)$ and then hypothesis (H2) implies that $\mathscr{L}_A(f)$ lies in $\mathcal{X}(\Omega)$ and $\mathscr{L}_A$ is a continuous operator.

Since $\mathcal{X}(\Omega)$ is a space of functions, it contains a canonical "positive cone", the set of positive functions, which is convex and invariant under dilation. By design, the transfer operator is positive in the sense that it maps the positive cone into itself. Typical expanding assumptions for T ensure that the positive cone is even mapped into a narrower cone, inducing a contraction on the set of positive directions endowed with a suitable distance (see e.g. [Ba00]). Instead of assuming such kind of hypothesis on T, we shall only assume the consequences that are usually drawn from them. Namely, we require $(T, \mathcal{X}(\Omega))$ to *satisfy a Ruelle–Perron–Frobenius theorem* (including a spectral gap) in the sense of the following two hypotheses.

(H3) There is an open set $\mathcal{U} \subset \mathcal{X}(\Omega)$ such that for all $A \in \mathcal{U}$, the transfer operator $\mathscr{L}_A$ has a positive maximal eigenvalue λ_A and a positive (bounded away from 0) eigenfunction $h_A \in \mathcal{X}(\Omega)$:

$$\mathscr{L}_A(h_A) = \lambda_A h_A,$$

and the dual operator $\mathscr{L}_A^*$ of $\mathscr{L}_A$ preserves the set of finite measures and has an eigenmeasure $\nu_A \in \mathcal{P}(\Omega)$ for the eigenvalue λ_A, in particular

$$\int \mathscr{L}_A(f)\,\mathrm{d}\nu_A = \lambda_A \int f\,\mathrm{d}\nu_A \quad \forall f \in \mathcal{X}(\Omega).$$

The fact that we only require an open set of potentials to yield a spectral gap does not cost any complications in our results and proofs, which are all local, but improves the potential for application. For example Cyr and Sarig [CS09] showed in the context of countable Markov chains that in very many cases a spectral gap occurs only for a dense open set of potentials. It is not straightforward to apply our framework to their work though, as their potentials do not live in the same function space their transfer operator is acting on. In the context of non-uniformly expanding maps, several results show a spectral gap for Hölder potentials with small variations (physically, a high temperature hypothesis)—see for example [AM06], and references cited in [BCV12].

Example 2.8. When all functions of $\mathcal{X}(\Omega)$ are continuous and T is finite-to-one and a local homeomorphism, $\mathscr{L}_A$ extends to all continuous functions and then the dual operator automatically acts on measures. If furthermore Ω is compact, then this action is continuous in the weak-$*$ topology.

(H4) For all $A \in \mathcal{U}$, there are positive constants D, δ such that for all $n \in \mathbb{N}$ and all $f \in \mathcal{X}(\Omega)$ such that $\int f \, d\nu_A = 0$, we have

$$\|\mathcal{L}_A^n(f)\| \leq D\lambda_A^n(1-\delta)^n\|f\|.$$

It follows in particular that λ_A is a simple eigenvalue and ν_A defines a natural (topological) complement to its eigendirection.

It is easy to see that $\mu_A = h_A \nu_A$ defines an invariant measure for T, and up to normalizing h_A we can assume μ_A is a probability measure which we will call the *Gibbs measure* of A.

Example 2.9. When T is expanding in a relatively general sense and $\mathcal{X}(\Omega)$ is a space of Hölder functions, (H3) and (H4) are proved in [KLS14] (the spectral gap is proved there for normalized potentials only, but see Remark 2.12 below).

2.1.3. Further hypotheses. Our first results will only use (H1) to (H4), but at some points we will need two further hypotheses.

The first one feels harmless (in the sense that it holds for most if not all relevant examples), but does not follow from the previous ones.

From Section 5 on, we will assume:

(H5) For all $A \in \mathcal{U}$ and all $f \in \mathcal{X}(\Omega)$, if f is non-negative and $\int f \, d\mu_A = 0$ then $f = 0$.

This notably ensures that not only is λ_A the unique, simple largest eigenvalue, but also h_A is the only non-negative eigenfunction.

Remark 2.10. If all functions in $\mathcal{X}(\Omega)$ are continuous, to ensure (H5) it is sufficient to assume that μ_A has full support for all A.

Example 2.11. Assume that T is continuous, all functions in $\mathcal{X}(\Omega)$ are continuous, and the only closed subsets $A \subset \Omega$ which are both forward and backward invariant (i.e. $T(A) = T^{-1}(A) = A$) are the empty set $\emptyset$ and the full space Ω. Then (H5) holds.

Indeed, since μ_A is an invariant measure, its support is a closed invariant subset of Ω. But (assuming without loss of generality that A is normalized, see below) the invariance under $\mathcal{L}_A^*$ and the fact that e^A is a positive function also implies that $\operatorname{supp} \mu_A$ is backward invariant, so that μ_A must have full support. The continuity of f then gives the conclusion.

In Section 7 we will use the following largeness hypothesis, meant to allow the use of convexity and to avoid degenerate cases such as $\mathcal{X}(\Omega) = \{\text{constants}\}$.

(H6) All transfer operators have a spectral gap (i.e. $\mathcal{U} = \mathcal{X}(\Omega)$) and all continuous functions (going to zero at infinity if Ω is not compact) $f : \Omega \to \mathbb{R}$ can be uniformly approximated by elements of $\mathcal{X}(\Omega)$.

(Note that we do not imply here that the functions in $\mathcal{X}(\Omega)$ are continuous themselves.)

2.2. *Normalization*

Among the potentials, of particular importance are the *normalized* ones, i.e. those potentials A such that $\mathscr{L}_A(\mathbf{1}) = \mathbf{1}$ (where $\mathbf{1}$ denotes the constant function with value 1), i.e. such that $\lambda_A = 1$ and $h_A = \mathbf{1}$. In other words, A is normalized when

$$\sum_{T(y)=x} e^{A(y)} = 1 \quad \forall x \in \Omega. \tag{2.1}$$

Two nice properties that give a first evidence for the relevance of this definition are that when A is normalized, first $\mathscr{L}_A$ is a left-inverse to the composition operator:

$$\mathscr{L}_A(f \circ T) = f \quad \forall f \in \mathcal{X}(\Omega),$$

and second $\mathscr{L}_A^*$ preserves the set of probability measures. One can then interpret $\mathscr{L}_A^*$ as a Markov chain, the numbers $e^{A(y)}$ representing the probability of transiting from x to y whenever $T(y) = x$; a realization of this Markov chain is a random reverse orbit of T.

As is well-known, the Ruelle–Perron–Frobenius theorem enables one to "normalize" a potential A, by writing

$$B = A + \log h_A - \log h_A \circ T - \log \lambda_A \in \mathcal{X}(\Omega).$$

Then one gets

$$\mathscr{L}_B(f) : x \mapsto \sum_{T(y)=x} e^{A(y)} \frac{h_A(y)}{\lambda_A h_A(x)} f(y), \qquad \mathscr{L}_B(f) = \frac{1}{\lambda_A h_A} \mathscr{L}_A(h_A f),$$

where (H3) ensures that h_A is bounded away from 0, and (H1) then ensures that $1/h_A$ is in $\mathcal{X}(\Omega)$. The transfer operators $\mathscr{L}_A$ and $\mathscr{L}_B$ are thus conjugate to each other up to a multiplicative constant λ_A, the conjugating operator being multiplication by h_A; in particular

$$\mathscr{L}_B(\mathbf{1}) = \frac{1}{\lambda_A h_A} \mathscr{L}_A(h_A) = \mathbf{1}.$$

This conjugacy shows that the Gibbs measure $\mu_A = h_A \nu_A$ is also the eigenprobability $\nu_B = \mu_B$ of $\mathscr{L}_B^*$.

Using the same computation as above, one sees that whenever two arbitrary potentials A, B are related as above, i.e. $B = A + g - g \circ T + c$ for some $g \in \mathcal{X}(\Omega)$ and $c \in \mathbb{R}$, their transfer operators and their duals are conjugate to each other up to a constant:

$$\mathscr{L}_B(\cdot) = e^c e^{-g} \mathscr{L}_A(e^g \cdot), \qquad \mathscr{L}_B^*(\cdot) = e^c e^g \mathscr{L}_A^*(e^{-g} \cdot)$$

where e^g is in $\mathcal{X}(\Omega)$, positive and bounded away from 0. It follows immediately that up to normalizing constants, $h_B = h_A e^{-g}$, $\nu_B = e^g \nu_A$ and $\lambda_B = e^c \lambda_A$. In particular we have $\mu_B = \mu_A$: both potentials define the same Gibbs measure. It is also straightforward to check that if moreover both A and B are normalized, then g must be a constant and c must be zero, so that $A = B$. In other words, we have a subspace

$$\mathcal{C} = \{g - g \circ T + c \mid g \in \mathcal{X}(\Omega),\ c \text{ a constant}\} \subset \mathcal{X}(\Omega)$$

such that each class of elements of $\mathcal{U}$ modulo $\mathcal{C}$ defines one Gibbs measure, and contains exactly one normalized potential (see Remark 2.13 below). One says that a function of the form $g - g \circ T$ is a *coboundary*, thus $\mathcal{C}$ is the space generated by coboundaries and constants.

Remark 2.12. The conjugacy between the transfer operator of the potential A and the transfer operator of its normalization $B = N(A)$ shows that a spectral gap for $\mathscr{L}_B$ implies the same spectral gap for $\mathscr{L}_A$ (with a different constant D, but the same δ). Indeed, if $\int f \, \mathrm{d}\nu_A = 0$ then $\int (f/h_A) \, \mathrm{d}\mu_A = 0$ and

$$\begin{aligned}\|\mathscr{L}_A^n(f)\| &= \|\lambda_A^n h_A \mathscr{L}_B^n(f/h_A)\| \leq \lambda_A^n \|h_A\| D(1-\delta)^n \|f/h_A\| \\ &\leq (D\|h_A\| \, \|1/h_A\|)\lambda_A^n (1-\delta)^n \|f\|.\end{aligned}$$

In particular, if hypothesis (H3) is satisfied, the spectral gap for normalized potentials implies (H4) for all potentials.

Remark 2.13. The same reasoning as in Remark 2.12 shows that $\mathcal{U}$ can be chosen invariant under all translations along $\mathcal{C}$, i.e. for all $A \in \mathcal{U}$ and $g - g \circ T + c \in \mathcal{C}$, we have $A + g - g \circ T + c \in \mathcal{U}$. *We shall always assume this invariance.*

All the above is very classical; our goal is now to study in more detail the following objects:

- the set $\mathcal{N} \subset \mathcal{U}$ of normalized potentials (which is not a linear subspace, see Remark 5.5),
- the *normalization map*

$$N : A \mapsto A + \log h_A - \log h_A \circ T - \log \lambda_A$$

  from $\mathcal{U}$ to $\mathcal{N}$,
- the quotient $\widehat{\mathcal{U}} = \mathcal{U}/\mathcal{C}$, and
- the Gibbs map $G : A \mapsto \mu_A$ from $\mathcal{U}$, seen as taking values either in $\mathcal{X}(\Omega)^*$ or in $\mathcal{P}_T(\Omega) \subset \mathcal{P}(\Omega)$.

The typical questions we want to answer are of differential-geometric flavor: is $\mathcal{N}$ a submanifold of $\mathcal{X}(\Omega)$? Are the maps N and G differentiable? How to endow $\mathcal{N}$ or $\widehat{\mathcal{U}}$ with a meaningful Riemannian metric? Can we then study gradient flows of natural functionals on these spaces?

Remark 2.14. The spectral gap hypothesis implies an exponential decay of correlation for functions in $\mathcal{X}(\Omega)$: indeed, if A is any potential in $\mathcal{U}$ and $f, g \in \mathcal{X}(\Omega)$ are such that $\int f \, \mathrm{d}\mu_A = 0$, we have

$$\begin{aligned}\left|\int f \cdot g \circ T^n \, \mathrm{d}\mu_A\right| &= \left|\int \mathscr{L}_{N(A)}^n (f \cdot g \circ T^n) \, \mathrm{d}\mu_A\right| = \left|\int \mathscr{L}_{N(A)}^n (f) \cdot g \, \mathrm{d}\mu_A\right| \\ &\leq \|\mathscr{L}_{N(A)}^n (f) \cdot g\|_\infty \leq C^{-2} \|\mathscr{L}_{N(A)}^n (f)\| \, \|g\| \\ &\leq C^{-2} D(1-\delta)^n \|f\| \, \|g\|.\end{aligned}$$

Remark 2.15. In typical situations, a normalized potential A can be recovered from the Gibbs measure as a Jacobian: for example, if T has inverse branches y_i near each $x \in \Omega$ which are local homeomorphisms with disjoint images, then

$$e^{A(y_i(x))} = \frac{\mathrm{d}\mu_A(y_i(x))}{\mathrm{d}\mu_A(x)}$$

in the sense that if $B = B(x, \varepsilon)$ is a small ball around x, the ratio of $\mu_A(y_i(B))$ to $\mu_A(B)$ goes to $e^{A(y_i(x))}$ as $\varepsilon \to 0$ (in doubt, integrate characteristic functions of balls with respect to one measure and change variables to verify this claim). Slight adaptations of this argument are needed for example for tent maps.

In general, it might a priori happen that two different normalized potentials A, A' have the same Gibbs measure. We will see much later in Remark 7.4 that our assumptions are sufficient to prevent this, and ensure perfect identifications between normalized potentials, mod $\mathcal{C}$ classes of potentials, and Gibbs measures.

2.3. Analytic maps and submanifolds

When working in infinite-dimensional spaces, just as differentiability has various definitions of varying strength (Gâteaux versus Fréchet), the analyticity of a map can be defined in several ways. Here, we take the strongest definition, recalled below.

First, recall that a closed linear subspace M in a Banach space $\mathcal{X}$ is said to be *topologically complemented*, or for short *complemented*, when there is a closed linear subspace N which is an algebraic complement. We shall only write $\mathcal{X} = M \oplus N$ when M and N are topological complements. The projection to M along N and the projection to N along M are then continuous, i.e. for all $x \in \mathcal{X}$, the decomposition $x = m + n$ with $m \in M$ and $n \in N$ exists, is unique, and m and n depend linearly continuously on x. Equivalently, M is complemented when it is the image, or the kernel, of a continuous linear projection $\mathcal{X} \to \mathcal{X}$.

Let $\mathcal{X}$ and $\mathcal{Y}$ be two Banach spaces, whose norms will both be denoted by $\|\cdot\|$. A continuous, symmetric, multilinear operator $a : \mathcal{X}^k \to \mathcal{Y}$ has an operator norm denoted by $|a|$; if ζ is a vector in $\mathcal{X}$, we denote by $\zeta^{(k)}$ the element $(\zeta, \dots, \zeta)$ of $\mathcal{X}^k$ and we have

$$\|a(\zeta^{(k)})\| \le |a|\, \|\zeta\|^k.$$

We shall say that a sequence $a_k : \mathcal{X}^k \to \mathcal{Y}$ of such k-ary operators ($k \ge 0$) determines a *series with positive radius of convergence* if the complex series

$$\sum_{k \ge 0} |a_k| z^k$$

has a positive radius of convergence in $\mathbb{C}$.

Let $\Phi : \mathcal{X} \supset U \to \mathcal{Y}$ be a map defined on an open subset of $\mathcal{X}$. We say that Φ is *analytic* if for each $x \in U$ there is a series of k-linear, symmetric, continuous operators $a_k : \mathcal{X}^k \to \mathcal{Y}$ with positive radius of convergence such that on an open subset of U the following identity holds:

$$\Phi(x + \zeta) = \sum_{k \ge 0} a_k(\zeta^{(k)}).$$

An analytic map is smooth (in particular, Fréchet differentiable and locally Lipschitz-continuous) and the operators a_k are uniquely defined by Φ. Most classical results hold in this context, in particular the inverse function theorem and the implicit function theorem (see [Ch85] and [Wh65]).

More precisely:

Theorem (Inverse function theorem). *Let $\Phi : \mathcal{X} \supset U \to \mathcal{Y}$ be an analytic map defined on an open set such that $D\Phi_x : \mathcal{X} \to \mathcal{Y}$ is a topological isomorphism for each x. Then for all $x \in U$ there are open neighborhoods V of x and W of $\Phi(x)$ such that Φ induces an analytic diffeomorphism $V \to W$.*

Theorem (Implicit function theorem). *Let $F : \mathcal{X} \supset U \to \mathcal{Y}$ be an analytic map defined on an open set such that $F(x) = 0$ for some $x \in U$, DF_x is onto $\mathcal{Y}$, and $\ker DF_x$ is complemented in $\mathcal{X}$. Then the level set $F^{-1}(0)$ is an analytic submanifold of $\mathcal{X}$ in a neighborhood V of x, i.e. there is a closed complemented subspace $\mathcal{S} \subset \mathcal{X}$ and an analytic diffeomorphism $\Phi : V \to W \subset \mathcal{X}$ such that $\Phi(F^{-1}(0) \cap V) = \mathcal{S} \cap W$.*

Writing $\mathcal{R}$ for a topological complement of $\ker DF_x$ in $\mathcal{X}$, we can further write $F^{-1}(0) \cap V$ as the graph of an analytic map $\ker DF_x \cap V \to \mathcal{R}$.

We shall also need an elementary result, which we state in a rather particular case for the sake of simplicity.

Lemma 2.16. *Let $\mathcal{X}, \mathcal{Y}$ be two Banach spaces, let $\Phi : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ be a continuous bilinear form and consider two analytic maps $f : U \to \mathcal{X}$ and $g : U \to \mathcal{Y}$ defined on the same open subset of $\mathcal{X}$. Then the map*

$$\Phi(f, g) : U \to \mathbb{R}, \qquad x \mapsto \Phi(f(x), g(x)),$$

is an analytic map.

Proof. Let $x \in U$ and let $f(x + \zeta) = \sum_k a_k(\zeta^{(k)})$ and $g(x + \zeta) = \sum_n b_n(\zeta^{(n)})$ be the local series expansions of f and g at x. Define

$$c_{k,n}(\zeta_1, \ldots, \zeta_{n+k}) := \frac{1}{(k+n)!} \sum_{\sigma \in \mathfrak{S}_{k+n}} \Phi\big(a_k(\zeta_1, \ldots, \zeta_k), b_n(\zeta_{1+n}, \ldots, \zeta_{k+n})\big),$$

$$d_j(\zeta_1, \ldots, \zeta_j) := \sum_{k+n=j} c_{k,n}(\zeta_1, \ldots, \zeta_j).$$

Then d_j is j-linear and symmetric, and

$$|d_j| \le \sum_{k+n=j} |c_{k,n}| \le \sum_{k+n=j} |\Phi|\,|a_k|\,|b_n|$$

so that $\sum |d_j| z^j$ has at least the convergence radius of $(\sum |a_k| z^k)(\sum |b_n| z^k)$. To conclude, observe that, since $d_j(\zeta^{(j)}) = \sum_{k+n=j} \Phi(a_k(\zeta^{(k)}), b_n(\zeta^{(n)}))$,

$$\Phi(f(x + \zeta), g(x + \zeta)) = \sum_j d_j(\zeta^{(k)})$$

for ζ in a neighborhood of 0 (the inversions between sums and limits holding thanks to the continuity in the Banach topology and the norm convergence of the series). □

2.4. Application to non-uniformly expanding circle maps: the crucial choice of potentials

In this last subsection, we give some pointers to situations in the literature where one can expect to use our results, and we discuss two basic tools to produce such situations. One important point is that often the choice of the space $\mathcal{X}(\Omega)$ is crucial.

2.4.1. A few references. The case of uniformly expanding maps, often considered with Hölder potentials, is well-known to have a spectral gap and it is almost impossible to list all relevant references. Let us mention again [PP90] as an entry point, and [KLS14] for a metric approach based on optimal transport. For maps with jumps (e.g. piecewise continuous interval maps), the space of Hölder potentials is not stable and some replacement is needed; it is common to use the space of bounded variation functions (see for example [Ba00]).

An interesting family of potentials, also suitable for piecewise defined maps, has been suggested by Liverani [Li13], where spectral gap is proved in the case of interval maps.

Some non-uniformly expanding maps, or even maps with cohabitation of expanding and mildly contracting behaviors, are known to also exhibit spectral gap for Hölder or smoother potentials which are close enough to being constant [VV10], [CV13].

2.4.2. Factors. When $T : \Omega \to \Omega$ is a topological factor of $S : \Lambda \to \Lambda$, i.e. there is a continuous onto map $\pi : \Lambda \to \Omega$ such that $T \circ \pi = \pi \circ S$, and if $\mathcal{X}(\Lambda)$ is a Banach algebra of functions on Λ, then one can transfer properties from S to T. Indeed, assume $S, \mathcal{X}(\Lambda)$ satisfies hypotheses (H1) to (H4) (with an open set $\mathcal{U}$ whose potentials yield a spectral gap) and define $\mathcal{X}(\Omega) = \{f : \Omega \to \mathbb{R} \mid f \circ \pi \in \mathcal{X}(\Lambda)\}$ and $\mathcal{V} = \{A : \Omega \to \mathbb{R} \mid A \circ \pi \in \mathcal{U}\}$. Then for each $A \in V$, the measure $\mu_A := \pi_\# \mu_{A\circ\pi}$ is easily seen to be invariant, and can be considered an analogue of a Gibbs measure (in some nice cases, one can view μ_A as a genuine Gibbs measure). Then all results on Gibbs measures for S translate to the measures μ_A, as for example

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varphi \mathrm{d}\mu_{A+t\zeta}\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Lambda \varphi\circ\pi \mathrm{d}\mu_{A\circ\pi+t\zeta\circ\pi}\bigg|_{t=0} = \langle \varphi\circ\pi, \zeta\circ\pi\rangle_{A\circ\pi}.$$

Consider the case when $T : \mathbb{S}^1 \to \mathbb{S}^1$ is a C^1 self-covering of degree 2 of the circle which is non-uniformly expanding with one neutral fixed point (i.e. $T'(x) > 1$ for all $x \neq 0$ and $T'(0) = 1$, $T(0) = 0$ where we identify an element of $\mathbb{R}$ with its class modulo 1). One can use for S the full one-sided shift over the alphabet $\{0, 1\}$. Indeed, let p be the non-null inverse image of 0 by T; then sending a sequence $x = (x_0, x_1, \dots)$ to the unique point $\pi(x) \in \mathbb{S}^1$ such that $T^i(\pi(x)) \in [0, p]$ if $x_i = 0$, and $T^i(\pi(x)) \in [p, 1]$ when $x_i = 1$, makes T a factor of S. If we then take for $\mathcal{X}$ a space of Hölder functions, then S has all the required properties to apply our work, which are then granted to T.

Compared to the case of uniformly expanding maps, the catch is that the usual metric on $\{0, 1\}^{\mathbb{N}}$ is not as nicely related to the metric on $\mathbb{S}^1$: π is not Hölder continuous. So, the potentials to be considered for T have a peculiar structure (notably, they must vary very slowly near 0 or near its inverse images by T^n when n is small).

2.4.3. Topological conjugacy. An even more obvious situation is when $(S, \Lambda, \mathcal{X}(\Lambda))$ satisfies our hypotheses, and (T, Ω) is topologically conjugate to (S, Λ), i.e. there is a *homeomorphism* $\pi : \Lambda \to \Omega$ such that $T \circ \pi = \pi \circ S$. Then we are in a "nice case" alluded to above: defining $\mathcal{X}(\Omega)$ as above, for every A and $f \in \mathcal{X}(\Lambda)$ we have $\mathscr{L}_{A\circ\pi}(f \circ \pi) = (\mathscr{L}_A f) \circ \pi$ and the spectral gap assumption (together with all other hypotheses) carries over from $\mathcal{X}(\Lambda)$ to $\mathcal{X}(\Omega)$.

When T is a non-uniformly expanding self-covering of the circle of degree 2 (say) with one neutral fixed point, then T is topologically conjugate to the model map $S : x \mapsto 2x \bmod 1$, for which the α-Hölder potentials are well-known to have a spectral gap. The conjugating map is uniquely defined as follows: the fixed point 0 of S is mapped to the unique fixed point of T, then the non-null inverse image p of 0 under S is mapped to its T-counterpart p', the two inverse images of p are mapped to the two inverse images of p' in the only monotonic way, and so on. The assumptions on T ensure that this process defines π on the countable dense set of dyadic points, and extends to a homeomorphism.

From this it follows that there is a suitable space $\{f : \mathbb{S}^1 \to \mathbb{R} \mid f \circ \pi \in \mathrm{Hol}_\alpha(\mathbb{S}^1)\}$ for which T satisfies all our hypotheses. Again, the catch is that this space is difficult to identify precisely, because π is wildly irregular, and the space might contain few nice (e.g. Hölder) functions. In particular, it may not contain the potential $1/|T'|$ related to the absolutely continuous invariant (side note: to answer this question amounts to a question about the irregular potential $1/|T' \circ \pi^{-1}|$ for the model map S).

The investigation of this kind of construction, and the identification of this space or nice subspaces thereof, will be a subject of further work.

3. Normalizing potentials

We will now consider the set of normalized potentials

$$\mathcal{N} = \{A \in \mathcal{U} \mid \mathscr{L}_A(\mathbf{1}) = \mathbf{1}\}$$

and the normalization map N that sends any potential A to its normalization:

$$N(A) = A - \log \lambda_A + \log h_A - \log h_A \circ T.$$

The map N can be described as the (non-linear) projection on $\mathcal{N}$ along

$$\mathcal{C} = \{g - g \circ T + c \mid g \in \mathcal{X}(\Omega), c \text{ a constant}\}.$$

We start from a simple lemma which will both prove useful and serve as an example of the use of convergent series in our study. We shall denote by $\ker \mu_A \subset \mathcal{X}(\Omega)$ the kernel of μ_A seen as a linear form, i.e.

$$\ker \mu_A := \left\{ f \in \mathcal{X}(\Omega) \;\middle|\; \int f \, \mathrm{d}\mu_A = 0 \right\}.$$

Lemma 3.1. *If A is a normalized potential in $\mathcal{U}$, then the operator $I - \mathscr{L}_A$ induces an isomorphism from* $\ker \mu_A$ *to itself, with inverse given by*

$$(I - \mathscr{L}_A)^{-1} = \sum_{k=0}^{\infty} \mathscr{L}_A^k : \ker \mu_A \to \ker \mu_A.$$

Proof. For all $f \in \mathcal{X}(\Omega)$, we have

$$\int (I - \mathscr{L}_A)(f)\,\mathrm{d}\mu_A = \int f\,\mathrm{d}\mu_A - \int f\,\mathrm{d}(\mathscr{L}_A^*\mu_A) = 0,$$

because μ_A is fixed by $\mathscr{L}_A^*$. It follows that $I - \mathscr{L}_A$ takes its values in $\ker \mu_A$.

For all $f \in \ker \mu_A$ and all n, we have

$$(I - \mathscr{L}_A)\Big(\sum_{k=0}^{n} \mathscr{L}_A^k(f)\Big) = f - \mathscr{L}_A^{n+1}(f).$$

By the spectral gap assumption, we know that $\sum \mathscr{L}_A^k(f)$ converges, it is bounded by $(D/\delta)\|f\|$, and $\mathscr{L}_A^{n+1}(f)$ goes to 0 as $n \to \infty$. We deduce that $\sum_{k=0}^{\infty} \mathscr{L}_A^k$ is well-defined and a right-inverse to $I - \mathscr{L}_A$, which is therefore onto $\ker \mu_A$.

By commutation the above shows that, for all $f \in \ker \mu_A$,

$$\sum_{k=0}^{n} \mathscr{L}_A^k(I - \mathscr{L}_A(f))$$

converges to f, so that we have defined an inverse to the restriction and corestriction of $I - \mathscr{L}_A$ to $\ker \mu_A$. □

This has useful consequences, which will be better phrased by introducing another operator related to A.

Definition 3.2. Given any *normalized* potential $A \in \mathcal{U}$, let $\mathscr{M}_A$ be the continuous linear operator on $\mathcal{X}(\Omega)$ defined by

$$\mathscr{M}_A(f) = -(I - \mathscr{L}_A)^{-1} \circ \mathscr{L}_A(f_A) = -\sum_{k=1}^{\infty} \mathscr{L}_A^k(f_A),$$

where as before $f_A := f - \int f\,\mathrm{d}\mu_A$.

Observe that $\mathscr{L}_A$ maps $\ker \mu_A$ to itself, so that $\mathscr{M}_A$ is indeed well-defined and takes its values in $\ker \mu_A$, and that $\mathscr{M}_A$ commutes with $\mathscr{L}_A$.

Proposition 3.3. *Let $A \in \mathcal{U}$ be a normalized potential. Then:*

(i) *given $f \in \mathcal{C}$, there is a unique decomposition $f = g - g \circ T + c$ with $g \in \ker \mu_A$ and c a constant, given by*

$$c = \int f\,\mathrm{d}\mu_A \quad \textit{and} \quad g = \mathscr{M}_A(f),$$

(ii) *the subspace $\mathcal{C}$ is closed in $\mathcal{X}(\Omega)$, so that $\widehat{\mathcal{X}} := \mathcal{X}(\Omega)/\mathcal{C}$ inherits a Banach space structure from $\|\cdot\|$,*

(iii) $\ker \mathscr{L}_A$ *and $\mathcal{C}$ are (topological) complements in $\mathcal{X}(\Omega)$,*

(iv) *$\mathscr{L}_A$ maps $\mathcal{C}$ onto $\mathcal{X}(\Omega)$.*

The decomposition in (iii) can be further refined:

$$\mathcal{X}(\Omega) = \underbrace{\ker \mathscr{L}_A \oplus \overbrace{\{\text{coboundaries}\}}^{\mathcal{C}}}_{\ker \mu_A} \oplus \{\text{constants}\}.$$

Proof. First observe that given any decomposition $f = g - g \circ T + c$ and any T-invariant probability measure μ, we have

$$\int f \, \mathrm{d}\mu = \int g \, \mathrm{d}\mu - \int g \, \mathrm{d}(T_{\#}\mu) + c = c$$

where $T_{\#}\mu$ is the usual pushforward of the measure μ with respect to T, so by invariance $T_{\#}\mu = \mu$. Thus c is uniquely defined, and applying the above equality to $\mu = \mu_A$ we get $c = \int f \, \mathrm{d}\mu_A$.

Let us then check that any $f \in \mathcal{C} \cap \ker \mathscr{L}_A$ must vanish. First, it is easy to see that $\ker \mathscr{L}_A \subset \ker \mu_A$:

$$\int f \, \mathrm{d}\mu_A = \int f \, \mathrm{d}(\mathscr{L}_A^* \mu_A) = \int \mathscr{L}_A(f) \, \mathrm{d}\mu_A = 0.$$

It follows that we can write $f = g - g \circ T$, so that

$$0 = \mathscr{L}_A(g - g \circ T) = \mathscr{L}_A(g) - g$$

and g is an eigenfunction of $\mathscr{L}_A$ for the eigenvalue 1. Therefore g is constant and $f = 0$.

To prove (i), we write $f = g_1 - g_1 \circ T + c$ for some g_1 and with $c = \int f \, \mathrm{d}\mu_A$. Setting $g = g_1 - \int g_1 \, \mathrm{d}\mu_A$, we still have $f = g - g \circ T + c$ and

$$\mathscr{L}_A(f - c) = \mathscr{L}_A(g - g \circ T) = \mathscr{L}_A(g) - g$$

since $\mathscr{L}_A$ is a left-inverse to the composition operator. Now, from $g \in \ker \mu_A$ it follows that $g = -(I - \mathscr{L}_A)^{-1} \mathscr{L}_A(f - c) = \mathscr{M}_A(f)$, as claimed.

To prove (ii), consider a sequence of functions $f_n \in \mathcal{C}$ which converges to $f \in \mathcal{X}(\Omega)$. Then using (i), we can write $f_n = g_n - g_n \circ T + c_n$ where g_n, c_n are images of f_n by continuous operators. In particular g_n and c_n have limits $g \in \mathcal{X}(\Omega)$ and $c \in \mathbb{R}$, so that $f = g - g \circ T + c \in \mathcal{C}$.

To prove (iii), since we already know that $\ker \mathscr{L}_A$ and $\mathcal{C}$ intersect trivially, we consider any $f \in \mathcal{X}(\Omega)$ and let $c := \int f \, \mathrm{d}\mu_A$ and $g = \mathscr{M}_A(f)$. We have

$$\mathscr{L}_A(g - g \circ T + c) = \mathscr{L}_A(g) - g + c\mathscr{L}_A(\mathbf{1}) = \mathscr{L}_A(f - c) + c = \mathscr{L}_A(f)$$

where the second equality follows from $(\mathscr{L}_A - I)\mathscr{M}_A = \mathscr{L}_A$ on $\ker \mu_A$. It follows that $\ell := f - (g - g \circ T + c)$ is an element of $\ker \mathscr{L}_A$. The decomposition

$$f = \ell + (g - g \circ T + c)$$

shows that $\mathcal{X}(\Omega) = \ker \mathscr{L}_A \oplus \mathcal{C}$, and since both spaces are closed, $\ker \mathscr{L}_A$ and $\mathcal{C}$ are complements.

To prove (iv), let $f \in \mathcal{X}(\Omega)$ and set $c := \int f \, \mathrm{d}\mu_A$ and $g := (I - \mathscr{L}_A)^{-1}(c - f)$. Now $g - g \circ T + c$ is an element of $\mathcal{C}$, and we have

$$\mathscr{L}_A(g - g \circ T + c) = \mathscr{L}_A(g) - g + c = -(I - \mathscr{L}_A)(g) + c = f - c + c = f. \qquad \square$$

We are now in a position to prove our first main result, that $\mathcal{N}$ is an analytic submanifold of $\mathcal{X}(\Omega)$. This result might be known, but we have not found a clear statement in the literature; related statements often involve a weaker definition of analyticity, the identification of the tangent space seems new, and we obtain the result without resorting to complex analysis as is usually done to prove the regularity of the eigendata of operators (see [PP90, Appendix V], where the weak definition of analyticity should be noted, and also [BS01, Section 3.3]). We shall in fact *deduce* from Theorem 3.4 that the leading eigenvalue and positive eigenfunction of $\mathscr{L}_A$ both depend analytically on A.

Theorem 3.4. *Under hypotheses (H1) to (H4), the set $\mathcal{N}$ of normalized potentials is an analytic submanifold of $\mathcal{U}$, and its tangent space at $A \in \mathcal{N}$ is $T_A\mathcal{N} = \ker \mathscr{L}_A$.*

Proof. This is a direct consequence of the implicit function theorem.

Let $F : \mathcal{U} \to \mathcal{X}(\Omega)$ be defined by

$$F(A)(x) = \mathscr{L}_A(\mathbf{1})(x) = \sum_{Ty=x} e^{A(y)}.$$

Then F is analytic, as follows from the analyticity of the exponential:

$$F(A+\zeta) = \sum_{k\geq 0} \frac{1}{k!} D^k F_A(\zeta),$$

where

$$D^k F_A(\zeta_1, \dots, \zeta_k)(x) := \sum_{Ty=x} e^{A(y)} \prod_{i=1}^{k} \zeta_i(y)$$

defines a series of continuous, symmetric k-linear operators with infinite radius of convergence. In more detail, we write

$$\begin{aligned} F(A+\zeta)(x) &= \sum_{Ty=x} e^{A(y)} e^{\zeta(y)} = \sum_{Ty=x} \lim_{K\to\infty} \sum_{k=0}^{K} e^{A(y)} \frac{1}{k!} \zeta(y)^k \\ &= \lim_{K\to\infty} \sum_{k=0}^{K} \frac{1}{k!} \sum_{Ty=x} e^{A(y)} \zeta(y)^k \end{aligned}$$

where the inversions are possible because the limit holds in norm (by the Banach algebra structure) and thanks to hypothesis (H2), requiring $f \mapsto (x \mapsto \sum_{Ty=x} f(y))$ to be a continuous linear operator. Then we observe that indeed the last term is $D^k F_A(\zeta, \dots, \zeta)$.

Now, given a potential A and a vector ζ both in $\mathcal{X}(\Omega)$, we have

$$DF_A(\zeta) = \sum_{Ty=x} e^{A(y)} \zeta(y) = \mathscr{L}_A(\zeta),$$

so that $DF_A = \mathscr{L}_A$; since we know from Proposition 3.3 that $\ker \mathscr{L}_A$ is complemented and $\mathscr{L}_A$ is onto $\mathcal{X}(\Omega)$, we can apply the implicit function theorem. □

We also directly get the analyticity of the normalization map as explained in the last paragraph of Section 2.3.

Theorem 3.5. *Under hypotheses (H1) to (H4), the normalization map $N : \mathcal{U} \to \mathcal{N}$ sending a potential to its normalized version is analytic. Moreover, its derivative DN_A at a point $A \in \mathcal{U}$ is the linear projection on $T_{N(A)}\mathcal{N} = \ker \mathscr{L}_{N(A)}$ in the direction of $\mathcal{C}$.*

Proof. See Figure 1 for a general picture of the various maps involved. Let $\Pi : \mathcal{X}(\Omega) \to \widehat{\mathcal{X}}$ be the quotient map; it is a continuous linear map, and in particular it is analytic. Its restriction $\Pi_{|\mathcal{N}}$ to the submanifold $\mathcal{N}$ is therefore an analytic map, and for all $A \in \mathcal{N}$ we have

$$D(\Pi_{|\mathcal{N}})_A = \Pi_{|T_A\mathcal{N}} = \Pi_{|\ker \mathscr{L}_A}.$$

Since $\ker \mathscr{L}_A$ and $\mathcal{C}$ are topological complements of each other, this differential is invertible with continuous inverse. The inverse function theorem then ensures that

$$\Pi_{|\mathcal{N}}^{-1} : \widehat{\mathcal{U}} \to \mathcal{N}$$

is well-defined and analytic. We get the desired result by observing that

$$N = \Pi_{|\mathcal{N}}^{-1} \circ \Pi. \qquad \square$$

Corollary 3.6. *Under hypotheses (H1) to (H4), the map $\Lambda : \mathcal{U} \to \mathbb{R}$ sending a potential to its leading eigenvalue, i.e. defined by $\Lambda(A) = \lambda_A$, is an analytic map. Moreover, for all $A \in \mathcal{U}$ and $\zeta \in \mathcal{X}(\Omega)$,*

$$D(\log \Lambda)_A(\zeta) = \int \zeta \, \mathrm{d}\mu_A.$$

Note that it will turn out that in our framework $\log \Lambda$ equals the pressure functional, so that this result also gives the derivative of the latter.

Proof of Corollary 3.6. Let μ be any invariant probability measure; from $N(A) = A - \log \Lambda(A) + \log H(A) - \log H(A) \circ T$ we get

$$\Lambda(A) = \exp\left(\int (A - N(A)) \, \mathrm{d}\mu \right),$$

which is analytic as composed of analytic maps.

Differentiating $\log \Lambda(A) = \int (A - N(A)) \, \mathrm{d}\mu$ with respect to A gives

$$D(\log \Lambda)_A(\zeta) = \int (\zeta - DN_A(\zeta)) \, \mathrm{d}\mu;$$

then specializing for $\mu = \mu_A$ and observing that $DN_A(\zeta) \in \ker \mu_A$ yields the desired formula. $\square$

Corollary 3.7. *Under hypotheses (H1) to (H4), the following maps are analytic:*

$$G : \mathcal{U} \to \mathcal{X}(\Omega)^*, \ A \mapsto \mu_A, \qquad H : \mathcal{U} \to \mathcal{X}(\Omega), \ A \mapsto h_A,$$

where h_A is the positive eigenfunction of $\mathscr{L}_A$ normalized by $\int h_A \, \mathrm{d}\nu_A = 1$ (recall ν_A is the eigenprobability of $\mathscr{L}_A^*$). In particular, for each $\varphi \in \mathcal{X}(\Omega)$, the map $G_\varphi : A \mapsto \int \varphi \, \mathrm{d}\mu_A$ is analytic.

As before, the dual space $\mathcal{X}(\Omega)^*$ is endowed with the dual (operator) norm.

Proof. Corollary 3.6 implies that $\mu_A = D(\log\Lambda)_A$ as a linear form defined on $\mathcal{X}(\Omega)$, so that the first part follows from the analyticity of $\log\Lambda$.

Fix any $A_0 \in \mathcal{N}$; then the positive eigenfunction $\tilde{H}(A)$ normalized by the conditon $\int \log\tilde{H}(A)\,\mathrm{d}\mu_{A_0} = 0$ is given by

$$\tilde{H}(A) = \exp\big(\mathscr{M}_{A_0}(A - N(A))\big)$$

(write again $N(A) = A - \log\Lambda(A) + \log\tilde{H}(A) - \log\tilde{H}(A)\circ T$). The map $\tilde{H}$ is thus analytic; denote by h_A the positive eigenfunction such that $\int h_A\,\mathrm{d}\nu_A = 1$, i.e. $\mathrm{d}\mu_A = h_A\,\mathrm{d}\nu_A$, and observe that since all eigenfunctions are multiples of one another,

$$\frac{h_A}{\tilde{H}(A)} = \frac{\int h_A\,\mathrm{d}\nu_A}{\int \tilde{H}(A)\,\mathrm{d}\nu_A}.$$

Now we can write

$$H(A) = \left(\int \frac{1}{\tilde{H}(A)}\,\mathrm{d}\mu_A\right)\tilde{H}(A), \tag{3.1}$$

which yields

$$\int H(A)\,\mathrm{d}\nu_A = \int \frac{h_A}{\tilde{H}(A)}\,\mathrm{d}\nu_A \cdot \int \tilde{H}(A)\,\mathrm{d}\nu_A = 1.$$

Applying Lemma 2.16 to $f = 1/\tilde{H}$, $g = G$ and Φ the pairing $\mathcal{X}(\Omega)\times\mathcal{X}(\Omega)^* \to \mathbb{R}$, we see that $\int (1/\tilde{H}(A))\,\mathrm{d}\mu_A$ depends analytically on A, so H is analytic. □

We have thus proved Corollary B from the introduction.

Corollaries 3.6 and 3.7 are classical in many cases, and were obtained under different assumptions and with different methods by Bomfim, Castro and Varandas [BCV12]; note that we notably do not assume the high-temperature regime (see their conditions (P) and (P′)), and once our framework is set, our proofs are very simple.

Let us end this section by proving that the spectral gap is locally uniform. This is usually obtained by perturbation theory, but also follows from calculus, and will be used later in the Central Limit Theorem.

Proposition 3.8. *The constants C and δ in hypothesis (H4) can be chosen locally uniform, i.e. for all $A \in \mathcal{U}$ there are a neighborhood $V \ni A$ and constants $D_V > 0$ and $\delta_V \in (0,1)$ such that for all $B \in V$, all $n \in \mathbb{N}$ and all $f \in \mathcal{X}(\Omega)$ such that $\int f\,\mathrm{d}\nu_B = 0$,*

$$\|\mathscr{L}_B^n(f)\| \le D_V\lambda_B^n(1-\delta_V)^n\|f\|.$$

Proof. For all $A \in \mathcal{U}$, we can decompose the transfer operator into $\mathscr{L}_A = \lambda_A\mathscr{P}_A + \mathscr{R}_A$ where $\mathscr{P}_A(f) = (\int f\,\mathrm{d}\nu_A)h_A$ is the projection of f on the leading eigenspace and $\mathscr{R}_A = \mathscr{L}_A\circ(\mathrm{Id} - \mathscr{P}_A)$ is the composition of the projection $f \mapsto f - (\int f\,\mathrm{d}\nu_A)h_A$ to $\ker\nu_A$ and of $\mathscr{L}_A$. As is well known and easily checked, $\mathscr{P}_A\circ\mathscr{R}_A = \mathscr{R}_A\circ\mathscr{P}_A = 0$, and thus $\mathscr{L}_A^n = \lambda_A^n\mathscr{P}_A + \mathscr{R}_A^n$. The spectral gap hypothesis (H4) shows that given A, there are n_0 and $\delta_1 \in (0,\delta)$ such that $|\mathscr{R}_A^{n_0}| < \lambda_A^{n_0}(1-\delta_1)^{n_0}$ (chosen to absorb the constant D). But Corollaries 3.6 and 3.7 show that $\mathscr{P}_B$ and thus $\mathscr{R}_B^{n_0}$ depend analytically on B. In particular $B \mapsto |\mathscr{R}_B^{n_0}|$ is continuous, and there is a neighborhood V of A on which this

norm is less than $\lambda_A^{n_0}(1-\delta_2)^{n_0}$ for some slightly smaller δ_2. By continuity of Λ, up to taking a smaller V we can assume that for all $B \in V$ we have $\lambda_B > \lambda_A(1-\delta_2/2)$, so that $|\mathscr{R}_B^{n_0}| < \lambda_B^{n_0}(1-\delta_3)^{n_0}$ on V for some yet smaller δ_3. Decompose any $n \in \mathbb{N}$ as $n = r + kn_0$ with $r < n$ and observe that for all $f \in \ker \nu_B$ (i.e. $\mathscr{P}_B(f) = 0$),

$$\mathscr{L}_B^n(f) = \mathscr{R}_B^r\big((\mathscr{R}_B^{n_0})^k(f)\big)$$

is bounded in norm by $D'\lambda_B^n(1-\delta_3)^n\|f\|$ where

$$D' = \max_{r=1,\dots,n-1} \|\mathscr{R}_B^r\|/(\lambda_B^r(A-\delta_3)^r)$$

is locally uniform. □

4. Differentiating the Gibbs map in the affine structure

There are at least two ways to endow the set $\mathcal{P}(\Omega)$ of probability measures with a kind of differential structure, i.e. to define what it means for a map such as the Gibbs map $G : A \mapsto \mu_A$ to be differentiable. In this section, we consider the *affine structure*, while in Section 6 we will consider the *Wasserstein structure*.

The affine structure is obtained simply by observing that $\mathcal{P}(\Omega)$ is a convex set in $\mathcal{X}(\Omega)^*$; "coordinates" are obtained by looking at integrals of test functions, so that G is often considered to be differentiable if

$$\forall A, \zeta, \varphi \in \mathcal{X}(\Omega): \quad \frac{\mathrm{d}}{\mathrm{d}t}\int \varphi \,\mathrm{d}\mu_{A+t\zeta}\bigg|_{t=0} \quad \text{exists.}$$

We will adopt here the definition of *Fréchet* differentiability for $G : \mathcal{X}(\Omega) \to \mathcal{X}(\Omega)^*$. It is stronger than the above one in three respects: we require that for each φ the directional derivatives at A can be collected as a *continuous linear* map $\mathcal{X}(\Omega) \to \mathbb{R}$, that all these linear maps for various φ can be collected as a continuous linear map $\mathcal{X}(\Omega) \to \mathcal{X}(\Omega)^*$, and that in the Taylor formula defining the derivative, the remainder is of the form $o(\|\varphi\|\,\|\zeta\|)$ (as $\zeta \to 0$). Note that at this point this strong definition is already ensured by the analyticity of G and we only want to get an explicit formula.

We shall now state our results, which will be proven in the next two subsections (first the main argument, then the ends of the proofs of each result). Recall that given a function φ, we denote by $\varphi_A := \varphi - \int \varphi \,\mathrm{d}\mu_A$ the projection of φ onto $\ker \mu_A$ along the space of constants.

Theorem 4.1. *Under hypotheses (H1) to (H4), for all $A \in \mathcal{U}$ there is a neighborhood V of 0 in $\mathcal{X}(\Omega)$ such that for all $\varphi \in \mathcal{X}(\Omega)$ and all $\zeta \in V$,*

$$\int \varphi \,\mathrm{d}\mu_{A+\zeta} - \int \varphi \,\mathrm{d}\mu_A = \int (I - \mathscr{L}_{N(A)})^{-1}(\varphi_A) \cdot DN_A(\zeta)\,\mathrm{d}\mu_A + O(\|\varphi\|\,\|\zeta\|^2).$$

Implicitly, the O-constant depends only on A (and of course V, Ω, T, $\mathcal{X}(\Omega)$) but not on φ or ζ. This result will be deduced from the following special case where the expression is simpler.

Theorem 4.2. *Under hypotheses (H1) to (H4), for all $A \in \mathcal{N}$, all φ having mean 0 with respect to μ_A, and all small enough ζ tangent to $\mathcal{N}$ at A,*

$$\int \varphi \, \mathrm{d}\mu_{A+\zeta} = \int (I - \mathscr{L}_A)^{-1}(\varphi) \cdot \zeta \, \mathrm{d}\mu_A + O(\|\varphi\| \, \|\zeta\|^2).$$

Writing G_φ for the composition of the evaluation at φ and the Gibbs map, i.e. $G_\varphi(A) = \int \varphi \, \mathrm{d}\mu_A$, we can recast the above formula as

$$D(G_\varphi)_A(\zeta) = \int (I - \mathscr{L}_A)^{-1}(\varphi) \cdot \zeta \, \mathrm{d}\mu_A \quad \text{when } A \in \mathcal{N},\ \zeta \in \ker \mathscr{L}_A,\ \varphi \in \ker \mu_A.$$

Using the series expression of $(I - \mathscr{L}_A)^{-1}$ and the fact that μ_A is fixed by $\mathscr{L}_A^*$ and that the transfer operator is a left-inverse to the composition operator, we can write this also as

$$D(G_\varphi)_A(\zeta) = \sum_{i=0}^{\infty} \int \varphi \cdot \zeta \circ T^i \, \mathrm{d}\mu_A. \tag{4.1}$$

This version has the advantage that it applies to test functions φ not necessarily in $\ker \mu_A$, because $\zeta \in \ker \mathscr{L}_A$ implies $\zeta \in \ker \mu_A$, and therefore adding a constant to φ does not change the value of the integrals.

We can rephrase Theorem 4.1 in a similar way, which will be used to define a metric on $\mathcal{X}(\Omega)$.

Corollary 4.3. *Under hypotheses (H1) to (H4), for all $A \in \mathcal{U}$ and $\zeta, \varphi \in \mathcal{X}(\Omega)$, if ζ is small enough then*

$$\begin{aligned} &\int \varphi \, \mathrm{d}\mu_{A+\zeta} - \int \varphi \, \mathrm{d}\mu_A \\ &\qquad = \int \varphi_A \zeta \, \mathrm{d}\mu_A + \sum_{i=1}^{\infty} \int (\varphi_A \cdot \zeta \circ T^i + \varphi_A \circ T^i \cdot \zeta) \, \mathrm{d}\mu_A + O(\|\varphi\| \, \|\zeta\|^2) \end{aligned}$$

where the sum converges and defines a continuous bilinear form.

Again, this means (here without normalizing conditions) that

$$D(G_\varphi)_A(\zeta) = \int \varphi_A \zeta \, \mathrm{d}\mu_A + \sum_{i=1}^{\infty} \int (\varphi_A \cdot \zeta \circ T^i + \varphi_A \circ T^i \cdot \zeta) \, \mathrm{d}\mu_A.$$

Observe that if T were invertible, the right-hand side above would read

$$\sum_{i \in \mathbb{Z}} \varphi_A \cdot \zeta \circ T^i \mathrm{d}\mu_A;$$

our setting does not apply directly to invertible maps (as $\mathcal{N}$ is reduced to a point in this case), but this feels reminiscent of the way one can use a coboundary to translate a potential for a two-sided subshift to a potential depending only on the past, thus relating to a one-sided subshift. Here, the non-normalized case yields an expression (Corollary 4.3) "depending on the past and the future" in some sense, while after normalization we get (4.1) which "only depends on the past". Unsurprisingly, $D(G_\varphi)_A$ will appear as the variance in the Central Limit Theorem for φ and μ_A (a "Green–Kubo formula").

4.1. The case of a pair of normalized potentials

To obtain Theorem 4.2, thanks to the regularity of the normalization map proved in the previous section, we are mostly reduced to estimating $\int \varphi \, \mathrm{d}(\mu_B - \mu_A)$ when $\varphi \in \mathcal{X}(\Omega)$ is fixed and A, B are *normalized* potentials which are close enough to each other. Up to adding a constant to φ, which does not change the value of the above integral, we assume that $\varphi \in \ker \mu_A$.

We first write (using the fact that μ_A and μ_B are fixed by $\mathscr{L}_A^*$ and $\mathscr{L}_B^*$ respectively)

$$\begin{aligned}\int \varphi \, \mathrm{d}(\mu_B - \mu_A) &= \int \mathscr{L}_B(\varphi) \, \mathrm{d}\mu_B - \int \mathscr{L}_A(\varphi) \, \mathrm{d}\mu_A \\ &= \int (\mathscr{L}_B(\varphi) - \mathscr{L}_A(\varphi)) \, \mathrm{d}\mu_B + \int \mathscr{L}_A(\varphi) \, \mathrm{d}(\mu_B - \mu_A). \qquad (4.2)\end{aligned}$$

Then, we observe that

$$\begin{aligned}(\mathscr{L}_B(\varphi) - \mathscr{L}_A(\varphi))(x) &= \sum_{T(y)=x} e^{A(y)} \varphi(y)(e^{B(y)-A(y)} - 1) \\ &= \mathscr{L}_A(\varphi \cdot (e^{B-A} - 1))(x),\end{aligned}$$

so that writing $R(x) = e^x - 1 - x \sim \frac{1}{2}x^2$, we get

$$\mathscr{L}_B(\varphi) - \mathscr{L}_A(\varphi) = \mathscr{L}_A(\varphi \cdot (B - A)) + \mathscr{L}_A(\varphi \cdot R(B - A)).$$

Thus

$$\begin{aligned}\int \varphi \, \mathrm{d}(\mu_B - \mu_A) &= \int \mathscr{L}_A(\varphi \cdot (B - A)) \, \mathrm{d}\mu_B + \int \mathscr{L}_A(\varphi \cdot R(B - A)) \, \mathrm{d}\mu_B \\ &\quad + \int \mathscr{L}_A(\varphi) \, \mathrm{d}(\mu_B - \mu_A) \\ &= \int \mathscr{L}_A(\varphi \cdot (B - A)) \, \mathrm{d}\mu_A + \int \mathscr{L}_A(\varphi \cdot (B - A)) \, \mathrm{d}(\mu_B - \mu_A) \\ &\quad + \int \mathscr{L}_A(\varphi \cdot R(B - A)) \, \mathrm{d}\mu_B + \int \mathscr{L}_A(\varphi) \, \mathrm{d}(\mu_B - \mu_A),\end{aligned}$$

so

$$\int \varphi \, \mathrm{d}(\mu_B - \mu_A) = \int \varphi \cdot (B - A) \, \mathrm{d}\mu_A + \int \mathscr{L}_A(\varphi) \, \mathrm{d}(\mu_B - \mu_A) + \mathscr{I}(\varphi, B) \qquad (4.3)$$

where $\mathscr{I}(\varphi, B) = \int \mathscr{L}_A(\varphi \cdot (B - A)) \, \mathrm{d}(\mu_B - \mu_A) + \int \mathscr{L}_A(\varphi \cdot R(B - A)) \, \mathrm{d}\mu_B$, which is linear in φ and which we now wish to bound by a multiple of $\|\varphi\| \, \|B - A\|^2$.

A first tool is the regularity of G.

Lemma 4.4. *The map $G : \mathcal{U} \to \mathcal{X}(\Omega)^*$ is locally Lipschitz: for all $A \in \mathcal{U}$ there exist a neighborhood $V \in \mathcal{U}$ of A and a constant C such that for all $\varphi \in \mathcal{X}(\Omega)$ and all $B \in V$,*

$$\left| \int \varphi \, \mathrm{d}(\mu_B - \mu_A) \right| \leq C \|\varphi\| \, \|B - A\|.$$

Proof. This follows from the analyticity of G obtained in Corollary 3.7. □

A second observation is that since $\mathcal{X}(\Omega)$ has a multiplicative norm, we get

$$\|R(B-A)\| = \left\| \sum_{k\geq 2} \frac{1}{k!}(B-A)^k \right\| \leq \sum_{k\geq 2} \frac{1}{k!}\|B-A\|^k = R(\|B-A\|) \leq C'\|B-A\|^2$$

when B is in any fixed neighborhood V of A.

Now, since $\|\cdot\|$ is assumed to control the sup norm and μ_B is a probability measure, whenever $B \in V$ we have

$$|\mathscr{I}(\varphi, B)| \leq C|\mathscr{L}_A|\,\|\varphi\|\,\|B-A\|^2 + C''|\mathscr{L}_A|\,\|\varphi\|\,\|R(B-A)\| \leq C'''\|\varphi\|\,\|B-A\|^2.$$

Now, applying (4.3) to its own second term repeatedly and recalling that $\mathscr{L}_A(\varphi)$ goes to zero thanks to the spectral gap assumption, we get

$$\begin{aligned}
\int \varphi \,\mathrm{d}(\mu_B - \mu_A) &= \int \varphi \cdot (B-A)\,\mathrm{d}\mu_A + \int \mathscr{L}_A(\varphi)\,\mathrm{d}(\mu_B - \mu_A) \\
&\quad + \mathscr{I}(\varphi, B) \\
&= \int (\varphi + \mathscr{L}_A(\varphi)) \cdot (B-A)\,\mathrm{d}\mu_A + \int \mathscr{L}_A^2(\varphi)\,\mathrm{d}(\mu_B - \mu_A) \\
&\quad + \mathscr{I}(\varphi + \mathscr{L}_A(\varphi), B) \\
&= \int \Big(\sum_{n\geq 0} \mathscr{L}_A^n(\varphi)\Big) \cdot (B-A)\,\mathrm{d}\mu_A + \mathscr{I}\Big(\sum_{n\geq 0} \mathscr{L}_A^n(\varphi), B\Big),
\end{aligned}$$

so

$$\int \varphi \,\mathrm{d}(\mu_B - \mu_A) = \int (I - \mathscr{L}_A)^{-1}(\varphi) \cdot (B-A)\,\mathrm{d}\mu_A + O(\|\varphi\|\,\|B-A\|^2), \qquad (4.4)$$

which is almost Theorem 4.2, except for the assumption that B is normalized.

4.2. *End of proofs*

Proof of Theorem 4.2. Since N is an analytic projection to $\mathcal{N}$ (i.e. N restricted to $\mathcal{N}$ is the identity), we have

$$N(A+\zeta) = A + \zeta + O(\|\zeta\|^2)$$

for all $A \in \mathcal{N}$ and all small enough $\zeta \in T_A\mathcal{N} = \ker \mathscr{L}_A$, with an implicit constant only depending on A.

Fix $A \in \mathcal{N}$, $\zeta \in \ker \mathscr{L}_A$ and $\varphi \in \ker \mu_A$, and set $B = N(A+\zeta)$. Using (4.4) with the normalized potentials A and B, we get

$$\begin{aligned}
\int \varphi \,\mathrm{d}(\mu_{A+\zeta} - \mu_A) &= \int \varphi \,\mathrm{d}(\mu_{A+\zeta} - \mu_B) + \int \varphi \,\mathrm{d}(\mu_B - \mu_A) \\
&= O\big(\|\varphi\|\,\|A+\zeta - N(A+\zeta)\|\big) \\
&\quad + \int (I - \mathscr{L}_A)^{-1}(\varphi) \cdot (B-A)\,\mathrm{d}\mu_A + O(\|\varphi\|\,\|B-A\|^2) \\
&= \int (I - \mathscr{L}_A)^{-1}(\varphi) \cdot \zeta \,\mathrm{d}\mu_A + O(\|\varphi\|\,\|\zeta\|^2)
\end{aligned}$$

for ζ small enough, and with an implicit constant that depends only on A. □

Proof of Theorem 4.1. Let $A \in \mathcal{U}$ and $\zeta, \varphi \in \mathcal{X}(\Omega)$ be arbitrary. Then we consider:

- $N(A)$, which is the normalized potential such that $\mu_{N(A)} = \mu_A$,
- $DN_A(\zeta)$, which is the projection of ζ onto $\ker \mathscr{L}_{N(A)}$ in the direction of $\mathcal{C}$,
- $\varphi_A = \varphi - \int \varphi \, \mathrm{d}\mu_A \in \ker \mu_A$,

and we apply Theorem 4.2 to this new potential, tangent vector, and test function. We obtain exactly the desired expression once we notice that

$$\begin{aligned} |\mu_{A+\zeta} - \mu_{N(A)+DN_A(\zeta)}| &= |\mu_{N(A+\zeta)} - \mu_{N(A)+DN_A(\zeta)}| \\ &= O\big(\|N(A+\zeta) - N(A) - DN_A(\zeta)\|\big) = O(\|\zeta\|^2). \end{aligned}$$ □

Proof of Corollary 4.3. We have to rewrite

$$\int (I - \mathscr{L}_{N(A)})^{-1} \varphi_A \cdot DN_A(\zeta) \, \mathrm{d}\mu_A.$$

We first observe that the final expression we aim for only involves A through the measure μ_A, so that we can as well replace A by $N(A)$, i.e. assume that A is normalized (this is just to avoid writing $\mathscr{L}_{N(A)}$ a dozen times).

We first write $(I - \mathscr{L}_A)^{-1} \varphi_A = \sum_{i \geq 0} \mathscr{L}_A^i \varphi_A$, and recall that $DN_A(\zeta)$ is the projection of ζ to $\ker \mathscr{L}_A$ along $\mathcal{C}$; this means that there is a function $g \in \mathcal{X}(\Omega)$ such that

$$DN_A(\zeta) = \zeta_A + g - g \circ T \tag{4.5}$$

(where $\zeta_A = \zeta - \int \zeta \, \mathrm{d}\mu_A \in \ker \mu_A$) and $\mathscr{L}_A(DN_A(\zeta)) = 0$. In particular, $\mathscr{M}_A(DN_A(\zeta)) = 0$; thus,

$$g = \mathscr{M}_A(DN_A(\zeta) - \zeta_A) = -\mathscr{M}_A(\zeta) = \sum_{i \geq 1} \mathscr{L}_A^i \zeta_A.$$

This leads us to

$$\begin{aligned} &\int (I - \mathscr{L}_A)^{-1} \varphi_A \cdot DN_A(\zeta) \, \mathrm{d}\mu_A \\ &\quad = \int \sum_{i \geq 0} \mathscr{L}_A^i \varphi_A \cdot \zeta_A \, \mathrm{d}\mu_A + \int \sum_{i \geq 0} \mathscr{L}_A^i \varphi_A \cdot g \, \mathrm{d}\mu_A - \int \sum_{i \geq 0} \mathscr{L}_A^i \varphi_A \cdot g \circ T \, \mathrm{d}\mu_A \\ &\quad = \int \sum_{i \geq 0} \mathscr{L}_A^i \varphi_A \cdot \zeta_A \, \mathrm{d}\mu_A + \int \sum_{i \geq 0} \mathscr{L}_A^i \varphi_A \cdot g \, \mathrm{d}\mu_A - \int \sum_{i \geq 0} \mathscr{L}_A^{i+1} \varphi_A \cdot g \, \mathrm{d}\mu_A \\ &\quad = \int \sum_{i \geq 0} \mathscr{L}_A^i \varphi_A \cdot \zeta_A \, \mathrm{d}\mu_A + \int \varphi_A \cdot g \, \mathrm{d}\mu_A \\ &\quad = \int \sum_{i \geq 0} \mathscr{L}_A^i \varphi_A \cdot \zeta_A \, \mathrm{d}\mu_A + \int \varphi_A \cdot \sum_{i \geq 1} \mathscr{L}_A^i \zeta_A \, \mathrm{d}\mu_A, \end{aligned}$$

and, by the invariance of μ_A under $\mathscr{L}_A^*$,

$$\begin{aligned} &\int (I - \mathscr{L}_A)^{-1} \varphi_A \cdot DN_A(\zeta) \, \mathrm{d}\mu_A \\ &\qquad = \int \varphi_A \zeta_A \, \mathrm{d}\mu_A + \sum_{i \geq 1} \int (\varphi_A \cdot \zeta_A \circ T^i + \varphi_A \circ T^i \cdot \zeta_A) \, \mathrm{d}\mu_A, \end{aligned}$$

where the sum converges (exponentially).

Finally, we observe that there is no use normalizing both φ and ζ, since for example $\int \varphi_A \zeta_A \, \mathrm{d}\mu_A = \int \varphi_A \zeta \, \mathrm{d}\mu_A$. All ζ_A can therefore be replaced by ζ, and we get the desired formula. □

5. A Riemannian metric on the space of normalized potentials

The goal of this section is to define and study a (weak) Riemannian metric on the space of Gibbs measures. More precisely, we construct a Riemannian metric on the manifold of normalized potentials, which corresponds equivalently to a Riemannian metric on the quotient space $\widehat{\mathcal{U}} = \mathcal{U}/\mathcal{C}$, and relates in various ways to dynamical quantities. Up to a conformal rescaling, this metric is very closely related to the metric defined by McMullen [McM08] (see also [BCS15] and references therein). Without entering into much detail, let us say that (in the context of rational maps acting on $\mathbb{C}$, in relation to the Weil–Petersson metric on the Teichmüller space) McMullen considers the "pressure metric" $g_A(\zeta)$ given at a potential A by the μ_A-variance of ζ, divided by the entropy of μ_A. We do not involve the entropy, so that the formula is more directly related to Section 4.

5.1. Weak and strong inner products on Banach spaces

Consider a positive symmetric bilinear form $\langle \cdot, \cdot \rangle$ on some Banach space $\mathcal{Y}$. There are two possible definitions of positive-definiteness. The first one is a copy-and-paste of the finite-dimensional definition: we require that

$$\forall y \neq 0 \in \mathcal{Y} : \quad \langle y, y \rangle > 0.$$

In this case, one says that $\langle \cdot, \cdot \rangle$ is *weakly positive-definite*. The second one is to require that the Banach norm $\|\cdot\|$ of $\mathcal{Y}$ controls $\langle \cdot, \cdot \rangle$ from below:

$$\exists C > 0, \forall y \in \mathcal{Y} : \quad \langle y, y \rangle \geq C \|y\|^2.$$

In this case, one says that $\langle \cdot, \cdot \rangle$ is *strongly positive-definite*; note that this condition implies weak positive-definiteness.

Most of the time, one is only interested in bilinear forms which are continuous with respect to the Banach topology of $\mathcal{Y}$. But if $\langle \cdot, \cdot \rangle$ is both continuous and strongly positive-definite, then its associated norm is equivalent to $\|\cdot\|$, and in particular $\mathcal{Y}$ must be isomorphic to a Hilbert space. Therefore, most Banach spaces have no continuous, strongly positive-definite inner product.

We shall say that $\langle \cdot, \cdot \rangle$ is an *inner product* if it is continuous and weakly positive-definite, and use the term *semi-definite inner product* for a merely continuous, positive semi-definite symmetric bilinear form. By a *Riemannian metric* on a smooth Banach manifold, we mean a field of inner products on the tangent spaces such that when translated in a chart, the inner product depends smoothly on the point, that is, it defines a smooth map from the domain of the chart to the Banach space of symmetric bilinear forms.

As a last remark, note that when $\langle \cdot, \cdot \rangle$ is an inner product inducing a complete norm, it endows $\mathcal{Y}$ with another structure of Banach space (more precisely, a Hilbert structure

of course). Then the the identity map $\mathcal{Y} \to \mathcal{Y}$ is a continuous bijection between the two Banach structures at hand, and is therefore an isomorphism. This implies in particular that $\langle \cdot, \cdot \rangle$ is strongly positive-definite. In other words, inner products which are not strongly positive-definite induce a norm which is never complete. This means that there will be a rather subtle interplay between the topology of $\mathcal{Y}$ and the measurements made from $\langle \cdot, \cdot \rangle$.

5.2. The variance metric

Now, we introduce our proposed metric. Its main properties are summed up in the following result.

Theorem 5.1. *Under hypotheses (H1) to (H5), there exists an analytic map from $\mathcal{U}$ to the space of continuous symmetric bilinear forms on $\mathcal{X}(\Omega)$, which maps any potential A to a semi-definite inner product $\langle \cdot, \cdot \rangle_A$ such that:*

(i) *$\langle \cdot, \cdot \rangle_A$ restricts to an inner product on $T_A\mathcal{N}$ for all $A \in \mathcal{N}$, thus inducing a Riemannian metric on $\mathcal{N}$,*

(ii) *this Riemannian metric coincides with the one obtained from $L^2(\mu_A)$:*

$$\forall A \in \mathcal{N}, \forall \eta, \zeta \in T_A\mathcal{N} : \quad \langle \eta, \zeta \rangle_A = \int \eta\zeta \, \mathrm{d}\mu_A,$$

(iii) *for all $A \in \mathcal{U}$, $\langle \cdot, \cdot \rangle_A$ induces a well-defined inner product on $\widehat{\mathcal{U}}$, thus inducing a Riemannian metric on this quotient space,*

(iv) *for all $A \in \mathcal{U}$ and $\zeta, \varphi \in \mathcal{X}(\Omega)$,*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \varphi \, \mathrm{d}\mu_{A+t\zeta} \bigg|_{t=0} = \langle \varphi, \zeta \rangle_A,$$

(v) *for all $A \in \mathcal{U}$ and $\varphi, \zeta \in \mathcal{X}(\Omega)$,*

$$\langle \varphi, \zeta \rangle_A = \int \varphi_A \zeta \, \mathrm{d}\mu_A + \sum_{i=1}^{\infty} \int (\varphi_A \cdot \zeta \circ T^i + \varphi_A \circ T^i \cdot \zeta) \, \mathrm{d}\mu_A$$

and

$$\langle \zeta, \zeta \rangle_A = \lim \frac{1}{n} \int \Big(\sum_{i=0}^{n-1} \zeta_A \circ T^i \Big)^2 \mathrm{d}\mu_A =: \mathrm{Var}(\zeta_A, \mu_A).$$

Of course, the metrics in $\mathcal{N}$ and $\widehat{\mathcal{U}}$ correspond to each other through the natural identification between these two spaces. There is really only one Riemannian metric, which can be viewed in two ways. Any of the above expressions could be taken as a definition of $\langle \cdot, \cdot \rangle_A$. Our point here is that these expressions define the same bilinear form, inducing an inner product on $T_A\mathcal{N}$. Note that the last item relates the metric to the Central Limit Theorem through the variance (or equivalently the sum of correlations)—see Section 5.3.

The end of this section is devoted to the proof of Theorem 5.1, so from now on we assume hypotheses (H1) to (H5).

Definition 5.2. For all $A \in \mathcal{U}$ and all $\eta, \zeta \in \mathcal{X}(\Omega)$, we set

$$\langle \eta, \zeta \rangle_A := D^2(\log \Lambda)_A(\eta, \zeta).$$

This definition gives the symmetry of $\langle \cdot, \cdot \rangle_A$ right away, and from Corollary 3.6 we deduce first that $A \mapsto \langle \cdot, \cdot \rangle_A$ is analytic, and second, since $G_\eta(A) = D(\log \Lambda)_A(\eta)$,

$$\langle \eta, \zeta \rangle_A = D(G_\eta)_A(\zeta) = D(G_\zeta)_A(\eta),$$

a symmetry which would not be obvious without the link with the leading eigenvalue.

As it is, $\langle \cdot, \cdot \rangle_A$ does not define an inner product, because it is not weakly positive-definite.

Proposition 5.3. *The symmetric form $\langle \cdot, \cdot \rangle_A$ is positive semi-definite, and for all $A \in \mathcal{N}$ and $\eta, \zeta \in T_A\mathcal{N}$ we have $\langle \eta, \zeta \rangle_A = \int \eta\zeta \, \mathrm{d}\mu_A$. Moreover, given $A, \zeta \in \mathcal{X}(\Omega)$, the following three statements are equivalent:*

(i) $\langle \zeta, \eta \rangle_A = 0$ *for all* $\eta \in \mathcal{X}(\Omega)$,
(ii) $\langle \zeta, \zeta \rangle_A = 0$,
(iii) ζ *is cohomologous to a constant, i.e.* $\zeta \in \mathcal{C}$.

Proof. First, Corollary 4.3 gives the following expressions:

$$\begin{aligned}\langle \eta, \zeta \rangle_A &= \int \eta_A \cdot \zeta_A \, \mathrm{d}\mu_A + \sum_{i \geq 1} \int (\eta_A \cdot \zeta_A \circ T^i + \eta_A \circ T^i \cdot \zeta_A) \, \mathrm{d}\mu_A \\ &= \int \eta_A \cdot \zeta_A \, \mathrm{d}\mu_A + \sum_{i \geq 1} \int \left(\mathscr{L}^i_{N(A)}(\eta_A) \cdot \zeta_A + \eta_A \cdot \mathscr{L}^i_{N(A)}(\zeta_A) \right) \mathrm{d}\mu_A.\end{aligned}$$

When $A \in \mathcal{N}$ and $\eta, \zeta \in T_A\mathcal{N} = \ker \mathscr{L}_A$ we therefore get $\eta_A = \eta$, $\zeta_A = \zeta$ and $\mathscr{L}_A(\eta) = \mathscr{L}_A(\zeta) = 0$, so that

$$\langle \eta, \zeta \rangle_A = \int \eta_A \cdot \zeta_A \, \mathrm{d}\mu_A + \sum_{i \geq 1} \int \left(\mathscr{L}^i_A(\eta_A) \cdot \zeta_A + \eta_A \cdot \mathscr{L}^i_A(\zeta_A) \right) \mathrm{d}\mu_A = \int \eta \cdot \zeta \, \mathrm{d}\mu_A.$$

It is clear that (i) implies (ii). Theorem 4.1 states that

$$\langle \eta, \zeta \rangle_A = \int (I - \mathscr{L}_{N(A)})^{-1}(\eta_A) \cdot DN_A(\zeta) \, \mathrm{d}\mu_A,$$

and as a consequence (iii) implies (i): if $\zeta \in \mathcal{C}$, since DN_A is a linear projector along $\mathcal{C}$, we have $DN_A(\zeta) = 0$ and ζ must be orthogonal to all vectors in $\mathcal{X}(\Omega)$.

Let us now show that (ii) implies (iii). Since $\langle \cdot, \cdot \rangle_A$ does not change if we add a constant or a coboundary to A (i.e. it only depends on μ_A), we can assume that A is normalized.

Suppose that ζ is isotropic, i.e. $\langle \zeta, \zeta \rangle_A = 0$. By Proposition 3.3, we decompose $\zeta = \zeta' + f$, where $\zeta' \in \ker \mathscr{L}_A$ and $f \in \mathcal{C}$. Then $\langle \zeta', \zeta' \rangle_A = \langle \zeta, \zeta \rangle_A$, since $\mathcal{C}$ is in the kernel of $\langle \cdot, \cdot \rangle_A$. Thus, $0 = \langle \zeta', \zeta' \rangle_A = \int \zeta'^2 \, \mathrm{d}\mu_A$. It follows from hypothesis (H5) that $\zeta' = 0$ and $\zeta = f \in \mathcal{C}$. The same reasoning shows that $\langle \cdot, \cdot \rangle_A$ is positive semi-definite. □

The last part of this proof is the only part where we use (H5).

Definition 5.4. Let $A \in \mathcal{U}$ and $[A] \in \widehat{\mathcal{U}} = \mathcal{U}/\mathcal{C}$ be its class modulo $\mathcal{C}$. For all $[\eta]$, $[\zeta]$ in $\widehat{\mathcal{X}} = \mathcal{X}(\Omega)/\mathcal{C}$, we define

$$\langle [\eta], [\zeta] \rangle_{[A]} = \langle \eta, \zeta \rangle_A,$$

which is well-defined by Proposition 5.3, i.e. does not depend on the chosen representatives in each class. If $A \in \mathcal{N}$, we still write $\langle \cdot, \cdot \rangle_A$ for the restriction of this inner product to $T_A\mathcal{N}$. Proposition 5.3 shows that both these products induce norms on the Banach spaces they are defined on ($\widehat{\mathcal{X}}$ and $T_A\mathcal{N} = \ker \mathscr{L}_A$, respectively). We denote both norms by $\|\cdot\|_A$, i.e.

$$\|\zeta\|_A = \sqrt{\langle \zeta, \zeta \rangle_A},$$

and we use this notation for general $\zeta \in \mathcal{X}(\Omega)$.

Let us now prove (in the usual way) statement (v) of Theorem 5.1. A direct computation gives

$$\begin{aligned}\int \Big(\sum_{i=0}^{n} \zeta_A \circ T^i\Big)^2 \mathrm{d}\mu_A &= \sum_{i,j=0}^{n-1} \int \zeta_A \circ T^i \cdot \zeta_A \circ T^j \,\mathrm{d}\mu_A \\ &= n \int \zeta_A^2 \,\mathrm{d}\mu_A + 2 \sum_{0 \le i < j \le n-1} \int \zeta_A \cdot \zeta_A \circ T^{j-i} \,\mathrm{d}\mu_A \\ &= n \int \zeta_A^2 \,\mathrm{d}\mu_A + 2 \sum_{k=1}^{n-1} (n-k) \int \zeta_A \cdot \zeta_A \circ T^k \,\mathrm{d}\mu_A.\end{aligned}$$

This equality only involves A through μ_A, so we can assume without loss of generality that A is normalized. Then

$$\begin{aligned}\frac{1}{n} \int \Big(\sum_{i=0}^{n-1} \zeta_A \circ T^i\Big)^2 \mathrm{d}\mu_A &= \int \zeta_A^2 \,\mathrm{d}\mu_A + 2 \sum_{k=1}^{n-1} \int \zeta_A \cdot \zeta_A \circ T^k \,\mathrm{d}\mu_A \\ &\quad - \frac{2}{n} \sum_{k=1}^{n-1} \int k \mathscr{L}_A^k(\zeta_A) \cdot \zeta_A \,\mathrm{d}\mu_A,\end{aligned}$$

where the last term is bounded in norm by $O((\|\zeta_A\|^2/n) \sum_{k=1}^{\infty} k\delta^k)$ with δ the spectral gap of $\mathscr{L}_A$. It follows that for all A, ζ we have

$$\mathrm{Var}(\zeta_A, \mu_A) = \|\zeta\|_A^2,$$

which, as is usual in common examples and as follows from Proposition 5.3, vanishes exactly when $\zeta \in \mathcal{C}$.

This concludes the proof of Theorem 5.1, and of Theorem D.

Remark 5.5. We have thus recovered in our setting the convexity of $\log \Lambda$, which—as is customary—will be interpreted in terms of "pressure" below.

A notable consequence is that the submanifold $\mathcal{N}$ does not contain any straight interval not reduced to a point. Indeed, assume that $(A + t\zeta)_{-\varepsilon < t < \varepsilon}$ is a straight interval

in $\mathcal{N}$; then first $\zeta \in T_A\mathcal{N}$, and second $\log \Lambda(A+t\zeta)$ is constantly 0 as t varies. Its second derivative must thus vanish, but at $t=0$ it is equal to $\langle \zeta, \zeta \rangle_A$, so that $\zeta \in \mathcal{C}$ by Proposition 5.3. Since $\mathcal{C}$ and $T_A\mathcal{N}$ are complements to each other, we have $\zeta = 0$ and the interval is reduced to a point.

5.3. The Central Limit Theorem

Let us now use Theorem 5.1 to show that variance really deserves its name in our general framework. The proof below is classical (the details are based on the presentation by Climenhaga [Cli13]); we include it nonetheless for the following reasons:

- the proof recalls why the second derivative of $\log \Lambda$ should be called the variance, and shows that varying the potential is an important tool even when one is only interested in one fixed Gibbs measure,
- we want to stress that the classical argument carries over to our setting, and
- we want to point out that there is no real need for complex potentials as long as one has strong enough integrability properties (here boundedness of elements of $\mathcal{X}(\Omega)$), as the moment-generating function can replace the characteristic function.

Theorem 5.6. *Under hypotheses (H1) to (H5), Birkhoff averages satisfy a Central Limit Theorem: Let $A \in \mathcal{U}$ and $\varphi \in \mathcal{X}(\Omega)$ be such that $\int \varphi \, \mathrm{d}\mu_A = 0$ and $\sigma^2 := \mathrm{Var}(\varphi, \mu_A) = \langle \varphi, \varphi \rangle_A \neq 0$. Let P be a random point on Ω with distribution μ_A, and consider the sequence of real random variables $X_k = \varphi(T^k(P))$ such that $X_0 + \cdots + X_{n-1}$ is the Birkhoff sum $S_n\varphi := \sum_{k=0}^{n-1} \varphi \circ T^k$ evaluated at P. Then the random variable*

$$\frac{X_0 + X_1 + \cdots + X_{n-1}}{\sqrt{n}}$$

converges in distribution to a centered normal distribution of variance σ^2.

The statement can be rewritten, as is more common in dynamics, to state that for all $c \in \mathbb{R}$,

$$\mu_A\left(\left\{x \in \Omega \;\middle|\; \frac{S_n\varphi(x)}{\sqrt{n}} < c\right\}\right) \to \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{c} e^{-\frac{t^2}{2\sigma^2}} \, \mathrm{d}t \quad \text{as } n \to \infty.$$

Proof. Since the statement is formulated in terms of μ_A only, we can assume A is normalized.

The convergence in distribution is equivalent to the convergence of the moment-generating function (which exists since $S_n\varphi$ is bounded for each n), so we only need to prove that for all $t \in \mathbb{R}$,

$$\psi_n(t) := \int e^{t\frac{S_n\varphi}{\sqrt{n}}} \, \mathrm{d}\mu_A \to \psi(t) := e^{t^2\sigma^2/2} \quad \text{as } n \to \infty$$

where ψ is the moment-generating function of the centered normal distribution of variance σ^2.

We can write the iterates of $\mathscr{L}_A$ as follows:

$$\mathscr{L}_A^n f(x) = \sum_{T^n(y)=x} e^{S_n A(y)} f(y),$$

which, applied to $A + t\varphi$ for any t, reads

$$\mathscr{L}_{A+t\varphi}^n f(x) = \sum_{T^n(y)=x} e^{S_n A(y)} e^{t S_n \varphi(y)} f(y) = \mathscr{L}_A^n (e^{t S_n \varphi} f)(x).$$

This yields

$$\psi_n(t) = \int e^{\frac{t}{\sqrt{n}} S_n \varphi} \, \mathrm{d}\mu_A = \int \mathscr{L}_A^n (e^{\frac{t}{\sqrt{n}} S_n \varphi}) \, \mathrm{d}\mu_A = \int \mathscr{L}_{A+\frac{t}{\sqrt{n}}\varphi}^n (\mathbf{1}) \, \mathrm{d}\mu_A.$$

Now, using the operator decomposition of the proof of Proposition 3.8 and a Taylor series, we have (for fixed t and increasing n)

$$\begin{aligned}\mathscr{L}_{A+\frac{t}{\sqrt{n}}\varphi}^n &= \Lambda\left(A + \frac{t}{\sqrt{n}}\varphi\right) \mathscr{P}_{A+\frac{t}{\sqrt{n}}\varphi} + \mathscr{R}_{A+\frac{t}{\sqrt{n}}\varphi}^n \\ &= \left(\Lambda(A) + \frac{t}{\sqrt{n}} D\Lambda_A(\varphi) + \frac{t^2}{2n} D^2\Lambda_A(\varphi,\varphi) + o\left(\frac{t^2}{n}\right)\right)^n (\mathscr{P}_A + o(1)) \\ &\quad + o((1-\delta)^n)\end{aligned}$$

where δ is a local uniform spectral gap as given by Proposition 3.8 and all o are in operator norm. Since $\Lambda(A) = 1$ and

$$D\Lambda_A(\varphi) = \Lambda(A) \cdot D(\log \Lambda)_A(\varphi) = 1 \cdot \int \varphi \, \mathrm{d}\mu_A = 0,$$

the important term in the Taylor series is determined by $D^2\Lambda_A(\varphi,\varphi) = \mathrm{Var}(\varphi, \mu_A) = \sigma^2$. Combining the previous two computations we get

$$\psi_n(t) \underset{n\to\infty}{\sim} \left(1 + \frac{t^2}{2n}\sigma^2\right)^n \to e^{t^2\sigma^2/2} = \psi(t),$$

which concludes the proof. □

6. Regularity of the Gibbs map: the Wasserstein structure

The development of optimal transportation and more precisely of the 2-Wasserstein distance has let an alternative differential structure for the set $\mathcal{P}(\Omega)$ emerge, notably driven by the work of Otto [O01], Benamou and Brenier [BB00] and Ambrosio, Gigli and Savaré [AGS05]. We shall rely on the formulation given by [Gi11], which allows one to define the differentiability of a map at a point (as opposed to more global notions, such as speed vectors defined almost everywhere). One could in principle consider the case when Ω is a Riemannian manifold, but for simplicity we shall restrict ourselves to $\Omega = \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ throughout this section.

6.1. Elements of optimal transportation

We will not give much detail on optimal transportation, but many references are available (see e.g. [Vi09] for a comprehensive source). Let us say that the 2-Wasserstein distance W_2 is a metric compatible with the weak topology, defined on $\mathcal{P}(\Omega)$ as the least cost needed to move one measure to another, when the cost of moving a unit mass is proportional to the *squared* distance between the starting point and the stopping point.

For each $\mu \in \mathcal{P}(\mathbb{S}^1)$, Gigli introduces a tangent space $T_\mu\mathcal{P}(\mathbb{S}^1)$ which may be only a metric cone, but turns out to be a Hilbert space in a number of cases. There are several possible definitions of such a tangent space (or cone), e.g. in terms of geodesics, in terms of measures on the tangent bundle, or in terms of vector fields on the manifold; the work of Gigli ties all these points of view together when μ belongs to a certain class of "nice" measures. In the present one-dimensional case, the relevant class to be considered is the set of atomless measures. Assuming $\mu \in \mathcal{P}(\mathbb{S}^1)$ has no atoms, one can consider as tangent space to $\mathcal{P}(\mathbb{S}^1)$ at μ the space

$$T_\mu\mathcal{P}(\mathbb{S}^1) := L^2_\nabla(\mu) := \overline{\{\nabla f \mid f \in C^\infty(\mathbb{S}^1, \mathbb{R})\}}^{L^2(\mu)}$$

of vector fields on $\mathbb{S}^1$ which are square-integrable with respect to μ and which are limits of gradients of smooth functions in $L^2(\mu)$ (note that the quotient structure of $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ makes it possible to identify all tangent spaces of $\mathbb{S}^1$ with $\mathbb{R}$, so that we can see vector fields as functions, and ∇f is simply f'). There is an obvious exponential map: given μ and $v \in T_\mu\mathcal{P}(\mathbb{S}^1)$ one sets $\exp_\mu(v) = (\mathrm{Id} + v)_\#\mu$, i.e. the mass at any point $x \in \mathbb{S}^1$ is moved to $x + v(x)$ mod 1. Then for each $v \in T_\mu\mathcal{P}(\mathbb{S}^1)$, one gets an exponential curve $(\exp_\mu(tv))_{t\in[0,\varepsilon)}$ which has the property that

$$W_2(\mu, \exp_\mu(tv)) = t\|v\|_\mu + o(t)$$

where $\|v\|_\mu$ is the $L^2(\mu)$-norm of v (here the fact that v can be approximated by gradients is crucial).

We will say that a curve $t \mapsto \mu_t$ from an interval to $\mathcal{P}(\mathbb{S}^1)$ is *Wasserstein-differentiable* at t_0 with tangent vector $v \in T_{\mu_{t_0}}\mathcal{P}(\mathbb{S}^1)$ whenever

$$W_2(\mu_{t_0+h}, \exp_{\mu_{t_0}}(hv)) = o(h).$$

Similarly, a map $H : \mathcal{Y} \to \mathcal{P}(\mathbb{S}^1)$ from a Banach space to the set of probability measures on $\mathbb{S}^1$ is *Wasserstein-differentiable* at a point $A \in \mathcal{Y}$ in a direction $\zeta \in \mathcal{Y}$ whenever there exists $v \in T_\mu\mathcal{P}(\mathbb{S}^1)$ such that

$$W_2\big(H(A + t\zeta), \exp_{H(A)}(tv)\big) = o(t),$$

i.e. the tangent vector v describes the first-order variations of H in the Wasserstein distance. Of course, one can define more stringent versions of this definition (Fréchet-like rather than Gâteaux-like), but since our result is negative, we get the strongest statement by sticking to the weakest definition.

When Ω is a manifold, in each of its variations (Gâteaux or Fréchet), Wasserstein differentiability is stronger than the corresponding variation of affine differentiability because of the *continuity equation* below; roughly, affine differentiability is about recording the "vertical" variations of the measure, i.e. the variation of the weight it gives to any given set, while Wasserstein differentiability is about recording the "horizontal" variations of the measure, i.e. how one should move the mass in the most economical way in order to obtain the given change of measure. The physical principle of mass preservation leads to the continuity equation, which in the present case $\Omega = \mathbb{S}^1$ has the following form:

Lemma 6.1. *Assume that $(\mu_t)_t$ is a curve of probability measures on $\mathbb{S}^1$ which is differentiable at 0 with tangent vector $v \in T_{\mu_0}\mathcal{P}(\mathbb{S}^1)$. Then for all smooth functions φ,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \varphi \,\mathrm{d}\mu_t\bigg|_{t=0} = \int \varphi' v \,\mathrm{d}\mu_0.$$

The most common version of the continuity equation is stated for curves of measures, with the above equality integrated over time. The proof of the present version is very simple and can be found in [Kl15a].

A curve $(x_t)_{t\in I}$ in a metric space is said to be *absolutely continuous* whenever there is a positive function $g \in L^1(I)$ such that for all $t_0, t_1 \in I$,

$$d(x_{t_0}, x_{t_1}) \leq \int_{t_0}^{t_1} g(s)\,\mathrm{d}s$$

(note when considering a curve (μ_t) in $\mathcal{P}(\Omega)$ that this notion has nothing to do with each measure μ_t being absolutely continuous or not!) In other words, an absolutely continuous curve is a curve whose speed exists almost everywhere and is integrable. A particular case is given by Lipschitz curves, whose speed is in L^∞; absolute continuity is therefore a very mild regularity condition. A Rademacher theorem holds in this setting: an absolutely continuous curve in $\mathcal{P}(\mathbb{S}^1)$ endowed with the 2-Wasserstein distance is differentiable at almost every time and satisfies the mean value theorem (see [AGS05]).

6.2. Roughness of the Gibbs map in the Wasserstein space

We are now in a position to state and prove the main result of this section, which shows that the Gibbs map is very far from being Wasserstein-smooth.

Theorem 6.2. *Assume T is $x \mapsto dx \bmod 1$ acting on $\mathbb{S}^1$ and $\mathcal{X}(\mathbb{S}^1)$ is the space of α-Hölder functions for some $\alpha \in (0, 1]$. If $(A_t)_{t\in I}$ is any smooth curve in $\mathcal{X}(\mathbb{S}^1)$, then its image curve $(\mu_{A_t})_{t\in I}$ under the Gibbs map is not (even locally) absolutely continuous in $(\mathcal{P}(\mathbb{S}^1), W_2)$ unless it is constant (i.e. unless $A_t \in A_0 + \mathcal{C}$ for all t).*

Recalling the interpretation of the Wasserstein metric W_2 above, we see that changing smoothly the potential changes smoothly the levels of the Gibbs measure (Theorem 4.1), but in a way that corresponds to brutal reallocations of mass (Theorem 6.2). This result should be compared with [KLS14, Corollary 1.3], where a Lipschitz-regularity result is

proved for the Gibbs map when $\mathcal{P}(\Omega)$ is endowed with the 1-Wasserstein distance (which however does not yield a differentiable structure).

The proof mostly relies on the following pointwise non-differentiability result.

Proposition 6.3. *Under the same assumption as in Theorem 6.2, consider the Gibbs map $G : \mathrm{Hol}_\alpha(\mathbb{S}^1) \to \mathcal{P}(\mathbb{S}^1)$ sending each A to μ_A. If G is Wasserstein-differentiable at any potential A in any direction ζ, then either μ_A is the Lebesgue measure (i.e. $A \in \mathcal{C}$) or the derivative vanishes (i.e. $W_2(\mu_{A+t\zeta}, \mu_A) = o(t)$).*

Proof. Assume that G is Wasserstein-differentiable at A in the direction ζ.

If φ is any smooth function, on the one hand the continuity equation gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \varphi \, \mathrm{d}\mu_{A+t\zeta} \bigg|_{t=0} = \int \varphi' v \, \mathrm{d}\mu_A$$

where $v \in L^2(\mu_A)$ is some vector field (which can be approximated by gradients in $L^2(\mu_A)$); on the other hand, Section 4 gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \varphi \, \mathrm{d}\mu_{A+t\zeta} \bigg|_{t=0} = \int (I - \mathscr{L}_A)^{-1}(\varphi_A) \cdot DN_A(\zeta) \, \mathrm{d}\mu_A .$$

We get two very different-looking linear forms in φ which both describe the variations of its integral. The proof will thus be complete as soon as we prove that unless μ_A is the Lebesgue measure, these two forms can agree only by vanishing.

For this, we use the following approximation lemma.

Lemma 6.4. *Let μ be a measure on $\mathbb{S}^1$ which is singular with respect to the Lebesgue measure and without atoms. Then for all $f \in L^2(\mu)$ and all $\beta < 1$ there is a sequence of smooth functions $\varphi_n : \mathbb{S}^1 \to \mathbb{R}$ such that $\varphi_n' \to f$ in $L^2(\mu)$ and $\varphi_n \to 0$ in $\mathrm{Hol}_\beta(\mathbb{S}^1)$.*

Proof. We first claim that when $I \subset [0,1]$ is an interval of length ℓ, $w : I \to \mathbb{R}$ is measurable and μ-essentially bounded by some number M, and $\varepsilon > 0$, there is a smooth function $\varphi : I \to \mathbb{R}$ such that φ and all its derivatives vanish at the endpoints of I, $\|\varphi\|_\infty \le \varepsilon$, $\|\varphi'\|_\infty \le M$, and $\int_I (\varphi' - w)^2 \, \mathrm{d}\mu \le \varepsilon^2 \ell^2$.

Let $\eta > 0$ be arbitrary, to be chosen later on. Since μ is concentrated on a λ-negligible set, there is a finite set of intervals $I_1, \dots, I_k \subset I$ with disjoint interiors whose total length is less than η and whose complement $J = I \setminus (I_1 \cup \cdots \cup I_k)$ has μ-mass less than η. Let w_1 be the function which:

- is constant on each I_i, with value the μ-average of w on I_i,
- is constant on J, with value such that $\int_I w_1 \, \mathrm{d}\lambda = 0$.

By taking η small enough and by dividing the intervals I_i into smaller intervals, we can ensure that $\int (w - w_1)^2 \, \mathrm{d}\mu$ is arbitrarily small.

Let w_2 be a smooth approximation of w_1 such that $\int (w - w_2)^2 \, \mathrm{d}\mu$ stays small, w_2 is bounded by M, $\int_I w_2 = 0$, and w_2 is zero on some neighborhoods of the endpoints of I (this last condition is easy to fulfill since μ has no atoms).

Define a smooth, M-Lipschitz function φ by

$$\varphi(x) := \int_a^x w_2(t)\,\mathrm{d}t$$

where $a = \min I$ is the starting point of I. Then $\varphi' = w_2$ is close to w in $L^2(\mu, I)$-norm and bounded above by M (though φ'' is extremely large), and φ and its derivatives vanish at both endpoints of I. The uniform norm of φ is then bounded by $M\eta$, and can thus be made arbitrarily small, proving the claim.

Now, given v and an integer n, choose a μ-essentially bounded function $\bar{v}$ which is $1/n$-close to v in $L^2(\mu)$, call M its essential bound, and choose ℓ small enough to ensure that $\ell^{1-\beta}M < 1/n$. Divide $\mathbb{S}^1$ into intervals of length ℓ and apply the claim to each of them. The boundary conditions enable us to glue the smooth functions defined on each interval into a smooth function φ_n defined on $\mathbb{S}^1$, such that φ_n' is M-bounded and $1/n$-close to $\bar{v}$ in $L^2(\mu)$ and $\|\varphi_n\|_\infty < 1/n$. For any $x, y \in \mathbb{S}^1$, when $|x - y| \le \ell$ we get

$$\frac{|\varphi_n(x) - \varphi_n(y)|}{|x - y|^\beta} \le \|\varphi_n'\|\,|x - y|^{1-\beta} \le M\ell^{1-\beta} \le \frac{1}{n},$$

and when $|x - y| \ge \ell$ we get

$$\frac{|\varphi_n(x) - \varphi_n(y)|}{|x - y|^\beta} \le \frac{|\varphi_n(x) - 0| + |0 - \varphi_n(y)|}{|x - y|^\beta} \le \frac{2M\ell}{|x - y|^\beta} \le 2M\ell^{1-\beta} \le 2/n.$$

This proves the lemma. □

Now we simply apply the lemma to $f = v$, and $\beta = \alpha$ if $\alpha < 1$, or any lower β otherwise (using the fact that the thermodynamical formalism holds for the current T with any β). The sequence φ_n of smooth functions provided by the lemma yields

$$\begin{aligned}
\int v^2\,\mathrm{d}\mu_A &= \lim_n \int v\varphi_n'\,\mathrm{d}\mu_A && (\varphi_n' \to v \text{ in } L^2(\mu_A))\\
&= \lim_n \frac{\mathrm{d}}{\mathrm{d}t}\int \varphi_n\,\mathrm{d}\mu_{A+t\zeta}\Big|_{t=0} && \text{(continuity equation)}\\
&= \lim_n \int (I - \mathscr{L}_A)^{-1}((\varphi_n)_A)\cdot DN_A(\zeta)\,\mathrm{d}\mu_A && \text{(Theorem 4.2)}\\
&= \int (I - \mathscr{L}_A)^{-1}(0)\cdot DN_A(\zeta)\,\mathrm{d}\mu_A = 0 && (\varphi_n \to 0 \text{ in } \mathrm{Hol}_\beta(\mathbb{S}^1))
\end{aligned}$$

(note that $\varphi \mapsto (I - \mathscr{L}_A)^{-1}(\varphi_A)$ is a continuous operator in the Hölder norm, which dominates the uniform norm).

Here T is a model circle map, so that μ_A has full support (in other words, hypothesis (H5) holds), and we deduce that v vanishes μ_A-almost everywhere and the Wasserstein derivative of $\mu_{A+t\zeta}$ vanishes. □

Proof of Theorem 6.2. If $(\mu_{A_t})_{t\in I}$ is absolutely continuous, it is differentiable almost everywhere and from Proposition 6.3 we deduce that at each t such that μ_{A_t} is differentiable and not Lebesgue, its derivative vanishes. The mean value inequality ensures that $(\mu_{A_t})_{t\in I}$ must then be constant. □

We end this section with some open questions. First, Proposition 6.3 leaves open the following.

Question 6.5. In the case of $T : x \mapsto dx \bmod 1$, is the Gibbs map differentiable at A when $A \in \mathcal{C}$ (i.e. when μ_A is the Lebesgue measure)?

Second, note that the analogue of Theorem 6.2 for the shift is true independently of the map G, since the 2-Wasserstein space of an ultrametric space such as $\mathcal{A}^{\mathbb{N}}$ contains no absolutely continuous curve at all (see [Kl15b]). The usual semi-conjugacy map π from a full one-sided shift to a model circle induces a bijection between their Gibbs measures, but the way these Gibbs measures vary with a change of potential may differ from the Wasserstein point of view, as π "closes the gaps" from the Cantor set, and these gaps are what prevents absolutely continuous curves there.

Indeed, the 2-Wasserstein space of a manifold contains plenty of absolutely continuous curves (it is even a geodesic space), so when Ω has a smooth structure, the irregularity of G with respect to the Wasserstein metric can be somewhat surprising. One then wonders how much it has to do with G, and how much with its image:

Question 6.6. Assume Ω is a manifold and T is smooth. Are there any non-constant, absolutely continuous curves $(\mu_t)_{t\in I}$ in $(\mathcal{P}(\Omega), W_2)$ such that μ_t is T-invariant for all t? What about the subset of Gibbs measures with α-Hölder potential?

In other words, we ask whether from the 2-Wasserstein point of view the set of T-invariant measures is a nice, somewhat smooth subset of the set of all probability measures that happens to be badly parametrized by G, or if it is itself a very irregular subset of $\mathcal{P}(\mathbb{S}^1)$ (one can think of the von Koch curve in $\mathbb{R}^2$ as an example of a connected, very irregular subset of a smooth space).

7. Application to equilibrium states

In this section we use the differential calculus developed above to study several classical optimization problems. From now on we assume all hypotheses, (H1) to (H6). In particular $\mathcal{U} = \mathcal{X}(\Omega)$: all transfer operators have a spectral gap.

7.1. Entropy and pressure

Given our broad framework, we shall use the following Legendre transform definition of entropy: for any T-invariant measure ν, we set

$$\mathrm{h}_{\mathcal{X}}(\nu) := \inf_{A\in\mathcal{X}} \left(\log \lambda_A - \int A \, \mathrm{d}\nu \right).$$

Note that this quantity a priori depends on the chosen function class $\mathcal{X}(\Omega)$; but in many cases it is in fact equal to the metric entropy of μ (see Remarks 7.5 and 7.8). The assumption (H6) ensures that $\mathcal{X}(\Omega)$ is quite large, preventing $\mathrm{h}_{\mathcal{X}}$ from being too degenerate.

Remark 7.1. The number $\log \lambda_A - \int A \, d\nu$ only depends on the class $[A]$ of A modulo $\mathcal{C}$ (adding a constant to A changes $\log \lambda_A$ and $\int A \, d\nu$ by the same additive constant, and adding a coboundary leaves both terms unchanged). In particular, one can rewrite

$$h_{\mathcal{X}}(\nu) = \inf_{A \in \mathcal{N}} \int (-A) \, d\nu$$

and observe that $A(y) = \log e^{A(y)}$ where

$$\mathbb{P}(x \to y) := \begin{cases} e^{A(y)} & \text{when } T(y) = x, \\ 0 & \text{otherwise,} \end{cases}$$

defines transition probabilities for a Markov chain on Ω supported on backward orbits of T. In other words, $h_{\mathcal{X}}(\nu)$ is the infimum of $\int (-\log \mathbb{P}(T(y) \to y)) \, d\nu(y)$ over Markov chains supported on backward orbits of T, such that transition probabilities depend on the endpoint, with a regularity specified by $\mathcal{X}(\Omega)$.

Together with such a definition of entropy naturally comes a dual quantity, the *pressure*: for any potential $B \in \mathcal{X}(\Omega)$ we set

$$\Pr(B) := \sup_{\mu \in \mathcal{P}_T(\Omega)} \left(h_{\mathcal{X}}(\mu) + \int B \, d\mu \right).$$

In many cases (e.g. shift in the Bernoulli space), this turns out to coincide with the classical topological pressure (see again Remarks 7.5 and 7.8). Here we will concentrate on the study of the above Legendrian formulations for these quantities, as they fit our framework most naturally.

One of our main concerns is to understand when and where the above infimum and supremum are attained; we therefore consider the families of functionals defined for $\mu, \nu \in \mathcal{P}_T(\Omega)$ and $A, B \in \mathcal{X}(\Omega)$ by

$$H_\nu(A) = \log \lambda_A - \int A \, d\nu, \qquad P_B(\mu) = h_{\mathcal{X}}(\mu) + \int B \, d\mu.$$

The functional P_B is defined for all T-invariant measures but we shall also study its restriction to Gibbs measures, considered as acting on potentials:

$$P_B(A) = h_{\mathcal{X}}(\mu_A) + \int B \, d\mu_A.$$

We will abuse notation and use the same name P_B for the map defined on invariant measures, the map defined on potentials, its restriction to normalized potentials and the map it induces on the quotient $\widehat{\mathcal{X}} = \mathcal{X}(\Omega)/\mathcal{C}$. The way we write the argument ($P_B(\mu_A)$, $P_B(A)$ or $P_B([A])$) will usually make the difference clear.

Since H_ν is $\mathcal{C}$-invariant, it induces a functional on the quotient $\widehat{\mathcal{X}}$, which we still denote by H_ν.

7.2. Classical equilibrium states and Legendre duality

7.2.1. The entropy functionals. We start with the study of the functionals H_ν.

Proposition 7.2. *For all $\nu \in \mathcal{P}_T(\Omega)$, the functional H_ν on $\mathcal{X}(\Omega)$ is analytic, with*

$$D(H_\nu)_A(\zeta) = \int \zeta \, \mathrm{d}\mu_A - \int \zeta \, \mathrm{d}\nu.$$

Moreover, the map $[A] \mapsto H_\nu([A])$ induced on $\widehat{\mathcal{X}}$ is strictly convex.

Proof. Let us recall that $\Lambda : \mathcal{X}(\Omega) \to (0, \infty)$ is the analytic functional defined by $\Lambda(A) = \lambda_A$, and that for all $A, \zeta \in \mathcal{X}(\Omega)$ we have $D(\log \Lambda)_A(\zeta) = \int \zeta \, \mathrm{d}\mu_A$ (Corollary 3.6). Since the second term in $H_\nu(A) = \log \lambda_A - \int A \, \mathrm{d}\nu$ is linear and thus analytic and equal to its derivative at any point, H_ν is analytic with $D(H_\nu)_A(\zeta) = \int \zeta \, \mathrm{d}\mu_A - \int \zeta \, \mathrm{d}\nu$. The second term is constant in A, and by Sections 4 and 5 the second derivative is given by

$$D^2(H_\nu)(\zeta, \eta) = DG_A(\eta)(\zeta) = \langle \eta, \zeta \rangle_A.$$

In other words, considering the functional H_ν induced on $\widehat{\mathcal{X}}$ we have $D^2(H_\nu) = \langle \cdot, \cdot \rangle_{[A]}$, which is weakly positive-definite, proving the strict convexity on $\widehat{\mathcal{X}}$. □

Note that we do not have *uniform* convexity (even locally) since the inner product is only weakly positive-definite (there are directions $[\zeta]$ with fixed size $\|[\zeta]\|$ such that the "convexity" $\|[\zeta]\|_{[A]}$ is arbitrarily small). Of course, H_ν is only weakly convex on $\mathcal{X}(\Omega)$ since it is constant along each fiber $A + \mathcal{C}$.

Proposition 7.2 now implies the following result.

Corollary 7.3. *Under hypotheses (H1) to (H6), if $\nu = \mu_B$ for some $B \in \mathcal{X}(\Omega)$, then $H_{\mu_B}([A])$ is uniquely minimized at $[A] = [B]$, and thus*

$$\mathrm{h}_{\mathcal{X}}(\mu_B) = \log \lambda_B - \int B \, \mathrm{d}\mu_B = - \int N(B) \, \mathrm{d}\mu_B.$$

When ν is not in the image of the Gibbs map, H_ν does not reach its infimum.

Note that a normalized B is non-positive and non-zero and has $\lambda_B = 1$, so that $\mathrm{h}_{\mathcal{X}}(\mu_B) > 0$ (use (H5) to get the strict inequality).

Proof. Hypothesis (H6) implies that $\mathcal{X}(\Omega)$ "separates measures", i.e.

$$\left(\forall \zeta \in \mathcal{X}(\Omega) : \int \zeta \, \mathrm{d}\mu = \int \zeta \, \mathrm{d}\nu \right) \Rightarrow \mu = \nu.$$

Using $D(H_\nu)_A(\zeta) = \int \zeta \, \mathrm{d}(\mu_A - \nu)$ we see that when ν is not in the image of the Gibbs map, H_ν has no critical point, hence no minimum; and when $\nu = \mu_B$, the critical points of H_{μ_B} are exactly the potentials A such that $\mu_A = \mu_B$. Going down to the quotient we get only one critical point $[B]$, and the strict convexity implies that this critical point is the unique minimizer. □

Remark 7.4. At first glance, it looks like we used that the Gibbs map $G : A \mapsto \mu_A$ is one-to-one in this proof, while we were only able to prove it in some cases in Remark 2.15. But in fact, the above proof rather *implies* the injectivity of G, since by strict convexity for all ν there can exist at most one critical point of H_ν on $\widehat{\mathcal{X}}$.

Remark 7.5. When $\nu = \mu_B$ is a Gibbs measure, we thus obtain

$$\mathrm{h}_{\mathcal{X}}(\mu_B) = \int (-\log e^{N(B)(y)})\, \mathrm{d}\mu_B(y)$$

where $e^{N(B)}$ can be interpreted as a transition probability, or as the Jacobian of T with respect to μ (Remark 2.15). This can be used for some $(\Omega, T, \mathcal{X}(\Omega))$ to show that $\mathrm{h}_{\mathcal{X}}(\mu_B)$ is equal to the metric entropy $h(\mu_B)$; in particular this is the case for the shift σ acting on the Bernoulli space $\mathcal{A}^{\mathbb{N}}$ with Hölder potentials (the Classical Thermodynamical Formalism in the sense of [PP90]).

In this case $(\sigma, \mathcal{A}^{\mathbb{N}}, \mathrm{Hol}_\alpha)$ the equality $\mathrm{h}_{\mathcal{X}}(\nu) = h(\nu)$ extends to any invariant probability ν. Indeed, by [Wa82, Theorem 9.12] for any σ-invariant probability ν on the Bernoulli space, the metric entropy $h(\nu)$ satisfies

$$h(\nu) = \inf_{A \in C^0(\mathcal{A}^{\mathbb{N}})} \left\{ P(A) - \int A \, \mathrm{d}\nu \right\}$$

where P is the topological pressure. As the topological pressure is a continuous function of the continuous potential A (see [Wa82, Theorem 9.7]) and the set of Hölder functions is dense in $C^0(\mathcal{A}^{\mathbb{N}})$, the infimum above can be restricted to Hölder potentials A. For Hölder potentials the pressure satisfies $P(A) = \log \lambda_A$ and this shows that $\mathrm{h}_{\mathcal{X}}(\nu) = h(\nu)$. Of course this reasoning applies to all cases when the topological pressure coincides with $\log \Lambda$ and the metric entropy is the Legendre dual of the pressure.

The analogous results is proved for Gibbs plans in [L+15a, Lemma 6]. An invariant probability is a particular case of a Gibbs plan (see [L+15a, (1)]), and this provides another proof for the equality of entropies.

7.2.2. The pressure functionals. We are now in a position to extend the following classical result to our general framework.

Theorem 7.6 (Gibbs measures are equilibrium states). *For all $B \in \mathcal{X}(\Omega)$, we have $\mathrm{Pr}(\mu_B) = \log \lambda_B$, and μ_B is the unique maximizer of $P_B(\mu) = \mathrm{h}_{\mathcal{X}}(\mu) + \int B \, \mathrm{d}\mu$ among all T-invariant probability measures.*

Proof. Simply observe that

$$P_B(\mu) - \log \lambda_B = \mathrm{h}_{\mathcal{X}}(\mu) + \int B \, \mathrm{d}\mu - \log \lambda_B = \inf_{A \in \mathcal{X}(\Omega)} H_\mu(A) - H_\mu(B).$$

Consider the functional $A \mapsto H_\mu(A) - H_\mu(B)$: it takes the value 0 at $A = B$, and by Corollary 7.3 this value is its infimum precisely when $\mu = \mu_B$. We deduce that $P_B(\mu_B) = \log \lambda_B$, and for any other measure $\mu \in \mathcal{P}_T(\Omega)$, $P_B(\mu) < \log \lambda_B$. □

Remark 7.7. The expression $\Pr(\mu_B) = \log \lambda_B$ shows that $\Pr$ and $\mathrm{h}_{\mathcal{X}}$ are really Legendre duals of each other, since we can now write $\mathrm{h}_{\mathcal{X}}(\mu) = \inf_A(\Pr(A) - \int A \,\mathrm{d}\mu)$.

Remark 7.8. We can deduce from that result that $\mathrm{h}_{\mathcal{X}}$ is the metric entropy and $\Pr$ the topological pressure whenever we know the latter to be equal to $\log \Lambda$ and the former to be its Legendre dual. In particular, this holds when T is the shift over a finite alphabet and $\mathcal{X} = \mathrm{Hol}_\alpha$, but of course in this case it is possible and more satisfactory to prove that $\mathrm{h}_{\mathcal{X}}$ and $\Pr$ are the classical quantities[6] and recover their interpretation in terms of eigenvalue and Legendre dual by the above.

Remark 7.9. As a particular case, the measure of maximal entropy is unique and equal to μ_0 where 0 is the zero of $\mathcal{X}(\Omega)$. One can then describe μ_0 as the stationary measure for the Markov chain on Ω defined by the normalized potential $N(0)$. If T is d-to-one, then $-\log d$ is obviously normalized and in the class of 0 modulo $\mathcal{C}$, so that the measure of maximal entropy is the stationary measure for the uniform random walk on backward orbits of T.

7.3. Gradients and gradient flows

7.3.1. Computation of some gradients. We can now use the metric $\langle\cdot,\cdot\rangle_A$ to define the gradients of the functionals $\mathrm{h}_{\mathcal{X}}$ and P_B.

Note that a weak Riemannian metric such as $\langle\cdot,\cdot\rangle_A$ does not give a gradient to all C^1 functionals: indeed, $\langle\cdot,\cdot\rangle_A$ induces a continuous, one-to-one map from the tangent space of $\mathcal{N}$ to its dual, but this map is not onto.[7] Only those functionals whose differentials belong to the image of this map will have a gradient.

First, the results of Section 4 and the very definition of the metric show that $G_\varphi : [A] \mapsto \int \varphi \,\mathrm{d}\mu_A$ defined on the quotient $\widehat{\mathcal{X}}$ has a gradient:

$$D(G_\varphi)_A(\zeta) = \langle \varphi, \zeta\rangle_A = \langle [\varphi], [\zeta]\rangle_{[A]}, \qquad \nabla G_\varphi([A]) = [\varphi].$$

Similarly, the function $A \mapsto \int \varphi \,\mathrm{d}\mu_A$ defined on $\mathcal{N}$ has a gradient at A, given by $DN_A(\varphi)$ (recall that the gradient must be a vector in $T_A\mathcal{N} = \ker \mathscr{L}_A$ and that DN_A is precisely the projection on this space along $\mathcal{C}$).

Then, we consider the map $A \mapsto \mathrm{h}_{\mathcal{X}}(\mu_A)$. As before, by abuse of notation we will also denote by $\mathrm{h}_{\mathcal{X}}$ this map, as well as its restriction to $\mathcal{N}$ and the map it induces on $\widehat{\mathcal{X}}$.

From $\mathrm{h}_{\mathcal{X}}(A) = -\int N(A) \,\mathrm{d}\mu_A$, the product rule yields

$$D(\mathrm{h}_{\mathcal{X}})_A(\zeta) = -\int DN_A(\zeta) \,\mathrm{d}\mu_A - \langle N(A), \zeta\rangle_A = \langle -A, \zeta\rangle_A$$

[6] For example one can proceed as in [L+15b], noting that we use the classical normalization here.

[7] If it were, by Banach's isomorphism theorem the map $\zeta \mapsto \langle \zeta, \cdot\rangle_A$ from $\mathcal{X}(\Omega)$ to its dual would be an isomorphism, which is equivalent to $\langle\cdot,\cdot\rangle_A$ being strongly positive-definite.

since $DN_A(\zeta) \in \ker \mathscr{L}_A \subset \ker \mu_A$ and $\mathcal{C} = \ker \langle \cdot, \cdot \rangle_A$. This computation shows further that $\mathrm{h}_{\mathcal{X}}$ (now considered as induced on $\widehat{\mathcal{X}}$ or restricted to $\mathcal{N}$) has a gradient:

$$\nabla \mathrm{h}_{\mathcal{X}}([A]) = -[A], \quad \text{or again} \quad \nabla \mathrm{h}_{\mathcal{X}}(A) = -DN_A(A) \ \text{ when } A \in \mathcal{N}.$$

Observing that $P_B(A) = \mathrm{h}_{\mathcal{X}}(A) + G_B(A)$ we have thus proved the following.

Proposition 7.10. *The maps G_φ, $\mathrm{h}_{\mathcal{X}}$ and P_B have gradients for the weak Riemannian metric $\langle \cdot, \cdot \rangle_A$, given by*

$$\begin{aligned} G_\varphi([A]) &= [\varphi], & \nabla G_\varphi(A) &= DN_A(\varphi), \\ \nabla \mathrm{h}_{\mathcal{X}}([A]) &= -[A], & \nabla \mathrm{h}_{\mathcal{X}}(A) &= -DN_A(A), \\ \nabla(P_B)([A]) &= [B - A], & \nabla(P_B)(A) &= DN_A(B - A), \end{aligned}$$

where the functionals are considered either on $\widehat{\mathcal{X}}$ (left column) or on $\mathcal{N}$ (right column).

7.3.2. Gradient flow. One particularly nice feature of the gradient of the pressure P_B computed in Section 7.3.1 is that it straightforwardly induces a gradient flow: for all $[A_0] \in \widehat{\mathcal{X}}$, there is a differentiable curve $[A_t]$ such that for all t,

$$\frac{\mathrm{d}}{\mathrm{d}t}[A_t] = \nabla(P_B)(A_t).$$

Indeed, a solution is given by

$$[A_t] = e^{-t}[A_0 - B] + [B].$$

Let us give a physical interpretation when T is the shift: we consider a system consisting of a $\mathbb{Z}$-lattice of particles; a potential A_0 then represents a combination of the interaction (and self-interaction) energy of the particles and of the temperature, and the Gibbs measure μ_{A_0} is an equilibrium state (which minimizes the "free energy" $-P_{A_0}$) and represents the macroscopic state of the system at equilibrium. Assume now that these interactions change instantly to be now described by the potential B. The gradient flow above is a natural and simple model for the evolution of the macroscopic state of the system, where the systems evolves "driven" by B. Note that in this interpretation, the state of the system out of equilibrium is an equilibrium state for a varying potential.

Remark 7.11. Let us consider a particular case, where the interactions are constant and only the temperature changes: $A_0 = (1/T_0)\varphi$ and $B = (1/T_1)\varphi$ for some $\varphi \in \mathcal{X}(\Omega)$; this corresponds to a system in contact with a heat bath whose temperature changes suddenly. According to our model, the system then evolves only in its temperature, because

$$[A_t] = e^{-t}[A_0 - B] + [B] = \left(e^{-t}\left(\frac{1}{T_0} - \frac{1}{T_1} \right) + \frac{1}{T_1} \right)[\varphi]$$

will be proportional to $[\varphi]$ for all t. Note that here, t should not be considered as time, since the speed of evolution of temperature would not be right. It might be possible to give a physical interpretation to the parameter t, or to rescale the functional P_B and the metric so as to obtain a physically sound evolution of the temperature.

Remark 7.12. Beware that this gradient flow really takes place on $\widehat{\mathcal{X}}$ (or equivalently, on $\mathcal{N}$): it is not defined on the whole of $\mathcal{X}(\Omega)$ because there the metric has a non-trivial kernel. Physically, this is to be expected because adding a constant or a coboundary to the potential does not change the minimizer of free energy (in the case of a coboundary, each interaction term between two particles is compensated by the exact opposite term between their right neighbors), and thus such an addition cannot be detected, has no physical influence.

Also, we cannot see this gradient flow as taking place in the set of invariant measures with the Wasserstein structure, because of Section 6: the Gibbs map is not differentiable, and when (A_t) is an integral curve of our gradient flow, the curve (μ_{A_t}) is not absolutely continuous (Theorem 6.2) and in particular not a gradient flow curve in the sense of Ambrosio, Gigli and Savaré [AGS05].

7.4. Prescribing integrals

In this section we study how one can find Gibbs measures with prescribed values for the integrals of a given set of test functions. This is both an application of the tools we introduced here (in particular, the weak metric of Section 5 makes the proof quite easy), and a main ingredient in the proof of existence and uniqueness of equilibrium states under linear constraints.

Fix a tuple $\Phi = (\varphi_1, \dots, \varphi_K) \in \mathcal{X}(\Omega)^K$ of test functions and denote by $\mathcal{P}_T$ the set of T-invariant probability measures. The *rotation vector* of an invariant measure μ is

$$\mathrm{rv}(\mu) := \left(\int \varphi_1 \,\mathrm{d}\mu, \dots, \int \varphi_K \,\mathrm{d}\mu \right) \in \mathbb{R}^K$$

and we want to study the set

$$\mathrm{Rot}(\Phi) := \{\mathrm{rv}(\mu) \mid \mu \in \mathcal{P}_T\}$$

of possible values of the rotation vector, and the freedom one has to prescribe the values of these integrals with respect to a *Gibbs* measure.

It is well-known and straightforward that $\mathrm{Rot}(\Phi)$ is convex; it must also be bounded since potentials are assumed to be bounded by (H1).

Observe that if the classes modulo $\mathcal{C}$ of the φ_k are linearly dependent, then their integrals with respect to any invariant measure must satisfy a linear relation. Let us be more specific: if $g - g\circ T + c$ is any element of $\mathcal{C}$ and μ is any T-invariant probability measure, then $\int (g - g \circ T + c)\,\mathrm{d}\mu = c$. Therefore, if there is a non-trivial relation $\sum x_k[\varphi_k] = 0$ then there are $g \in \mathcal{X}$ and $c \in \mathbb{R}$ such that $\sum x_k\varphi_k = g - g\circ T + c$ and for all $\mu \in \mathcal{P}_T$ we get the relation $\sum x_k \int \varphi_k \,\mathrm{d}\mu = c$, constraining the vector of integrals to an affine subspace of $\mathbb{R}^K$. But this constraint on the rotation vector can be worked out from the φ_k, and one can restrict to a maximal subset $S \subset \{1, \dots, K\}$ such that the family $([\varphi_k])_{k\in S}$ is linearly independent. Then the corresponding integrals will determine the integrals of all φ_k. This procedure reduces the problem to the case when the $[\varphi_k]$ are linearly independent, which we will always assume in what follows.

We then get the following (which does not pretend to much originality, see [KW14] and [Je01]; note that our proof is close to the one by Kucherenko and Wolf, but the metric $\langle\cdot,\cdot\rangle_A$ makes the injectivity of the Jacobian obvious and we use a differential-geometric argument to show that the map is onto).

Theorem 7.13. *Assume hypotheses (H1) to (H6) and let $\Phi = (\varphi_1, \dots, \varphi_K) \in \mathcal{X}(\Omega)^K$ be such that the classes $[\varphi_1], \dots, [\varphi_K]$ modulo $\mathcal{C}$ are linearly independent. Then for all $B \in \mathcal{X}(\Omega)$, the map*

$$\mathbb{R}^K \to \operatorname{int}\operatorname{Rot}(\Phi), \qquad (a_1, \dots, a_K) \mapsto \operatorname{rv}(\mu_{B+a_1\varphi_1+\cdots+a_K\varphi_K}),$$

is an analytic diffeomorphism; in particular $\operatorname{Rot}(\Phi)$ has non-empty interior and all its interior values are achieved by Gibbs measures.

Proof. Consider the analytic maps

$$I : \mathbb{R}^K \to \mathcal{X}(\Omega), \qquad \bar\alpha = (\alpha_1, \dots, \alpha_K) \mapsto B + \sum \alpha_k \varphi_k,$$

where B is any fixed potential,

$$J : \mathcal{X}(\Omega) \to \mathbb{R}^K, \qquad A \mapsto \left(\int \varphi_1 \, \mathrm{d}\mu_A, \dots, \int \varphi_K \, \mathrm{d}\mu_A \right),$$

and their composition $L = J \circ I : \mathbb{R}^K \to \mathbb{R}^K$. We also denote by L_k the k-th component of L, i.e. $L_k(\bar\alpha) = \int \varphi_k \, \mathrm{d}\mu_{I(\bar\alpha)}$.

The differential of L is given by Sections 4 and 5:

$$\frac{\partial L_k}{\partial x_j}(\bar\alpha) = \langle [\varphi_k], [\varphi_j] \rangle_{I(\bar\alpha)}.$$

This defines a Gram matrix, which is invertible since the $[\varphi_k]$ are linearly independent; it follows from the local inverse function theorem that L is a local diffeomorphism.

If B is any potential, this implies that $L(B)$ is in the interior of the image of L, in particular in the interior of $\operatorname{Rot}(\Phi)$ (which must thus be non-empty).

What is left to prove is that L is a *global* diffeomorphism from $\mathbb{R}^K$ to $\operatorname{int}\operatorname{Rot}(\Phi)$. Since that interior is diffeomorphic to $\mathbb{R}^K$, a theorem of [Go72] reduces this to proving that L is proper when its codomain is taken to be $\operatorname{int}\operatorname{Rot}(\Phi)$, i.e. whenever a sequence $\bar x^{(n)}$ escapes from compacts of $\mathbb{R}^K$, the points $L(\bar x^{(n)})$ escape from compacts of $\operatorname{int}\operatorname{Rot}(\Phi)$. In other words, we want to prove that if $\bar x^{(n)} \to \infty$ and $L(\bar x^{(n)})$ converges, the limit lies on $\partial \operatorname{Rot}(\Phi)$.

Now, if $\bar x^{(n)} \to \infty$ and $L(\bar x^{(n)})$ converges, up to taking a subsequence we can assume that $\bar x_{(n)} = t_n \bar u + o(t_n)$ where (t_n) is a diverging sequence of positive numbers, and $\bar u$ is a unit vector in $\mathbb{R}^K$ (this is simply the compactness of the unit sphere).

Observe that if $\bar x$ is a boundary point of $\operatorname{Rot}(\Phi)$ and $\Phi = \sum y_k e_k^*$ (where (e_k^*) is the canonical dual basis) is a linear form on $\mathbb{R}^K$ whose maximum on $\operatorname{Rot}(\Phi)$ is reached at $\bar x$, then

$$\Phi(\bar x) = \max \left\{ \int \sum y_k \varphi_k \, \mathrm{d}\mu \;\middle|\; \mu \in \mathcal{P}_T \right\},$$

and conversely points maximizing a linear form must lie on the boundary.

Back to $L(\bar{x}^{(n)})$, we have $I(\bar{x}^{(n)}) = t_n\varphi_{\bar{u}} + o(t_n)$ where $\varphi_{\bar{u}} = \sum u_k\varphi_k$. The variational principle tells us that $\mu_{I(\bar{x}^{(n)})}$ maximizes $\mathrm{h}_{\mathcal{X}}(\mu) + \int (t_n\varphi_{\bar{u}} + o(t_n))\,\mathrm{d}\mu$, and it follows that the accumulation points of this sequence of measures are all maximizing measures of $\varphi_{\bar{u}}$. This precisely means that the limit of $L(\bar{x}^{(n)})$ is a boundary point, and we are done. □

As a by-product of this result, we get the following.

Corollary 7.14. *Under hypotheses (H1) to (H6), if $\mathcal{X}(\Omega)$ is separable,*[8] *then the set $G(\mathcal{X}(\Omega))$ of Gibbs measures is weakly dense in $\mathcal{P}_T(\Omega)$.*

Proof. By separability, let $(\varphi_k)_{k\in\mathbb{N}}$ be a sequence dense in $\mathcal{X}(\Omega)$. Then for every continuous $f : \Omega \to \mathbb{R}$ tending to zero at infinity there is a subsequence $(\varphi_{k_i})_{i\in\mathbb{N}}$ which converges to f uniformly.

Let $\mu \in \mathcal{P}_T(\Omega)$; from Theorem 7.13, for each $K \in \mathbb{N}$ there is a potential $A_K \in \mathcal{X}(\Omega)$ such that

$$\left|\int \varphi_k \,\mathrm{d}\mu_{A_K} - \int \varphi_k \,\mathrm{d}\mu\right| < \frac{1}{K} \quad \forall k \in \{1, \dots, K\}.$$

Given any continuous $f : \Omega \to \mathbb{R}$ tending to zero at infinity and any $\varepsilon > 0$, there is some k_0 such that $\|f - \varphi_{k_0}\|_\infty \le \varepsilon$. For all $K \ge \max(k_0, 1/\varepsilon)$ we thus have

$$\begin{aligned}\left|\int f \,\mathrm{d}\mu_{A_K} - \int f \,\mathrm{d}\mu\right| &\le \left|\int f \,\mathrm{d}\mu_{A_K} - \int \varphi_{k_0} \,\mathrm{d}\mu_{A_K}\right| + \left|\int \varphi_{k_0} \,\mathrm{d}\mu_{A_K} - \int \varphi_{k_0} \,\mathrm{d}\mu\right| \\ &\quad + \left|\int \varphi_{k_0} \,\mathrm{d}\mu - \int f \,\mathrm{d}\mu\right| \\ &< 3\varepsilon.\end{aligned}$$

Letting $\varepsilon \to 0$, we see that $\int f \,\mathrm{d}\mu_{A_K} \to \int f \,\mathrm{d}\mu$, so (μ_{A_K}) converges weakly to μ. □

7.5. Optimization under constraints

Our goal here is to optimize the P_B functionals (for example, the entropy $\mathrm{h}_{\mathcal{X}}$) on natural subsets of invariant measures, obtained by constraining the integrals of some functions. These questions have been considered by Jenkinson [Je01] in the case of entropy and by Kucherenko and Wolf [KW14, KW15], with somewhat different assumptions and methods. We believe that part of our claims are more explicit in some issues.

As before, we fix test functions $\Phi = (\varphi_1, \dots, \varphi_K) \in \mathcal{X}(\Omega)^K$ and we consider the set $\mathcal{P}_T[\Phi]$ of T-invariant measures μ such that $\int \varphi_k \,\mathrm{d}\mu = 0$ for all k; among them are the Gibbs measures whose normalized potential lies in

$$\mathcal{N}[\Phi] := \left\{ A \in \mathcal{N} \;\middle|\; \forall k : \int \varphi_k \,\mathrm{d}\mu_A = 0 \right\}.$$

We will also denote by $\widehat{\mathcal{X}}[\Phi]$ the set of classes $[A] \in \widehat{\mathcal{X}} = \mathcal{X}(\Omega)/\mathcal{C}$ such that $A \in \mathcal{N}[\Phi]$.

[8] Or more generally if in (H6) the approximation can be obtained from a fixed countable subset of $\mathcal{X}$.

With these notations, we will prove the following constrained (or "localized") version of the variational principle.

Theorem 7.15. *Assume hypotheses (H1) to (H6) and let* $\Phi = (\varphi_1, \dots, \varphi_K) \in \mathcal{X}(\Omega)^K$ *be such that the* $[\varphi_k]$ *are linearly independent, and* 0 *is an interior vector of* $\mathrm{Rot}(\Phi)$. *For each* $B \in \mathcal{X}(\Omega)$ *denote by* B_0 *the unique element* $B_0 = B + a_1\varphi_1 + \cdots + a_K\varphi_K$ *such that* $[B_0] \in \widehat{\mathcal{X}}[\Phi]$ (*Theorem 7.13*). *Then* μ_{B_0} *uniquely maximizes* P_B *over* $\mathcal{P}_T[\Phi]$, *and the value of the maximum is* $P_B(B_0) = \log \lambda_{B_0}$.

Proof. We simply observe that for all $\mu \in \mathcal{P}_T[\Phi]$ we have

$$P_B(\mu) = \mathrm{h}_{\mathcal{X}}(\mu) + \int (B_0 - a_1\varphi_1 - \cdots - a_K\varphi_K)\,\mathrm{d}\mu = \mathrm{h}_{\mathcal{X}}(\mu) + \int B_0\,\mathrm{d}\mu = P_{B_0}(\mu).$$

Applying Theorem 7.6 to P_{B_0} we see that $P_B(\mu_{B_0}) = P_{B_0}(\mu_{B_0}) = \log \lambda_{B_0}$ is greater than $P_B(\mu) = P_{B_0}(\mu)$ whenever $\mu \neq \mu_{B_0}$ is in $\mathcal{P}_T[\Phi]$. □

We can use this to recover in our setting another result from [KW14].

Corollary 7.16. *Assume hypotheses (H1) to (H6), let* $\Phi = (\varphi_1, \dots, \varphi_K) \in \mathcal{X}(\Omega)^K$ *be such that the* $[\varphi_k]$ *are linearly independent, and for* $w \in \mathrm{int}\,\mathrm{Rot}(\Phi)$ *define*

$$H(w) = \sup\{\mathrm{h}_{\mathcal{X}}(\mu) \mid \mathrm{rv}(\mu) = w\}.$$

Then H *is a positive, analytic map.*

Proof. By Theorem 7.13 there are unique analytic functions $a_k : \mathrm{int}\,C \to \mathbb{R}$ such that

$$\mathrm{rv}(\mu_{a_1(w)\varphi_1 + \cdots + a_K(w)\varphi_K}) = w \quad \forall w.$$

Setting $A(w) = a_1(w)\varphi_1 + \cdots + a_K(w)\varphi_K$ and applying Theorem 7.15 to $(\varphi_1 - w_1, \dots, \varphi_K - w_K)$ we obtain

$$H(w) = \mathrm{h}_{\mathcal{X}}(\mu_{A(w)}) = \log \Lambda(A(w)) - a_1(w)w_1 - \cdots - a_K(w)w_K,$$

proving the claim. □

Remark 7.17. Assume that T is the shift over a finite alphabet and $\mathcal{X} = \mathrm{Hol}_\alpha$ (recall that $\mathrm{h}_{\mathcal{X}}(\mu)$ is the metric entropy in this case, Remark 7.5). Let n be any positive integer, and let $\Phi = (\varphi_1, \dots, \varphi_K)$ and B be Hölder functions that only depend on the first n coordinates, and such that $\widehat{\mathcal{X}}[\Phi]$ is non-empty.

Then we claim that there is a unique measure maximizing $P_B(\mu)$ among all elements of $\mathcal{P}_T[\Phi]$, and that this measure is an $(n-1)$-step Markov measure (i.e. a Gibbs measure μ_A such that $N(A)$ only depends on the first n coordinates). In particular, applying this to $B = 0$, we see that there is an $(n-1)$-step Markov measure maximizing the entropy subject to any finite set of robustly simultaneously satisfiable constraints $\int \varphi_k\,\mathrm{d}\mu = 0$ whenever the φ_k are constant on cylinders of depth n.

Proof. The only point that does not follow immediately from Theorem 7.15 is that μ_A is n-Markov. But we know that we can take $A = B + \sum x_k\varphi_k$ for some (x_k); notice that this A might not be normalized, but is constant on each depth-n cylinder.

Now, $\mathscr{L}_A$ preserves the subspace of $\mathcal{X}(\Omega)$ made up of functions that only depend on the first $n-1$ coordinates. In particular, for all N the function $\mathscr{L}_A^N(\mathbf{1})$ only depends on the first $n-1$ coordinates. Since this is a closed space, the leading eigenfunction h_A only depends on the first $n-1$ coordinates, and $h_A \circ T$ only depends on the first n coordinates.

Now $N(A) = A + \log h_A - \log h_A \circ T - \log \lambda_A$ only depends on the first n coordinates, which precisely means that μ_A is $(n-1)$-step Markov. □

Let us give a couple of examples, which we will not make as general as possible but which will be very explicit. Let $\Omega = \{0,1\}^{\mathbb{N}}$, T be the shift and $\mathcal{X}(\Omega)$ be a space of Hölder functions for one of the usual metrics of Ω. Given any finite word ω, let $\omega*$ be the cylinder defined by ω, i.e. the set of words starting with ω.

Example 7.18. Among shift-invariant measures μ such that $\mu(0*) = .9$, the Bernoulli measure of parameter .9 (i.e. the distribution of the word $\alpha_1\alpha_2\ldots$ where the α_j are i.i.d. random variables taking the value 0 with probability .9) maximizes entropy.

Indeed, from Remark 7.17 we know that there is a Bernoulli measure realizing this maximum, and the Bernoulli measure with parameter .9 is the only one to satisfy the constraint.

Example 7.19. Among shift-invariant measures μ such that $\mu(01*) = 2\mu(11*)$, the Markov measure associated to the transition probabilities

$$\begin{aligned} \mathbb{P}(0 \to 0) &= 1-a, & \mathbb{P}(0 \to 1) &= a, \\ \mathbb{P}(1 \to 0) &= 2/3, & \mathbb{P}(1 \to 1) &= 1/3, \end{aligned}$$

where a is the only real solution to

$$(1-a)^5 = \tfrac{4}{27}a^2 \qquad (a \simeq 0.487803),$$

maximizes entropy.

It is easily seen that the constraint is satisfiable by a Markov measure, in particular by a Gibbs measure, thus we can apply Remark 7.17 to $B = 0$, $K = 1$ and $\varphi = \mathbf{1}_{10*} - 2\cdot\mathbf{1}_{11*}$ where $\mathbf{1}_S$ is the indicator function of the set S.

The constraint easily translates into $\mathbb{P}(1 \to 0) = 2/3$, and we define $a = \mathbb{P}(0 \to 1)$. We know that the Gibbs entropy maximizing measure is given by a potential of the form $A = x\varphi$ where $x \in \mathbb{R}$; to translate this into transition probabilities, we only have to normalize A:

$$N(A) = x\varphi + \log h - \log h \circ T + \log \lambda$$

where $\lambda \in \mathbb{R}$ and h only depends on the first coordinates and matters only up to a multiplicative constant; we thus define $\alpha = h(0*)/h(1*)$. Letting $\eta = e^x$, we then recover the transition probabilities as follows:

$$\begin{aligned} \mathbb{P}(0 \to 0) &= e^{N(A)(00*)} = \lambda, & \mathbb{P}(1 \to 0) &= e^{N(A)(01*)} = \alpha\lambda, \\ \mathbb{P}(0 \to 1) &= e^{N(A)(10*)} = \eta\alpha^{-1}\lambda, & \mathbb{P}(1 \to 1) &= e^{N(A)(11*)} = \eta^{-2}\lambda. \end{aligned}$$

We then have to solve the system

$$\begin{cases} 1-a=\lambda, \\ a=\eta\alpha^{-1}\lambda, \\ 2/3=\alpha\lambda, \\ 1/3=\eta^{-2}\lambda. \end{cases}$$

This will give the only η such that μ_A with the above A satisfies the constraint, and from Remark 7.17 we know that μ_A maximizes entropy under this constraint; then the corresponding value of a gives the transition probability we seek. Note that while we have some computation to do, we do not have to estimate the actual entropy of Markov measures, nor do we have to compute the eigendata of $\mathscr{L}_A$ directly.

The above system is easily solved by substitution: $\lambda = 1-a$, then $\alpha = 2/(3(1-a))$, $\eta = 2a/(3(1-a)^2)$ and finally the last equation yields $[2a/(3(1-a)^2)]^2 = 3(1-a)$, so that $(1-a)^5 = \frac{4}{27}a^2$.

8. Explicit computations for a restricted model

In this section we explicitly show an example of the construction of Section 5 and some of its consequences. The dynamic we consider is the shift acting on the space $\{1,2\}^{\mathbb{N}}$. We choose $\mathcal{X}$ to be the space of α-Hölder functions for any α, and denote by $\mathcal{X}_2$ the subset of potentials which depend only on the first two coordinates (of elements in $\{1,2\}^{\mathbb{N}}$). Note that we formally cannot take $\mathcal{X}_2$ as our full space of potentials, since it is not invariant under composition with T. It is easy to check that, in this setting, (H1)–(H6) are satisfied (or one can find all the details in [PP90]).

8.1. A positively curved metric

If a potential $A \in \mathcal{X}_2$ depends just on two coordinates then we can write A_{ij} for the value of A evaluated on the cylinder $ij*$ (i.e. the elements of $\{1,2\}^{\mathbb{N}}$ of the type $ij\ldots$), and we shall identify $\mathcal{X}_2$ with the space of 2 by 2 real matrices. The value of e^A is well-defined on such a cylinder, and the action of the operator $\mathscr{L}_A$ on potentials φ depending only on the first coordinate explicitly reads

$$(\mathscr{L}_A\phi(1*), \mathscr{L}_A\phi(2*)) = \begin{pmatrix}\phi(1*) & \phi(2*)\end{pmatrix}\begin{pmatrix} e^{A_{11}} & e^{A_{12}} \\ e^{A_{21}} & e^{A_{22}}\end{pmatrix}.$$

We can thus think of the operator $\mathscr{L}_A$ as acting on a function as left multiplication by a matrix, and we shall denote by L the map

$$L : A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{pmatrix} \mapsto L_A = \begin{pmatrix} e^{A_{11}} & e^{A_{12}} \\ e^{A_{21}} & e^{A_{22}}\end{pmatrix}$$

which exponentiates each coordinate of the matrix A, and identify freely $\mathscr{L}_A$ and L_A.

To normalize the potential, that is, to find the potential $\overline{A} := N(A)$ differing from A by a coboundary and a constant such that $\mathscr{L}_{\overline{A}}(1) = 1$, we can apply the Perron–Frobenius

theorem[9] to the matrix L_A and solve with respect to the maximal eigenvalue and the left eigenvector, i.e. $\ell L_A = \lambda_A \ell$. After normalization, we obtain

$$\overline{A}(i,j) = \frac{e^{A(i,j)}\ell_i}{\lambda_A \ell_j}.$$

From now on, we will assume that A is normalized and avoid the notation $\overline{A}$.

We observe that the set $\mathcal{N}_2 := N(\mathcal{X}_2)$ of normalized potentials depending on two coordinates is defined by the equations

$$\begin{cases} e^{A_{11}} + e^{A_{21}} = 1, \\ e^{A_{12}} + e^{A_{22}} = 1, \end{cases}$$

so that $L(\mathcal{N}_2)$ is the set of 2 by 2 column stochastic matrices, denoted by $\mathcal{S}_2$.

To sum up, a normalized potential in $\mathcal{N}_2$ can be represented by the matrix of its values on cylinders, subject to a non-linear system of constraints, or as a column stochastic matrix after coordinatewise exponentiation. We thus obtain a natural chart $S : [0,1] \times [0,1] \to \mathcal{S}_2$ by setting

$$S(x,y) = \begin{pmatrix} x & 1-y \\ 1-x & y \end{pmatrix}$$

where $x, y \in (0,1)$ can be thought of as transition probabilities $\mathbb{P}(1 \to 1)$ and $\mathbb{P}(2 \to 2)$, respectively.

This parametrization has the advantage that $\mathcal{S}_2$ is (an open subset of) an affine subspace of $M_{2,2}(\mathbb{R})$: it has the same tangent space at each point, a basis of which is given by

$$\frac{\partial S}{\partial x} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \quad \frac{\partial S}{\partial y} = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}.$$

A tangent vector ψ to $\mathcal{S}_2$ at $S(x,y)$ will be written as (ψ_1, ψ_2) in this basis, so that the corresponding parametrized line γ can be expressed for $s \in \mathbb{R}$ sufficiently small by

$$\gamma_{S(x,y),\psi}(s) = \begin{pmatrix} x + s\psi_1 & 1 - y - s\psi_2 \\ 1 - x - s\psi_1 & y + s\psi_2 \end{pmatrix} \in \mathcal{S}_2.$$

The expression above is very readable, though does not allow us to compute the metric of $T_A\mathcal{N}_2$ right away. However, given $A \in \mathcal{N}_2$ and $\xi \in T_A\mathcal{N}_2$, by Theorem 5.1(ii), we have

$$\langle \zeta, \zeta \rangle_A = \int \zeta^2 \, \mathrm{d}\mu_A.$$

It will thus be convenient to work both in $\mathcal{N}_2$ where the functional interpretation of matrices and vectors is clear, and in $\mathcal{S}_2$ where the Gibbs measures naturally appear. If we consider a variation $\exp(A_{ij} + s\zeta_{ij})$ and differentiate at zero, we find that the system

$$\begin{cases} e^{A_{11}}\zeta_{11} + e^{A_{21}}\zeta_{21} = 0, \\ e^{A_{12}}\zeta_{12} + e^{A_{22}}\zeta_{22} = 0, \end{cases}$$

[9] See for example [Ga59] for the exact statement.

defines $T_A\mathcal{N}_2 \subset \mathcal{X}_2$ (in particular we see that this tangent plane depends on the point A). If $L_A = S(x, y)$ and ψ corresponds to ζ in $T_{L_A}\mathcal{S}_2$, i.e. $\psi = DL_A(\zeta)$, then

$$\begin{pmatrix} \zeta_{11} & \zeta_{12} \\ \zeta_{21} & \zeta_{22} \end{pmatrix} = \begin{pmatrix} \frac{\psi_1}{x} & \frac{-\psi_2}{1-y} \\ \frac{-\psi_1}{1-x} & \frac{\psi_2}{y} \end{pmatrix}.$$

Now the matrix $S(x, y)$ has a right eigenvector

$$\pi = (\pi(1*), \pi(2*)) = \left(\frac{1-y}{2-x-y}, \frac{1-x}{2-x-y} \right),$$

which is the invariant measure on $\{1, 2\}$ of the Markov chain defined by A, and the measures of cylinders with respect to μ_A are

$$\begin{aligned} \mu(11*) &= \mathbb{P}(1 \to 1)\pi(1), \qquad \mu(21*) = \mathbb{P}(1 \to 2)\pi(1), \\ \mu(12*) &= \mathbb{P}(2 \to 1)\pi(2), \qquad \mu(22*) = \mathbb{P}(2 \to 2)\pi(2). \end{aligned}$$

It is now easy to compute the metric:

$$\begin{aligned} \int \zeta^2 \, \mathrm{d}\mu_A = \sum_{i,j} \zeta_{i,j}^2 \mu(ij*) &= \frac{\psi_1^2}{x^2} \frac{x(1-y)}{2-x-y} + \frac{\psi_2^2}{(1-y)^2} \frac{(1-y)(1-x)}{2-x-y} \\ &\quad + \frac{\psi_1^2}{(1-x)^2} \frac{(1-x)(1-y)}{2-x-y} + \frac{\psi_2^2}{y^2} \frac{y(1-x)}{2-x-y} \\ &= \frac{1}{2-x-y} \left(\frac{1-y}{x(1-x)} \psi_1^2 + \frac{1-x}{y(1-y)} \psi_2^2 \right). \end{aligned}$$

Proposition 8.1. *The restriction g of the variance metric $\langle \cdot, \cdot \rangle_A$ to $\mathcal{N}_2$ is given in the chart S by*

$$g_A = \begin{pmatrix} \frac{(1-y)}{x(1-x)(2-x-y)} & 0 \\ 0 & \frac{(1-x)}{y(1-y)(2-x-y)} \end{pmatrix} \tag{8.1}$$

(which is positive-definite for all $x, y \in (0, 1) \times (0, 1)$).

This means that for $L_A = S(x, y)$ and $\psi = DL_A(\zeta)$ we have $|\zeta|_A^2 = (\psi_1 \ \ \psi_2) g_A \binom{\psi_1}{\psi_2}$.

Remark 8.2. Observe that, not incidentally, by recalling the proof of Proposition 5.3 we could have computed $\langle \zeta, \zeta \rangle_A$ in a more roundabout way by using the equation contained there,

$$\langle \zeta, \zeta \rangle_A = D^2(\log \Lambda)_A(\zeta, \zeta) = D(G_\zeta)_A(\zeta).$$

As a side effect we easily obtain

$$D^2(\log \Lambda)_A(\zeta, \zeta) = \frac{x(1-y)}{(1-x)(2-x-y)} \zeta_{11}^2 + \frac{(1-x)y}{(1-y)(2-x-y)} \zeta_{22}^2. \tag{8.2}$$

We see for example that when x or y goes to 0, the pressure becomes very flat (as opposed to very convex, i.e. its Hessian goes to zero).

From the metric tensor, we compute the curvature at each point. For simplicity, if we let $g_A = \left(\begin{smallmatrix} E & 0 \\ 0 & G \end{smallmatrix}\right)$ then we use the explicit formula for the curvature

$$K(A) = -\frac{1}{2\sqrt{EG}}\left\{\left(\frac{E_y}{\sqrt{EG}}\right)_y + \left(\frac{G_x}{\sqrt{EG}}\right)_x\right\}$$

where subscripts indicate partial derivatives with respect to the indicated variables. The expression simplifies greatly (see Section 8.3):

Corollary 8.3. *When $L_A = S(x, y)$, the Gaussian curvature of g at A is given by*

$$K(A) = \frac{1}{2 - x - y}.$$

Remark 8.4. In the case at hand, the curvature is always strictly positive. In fact, it is even bounded away from 0, so that $\mathcal{N}_2$ endowed with g is not complete (indeed, if g were complete then the Bonnet–Myers theorem would imply that $\mathcal{N}_2$ is compact).

8.2. Rescaling the metric

In the previous reasoning we considered the Riemannian norm $\langle \zeta, \zeta \rangle_A$ of a tangent vector ζ at the potential A given by the asymptotic variance, as in Theorem 5.1. We wonder how rescaling the metric by the entropy would affect such curvature, based on previous work by McMullen [McM08]. Given the eigenvector π of the previous section (which corresponds to the eigenmeasure), the entropy is given as a function of x, y by

$$\begin{aligned} h(x, y) = &- \frac{1 - y}{2 - x - y}\big(x \log(x) + (1 - x)\log(1 - x)\big) \\ &- \frac{1 - x}{2 - x - y}\big((1 - y)\log(1 - y) + y \log(y)\big). \end{aligned}$$

This function is always positive on $(0, 1) \times (0, 1)$ and is 0 in the limit to the vertex $(0, 0)$ and the edges $\{1\} \times [0, 1]$ and $[0, 1] \times \{1\}$ (Figure 2). Note that there is a strong asymmetry between the cases $x = 0$ and $x = 1$ (similarly for y), as $x = 1$ means the Markov chain gets stuck at state 1, while $x = 0$ means the random walk is always repelled away from state 1, but then can either stay at 2 or come back to 1, leaving enough uncertainty to yield positive entropy.

We rescale the metric associated to the matrix g_A of the previous section to a new $\tilde{g}_A$ in the interior of the square by setting $\tilde{g}_A = \left(\begin{smallmatrix} E/h & 0 \\ 0 & G/h \end{smallmatrix}\right)$ where h is the entropy functional. We denote by K the curvature associated to the metric g, and by $\widetilde{K}$ the one associated to $\tilde{g}$.

After a little juggling with the equations, for a strictly positive function $h(x, y)$ one gets

$$\frac{\widetilde{K}}{h} = K + \frac{1}{2\sqrt{EG}}\left(\left(\frac{\sqrt{E}}{\sqrt{G}}\,\frac{h_y}{h}\right)_y + \left(\frac{\sqrt{G}}{\sqrt{E}}\,\frac{h_x}{h}\right)_x\right).$$

The explicit expression of $\widetilde{K}$ is long and tedious to handle. We use the software Maxima both to do the necessary symbolic manipulation and to plot the graph (Figure 3). In the case of some subshifts related to Fuchsian groups, McMullen showed that this pre-

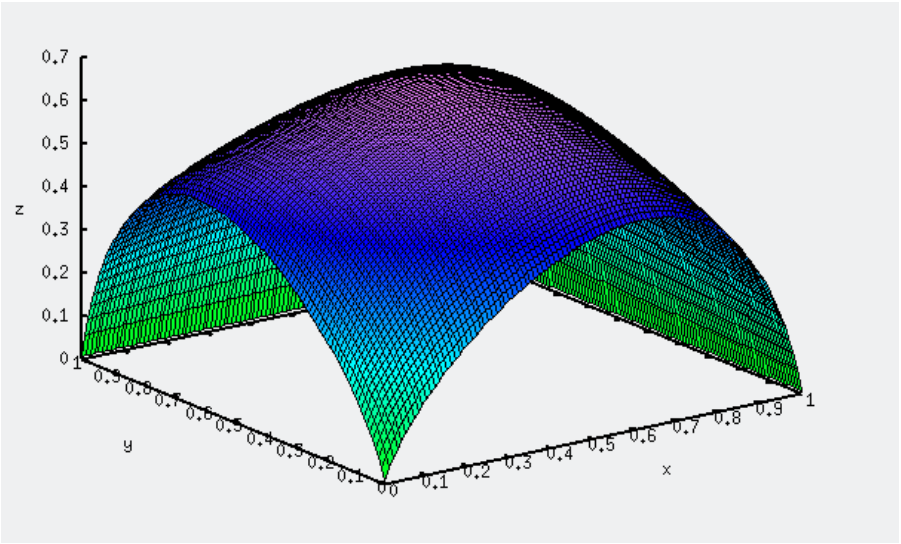

Fig. 2. The entropy in (x, y) coordinates.

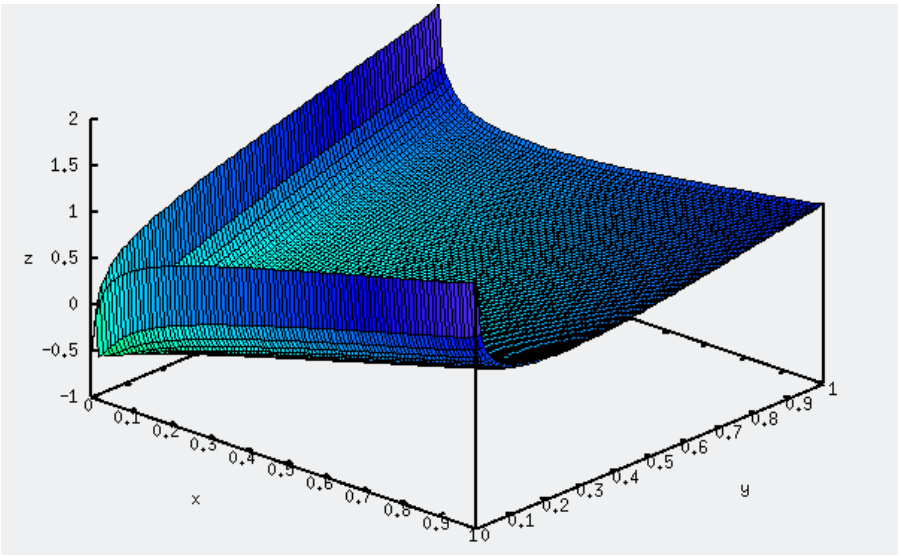

Fig. 3. Curvature of the variance metric with McMullen's normalization.

cise scaling of the metric identifies with the Weil–Petersson metric on Teichmüller space, which is known to be of negative Ricci curvature. One could thus expect that in our 2-dimensional case, where Ricci curvature is (up to a constant) Gauss curvature, $\tilde{g}$ should have negative Gauss curvature. However, this turns out not to be the case: $\widetilde{K}$ takes both positive and negative values.

8.3. Intermediate steps

From [DC76, Section 4], to explicitly compute the curvature we make the following steps. Observe that

$$E_y = -\frac{1}{x(2-x-y)^2} \quad \text{and} \quad G_x = -\frac{1}{y(2-x-y)^2}.$$

Moreover,

$$\sqrt{E\,G} = \frac{1}{\sqrt{x\,y}(2-x-y)}.$$

Therefore,

$$\frac{\partial}{\partial y}\left(\frac{E_y}{\sqrt{EG}}\right) = -\frac{\partial}{\partial y}\left(\frac{\sqrt{y}}{\sqrt{x}}\,\frac{1}{2-x-y}\right) = -\frac{1}{2\sqrt{xy}}\,\frac{2-x+y}{(2-x-y)^2}$$

and similarly

$$\frac{\partial}{\partial x}\left(\frac{G_x}{\sqrt{EG}}\right) = -\frac{1}{2\sqrt{xy}}\,\frac{2+x-y}{(2-x-y)^2}.$$

Finally,

$$\begin{aligned}\left(\frac{E_y}{\sqrt{EG}}\right)_y + \left(\frac{G_x}{\sqrt{EG}}\right)_x &= -\frac{1}{2\sqrt{xy}}\,\frac{2-x+y}{(2-x-y)^2} - \frac{1}{2\sqrt{xy}}\,\frac{2+x-y}{(2-x-y)^2} \\ &= -\frac{2}{\sqrt{xy}}\,\frac{1}{(2-x-y)^2}.\end{aligned}$$

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