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The algebra of conformal blocks

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Abstract. For each simply connected, simple complex group G we show that the direct sum of all vector bundles of conformal blocks on the moduli stack $\mathcal{M}_{g,n}$ of stable marked curves carries the structure of a flat sheaf of commutative algebras. The fiber of this sheaf over a smooth marked curve $(C, \vec{p})$ agrees with the Cox ring of the moduli of quasi-parabolic principal G-bundles on $(C, \vec{p})$. We use the factorization rules on conformal blocks to produce flat degenerations of these algebras. In the $\mathrm{SL}_2(\mathbb{C})$ case, these degenerations result in toric varieties which appear in the theory of phylogenetic statistical varieties, and the study of integrable systems in the moduli of rank 2 vector bundles. We conclude with a combinatorial proof that the Cox ring of the moduli stack of quasi-parabolic principal $\mathrm{SL}_2(\mathbb{C})$-bundles over a generic curve is generated by conformal blocks of levels 1 and 2 with relations generated in degrees 2, 3, and 4.

Keywords. Conformal blocks, principal bundles, phylogenetics

Contents

1. Introduction

Let $(C, \vec{p})$ be a smooth, complex, projective curve with distinct marked points $\vec{p} = \{p_1, \dots, p_n\} \subset C$, and let G be a simple, simply connected complex group. We fix a Borel subgroup B and choose a parabolic Λ_i containing B for each p_i. A *quasi-parabolic principal G-bundle* on C of type $\vec{\Lambda} = \{\Lambda_1, \dots, \Lambda_n\}$ is a principal G-bundle $E \to C$ together with a choice of a point ρ_i in the fiber over p_i of the associated bundle $E \times_G (G/\Lambda_i)$ (equivalently, a choice of right Λ_i-orbit in the fiber of E over p_i). We study the Cox ring, or total coordinate ring, of the moduli stack $\mathcal{M}_{C,\vec{p}}(\vec{\Lambda})$ of these objects.

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The Cox ring associated to this moduli problem is the direct sum of all the spaces of global sections of line bundles on the stack taken over the torsion free part of the Picard group. The group $\mathrm{Pic}(\mathcal{M}_{C,\vec{p}}(\vec{\Lambda}))$ is computed by Laszlo and Sorger [LS97], who show it is a free Abelian group:

$$\mathrm{Pic}(\mathcal{M}_{C,\vec{p}}(\vec{\Lambda})) = \mathcal{X}(\Lambda_1) \times \cdots \times \mathcal{X}(\Lambda_n) \times \mathbb{Z}. \tag{1}$$

Here $\mathcal{X}(\Lambda_i)$ is the group of characters of the parabolic subgroup $\Lambda_i \subset G$. A celebrated result of Faltings [Fal94], Kumar, Narasimhan and Ramanathan [KNR94], Beauville, Laszlo and Sorger [BL94], [BLS98], [Sor99], and Pauly [Pau96] identifies global sections of line bundles on $\mathcal{M}_{C,\vec{p}}(\vec{\Lambda})$ with the spaces of conformal blocks from the Wess–Zumino–Novikov–Witten (WZNW) model of conformal theory. For a fixed curve $(C, \vec{p})$, there is one such space $\mathcal{V}^{\dagger}_{C,\vec{p}}(\vec{\lambda}, L)$ for each choice of dominant weights $\lambda_i \in \mathcal{X}(\Lambda_i)$ and a non-negative integer $L \in \mathbb{Z}_{\geq 0}$:

$$H^0(\mathcal{M}_{C,\vec{p}}(\vec{\Lambda}), \mathcal{L}(\vec{\lambda}, L)) = \mathcal{V}^{\dagger}_{C,\vec{p}}(\vec{\lambda}, L). \tag{2}$$

The Cox ring of $\mathcal{M}_{C,\vec{p}}(\vec{\Lambda})$ is therefore the sum of all the spaces of conformal blocks with compatible parabolic data:

$$\mathrm{Cox}(\mathcal{M}_{C,\vec{p}}(\vec{\Lambda})) = \bigoplus_{\vec{\lambda},L} H^0(\mathcal{M}_{C,\vec{p}}(\vec{\Lambda}), \mathcal{L}(\vec{\lambda}, L)) = \bigoplus_{\vec{\lambda},L} \mathcal{V}^{\dagger}_{C,\vec{p}}(\vec{\lambda}, L). \tag{3}$$

The main theorem of this paper produces a family of flat degenerations of $\mathrm{Cox}(\mathcal{M}_{C,\vec{p}}(\vec{\Lambda}))$ from the combinatorial properties of the WZNW theory. We state this theorem for $\Lambda_i = B \subset G$ a Borel subgroup, denoted $\mathcal{M}_{C,\vec{p}}(G)$, as all other cases are implied by this case. In what follows, Γ is a graph with non-leaf vertex set $V(\Gamma)$ and edge set $E(\Gamma)$.

Theorem 1.1. *For $(C, \vec{p})$ a marked stable curve, there is a flat degeneration of $\mathrm{Cox}(\mathcal{M}_{C,\vec{p}}(G))$ for every trivalent graph Γ with first Betti number $g = \mathrm{genus}(C)$ and $n = |\vec{p}|$ leaves:*

$$\mathrm{Cox}(\mathcal{M}_{C,\vec{p}}(G)) \Rightarrow \Big[\bigotimes_{v \in V(\Gamma)} \mathrm{Cox}(\mathcal{M}_{0,3}(G))\Big]^{T_\Gamma}. \tag{4}$$

Here T_Γ is a product of $|E(\Gamma)| - n$ tori $T \times \mathbb{C}^*$, where $T \subset G$ is a maximal torus.

For a description of the action of T_Γ on $\bigotimes_{v \in V(\Gamma)} \mathrm{Cox}(\mathcal{M}_{0,3}(G))$ see Section 3. Theorem 1.1 is a "ringification" of the *factorization rules* for conformal blocks (see Section 3); and in keeping with the flavor of factorization, many algebraic properties of the Cox ring of $\mathcal{M}_{C,\vec{p}}(G)$ can be understood in terms of the 3-pointed, genus 0 case. For example, in the $\mathrm{SL}_2(\mathbb{C})$ case (discussed in more detail below), Theorem 1.1 is used to show that $\mathrm{Cox}(\mathcal{M}_{C,\vec{p}}(\mathrm{SL}_2(\mathbb{C}))$ is generically finitely generated (Theorems 1.3 and 1.5). Furthermore, it can be shown that $\mathrm{Cox}(\mathcal{M}_{C,\vec{p}}(\mathrm{SL}_2(\mathbb{C}))$ is a Gorenstein algebra using an argument along the lines of [Man13, Theorem 7.3].

Let $R_{C,\vec{p}}(\vec{\lambda}, L) = \bigoplus_{N\geq 0} H^0(\mathcal{M}_{C,\vec{p}}(N\vec{\Lambda}), N\mathcal{L}(\vec{\lambda}, L)) = \bigoplus_{N\geq 0} \mathcal{V}^{\dagger}_{C,\vec{p}}(N\vec{\lambda}, NL)$ denote the projective coordinate ring corresponding to the line bundle $\mathcal{L}(\vec{\lambda}, L)$ on $\mathcal{M}_{C,\vec{p}}(\vec{\Lambda})$; note that this is a graded subalgebra of $\mathrm{Cox}(\mathcal{M}_{C,\vec{p}}(\vec{\Lambda}))$. The corresponding coarse moduli space $\mathrm{Proj}(R_{C,\vec{p}}(\vec{\lambda}, L))$ is denoted $M_{C,\vec{p}}(\vec{\lambda}, L)$. The celebrated Verlinde formula [Ver88], [Fal94], [Bea96] calculates the dimension of the space $H^0(\mathcal{M}_{C,\vec{p}}(\vec{\Lambda}), \mathcal{L}(\vec{\lambda}, L))$ of global sections. We view Theorem 1.1 as a first step toward developing a polyhedral rule for computing the Verlinde formula, which should correspond to toric degenerations of the coarse moduli $M_{C,\vec{p}}(\vec{\lambda}, L)$.

A toric degeneration of an algebra is a flat family of algebras over some base, with a special fiber equal to the semigroup algebra of a normal affine semigroup. Presentation results, and certain algebraic properties (e.g. Gorenstein, Koszul), can be easier to prove on an algebra with a toric degeneration, as these properties are controllable under flat degeneration and are more readily established by combinatorial means on the special fiber of the degeneration. Theorem 1.1 reduces the problem to finding such a degeneration for $R_{C,\vec{p}}(\vec{\lambda}, L)$ or $\mathrm{Cox}(\mathcal{M}_{C,\vec{p}}(G))$ to finding a degeneration for $\mathrm{Cox}(\mathcal{M}_{0,3}(G))$ which respects the multigrading by dominant weights. When $G = \mathrm{SL}_2(\mathbb{C})$, the algebra $\mathrm{Cox}(\mathcal{M}_{0,3}(\mathrm{SL}_2(\mathbb{C})))$ is already an affine semigroup algebra, so in this case our degenerations are toric. The relevant affine semigroup algebras are the graded algebras associated to the following polytopes.

Definition 1.2. For Γ a trivalent graph, let P_Γ be the polytope of weightings $w : E(\Gamma) \to \mathbb{R}_{\geq 0}$ which satisfy the following properties at each internal vertex $v \in V(\Gamma)$.

(1) The sum of the three weights incident on a vertex is $w_1(v) + w_2(v) + w_3(v) \leq 2$.
(2) These three weights satisfy the triangle inequalities, $|w_1(v) - w_3(v)| \leq w_2(v) \leq w_1(v) + w_3(v)$.

Also see [Buc12] for a description of these polytopes. We consider the P_Γ and their Minkowski sums with respect to the lattice $\mathcal{L}_\Gamma \subset \mathbb{R}^{E(\Gamma)}$ defined by the condition that all edges are weighted with integers, and $w_1(v)+w_2(v)+w_3(v) \in 2\mathbb{Z}$ for all $v \in V(\Gamma)$. An affine semigroup is obtained from P_Γ by considering the lattice points in the Minkowski sums $L\circ P_\Gamma = \{u_1 + \cdots + u_L \mid u_i \in P_\Gamma\}$. The union $S(P_\Gamma) = \coprod_{L\geq 0} L\circ P_\Gamma \cap \mathcal{L}_\Gamma$ is naturally graded, and it is easy to see that if $u_1 \in L\circ P_\Gamma$ and $u_2 \in K\circ P_\Gamma$, then $u_1 + u_2 \in (L + K)\circ P_\Gamma$, where the operation $+$ is sum of integer valued functions on $E(\Gamma)$. Let $\mathbb{C}[P_\Gamma]$ be the affine semigroup algebra obtained from the graded affine semigroup $S(P_\Gamma)$; note that $\mathbb{C}[P_\Gamma]$ comes with a distinguished basis in bijection with the graded set of lattice points. We show the following in Section 5.

Theorem 1.3. *Let $(C, \vec{p})$ be an n-marked smooth, projective curve of genus g, and Γ a trivalent graph with first Betti number g and n leaves. There is a flat degeneration*

$$\mathrm{Cox}(\mathcal{M}_{C,\vec{p}}(\mathrm{SL}_2(\mathbb{C}))) \Rightarrow \mathbb{C}[P_\Gamma]. \tag{5}$$

Recall that dominant weights of $\mathrm{SL}_2(\mathbb{C})$ are non-negative integers, therefore we may associate a projective coordinate ring of the moduli of quasi-parabolic principal $\mathrm{SL}_2(\mathbb{C})$-bundles $R_{C,\vec{p}}(\vec{r}, L)$ to the data $(\vec{r}, L) \in \mathbb{Z}^{n+1}_{\geq 0}$. As a corollary of Theorem 1.3 we also obtain explicit toric degenerations of spaces $M_{C,\vec{p}}(\vec{r}, L) = \mathrm{Proj}(R_{C,\vec{p}}(\vec{r}, L))$.

Definition 1.4. let $P_\Gamma(\vec{r}, L)$ be the polytope obtained as the fiber over $\vec{r}$ for the map $L \circ P_\Gamma \to \mathbb{R}^n$, computed by forgetting all weights except those on the leaf-edges.

Let $\mathbb{C}[P_\Gamma(\vec{r}, L)]$ be the affine semigroup defined by the polytope $P_\Gamma(\vec{r}, L)$. For $G = \mathrm{SL}_2(\mathbb{C})$, $(C, \vec{p})$ a marked smooth projective curve, and Γ a trivalent graph with compatible information. There is a flat degeneration

$$R_{C,\vec{p}}(\vec{r}, L)) \Rightarrow \mathbb{C}[P_\Gamma(\vec{r}, L)]. \tag{6}$$

Theorem 1.3 is utilized in [Man10b] and [Man12] to prove that the projective coordinate ring of the square of any effective line bundle on the moduli stack $\mathcal{M}_{C,\vec{p}}(\mathrm{SL}_2(\mathbb{C}))$ is generated by its degree 1 elements, and is a Koszul algebra for generic $(C, \vec{p})$. This result follows from the analysis of $P_\Gamma(\vec{r}, L)$ for particular well-chosen trivalent graphs. We follow a similar strategy here; by studying a particular polytope $P_{\Gamma_{g,n}}$, we prove the following.

Theorem 1.5. *For generic $(C, \vec{p})$, the algebra $\mathrm{Cox}(\mathcal{M}_{C,\vec{p}}(\mathrm{SL}_2(\mathbb{C})))$ is generated by conformal blocks of level $L = 1, 2$. The corresponding ideal of relations is generated in levels* $2, 3, 4$.

We prove Theorem 1.5 by establishing these properties for the algebra $\mathbb{C}[P_{\Gamma_{g,n}}]$ in Section 5; the theorem then holds for $\mathrm{Cox}(\mathcal{M}_{C,\vec{p}}(\mathrm{SL}_2(\mathbb{C})))$ because of general properties of flat families of algebras. In particular, the degrees of generators and relations needed to present a particular algebra in a flat family bound such degrees generically; for more discussion on this point see [Man12, Theorem 1.11].

Theorem 1.5 is a simultaneous generalization of theorems of Castravet and Tevelev [CT06], Sturmfels and Xu [SX10], and Abe [Abe10]. Castravet and Tevelev, along with Sturmfels and Xu, treat the case $g = 0$, and show that $L = 1$ conformal blocks generate with quadratic relations by utilizing a theorem of Bauer [Bau91] and results of Buczyńska and Wiśniewski [BW07] discussed below. Abe treats the case $n = 0$, and shows that $L = 1$ conformal blocks generate by combining a proof that $L = 1, 2$ suffice with a result of Beauville [Bea91] which establishes that the $L = 1$ component generates $L = 2$.

Theorem 1.5 should be of interest in the emerging field of Newton–Okounkov bodies [KK12], [LM09], [HK12]. We show in Proposition 5.4 that the polytope P_Γ is a Newton–Okounkov body of the scheme $\mathrm{Proj}(V_{C,\vec{p}}(\mathrm{SL}_2(\mathbb{C})))$, when $(C, \vec{p})$ is a stable curve of type Γ.

1.1. Organization and methods

Theorem 1.1 is a consequence of the commutative algebra analogues of classical results from the theory of conformal blocks proved in [TUY89]. In what follows, $\bar{\mathcal{M}}_{g,n}$ denotes the Deligne–Mumford stack of stable n-pointed curves of genus g (see [DM69]).

(1) (Flatness) The spaces $\mathcal{V}^\dagger_{C,\vec{p}}(\vec{\lambda}, L)$ fit together into a vector bundle $\mathcal{V}^\dagger(\vec{\lambda}, L)$ over $\bar{\mathcal{M}}_{g,n}$.

(2) (Factorization) For $\tilde{C}$ the partial normalization of a stable curve C at a double point q, there is an isomorphism of vector spaces

$$\mathcal{V}^{\dagger}_{C,\vec{p}}(\vec{\lambda}, L) \xleftarrow{\cong} \bigoplus_{\alpha \in \Delta_L} \mathcal{V}^{\dagger}_{\tilde{C},\vec{p},q_1,q_2}(\vec{\lambda}, \alpha, \alpha^*, L);$$

here the point q_1 is always assigned the dual weight α^* to the weight α assigned to its partner q_2 (see Figure 1).

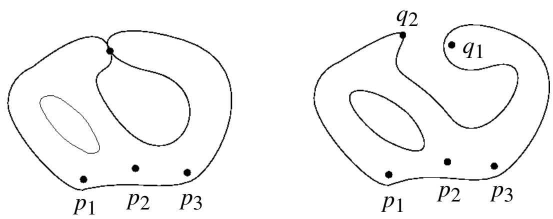

Fig. 1. Normalization of a triple marked stable genus 2 curve.

In Section 2 we build a multiplication operation on the direct sum $\mathcal{V}^{\dagger}(G) = \bigoplus_{\vec{\lambda},L} \mathcal{V}^{\dagger}(\vec{\lambda}, L)$ from elements of Kac–Moody representation theory. The following is a consequence of this construction.

Proposition 1.6. *For any simple Lie algebra $\mathfrak{g}$ with associated simple, simply connected reductive group G, the direct sum of vector bundles*

$$\mathcal{V}^{\dagger}(G) = \bigoplus_{\vec{\lambda},L} \mathcal{V}^{\dagger}(\vec{\lambda}, L) \tag{7}$$

has the structure of a flat sheaf of algebras on $\bar{\mathcal{M}}_{g,n}$. Over a smooth marked curve $(C, \vec{p})$, multiplication on this sheaf agrees with multiplication of global sections on the corresponding line bundles over the moduli $\mathcal{M}_{C,\vec{p}}(G)$:

$$\mathcal{V}^{\dagger}_{C,\vec{p}}(G) \cong \mathrm{Cox}(\mathcal{M}_{C,\vec{p}}(G)). \tag{8}$$

We call $\mathcal{V}^{\dagger}_{C,\vec{p}}(G)$ the *algebra of conformal blocks over* $(C, \vec{p})$. The global object $\mathcal{V}^{\dagger}(G)$ relates the Cox ring of $\mathcal{M}_{C,\vec{p}}(G)$ to the algebra of conformal blocks for a non-smooth, stable curve by a flat family.

Recall that $\bar{\mathcal{M}}_{g,n}$ is stratified by the stability type of the curves $(C, \vec{p})$; this is the source of the graph combinatorics in Theorem 1.1. The strata of $\bar{\mathcal{M}}_{g,n}$ are indexed by connected graphs Γ with n labeled leaves. Each internal vertex of the graph is labeled with a number g_i, thought of as the "internal genus" of that vertex (see Figure 2). The genus g of the whole graph is computed by summing these numbers and adding the first Betti number of Γ. An internal vertex corresponds to a smooth component in the normalization of a representative curve of the stratum, and leaves correspond to marked points. The lowest strata of $\bar{\mathcal{M}}_{g,n}$ are isolated points indexed precisely by trivalent graphs. In Section 3, we utilize the factorization rules of conformal blocks to degenerate the algebra over non-smooth, stable curves.

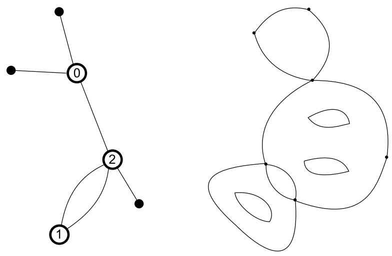

Fig. 2. The graph of a stable curve type.

Proposition 1.7. *Let $(C, \vec{p})$ be a stable curve of stability type Γ, with smooth normalization $(\tilde{C}, \vec{p}, \vec{q}_1, \vec{q}_2)$. There is a flat degeneration*

$$\mathcal{V}^{\dagger}_{C,\vec{p}}(G) \Rightarrow [\mathcal{V}^{\dagger}_{\tilde{C},\vec{p},\vec{q}_1,\vec{q}_2}(G)]^{T_\Gamma}. \tag{9}$$

Here $T_\Gamma = (T \times \mathbb{C}^*)^{|E(\Gamma)|-|\vec{p}|}$, where $T \subset G$ is a maximal torus.

Taken together, Propositions 1.6 and 1.7 prove Theorem 1.1.

1.2. The relationship with configuration spaces

There is a natural map relating the space of conformal blocks with labels $\vec{\lambda}$ to the space of $\mathfrak{g}$-invariants in $V(\vec{\lambda}^*) = V(\lambda_1^*) \otimes \cdots \otimes V(\lambda_n^*)$. This is called the *correlation map* (see the book of Ueno [Uen97] for more information on its properties):

$$F_{C,\vec{p}} : \mathcal{V}^{\dagger}_{C,\vec{p}}(\vec{\lambda}, L) \to \mathrm{Hom}_{\mathbb{C}}(V(\vec{\lambda})/\mathfrak{g}V(\vec{\lambda}), \mathbb{C}) \cong V(\vec{\lambda}^*)^{\mathfrak{g}}. \tag{10}$$

When the genus of the curve C is 0, the map $F_{C,\vec{p}}$ is injective, allowing us to make certain aspects of conformal blocks more concrete by relating them to vector spaces from classical representation theory. Let $U \subset G$ be the maximal unipotent subgroup contained in B, and let $\mathfrak{A}_n$ be the algebra of invariants in the tensor product $\mathbb{C}[G/U]^{\otimes n}$ with respect to the left diagonal action by G. The space of invariants $V(\vec{\lambda}^*)^{\mathfrak{g}}$ is a subspace of $\mathfrak{A}_n$ and is also the space of global sections of a line bundle $\mathcal{L}(\vec{\lambda})$ on the space of configurations $M_{\vec{\lambda}}$ (see Subsection 4.3). Furthermore, the projective coordinate ring $\mathfrak{A}_{\vec{\lambda}} = \bigoplus_{N\geq 0} H^0(M_{\vec{\lambda}}, \mathcal{L}(\vec{\lambda})^{\otimes N})$ is always a subalgebra of $\mathfrak{A}_n$. In Sections 2 and 4 we show that the correlation map can be enhanced to a map of algebras $F_{C,\vec{p}} : \mathcal{V}^{\dagger}_{C,\vec{p}}(G) \to \hat{\mathfrak{A}}_n$, where $\hat{\mathfrak{A}}_n$ is a certain Rees algebra of $\mathfrak{A}_n$ (see Section 4). This relationship then passes to a graded inclusion $R_{C,\vec{p}}(\vec{\lambda}, L) \subset \mathfrak{A}_{\vec{\lambda}}$. In Section 4 we then relate the degenerations constructed in Theorem 1.1 to degenerations of configuration spaces constructed in [Man10a] and [Man14, Proposition 4.9].

Theorem 1.8. *Let $(C, \vec{p})$ be a marked, stable, genus 0 curve of type $\mathcal{T}$. Then the degeneration from Theorem 1.7 on $\mathcal{V}^{\dagger}_{C,\vec{p}}(G)$ extends to a degeneration on $\hat{\mathfrak{A}}_n$ corresponding to $\mathcal{T}$ under the correlation morphism $F_{C,\vec{p}}$. Furthermore, the induced degeneration of $R_{C,\vec{p}}(\vec{\lambda}, L)$ extends to a degeneration on $\mathfrak{A}_{\vec{\lambda}}$.*

For $\mathfrak{g} = \mathrm{sl}_2(\mathbb{C})$, the algebra $\mathfrak{A}_n$ is the projective coordinate ring of the Grassmannian variety $\mathrm{Gr}_2(\mathbb{C}^n)$. In [SS04], Speyer and Sturmfels describe the tropical variety $T^{n,2}$ of $\mathrm{Gr}_2(\mathbb{C}^n)$ with respect to the embedding in $\mathbb{P}^{\binom{n}{2}-1}$ given by the Plücker generators of its coordinate ring. They show that $T^{n,2}$ has a maximal face for each trivalent tree with n ordered leaves. As $T^{n,2}$ is a subfan of the Gröbner fan of $\mathrm{Gr}_2(\mathbb{C}^n)$, each point $w \in T^{n,2}$ defines an initial ideal I_w of the ideal of relations on the Plücker generators. The degeneration of $\mathfrak{A}_n$ associated to a trivalent tree $\mathcal{T}$ with n ordered leaves constructed in Theorem 1.8 is then presented by an initial ideal I_w from the associated face of $T^{n,2}$ (see [HMM11], [HMSV09]). These degenerations are used by Howard, Millson, Snowden, and Vakil [HMSV09] to study the moduli space $M_{\vec{r}}$ of weighted $\vec{r}$-weighted ordered point arrangements on $\mathbb{P}^1$, for $\vec{r} \in \mathbb{Z}^n_{\geq 0}$. The appearance of trees $\mathcal{T}$ in the context of degenerations of $M_{\vec{r}}$, and Sturmfels and Xu's work on $\mathcal{M}_{\mathbb{P}^1,\vec{p}}(\mathrm{SL}_2(\mathbb{C}))$ led Millson to conjecture the following [Mil09].

Conjecture 1.9 (Millson). *For a curve C of genus 0 and a trivalent tree $\mathcal{T}$ with n leaves, consider the degeneration of the ring $\mathfrak{A}_{\vec{r}}$ induced by the construction by Speyer and Sturmfels associated to the tree $\mathcal{T}$. Then the induced degeneration on $R_{C,\vec{p}}(\vec{r}, L)$ is isomorphic to $\mathbb{C}[P_{\mathcal{T}}(\vec{r}, L)]$.*

The following is a consequence of Theorem 1.8.

Corollary 1.10. *For a curve C of genus 0 and a trivalent tree $\mathcal{T}$ with n leaves, the induced degeneration on $R_{C,\vec{p}}(\vec{r}, L)$ above is isomorphic to $\mathbb{C}[P_{\mathcal{T}}(\vec{r}, L)]$ when $(C, \vec{p})$ is the stable curve of type $\mathcal{T}$.*

1.3. Phylogenetics and conformal blocks

Along with their relationship to $\mathrm{SL}_2(\mathbb{C})$ conformal blocks, the affine semigroup algebras $\mathbb{C}[P_\Gamma]$ make an appearance in mathematical biology. The scheme $\mathrm{Proj}(\mathbb{C}[P_\Gamma])$ is a statistical model based on the Jukes–Cantor binary model of phylogenetics [BW07], [Buc12], [BBKM13], [SX10]. In [Buc12] and [BW07], Buczyńska and Wiśniewski show geometrically that the Hilbert functions of the algebras $\mathbb{C}[P_\Gamma]$ only depend on the number of leaves and the first Betti number of Γ. The visually appealing method employed by Buczyńska and Wiśniewski to obtain this result is to construct pairwise deformations between algebras associated to combinatorial alterations of the underyling graphs (see Figure 3). In a sense, Theorem 1.3 “fills in” this picture over $\bar{\mathcal{M}}_{g,n}$.

The commutative algebra $\mathbb{C}[P_\Gamma]$ is also studied by Buczyńska, Buczyński, Kubjas, and Michałek [BBKM13], who prove the following theorem.

Theorem 1.11 (BBKM). *Let Γ be a graph with first Betti number g and n leaves. Then $\mathbb{C}[P_\Gamma]$ is generated in degree $\leq g + 1$. There exist graphs where this bound is attained.*

Although this was not the focus of their work, note that the result in [BBKM13] and Theorem 1.3 imply that $\mathrm{Cox}(\mathcal{M}_{C,\vec{p}}(\mathrm{SL}_2(\mathbb{C})))$ is generated by conformal blocks of level $\leq g + 1$ for $(C, \vec{p})$ generic, and Theorem 1.5 shows that this bound can be lowered

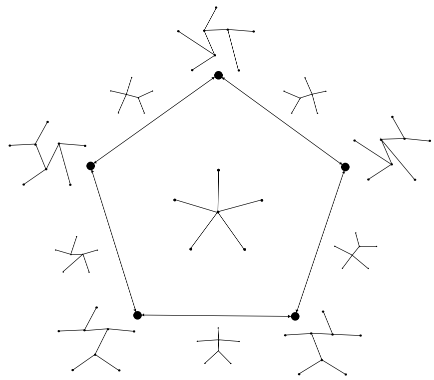

Fig. 3. Combinatorial alterations between trees with five ordered leaves.

to 2. The [BBKM13] result also shows that the special fibers $\mathcal{V}^{\dagger}_{C,\vec{p}}(\mathrm{SL}_2(\mathbb{C}))$ for $(C, \vec{p})$ of trivalent graph type Γ can all be generated in degree $\leq g + 1$.

The connection between phylogenetics and moduli of principal $\mathrm{SL}_2(\mathbb{C})$-bundles was first made in the $g = 0$ case in [SX10]. Sturmfels and Xu show that the binomial ideal defining $\mathbb{C}[P_{\mathcal{T}}]$ for $\mathcal{T}$ a tree coincides with an initial ideal of an ideal presenting the algebra $\mathrm{Cox}(\mathcal{M}_{C,\vec{p}}(\mathrm{SL}_2(\mathbb{C})))$. We note that this connection can also be made directly through the combinatorics of $\mathrm{sl}_2(\mathbb{C})$ conformal blocks. In particular, the quantum Clebsch–Gordan rule (see Section 5) and the factorization rules (see Section 3) establish directly that the lattice points of the polytope $P_\Gamma(\vec{r}, L)$ count the dimension of the space $\mathcal{V}^{\dagger}_{C,\vec{p}}(\vec{r}, L)$ when the genus of C is $\beta_1(\Gamma)$ and $|\vec{p}|$ is the number of leaves of Γ.

1.4. Remarks on integrable systems in the moduli of principal bundles

The polytopes P_Γ and $P_\Gamma(\vec{r}, L)$ appear in work of Hurtubise and Jeffrey [HJ00] and Jeffrey and Weitsman [JW92], on the integrable systems in the moduli of bundles associated to the Goldman flows on those spaces. In particular, in [HJ00], Hurtubise and Jeffrey note that the presence of a dense, open integrable system in the moduli space with momentum image P_Γ almost gives a proof of the Verlinde formula. They reason that if the moduli space were toric, then the Verlinde formula could be computed by counting the lattice points in P_Γ, which coincides with the expected dimension. Theorem 1.3 enhances this picture by showing that the toric variety associated to P_Γ is a flat degeneration of the moduli space. It would be interesting to relate the degeneration constructed in Theorem 1.3 and the integrable system studied in [HJ00] along the lines of work of Kaveh and Harada [HK12] (see also Subsection 5.2).

1.5. Notation

Here we collect some frequently used notation.

G	a simple, simply connected affine group over $\mathbb{C}$
$\mathfrak{g}$	the Lie algebra of G

$\lambda \in \Delta$	a dominant weight of $\mathfrak{g}$ in a Weyl chamber
$V(\lambda)$	the irreducible representation of $\mathfrak{g}$ associated to λ
$\hat{\mathfrak{g}}$	the affine Kac–Moody algebra of $\mathfrak{g}$
$\mathcal{H}(\lambda, L)$	the integrable highest weight representation of $\hat{\mathfrak{g}}$ associated to λ, L
$(C, \vec{p})$	a curve with marked points
$\mathcal{V}^{\dagger}_{C,\vec{p}}(\vec{\lambda}, L)$	the space of conformal blocks associated to the data C, $\vec{p}$, $\vec{\lambda}$, L
$\mathcal{V}^{\dagger}_{C,\vec{p}}(G)$	the algebra of conformal blocks on $[C, \vec{p}\,]$ for the group G
$\mathcal{M}_{C,\vec{p}}(G)$	the moduli stack of quasi-parabolic principal G-bundles on $[C, \vec{p}\,]$
$M_{C,\vec{p}}(\vec{\lambda}, L)$	the moduli space of semistable parabolic G-bundles on $[C, \vec{p}\,]$
$\bar{\mathcal{M}}_{g,n}$	the moduli stack of stable curves of genus g with n marked points
Γ	a graph
$E(\Gamma)$	the edges of Γ
$V(\Gamma)$	the non-leaf vertices of Γ

2. The sheaf of conformal blocks

In this section we construct the multiplication operation on the sheaf of conformal blocks, and show that its specialization at a smooth marked curve $(C, \vec{p}\,)$ is equal to multiplication of global sections of line bundles on the moduli $\mathcal{M}_{C,\vec{p}}(G)$. We thank Eduard Looijenga for the remarks he provided on his construction of the sheaf $\mathcal{V}^{\dagger}(\vec{\lambda}, L)$ of conformal blocks on $\bar{\mathcal{M}}_{g,n}$. For a simple Lie algebra $\mathfrak{g}$ we fix a Cartan subalgebra, a system of positive roots, and we let Δ denote the corresponding Weyl chamber.

2.1. Construction of the sheaf of conformal blocks

We refer the reader to the accounts of this construction in [Bea96], [Kum87], [KNR94], [Loo], [SU99], [TUY89] and [Fak12]. Let θ be the longest root, with associated Cartan element $\theta^{\vee}$; the level L alcove $\Delta_L \subset \Delta$ is the simplex defined by the condition $\lambda(\theta^{\vee}) \leq L$. For each dominant weight $\lambda \in \Delta_L$ there is an integrable highest weight module $\mathcal{H}(\lambda, L)$ of the affine Kac–Moody algebra $\hat{\mathfrak{g}}$ (see [Bea96, 1.5]). Recall that $\hat{\mathfrak{g}}$ contains a subalgebra naturally isomorphic to $\mathfrak{g}$. The module $\mathcal{H}(\lambda, L)$ likewise contains a $\mathfrak{g}$-highest weight vector v_{λ} which generates the irreducible $\mathfrak{g}$-representation $V(\lambda)$.

The Kac–Moody algebra $\hat{\mathfrak{g}}$ is a central extension of $\mathfrak{g} \otimes \mathbb{C}((t))$, where the additional central element acts on $\mathcal{H}(\lambda, L)$ with weight L. Let $\hat{\mathfrak{g}}_n$ be the Lie algebra $\sum_{i=1}^{n} \hat{\mathfrak{g}}$ with central elements identified; this algebra acts on tensor products of integrable highest weight modules with a common level: $\mathcal{H}(\vec{\lambda}, L) = \mathcal{H}(\lambda_1, L) \otimes \cdots \otimes \mathcal{H}(\lambda_n, L)$. For a connected, stable curve $(C, \vec{p}\,)$, there is an associated Lie algebra $\hat{\mathfrak{g}}[C, \vec{p}\,] = \mathfrak{g} \otimes \mathbb{C}[C \setminus \{\vec{p}\}]$. By fixing a local parameter t_i at each marked point p_i we obtain a map $\hat{\mathfrak{g}}[C, \vec{p}\,] \to \bigoplus_{i=1}^{n} \mathfrak{g} \otimes \mathbb{C}((t_i))$ by power-series expansion. By the Residue Theorem (see [Bea96, Part I]), this map extends to the central extension $\hat{\mathfrak{g}}_n$, so $\mathcal{H}(\vec{\lambda}, L)$ can be considered as a representation of $\hat{\mathfrak{g}}[C, \vec{p}\,]$.

Now we sheafify this construction following [Loo] and [Fak12]. Let S be a smooth affine variety over $\mathbb{C}$. By a *stable curve* of genus g over S with n marked points we mean a proper, flat map $\pi : C \to S$ with fibers equal to genus g curves with at worst double point singularities, and n pairwise non-intersecting sections $p_1, \ldots, p_n : S \to C$ with images contained in the smooth locus, such that $C \setminus \bigcup p_i(S)$ is affine over S. We assume we have specified isomorphisms $\psi_i : \hat{\mathcal{O}}_{C,p_i(S)} \to H^0(S, \mathcal{O}_S)[[t]]$.

We require the following sheaves of Lie algebras over $\mathcal{O}_S$: $\hat{\mathfrak{g}}_n(S) = \hat{\mathfrak{g}}_n \otimes \mathcal{O}_S$, $\mathfrak{g}(S) = \mathfrak{g} \otimes \mathcal{O}_S$, and $\hat{\mathfrak{g}}(C, \vec{p}) = \mathfrak{g} \otimes \pi_*[\mathcal{O}_{C \setminus \bigcup p_i(S)})]$. The algebra $\mathfrak{g}(S)$ can be realized as a Lie subalgebra of $\hat{\mathfrak{g}}(C, \vec{p})$, and the fiber of $\hat{\mathfrak{g}}(C, \vec{p})$ at s is equal to $\hat{\mathfrak{g}}[\pi^{-1}(s), \vec{p}(s)]$. Furthermore, by using the ψ_i, we realize $\hat{\mathfrak{g}}(C, \vec{p})$ as a Lie subalgebra of $\hat{\mathfrak{g}}_n(S)$. We also require the sheafified highest weight representations $\mathcal{H}_S(\vec{\lambda}, L) = \mathcal{H}(\vec{\lambda}, L) \otimes \mathcal{O}_S$ and $V_S(\vec{\lambda}) = V(\vec{\lambda}) \otimes \mathcal{O}_S$. The sheaf $\mathcal{H}_S(\vec{\lambda}, L)$ is a $\hat{\mathfrak{g}}_n(S)$-module, and therefore also a $\hat{\mathfrak{g}}(C, \vec{p})$-module, and the sheaf $V_S(\vec{\lambda})$ is likewise a $\mathfrak{g}(S)$-submodule of $\mathcal{H}_S(\vec{\lambda}, L)$.

For any Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ and representation M there is a space of invariants:

$$\mathrm{Hom}_{\mathbb{C}}(M/\mathfrak{g}M, \mathbb{C}) \cong (M^*)^{\mathfrak{g}}. \tag{11}$$

The same construction may be applied to sheaves of Lie algebras and representations over a scheme S, with $\mathrm{Hom}_{\mathcal{O}_S}(-,-)$ the sheaf of morphisms, and $\mathcal{O}_S$ as a dualizing object.

Definition 2.1. The sheaf of *vacua* or *conformal blocks*, $\mathcal{V}^{\dagger}_{C,\vec{p}}(\vec{\lambda}, L)$, is defined to be the following sheaf of invariants:

$$\mathcal{V}^{\dagger}_{C,\vec{p}}(\vec{\lambda}, L) = \mathrm{Hom}_{\mathcal{O}_S}\big(\mathcal{H}_S(\vec{\lambda}, L)/\hat{\mathfrak{g}}(C, \vec{p})\mathcal{H}_S(\vec{\lambda}, L), \mathcal{O}_S\big). \tag{12}$$

Note that $\mathcal{V}^{\dagger}_{C,\vec{p}}(\vec{\lambda}, L)$ is naturally a subsheaf of the dual $\mathcal{H}_S(\vec{\lambda}, L)^*$. Taking a single fiber $\pi^{-1}(s)$ of π we have the vector space of conformal blocks,

$$\mathcal{V}^{\dagger}_{\pi^{-1}(s),\vec{p}(s)}(\vec{\lambda}, L) = \mathrm{Hom}_{\mathbb{C}}\big(\mathcal{H}(\vec{\lambda}, L)/\hat{\mathfrak{g}}[\pi^{-1}(s), \vec{p}(s)]\mathcal{H}(\vec{\lambda}, L), \mathbb{C}\big). \tag{13}$$

Over a general smooth base scheme one can always choose the isomorphisms ψ_i Zariski locally. Furthermore, a description of these sheaves can be given which does not depend on the choice of the ψ_i (see [Loo] and [Bea96, 1.7]). The corresponding sheaves $\mathcal{V}^{\dagger}(\vec{\lambda}, L)$ on the moduli stack $\mathcal{M}_{g,n}$ are proved to be locally free and coherent in [Uen97] and [TUY89] (see also [Loo]).

2.2. *Multiplication of conformal blocks*

Now we define the multiplication operation on sheaves of conformal blocks. We let $C_{\lambda;\gamma} : V(\lambda + \gamma) \to V(\lambda) \otimes V(\gamma)$ be the $\mathfrak{g}$-intertwiner which sends the highest weight vector $v_{\lambda+\gamma}$ to $v_\lambda \otimes v_\gamma$. This operation also makes sense on integrable $\hat{\mathfrak{g}}$-representations, so by abuse of notation $C_{\lambda;\gamma} : \mathcal{H}(\lambda+\gamma, L+K) \to \mathcal{H}(\lambda, L) \otimes \mathcal{H}(\gamma, K)$ is the map defined in the same way (see [Kum87, 1.6]). Notice that these maps coincide under the restriction to the subspace $V(\lambda+\gamma) \subset \mathcal{H}(\lambda+\gamma, L+K)$. Furthermore, the inclusions $i : V(\lambda) \to \mathcal{H}(\lambda, L)$ and $\mathfrak{g} \subset \hat{\mathfrak{g}}[C, \vec{p}]$ induce the correlation maps $F : \mathcal{V}^{\dagger}_{C,\vec{p}}(\vec{\lambda}, L) \to [V(\vec{\lambda})]^{\mathfrak{g}}$. We let $\bar{I}$ denote the natural inclusion $\bar{I} : \mathcal{V}^{\dagger}_{C,\vec{p}}(\vec{\lambda}, L) \to \mathcal{H}_S(\vec{\lambda}, L)^*$, and I denote the inclusion $I : [V_S(\vec{\lambda})^*]^{\mathfrak{g}(S)} \to V_S(\vec{\lambda})^*$. As a consequence of these definitions the following diagram commutes:

$$\begin{tikzcd} \mathcal{V}^{\dagger}_{C,\vec{p}}(\vec{\lambda}, L) \arrow[r, "\bar{I}"] & \mathcal{H}_S(\vec{\lambda}, L)^* \\ {[V_S(\vec{\lambda})^*]^{\mathfrak{g}(S)}} \arrow[u, "F"] \arrow[r, "I"] & V_S(\vec{\lambda})^* \arrow[u, "\vec{i}^*"'] \end{tikzcd}$$

The map $C^*_{\vec{\lambda};\vec{\gamma}} : \mathcal{H}_S(\vec{\lambda}, L)^* \otimes_{\mathcal{O}_S} \mathcal{H}_S(\vec{\gamma}, K)^* \to \mathcal{H}_S(\vec{\lambda}+\vec{\gamma}, K+L)^*$ formed by dualizing $C_{\vec{\lambda};\vec{\gamma}}$ is the global section multiplication operation on the projective coordinate rings of Kac–Moody flag varieties by the Kac–Moody version of the Borel–Bott–Weil theorem of Kumar [Kum87]. As a consequence, this map defines a commutative and associative multiplication on the direct sum of the $\mathcal{H}_S(\vec{\lambda}, L)^*$, and restricts to such an operation on the sheaves of vacua. We define $C^*_{\vec{\lambda};\vec{\gamma}} : [V_S(\vec{\lambda})]^{\mathfrak{g}(S)} \otimes [V_S(\vec{\gamma})]^{\mathfrak{g}(S)} \to [V_S(\vec{\lambda} + \vec{\gamma})]^{\mathfrak{g}(S)}$ using the same recipe. The following lemma is then immediate.

Lemma 2.2. *The multiplication maps $C^*_{\vec{\lambda};\vec{\gamma}}$ commute with the correlation map F.*

When the genus g is 0, the map F is a monomorphism by an observation of Tsuchiya, Ueno and Yamada [TUY89]. Everything here commutes with specialization, so we also obtain the following diagram:

$$\begin{tikzcd} \mathcal{V}^{\dagger}_{\pi^{-1}(s),\vec{p}(s)}(\vec{\lambda}, L) \otimes \mathcal{V}^{\dagger}_{\pi^{-1}(s),\vec{p}(s)}(\vec{\gamma}, K) \arrow[r, "C^*_{\vec{\lambda};\vec{\gamma}}"] \arrow[d, "F\otimes F"'] & \mathcal{V}^{\dagger}_{\pi^{-1}(s),\vec{p}(s)}(\vec{\lambda}+\vec{\gamma}, L+K) \arrow[d, "F"] \\ {[V(\vec{\lambda})^*]^{\mathfrak{g}} \otimes [V(\vec{\gamma})^*]^{\mathfrak{g}}} \arrow[r, "C^*_{\vec{\lambda};\vec{\gamma}}"] & {[V(\vec{\gamma}+\vec{\lambda})^*]^{\mathfrak{g}}} \end{tikzcd}$$

This multiplication operation also works well with the alternative constructions of the spaces of conformal blocks given in [Bea96]. We fix an $n+1$-marked curve $(C, \vec{p}, q)$. By identifying highest weight vectors, we get the following diagram of $\mathfrak{g}$-representations:

$$V(0) \otimes \mathcal{H}(\vec{\lambda}, L) \to \mathcal{H}(0, L) \otimes \mathcal{H}(\vec{\lambda}, L) \leftarrow \mathcal{H}(0, L) \otimes V(\vec{\lambda}).$$

The space on the left is a $\hat{\mathfrak{g}}[C, \vec{p}]$-representation, the middle is a $\hat{\mathfrak{g}}[C, \vec{p}, q]$-representation, and the space on the right is a $\hat{\mathfrak{g}}[C, q]$-representation where the action on $V(\lambda_i)$ is by evaluation at p_i. The following is a ringification of a result which appears in [Bea96].

Proposition 2.3. *Let C be a stable curve. The following are isomorphisms of algebras over $\mathbb{C}$:*

$$\bigoplus_{\vec{\lambda},L} \mathcal{V}^{\dagger}_{C,\vec{p}}(\vec{\lambda}, L) \to \bigoplus_{\vec{\lambda},L} \mathcal{V}^{\dagger}_{C,\vec{p},q}(0, \vec{\lambda}, L) \leftarrow \bigoplus_{\vec{\lambda},L} [\mathcal{H}(0, L) \otimes V(\vec{\lambda})]^{\mathfrak{g}[C,q]}.$$

Proof. By a theorem in [Bea96] the morphism on the right is an isomorphism of vector spaces, and by *vacuum propagation* (see [TUY89], [SU99], [Bea96] and [NT05]) the morphism on the left is also an isomorphism of vector spaces. Both maps are defined by identifying highest weight vectors, then dualizing; this gives a diagram of rings, with graded components,

$$[V(0)\otimes \mathcal{H}(\vec{\lambda})]^* \leftarrow [\mathcal{H}(0,L)\otimes \mathcal{H}(\vec{\lambda},L)]^* \rightarrow [\mathcal{H}(0,L)\otimes V(\vec{\lambda})]^*.$$

Taking Lie algebra invariants picks out subspaces which are preserved by multiplication. □

Beauville's result gives the identification $\mathcal{V}^{\dagger}_{C,\vec{p}}(\vec{\lambda},L) = [\mathcal{H}(0,L)^*\otimes V(\vec{\lambda})^*]^{\hat{\mathfrak{g}}[C,q]}$ (see also [LS97]).

2.3. *Moduli of quasi-parabolic principal bundles*

For what follows we refer the reader to the work of Kumar, Kumar–Narasimhan–Ramanathan, Laszlo–Sorger, and Pauly [Kum87], [KNR94], [LS97], [Sor99], [Pau96]. The theorem below can be found in [KNR94], [LS97] and in [Sor99] for the exceptional groups.

Theorem 2.4. *The moduli stack $\mathcal{M}_{C,\vec{p}}(\vec{\Lambda})$ of quasi-parabolic G-bundles on C smooth, with parabolic structure $\vec{\Lambda}$ at the marked points, carries a line bundle $\mathcal{L}(\vec{\lambda},L)$, where λ_i is a dominant weight in the face of Δ associated to Λ_i. The global sections of this line bundle are identified with a space of conformal blocks:*

$$H^0(\mathcal{M}_{C,\vec{p}}(\vec{\Lambda}),\mathcal{L}(\vec{\lambda},L)) \cong [\mathcal{H}(0,L)^*\otimes V(\vec{\lambda}^*)]^{\mathfrak{g}[C,q]}. \tag{14}$$

The stack $\mathcal{M}_{C,\vec{p}}(\vec{\Lambda})$ is obtained as a quotient of the ind-variety $Q\times G/\Lambda_1\times\cdots\times G/\Lambda_n$ by the ind-group $G(\mathbb{C}[C\setminus q])$ for $q\in C$, where Q is the affine Grassmannian variety. This space is constructed as a quotient $Q = L(G)/L^+(G)$, where $L(G)$ is the loop group of G. Let $\hat{\mathcal{O}}_q$ be the formal completion of the local ring at q, and let $\mathfrak{k}_q$ be the quotient field of $\hat{\mathcal{O}}_q$. Then $L(G) = G(\mathfrak{k}_q)$, and $L^+(G) = G(\hat{\mathcal{O}}_q)$. The space $Q\times G/\Lambda_1\times\cdots\times G/\Lambda_n$ carries line bundles $L(L,\vec{\lambda})$ with global section spaces equal to $\mathcal{H}(0,L)^*\otimes V(\vec{\lambda}^*)$. By the Borel–Bott–Weil theorem in the Kac–Moody setting proved by Kumar [Kum87], multiplication of global sections is computed with the maps $C^*_{\lambda;\gamma}$ from the previous subsection.

Proposition 2.5. *For $(C,\vec{p})\in\mathcal{M}_{g,n}\subset\bar{\mathcal{M}}_{g,n}$ there is a monomorphism of multigraded rings*

$$h_{\vec{\Lambda}}:\mathrm{Cox}(\mathcal{M}_{C,\vec{p}}(\vec{\Lambda}))\rightarrow \mathcal{V}^{\dagger}_{C,\vec{p}}(G). \tag{15}$$

The image of this monomorphism is the direct sum of the conformal blocks $\mathcal{V}^{\dagger}_{C,\vec{p}}(\vec{\lambda},L)$ with λ_i a dominant weight in the face of Δ associated to Λ_i. This is an isomorphism when all Λ_i are Borel subgroups.

Proof. As mentioned in the introduction, in [LS97] Laszlo and Sorger identify the Picard group of $\mathcal{M}_{C,\vec{p}}(\vec{\Lambda})$:

$$\mathrm{Pic}(\mathcal{M}_{C,\vec{p}}(\vec{\Lambda})) = \mathcal{X}(\Lambda_1)\times\cdots\times\mathcal{X}(\Lambda_n)\times\mathbb{Z}, \tag{16}$$

where $\mathcal{X}(\Lambda_i)$ is the character group of Λ_i. For any line bundle $\mathcal{L}(\vec{\lambda},L)$ on $\mathcal{M}_{C,\vec{p}}(\vec{\Lambda})$ there is an isomorphism between the sections of $\mathcal{L}(\vec{\lambda},L)$ and the $G(\mathbb{C}[C\setminus q])$-equivariant sections of the pullback bundle on $Q\times G/\Lambda_1\times\cdots\times G/\Lambda_n$ by a standard theorem on quotient stacks [LS97]. By Borel–Bott–Weil (standard and Kac–Moody versions), such a line bundle is effective only if each λ_i is dominant and L is non-negative. In this case we have concluded above that the global sections are spaces of conformal blocks and that the multiplication operation on the section spaces is computed by the multiplication operation in $\mathcal{V}^{\dagger}_{C,\vec{p}}(G)$. □

3. Filtrations of the algebra of conformal blocks

In the previous section we built the flat sheaf $\mathcal{V}^{\dagger}(G)$ of algebras. This allows us to relate $\mathrm{Cox}(\mathcal{M}_{C,\vec{p}}(\vec{\Lambda}))$ to the algebra $\mathcal{V}^{\dagger}_{C',\vec{q}}(G)$ for $(C',\vec{q})$ a singular curve. In this section we use the factorization map of Tsuchiya–Ueno–Yamada to define a degeneration $\mathcal{V}^{\dagger}_{C,\vec{p}}(G) \Rightarrow [\bigotimes_{v\in V(\Gamma)} \mathcal{V}^{\dagger}_{0,3}(G)]^{T_\Gamma}$ for a singular curve $(C,\vec{p})$ of type Γ, and prove Theorem 1.1. We begin with a discussion of the factorization isomorphism. For each dominant weight $\lambda \in \Delta$ and its dual λ^*, and a choice of highest weight vector $v_\lambda \in V(\lambda)$ and lowest weight vector $\hat{v}_{\lambda^*} \in V(\lambda^*)$, let $F_\lambda : V(\lambda)\otimes V(\lambda^*) \to \mathbb{C}$ be the unique equivariant map such that $F_\lambda(v_\lambda \otimes \hat{v}_{\lambda^*}) = 1$. The map F_λ gives an isomorphism

$$\mathrm{Hom}_{\mathbb{C}}(V(\lambda), V(\lambda)) \cong V(\lambda)\otimes V(\lambda^*), \tag{17}$$

where $\sum_i x_i \otimes y_i$ acts on $v \in V(\lambda)$ as $\sum_i x_i \otimes F_\lambda(y_i \otimes v)$. Let $O_{\lambda,\lambda^*} \in V(\lambda)\otimes V(\lambda^*)$ represent the identity under this isomorphism. This element defines a $\mathfrak{g}$-linear map

$$V(\vec{\lambda}) \xrightarrow{\rho_\alpha} V(\vec{\lambda}) \otimes V(\alpha) \otimes V(\alpha^*),$$

which sends X to $X \otimes O_{\alpha,\alpha^*}$. The map ρ_α also makes sense for integrable highest weight representations of $\hat{\mathfrak{g}}$, and we can define $\rho_\alpha : \mathcal{H}(\vec{\lambda}, L) \to \mathcal{H}(\vec{\lambda}, \alpha, \alpha^*, L) = \mathcal{H}(\vec{\lambda}, L) \otimes \mathcal{H}(\alpha, \alpha^*, L)$.

We fix a stable curve C, with singular point $q \in C$, and we let $\tilde{C}$ be the partial normalization of C at q with two new marked points $q_1, q_2 \in \tilde{C}$. The modules $\mathcal{H}(\vec{\lambda}, L)$ and $\mathcal{H}(\vec{\lambda}, \alpha, \alpha^*, L)$ are viewed as representations of $\hat{\mathfrak{g}}[C, \vec{p}]$ and $\hat{\mathfrak{g}}[\tilde{C}, \vec{p}, q_1, q_2]$, respectively. Taking dual spaces, and then invariants by these Lie algebras, yields the following map, which is shown to be injective in [TUY89]:

$$\mathcal{V}^{\dagger}_{C,\vec{p}}(\vec{\lambda}, L) \xleftarrow{\hat{\rho}_\alpha} \mathcal{V}^{\dagger}_{\tilde{C},\vec{p},q_1,q_2}(\vec{\lambda}, \alpha, \alpha^*, L)$$

This operation can be performed with any finite number of nodal singular points $\vec{q}$. Summing over all $\vec{\alpha} \in \Delta_L^m$ gives the factorization isomorphism. As we show below, the extra dominant weight data $\vec{\alpha} \in \Delta_L^m$ brought out by factorization defines a filtration on $\mathcal{V}^{\dagger}_{C,\vec{p}}(G)$.

Consider the tensor product decomposition

$$V(\alpha)\otimes V(\beta) \cong \bigoplus W^{\eta}_{\alpha,\beta} \otimes V(\eta), \tag{18}$$

where $W^{\eta}_{\alpha,\beta} = \mathrm{Hom}_{\mathfrak{g}}(V(\eta), V(\alpha)\otimes V(\beta))$. We have the following identities:

$$\begin{aligned} V(\alpha)\otimes V(\beta)\otimes V(\alpha^*)\otimes V(\beta^*) &\cong \mathrm{Hom}_{\mathbb{C}}(V(\alpha)\otimes V(\beta), V(\alpha)\otimes V(\beta)) \\ &\cong \mathrm{Hom}_{\mathbb{C}}\Big(\bigoplus W^{\eta}_{\alpha,\beta}\otimes V(\eta), \bigoplus W^{\eta}_{\alpha,\beta}\otimes V(\eta)\Big) \\ &\cong \Big[\bigoplus W^{\eta}_{\alpha,\beta}\otimes V(\eta)\Big] \otimes \Big[\bigoplus W^{\eta^*}_{\alpha^*,\beta^*}\otimes V(\eta^*)\Big]. \end{aligned} \tag{19}$$

For any two maps $f \otimes g \in W^{\eta}_{\alpha,\beta} \otimes W^{\eta^*}_{\alpha^*,\beta^*}$, the map $F_\alpha \otimes F_\beta \circ (f \otimes g) : V(\eta) \otimes V(\eta^*) \to \mathbb{C}$ must be a multiple $F^{\eta}_{\alpha,\beta}(f, g)$ of F_η. This assignment defines a bilinear map $F^{\eta}_{\alpha,\beta} : W^{\eta}_{\alpha,\beta} \otimes W^{\eta^*}_{\alpha^*,\beta^*} \to \mathbb{C}$. We let $I^{\eta}_{\alpha,\beta}$ represent the identity map under the induced isomorphism $W^{\eta}_{\alpha,\beta} \otimes W^{\eta^*}_{\alpha^*,\beta^*} = \mathrm{Hom}(W^{\eta}_{\alpha,\beta}, W^{\eta}_{\alpha,\beta})$. By definition we have $F_\alpha \otimes F_\beta = \sum F^{\eta}_{\alpha,\beta} \otimes F_\eta$, and therefore

$$O_{\alpha,\alpha^*} \otimes O_{\beta,\beta^*} = \sum I^{\eta}_{\alpha,\beta} \otimes O_{\eta,\eta^*}. \tag{20}$$

We let $f_\eta : W^{\eta}_{\alpha,\beta} \otimes V(\eta) \to V(\alpha) \otimes V(\beta)$ denote the inclusion defined by the direct sum decomposition, with $f_{\eta^*} : W^{\eta^*}_{\alpha^*,\beta^*} \otimes V(\eta^*) \to V(\alpha^*) \otimes V(\beta^*)$ the corresponding maps on the dual representations. By our choices above, we have $f_{\alpha+\beta} = C_{\alpha;\beta}$, and it is a straightforward calculation to verify that $f_{\alpha^*+\beta^*} = C_{\alpha^*;\beta^*}$. We recall the Verma $\hat{\mathfrak{g}}$-module $\bar{V}(\lambda, L)$ from [Bea96, 1.6] and [Kum87, 1.5]. The f_η give maps

$$W^{\eta}_{\alpha,\beta} \otimes \bar{V}(\eta, K+L) \xrightarrow{\bar{f}_\eta} \mathcal{H}(\alpha, L) \otimes \mathcal{H}(\beta, K),$$

and the identity (20) above implies that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{H}(\vec{\lambda}, L) \otimes \mathcal{H}(\vec{\gamma}, K) \arrow[r, "\rho_\alpha \otimes \rho_\beta"] & {[\mathcal{H}(\vec{\lambda}, L) \otimes \mathcal{H}(\alpha, \alpha^*, K)] \otimes [\mathcal{H}(\vec{\gamma}, K) \otimes \mathcal{H}(\beta, \beta^*, L)]} \\
\mathcal{H}(\vec{\lambda}+\vec{\gamma}, K+L) \arrow[u, "C_{\vec{\lambda};\vec{\gamma}}"] \arrow[r, "\sum_\eta (I^{\eta}_{\alpha,\beta} \otimes \rho_\eta)"] & \bigoplus_\eta \mathcal{H}(\vec{\lambda}+\vec{\gamma}, K+L) \otimes W^{\eta}_{\alpha,\beta} \otimes W^{\eta^*}_{\alpha^*,\beta^*} \otimes \bar{V}(\eta, K+L) \otimes \bar{V}(\eta^*, K+L) \arrow[u, "\sum_\eta C_{\vec{\lambda};\vec{\gamma}} \otimes \bar{f}_\eta \otimes \bar{f}_{\eta^*}"']
\end{tikzcd} \tag{21}$$

Here the sum is over all dominant weights η which are smaller than $\alpha + \beta$ in the dominant weight ordering. The bottom map sends a vector Y to $\sum Y \otimes I^{\eta}_{\alpha,\beta} \otimes O_{\eta,\eta^*}$, so we replace $\bar{f}_\eta \otimes \bar{f}_{\eta^*}$ with $\phi_{\eta,\eta^*}(X) = \bar{f}_\eta \otimes \bar{f}_{\eta^*}(I^{\eta}_{\alpha,\beta} \otimes X)$. For $\eta = \alpha + \beta$, by definition,

$$\bar{f}_{\alpha+\beta} \otimes \bar{f}_{\alpha^*+\beta^*} = \phi_{\alpha+\beta,\alpha^*+\beta^*} = C_{\alpha,\alpha^*;\beta,\beta^*}. \tag{22}$$

Proposition 3.1. *The following diagram commutes:*

$$\begin{tikzcd}
\mathcal{V}^{\dagger}_{C,\vec{p}}(\vec{\lambda}, L) \otimes \mathcal{V}^{\dagger}_{C,\vec{p}}(\vec{\gamma}, K) \arrow[d, "C^*_{\vec{\lambda};\vec{\gamma}}"'] & \mathcal{V}^{\dagger}_{\tilde{C},\vec{p},q_1,q_2}(\vec{\lambda}, \alpha, \alpha^*, L) \otimes \mathcal{V}^{\dagger}_{\tilde{C},\vec{p},q_1,q_2}(\vec{\gamma}, \beta, \beta^*, K) \arrow[l, "\hat{\rho}_\alpha \otimes \hat{\rho}_\beta"'] \arrow[d, "{[\sum_\eta C_{\vec{\lambda};\vec{\gamma}} \otimes \phi_{\eta,\eta^*}]^*}"] \\
\mathcal{V}^{\dagger}_{C,\vec{p}}(\vec{\lambda}+\vec{\gamma}, L+K) & \bigoplus_\eta \mathcal{V}^{\dagger}_{\tilde{C},\vec{p},q_1,q_2}(\vec{\lambda}+\vec{\gamma}, \eta, \eta^*, L+K) \arrow[l, "\sum_\eta \hat{\rho}_\eta"']
\end{tikzcd}$$

Proof. We may dualize diagram (21) to obtain

$$\begin{tikzcd}
{[\mathcal{H}(\vec{\lambda}, L)]^* \otimes [\mathcal{H}(\vec{\gamma}, K)]^*} \arrow[d, "C^*_{\vec{\lambda}+\vec{\gamma}}"'] & {[\mathcal{H}(\vec{\lambda}, L) \otimes \mathcal{H}(\alpha, \alpha^*, K)]^* \otimes [\mathcal{H}(\vec{\gamma}, K) \otimes \mathcal{H}(\beta, \beta^*, L)]^*} \arrow[l, "\hat{\rho}_\alpha \otimes \hat{\rho}_\beta"'] \arrow[d, "{[\sum_\eta C_{\vec{\lambda};\vec{\gamma}} \otimes \phi_{\eta,\eta^*}]^*}"] \\
{[\mathcal{H}(\vec{\lambda}+\vec{\gamma}, K+L)]^*} & \bigoplus_\eta [\mathcal{H}(\vec{\lambda}+\vec{\gamma}, K+L) \otimes \bar{V}(\eta, K+L) \otimes \bar{V}(\eta^*, K+L)]^* \arrow[l, "\sum_\eta \hat{\rho}_\eta"']
\end{tikzcd}$$

We must determine what happens to the spaces of conformal blocks inside each of the vector spaces in this diagram. Here $C^*_{\vec{\lambda}+\vec{\gamma}}$ maps $\mathcal{V}^\dagger_{C,\vec{p}}(\vec{\lambda}, L) \otimes \mathcal{V}^\dagger_{C,\vec{p}}(\vec{\gamma}, K)$ to $\mathcal{V}^\dagger_{C,\vec{p}}(\vec{\lambda}+\vec{\gamma}, K + L)$ and $[\sum_\eta C_{\vec{\lambda};\vec{\gamma}} \otimes \phi_{\eta,\eta}]^*$ maps $\mathcal{V}^\dagger_{\tilde{C},\vec{p},q_1,q_2}(\vec{\lambda}, \alpha, \alpha^*, K) \otimes \mathcal{V}^\dagger_{C,\vec{p},q_1,q_2}(\vec{\gamma}, \beta, \beta^*, L)$ to $[\mathcal{H}(\vec{\lambda} + \vec{\gamma}, L + K) \otimes \bar{V}(\eta) \otimes \bar{V}(\eta^*)]^{\mathfrak{g}(C\setminus\vec{p},q_1,q_2)}$, respectively, because both maps are obtained by dualizing and taking invariants. Furthermore, the map $\hat{\rho}_\alpha \otimes \hat{\rho}_\beta$ sends vectors from $\mathcal{V}^\dagger_{\tilde{C},\vec{p},q_1,q_2}(\vec{\lambda}, \alpha, \alpha^*, K) \otimes \mathcal{V}^\dagger_{C,\vec{p},q_1,q_2}(\vec{\gamma}, \beta, \beta^*, L)$ into $\mathcal{V}^\dagger_{C,\vec{p}}(\vec{\lambda}, L) \otimes \mathcal{V}^\dagger_{C,\vec{p}}(\vec{\gamma}, K)$ by the factorization theorem.

In order to analyze the bottom arrow of the diagram, we first consider the projection map,

$$\pi_{\eta,K+L} : \bar{V}(\eta, K + L) \to \mathcal{H}(\eta, K + L) \tag{23}$$

of highest weight $\hat{\mathfrak{g}}$-modules. The map $[\mathrm{Id} \otimes \pi_{\eta,K+L} \otimes \pi_{\eta^*,K+L}]^* \circ \hat{\rho}_\eta$, which takes $[\mathcal{H}(\vec{\lambda} + \vec{\gamma}, K + L) \otimes \mathcal{H}(\eta, K + L) \otimes \mathcal{H}(\eta^*, K + L)]^*$ to $[\mathcal{H}(\vec{\lambda} + \vec{\gamma}, K + L)]^*$, is by definition equal to the map used in the proof of the factorization rules. This implies that it takes the space $\mathcal{V}^\dagger_{\tilde{C},\vec{p},q_1,q_2}(\vec{\lambda}+\vec{\gamma}, \eta, \eta^*, L+K)$ to $\mathcal{V}^\dagger_{C,\vec{p}}(\vec{\lambda}+\vec{\gamma}, L+K)$. The picture is then completed by a theorem of Beauville [Bea96, proof of Proposition 2.3], which asserts the following equality for any smooth curve $\tilde{C}$, induced by the maps $\pi_{\eta,K+L}$:

$$\begin{aligned} V^\dagger_{\hat{\mathfrak{g}}[\tilde{C},\vec{p},\vec{q}]}[\mathcal{H}(\vec{\alpha}, L) \otimes \bar{V}(\vec{\beta}, L)] &\cong V^\dagger_{\hat{\mathfrak{g}}[\tilde{C},\vec{p},\vec{q}]}[\mathcal{H}(\vec{\alpha}, L) \otimes \mathcal{H}(\vec{\beta}, L)] \\ &= \mathcal{V}^\dagger_{\tilde{C},\vec{p},\vec{q}}(\vec{\alpha}, \vec{\beta}, L). \end{aligned} \tag{24}$$

□

The diagram of Proposition 3.1 only represents the case of a curve with one singularity, but the general case follows by the same methods. For any tensor product of elements $\chi_1 \otimes \chi_2 \in \mathcal{V}^\dagger_{\tilde{C},\vec{p},q_1,q_2}(\vec{\lambda}, \alpha, \alpha^*, L) \otimes \mathcal{V}^\dagger_{\tilde{C},\vec{p},q_1,q_2}(\vec{\gamma}, \beta, \beta^*, K)$, the multiplication $\chi_1 \times \chi_2$ can be expanded as follows:

$$\chi_1 \times \chi_2 = C^*_{\vec{\lambda};\vec{\gamma}}(\hat{\rho}_\alpha(\chi_1) \otimes \hat{\rho}_\beta(\chi_2)) = \hat{\rho}_{\alpha+\beta} \circ C^*_{\vec{\lambda};\vec{\gamma},\alpha,\beta,\alpha^*,\beta^*}(\chi_1 \otimes \chi_2) + \sum \chi_\eta. \tag{25}$$

Recall that there is a partial ordering $\prec$ on dominant weights such that $\lambda \prec \gamma$ if and only if $\gamma - \lambda$ is a sum of positive roots. The χ_η in the expansion of $\chi_1 \times \chi_2$ are the summands from components of the direct sum with $\eta \prec \alpha + \beta$ as dominant weights, in particular multiplication in $\mathcal{V}^\dagger_{C,\vec{p}}$ is multiplication in $\mathcal{V}^\dagger_{\tilde{C},\vec{p},q_1,q_2}(G)$ with additional "lower" terms. We use this observation to build an algebra filtration on $\mathcal{V}^\dagger_{C,\vec{p}}(G)$. Recall that any coweight θ in the dual Weyl chamber $\Delta^\vee$ of Δ has the property that $\theta(\gamma) \geq \theta(\lambda)$ when $\lambda \prec \gamma$, and furthermore this inequality is strict when θ is taken from the interior of $\Delta^\vee$.

Definition 3.2. We define a *coweighting* of the normalized curve $(\tilde{C}, \vec{p}, \vec{q})$ to be an assignment of coweights from the dual Weyl chamber $\theta \in \Delta^\vee$ to the new marked points $\vec{q}$, such that identified points q_1, q_2 are assigned dual coweights (see Figure 4). We define a *weighting* of $(\tilde{C}, \vec{p}, \vec{q})$ analogously.

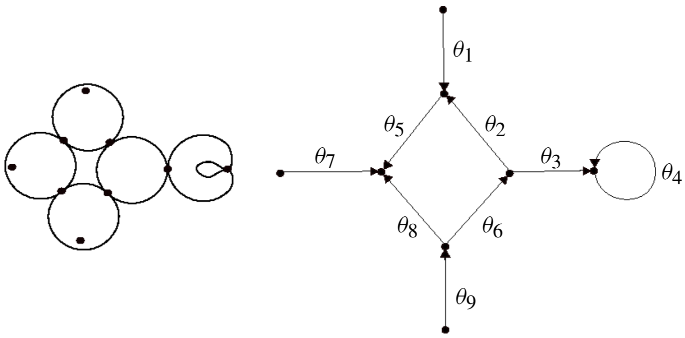

Fig. 4. A coweighting.

Coweightings and coweightings of $(\tilde{C}, \vec{p}, \vec{q})$ can be visualized as assignments of coweights to the non-leaf edges of the graph Γ which labels the stratum of $\bar{\mathcal{M}}_{g,n}$ containing C. For any non-leaf edge $e \in E(\Gamma)$ there are two points $q_1, q_2 \in \vec{q}$, and therefore two dual weights α, α^* assigned to this edge. We can encode this data by specifying an orientation $q_1 \to q_2$ for each $e \in E(\Gamma)$ and recording one of the weights; it is then understood that the weight recorded is that on q_1, and the weight on q_2 is obtained by duality. Now we can denote these objects by a pair $(\Gamma, \vec{\theta})$, where Γ is the graph with a choice of orientation, and $\vec{\theta}$ is an assignment of a single coweight (respectively weight) to each non-leaf edge in Γ. By extending the natural pairing between weights and coweights bilinearly we can view coweightings of $(\tilde{C}, \vec{p}, \vec{q})$ as linear functions on weightings of $(\tilde{C}, \vec{p}, \vec{q})$:

$$(\Gamma, \vec{\theta}) \circ (\Gamma, \vec{\alpha}) = \sum_{e \in \mathrm{Edge}(\Gamma)} \theta_e(\alpha_e). \tag{26}$$

Now we describe a class of filtrations on the algebra of conformal blocks. Fix a partial normalization $(\tilde{C}, \vec{p}, \vec{q})$ of $(C, \vec{p})$, and choose once and for all an orientation of the edges of the associated Γ. Let $\mathcal{V}^{\dagger}_{C,\vec{p}}(\vec{\lambda}, L)_{\vec{\alpha}} \subset \mathcal{V}^{\dagger}_{C,\vec{p}}(\vec{\lambda}, L)$ denote the subspace $\mathcal{V}^{\dagger}_{\tilde{C},\vec{p},\vec{q}}(\vec{\lambda}, \vec{\alpha}, \vec{\alpha}^*, L)$ obtained by factorization. Notice that there is a natural weighting of $(\tilde{C}, \vec{p}, \vec{q})$ associated to each of these subspaces, namely $(\Gamma, \vec{\alpha})$. Now, by choosing a coweighting $(\Gamma, \vec{\theta})$, and using the pairing above, we obtain a filtration $\mathcal{F}_{\Gamma,\vec{\theta}}$ on $\mathcal{V}^{\dagger}_{C,\vec{p}}(G)$ by giving $\chi \in \mathcal{V}^{\dagger}_{C,\vec{p}}(\vec{\lambda}, L)_{\vec{\alpha}}$ filtration level $(\Gamma, \vec{\theta}) \circ (\Gamma, \vec{\alpha})$.

By considering equation (25) above, we see that the product $\chi_1 \times \chi_2$ and $\hat{\rho}_{\alpha+\beta} \circ C^*_{\vec{\lambda},\alpha,\alpha^*;\vec{\gamma},\beta,\beta^*}(\chi_1 \otimes \chi_2)$ always have the same filtration level, whereas the filtration level of a lower summand χ_η is always less than or equal to this value. For coweightings where all coweights have been chosen in the interior of the dual Weyl chamber of $\mathfrak{g}$, the terms χ_η are always given a strictly smaller filtrational level than $\chi_1 \times \chi_2$.

Proposition 3.3. *The filtration $\mathcal{F}_{\Gamma,\vec{\theta}}$ for $(\Gamma, \vec{\theta})$ respects multiplication on the ring $\mathcal{V}^{\dagger}_{C,\vec{p}}(G)$. If the components of $\vec{\theta}$ are strictly positive on all positive roots, then the image of the associated graded multiplication map*

$$\mathcal{V}^{\dagger}_{C,\vec{p}}(\vec{\lambda}, L)_{\vec{\alpha}} \otimes \mathcal{V}^{\dagger}_{C,\vec{p}}(\vec{\gamma}, K)_{\vec{\beta}} \\ \to \mathcal{F}^{\leq \vec{\theta}(\vec{\lambda}, \vec{\alpha}+\vec{\beta}, \vec{\alpha}^*+\vec{\beta}^*)}_{\Gamma,\vec{\theta}}(\mathcal{V}^{\dagger}_{C,\vec{p}}(G))/\mathcal{F}^{< \vec{\theta}(\vec{\lambda}, \vec{\alpha}+\vec{\beta}, \vec{\alpha}^*+\vec{\beta}^*)}_{\Gamma,\vec{\theta}}(\mathcal{V}^{\dagger}_{C,\vec{p}}(G)) \tag{27}$$

can be canonically identified with $\mathcal{V}^{\dagger}_{C,\vec{p}}(\vec{\lambda}+\vec{\gamma}, K+L)_{\vec{\alpha}+\vec{\beta}}$. *Moreover this map coincides with multiplication in the algebra* $\mathcal{V}^{\dagger}_{\tilde{C},\vec{p},\vec{q}}(G)$.

Proof. This all follows from Proposition 3.1 above and the observation that $\chi_1 \times \chi_2$ and $\hat{\rho}_{\vec{\alpha}+\vec{\beta}} \circ C^*_{\vec{\lambda};\vec{\gamma},\vec{\alpha}+\vec{\beta},\vec{\alpha}^*+\vec{\beta}^*}(\chi_1 \otimes \chi_2)$ always have the same filtration level. □

Proposition 3.3 shows that if the coweighting $(\Gamma, \vec{\theta})$ is generic, the associated graded algebra $\mathrm{gr}_{\mathcal{F}_{\Gamma,\vec{\theta}}}(\mathcal{V}^{\dagger}_{C,\vec{p}})$ is isomorphic to the subalgebra of $\mathcal{V}^{\dagger}_{\tilde{C},\vec{p},\vec{q}}(G)$ formed by the conformal block spaces $\mathcal{V}^{\dagger}_{\tilde{C},\vec{p},\vec{q}}(\vec{\lambda}, \vec{\alpha}, \vec{\alpha}^*, L)$—namely those with dual dominant weights assigned to paired points q_1, q_2 , and all levels L on different connected components of $\tilde{C}$ equal. This subalgebra can be realized as the algebra of invariants with respect to a certain torus. For each oriented edge $e \in E(\Gamma)$ (respectively, ordered pair of points $q_1, q_2 \in \tilde{C}$ associated to a normalized singularity) there is an action of a maximal torus $T \subset G$ on $\mathcal{V}^{\dagger}_{\tilde{C},\vec{p},\vec{q}}(G)$ whose character on the subspace $\mathcal{V}^{\dagger}_{\tilde{C},\vec{p},\vec{q}}(\vec{\lambda}, \vec{\alpha}, L)$ is the difference $\alpha_1 - \alpha_2^*$, where α_i is the label of q_i. Similarly, if $\tilde{C}$ has multiple connected components, there is an action of $\mathbb{C}^*$ for each edge $e \in E(\Gamma)$ whose character is the difference between the levels on the spaces of conformal blocks associated to the components connected by e. We let T_Γ be the product of all such $T \times \mathbb{C}^*$. The algebra $(\mathcal{V}^{\dagger}_{\tilde{C},\vec{p},\vec{q}}(G))^{T_\Gamma}$ of invariants is then the required sum of spaces of conformal blocks.

Next we show that the associated graded algebra $\mathrm{gr}_{\mathcal{F}_{\Gamma,\vec{\theta}}}(\mathcal{V}^{\dagger}_{C,\vec{p}})$ can be realized as a flat degeneration of $\mathcal{V}^{\dagger}_{C,\vec{p}}$. We may form the Reese algebra $R_{(\Gamma,\vec{\theta})}[\mathcal{V}^{\dagger}_{C,\vec{p}}(G)] \subset \mathcal{V}^{\dagger}_{C,\vec{p}}(G)[t]$, defined as

$$R_{(\Gamma,\vec{\theta})}[\mathcal{V}^{\dagger}_{C,\vec{p}}(G)] = \bigoplus_{N \geq 0} \mathcal{F}^{\leq N}_{\Gamma,\vec{\theta}}[\mathcal{V}^{\dagger}_{C,\vec{p}}(G)]. \tag{28}$$

The Reese algebra of a filtration is a standard construction in commutative algebra; it is naturally a $\mathbb{C}[t]$-algebra, where t acts by sending an element in filtration level N to its identical copy in level $N+1$. The following properties are standard (see e.g. [AB04]):

- $R_{(\Gamma,\vec{\theta})}[\mathcal{V}^{\dagger}_{C,\vec{p}}(G)]$ is flat over $\mathbb{C}[t]$,
- $\frac{1}{t} R_{(\Gamma,\vec{\theta})}[\mathcal{V}^{\dagger}_{C,\vec{p}}(G)] \cong \mathcal{V}^{\dagger}_{C,\vec{p}}(G)[t, 1/t]$,
- $R_{(\Gamma,\vec{\theta})}[\mathcal{V}^{\dagger}_{C,\vec{p}}(G)]/t$ is the associated graded algebra $(\mathcal{V}^{\dagger}_{\tilde{C},\vec{p},\vec{q}}(G))^{T_\Gamma}$.

This construction gives a flat degeneration with generic fiber $\mathcal{V}^{\dagger}_{C,\vec{p}}(G)$, and special fiber equal to the subalgebra $(\mathcal{V}^{\dagger}_{\tilde{C},\vec{p},\vec{q}}(G))^{T_\Gamma}$ of $\mathcal{V}^{\dagger}_{\tilde{C},\vec{p},\vec{q}}(G)$. This proves Theorems 1.7 and 1.1.

Remark 3.4. The coordinate ring of the group G has the following decomposition into isotypical $G \times G$ components (see [Gro97, Theorem 12.9]):

$$\mathbb{C}[G] = \bigoplus_{\lambda \in \Delta} V(\lambda, \lambda^*). \tag{29}$$

The maps ϕ_{η,η^*} are the components of the dual of the multiplication map $m : \mathbb{C}[G] \otimes \mathbb{C}[G] \to \mathbb{C}[G]$ on this coordinate ring. Using Proposition 2.3 and a suitable modification of diagram (21), we can construct an isomorphism of algebras out of the factorization maps $\rho_\alpha : \mathcal{H}(0, L) \otimes V(\vec{\lambda}) \to \mathcal{H}(0, L) \otimes V(\vec{\lambda}, \alpha, \alpha^*)$:

$$\rho : \mathcal{H}(0, L) \otimes V(\vec{\lambda}) \to \bigoplus_{\alpha \in \Delta_L} \mathcal{H}(0, L) \otimes V(\vec{\lambda}, \alpha, \alpha^*), \tag{30}$$

$$\begin{aligned} \rho^* : [\mathbb{C}[Q] \otimes \mathbb{C}[G/U]^{\otimes n} \otimes \mathbb{C}[G]]^{\hat{\mathfrak{g}}[\tilde{C},p,q_1,q_2]} \\ \xrightarrow{\cong} [\mathbb{C}[Q] \otimes \mathbb{C}[G/U]^{\otimes n}]^{\hat{\mathfrak{g}}[C,p]} = \mathcal{V}^\dagger_{C,\vec{p}}(G). \end{aligned} \tag{31}$$

Here $\mathbb{C}[Q] = \bigoplus_{L \geq 0} \mathcal{H}(0, L)^*$ is the total coordinate ring of the affine Grassmannian variety, and $\mathbb{C}[G/U] = \bigoplus_{\lambda \in \Delta} V(\lambda)$ is the coordinate ring of the quotient of G by a maximal unipotent subgroup. Let $G /\!\!/ U = \mathrm{Spec}(\mathbb{C}[G/U])$ be the GIT quotient of G by U; notice that the grading by dominant weight corresponds to a residual T-action on $G /\!\!/ U$. The degeneration constructed in this section is then induced by the so-called horospherical contraction of the group scheme G [Pop86]. This is a flat $G \times G$ degeneration of G to the scheme $[G /\!\!/ U \times G /\!\!/ U]/T$, where the T-action is defined through the residual $T \times T$-action on $G /\!\!/ U \times G /\!\!/ U$ so that $V(\lambda) \otimes V(\lambda^*) \subset \mathbb{C}[G /\!\!/ U] \otimes C[G /\!\!/ U]$ is invariant.

Remark 3.5. The association of a filtration (actually a valuation) on $\mathcal{V}^\dagger_{C,\vec{p}}(G)$ to a coweighting $(\Gamma, \vec{\theta})$ suggests that coweightings should play a role in the tropical theory of the moduli of principal bundles [Pay09].

Example 3.6. We compute an example for the curve in Figure 5. Explicitly we have

$$\begin{aligned} (\mathcal{V}^\dagger_{\tilde{C}',\vec{q}}(G))^T &= (\mathcal{V}^\dagger_{\mathbb{P}^1,q_1,q_2,q_3}(G) \otimes \mathcal{V}^\dagger_{\mathbb{P}^1,q_4,q_5,q_6}(G))^T \\ &= \bigoplus_{L,\alpha,\beta,\gamma} [\mathcal{V}^\dagger_{\mathbb{P}^1,q_1,q_2,q_3}(\alpha, \alpha^*, \beta, L) \otimes \mathcal{V}^\dagger_{\mathbb{P}^1,q_4,q_5,q_6}(\beta^*, \gamma, \gamma^*, L)]. \end{aligned} \tag{32}$$

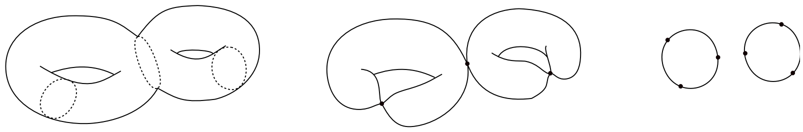

Fig. 5. Curve of genus 2, stable curve of genus 2 and disjoint union of two triple marked curves of genus 0.

Multiplication is computed componentwise over the tensor product. In the case $\mathfrak{g} = \mathrm{sl}_2(\mathbb{C})$, this ring is the semigroup of weightings on the graph pictured in Figure 6, where the middle edge is always weighted even and less than or equal to twice either of the loop edges, and the sum of twice either loop edge and the middle edge is bounded by the level.

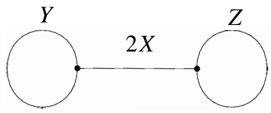

Fig. 6. Weighted genus 2 graph.

4. Correlation and the genus 0 case

In this section we prove Theorem 1.8, which relates the degeneration on $\mathcal{V}^{\dagger}_{C,\vec{p}}(G)$ to a similar construction on an algebra $\hat{\mathfrak{A}}_n$ which we now describe. Recall that $\mathfrak{A}_n$ is the algebra of left diagonal G-invariants in $\mathbb{C}[G/U]^{\otimes n}$, and let $M_n = \mathrm{Spec}(\mathfrak{A}_n)$. The coordinate ring of G/U is known to be a multiplicity-free direct sum of the irreducible representations of G (see e.g. [PV72]) with multiplication computed by the maps $C^*_{\lambda;\gamma} : V(\lambda^*) \otimes V(\gamma^*) \to V(\lambda^* + \gamma^*)$. It follows that the coordinate ring $\mathfrak{A}_n$ of this space is the graded direct sum $\bigoplus_{\vec{\lambda}\in\Delta^n} [V(\vec{\lambda})]^{\mathfrak{g}}$. The algebra $\hat{\mathfrak{A}}_n$ is defined to be the following Rees algebra of $\mathfrak{A}_n$:

$$\hat{\mathfrak{A}}_n = \bigoplus_{\vec{\lambda}\in\Delta^n,\, \lambda_i(\theta^\vee)\le L} [V(\vec{\lambda})]^{\mathfrak{g}} t^L. \tag{33}$$

Now we may observe that there is a correlation map $F_{C,\vec{p}} : \mathcal{V}^{\dagger}_{C,\vec{p}}(G) \to \hat{\mathfrak{A}}_n$ which is a map of algebras by Lemma 2.2.

4.1. The correlation morphism

The invariant spaces $[V(\vec{\lambda})]^{\mathfrak{g}}$ also have a factorization property, so a simplified version of the construction in Section 3 applies to the algebra $\mathfrak{A}_n$ (see also [Man10a]).

Proposition 4.1. *Let $S_1 \cup S_2 = [n]$ be a partition of n indices, and let $\mathcal{T}$ be the tree with leaves in bijection with $[n]$ and one interior edge defined by this partition. There is a factorization isomorphism $\sum \rho_\eta : \bigoplus_\eta [V(\vec{\lambda}_1, \eta)]^{\mathfrak{g}} \otimes [V(\eta^*, \vec{\lambda}_2)]^{\mathfrak{g}} \to [V(\vec{\lambda}_1, \vec{\lambda}_2)]^{\mathfrak{g}}$. This defines a filtration $\mathcal{F}_{\mathcal{T},\vec{\theta}}$ on $\hat{\mathfrak{A}}_n$ for each coweighting of the tree $\mathcal{T}$. If the components of $\vec{\theta}$ are strictly positive on all positive roots, then the associated graded algebra is isomorphic to $[\hat{\mathfrak{A}}_{|S_1|} \otimes \hat{\mathfrak{A}}_{|S_2|}]^{T\times\mathbb{C}^*}$.*

Proof. Apply the steps of Section 3. □

In Section 2 we showed that there is a 1-1 map $F_{C,\vec{p}} : \mathcal{V}^{\dagger}_{C,\vec{p}}(G) \to \hat{\mathfrak{A}}_n$ when C is a genus 0 stable curve. Next we show that over a stable curve of type $\mathcal{T}$, the correlation morphism $F_{C,\vec{p}} : \mathcal{V}^{\dagger}_{C,\vec{p}}(G) \to \hat{\mathfrak{A}}_n$ intertwines the factorization of conformal blocks with the branching decomposition defined by $\mathcal{T}$ on $\hat{\mathfrak{A}}_n$. Let $(C, \vec{p}_1, \vec{p}_2)$ be a stable curve of type $\mathcal{T}$, with $\tilde{C} = C_1 \cup C_2$. We have a commuting square of $\mathfrak{g}$-representations

$$\begin{tikzcd}
\mathcal{H}(\vec{\lambda}, L) \arrow[r, "\rho_\eta"] & \mathcal{H}(\vec{\lambda}_1, \eta, L) \otimes \mathcal{H}(\eta^*, \vec{\lambda}_2, L) \\
V(\vec{\lambda}) \arrow[u, "i"] \arrow[r, "\rho_\eta"] & V(\vec{\lambda}_1, \eta) \otimes V(\eta^*, \vec{\lambda}_2) \arrow[u, "i\otimes i"']
\end{tikzcd}$$

We may dualize this diagram, and take invariants with respect to the Lie algebras $\hat{\mathfrak{g}}[C, \vec{p}_1, \vec{p}_2]$ on the top left, $\hat{\mathfrak{g}}[C_1 \setminus \vec{p}_1] \oplus \hat{\mathfrak{g}}[C_2 \setminus \vec{p}_2]$ on the top right, $\mathfrak{g}$ on the bottom left, and $\mathfrak{g} \oplus \mathfrak{g}$ on the bottom right. The top morphism $\hat{\rho}_\eta : [\mathcal{H}(\vec{\lambda}_1, \eta, L) \otimes \mathcal{H}(\eta^*, \vec{\lambda}_2, L)]^* \to \mathcal{H}(\vec{\lambda}, L)^*$ on the dual spaces takes the graded component $\mathcal{V}^\dagger_{C_1,\vec{p}_1,q_1}(\vec{\lambda}_1, \eta, L) \otimes \mathcal{V}^\dagger_{C_2,\vec{p}_2,q_2}(\eta^*, \vec{\lambda}_2, L)$ into $\mathcal{V}^\dagger_{C,\vec{p}_1,\vec{p}_2}(\vec{\lambda}, L)$ by the factorization rules. The bottom morphism $\hat{\rho}_\eta : [V(\vec{\lambda}_1, \eta) \otimes V(\eta^*, \vec{\lambda}_2)^*]^{\mathfrak{g}} \to V(\vec{\lambda})^*$ takes $[V(\vec{\lambda}_1, \eta)] \otimes [V(\eta^*, \vec{\lambda}_2)]^{\mathfrak{g}}$ to $[V(\vec{\lambda})]^{\mathfrak{g}}$ by Proposition 4.1. The fact that these invariant spaces are connected by the appropriate morphisms, along with the commutativity of the dual diagram, implies that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{V}^\dagger_{C,\vec{p}_1,\vec{p}_2}(\vec{\lambda}, L) \arrow[d, "F_{C,\vec{p}_1,\vec{p}_2}"'] & \mathcal{V}^\dagger_{C_1,\vec{p}_1,q_1}(\vec{\lambda}_1, \eta, L) \otimes \mathcal{V}^\dagger_{C_2,\vec{p}_2,q_2}(\eta^*, \vec{\lambda}_2, L) \arrow[l, "\hat{\rho}_\eta"'] \arrow[d, "F_{C_1,\vec{p}_1,q_1} \otimes F_{C_2,q_2\vec{p}_2}"] \\
{[V(\vec{\lambda})^*]^{\mathfrak{g}}} & {[V(\vec{\lambda}_1, \eta)^*]^{\mathfrak{g}} \otimes [V(\eta^*, \vec{\lambda}_2)^*]^{\mathfrak{g}}} \arrow[l, "\hat{\rho}_\alpha"']
\end{tikzcd}$$

This shows that the direct sum decompositions of $\mathcal{V}^\dagger_{C,\vec{p}_1,\vec{p}_2}(\vec{\lambda}, L)$ and $[V(\vec{\lambda})]^{\mathfrak{g}}$ from the factorization rules are compatible, and implies that the filtrations on the branching algebras and the algebras of conformal blocks agree. Theorem 1.8 follows by induction.

Remark 4.2. Whenever $\tilde{C}$ is a disjoint union $C_1 \cup C_2$, the diagram above commutes. This implies that a version of Theorem 1.8 is true for general genus, except that the correlation $F_{C,\vec{p}}$ is no longer a monomorphism.

4.2. The case (0, 3)

For a genus 0, triple marked curve there is no moduli, $\mathcal{M}_{0,3} = \{pt\}$, so the algebra of conformal blocks is unique. In this case, conformal blocks have a purely representation-theoretic description as a subspace of the space of invariants. See [TUY89] for the following.

Proposition 4.3. *The space $\mathcal{V}^\dagger_{0,3}(\lambda, \gamma, \mu, L) \subset [V(\lambda^*) \otimes V(\gamma^*) \otimes V(\mu^*)]^{\mathfrak{g}}$ has the following description. Consider the factorizations of $V(\lambda^*)$, $V(\gamma^*)$ and $V(\mu^*)$ as $\mathrm{sl}_2(\mathbb{C})$-representations with respect to the longest root $\theta : \mathrm{sl}_2(\mathbb{C}) \to \mathfrak{g}$: $V(\lambda^*) = \bigoplus W(\lambda^*, i) \otimes V(i)$, $V(\gamma^*) = \bigoplus W(\gamma^*, j) \otimes V(j)$, $V(\mu^*) = \bigoplus W(\mu^*, k) \otimes V(k)$. Let $W(\lambda^*, \gamma^*, \mu^*, L)$ be the subspace of $V(\lambda^*) \otimes V(\gamma^*) \otimes V(\mu^*)$ of components $V(i) \otimes V(j) \otimes V(k)$ with $i + j + k \leq 2L$. Then*

$$\mathcal{V}^\dagger_{0,3}(\lambda, \gamma, \mu, L) = W(\lambda^*, \gamma^*, \mu^*, L) \cap [V(\lambda^*) \otimes V(\gamma^*) \otimes V(\mu^*)]^{\mathfrak{g}}. \tag{34}$$

Remark 4.4. The degeneration constructed in Section 3 could be completed to a toric degeneration given any T^3-invariant toric degeneration of $\mathcal{V}^\dagger_{0,3}(G)$. A sufficient condition for the existence of such a filtration would be the existence of a basis $B(\lambda, \gamma, \mu)$ of each space $[V(\lambda) \otimes V(\gamma) \otimes V(\mu)]^{\mathfrak{g}}$ with the following properties:

(1) The bases $B(\lambda, \gamma, \mu)$ have a “lower-triangular multiplication" property with respect to the multiplication in $\mathfrak{A}_3$.

(2) The intersection $B(\lambda, \gamma, \mu) \cap \mathcal{V}^{\dagger}_{0,3}(\lambda, \gamma, \mu, L) \subset [V(\lambda^*) \otimes V(\gamma^*) \otimes V(\mu^*)]^{\mathfrak{g}}$ is a basis for each L.

Lusztig's dual canonical basis satisfies the first property above, and the resulting degenerations are explored in [Man10a]. In [Man13] we use this technique to build toric degenerations of the algebra of conformal blocks when $G = \mathrm{SL}_3(\mathbb{C})$.

4.3. *Projective coordinate rings*

Recall that the coordinate ring $\mathfrak{A}_n$ is the multiplicity-free direct sum of the invariant spaces $V^{\dagger}_{\mathfrak{g}}(V(\vec{\lambda}))$, and that this algebra is graded by the tuples of dominant weights $\vec{\lambda} \in \Delta^n$. This grading corresponds to a right action of T^n on M_n. We let $M_{\vec{\lambda}}$ be the GIT qoutient of M_n by T^n with respect to the character defined by the tuple $\vec{\lambda}$; this is the space of configurations associated to $\vec{\lambda}$. The projective coordinate ring naturally associated to $M_{\vec{\lambda}}$ by this construction is then the direct sum $\mathfrak{A}_{\vec{\lambda}} \subset \mathfrak{A}_n$ of the invariant spaces $[V(K\vec{\lambda}^*)]^{\mathfrak{g}}$ for $K \geq 0$.

All of the techniques we have used to study algebras of conformal blocks and branching algebras are carried out on graded pieces of these algebras. Because of this, much of what we say can be extended to nice graded subalgebras; in particular these statements apply to the projective coordinate ring of the coarse moduli spaces $\mathcal{M}_{C,\vec{p}}(\vec{\lambda}, L)$. In particular, the correlation map $F_{C,\vec{p}} : \mathcal{V}^{\dagger}_{C,\vec{p}}(G) \to \hat{\mathfrak{A}}_n$ induces a map on projective coordinate rings:

$$F^{\vec{\lambda},L}_{C,\vec{p}} : R_{C,\vec{p}}(\vec{\lambda}, L) \to \mathfrak{A}_{\vec{\lambda}}. \tag{35}$$

When $g = 0$, this map is a monomorphism, in which case we can deduce the following.

Proposition 4.5. *For $g = 0$ and $L \gg 0$ the map $F^{\vec{\lambda},L}_{C,\vec{p}}$ above is an isomorphism.*

Proof. The algebra $\mathfrak{A}_{\vec{\lambda}}$ is finitely generated, say by the spaces $[V(N_i\vec{\lambda}^*)]^{\mathfrak{g}}$ for $i = 1, \dots, k$. Each of these spaces is filtered by the spaces of conformal blocks $\mathcal{V}^{\dagger}_{C,\vec{p}}(\vec{\lambda}_i, L)$. It follows that if L is chosen to be sufficiently large, then $\mathcal{V}^{\dagger}_{C,\vec{p}}(\vec{\lambda}_i, L) = [V(N_i\vec{\lambda}^*)]^{\mathfrak{g}}$, so that $R_{C,\vec{p}}(\vec{\lambda}, L) \cong \mathfrak{A}_{\vec{\lambda}}$. □

Compare this proposition with [TW03, Remark 4.3]. The algebra $R_{C,\vec{p}}(\vec{\lambda}, L)$ is the projective coordinate ring of $M^{\mathrm{ss}}_{C,\vec{p}}(\vec{\lambda}, L)$, the coarse moduli space of semistable bundles, where the semistability condition is determined by the data $(\vec{\lambda}, L)$. Degenerations associated to labeled trees carry over to these algebras as well. This implies that for large L, a toric degeneration of $M_{\vec{\lambda}}$ gives a toric degeneration of a ring of generalized theta functions. Toric degenerations of $\mathfrak{A}_{\vec{\lambda}}$ can be constructed from a toric degeneration of $\mathfrak{A}_n$ [Man10a], [Man14].

5. The case $\mathfrak{g} = \mathrm{sl}_2(\mathbb{C})$

The results of this section should be of independent interest for readers interested in the $\mathbb{Z}/2\mathbb{Z}$ group-based phylogenetic statistical models [BBKM13], [Buc12], [BW07]. We refer the reader to [KM14] for the connection between conformal blocks and other group-based models.

5.1. Proof of Theorem 1.3

For dominant $\mathrm{sl}_2(\mathbb{C})$-weights $r_1, r_2, r_3 \in \mathbb{Z}_{\geq 0}$, the spaces $[V(r_1) \otimes V(r_2) \otimes V(r_3)]^{\mathrm{sl}_2(\mathbb{C})}$ are multiplicity free, and are non-trivial when $r_1 + r_2 + r_3 \in 2\mathbb{Z}$ and $|r_1 - r_3| \leq r_2 \leq r_1 + r_3$; these are known as the *Clebsch–Gordan conditions*. Proposition 4.3 implies that the spaces $\mathcal{V}^{\dagger}_{0,3}(r_1, r_2, r_3, L)$ are also multiplicity free, and that they are non-trivial when $r_1 + r_2 + r_3 \leq 2L$ and the Clebsch–Gordan conditions are satisfied; these are known as the *quantum Clebsch–Gordan conditions*. The following is a consequence of the fact that the weights (r_1, r_2, r_3, L) define a multigrading of $\mathcal{V}^{\dagger}_{0,3}(\mathrm{SL}_2(\mathbb{C}))$.

Proposition 5.1. *For $G = \mathrm{SL}_2(\mathbb{C})$ the algebra $\mathcal{V}^{\dagger}_{0,3}(\mathrm{SL}_2(\mathbb{C}))$ is isomorphic to the graded affine semigroup algebra associated to the polytope $P_3 = \mathrm{conv}\{(0,0,0), (1,1,0), (1,0,1), (0,1,1)\} \subset \mathbb{R}^3$, with respect to the lattice determined by $r_i \in \mathbb{Z}$, $r_1 + r_2 + r_3 \in 2\mathbb{Z}$.*

Now we describe the algebra $\bigotimes_{v \in V(\Gamma)} \mathcal{V}^{\dagger}_{0,3}(\mathrm{SL}_2(\mathbb{C})) = \mathbb{C}[P_3^{V(\Gamma)}]$ and its T_Γ-invariant subalgebra. We have associated a copy of P_3 to each vertex $v \in V(\Gamma)$, and likewise each entry r_1, r_2, r_3 of a point in this P_3 is assigned to an edge incident to v. The isotypical spaces of the T_Γ-action on $\mathbb{C}[P_3^{V(\Gamma)}]$ are each spanned by a lattice point $\vec{w} \in P_3^{V(\Gamma)}$. The character of the $T \times \mathbb{C}^*$-action associated to a given edge $e \in E(\Gamma)$ on a $\vec{w}$ returns the difference $L_v(\vec{w}) - L_u(\vec{w})$ of the levels on the end points $\{u, v\}$ of e, and the difference $w_v(e) - w_u(e)$ between the weights assigned to e by the v and u components of $\vec{w}$. Taking the torus invariants $[\mathbb{C}[P_3]^{\otimes |V(\Gamma)|}]^{T_\Gamma}$ therefore picks out exactly those lattice points with components from the same Minkowski sum $L \circ P_3$, which consistently weight the edges of Γ (see Figure 7). By definition, the invariant subalgebra is the affine semigroup algebra $\mathbb{C}[P_\Gamma]$; this proves Theorem 1.3.

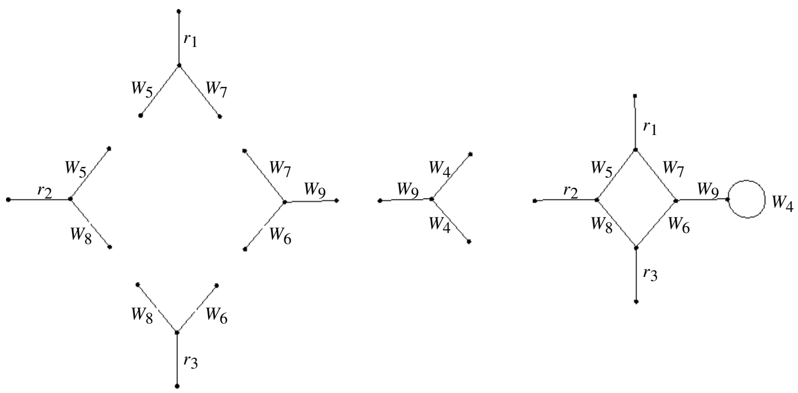

Fig. 7. A T_Γ-invariant.

5.2. P_Γ is a Newton–Okounkov body

As an application of Propositions 3.1 and 3.3 we interpret the polytope P_Γ as a Newton–Okounkov body for the algebra of conformal blocks over the marked curve of type Γ. The results of this subsection will not be needed elsewhere in the paper. For background on the theory of Newton–Okounkov bodies we refer the reader to the papers of Kaveh and Khovanskii [KK12] and Lazarsfeld and Mustaţă [LM09]. A Newton–Okounkov body P of a graded algebra $A = \bigoplus_{L\geq 0} A_L$ is a combinatorial invariant associated to a choice of a rank $a+1$ valuation $v : A \to \mathbb{Z}^{a+1}$ (see [KK12, Definition 2.8]) which refines the grading on A. Here $a+1$ is the Krull dimension of A, and $\mathbb{Z}^{a+1}$ is given a lexicographic ordering which prioritizes the grading component. In particular, we assume that $v(f) = (\dots, \deg(f)) \in \mathbb{Z}^{a+1}$ for f homogeneous. The image $v(A) \subset \mathbb{Z}^{a+1}$ is an affine semigroup (see [KK12, Proposition 2.10]), and its convex hull $\bar{P} \subset \mathbb{R}^{a+1}$ is a convex cone. The slice $P = \bar{P} \cap \mathbb{R}^a \times \{1\} \subset \mathbb{R}^{a+1}$ is called the *Newton–Okounkov body* of A associated to v.

Now we fix $A = \mathcal{V}^\dagger_{C,\vec{p}}(\mathrm{SL}_2(\mathbb{C}))$, where $(C, \vec{p})$ is the stable curve of type Γ. The following proposition is a direct consequence of Propositions 3.1 and 3.3.

Proposition 5.2. *As a vector space, the algebra $\mathcal{V}^\dagger_{C,\vec{p}}(\mathrm{SL}_2(\mathbb{C}))$ is a direct sum of one-dimensional spaces $\mathbb{C}\phi_{w,L}$, where w is a lattice point in the L-th Minkowski sum $L\circ P_\Gamma$. The product $\phi_{w,L}\phi_{w',K}$ is a sum of the form $\phi_{w+w',L+K} + \sum_{u<w+w'} C_u\phi_{u,L+K}$, where the partial ordering $u < w+w'$ means that $u(e) \leq w(e)+w'(e)$ for each edge $e \in E(\Gamma)$ and there is some edge where this inequality is strict.*

By placing a total ordering $\ll$ on the edge set $E(\Gamma)$ we can produce a lexicographic ordering $\prec$ on the basis members $\phi_{w,L}$ as follows. We say $\phi_{w,L} \prec \phi_{w',K}$ if $(w(e_1), \dots, w(e_{|E(\Gamma)|}), L)$ is less than $(w'(e_1), \dots, w(e_{|E(\Gamma)|}), K)$ in the lexicographic ordering, where the set $E(\Gamma)$ has been placed in bijection with the integers $\{1, \dots, |E(\Gamma)|\}$ using $\ll$. Using $\prec$, we define a filtration on $\mathcal{V}^\dagger_{C,\vec{p}}(\mathrm{SL}_2(\mathbb{C}))$ by the vector spaces $F_\Gamma^{\preceq w} = \bigoplus_{w'\preceq w} \mathbb{C}\phi_{w'}$. Proposition 5.2 (along with Propositions 3.1 and 3.3) implies the following proposition.

Proposition 5.3. *The filtration F_Γ is an algebra filtration of $\mathcal{V}^\dagger_{C,\vec{p}}(\mathrm{SL}_2(\mathbb{C}))$. The associated graded algebra of the filtration F_Γ is isomorphic to the affine semigroup algebra $\mathbb{C}[P_\Gamma]$.*

As $\mathbb{C}[P_\Gamma]$ is a domain, the filtration F_Γ can be used to define a valuation v_Γ (see [Man14, Proposition 3.2]):

$$v_\Gamma(f) = \min\{w \mid f \in F_\Gamma^{\preceq w}\}, \qquad f \in \mathcal{V}^\dagger_{C,\vec{p}}(\mathrm{SL}_2(\mathbb{C})). \tag{36}$$

Proposition 5.4. *The Newton–Okounkov body of $\mathcal{V}^\dagger_{C,\vec{p}}(\mathrm{SL}_2(\mathbb{C}))$ with respect to the valuation v_Γ is the polytope P_Γ.*

Proof. The image of v_Γ is the set of all (w, L) for w a lattice point in $L\circ P_\Gamma$. It follows that the level 1 slice can be identified with the closure of the set of all $\frac{1}{L}w$; this is P_Γ itself. $\square$

Remark 5.5. An identical construction to the one given above shows that the polytope $P_\Gamma(\vec{r}, L)$ studied in [Man10b] and [Man12] is a Newton–Okounkov body for $R_{C,\vec{p}}(\vec{r}, L)$, where $(C, \vec{p})$ is the stable curve of type Γ.

For each connected, trivalent graph Γ with $\beta_1(\Gamma) = g$ and n leaves, Theorem 1.3 produces a tool to study $\mathcal{V}^\dagger_{C,\vec{p}}(\mathrm{SL}_2(\mathbb{C}))$ for generic $(C, \vec{p})$. We prove Theorem 1.5 by focusing on the algebra $\mathbb{C}[P_{\Gamma_{g,n}}]$ attached to a particular graph $\Gamma_{g,n}$ (see Figure 8).

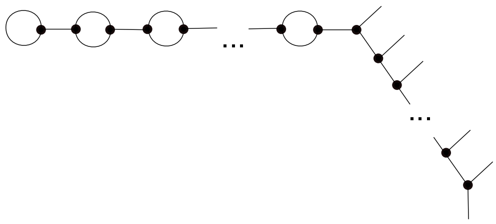

Fig. 8. The graph $\Gamma_{g,n}$.

5.3. *Generators*

The following three lemmas show that $P_{\Gamma_{g,n}}$ and the second Minkowski sum $2\circ P_{\Gamma_{g,n}}$ suffice to generate $\mathbb{C}[P_{\Gamma_{g,n}}]$.

Lemma 5.6. *The case* (g, n) *follows from cases* $(g, 1)$ *and* $(0, n+1)$

Lemma 5.7. *The multiplication map* $\pi : P_{\Gamma_{g,1}} \times 2L\circ P_{g,1} \to (2L+1)\circ P_{\Gamma_{g,1}}$ *is surjective.*

Lemma 5.8. *The multiplication map* $\pi : [2\circ P_{\Gamma_{g,1}}]^{\times L} \to 2L\circ P_{\Gamma_{g,1}}$ *is surjective.*

In [BW07] Buczyńska and Wiśniewski show that $\mathbb{C}[P_{\Gamma_{0,n}}]$ is generated by the lattice points of $P_{\Gamma_{0,n}}$, and that the ideal of relations on these generators has a quadratic, square-free Gröbner basis. It follows that $P_{0,n}$ is generated by weightings which weight all edges $e \in E(\Gamma_{0,n})$ with 0 or 1.

5.4. *Proof of Lemma 5.6*

We prove Lemma 5.6 with a toric fiber product argument. The weight on any edge of $\Gamma_{g,n}$ coming from a lattice point of $L\circ P_{\Gamma_{g,n}}$ must be less than or equal to L: this follows directly from the defining inequalities. The lattice under consideration forces each horizontal edge in $\Gamma_{g,1}$ to be weighted with an even number, including the leaf edge. We consider an element of $L\circ P_{\Gamma_{g,n}}$ $(g > 0)$ as an element of $L\circ P_{\Gamma_{g,1}}$ glued to an element of $L\circ P_{\Gamma_{0,n+1}}$ (see Figure 9).

Let $\omega \in L\circ P_{\Gamma_{g,n}}$, and let $\omega^g \in L\circ P_{\Gamma_{g,1}}$ and $\omega^0 \in L\circ P_{\Gamma_{0,n+1}}$ be the restrictions of ω to the copies of $\Gamma_{g,1}$ and $\Gamma_{0,n+1}$ in $\Gamma_{g,n}$ respectively. By the theorem of Buczyńska and Wiśniewski, ω^0 decomposes into L weightings of $\Gamma_{0,n+1}$:

$$\omega^0 = \eta_1 + \cdots + \eta_L. \tag{37}$$

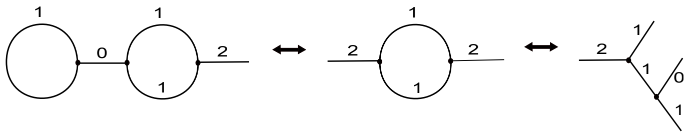

Fig. 9

Now suppose that the generation statement of Theorem 1.1 holds for $P_{\Gamma_{g,1}}$; then ω^g likewise decomposes into elements of degree 1 and 2:

$$\omega^g = [\alpha_1 + \cdots + \alpha_k] + [\beta_1 + \cdots + \beta_m]. \tag{38}$$

Here $2m+k = L$. Due to the restrictions imposed by the lattice, the α_i all have weight 0 on the edge shared by $\Gamma_{g,1}$ and $\Gamma_{0,n+1}$. We let $\eta_1, \ldots, \eta_{L'}$ and $\beta_1, \ldots, \beta_{m'}$ be the elements with a non-zero weight on this edge in $P_{\Gamma_{0,n+1}}$ and $P_{\Gamma_{g,1}}$ respectively. We must have $L' = 2m'$, so we may pair up the elements $\eta_1 + \eta_2, \ldots$ to make m' elements in $2 \circ P_{\Gamma_{0,n+1}}$, and we can glue these to the β_i along the shared edge. What remains must be exactly k' η_i's and k' α_i's, both with 0 along the shared edge, which can then likewise be glued along the edge in any order.

5.5. *Proof of Lemma 5.7*

We prove Lemma 5.7 with an analysis of weightings on the component graphs in Figure 10. We define the graded semigroups B_1 and B_2 accordingly; they are composed of those weightings on the pictured graphs which are even on the leaf edges. We let o_1 and o_2 be elements which weight the edges in a loop with 1 and all other edges with 0, in B_1 and B_2 respectively. The following proposition is the crux of Lemmas 5.7 and 5.8.

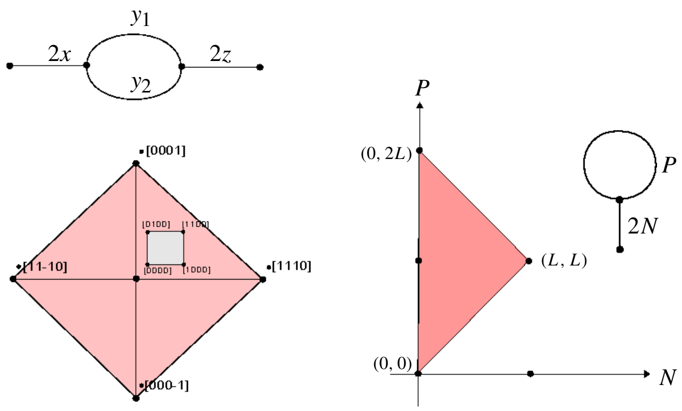

Fig. 10. Polytopes of weightings from $B_1(2L)$ and $B_2(2)$.

Proposition 5.9. *The semigroups B_1 and B_2 are generated by elements of degrees* 1 *and* 2 (*these are the elements appearing in Figures 11 and 12 respectively*).

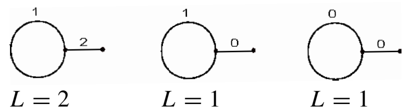

Fig. 11

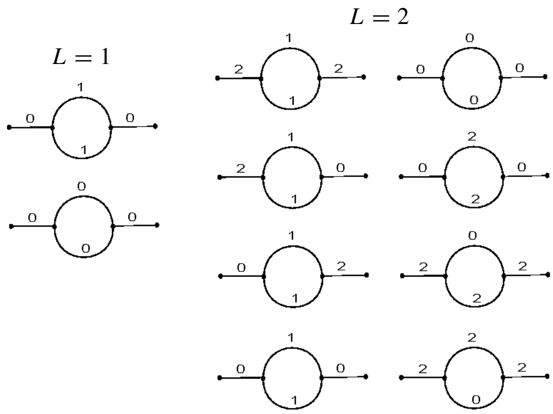

Fig. 12

Proof. As B_1 and B_2 are both the semigroup of lattice points in a convex cone, we directly verify that the elements of degrees 1 and 2 generate by checking that they constitute a Hilbert basis using the software package Macaulay 2 [GS] or 4ti2 [tt]. This software uses the project-and-lift algorithm [Hem02]. □

It follows that all generators weight the leaf edges of either graph with either 0 or 2. We prove Lemma 5.7 by pulling all copies of o_1 and o_2 off the loops in $\Gamma_{g,1}$. If a loop has a vertex with incident entries adding to $2L + 1$ we can extract an o_1 or o_2 from that loop, otherwise we leave it alone. The resulting collection of loops is a member of $P_{\Gamma_{g,n}}$, and the remainder is an element of $2L \circ P_{\Gamma_{g,n}}$ (see Figure 13).

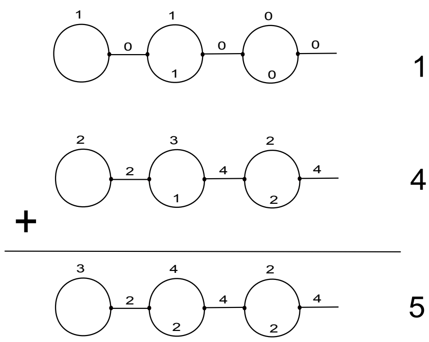

Fig. 13

5.6. *Proof of Lemma 5.8*

A weighting of $\Gamma_{g,1}$ automatically induces a weighting of $\Gamma_{g-1,1}$ and an element of B_2. Now we are in a position to use an argument similar to the proof of Lemma 5.6. The theorem is true for $\mathbb{C}[P_{\Gamma_{1,1}}] = \mathbb{C}[B_1]$; suppose that it holds for $\Gamma_{g-1,1}$. For an element $\omega \in 2L \circ P_{\Gamma_{g,1}}$ we consider the restrictions ω_1, ω_2 to $\Gamma_{g-1,1}$ and B_2 respectively. By induction, ω_1 and ω_2 can be factored into L elements of $2 \circ P_{\Gamma_{g-1,1}}$ and $B_2(2)$, respectively:

$$\omega_1 = \alpha_1 + \cdots + \alpha_L, \qquad \omega_2 = \beta_1 + \cdots + \beta_L. \tag{39}$$

The number of α_i which weight the edge shared by $\Gamma_{g-1,1}$ and B_2 with a 2 must equal the number of β_i which also weight this edge 2. Elements which weight this edge the same can therefore be matched up; this proves Lemma 5.8.

5.7. *Relations*

We will describe relations for the building block semigroups B_1, B_2; then we show that these relation results are stable as these building blocks are glued together. Analysis of the building block semigroups gives the $(g, 1)$ case, which we combine with results in [BW07] to handle the (g, n) case.

The semigroup algebra $\mathbb{C}[B_1]$ is generated by two elements of level 1 and an element of level 2, and no relations hold between these elements. The generators of the semigroup algebra $\mathbb{C}[B_2]$ are given in Figure 11; a Markov basis for the toric ideal vanishing on these generators can be computed using the software package 4ti2 [tt]. This computation shows that the relations for $\mathbb{C}[B_2]$ are generated by those of levels 2, 3, and 4; we omit the details.

5.8. *Relations for* $\mathbb{C}[P_{\Gamma_{g,1}}]$

We describe relations for $\mathbb{C}[P_{\Gamma_{g,1}}]$ by building relations for this semigroup out of those of B_1 and B_2. The theorem holds for B_1 because this semigroup has unique factorization; this is the base case of our induction. To treat $P_{\Gamma_{g,1}}$, we consider it as a toric fiber product of $P_{\Gamma_{g-1,1}}$ with B_2 over the semigroup of non-negative integer points (n, L) with $2n \leq L$. This semigroup is generated by $(0, 1)$ and $(1, 2)$, and also has unique factorization. The maps from $P_{\Gamma_{g-1,1}}$ and B_2 to this semigroup are computed by sending a lattice point to the pair given by half the weight on the edge shared by the supporting graphs of these semigroups, and the level of the weighting. Given an element $\omega \in P_{\Gamma_{g,1}}$ and two factorizations, we first collect level 1 terms into level 2 terms; this gives factorizations into only level 2 elements in the even level case, and level 2 elements except one level 1 element in the odd case. Note that in doing so, we have used at most level 2 relations.

Now we consider the restriction of this relation to $P_{\Gamma_{g-1,1}}$. By induction, there is a way to transform $\sum \beta_i^*|_{g-1,1}$ into $\sum \alpha_j^*|_{g-1,1}$ using degree 2, 3, 4 relations. We claim that each of these relations can be lifted to a relation in $P_{\Gamma_{g,1}}$. Any such relation must preserve the list of values along the shared edge; this means that we can “unglue” the B_2 side of the weightings involved in the relation, perform the relation on the $(g-1, 1)$ side,

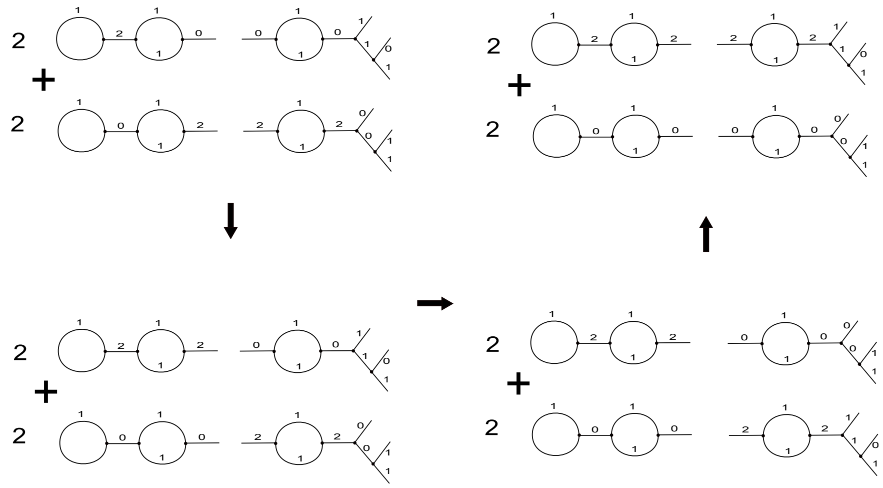

Fig. 14

and glue the B_2 weightings back on in any way consistent with their values along the shared edge to obtain a relation on $P_{\Gamma_{g,1}}$ (see Figure 14).

This proves that we can transform $\sum \beta_i^*$ to $\sum \alpha_j^*$ "along $P_{\Gamma_{g-1,1}}$." But we can now apply the same argument to perform this transformation over B_2, using the relations above. This completes the proof of the relation statement in Theorem 1.5 for $(g, 1)$.

5.9. Relations for $\mathbb{C}[P_{\Gamma_{g,n}}]$

The same argument used in the case $P_{\Gamma_{g,1}}$ to extend relations from $P_{\Gamma_{g-1,1}}$ to $P_{\Gamma_{g,1}}$ can be used to show that any relation on $P_{\Gamma_{g,1}}$ can be extended to a relation on $P_{\Gamma_{g,n}}$. From this it follows that given two normalized ways to represent an element $\omega = \sum \beta_i^* = \sum \alpha_j^*$, one can be transformed to the other "over $P_{\Gamma_{g,1}}$" with relations of degree 2, 3, 4. Now the same argument can be applied over the edge which separates $P_{\Gamma_{0,n+1}}$ from $P_{\Gamma_{g,1}}$ using the degree 2 $P_{\Gamma_{0,n+1}}$ relations discovered in [BW07]. Our analysis takes care of Theorem 1.3 for $n > 0$; the $n = 0$ case is handled by noting that $\mathbb{C}[P_{\Gamma_{g,0}}] \subset \mathbb{C}[P_{\Gamma_{g,1}}]$ is spanned by the set of elements with 0 weighting the leaf, and all of our techniques specialize to this case without alteration.

5.10. Best possible generation of $\mathbb{C}[P_\Gamma]$

We sketch how our quadratic generation result is in a certain sense best possible for the semigroup algebras $\mathbb{C}[P_\Gamma]$ when $g > 0$. We say a trivalent graph Γ' is a *trivalent minor* of a trivalent graph Γ if Γ' can be obtained from a subgraph of Γ by replacing pairs of edges joined at a bivalent vertex with a single edge, and any leaf of Γ' is also a leaf of Γ.

Proposition 5.10. *If* Γ *contains a trivalent minor isomorphic to one of the graphs depicted in Figure 15, then* $\mathbb{C}[P_\Gamma]$ *has indecomposable elements of degree* ≥ 2.

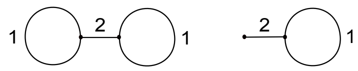

Fig. 15. Trivalent minors with indecomposable weightings

This proposition holds for the graphs depicted in Figure 15, and any indecomposable element on a trivalent minor of a graph Γ gives an indecomposable on Γ by extending this element by 0 to the rest of Γ. Notice that any trivalent graph with $g, n \geq 1$ contains the graph on the right in Figure 15 as a trivalent minor; for this reason we focus on Γ with $n = 0$. For $g = 2, 3, 4$, it can be shown that the graphs depicted in Figure 16 do not contain either of the graphs in Figure 15, and that these are the only trivalent graphs with this property. Any trivalent graph Γ with $g \geq 5$ which does not have either graph in Figure 15 as a trivalent minor must have the graph on the right in Figure 16 as a $g = 4$ minor. Any extension of this graph to a trivalent $g = 5$ graph can then be checked to have the graph on the left in Figure 15 as a trivalent minor. This shows our results are best possible, excluding the cases $n = 0$, $g = 2, 3, 4$.

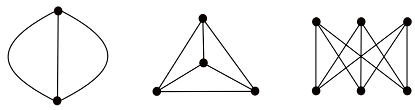

Fig. 16

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