J. Eur. Math. Soc. 20, 2333–2356

DOI 10.4171/JEMS/813

Jörg Brüdern · Trevor D. Wooley

Arithmetic harmonic analysis for smooth quartic Weyl sums: three additive equations

Received July 24, 2015

Abstract. We establish the non-singular Hasse principle for systems of three diagonal quartic equations in 32 or more variables, subject to a certain rank condition. Our methods employ the arithmetic harmonic analysis of smooth quartic Weyl sums and also a new estimate for their tenth moment.

Keywords. Quartic Diophantine equations, Hardy–Littlewood method

1. Introduction

In recent years, investigations concerning the solubility of systems of diagonal Diophantine equations via the circle method have been enriched through the use of such unconventional elements as thin averages of Fourier coefficients only partially of arithmetic nature [5], and moment estimates of odd order [6]. These innovations have been applied in several instances to surmount the barrier imposed by the classical scaling principle for suitably entangled systems of diagonal equations. This principle suggests that the number of variables required to solve a system should grow in proportion to the number of its equations. In particular, the recent work of the authors [6] concerning pairs of diagonal quartic equations applies estimates for cubic moments of Fourier coefficients to show that 22 variables suffice to establish the Hasse principle. While the corresponding conclusion for a single quartic equation is available only when the number of variables is at least 12 (this may be established in a manner similar to [14, Theorem 1.2]), our work [6] employs, on average, only 11 variables per equation. We now develop such ideas further, and provide a flexible approach to the control of large values of Fourier coefficients associated with quartic Weyl sums. Once the arithmetic problem at hand is transformed into one in which only Fourier coefficients are present, one is at liberty to consider fractional numbers of variables, as well as fractional numbers of equations. We illustrate the potential of such ideas by investigating the Hasse principle for systems involving three diagonal quartic forms.

J. Brüdern: Mathematisches Institut, Bunsenstrasse 3–5, D-37073 Göttingen, Germany; e-mail: bruedern@uni-math.gwdg.de

T. D. Wooley: School of Mathematics, University of Bristol, University Walk, Clifton, Bristol BS8 1TW, United Kingdom; e-mail: matdw@bristol.ac.uk

Mathematics Subject Classification (2010): 11D72, 11P55, 11E76

Consider a matrix $(a_{ij}) \in \mathbb{Z}^{3\times s}$ with the associated system of equations

$$\sum_{j=1}^{s} a_{ij} x_j^4 = 0 \quad (1 \le i \le 3). \tag{1.1}$$

We verify the Hasse principle for systems of the shape (1.1), subject to a suitable rank condition on (a_{ij}), whenever $s \ge 32$. This should be compared with the conclusion from [3, Theorem 1], which would furnish the Hasse principle for systems of r diagonal quartic equations in s variables only when $s > 12r$. While the latter conclusion is consistent with the classical scaling principle mentioned above, our new result concerning the system (1.1) employs an average of only $10\frac{2}{3}$ variables per equation, and is even more economical than our earlier results [6] for pairs of equations. The new methods of this paper also improve the latter work, covering essentially all of those cases in 22 or more variables that had previously defied resolution (see [7]).

In order to give a precise statement of our result, we introduce some notation. When $s \ge 3$ and any collection of three columns of the matrix (a_{ij}) is linearly independent, we refer to (a_{ij}) as being *highly non-singular*. We say that the matrix of coefficients (a_{ij}) is *propitious* when $s \ge 32$ and it has the block structure $(A_0, A_1, \dots, A_7, B)$, in which $A_l \in \mathbb{Z}^{3\times 4}$ is highly non-singular for each l, and $B \in \mathbb{Z}^{3\times(s-32)}$. Note that the set of $3 \times s$ matrices with $s \ge 32$ which fail to be propitious is very thin. Indeed, typical $3 \times s$ matrices are highly non-singular, and hence also propitious when $s \ge 32$. Finally, given a positive number P, we denote by $\mathcal{N}(P)$ the number of integral solutions $\mathbf{x}$ of (1.1) with $|x_j| \le P$ $(1 \le j \le s)$.

Theorem 1.1. *Let $s \ge 32$, and suppose that $(a_{ij}) \in \mathbb{Z}^{3\times s}$ is propitious. Then provided that the system (1.1) has non-singular real and p-adic solutions for each prime number p, one has $\mathcal{N}(P) \gg P^{s-12}$.*

We remark that [1, Theorem 1] guarantees the existence of a non-zero p-adic solution of the Diophantine system (1.1) provided only that $s \ge 25$ and $p \ge 2^{16}$. A familiar p-adic compactness argument (see [8, Theorem 4]) allows one to deduce that for a propitious system the p-adic solubility hypothesis in Theorem 1.1 is void for all $p \ge 2^{16}$, and one may determine whether or not it possesses non-trivial integral solutions with a finite computation.

The novel arithmetic harmonic analysis associated with our proof of Theorem 1.1 depends on the fourth power moment of certain Fourier coefficients. For a continuous function $H : \mathbb{R} \to [0, \infty)$ of period 1, let

$$c(n) = \int_0^1 H(\alpha) e(-n\alpha)\, \mathrm{d}\alpha,$$

where as usual we write $e(z)$ for $e^{2\pi i z}$. We relate the correlation

$$\int_{[0,1)^3} H(\alpha_1)H(\alpha_2)H(\alpha_3)H(-\alpha_1-\alpha_2-\alpha_3)\, \mathrm{d}\boldsymbol{\alpha}$$

to the moment $\sum_{n\in\mathbb{Z}} |c(n)|^4$, and bound the latter by using large values estimates for Fourier coefficients.

Choose a number $\delta \in [0, 1)$, and let

$$g_\delta(\alpha; P, R) = \sum_{\substack{x \in \mathcal{A}(P,R) \\ \delta P < x \leq P}} e(\alpha x^4),$$

where $\mathcal{A}(P, R)$ denotes the set of numbers $n \in [1, P]$, all of whose prime divisors are at most R. When there is no doubt about the choice of parameters, we abbreviate $g_\delta(\alpha; P, R)$ to $g(\alpha)$. We take $R = P^\eta$ and $H(\alpha) = |g(\alpha)|^{8-\nu}$, where η and ν are sufficiently small positive numbers. An application of Hölder's inequality conveys us from the above correlation to the mean value

$$\int_{[0,1)^3} |g(\alpha_1)g(\alpha_2)|^8 |g(\alpha_3)g(\alpha_1 + \alpha_2 + \alpha_3)|^{8-2\nu} \,\mathrm{d}\boldsymbol{\alpha}.$$

Here the presence of even exponents offers the possibility of replacing the smooth Weyl sum $g(\alpha)$ by its classical cousin

$$f_\delta(\alpha; P) = \sum_{\delta P < x \leq P} e(\alpha x^4),$$

and one perceives the potential for applying the Hardy–Littlewood method to achieve an essentially optimal estimate. In this way, in §5 we obtain the estimate contained in the following theorem, which provides just adequate space for a subsequent application of the circle method to establish Theorem 1.1.

Theorem 1.2. *Suppose that a_i, b_i $(1 \leq i \leq 3)$ are non-zero integers, and that $\delta \in (0, 1)$. Then, whenever η and ν are sufficiently small positive numbers and $1 \leq R \leq P^\eta$, one has*

$$\int_{[0,1)^3} |g(a_1\alpha_1)g(a_2\alpha_2)g(a_3\alpha_3)g(b_1\alpha_1 + b_2\alpha_2 + b_3\alpha_3)|^{8-\nu} \,\mathrm{d}\boldsymbol{\alpha} \ll P^{20-4\nu}.$$

Our proof of Theorem 1.2 involves an analysis of the large values of the Fourier coefficients

$$\int_0^1 |g(\alpha)|^{8-\nu} e(-n\alpha) \,\mathrm{d}\alpha,$$

and this is made to depend on a tenth moment of $g(\alpha)$. Unfortunately, available estimates for this tenth moment would fall woefully short of the strength required to press the method home. We therefore reconfigure and enhance earlier analyses of quartic smooth Weyl sums due to Vaughan [14] and the present authors [4]. In this context, we refer to the number Δ_t as an *admissible exponent* for the positive even integer t if there exists a positive number η such that, whenever $1 \leq R \leq P^\eta$, one has

$$\int_0^1 |f_0(\alpha; P)^2 g_0(\alpha; P, R)^{t-2}| \,\mathrm{d}\alpha \ll P^{t-4+\Delta_t}. \tag{1.2}$$

Note that, in such circumstances, it follows from orthogonality and a consideration of the underlying Diophantine equations that

$$\int_0^1 |g_0(\alpha; P, R)|^t \,\mathrm{d}\alpha \ll P^{t-4+\Delta_t}. \tag{1.3}$$

Theorem 1.3. *The number* $\Delta_{10} = 0.1991466$ *is an admissible exponent.*

We remark that, by applying the methods of Vaughan [14], the authors [4] obtained the admissible exponent 0.213431 in place of 0.1991466, thus improving on the earlier work of Vaughan [14, 15], which, when appropriately combined, delivers the bound (1.3) in the case $t = 10$ with $\Delta_{10} = 0.2142036$. For the application considered here it is vital to have at hand admissible exponents Δ_8 and Δ_{10} with $\Delta_8 + 2\Delta_{10} < 1$. In our earlier work [4] we showed that $\Delta_8 = 0.594193$ is admissible. With the numerical value for Δ_{10} provided by Theorem 1.3, we obtain $\Delta_8 + 2\Delta_{10} < 0.9925$, leaving barely any space to spare in the precision to which we estimate the tenth moment.[1]

Our basic parameter is P, a sufficiently large positive number. In this paper, implicit constants in Vinogradov's notation $\ll$ and $\gg$ may depend on s and ε, as well as ambient coefficients stemming from Diophantine systems such as (1.1). We make frequent use of vector notation in the form $\mathbf{x} = (x_1, \ldots, x_r)$. Here, the dimension r depends on the course of the argument. Occasionally, we abbreviate systems of inequalities $0 \le a_i \le q$ $(1 \le i \le r)$ to $0 \le \mathbf{a} \le q$, and use $(q, \mathbf{a})$ as a shorthand for the largest factor common to the integers $a_1, \ldots, a_r$ and the natural number q. Whenever ε appears in a statement, either implicitly or explicitly, we assert that the statement holds for each $\varepsilon > 0$. Whenever R appears in a statement, it is asserted that there exists a number $\eta > 0$ such that this statement is true for all $1 \le R \le P^\eta$. Whenever ε occurs in a statement involving also R, then we allow η to depend on ε. Note that our conventions allow us, for example, to conclude that $R^9 \ll P^\varepsilon$.

2. The tenth moment of smooth quartic Weyl sums

In this section, we shall be occupied with the verification of Theorem 1.3. The new ingredient in our treatment is an approach to the exponential sum associated with the difference polynomial

$$\Psi(z, h, m) = m^{-4}\big((z + hm^4)^4 - (z - hm^4)^4\big) = 8hz(z^2 + h^2m^8)$$

that diverges from previous work in several respects. Some notation is required to describe the novel features in detail. Whenever $0 \le \theta \le 1/4$, we put

$$M = P^\theta, \qquad H = PM^{-4}, \qquad Q = PM^{-1},$$

and introduce the sum

$$E_0(\alpha) = \sum_{1 \le h \le H} \sum_{1 \le l \le 2P} \sum_{M < m_1 < m_2 \le MR} e(8\alpha l h^3 (m_1^8 - m_2^8)).$$

Our first auxiliary lemma supplies an estimate for the mean square of $E_0(\alpha)$.

[1] Ford [9] and Israilov and Allakov [11] have recorded exponents Δ_8 and Δ_{10} that are smaller than those obtained here. These works are erroneous. See also [10].

Lemma 2.1. *One has*

$$\int_0^1 |E_0(\alpha)|^2 \, d\alpha \ll P^{1+\varepsilon} H M^2.$$

Proof. The integral on the left hand side of the proposed estimate is equal to the number of solutions of the Diophantine equation

$$l_1 h_1^3 (m_1^8 - m_2^8) = l_2 h_2^3 (m_3^8 - m_4^8),$$

in which, for $j = 1$ and 2, the variables are subject to the conditions

$$1 \le l_j \le 2P, \quad 1 \le h_j \le H, \quad M < m_{2j-1} < m_{2j} \le MR.$$

There are $O(PHM^2R^2)$ choices for l_2, h_2, m_3, m_4, and for each such choice, the numbers l_1, h_1 and $m_1^8 - m_2^8$ are divisors of the non-zero integer $l_2 h_2^3 (m_3^8 - m_4^8)$. A familiar estimate for the number of divisors now shows that the number of choices for l_1, h_1, m_1 and m_2 is bounded by $O(P^\varepsilon)$, and the lemma follows. □

We emphasise here and in what follows our use of the conventions concerning the parameters R and ε described in the penultimate paragraph of §1. In plain language, Lemma 2.1 asserts that for each $\varepsilon > 0$, there exists a positive number $\eta = \eta(\varepsilon)$ having the property that, whenever $1 \le R \le P^\eta$, then $\int_0^1 |E_0(\alpha)|^2 \, d\alpha \ll P^{1+\varepsilon} H M^2$.

The next lemma is the key to our new tenth moment estimate.

Lemma 2.2. *One has*

$$\int_0^1 |E_0(\alpha)^2 f_0(\alpha; 2Q)^4| \, d\alpha \ll Q^{5+\varepsilon}.$$

Proof. By Weyl's differencing technique [16, Lemma 2.3], one finds that

$$|f_0(\alpha; 2Q)|^4 \ll Q^3 + Q \sum_{0<|n|\le 32Q^4} c(n) e(\alpha n),$$

where the coefficients $c(n)$ are certain rational numbers satisfying $c(n) \ll |n|^\varepsilon$. Write

$$\varrho(n) = \int_0^1 |E_0(\alpha)|^2 e(\alpha n) \, d\alpha.$$

Then it follows that

$$\int_0^1 |E_0(\alpha)^2 f_0(\alpha; 2Q)^4| \, d\alpha \ll Q^3 \varrho(0) + Q \sum_{0<|n|\le 32Q^4} c(n) \varrho(n).$$

By orthogonality, one has $\varrho(n) \ge 0$. Furthermore, Lemma 2.1 supplies the bound $\varrho(0) \ll P^{1+\varepsilon} H M^2$. Thus, we deduce that

$$\begin{aligned}\int_0^1 |E_0(\alpha)^2 f_0(\alpha; 2Q)^4| \, d\alpha &\ll Q^3 P^{1+\varepsilon} H M^2 + Q P^\varepsilon \sum_{n\in\mathbb{Z}} \varrho(n) \\ &\ll P^\varepsilon (Q^3 PHM^2 + Q E_0(0)^2) \ll Q^{5+2\varepsilon}.\end{aligned}$$

□

Lemma 2.3. *The exponents* $\Delta_8 = 0.594193$ *and* $\Delta_{12} = 0$ *are admissible.*

Proof. The desired conclusion concerning Δ_8 follows from [4, Theorem 2] and the discussion surrounding the table of exponents in [4, p. 393]. Meanwhile, the upper bound (1.2) when $t = 12$ is a consequence of [14, Lemma 5.2]. □

We initiate our estimation of the tenth moment by choosing an admissible value for Δ_{10}. That such values exist follows from the trivial bounds for f_0 and g_0. For the rest of this section, we work with the sums $g_0(\alpha; P, R)$ and $f_0(\alpha; P)$ only, and abbreviate these to $g(\alpha)$ and $f(\alpha)$, respectively. We put $g_\flat(\alpha) = g_0(\alpha; 2Q, R)$, and for the sake of concision, for positive even integers t, we write

$$U_t = \int_0^1 |g_\flat(\alpha)|^t \, d\alpha.$$

Further, we require the exponential sum

$$F_1(\alpha) = \sum_{1\le h\le H} \sum_{M<m\le MR} \sum_{1\le z\le 2P} e(8\alpha hz(z^2 + h^2 m^8)).$$

Lemma 2.4. *Suppose that* Δ_8 *and* Δ_{10} *are admissible exponents satisfying*

$$\tfrac{1}{2} < \Delta_8 < \tfrac{3}{5}, \quad \tfrac{1}{10} < \Delta_{10} < \tfrac{1}{4} \quad \textit{and} \quad \tfrac{3}{2}\Delta_8 - \tfrac{5}{7} < \Delta_{10} < 2\Delta_8 - \tfrac{27}{28}.$$

Put

$$\theta = \max\left\{ \frac{3}{17}, \frac{7 + 2\Delta_{10} - 3\Delta_8}{33 + 2\Delta_{10} - 3\Delta_8} \right\},$$

and define $\Delta'_{10} = \Delta_8(1-\theta) + 4\theta - 1$. *Then whenever* $\Delta > \Delta'_{10}$, *the exponent* Δ *is admissible for* $t = 10$.

Proof. Our starting point is an application of a suitable version of the *fundamental lemma* in the iterative method (compare [14, Lemma 2.1]). Thus, as a consequence of [20, Lemma 2.3] in combination with the argument of the proof of [20, Lemma 3.1] (see [17, Lemma 2.1]),

$$\int_0^1 |f(\alpha)^2 g(\alpha)^8| \, d\alpha \ll P^\varepsilon M^7 (PMQ^{4+\Delta_8} + T) \tag{2.1}$$

where

$$T = \int_0^1 F_1(\alpha) |g_\flat(\alpha)|^8 \, d\alpha. \tag{2.2}$$

By Cauchy's inequality,

$$|F_1(\alpha)|^2 \le HMR \sum_{1\le h\le H} \sum_{M<m\le MR} \Big| \sum_{1\le z\le 2P} e(\alpha\Psi(z,h,m)) \Big|^2.$$

Here, we open the square and rewrite it as a double sum over z_1 and z_2, say. The substitutions $z = z_1 + z_2$ and $l = z_1 - z_2$ then yield

$$\Big|\sum_{1\le z\le 2P} e(\alpha\Psi(z,h,m))\Big|^2 = \sum_{|l|\le 2P}\sum_{z\in\mathcal{B}(l)} e\big(\alpha hl(6z^2+2l^2+8h^2m^8)\big),$$

in which $\mathcal{B}(l)$ denotes the set of all integers z with $1 \le z \pm l \le 4P$ and $z \equiv l \bmod 2$. Separation of the term $l = 0$ delivers the inequality

$$|F_1(\alpha)|^2 \ll P^{1+\varepsilon}H^2M^2 + P^{\varepsilon}HM \sum_{\substack{1\le h\le H\\ 1\le l\le 2P}} \Big|\sum_{M<m\le MR}\sum_{z\in\mathcal{B}(l)} e(\alpha hl(6z^2+8h^2m^8))\Big|.$$

Yet another application of Cauchy's inequality now produces the bound

$$|F_1(\alpha)|^2 \ll P^{1+\varepsilon}H^2M^2 + P^{\varepsilon}HM(D(\alpha)E(\alpha))^{1/2},$$

in which

$$D(\alpha) = \sum_{1\le h\le H}\sum_{1\le l\le 2P}\Big|\sum_{z\in\mathcal{B}(l)} e(6\alpha hlz^2)\Big|^2,$$

$$E(\alpha) = \sum_{1\le h\le H}\sum_{1\le l\le 2P}\Big|\sum_{M<m\le MR} e(8\alpha lh^3m^8)\Big|^2.$$

On substituting the last inequality for $|F_1(\alpha)|^2$ into (2.2), we infer that

$$T \ll P^{1/2+\varepsilon}HMQ^{4+\Delta_8} + P^{\varepsilon}(HM)^{1/2}T_1, \tag{2.3}$$

where

$$T_1 = \int_0^1 (D(\alpha)E(\alpha))^{1/4}|g_\flat(\alpha)|^8\,\mathrm{d}\alpha. \tag{2.4}$$

We apply the Hardy–Littlewood method to estimate T_1. For integers a, q with $0 \le a \le q \le P$ and $(a,q) = 1$, let $\mathfrak{N}(q,a)$ denote the set of all $\alpha \in [0,1)$ with $|q\alpha - a| \le PQ^{-4}$, and let $\mathfrak{N}$ denote the union of these intervals. Note that this union is disjoint. Define the function $\Omega : [0,1) \to [0,1]$ by

$$\Omega(\alpha) = (q + Q^4|q\alpha - a|)^{-1} \qquad (\alpha \in \mathfrak{N}(q,a)),$$

and put $\Omega(\alpha) = 0$ when $\alpha \notin \mathfrak{N}$.

By Dirichlet's theorem on Diophantine approximation, whenever $\alpha \in [0,1)$, there are integers a, q with $0 \le a \le q \le Q^4P^{-1}$, $(a,q) = 1$ and $|q\alpha - a| \le PQ^{-4}$. Moreover, although our sum $D(\alpha)$ differs in detail from that used by Vaughan [14] in his equation (3.2), the proof of [14, Lemma 3.1] applies to our sum as well and yields the same estimate. We therefore conclude that the bound

$$D(\alpha) \ll P^{2+\varepsilon}H + P^{3+\varepsilon}H\Omega(\alpha)$$

holds for all $\alpha \in [0, 1)$. Consequently, we deduce from (2.4) that

$$T_1 \ll P^{1/2+\varepsilon} H^{1/4} T_2 + P^{3/4+\varepsilon} H^{1/4} T_3, \tag{2.5}$$

where

$$T_2 = \int_0^1 E(\alpha)^{1/4} |g_\flat(\alpha)|^8 \, \mathrm{d}\alpha \quad \text{and} \quad T_3 = \int_{\mathfrak{N}} (\Omega(\alpha) E(\alpha))^{1/4} |g_\flat(\alpha)|^8 \, \mathrm{d}\alpha.$$

The estimation of T_2 will involve the application of Lemma 2.2. An inspection of the definitions of $E_0(\alpha)$ and $E(\alpha)$ reveals that

$$E(\alpha) \ll |E_0(\alpha)| + PHMR. \tag{2.6}$$

As a first bound for T_2, we then have

$$T_2 \ll P^\varepsilon (PHM)^{1/4} U_8 + \int_0^1 |E_0(\alpha)|^{1/4} |g_\flat(\alpha)|^8 \, \mathrm{d}\alpha.$$

Let

$$V = \int_0^1 |E_0(\alpha)^2 g_\flat(\alpha)^4| \, \mathrm{d}\alpha.$$

Then, a further application of Hölder's inequality yields the bound

$$T_2 \ll P^\varepsilon (PHM)^{1/4} U_8 + U_8^{5/8} U_{10}^{1/4} V^{1/8}.$$

By orthogonality, the mean value V counts the number of integral solutions of an associated Diophantine equation. By relaxing the conditions on the variables associated with the exponential sum $g_\flat(\alpha)$, and reversing course on the application of orthogonality, we infer via Lemma 2.2 that

$$V \leq \int_0^1 |E_0(\alpha)^2 f_0(\alpha; 2Q)^4| \, \mathrm{d}\alpha \ll (PHM^2)^2 Q^{1+\varepsilon}, \tag{2.7}$$

and so by applying (1.3), we deduce that

$$P^{1/2} H^{1/4} T_2 \ll P^{3/4+\varepsilon} (HM)^{1/2} Q^4 (M^{-1/4} Q^{\Delta_8} + Q^{(5\Delta_8 + 2\Delta_{10} + 1)/8}).$$

However, the hypothesis $\Delta_{10} > \frac{3}{2}\Delta_8 - \frac{5}{7}$ ensures that

$$\frac{3\Delta_8 - 2\Delta_{10} - 1}{3\Delta_8 - 2\Delta_{10} + 1} < \frac{7 + 2\Delta_{10} - 3\Delta_8}{33 + 2\Delta_{10} - 3\Delta_8}.$$

Hence we have

$$\theta > \frac{3\Delta_8 - 2\Delta_{10} - 1}{3\Delta_8 - 2\Delta_{10} + 1},$$

so that

$$M^2 \geq Q^{3\Delta_8 - 2\Delta_{10} - 1}.$$

We thus conclude that

$$P^{1/2}H^{1/4}T_2 \ll P^{3/4+\varepsilon}(HM)^{1/2}Q^{4+(5\Delta_8+2\Delta_{10}+1)/8}. \tag{2.8}$$

As our first step in estimating T_3, we apply (2.6) to deduce that

$$T_3 \ll T_4 + (PHMR)^{1/4}T_5,$$

where

$$T_4 = \int_{\mathfrak{N}} (\Omega(\alpha)|E_0(\alpha)|)^{1/4}\,|g_\flat(\alpha)|^8\,\mathrm{d}\alpha \quad \text{and} \quad T_5 = \int_{\mathfrak{N}} \Omega(\alpha)^{1/4}|g_\flat(\alpha)|^8\,\mathrm{d}\alpha.$$

Write

$$W = \int_{\mathfrak{N}} \Omega(\alpha)|g_\flat(\alpha)|^4\,\mathrm{d}\alpha.$$

Then an application of [2, Lemma 2] confirms the estimate

$$W \ll Q^{\varepsilon-4}(PQ^2 + Q^4) \ll Q^{\varepsilon}.$$

Hölder's inequality therefore combines with (1.3), (2.7) and Lemma 2.3 to give

$$T_4 \le V^{1/8}W^{1/4}U_{10}^{1/2}U_{12}^{1/8} \ll P^{\varepsilon}\big((PHM^2)^2Q\big)^{1/8}(Q^{6+\Delta_{10}})^{1/2}(Q^8)^{1/8},$$

whence

$$P^{3/4+\varepsilon}H^{1/4}T_4 \ll P^{1+\varepsilon}(HM)^{1/2}Q^{4+(4\Delta_{10}+1)/8}. \tag{2.9}$$

In like manner, another application of Hölder's inequality yields the bound

$$T_5 \ll W^{1/4}U_8^{1/4}U_{10}^{1/2} \ll P^{\varepsilon}(Q^{4+\Delta_8})^{1/4}(Q^{6+\Delta_{10}})^{1/2},$$

whence

$$P^{1+\varepsilon}H^{1/2}M^{1/4}T_5 \ll P^{1+\varepsilon}(HM)^{1/2}Q^4(M^{-1/4}Q^{(\Delta_8+2\Delta_{10})/4}). \tag{2.10}$$

By combining (2.9) and (2.10), we conclude that

$$P^{3/4}H^{1/4}T_3 \ll P^{1+\varepsilon}(HM)^{1/2}Q^4(Q^{(4\Delta_{10}+1)/8} + M^{-1/4}Q^{(\Delta_8+2\Delta_{10})/4}).$$

The hypotheses of the statement of the lemma imply that

$$\theta \ge \frac{3}{17} > \frac{1}{11} \ge \frac{2\Delta_8 - 1}{2\Delta_8 + 1},$$

so that $M \ge Q^{\Delta_8 - 1/2}$. Thus we conclude that

$$P^{3/4}H^{1/4}T_3 \ll P^{1+\varepsilon}(HM)^{1/2}Q^{4+(4\Delta_{10}+1)/8}. \tag{2.11}$$

We may now collect together our various estimates, first combining (2.5), (2.8) and (2.11), and substituting the result into (2.3) to obtain the bound

$$T \ll P^{1+\varepsilon}MHQ^{4+\Delta_8}(P^{-1/2} + P^{-1/4}Q^{(1+2\Delta_{10}-3\Delta_8)/8} + Q^{(1+4\Delta_{10}-8\Delta_8)/8}).$$

Since $\theta \geq \frac{3}{17} > \frac{1}{8}$, one has $HP^{-1/2} \leq 1$, and the bound

$$HP^{-1/4}Q^{(1+2\Delta_{10}-3\Delta_8)/8} \leq 1$$

follows in its turn from the hypothesis that

$$\theta \geq \frac{7+2\Delta_{10}-3\Delta_8}{33+2\Delta_{10}-3\Delta_8}.$$

Meanwhile, since we suppose that $\Delta_{10} < 2\Delta_8 - \frac{27}{28}$, one finds from the hypothesis $\theta \geq \frac{3}{17}$ that

$$HQ^{(1+4\Delta_{10}-8\Delta_8)/8} < HQ^{-5/14} \leq 1.$$

We therefore deduce that $T \ll P^{1+\varepsilon}MQ^{4+\Delta_8}$, and on substituting into (2.1), we obtain the bound

$$\int_0^1 |f(\alpha)^2 g(\alpha)^8|\,\mathrm{d}\alpha \ll P^{1+\varepsilon}M^8Q^{4+\Delta_8} = P^{6+\Delta'_{10}+\varepsilon},$$

where $\Delta'_{10} = \Delta_8(1-\theta)+4\theta-1$. It follows that whenever $\Delta > \Delta'_{10}$, then Δ is admissible for $t = 10$, and so the proof of the lemma is complete. □

We are now equipped to describe the iteration that yields the admissible exponent recorded in Theorem 1.3. We recall from Lemma 2.3 that the exponent $\Delta_8 = 0.594193$ is admissible. Also, from the work of Vaughan [14] and the authors [4], there exists an admissible exponent Δ_{10} smaller than 0.22. Suppose then that an admissible exponent Δ_{10} has been established satisfying

$$0.2241\ldots = 2\Delta_8 - \tfrac{27}{28} > \Delta_{10} > \tfrac{3}{2}\Delta_8 - \tfrac{5}{7} = 0.1770\ldots.$$

It follows that Lemma 2.4 then applies with

$$\theta = \frac{7+2\Delta_{10}-3\Delta_8}{33+2\Delta_{10}-3\Delta_8},$$

and that any exponent Δ'_{10} exceeding $\Delta_8(1-\theta)+4\theta-1$ is also admissible. On iterating this treatment, one finds a decreasing sequence of admissible exponents converging to the larger root Δ^*_{10} of the equation

$$\Delta^*_{10} = \Delta_8 - 1 + (4-\Delta_8)\frac{7+2\Delta^*_{10}-3\Delta_8}{33+2\Delta^*_{10}-3\Delta_8}.$$

On using the value for Δ_8 recorded in Lemma 2.3, one readily confirms that Δ^*_{10} satisfies the equation

$$2(\Delta^*_{10})^2 + (27-3\Delta_8)\Delta^*_{10} + 5 - 17\Delta_8 = 0,$$

whence

$$\Delta^*_{10} = \tfrac{1}{4}\big(3\Delta_8 - 27 + \sqrt{689 - 26\Delta_8 + 9\Delta_8^2}\big) = 0.199146547\ldots.$$

Given any positive number δ, this iteration yields an admissible exponent $\Delta_{10}^{\dagger}$, satisfying $\Delta_{10}^{*} < \Delta_{10}^{\dagger} < \Delta_{10}^{*} + \delta$, after a number of iterations bounded solely in terms of δ. Consequently, keeping in mind our conventions concerning ε and R, we have

$$\int_0^1 |f(\alpha)^2 g(\alpha)^8|\,\mathrm{d}\alpha \ll P^{6+\Delta_{10}^{*}+\varepsilon}.$$

We deduce that the exponent Δ_{10} is admissible whenever $\Delta_{10} > \Delta_{10}^{*}$, and thus we arrive at the conclusion of Theorem 1.3.

3. Large values estimates

Our next task is to provide a proof of the mixed fractional moment estimate recorded in Theorem 1.2. Within this and the next two sections, we fix a choice of $\delta \in (0, 1)$ once and for all, and then adumbrate $f_\delta(\alpha; P)$ to $f(\alpha)$ and $g_\delta(\alpha; P, R)$ to $g(\alpha)$. Finally, according to Theorem 1.3 and Lemma 2.3, we are at liberty to suppose that Δ_8 and Δ_{10} are admissible exponents satisfying the inequalities

$$\Delta_8 \le 0.594193 \quad \text{and} \quad \Delta_{10} \le 0.1991466.$$

When $0 \le \tau \le 1$, we define the Fourier coefficient

$$\psi_\tau(n) = \int_0^1 |g(\alpha)|^{8-\tau} e(-n\alpha)\,\mathrm{d}\alpha. \tag{3.1}$$

Notice here that since $|g(\alpha)|^{8-\tau}$ is real and even, the Fourier coefficient $\psi_\tau(n)$ is necessarily real. Our goal in Lemmata 4.1 and 4.2 will be to estimate the fourth moment

$$\sum_{|n|\le P^5} |\psi_\tau(n)|^4$$

together with some of its relatives. This we achieve by dividing the range of summation into dyadic intervals according to the size of the Fourier coefficients. With such ideas in mind, when $T > 0$, we write

$$M_\tau(T) = \sum_{\substack{|n|\le P^5 \\ T<|\psi_\tau(n)|\le 2T}} |\psi_\tau(n)|^4. \tag{3.2}$$

By applying the triangle inequality to (3.1) in combination with Hölder's inequality, one obtains the bound

$$|\psi_\tau(n)| \le \psi_\tau(0) \le \left(\int_0^1 |g(\alpha)|^8\,\mathrm{d}\alpha\right)^{1-\tau/8} \ll P^{4+\Delta_8},$$

and thus we may restrict attention to values of T with $T \le P^5$.

We now seek to bound $M_\tau(T)$ when $1 \le T \le P^5$. Define $\mathcal{Z}_T$ to be the set of integers n with $|n| \le P^5$ such that $T < |\psi_\tau(n)| \le 2T$, and write $Z_T = \text{card}(\mathcal{Z}_T)$. For each $n \in \mathcal{Z}_T$, we take $\omega_n = 1$ when $\psi_\tau(n) > 0$, and $\omega_n = -1$ when $\psi_\tau(n) < 0$, and then define

$$K_T(\alpha) = \sum_{n\in\mathcal{Z}_T} \omega_n e(-n\alpha).$$

Thus we have

$$\begin{aligned}\int_0^1 |g(\alpha)|^{8-\tau} K_T(\alpha)\,\mathrm{d}\alpha &= \sum_{n\in\mathcal{Z}_T} \omega_n \int_0^1 |g(\alpha)|^{8-\tau} e(-n\alpha)\,\mathrm{d}\alpha \\ &= \sum_{n\in\mathcal{Z}_T} |\psi_\tau(n)| > TZ_T. \end{aligned} \tag{3.3}$$

Before announcing our basic large values estimates, we recall that as an immediate consequence of [12, Lemma 2.1], one has

$$\int_0^1 |g(\alpha)^4 K_T(\alpha)^2|\,\mathrm{d}\alpha \le \int_0^1 |f(\alpha)^4 K_T(\alpha)^2|\,\mathrm{d}\alpha \ll P^3 Z_T + P^{2+\varepsilon} Z_T^{3/2}. \tag{3.4}$$

Finally, we introduce the exponents

$$\kappa_1(\tau) = 11 - (2 - \Delta_{10})\tau, \tag{3.5}$$

$$\kappa_2(\tau) = 19 + \Delta_8 + 2\Delta_{10} - (4 - 2\Delta_8 + 2\Delta_{10})\tau. \tag{3.6}$$

Lemma 3.1. *Let* $0 < \tau \le 1$ *and* $1 \le T \le P^5$. *Then*

$$Z_T \ll P^\varepsilon (P^{\kappa_r(\tau)} T^{-2r} + P^{2\kappa_r(\tau)-2} T^{-4r}) \quad (r = 1, 2).$$

Proof. In the looming discussion we drop mention of T and τ from our various notations. When $r \in \{1, 2\}$ and $s \in \mathbb{N}$ is even, define

$$I_r = \int_0^1 |g(\alpha)^4 K(\alpha)^{2r}|\,\mathrm{d}\alpha \quad \text{and} \quad J_s = \int_0^1 |g(\alpha)|^s\,\mathrm{d}\alpha.$$

As an immediate consequence of (1.3), Lemma 2.3 and Theorem 1.3, one has

$$J_8 \ll P^{4+\Delta_8}, \quad J_{10} \ll P^{6+\Delta_{10}} \quad \text{and} \quad J_{12} \ll P^8. \tag{3.7}$$

Then an application of Hölder's inequality shows in the first instance that

$$\int_0^1 |g(\alpha)|^{8-\tau} K(\alpha)\,\mathrm{d}\alpha \le I_1^{1/2} J_{10}^{\tau/2} J_{12}^{(1-\tau)/2},$$

and by means of (3.3), (3.4) and (3.7), we infer the bound

$$\begin{aligned} TZ < \int_0^1 |g(\alpha)|^{8-\tau} K(\alpha)\,\mathrm{d}\alpha &\ll P^\varepsilon (P^3 Z + P^2 Z^{3/2})^{1/2} (P^{6+\Delta_{10}})^{\tau/2} (P^8)^{(1-\tau)/2} \\ &\ll P^\varepsilon \big((P^{\kappa_1(\tau)} Z)^{1/2} + (P^{2\kappa_1(\tau)-2} Z^3)^{1/4} \big). \end{aligned}$$

The claimed estimate with $r = 1$ follows on disentangling this bound.

Meanwhile, another application of Hölder's inequality yields

$$\int_0^1 |g(\alpha)|^{8-\tau} K(\alpha)\,\mathrm{d}\alpha \le I_2^{1/4} J_8^{(1+2\tau)/4} J_{10}^{(1-\tau)/2}.$$

Hence, by applying a trivial estimate for $K(\alpha)$ in combination with (3.3), (3.4) and (3.7), we infer that

$$\begin{aligned} TZ &\ll P^{\varepsilon}(P^3Z^3 + P^2Z^{7/2})^{1/4}(P^{4+\Delta_8})^{(1+2\tau)/4}(P^{6+\Delta_{10}})^{(1-\tau)/2} \\ &\ll P^{\varepsilon}\big((P^{\kappa_2(\tau)}Z^3)^{1/4} + (P^{2\kappa_2(\tau)-2}Z^7)^{1/8}\big). \end{aligned}$$

The claimed estimate with $r = 2$ follows on disentangling this bound. □

We require large values estimates of similar type for related mean values associated with a restriction to a set of minor arcs. Define the major arcs $\mathfrak{M}$ to be the union of the intervals

$$\mathfrak{M}(q,a) = \{\alpha \in [0,1) : |q\alpha - a| \le P^{-7/2}\}, \tag{3.8}$$

with $0 \le a \le q \le P^{1/2}$ and $(a,q) = 1$, and then put $\mathfrak{m} = [0,1) \setminus \mathfrak{M}$. When $\mathfrak{B} \subseteq [0,1)$ is measurable, we define

$$\Psi_{\mathfrak{B}}(n) = \int_{\mathfrak{B}} |f(\alpha)^2 g(\alpha)^6| e(-n\alpha)\,\mathrm{d}\alpha,$$

and when $T > 0$, define

$$M_0^*(T) = \sum_{\substack{|n| \le P^5 \\ T < |\Psi_{\mathfrak{m}}(n)| \le 2T}} |\Psi_{\mathfrak{m}}(n)|^4. \tag{3.9}$$

Define $\mathcal{Z}_0$ to be the set of integers n with $|n| \le P^5$ for which $T < |\Psi_{\mathfrak{m}}(n)| \le 2T$, and write $Z_0 = Z_0(T)$ for $\mathrm{card}(\mathcal{Z}_0)$. For each $n \in \mathcal{Z}_0$, we take $\omega_n = 1$ when $\Psi_{\mathfrak{m}}(n) > 0$, and we put $\omega_n = -1$ when $\Psi_{\mathfrak{m}}(n) < 0$. Also, we define

$$K_0(\alpha) = \sum_{n \in \mathcal{Z}_0} \omega_n e(-n\alpha).$$

Then, as in (3.3), one obtains

$$\int_{\mathfrak{m}} |f(\alpha)^2 g(\alpha)^6| K_0(\alpha)\,\mathrm{d}\alpha = \sum_{n \in \mathcal{Z}_0} |\Psi_{\mathfrak{m}}(n)| > TZ_0. \tag{3.10}$$

Before announcing our large values estimates for $\Psi_{\mathfrak{m}}(n)$, we recall the definitions (3.5) and (3.6) of $\kappa_1(\tau)$ and $\kappa_2(\tau)$.

Lemma 3.2. *Let* $1 \le T \le P^5$. *Then*

$$Z_0 \ll P^{\varepsilon}(P^{\kappa_1(0)+\Delta_{10}-1/4}T^{-2} + P^{2\kappa_1(0)+2\Delta_{10}-5/2}T^{-4})$$

and

$$Z_0 \ll P^{\varepsilon}(P^{\kappa_2(0)}T^{-4} + P^{2\kappa_2(0)-2}T^{-8}).$$

Proof. Define

$$\mathcal{J} = \int_{\mathfrak{m}} |f(\alpha)^2 g(\alpha)^{10}|\, d\alpha.$$

An enhanced version of Weyl's inequality (see [13, Lemma 3]) shows that

$$\sup_{\alpha \in \mathfrak{m}} |f(\alpha)| \ll P^{7/8+\varepsilon},$$

and so we deduce via (3.7) that

$$\mathcal{J} \le \Big(\sup_{\alpha \in \mathfrak{m}} |f(\alpha)|\Big)^2 J_{10} \ll P^{8+\Delta_{10}-1/4+\varepsilon}.$$

An application of Schwarz's inequality shows that

$$\int_{\mathfrak{m}} |f(\alpha)^2 g(\alpha)^6| K_0(\alpha)\, d\alpha \le \left(\int_0^1 |f(\alpha) g(\alpha) K_0(\alpha)|^2\, d\alpha \right)^{1/2} \mathcal{J}^{1/2}.$$

In view of (3.4) and (3.10), another application of Schwarz's inequality yields

$$T Z_0 < \int_{\mathfrak{m}} |f(\alpha)^2 g(\alpha)^6| K_0(\alpha)\, d\alpha \ll P^{\varepsilon} (P^3 Z_0 + P^2 Z_0^{3/2})^{1/2} (P^{8+\Delta_{10}-1/4})^{1/2}.$$

The first of the claimed estimates follows by disentangling this bound. For the second we proceed just as in the proof of Lemma 3.1 in the case $r = 2$, noting that the mean value estimates for J_8 and J_{10} should in this instance be replaced by the estimates

$$\int_0^1 |f(\alpha)^2 g(\alpha)^6|\, d\alpha \ll P^{4+\Delta_8} \quad \text{and} \quad \int_0^1 |f(\alpha)^2 g(\alpha)^8|\, d\alpha \ll P^{6+\Delta_{10}},$$

available via (1.2). This completes the proof of the lemma. □

4. Fourier coefficients and their moments

Our goal in this section is the proof of an estimate for a certain mixed moment of Fourier coefficients associated with quartic Weyl sums. This we achieve by employing our large values estimates of the previous section so as to bound the quantities $M_\tau(T)$ and $M_0^*(T)$ defined in (3.2) and (3.9). We proceed in stages. In what follows, we make use of a positive number τ satisfying

$$40\tau \le \min\{1 - 4\Delta_{10}, 1 - 2\Delta_{10} - \Delta_8\}. \tag{4.1}$$

Lemma 4.1. *Suppose that τ is a positive number satisfying (4.1). Then*

$$\sum_{|n| \le P^5} |\psi_\tau(n)|^4 \ll P^{20-\tau} \quad \textit{and} \quad \sum_{|n| \le P^5} |\Psi_{\mathfrak{m}}(n)|^4 \ll P^{20-9\tau}.$$

Proof. Observe first that

$$\sum_{\substack{|n|\le P^5\\ |\psi_\tau(n)|>1}} |\psi_\tau(n)|^4 \le \sum_{\substack{l=0\\ 2^l\le P^5}}^{\infty} M_\tau(2^l).$$

Thus, for some number T with $1 \le T \le P^5$, one has

$$\sum_{|n|\le P^5} |\psi_\tau(n)|^4 \ll P^5 + (\log P)M_\tau(T) \ll P^5 + P^\varepsilon T^4 Z_T.$$

Should T satisfy the bound $T \le P^{9/2}$, then it follows from the estimate supplied by Lemma 3.1 with $r = 1$ that

$$\sum_{|n|\le P^5} |\psi_\tau(n)|^4 \ll P^5 + P^\varepsilon(P^{\kappa_1(\tau)}T^2 + P^{2\kappa_1(\tau)-2}) \ll P^{20-(2-\Delta_{10})\tau+\varepsilon} \ll P^{20-\tau}.$$

Meanwhile, when $P^{9/2} < T \le P^5$, we discern from Lemma 3.1 with $r = 2$ that

$$\begin{aligned}\sum_{|n|\le P^5} |\psi_\tau(n)|^4 &\ll P^5 + P^\varepsilon(P^{\kappa_2(\tau)} + P^{2\kappa_2(\tau)-2}T^{-4})\\ &\ll P^\varepsilon(P^{\kappa_2(\tau)} + P^{2\kappa_2(\tau)-20}).\end{aligned}$$

In view of our hypotheses concerning τ, one finds that

$$\kappa_2(\tau) \le 19 + \Delta_8 + 2\Delta_{10} \le 20 - 40\tau$$

and

$$2\kappa_2(\tau) - 20 \le 18 + 2\Delta_8 + 4\Delta_{10} \le 20 - 80\tau.$$

The first conclusion of the lemma is now immediate.

In like manner, one finds that for some number T with $1 \le T \le P^5$, one has

$$\sum_{|n|\le P^5} |\Psi_{\mathfrak{m}}(n)|^4 \ll P^5 + (\log P)M_0^*(T) \ll P^5 + P^\varepsilon T^4 Z_0(T).$$

Should one have $T \le P^{9/2}$, it follows from the first estimate of Lemma 3.2 that

$$\begin{aligned}\sum_{|n|\le P^5} |\Psi_{\mathfrak{m}}(n)|^4 &\ll P^5 + P^\varepsilon(P^{11+\Delta_{10}-1/4}T^2 + P^{20+2\Delta_{10}-1/2})\\ &\ll P^\varepsilon(P^{20+\Delta_{10}-1/4} + P^{20+2\Delta_{10}-1/2}).\end{aligned}$$

Meanwhile, when $P^{9/2} < T \le P^5$, the second estimate of Lemma 3.2 yields

$$\begin{aligned}\sum_{|n|\le P^5} |\Psi_{\mathfrak{m}}(n)|^4 &\ll P^5 + P^\varepsilon(P^{\kappa_2(0)} + P^{2\kappa_2(0)-2}T^{-4})\\ &\ll P^\varepsilon(P^{19+\Delta_8+2\Delta_{10}} + P^{18+2\Delta_8+4\Delta_{10}}).\end{aligned}$$

Thus, in all cases, our hypotheses concerning τ ensure that

$$\sum_{|n|\le P^5} |\Psi_{\mathfrak{m}}(n)|^4 \ll P^{20-10\tau+\varepsilon},$$

and the second conclusion of the lemma follows. □

Lemma 4.2. *Let a and b be non-zero integers, and suppose that the positive number τ satisfies* (4.1). *Then*

$$\sum_{|n|\le P^5} \psi_0(an)^2\psi_\tau(bn)^2 \ll P^{20-2\tau}.$$

Proof. By orthogonality, the Fourier coefficient $\psi_0(n)$ counts the number of integral solutions of the Diophantine equation

$$\sum_{i=1}^{4}(x_i^4-y_i^4)=n,$$

with $x_i, y_i \in \mathcal{A}(P,R)$ and $\delta P < x_i, y_i \le P$ $(1\le i\le 4)$. Thus, in particular, one has $\psi_0(an)=0$ whenever $|n|>P^{9/2}$. Moreover, by relaxing the condition on x_1 and y_1 so that these variables are no longer required to lie in $\mathcal{A}(P,R)$, and again invoking orthogonality, it is apparent that

$$\psi_0(n)=\int_0^1 |g(\alpha)|^8 e(-n\alpha)\,\mathrm{d}\alpha \le \int_0^1 |f(\alpha)^2 g(\alpha)^6| e(-n\alpha)\,\mathrm{d}\alpha = \Psi_{[0,1)}(n),$$

whence $\psi_0(n)\le \Psi_{\mathfrak{m}}(n)+\Psi_{\mathfrak{M}}(n)$. Thus,

$$\psi_0(n)^2 \le 2(\Psi_{\mathfrak{m}}(n)^2+\Psi_{\mathfrak{M}}(n)^2),$$

and we deduce that

$$\sum_{|n|\le P^5}\psi_0(an)^2\psi_\tau(bn)^2 \ll \Xi(\mathfrak{M})+\Xi(\mathfrak{m}), \tag{4.2}$$

where

$$\Xi(\mathfrak{B})=\sum_{|n|\le P^{9/2}}\Psi_{\mathfrak{B}}(an)^2\psi_\tau(bn)^2.$$

On the one hand, by Cauchy's inequality and Lemma 4.1, we have

$$\begin{aligned}\Xi(\mathfrak{m}) &\le \Big(\sum_{|n|\le P^5}\Psi_{\mathfrak{m}}(n)^4\Big)^{1/2}\Big(\sum_{|n|\le P^5}\psi_\tau(n)^4\Big)^{1/2}\\ &\ll (P^{20-9\tau})^{1/2}(P^{20-\tau})^{1/2} \ll P^{20-5\tau}.\end{aligned}$$

On the other hand, the adjuvant Lemma 9.2 provided in the appendix combines with the triangle inequality to give $\Psi_{\mathfrak{M}}(an)=O(P^4)$. Thus, as a consequence of Bessel's inequality, one has

$$\Xi(\mathfrak{M}) \ll (P^4)^2\sum_{|n|\le P^5}\psi_\tau(bn)^2 \ll P^8\int_0^1 |g(\alpha)|^{16-2\tau}\,\mathrm{d}\alpha.$$

By (1.3) and Lemma 2.3, we now infer that

$$\Xi(\mathfrak{M}) \ll P^{12-2\tau}\int_0^1 |g(\alpha)|^{12}\,\mathrm{d}\alpha \ll P^{20-2\tau},$$

and thus it follows that $\Xi(\mathfrak{M})+\Xi(\mathfrak{m}) \ll P^{20-2\tau}$. The conclusion of the lemma is now immediate from (4.2). □

5. The transition to moments of smooth Weyl sums

In this section we establish Theorem 1.2. With this end in view, we put

$$\psi_\tau(m;l)=\begin{cases}\psi_\tau(m/l) & \text{when } l\mid m,\\ 0 & \text{otherwise.}\end{cases}$$

Suppose that a_i, b_i $(1\le i\le 3)$ are non-zero integers. When ν is a sufficiently small positive number, we write

$$I_\nu(\mathbf{a},\mathbf{b})=\int_{[0,1)^3}|g(a_1\alpha_1)g(a_2\alpha_2)g(a_3\alpha_3)g(b_1\alpha_1+b_2\alpha_2+b_3\alpha_3)|^{8-\nu}\,\mathrm{d}\boldsymbol{\alpha}. \tag{5.1}$$

An application of Schwarz's inequality reveals that

$$I_\nu(\mathbf{a},\mathbf{b})\le W_0^{1/2}W_1^{1/2},$$

where

$$W_0=\int_{[0,1)^3}|g(a_1\alpha_1)g(a_2\alpha_2)|^{8-2\nu}|g(a_3\alpha_3)g(b_1\alpha_1+b_2\alpha_2+b_3\alpha_3)|^{8}\,\mathrm{d}\boldsymbol{\alpha},$$
$$W_1=\int_{[0,1)^3}|g(a_3\alpha_3)g(b_1\alpha_1+b_2\alpha_2+b_3\alpha_3)|^{8-2\nu}|g(a_1\alpha_1)g(a_2\alpha_2)|^{8}\,\mathrm{d}\boldsymbol{\alpha}.$$

Observe that in the mean value W_1, the substitution

$$\alpha_1=b_2\beta_3,\qquad \alpha_2=\beta_2-b_3\beta_1-b_1\beta_3,\qquad \alpha_3=b_2\beta_1$$

implies that $b_2\beta_2=b_1\alpha_1+b_2\alpha_2+b_3\alpha_3$, and hence, by periodicity modulo 1 of the integrand, we obtain the relation

$$W_1\ll\int_{[0,1)^3}|g(a_3b_2\beta_1)g(b_2\beta_2)|^{8-2\nu}|g(a_1b_2\beta_3)g(a_2\beta_2-a_2b_3\beta_1-a_2b_1\beta_3)|^{8}\,\mathrm{d}\boldsymbol{\alpha}.$$

By invoking symmetry, we therefore discern that there are non-zero integers c_i, d_i $(1\le i\le 3)$, depending at most on $\mathbf{a}$ and $\mathbf{b}$, for which

$$I_\nu(\mathbf{a},\mathbf{b})\ll\int_{[0,1)^3}|g(c_1\alpha_1)g(c_2\alpha_2)|^{8-2\nu}|g(c_3\alpha_3)g(d_1\alpha_1+d_2\alpha_2+d_3\alpha_3)|^{8}\,\mathrm{d}\boldsymbol{\alpha}.$$

Next, since $|g(\theta)|=|g(-\theta)|$ and

$$|g(\theta)|^8=\sum_{|n|\le 4P^4}\psi_0(n)e(n\theta),$$

we find that

$$\begin{aligned}I_\nu(\mathbf{a},\mathbf{b})&\ll\sum_{|n|\le 4P^4}\psi_0(n)\int_{[0,1)^3}|g(c_1\alpha_1)g(c_2\alpha_2)|^{8-2\nu}|g(c_3\alpha_3)|^8e(-n\mathbf{d}\cdot\boldsymbol{\alpha})\,\mathrm{d}\boldsymbol{\alpha}\\&=\sum_{|n|\le 4P^4}\psi_0(n)\psi_{2\nu}(nd_1;c_1)\psi_{2\nu}(nd_2;c_2)\psi_0(nd_3;c_3).\end{aligned}$$

Thus, on employing the inequality $|z_1 z_2| \leq 2(|z_1|^2 + |z_2|^2)$, we obtain

$$I_\nu(\mathbf{a}, \mathbf{b}) \ll \sum_{|n|\leq 4P^4} \big(\psi_0(n)^2 + \psi_0(nd_3; c_3)^2\big)\big(\psi_{2\nu}(nd_1; c_1)^2 + \psi_{2\nu}(nd_2; c_2)^2\big)$$
$$\ll \sum_{|n|\leq 4P^4} \psi_0(k_1 n; l_1)^2 \psi_{2\nu}(k_2 n; l_2)^2,$$

for suitable non-zero integers k_i, l_i. Hence, we conclude from Lemma 4.2 that

$$I_\nu(\mathbf{a}, \mathbf{b}) \ll \sum_{|n|\leq 4P^4} \psi_0(k_1 n)^2 \psi_{2\nu}(k_2 n)^2 \ll P^{20-4\nu}.$$

This completes the proof of Theorem 1.2

6. Prelude to the circle method

We assume the hypotheses of Theorem 1.1, and in particular suppose that $s \geq 32$. With the column vectors $(a_{ij})_{1\leq i\leq 3} \in \mathbb{Z}^3 \setminus \{\mathbf{0}\}$, we associate the ternary forms

$$\Lambda_j = \sum_{i=1}^{3} a_{ij}\alpha_i \quad (1 \leq j \leq s),$$

and the linear forms $L_i(\boldsymbol{\gamma})$ $(1 \leq i \leq 3)$ defined for $\boldsymbol{\gamma} \in \mathbb{R}^s$ by

$$L_i(\boldsymbol{\gamma}) = \sum_{j=1}^{s} a_{ij}\gamma_j.$$

The hypotheses of Theorem 1.1 ensure that there is a non-singular real solution of the system (1.1). By invoking homogeneity, therefore, one finds that there exists a real solution $\mathbf{x} = \boldsymbol{\theta}$ in $[0, 1)^s$ for which the $3 \times s$ matrix $(4a_{l,j}\theta_j^3)$ has maximal rank. Hence, there exist distinct indices j_1, j_2 and j_3 for which the 3×3 matrix formed with the columns indexed by j_1, j_2 and j_3 is non-singular. The solution set of the system of equations (1.1) remains unchanged if one replaces any one of its equations by the equation obtained by adding to it any multiple of another equation. Thus, by appropriate elementary row operations on the matrix $(a_{l,j})$ of coefficients, there is no loss of generality in supposing that the system (1.1) takes the form

$$a_{l,j_l} x_{j_l}^4 = -\sum_{\substack{j=1\\ j\notin\{j_1,j_2,j_3\}}}^{s} a_{l,j} x_j^4 \quad (1 \leq l \leq 3) \tag{6.1}$$

with $a_{l,j_l} \neq 0$ $(1 \leq l \leq 3)$. An application of the inverse function theorem consequently confirms that whenever $\Delta > 0$ is sufficiently small, the simultaneous equations

$$a_{l,j_l} x_{j_l}^4 = -\sum_{\substack{j=1\\ j\notin\{j_1,j_2,j_3\}}}^{s} a_{l,j} (\theta_j + \Delta)^4 \quad (1 \leq l \leq 3)$$

remain soluble for x_{l,j_l} with $x_{l,j_l} > 0$. In this way we see that the system (1.1) possesses a non-singular real solution $\boldsymbol{\theta}$ satisfying $\boldsymbol{\theta} \in (0,1)^s$. Now we choose a positive number δ with the property that $\boldsymbol{\theta} \in (\delta,1)^s$, and fix this value of δ throughout the remaining sections of this paper. In addition, we fix $\eta > 0$ and $\nu > 0$ to be sufficiently small in the context of Theorem 1.2.

Next, define

$$G_0(\boldsymbol{\alpha}) = \prod_{j=1}^{32} g(\Lambda_j) \quad \text{and} \quad G(\boldsymbol{\alpha}) = \prod_{j=1}^{s} g(\Lambda_j).$$

Here and later, we write $g(\alpha) = g_\delta(\alpha; P, R)$. By orthogonality, one has

$$\mathcal{N}(P) \geq \int_{[0,1)^3} G(\boldsymbol{\alpha})\, \mathrm{d}\boldsymbol{\alpha}.$$

The Hardy–Littlewood dissection is defined as follows. We put $L = \log\log P$, take $Q = L^{40}$, and when $b_l \in \mathbb{Z}$ $(1 \leq l \leq 3)$ and $q \in \mathbb{N}$ we define

$$\mathfrak{N}(q, \mathbf{b}) = \{\boldsymbol{\alpha} \in [0,1)^3 : |\alpha_l - b_l/q| \leq QP^{-4} \ (1 \leq l \leq 3)\}.$$

We then take $\mathfrak{N}$ to be the union of the boxes $\mathfrak{N}(q,\mathbf{b})$ with $0 \leq \mathbf{b} \leq q \leq Q$ and $(q,\mathbf{b}) = 1$. Finally, we put $\mathfrak{n} = [0,1)^3 \setminus \mathfrak{N}$.

The contribution of the major arcs $\mathfrak{N}$ in this dissection satisfies

$$\int_{\mathfrak{N}} G(\boldsymbol{\alpha})\, \mathrm{d}\boldsymbol{\alpha} \gg P^{s-12}, \tag{6.2}$$

a fact we confirm in §8. Meanwhile, in §7 we show that

$$\int_{\mathfrak{n}} G(\boldsymbol{\alpha})\, \mathrm{d}\boldsymbol{\alpha} = o(P^{s-12}). \tag{6.3}$$

The desired conclusion $\mathcal{N}(P) \gg P^{s-12}$ is immediate from (6.2) and (6.3) on noting that $[0,1)^3$ is the disjoint union of $\mathfrak{N}$ and $\mathfrak{n}$.

7. The minor arc treatment

In this section we establish the minor arcs bound (6.3). We start with an inspection of the proof of [21, Lemma 8.1]. This shows that there exist positive numbers B and C with the following property. Suppose that P is a large real number, and that γ is a real number with $P^{-B} < \gamma \leq 1$. Then, whenever $|g(\alpha)| \geq \gamma P$, there exist integers a and q with

$$(a,q) = 1, \quad 1 \leq q \leq C\gamma^{-12} \quad \text{and} \quad |q\alpha - a| \leq C\gamma^{-12}P^{-4}.$$

Note that, whenever $|G(\boldsymbol{\alpha})| \geq P^s L^{-1}$, then $|g(\Lambda_j)| \geq PL^{-1}$ for $1 \leq j \leq s$. Hence, there exist integers c_j and q_j with $1 \leq q_j \leq L^{13}$, $(c_j, q_j) = 1$ and $|q_j\Lambda_j - c_j| \leq L^{13}P^{-4}$ $(1 \leq j \leq s)$. By considering the indices j_1, j_2, j_3, one finds that there exist $b_l \in \mathbb{Z}$

$(1 \le l \le 3)$ and $q \in \mathbb{N}$ with $0 \le \mathbf{b} \le q \le L^{40}$, $(q, \mathbf{b}) = 1$ and $|\alpha_l - b_l/q| \le L^{40} P^{-4}$ $(1 \le l \le 3)$. Hence $\boldsymbol{\alpha} \in \mathfrak{N}$. This shows that

$$\sup_{\boldsymbol{\alpha} \in \mathfrak{n}} |G(\boldsymbol{\alpha})| \ll P^s L^{-1}.$$

On applying a trivial estimate for excessive factors $g(\alpha)$, therefore, we obtain

$$\begin{aligned}\int_{\mathfrak{n}} G(\boldsymbol{\alpha})\,\mathrm{d}\boldsymbol{\alpha} &\ll \Big(\sup_{\boldsymbol{\alpha} \in \mathfrak{n}} |G(\boldsymbol{\alpha})|\Big)^{\nu} \int_{[0,1)^3} |G(\boldsymbol{\alpha})|^{1-\nu}\,\mathrm{d}\boldsymbol{\alpha} \\ &\ll (P^{s-32})^{1-\nu} (P^s L^{-1})^{\nu} \int_{[0,1)^3} |G_0(\boldsymbol{\alpha})|^{1-\nu}\,\mathrm{d}\boldsymbol{\alpha}.\end{aligned}$$

Further, by applying Hölder's inequality, we obtain

$$\int_{[0,1)^3} |G_0(\boldsymbol{\alpha})|^{1-\nu}\,\mathrm{d}\boldsymbol{\alpha} \le \prod_{l=0}^{7} \bigg(\int_{[0,1)^3} |g(\Lambda_{4l+1}) \dots g(\Lambda_{4l+4})|^{8-8\nu}\,\mathrm{d}\boldsymbol{\alpha} \bigg)^{1/8}.$$

On noting that $g(\alpha)$ has period 1, a change of variables confirms that for each l with $0 \le l \le 7$ there are non-zero integers a_i, b_i $(1 \le i \le 3)$ such that, in the notation introduced in (5.1), one has

$$\int_{[0,1)^3} |g(\Lambda_{4l+1}) \dots g(\Lambda_{4l+4})|^{8-8\nu}\,\mathrm{d}\boldsymbol{\alpha} \ll I_{8\nu}(\mathbf{a}, \mathbf{b}).$$

Hence, by Theorem 1.2, one concludes that

$$\int_{\mathfrak{n}} G(\boldsymbol{\alpha})\,\mathrm{d}\boldsymbol{\alpha} \ll (P^{s-32})^{1-\nu} (P^s L^{-1})^{\nu} (P^{20-32\nu}) \ll P^{s-12} L^{-\nu}.$$

This inequality is a quantitative form of (6.3).

8. The major arcs analysis

The analysis of the major arcs is largely standard. Define

$$\begin{aligned} S(q,a) &= \sum_{r=1}^{q} e(ar^4/q), \qquad T(q,\mathbf{c}) = q^{-s} \prod_{j=1}^{s} S(q, \Lambda_j(\mathbf{c})), \\ \mathfrak{A}(q) &= \sum_{\substack{1 \le \mathbf{c} \le q \\ (q,\mathbf{c})=1}} T(q,\mathbf{c}), \qquad \mathfrak{S}(X) = \sum_{1 \le q \le X} \mathfrak{A}(q). \end{aligned}$$

Also, put

$$v(\theta) = \int_{\delta P}^{P} e(\theta \gamma^4)\,\mathrm{d}\gamma \quad \text{and} \quad V(\boldsymbol{\gamma}) = \prod_{j=1}^{s} v(\Lambda_j(\boldsymbol{\gamma})).$$

Write $\mathcal{B}(X) = [-XP^{-4}, XP^{-4}]^3$ and define

$$\mathfrak{J}(X) = \int_{\mathcal{B}(X)} V(\boldsymbol{\gamma})\,\mathrm{d}\boldsymbol{\gamma}.$$

Standard arguments ([14, Lemma 5.4] and [19, Lemma 8.5]) show that there is a positive number ϱ having the property that whenever $\boldsymbol{\alpha} \in \mathfrak{N}(q, \mathbf{b}) \subseteq \mathfrak{N}$, one has

$$G(\boldsymbol{\alpha}) - \varrho T(q, \mathbf{c}) V(\boldsymbol{\alpha} - \mathbf{b}/q) \ll P^s (\log P)^{-1/2}.$$

Integrating over $\mathfrak{N}$, we infer that

$$\int_{\mathfrak{N}} G(\boldsymbol{\alpha})\,\mathrm{d}\boldsymbol{\alpha} = \varrho \mathfrak{S}(Q)\mathfrak{J}(Q) + O(P^{s-12}(\log P)^{-1/4}). \tag{8.1}$$

Lemma 8.1. *Under the hypotheses of Theorem 1.1, the limit* $\mathfrak{S} = \lim_{X\to\infty} \mathfrak{S}(X)$ *exists,* $\mathfrak{S} - \mathfrak{S}(X) \ll X^{-1/2}$*, and* $\mathfrak{S} \gg 1$.

Proof. Recall that [16, Theorem 4.2] gives $q^{-1}S(q, a) \ll q^{-1/4}(q, a)^{1/4}$. Hence, on writing $u_j = (q, \Lambda_j(\mathbf{c}))$, we obtain

$$T(q, \mathbf{c}) \ll q^{-8}(u_1 \dots u_{32})^{1/4}.$$

By applying the elementary inequality $|z_1 \dots z_n| \le |z_1|^n + \dots + |z_n|^n$ twice, one finds that

$$(u_1 \dots u_{32})^{1/4} \ll \sum_{l=0}^{7} \sum_{(\mathfrak{a},\mathfrak{b},\mathfrak{c}) \in \mathcal{S}_l} |u_{\mathfrak{a}} u_{\mathfrak{b}} u_{\mathfrak{c}}|^{8/3},$$

where $\mathcal{S}_l$ denotes the set of triples $(\mathfrak{a}, \mathfrak{b}, \mathfrak{c})$ of integers with

$$4l < \mathfrak{a} < \mathfrak{b} < \mathfrak{c} \le 4l + 4.$$

Thus

$$\mathfrak{A}(q) \ll q^{-8} \max_{0\le l\le 7} \max_{(\mathfrak{a},\mathfrak{b},\mathfrak{c}) \in \mathcal{S}_l} \sum_{\substack{1\le \mathbf{c}\le q \\ (q,\mathbf{c})=1}} |u_{\mathfrak{a}} u_{\mathfrak{b}} u_{\mathfrak{c}}|^{8/3}.$$

By symmetry, we may suppose that the maximum here occurs when $l = 0$ and $(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}) = (1, 2, 3)$. The argument following from equation (95) to the end of the proof of Lemma 23 in Davenport and Lewis [8] then shows that

$$\mathfrak{A}(q) \ll q^{-8} \sum_{\substack{u_1|q \ u_2|q \ u_3|q \\ (u_1,u_2,u_3) \ll 1}} \sum \sum (u_1u_2u_3)^{8/3} q^3/(u_1u_2u_3).$$

Since $u_1u_2u_3 \ll q^2$, an elementary estimate for the divisor function yields the bound $\mathfrak{A}(q) \ll q^{\varepsilon-5/3}$. Hence $\lim_{X\to\infty} \mathfrak{S}(X)$ exists, and $\mathfrak{S} - \mathfrak{S}(X) \ll X^{\varepsilon-2/3}$. The remaining conclusions follow as in [8, Lemma 31]. □

Lemma 8.2. *Under the hypotheses of Theorem 1.1, the limit* $\mathfrak{J} = \lim_{X\to\infty} \mathfrak{J}(X)$ *exists,* $\mathfrak{J} - \mathfrak{J}(X) \ll P^{s-12}X^{-1}$*, and* $\mathfrak{J} \gg P^{s-12}$.

Proof. Write $\widehat{\mathcal{B}}(X)$ for $\mathbb{R}^3 \setminus \mathcal{B}(X)$, and recall the prearrangement of indices implicit in (6.1). Then a direct modification of the argument of [8, Lemma 30], following an analysis similar to that of Lemma 8.1, confirms that, for a suitable positive number $\Theta = \Theta(\mathbf{c})$,

$$\int_{\widehat{\mathcal{B}}(X)} |v(\Lambda_1(\boldsymbol{\gamma}))\dots v(\Lambda_{32}(\boldsymbol{\gamma}))|\,\mathrm{d}\boldsymbol{\gamma} \ll P^{32}\int_{\widehat{\mathcal{B}}(\Theta X)}\prod_{i=1}^{3}(1+P^4|\xi_i|)^{-8/3}\,\mathrm{d}\boldsymbol{\xi}.$$

By applying trivial bounds for the additional factors $v(\Lambda_j(\boldsymbol{\gamma}))$ for $j > 32$, we therefore conclude that

$$\int_{\mathbb{R}^3\setminus\mathcal{B}(X)} V(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} \ll P^{s-32}(P^{20}X^{-1}) \ll P^{s-12}X^{-1}.$$

In particular, the limit $\mathfrak{J} = \lim_{X\to\infty}\mathfrak{J}(X)$ exists, and one has $\mathfrak{J} - \mathfrak{J}(X) \ll P^{s-12}X^{-1}$. By the argument concluding the proof of [8, Lemma 30], one finds via Fourier's integral theorem that $\mathfrak{J} \gg P^{s-12}$. □

Subject to the hypotheses of Theorem 1.1, the conclusions of Lemmata 8.1 and 8.2 combine with (8.1) to deliver the lower bound (6.2). In view of the discussion concluding §6, this establishes Theorem 1.1.

9. Appendix: an adjuvant lemma

Before announcing our adjuvant pruning lemma, for $k \geq 4$ we define the multiplicative function $w_k(q)$ by defining, for each prime number p,

$$w_k(p^{uk+v}) = \begin{cases} kp^{-u-1/2} & \text{when } u \geq 0 \text{ and } v = 1,\\ p^{-u-1} & \text{when } u \geq 0 \text{ and } 2 \leq v \leq k.\end{cases}$$

Lemma 9.1. *Suppose that $k \geq 4$. Let $\mathfrak{K}$ denote the union of the intervals*

$$\mathfrak{K}(q,a) = \{\alpha \in [0,1) : |q\alpha - a| \leq P^{1-k}\}$$

with $0 \leq a \leq q \leq P$ and $(a,q) = 1$. Let ω be a real number with $\omega > 1$, and define the function $\Upsilon_\omega(\alpha)$ for $\alpha \in \mathfrak{K}$ by taking

$$\Upsilon_\omega(\alpha) = w_k(q)^{2\omega}(1 + P^k|\alpha - a/q|)^{-\omega},$$

when $\alpha \in \mathfrak{K}(q,a) \subseteq \mathfrak{K}$. Also, let t be a real number with $t \geq \lfloor k/2\rfloor$. Then for any subset $\mathcal{A}$ of $[1,P]\cap\mathbb{Z}$, one has

$$\int_{\mathfrak{K}} \Upsilon_\omega(\alpha)\Big|\sum_{x\in\mathcal{A}} e(\alpha x^k)\Big|^{2t}\,\mathrm{d}\alpha \ll P^{2t-k}.$$

Proof. We follow the proof of [18, Lemma 5.4] as far as [18, equation (5.8)], mutatis mutandis, reaching the estimate

$$\int_{\mathfrak{K}} \Upsilon_\omega(\alpha)\Big|\sum_{x\in\mathcal{A}} e(\alpha x^k)\Big|^{2t}\,\mathrm{d}\alpha \ll P^{2t-k}\sum_{1\le q\le P} w_k(q)^{2\omega}\sigma(q), \tag{9.1}$$

where

$$\sigma(q)=\sum_{r|q} r w_k(r)^{2t},$$

given in [18, equation (5.9)]. Following the argument concluding the proof of [18, Lemma 5.4], we find that

$$\begin{aligned} w_k(p)^{2\omega}\sigma(p) &\ll_k p^{-\omega},\\ w_k(p^{uk+1})^{2\omega}\sigma(p^{uk+1}) &\ll_k p^{-u-\omega+1/k} \qquad (u\ge 1),\\ w_k(p^{uk+v})^{2\omega}\sigma(p^{uk+v}) &\ll_k p^{-u-\omega} \qquad (u\ge 0 \text{ and } 2\le v\le k)\end{aligned}$$

whence

$$\sum_{1\le q\le P} w_k(q)^{2\omega}\sigma(q)\le \prod_{p\le P}(1+Ap^{-\omega}),$$

for a suitable $A=A(k,\omega)>0$. Since $\omega>1$, the desired conclusion now follows from (9.1). □

We apply this lemma when $k=4$ to confirm the following estimate that we announce in the notation of §4. In particular, we recall the definition of the major arcs $\mathfrak{M}$ given via (3.8).

Lemma 9.2. *One has*

$$\int_{\mathfrak{M}} |f(\alpha)^2 g(\alpha)^6|\,\mathrm{d}\alpha \ll P^4.$$

Proof. By reference to [16, Theorem 4.1] and its sequel in [16], one finds that when $\alpha\in\mathfrak{M}(q,a)\subseteq\mathfrak{M}$, then

$$f(\alpha)\ll Pw_4(q)(1+P^4|\alpha-a/q|)^{-1}+O(P^{1/4+\varepsilon})\ll Pw_4(q)(1+P^4|\alpha-a/q|)^{-1}.$$

Hence, as an immediate consequence of Lemma 9.1, one obtains

$$\int_{\mathfrak{M}} |f(\alpha)^{8/3} g(\alpha)^4|\,\mathrm{d}\alpha \ll P^{8/3}\int_{\mathfrak{M}} \Upsilon_{4/3}(\alpha)|g(\alpha)|^4\,\mathrm{d}\alpha \ll P^{8/3}.$$

We recall that Lemma 2.3 shows $\Delta_{12}=0$ to be an admissible exponent. Then it follows via Hölder's inequality that

$$\begin{aligned}\int_{\mathfrak{M}} |f(\alpha)^2 g(\alpha)^6|\,\mathrm{d}\alpha &\le \left(\int_{\mathfrak{M}} |f(\alpha)^{8/3} g(\alpha)^4|\,\mathrm{d}\alpha\right)^{3/4}\left(\int_0^1 |g(\alpha)|^{12}\,\mathrm{d}\alpha\right)^{1/4}\\ &\ll (P^{8/3})^{3/4}(P^8)^{1/4}\ll P^4.\end{aligned}$$

This completes the proof of the lemma. □

Acknowledgments. The authors are grateful to the referee for useful comments.

The authors acknowledge support by Akademie der Wissenschaften zu Göttingen and Deutsche Forschungsgemeinschaft.

References

[1] Atkinson, O. D., Brüdern, J., Cook, R. J.: Simultaneous additive congruences to a large prime modulus. Mathematika **39**, 1–9 (1992) Zbl 0774.11016 MR 1176464

[2] Brüdern, J.: A problem in additive number theory. Math. Proc. Cambridge Philos. Soc. **103**, 27–33 (1988) Zbl 0655.10041 MR 0913447

[3] Brüdern, J., Cook, R. J.: On simultaneous diagonal equations and inequalities. Acta Arith. **62**, 125–149 (1992) Zbl 0774.11015 MR 1183985

[4] Brüdern, J., Wooley, T. D.: On Waring's problem: two cubes and seven biquadrates. Tsukuba J. Math. **24**, 387–417 (2000) Zbl 1003.11045 MR 1818095

[5] Brüdern, J., Wooley, T. D.: The Hasse principle for pairs of diagonal cubic forms. Ann. of Math. (2) **166**, 865–895 (2007) Zbl 1171.11053 MR 2373375

[6] Brüdern, J., Wooley, T. D.: Cubic moments of Fourier coefficients and pairs of diagonal quartic forms. J. Eur. Math. Soc. **17**, 2887–2901 (2015) Zbl 06514543 MR 3420525

[7] Brüdern, J., Wooley, T. D.: Pairs of diagonal quartic forms: the non-singular Hasse principle. In preparation

[8] Davenport, H., Lewis, D. J.: Simultaneous equations of additive type. Philos. Trans. Roy. Soc. London Ser. A **264**, 557–595 (1969) Zbl 0207.35304 MR 0245542

[9] Ford, K. B.: The representation of numbers as sums of unlike powers. II. J. Amer. Math. Soc. **9**, 919–940 (1996) Zbl 0866.11054 MR 1325794

[10] Ford, K. B.: Addendum and corrigendum to "The representation of numbers as sums of unlike powers. II". J. Amer. Math. Soc. **12**, 1213 (1999) Zbl 0866.11054 MR 1699354

[11] Israilov, M. I., Allakov, I. A.: On the sum of kth powers of natural numbers. Trudy Mat. Inst. Steklova **207**, 172–179 (1994) (in Russian) Zbl 0895.11038 MR 1401811

[12] Kawada, K., Wooley, T. D.: Davenport's method and slim exceptional sets: the asymptotic formulae in Waring's problem. Mathematika **56**, 305–321 (2010) Zbl 1238.11091 MR 2678031

[13] Vaughan, R. C.: On Waring's problem for smaller exponents. II. Mathematika **33**, 6–22 (1986) Zbl 0601.10037 MR 0859494

[14] Vaughan, R. C.: A new iterative method in Waring's problem. Acta Math. **162**, 1–71 (1989) Zbl 0665.10033 MR 0981199

[15] Vaughan, R. C.: A new iterative method in Waring's problem II. J. London Math. Soc. (2) **39**, 219–230 (1989) Zbl 0677.10035 MR 0991657

[16] Vaughan, R. C.: The Hardy–Littlewood Method. 2nd ed., Cambridge Univ. Press, Cambridge (1997) Zbl 0868.11046 MR 1435742

[17] Vaughan, R. C., Wooley, T. D.: Further improvements in Waring's problem. Acta Math. **174**, 147–240 (1995) Zbl 0849.11075 MR 1351319

[18] Vaughan, R. C., Wooley, T. D.: Further improvements in Waring's problem, IV: Higher powers. Acta Arith. **94**, 203–285 (2000) Zbl 0972.11092 MR 1776896

[19] Wooley, T. D.: On simultaneous additive equations, II. J. Reine Angew. Math. **419**, 141–198 (1991) Zbl 0721.11011 MR 1116923

[20] Wooley, T. D.: Large improvements in Waring's problem. Ann. of Math. (2) **135**, 131–164 (1992) Zbl 0754.11026 MR 1147960

[21] Wooley, T. D.: On Diophantine inequalities: Freeman's asymptotic formulae. In: Proceedings of the Session in Analytic Number Theory and Diophantine Equations (Bonn, 2002), D. R. Heath-Brown and B. Z. Moroz (eds.), Bonner Math. Schriften 360, art. 30, 32 pp. (2003) Zbl 1196.11055 MR 2075639