J. Eur. Math. Soc. 20, 2589–2606

DOI 10.4171/JEMS/819

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Vanishing theorems for mixed Hodge modules and applications

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Received August 24, 2015

Abstract. We prove two Kawamata–Viehweg type vanishing theorems with coefficients, one for polarisable variations of Hodge structures (PVHSs) and the other for general mixed Hodge modules. They generalise previous vanishing theorems of Kawamata, Viehweg, Esnault and Viehweg, Illusie, Saito, and Popa.

The two theorems are complementary. One applies to all mixed Hodge modules, but only covers (of necessity) the lowest Hodge graded piece. The other applies only to Deligne's canonical extension of PVHSs across simple normal crossings divisors, but covers all the Hodge graded pieces.

As application, we obtain new results on the cohomology of Shimura varieties and an alternative proof of a known result in birational geometry: (i) Using the one for PVHSs, we obtain new vanishing theorems for the cohomology of arbitrary Shimura varieties (including the ones attached to exceptional groups) with automorphic coefficients, extending works of Faltings, Li and Schwermer, and our own with Lan. (ii) The vanishing theorem for mixed Hodge modules contains, as a special case, Kollár's vanishing theorem for the higher direct images of dualising sheaves and big and nef line bundles.

Keywords. Mixed Hodge modules, vanishing theorems, Shimura varieties

1. Introduction

First we briefly review the vanishing theorems of Kodaira, Nakano, Kawamata, Viehweg, and Esnault, which we generalise. We then state the two main vanishing theorems and two applications.

Let X be a complex projective smooth variety of dimension n. Then for any ample line bundle L on X, the Kodaira vanishing theorem shows

$$H^i(X, L^{-1}) = 0 \quad \text{for all } i < n. \tag{1.1}$$

It was later generalised by Nakano, with differential forms of higher order allowed:

$$H^i(X, L^{-1} \otimes \Omega_X^j) = 0 \quad \text{whenever } i + j < n. \tag{1.2}$$

These vanishing theorems and their generalisations have played an essential role in birational geometry (see e.g. [4]). Recently, one has also seen new applications of vanishing

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Mathematics Subject Classification (2010): Primary 14F17; Secondary 14G35

theorems to the theory of automorphic forms; see e.g. the work of Stroh [23] and also our own with Lan [14], [15].

In these birational and arithmetic applications, one often needs to allow line bundles that are *big and nef*, but not necessarily ample: examples include the pullback of an ample line bundle by any generically finite morphism (such as blowing up), and the canonical bundle on smooth projective toroidal compactifications of Siegel modular varieties parametrising polarised abelian varieties with level structure.

For big and nef line bundles L, the Kawamata–Viehweg vanishing theorem [9], [24] shows that (1.1) still holds, while the stronger statement (1.2) does not any longer. However, a *logarithmic* modification of (1.2) persists. Namely, suppose there is an effective divisor D' supported on a simple normal crossings divisor D such that

$$L^{\nu}(-D') \quad \text{is ample for all } \nu \gg 0. \tag{1.3}$$

This condition implies that L is big and nef, and conversely, one may always arrive at such a situation, by Hironaka's embedded resolution of singularities.[1] Moreover, the condition is often satisfied in 'natural' examples, such as the aforementioned toroidal compactifications, with D the toroidal boundary.

In this case, Esnault and Viehweg proved (see [4, Prop. 11.5] or [3])

$$H^i(X, L^{-1} \otimes \Omega_X^j(\log D)) = 0 \quad \text{whenever } i + j < n. \tag{1.4}$$

This is interesting for two reasons. First, Esnault and Viehweg *deduce* the Kawamata–Viehweg vanishing (1.1) from (1.4). Secondly, in arithmetic geometry, the logarithmic bundles and their Grothendieck–Serre duals often have intrinsic interest, e.g. in terms of cusp forms on the Siegel modular varieties.

The first goal of this article is to generalise these theorems to include *nontrivial coefficients*. These coefficients include: (a) the relative cohomology of a projective smooth morphism, which gives a polarisable variation of Hodge structures in the sense of Griffiths; this subclass contains automorphic bundles on the Siegel modular varieties; and (b) the (total) direct image of the dualising sheaf under a projective morphism, as considered by Kollár [10], [11].

Here is the first main theorem (Theorem 2.1) of this article, covering the cases (a):

Theorem 1. *Let L be a big and nef line bundle satisfying* (1.3) *and let* $(\widetilde{M}, F)$ *be the canonical extension* (*of Deligne, see* §2) *of a polarisable variation of Hodge structures on $X - D$. Then*

$$H^q\big(X, L^{-1} \otimes_{\mathscr{O}_X} \mathrm{Gr}_p^F(\mathrm{DR}_{(X,D)}(\widetilde{M}, F))\big) = 0 \quad \textit{for all } q < 0 \textit{ and } p \in \mathbb{Z},$$

where the logarithmic de Rham complex is concentrated in degrees $[-\dim X, 0]$.

[1] In fact, L is big and nef if and only if there exists an effective divisor D' such that $L^{\nu}(-D')$ is ample for all $\nu \gg 0$. One uses Hironaka's theorem to secure (after a succession of embedded resolutions) a divisor D' that has simple normal crossings; cf. the proof of Theorem 3.2.

Applied to the constant variation $\mathbb{Q}_{X-D}$, this result gives (1.4). More generally, when applied to the relative cohomology of a semistable morphism $(Y, E) \to (X, D)$, it recovers our previous vanishing theorem [15, Cor. 3.2.8], which was in turn a generalisation of Illusie's [7, Cor. 4.17].

Now we turn to Saito's theory of mixed Hodge modules [21], the most comprehensive framework to date for coefficient systems. Even for the proof of Theorem 1, whose statement does not involve mixed Hodge modules, we need to rely on Saito's theory—see the discussion at the end of the introduction.

For mixed Hodge modules and *ample* line bundles, Saito has proved a definitive form of Kodaira–Nakano type vanishing theorem. For any mixed Hodge module M and any ample line bundle L on any complex projective scheme X, he proved [21, Th. 2.33]

$$H^q(X, L^{-1} \otimes \mathrm{Gr}_p^F(\mathrm{DR}(M))) = 0 \quad \text{for all } q < 0 \text{ and } p \in \mathbb{Z}, \tag{1.5}$$

and[2] its dual form:

$$H^q(X, L \otimes \mathrm{Gr}_p^F(\mathrm{DR}(M))) = 0 \quad \text{for all } q > 0 \text{ and } p \in \mathbb{Z}. \tag{1.6}$$

In case X is smooth and $M = \mathbb{Q}_X[n]$ is trivial, the lowest graded piece Gr_p^F for the Hodge filtration F gives the Kodaira vanishing, while all the graded pieces together give the Nakano vanishing. We remark that Schnell [22] has recently obtained an alternative proof of Saito's theorem.

In generalising (1.5) and (1.6) for big and nef line bundles, one finds that the rift between Kodaira's and Nakano's vanishing theorems reemerges, which leads to two different directions:

(i) In the direction of Kodaira's vanishing (1.1), one looks for a vanishing theorem that applies to all mixed Hodge modules, but covers only the *lowest* Hodge graded piece (inevitably). Such a statement was proved by Popa [18, Th. 11.1], under a condition of "transversality" on M and L.

In this article, we *remove this condition completely*, and thereby obtain a general vanishing theorem of Kawamata–Viehweg type (Theorem 3.2). The following is a simpler but essential case of Theorem 3.2, which requires less notation to state (see §3 for more details):

Theorem 2. *Let X be an irreducible projective scheme over $\mathbb{C}$, L a big and nef line bundle on X, and M the intermediate extension to X of a PVHS on a smooth dense open subset of X. Then for the lowest nonzero Hodge graded piece $S_X(M)$ of* DR(M)*, we have*

$$H^q(X, L \otimes S_X(M)) = 0 \quad \textit{for all } q > 0.$$

[2] In case X is singular, the de Rham complex $\mathrm{DR}(M)$ is defined using the mixed Hodge module version of Kashiwara's equivalence: for any embedding j of X into a smooth variety Y, M gives rise to a mixed Hodge module M_Y on Y with support in X (unique up to isomorphism); then $\mathrm{DR}(M) = \mathrm{DR}_Y(M_Y)$. This complex, supported on X, is independent of the choice of j up to quasiisomorphism.

Recently Lei Wu [25] announced that he can also remove the tranversality condition, using different methods.

(ii) In the direction of Nakano's vanishing (1.2) modified into Esnault and Viehweg's vanishing (1.4), we prove, in the same geometric situation, a theorem that covers *all* the Hodge graded pieces at once, for the canonical extension of *any* polarisable variation of Hodge structures: this is Theorem 1, mentioned above.

Applications. (A) From Theorem 1, we obtain an extension of our results [14], [15] to arbitrary Shimura varieties. For one that is easier to state, we get an algebro-geometric proof of the following, which was first obtained by Faltings in the compact case [5], and by Li and Schwermer in general [16], [17]:

Theorem 3. *Let* $(G, \mathscr{X})$ *be an arbitrary Shimura datum, and let* $X = \mathrm{Sh}_K(G, \mathscr{X})$ *be the smooth Shimura variety attached to a neat level subgroup* K*. Consider the local system* V_μ *attached to a sufficiently regular weight* μ *corresponding to an algebraic representation of* G*. Then*

$$H^i(X, V_\mu) = 0 \quad \text{for all } i < \dim X \quad \text{and} \quad H^i_c(X, V_\mu) = 0 \quad \text{for all } i > \dim X.$$

In particular, if X *is compact, then the cohomology is concentrated in the middle degree.*

Our results are sharper than this, and apply to all weights and to coherent (and not just Betti or de Rham) cohomology. This way we obtain new results, not previously covered by Faltings nor by Li and Schwermer: see Theorem 4.1 below, which requires more notation to state, recalled in §4.

(B) Theorem 2 contains the following theorem of Kollár as a special case:

Theorem 4 (Kollár). *Let* $f : X \to Y$ *be a surjective morphism from an irreducible projective smooth complex variety* X *to a projective complex variety* Y*, and let* L *be a big and nef line bundle on* Y*. Then*

$$H^j(Y, L \otimes R^i f_* \omega_X) = 0 \quad \text{for all } j > 0 \text{ and } i \geq 0,$$

where $\omega_X = \Omega^{\dim X}_{X/\mathbb{C}}$ *is the dualising sheaf.*

Use of mixed Hodge modules. In order to obtain these applications, which only concern coherent sheaves and local systems, we still rely on Saito's theory for proof, in two ways:

(1) Let (M, F) be a polarisable variation of pure Hodge structures on the complement U of a simple normal crossings divisor (SNCD) D on a projective smooth variety X, and let $(\widetilde{M}, F)$ be the canonical extension (of Deligne, see §2). Then the Hodge-to-de-Rham spectral sequence

$$E_1^{p,q} = H^{p+q}(X, \mathrm{Gr}^p_F(\mathrm{DR}_{(X,D)}(\widetilde{M}, F))) \Rightarrow H^{p+q}(X, \mathrm{DR}_{(X,D)}(\widetilde{M}))$$

for the logarithmic de Rham complex of $(\widetilde{M}, F)$, degenerates at E_1. The only known proof of this fact for general variations (even for general Gauss–Manin systems) is obtained by using Saito's direct image theorem.[3]

Our proof of Theorem 1 depends on this fact and a slight generalisation ("twist") of it.

(2) Kollár made conjectures [11] on the existence of certain coherent sheaves associated with any generic polarisable variation of Hodge structures, satisfying properties of vanishing, torsionfreeness, and decomposition. These were later confirmed by Saito [19]: the lowest Hodge graded piece of the Hodge module that is obtained as the intermediate extension of the PVHS satisfies these properties. Our generalisation of Kollár's vanishing theorem depends on this.

Plan. Sections 2 and 3 are devoted to the proof of the vanishing theorems. In §3, we also obtain the generalisation of Kollár's vanishing theorem to big and nef line bundles. Applications to automorphic forms and Shimura varieties are given in §4.

Notation and conventions. We often denote a mixed Hodge module simply by the underlying perverse sheaf with $\mathbb{Q}$-structure or by the underlying filtered $\mathscr{D}$-module.

Left $\mathscr{D}$-modules are used. Saito [19] and Popa [18] employ *right* $\mathscr{D}$-modules, which necessitates changes in the indexing of the Hodge filtration and in the de Rham complex. We also use (as does Saito) Griffiths' notation of decreasing Hodge filtration for variations of Hodge structures; as usual, superscript $F^\bullet$ denotes a decreasing filtration, while $F_\bullet$ an increasing one.

2. Vanishing theorems for canonical extensions of PVHSs

Let X be a projective smooth complex variety of pure dimension n, $D = \bigcup_j D_j$ a simple normal crossings divisor with complement $U = X - D$, and (M, F) the filtered $\mathscr{D}_U$-module underlying a polarisable variation of pure Hodge structures on U. Denote by $(\widetilde{M}, F)$ *the canonical extension* of Deligne, that is, the locally free sheaf of $\mathscr{O}_X$-modules with an integrable connection with logarithmic singularities along D, whose residue maps have eigenvalues in $[0, 1)$.

Then $\widetilde{M}(*D)$, the $\mathscr{D}$-module of sections meromorphic along D, underlies the mixed Hodge module j_*M, and we have a filtered quasiisomorphism

$$\mathrm{DR}_{(X,D)}(\widetilde{M}, F) \simeq \mathrm{DR}(\widetilde{M}(*D), F), \tag{2.1}$$

where the left hand side denotes the complex

$$\widetilde{M} \to \Omega^1_X(\log D) \otimes \widetilde{M} \to \cdots \to \Omega^n_X(\log D) \otimes \widetilde{M}$$

[3] For special $(\widetilde{M}, F)$, this can be proved without using mixed Hodge modules. When D is empty, it follows from a theorem of Deligne: see Zucker's [26, Cor. 2.11]. (See also Appendix D by Timmerscheidt to [4] by Esnault and Viehweg.) Also, if $(\widetilde{M}, F)$ is the Gauss–Manin system of a *semistable* morphism (which forces the local monodromy to be *unipotent*, by a theorem of Katz [8]), then the statement follows from a theorem of Illusie [7].

concentrated in degrees $[-n, 0]$ and filtered by

$$F^\bullet \widetilde{M} \to \Omega^1_X(\log D) \otimes F^{\bullet-1}\widetilde{M} \to \cdots \to \Omega^n_X(\log D) \otimes F^{\bullet-n}\widetilde{M}.$$

See [21, §3.b] for more details.

Now let L be an invertible sheaf on X, $D' = \sum e_j D_j$ an effective divisor whose support is contained in D, and $\nu_0 \geq 0$ an integer such that

$$L^{\otimes\nu}(-D') \text{ is ample for all } \nu \geq \nu_0; \tag{2.2}$$

in particular, L is big and nef.[4]

Theorem 2.1. *With the notation and assumptions as above, we have*

$$H^q\big(X, L^{-1} \otimes_{\mathscr{O}_X} \mathrm{Gr}^F_*(\mathrm{DR}_{(X,D)}(\widetilde{M}, F))\big) = 0 \quad \textit{for all } q < 0. \tag{2.3}$$

Remark 2.2. This generalises a previous vanishing theorem of ours [15, Cor. 3.2.8] in two ways: the polarisable variation of Hodge structures M is no longer required to be a Gauss–Manin system (i.e., the relative cohomology of a projective smooth family), nor is it required to have unipotent local monodromy.

We present two slightly different proofs: The main one, given immediately below, follows the method of Esnault and Viehweg closely and uses as additional ingredients Saito's theory of mixed Hodge modules (especially his direct image theorem) and a simple observation on the Hodge graded pieces of the de Rham complex of tensor products (which was also exploited in [15]). The other proof follows a more 'topological' method (using perverse sheaves instead of coherent sheaves), and is closer to Ramanujam's method. For the latter, see Remark 2.5.

Proof of Theorem 2.1. Replacing D' and ν_0 if necessary (but without changing D), we may assume furthermore that

$$L^{\otimes\nu}(-D') \text{ is very ample for all } \nu \geq \nu_0$$

(see [4, condition (**) on p.130] and also [15, Lemma 3.20]).

Let Z be a sufficiently general section of the very ample line bundle $L^{\otimes N}(-D')$ for an integer $N \geq \nu_0$, so that $D \cup Z$ and $D \cap Z$ are simple normal crossings divisors on X and Z, respectively (see [15, §2.1] for details of this Bertini type argument). Then using the Poincaré residue map, we obtain a short exact sequence of filtered logarithmic de Rham complexes:

$$0 \to \mathrm{DR}_{(X,D)}(\widetilde{M}, F) \to \mathrm{DR}_{(X,D+Z)}(\widetilde{M}, F) \to \mathrm{DR}_{(Z,D\cap Z)}(\widetilde{M}|_Z, F|_Z[-1]) \to 0,$$

[4] We remind the reader that the condition is slightly stronger than L being big and nef, in that for the latter one allows D' with arbitrary singularities. As is known, and will be reviewed in the proof of Theorem 3.2, one can arrive at this geometrically favourable situation by use of Hironaka's embedded resolution.

hence an exact sequence of the associated graded

$$0 \to \mathrm{Gr}^*(\mathrm{DR}_{(X,D)}(\widetilde{M}, F)) \to \mathrm{Gr}^*(\mathrm{DR}_{(X,D+Z)}(\widetilde{M}, F)) \\ \to \mathrm{Gr}^*(\mathrm{DR}_{(Z,D\cap Z)}(\widetilde{M}|_Z, F|_Z[-1])) \to 0,$$

therefore an exact sequence

$$H^{q-1}\big(Z, L^{-1}|_Z \otimes \mathrm{Gr}^*(\mathrm{DR}_{(Z,D\cap Z)}(\widetilde{M}|_Z, F|_Z[-1]))\big) \\ \to H^q\big(X, L^{-1} \otimes \mathrm{Gr}^*(\mathrm{DR}_{(X,D)}(\widetilde{M}, F))\big) \\ \to H^q\big(X, L^{-1} \otimes \mathrm{Gr}^*(\mathrm{DR}_{(X,D+Z))}(\widetilde{M}, F))\big).$$

By induction on $n = \dim X$, we may assume that the first group vanishes; note that $L|_Z$ satisfies the condition (2.2) for the divisor $D'|_Z$, which is supported on the simple normal crossings divisor $D \cap Z$.

Therefore, one is reduced to proving

$$H^q\big(X, L^{-1} \otimes_{\mathscr{O}_X} \mathrm{Gr}^*(\mathrm{DR}_{(X,D+Z)}(\widetilde{M}, F))\big) = 0 \quad \text{for all } q < 0. \tag{2.4}$$

Now recall the construction by Esnault and Viehweg [4] of the logarithmic connections $\nabla^{(i)}$ on the line bundles $\mathscr{L}^{(i)^{-1}}$. As a line bundle, we have

$$\mathscr{L}^{(i)^{-1}} = L^{\otimes -i}\left(\left\lfloor \frac{i(D'+Z)}{N} \right\rfloor\right),$$

and the residue of $\nabla^{(i)}$ along any branch D_j of D is the fractional part of $ie_j/N \in \mathbb{Q}$:

$$\left\{\frac{ie_j}{N}\right\} = \frac{ie_j}{N} - \left\lfloor \frac{ie_j}{N} \right\rfloor \in [0, 1), \tag{2.5}$$

while the residue along Z is equal to $\{i/N\} \in [0, 1)$ (see [4, Th. 3.2]).

Recall also that the direct sum of these bundles has the description [4, Cor. 3.16]

$$\bigoplus_{i=0}^{N-1} \mathscr{L}^{(i)^{-1}} \simeq \pi_* \mathscr{O}_Y \tag{2.6}$$

where $\pi : Y \to X$ is the cyclic cover[5] that is "obtained by taking the Nth root out of $D' + Z$".[6] In other words, each $(\mathscr{L}^{(i)^{-1}}, \nabla^{(i)})$ is the canonical extension[7] of its restriction

[5] If Z meets D nontrivially, that is, if $n \geq 2$ and $D \neq \emptyset$, then Y is singular. But it only has rational singularities, and a resolution does not affect the direct image of the structure sheaf.

[6] This is the terminology used by Esnault and Viehweg [4, p. 22]. As remarked by a referee, a slightly more precise expression would be "obtained by using *the* Nth root L of $\mathscr{O}_X(D' + Z)$".

[7] Here the individual $\mathscr{L}^{(i)^{-1}}(*(D + Z))$ cannot in general be considered as a mixed Hodge module, for want of a $\mathbb{Q}$-structure on the perverse sheaf; the best one can get is a $\mathbb{Q}(\mu_N)$-structure. However, Deligne's canonical extension of an integrable connection is defined without reference to any $\mathbb{Q}$-structure, if present.

to $X-(D\cup Z)$, and their sum is the canonical extension of the $\mathscr{D}$-module underlying the polarisable variation of Hodge structures

$$\pi_*\mathbb{Q}_{Y-\pi^{-1}(D\cup Z)}[n]$$

of pure weight 0 and with the trivial Hodge filtration on $X-(D\cup Z)$; note that, over this last open subset, π is finite, étale and Galois with group $\mathbb{Z}/N$.

It also follows that the partial sum

$$C := \bigoplus_{i=1}^{N-1} \mathscr{L}^{(i)^{-1}} = \operatorname{Coker}(\mathscr{O}_X \to \pi_*\mathscr{O}_Y)$$

underlies over the open $X-(D\cup Z)$ a direct summand of $\pi_*\mathbb{Q}_{Y-\pi^{-1}(D\cup Z)}[n]$ complementary to the constant variation $\mathbb{Q}_{X-(D\cup Z)}[n]$.

We now assume an additional condition on N:

(A) Let $\epsilon \in (0,1]$ be largest such that all the eigenvalues of the residue maps of $\widetilde{M}$ along all the components of $D\cup Z$ lie in the interval $[0, 1-\epsilon]$. Further, $N \geq \max(\nu_0, 2)$ and N is so large that the numbers e_j/N all fall in $[0,\epsilon)$.

The condition implies that all the eigenvalues of the residues in (2.5) of $\mathscr{L}^{(1)^{-1}}$ lie in $[0,\epsilon)$ and that the residue along Z, which is $1/N$, belongs to the interval $(0,1)$. It also implies that

$$\mathscr{L}^{(1)^{-1}} = L^{-1}.$$

Consider the quasicoherent module with integrable connection given by the tensor product

$$T := \widetilde{M}(*(D+Z)) \otimes_{\mathscr{O}_X} C(*(D+Z)),$$

which underlies the mixed Hodge module

$$j'_*(M|_{X-(D\cup Z)} \otimes C|_{X-(D\cup Z)})$$

where $j' : X-(D\cup Z) \to X$ denotes the inclusion, and the tensor product in the parentheses is that of PVHSs on $X-(D\cup Z)$. (We note that the restrictions to $X-(D\cup Z)$ of $\mathscr{L}^{(i)^{-1}}$ and L^{-i} are naturally isomorphic. It is a bit easier to construct an integrable connection with logarithmic singularities on the latter; the corresponding finite covering, not normal in general, is denoted by Y' in the construction [4, §3.5].)

Now we apply Saito's direct image theorem [21, 2.14] to the structure morphism of X to Spec $\mathbb{C}$ and the mixed Hodge module T. Concretely, it means that the spectral sequence

$$E_1^{p,q} = H^{p+q}(X, \mathrm{Gr}_{-p}(\mathrm{DR}(T,F))) \Rightarrow H^{p+q}(X, \mathrm{DR}(T)) \tag{2.7}$$

degenerates at E_1. We then decompose T into $N-1$ filtered $\mathscr{D}$-modules, and look at the direct summand of the spectral sequence corresponding to $i=1$. By the condition (A) above, the tensor product connection with logarithmic singularities

$$\widetilde{M} \otimes_{\mathscr{O}_X} \mathscr{L}^{(1)^{-1}}$$

is still the canonical extension of its restriction to $X - (D \cup Z)$: the point is that the eigenvalues of the residues remain in the range $[0, 1)$.

Therefore by using the filtered quasiisomorphism (2.1),[8] we see that the spectral sequence

$$\begin{aligned} E_1^{p,q} &= H^{p+q}\big(X, \mathrm{Gr}_{-p}(\mathrm{DR}_{(X,D+Z)}(\widetilde{M} \otimes \mathscr{L}^{(1)^{-1}}, F))\big) \\ &\Rightarrow H^{p+q}\big(X, \mathrm{DR}_{(X,D+Z)}(\widetilde{M} \otimes \mathscr{L}^{(1)^{-1}})\big) \end{aligned} \tag{2.8}$$

also degenerates at E_1.

Remark 2.3. As noted by a referee, one may use the theory of mixed Hodge modules with $\mathbb{R}$-coefficients (rather than $\mathbb{Q}$-coefficients). Then one only needs to use the summand T' of T, corresponding to the factors with indices $i = 1$ and $i = N - 1$. The perverse sheaf underlying T' is defined over the maximal totally real subfield of $\mathbb{Q}(\mu_N)$.

Since the Hodge filtration on $\mathscr{L}^{(1)^{-1}}$ is trivial (i.e., its associated graded is concentrated in degree zero), we have

$$\mathrm{Gr}_{-p}(\mathrm{DR}_{(X,D+Z)}(\widetilde{M} \otimes \mathscr{L}^{(1)^{-1}}, F)) \simeq \mathscr{L}^{(1)^{-1}} \otimes_{\mathscr{O}_X} \mathrm{Gr}_{-p}(\mathrm{DR}_{(X,D+Z)}(\widetilde{M}, F)). \tag{2.9}$$

For any integer $a > 0$, the line bundle $\mathscr{O}_X(-aZ)$ (that is, the ideal sheaf of aZ) has an integrable connection with logarithmic singularities along Z, whose residue along Z is equal to a (see [4, Lemma 2.7]). On the other hand, $\widetilde{M}$ has zero residue along Z, while $\mathscr{L}^{(1)^{-1}}$ has residue $1/N \in (0, 1)$ along Z.

Therefore, by a repeated application of [4, Lemma 2.10], we get a quasiisomorphism

$$\mathrm{DR}_{(X,D+Z)}(\widetilde{M} \otimes \mathscr{L}^{(1)^{-1}}) \simeq \mathrm{DR}_{(X,D+Z)}(\widetilde{M} \otimes \mathscr{L}^{(1)^{-1}}(-aZ)), \tag{2.10}$$

without filtration. By putting the trivial filtration F on $\mathscr{O}(-aZ)$ concentrated in degree 0 (but we do *not* consider it as underlying a mixed Hodge module), we get the spectral sequence associated with any filtered complex

$$\begin{aligned} E_1^{p,q} &= H^{p+q}\big(X, \mathrm{Gr}_{-p}(\mathrm{DR}_{(X,D+Z)}(\widetilde{M} \otimes \mathscr{L}^{(1)^{-1}}(-aZ), F))\big) \\ &\Rightarrow H^{p+q}\big(X, \mathrm{DR}_{(X,D+Z)}(\widetilde{M} \otimes \mathscr{L}^{(1)^{-1}}(-aZ))\big) \end{aligned} \tag{2.11}$$

and the identification similar to (2.9):

$$\begin{aligned} &\mathrm{Gr}_{-p}\big(\mathrm{DR}_{(X,D+Z)}(\widetilde{M} \otimes \mathscr{L}^{(1)^{-1}}(-aZ), F)\big) \\ &\qquad \simeq \mathscr{L}^{(1)^{-1}}(-aZ) \otimes_{\mathscr{O}_X} \mathrm{Gr}_{-p}(\mathrm{DR}_{(X,D+Z)}(\widetilde{M}, F)). \end{aligned} \tag{2.12}$$

[8] We do *not* in general have a filtered quasiisomorphism between the summand of $\mathrm{DR}(T)$ corresponding to the index $i > 1$ and the logarithmic de Rham complex for $\widetilde{M} \otimes \mathscr{L}^{(i)^{-1}}$, unless M has unipotent local monodromy everywhere.

By combining the degeneration of the spectral sequence (2.8), the (perhaps nondegenerate) spectral sequence (2.11), the identifications (2.9) and (2.12), and the quasiisomorphism (2.10), we get the inequality

$$\begin{aligned}\dim_{\mathbb{C}} H^q\big(X, \mathscr{L}^{(1)^{-1}} \otimes \operatorname{Gr}_*(\mathrm{DR}_{(X,D+Z)}(\widetilde{M}, F))\big) \\ &= \dim_{\mathbb{C}} H^q\big(X, \mathrm{DR}_{(X,D+Z)}(\widetilde{M} \otimes \mathscr{L}^{(1)^{-1}})\big) \\ &= \dim_{\mathbb{C}} H^q\big(X, \mathrm{DR}_{(X,D+Z)}(\widetilde{M} \otimes \mathscr{L}^{(1)^{-1}}(-aZ))\big) \\ &\leq \dim_{\mathbb{C}} H^q\big(X, \mathscr{L}^{(1)^{-1}}(-aZ) \otimes \operatorname{Gr}_*(\mathrm{DR}_{(X,D+Z)}(\widetilde{M}, F))\big).\end{aligned}$$

If $q < 0$ and $a \gg 0$, Serre's vanishing theorem applied to the ample divisor Z shows that the last cohomology group vanishes. (Note that the graded pieces of the logarithmic de Rham complex are complexes of locally free sheaves concentrated in degrees $[-n, 0]$.) It only remains to note that $\mathscr{L}^{(1)^{-1}} \simeq L^{-1}$ from the condition (A), and we get (2.4).

This completes the proof of Theorem 2.1. □

Remark 2.4. The result extends to the canonical extensions of (graded-)polarisable variations of Hodge structures M on $X - D$.

Remark 2.5. In the last step of the proof, one may alternatively apply Artin's vanishing theorem for perverse sheaves on affine varieties, instead of the Serre vanishing theorem. Here is a sketch of the argument, suggested by an anonymous referee.

Let $V = X - Z$ and factor the inclusion j' of $U = X - (Z \cup D)$ into X in two steps:

$$U \overset{j}{\hookrightarrow} V \overset{k}{\hookrightarrow} X.$$

Then for the filtered module with integrable connection $M' := (L^{-1} \otimes M)|_U$ on U, we have natural isomorphisms of filtered regular holonomic $\mathscr{D}$-modules:

$$k_! j_* M' = k_* j_* M' = j'_* M'.$$

Here we use the fact, implied by the condition (A), that the local monodromy around Z does not have eigenvalue 1.

By definition, we then have

$$\begin{aligned} H^q(X, \mathrm{DR}(j'_* M')) &= H^q(V, \mathrm{DR}(j_*(M'))), \\ H^q(X, \mathrm{DR}(k_! j_* M')) &= H^q_c(V, \mathrm{DR}(j_*(M'))) \end{aligned}$$

for all q. Since V is an affine variety and $j_* M'$ is a perverse sheaf (note that j is an affine open immersion), Artin's vanishing theorem [1, Th. 4.1.1 et Cor. 4.1.2] gives

$$H^q(X, \mathrm{DR}(j'_* M')) = 0 \quad \text{for all } q \neq 0.$$

On the other hand, we have also seen that the spectral sequence associated with the filtred complex $\mathrm{DR}(j'_* M')$ degenerates at E_1. Hence

$$H^q(X, \operatorname{Gr}^*(\mathrm{DR}(j'_* M'))) = 0 \quad \text{for all } q \neq 0,$$

which implies (2.4) for all $q \neq 0$. (But of course, one cannot replace $q < 0$ with $q \neq 0$ in the statement of Theorem 2.1.)

The presence of these two different but parallel methods, one relying on Artin's vanishing theorem on the constructible side and the other on Serre's vanishing theorem on the coherent side (via $\mathscr{D}$-modules), is perhaps not surprising. As remarked by Schnell in the introduction to [22], it traces back (at least) to the two different approaches taken by Ramanujam and by Esnault and Viehweg. We add that Saito's proof of Kodaira type vanishing theorems in [21] makes use of these ideas.

Now we consider the dual form. We have a canonical integrable connection $\widetilde{M}^\vee(-D)$ (see [4, §2]), whose residue maps have eigenvalues in (0, 1]. It is also equipped with a canonical Hodge filtration [21, §3.b], in such a way that the logarithmic de Rham complex of $(\widetilde{M}^\vee(-D), F)$ is filtered quasiisomorphic to the extension by zero $j_!$ of the Hodge module $(M^\vee, F)$ on U (see esp. [21, (3.10.9)]).

Corollary 2.6. *Keep the notation and assumptions of Theorem 2.1. Then*

$$H^q\big(X, L \otimes_{\mathscr{O}_X} \mathrm{Gr}_*(\mathrm{DR}_{(X,D)}(\widetilde{M}^\vee(-D), F))\big) = 0 \quad \textit{for all } q > 0.$$

If, in addition, M has unipotent *local monodromy, then*

$$H^q\big(X, L \otimes_{\mathscr{O}_X} \mathrm{Gr}_*(\mathrm{DR}_{(X,D)}(\widetilde{M}(-D), F))\big) = 0 \quad \textit{for all } q > 0. \tag{2.13}$$

Proof. The Kodaira–Spencer complexes (of $\mathscr{O}_X$-modules and linear maps)

$$\mathrm{Gr}^*(\mathrm{DR}_{(X,D)}(\widetilde{M}, F)) \quad \text{and} \quad \mathrm{Gr}^*(\mathrm{DR}_{(X,D)}(\widetilde{M}^\vee(-D), F))$$

are $\mathscr{O}_X$-dual to each other, with values in $\Omega_X^n[n]$. The first statement thus follows from Grothendieck duality and Theorem 2.1.

If $\widetilde{M}$ has nilpotent residues, then $(\widetilde{M})^\vee$ is itself the canonical extension of $M^\vee$, so we get the second statement by applying Theorem 2.1 to the dual of M and then taking the Grothendieck dual. □

We also state the results in terms of mixed Hodge modules:

Corollary 2.7. *Keep the notation and assumptions of Theorem 2.1. Then*

$$H^q\big(X, L^{-1} \otimes_{\mathscr{O}_X} \mathrm{Gr}_*^F(\mathrm{DR}(j_*(M, F)))\big) = 0 \quad \textit{for all } q < 0,$$
$$H^q\big(X, L \otimes_{\mathscr{O}_X} \mathrm{Gr}_*^F(\mathrm{DR}(j_!(M, F)))\big) = 0 \quad \textit{for all } q > 0.$$

3. Vanishing theorems for mixed Hodge modules

In this section, we follow the direction of the Kodaira vanishing theorem (as opposed to the Nakano vanishing theorem) and deduce Theorem 3.2, for the lowest Hodge grade piece of a mixed Hodge module, from Theorem 2.1. This deduction relies on some fundamental theorems in Saito's theory of mixed Hodge modules, in addition to the more standard technique of embedded resolutions.

The statement (3.1) below was first proved by Popa for mixed Hodge modules strictly supported on X that satisfy a "transversality" condition, namely, that the augmented base locus of the line bundle should be disjoint from the singular locus of the mixed Hodge module. Recently Lei Wu announced [25] that he can also remove the transversality condition, using different methods.

We note in passing that in the situation of §2, this transversality condition is usually not satisfied: in practice, the two loci often coincide, and are equal to the divisor D.

In order to state the theorem, we first recall the central object in Saito's article on Kollár's conjecture [19] (in which *right* $\mathscr{D}$-modules are used). For a nonzero mixed Hodge module M on a quasiprojective scheme over $\mathbb{C}$, we denote by $p(M)$ the least integer p such that $\mathrm{Gr}^F_p(\mathrm{DR}(M))$ is reduced to a single coherent sheaf concentrated in degree 0. Then we write

$$S_X(M) := \mathrm{Gr}^F_{p(M)}(\mathrm{DR}(M)).$$

This has the following description in terms of Kashiwara's equivalence. Let $X \hookrightarrow Y$ be a closed embedding of X into an irreducible smooth variety (e.g. a Zariski open subset of a projective space), so that M corresponds to a mixed Hodge module M_Y on Y with support in X. Let $p(M, Y)$ be the smallest integer p such that $\mathrm{Gr}^F_p(M, Y) \neq 0$. Then

$$S_X(M) \simeq \mathrm{Gr}^F_{p(M,Y)} M_Y \otimes_{\mathscr{O}_Y} \Omega_Y^{\dim Y}.$$

(See [19] for a more detailed explanation.) The following fact, which will be used below, is trivial in view of this description.

Lemma 3.1. *Let* $0 \to M' \to M \to M'' \to 0$ *be an exact sequence of nonzero mixed Hodge modules. Then* $p(M) = \min(p(M'), p(M''))$.

In the proof and the ensuing technical proposition, we will also make use of the *highest* Hodge graded piece. We follow Saito's notation (see [19, §3]), and write

$$Q_X(M) := \mathrm{Gr}^F_{q(M)}(\mathrm{DR}(M)),$$

where $q(M)$ is the largest integer such that $\mathrm{Gr}^F_q(\mathrm{DR}(M)) \neq 0$.

Theorem 3.2. *Let X be a projective scheme over $\mathbb{C}$, L a line bundle on X, and M a nonzero mixed Hodge module on X.*

Suppose that, for any $w \in \mathbb{Z}$ and any direct summand N in the decomposition by strict support of the weight graded piece $\mathrm{Gr}^W_w(M)$ such that $p(N) = p(M)$, L restricts to a big and nef line bundle on the strict support of N.

(*This condition is satisfied, notably, when L is big and nef on an irreducible X and M is the intermediate extension of a polarisable variation of pure Hodge structures on a smooth open dense subset of X.*)

Then

$$H^q(X, L \otimes S_X(M)) = 0 \quad \textit{for all } q > 0. \tag{3.1}$$

Remark 3.3. The condition on L (of "positivity with respect to M") is indeed necessary. For example, let C be an elliptic curve embedded in $\mathbb{P}^2$, S the cone of C in $\mathbb{P}^3$, $\pi : X \to S$ the ruled surface obtained by blowing up S at the vertex, and $E \simeq C$ the exceptional divisor. Then the pullback $L := \pi^*\mathscr{O}_S(1)$ is a big and nef line bundle on X, but $L|_E$ is *trivial*. Thus (3.1) is false for the polarisable Hodge module $M = \mathbb{Q}_E[1]$, since $S_X(M) \simeq \Omega^1_E$.

Proof of Theorem 3.2

Step 1. Reduction to the case of pure Hodge modules with strict support. Since $\mathrm{Gr}^F_p(\mathrm{DR}(\cdot))$ is an exact functor on mixed Hodge modules, and since Lemma 3.1 applied to the weight filtration on M yields

$$p(\mathrm{Gr}^W_w(M)) \geq p(M) \quad \text{for all } w \in \mathbb{Z} \text{ with } \mathrm{Gr}^W_w(M) \neq 0,$$

we are reduced to the case of M pure.

Now by applying Saito's decomposition theorem by strict support, we may assume that X is also integral, and that M is the intermediate extension of a polarisable variation of pure Hodge structures M_0 on a smooth open dense subset U, which we may shrink further whenever necessary:

$$M = j_{!*}(M_0).$$

Therefore we may assume that U is the complement of an effective Cartier divisor D_1. By the condition on L in the statement, we may also assume that L is big and nef on X.

Step 2: Embedded resolutions. We first choose a resolution of singularities $\tau : Y \to X$, obtained by successive blowups with smooth centers contained in the inverse images of

$$\mathrm{Sing}(D_1) \cup \mathrm{Sing}(X) \subseteq D_1,$$

such that the reduced divisor $E_1 := (\tau^*(D_1))_{\mathrm{red}}$ (which contains the exceptional divisor, by construction) is a simple normal crossings divisor. Note that τ maps $Y - E_1$ isomorphically onto U.

Then, because $\tau^*(L)$ is also big and nef on Y, there exists an effective divisor E_2 on Y such that

$$(\tau^*L)^{\otimes \nu}(-E_2) \text{ is ample for all } \nu \gg 0.$$

Again by a sequence of blowups with smooth centers from the singular locus of $E_1 + E_2$, we obtain a modification $\sigma : Z \to Y$ such that

(1) the reduced inverse image $F := (\sigma^*(E_1 + E_2))_{\mathrm{red}}$ is a simple normal crossings divisor,
(2) there exists an effective divisor F'' with support contained in F such that

$$((\tau \circ \sigma)^*L)^{\otimes \nu}(-F'') \text{ is ample for all } \nu \gg 0.$$

(To see the latter, note that the ideal sheaf of the exceptional divisor in the blowup of any smooth variety along any smooth subvariety is a relatively ample line bundle [6, II.8.24].)

Note that σ maps $Z - F$ isomorphically onto an open subscheme of $Y - E_1$, which is in turn mapped isomorphically onto U by τ. Therefore, we may replace $U \subseteq X$ with a dense open subscheme so that the resolution

$$\rho := \tau \circ \sigma : Z \to X$$

restricts to an isomorphism of $Z - F$ onto U. Note that this does not change $M = j_{!*}(M_0)$. Thus U can be considered as an open subscheme of Z as well.

Step 3: Saito's stability theorem and reduction to the SNCD case. Denoting by j' the inclusion of U in Z, we have the Hodge module

$$M' := j'_{!*}(M_0),$$

obtained as the intermediate extension of a polarisable variation of pure Hodge structures on the complement of a normal crossings divisor.

By the theorem of stability of Saito [19, Th. 1.1 and Th. 3.2], we have

$$\mathbb{R}\rho_* S_Z(M') \simeq S_X(M).$$

Hence by the projection formula,

$$H^q(X, L \otimes S_X(M)) \simeq H^q(Z, \rho^*(L) \otimes S_Z(M')),$$

and (3.1) for $(M', \rho^* L, Z)$ implies the same for (M, L, X).

Step 4: Duality and application of Theorem 2.1. Let $N_0 = M_0^\vee$ be the dual variation of Hodge structures of M_0 on U, and N' the intermediate extension of N_0 to Z. Then over Z, the coherent sheaf $S_Z(M')$ is locally free of finite rank, and has Grothendieck dual

$$\mathbb{R}\, \underline{\mathrm{Hom}}_{\mathcal{O}_Z}(S_Z(M'), \Omega^n_Z[n]) \simeq Q_Z(N')$$

in the notation of [19, (3.1.1)], by the remark immediately following it. Therefore, by Grothendieck duality, it is enough to prove

$$H^q(Z, (\rho^* L)^{-1} \otimes Q_Z(N')) = 0 \quad \text{for all } q < 0.$$

Now by Proposition 3.4 below and the quasiisomorphism (2.1), we have an isomorphism

$$Q_Z(N') \simeq \mathrm{Gr}_F^{\mathrm{lowest}}(\widetilde{N}_0)[n] = \mathrm{Gr}_F^{\mathrm{lowest}}(\mathrm{DR}_{(Z,F)}(\widetilde{N}_0, F)).$$

(Here we use the lowest graded piece in the *decreasing* Hodge filtration, as does Saito in [19, §3].) Therefore we get the desired vanishing from Theorem 2.1. □

The following technical proposition, which was crucially used in the proof of Theorem 3.2, is essentially due to Saito. We state it as an independent proposition, since it may be a useful tool, in general, for reducing a problem to the case of SNCD compactifications (via embedded resolutions).

Proposition 3.4. *Let (M, F) be a polarisable variation of pure Hodge structures on the complement U of a simple normal crossings divisor D on a smooth complex variety X. Then the natural maps*

$$j_! M \to j_{!*} M \to j_* M$$

induce natural isomorphisms

$$S_X(j_! M) \simeq S_X(j_{!*} M) \quad \textit{and} \quad Q_X(j_{!*} M) \simeq Q_X(j_* M).$$

Proof. In other words, the highest (resp. lowest) Hodge graded piece of the intermediate extension $j_{!*}(M, F)$ is equal to the highest Hodge graded piece of the derived direct image $j_*(M, F)$ (resp. the lowest Hodge graded piece of the extension by zero $j_!(M, F)$).

The first identification follows from Saito's formula [19, (3.1.1)], and ultimately [20, Prop. 3.2.2(ii)] (applied to the graph embedding attached to the local defining function of the divisor), since $\partial_t : \mathrm{Gr}^V_{-1} \to \mathrm{Gr}^V_0$ is bijective (resp. surjective) in the case of $j_!(M, F)$ (resp. $j_{!*}(M, F)$).

The second identification then follows from the first by duality and the compatibility of the three dualising functors, (i) in the category of mixed Hodge modules $\mathrm{MHM}(X)$; (ii) in the bounded derived category of holonomic filtered differential complexes (with finite filtration) $D^{\mathrm{b}}_{\mathrm{hol}} F^f(\mathscr{O}_X, \mathrm{Diff})$; and (iii) in the bounded derived category of $\mathscr{O}_X$-modules with coherent cohomology sheaves $D^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{O}_X)$. ◻

From the theorem we easily deduce the following corollary, which extends Kollár's vanishing theorem [10, Th. 2.1(iii)], where L is no longer required to be ample. However, we note that this was first proved by Kollár himself already in [12, 10.19].

Corollary 3.5. *Let $f : X \to Y$ be a surjective morphism from an irreducible projective smooth complex variety X onto a projective complex variety Y, and let L be a big and nef line bundle on Y. Then*

$$H^q(Y, L \otimes R^i f_* \omega_X) = 0 \quad \textit{for all } q > 0 \textit{ and } i \geq 0,$$

where ω_X is the dualising $(=$ canonical$)$ sheaf.

Proof. Saito's stability theorem [19, Th. 1.1] also shows that $R^i f_* \omega_X$ is, when nonzero, the lowest Hodge graded piece $S_Y(M)$ of the de Rham complex of a pure Hodge module M strictly supported on Y. Thus we may apply Theorem 3.2. ◻

4. Applications to automorphic forms

The cohomology of Shimura varieties (locally symmetric varieties of hermitian type) has long been studied from various viewpoints, and with various motivations. Here we mention only two lines of results that are most directly related to our work, referring the interested reader to [16, §2] and [5], in which related work of Matsushima, Murakami, Kumaresan, Vogan and Zuckerman is reviewed.

(A) Faltings [5] proved various decomposition and vanishing theorems, which give a complete picture in the compact case, and a partial one in the noncompact case. He uses Hodge theory and his dual BGG construction, and covers coherent cohomology as well as Betti and de Rham cohomology.
(B) Li and Schwermer [16], [17], on the other hand, can treat more general locally symmetric varieties, with or without hermitian structure. However, in the hermitian cases, they do not cover the cases of coherent cohomology groups that Faltings' results do.

In our work with Kai-Wen Lan [14], [15], we proved vanishing theorems for the cohomology (coherent, Betti and de Rham) of *PEL type* Shimura varieties with automorphic coefficients. Our methods were closer to Faltings', and we obtained vanishing theorems for coherent cohomology even in the noncompact case (but still of PEL type).

The basic ingredients were: (1) a then new vanishing theorem for certain Gauss–Manin systems and big and nef line bundles; (2) Faltings' dual BGG construction; and (3) a geometric interpretation of a weight being sufficiently regular.

The ingredients (2) and (3) carry over to the case of arbitrary Shimura varieties without too much difficulty. However, there is a serious obstacle in generalising (1), which makes the general case qualitatively different.

Namely, there is no known way of realising the automorphic bundles over general Shimura varieties by using the relative cohomology (Gauss–Manin system) of *any* projective smooth family—in fact, one suspects that it may not be possible at all. In the case of PEL type Shimura varieties, the universal abelian schemes that "come with" the Shimura varieties (interpreted as moduli spaces) can be used for their geometric realisation: see, e.g., [15, §5].

We can overcome this difficulty by using Theorem 2.1 and Corollary 2.6: the automorphic local systems in question are still variations of pure Hodge structures, and we can use a lemma of Deligne's (see [2, Lemme 2.8]) in order to verify the condition of polarisability.

With this, all the results for characteristic-zero coefficients in [15] can be generalised, *mutatis mutandis*, to all the Shimura varieties in complete generality. For one, the vanishing theorem mentioned in the Introduction is obtained in the way [15, Th. 8.18] was obtained in the PEL type case.

New results, not covered in the previous works of Faltings, Li and Schwermer, are obtained for the *coherent* cohomology groups of automorphic bundles with *general* weights. In order to state them, we need to set up a minimum amount of notation. (For details, see Faltings [5], or our work with Lan [15], in which references to the original works of Harris, Milne, Mumford and Pink are provided.)

Let $(G, \mathscr{X})$ be an arbitrary Shimura datum, and P the parabolic subgroup stabilising an element $h \in \mathscr{X}$, with Levi subgroup M. We denote by $W_{G_{\mathbb{C}}}$ and $W_{M_{\mathbb{C}}}$ the respective Weyl groups, and by $W^{M_{\mathbb{C}}}$ the set of elements in $W_{G_{\mathbb{C}}}$ mapping $G_{\mathbb{C}}$-dominant weights to $M_{\mathbb{C}}$-dominant weights. For a neat level subgroup K of $G(\mathbb{A}_f)$, let

$$X = \mathrm{Sh}_K(G, \mathscr{X}) = G(\mathbb{Q})\backslash(G(\mathbb{A}_f) \times \mathscr{X})/K$$

be the Shimura variety attached to it, and let $\overline{X} = \overline{X}_{\Sigma}$ be the toroidal compactification of X associated with a projective smooth cone decomposition Σ. Finally, we denote by

W_ν the automorphic coherent vector bundle on X associated with a dominant weight ν of $M_{\mathbb{C}}$, and by superscript 'can' and 'sub' the canonical and subcanonical extensions of the bundle to $\overline{X}$, respectively.

Now we can state the following generalisation of [15, Cor. 7.24]:

Theorem 4.1. *For any dominant weight μ of $G_{\mathbb{C}}$, any Weyl group element $w \in W^{M_{\mathbb{C}}}$, and any positive parallel dominant weight ν of $M_{\mathbb{C}}$, we have*

$$H^i(\overline{X}, (W^\vee_{w\cdot[\mu]+\nu})^{\mathrm{can}}) = 0 \quad \textit{for all } i < \dim X - \ell(w), \tag{4.1}$$

$$H^i(\overline{X}, (W^\vee_{w\cdot[\mu]-\nu})^{\mathrm{sub}}) = 0 \quad \textit{for all } i > \dim X - \ell(w). \tag{4.2}$$

The proof of Theorem 4.1 is parallel to that of [15, Cor. 7.24], given the new vanishing theorems (Theorem 2.1 and Corollary 2.6). There are some group-theoretic subtleties of combinatorial nature to be sorted out for all reductive groups giving rise to Shimura varieties in complete generality. For the necessary details and some explicit examples, we refer the reader to a recent work of K.-W. Lan [13].

Acknowledgments. The author has benefited from conversations with several mathematicians. He is particularly grateful to H. Esnault and L. Illusie for their comments on the Hodge to de Rham spectral sequence; to N. Katz, S. Morel and Z. Yun for discussions on technical points in §2; to J. Kollár for his comments and question on our result in §3; to M. Popa and Ch. Schnell for pointing out a mistake in §3 of an earlier draft; to K.-W. Lan for his comments on §4; and to S. Dasgupta and M. Kim for their comments on the exposition.

The author is also truly grateful to the anonymous referee who gave numerous helpful, substantial and clarifying comments, especially regarding Theorem 2.1, Remark 2.5 and Proposition 3.4.

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