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Globalization of supercuspidal representations over function fields and applications

Abstract. Let H be a connected reductive group defined over a non-archimedean local field F of characteristic $p > 0$. Using Poincaré series, we globalize supercuspidal representations of H_F in such a way that we have control over ramification at all other places, and such that the notion of distinction with respect to a unipotent subgroup (indeed more general subgroups) is preserved. In combination with the work of Vincent Lafforgue on the global Langlands correspondence, we present some applications, such as the stability of Langlands–Shahidi γ-factors and the local Langlands correspondence for classical groups.

Keywords. Globalization, supercuspidal representations, function fields, local Langlands correspondence

1. Statement of results

In this paper, we present a useful globalization result for supercuspidal representations over a non-archimedean local field of characteristic $p > 0$.

Theorem 1.1. *Suppose we are given the following data:*

- $k = \mathbb{F}_q(Y)$ *is the global function field of an absolutely irreducible smooth projective curve Y over a finite field $\mathbb{F}_q$, with associated ring of adeles $\mathbb{A}$;*
- S_0 *is a nonempty finite set of places of k;*
- H *is a smooth connected reductive group over k with Z the identity component of its center;*
- $N \subset H$ *is a (possibly trivial) smooth connected k-split unipotent subgroup over k;*
- $\chi = \prod_v \chi_v : N(\mathbb{A}) \to \mathbb{C}^\times$ *is a (possibly trivial) unitary character trivial on $N(k)$;*
- $\omega = \prod_v \omega_v$ *is a character of $Z(k)\backslash Z(\mathbb{A})$;*
- *for each $v_0 \in S_0$, π_{v_0} is a supercuspidal representation of $H(k_{v_0})$ which is $(Z(k_{v_0}), \omega_{v_0})$- and $(N(k_{v_0}), \chi_{v_0})$-distinguished, i.e.*

$$\mathrm{Hom}_{Z(k_{v_0}) \cdot N(k_{v_0})}(\pi_{v_0}, \omega_{v_0} \otimes \chi_{v_0}) \neq 0.$$

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Then there exists a cuspidal representation Π of $H(\mathbb{A})$ satisfying:

  (i) *for all $v_0 \in S_0$, $\Pi_{v_0} \cong \pi_{v_0}$;*
 (ii) *for all $v \notin S_0$, Π_v is a constituent of a principal series representation induced from a minimal parabolic subgroup of H_v and whose restriction to the derived group H_v^{der} has depth 0;*
(iii) *Π has central character ω and nonzero automorphic (N, χ)-period.*

We make a few remarks:

– If N is trivial, then the local condition and global conclusion about (N, χ) are vacuous and thus (N, χ) can be suppressed.
– If H is quasi-split over k, N is a maximal unipotent subgroup of H and χ is a generic character, then (iii) says that Π is globally χ-generic. Moreover, (ii) implies that for $v \notin S_0$, Π_v is induced from the Borel subgroup.
– One has to be careful in working with unipotent subgroups over nonperfect fields, even if one is working with smooth connected groups, as these may not be k-split (i.e. successive extensions of the additive group $\mathbb{G}_{\mathrm{a}}$). For these subtleties, the reader can consult [O, Chap. 5] or [CGP, Appendix B]. In this paper, we shall only consider smooth connected k-split unipotent groups and these are isomorphic to affine spaces as algebraic varieties. We shall abbreviate the terminology by simply referring to these as unipotent groups without further comment.

We should mention that the cuspidal representation in Theorem 1.1 is constructed by means of Poincaré series. Such globalization results were first proved by Henniart [H] and extended by Vignéras [V] (over arbitrary global fields) and Shahidi [Sh2] (over number fields). The recent paper of Moy–Muić [MM] further refines this series of results over number fields, allowing one to globalize nonsupercuspidal representations (under certain hypotheses). There is also an analogous globalization result due to S. W. Shin [Shin] over totally real fields, proved using the Arthur trace formula. However, in all these versions, one loses control of the local component of the cuspidal representation at one place of k, typically an archimedean place. Our theorem, on the other hand, gives rather good control at all places. The proof of our theorem is inspired by [HL3, Theorem 3.3], which is a special case of Theorem 1.1 in the context of GL_n. The slight improvement over the treatment in [HL3, Theorem 3.3] is that we make no use of the fact/hypothesis that supercuspidal representations can be constructed by compact induction. For the case of generic representations of quasi-split reductive groups mentioned in the remark above, a proof can also be found in [L2, §4.1].

The following corollary of Theorem 1.1 is useful in practice.

Corollary 1.2. *Let F be a local field of characteristic $p > 0$ and let H_F be a connected reductive group over F with Z_F the identity component of its center and N_F the unipotent radical of a parabolic F-subgroup $P_F = M_F \cdot N_F$ (so N_F is possibly trivial here). Assume that χ_F is a unitary character of N_F which lies in an open M_F-orbit. Suppose that $\pi_1, \ldots, \pi_a$ is a collection of supercuspidal representations of H_F which have the same central character under Z_F and which are (N_F, χ_F)-distinguished (and hence distinguished with respect to any character in the same M_F-orbit as χ_F). Then there exist*

- *a global function field k, with a finite set $\{v_1, \ldots, v_a\}$ of places and isomorphisms $k_{v_i} \cong F$;*
- *a connected reductive k-group H with isomorphisms $H_{v_i} \cong H_F$, containing a parabolic k-subgroup $P = M \cdot N$ such that $P_{v_i} \cong P_F$;*
- *a unitary character χ of $N(\mathbb{A})$ trivial on $N(k)$ such that χ_{v_i} and χ_F lie in the same M_F-orbit for each i;*
- *a cuspidal automorphic representation Π of $H(\mathbb{A})$ which is globally (N, χ)-distinguished, with $\Pi_{v_i} \cong \pi_i$ for $i = 1, \ldots, a$, and with Π_v contained in a principal series representation induced from a minimal parabolic subgroup for all other v, such that Π_v is of depth 0 when restricted to H_v^{der}.*

The main point of the corollary is that only local data is given, and so one needs to global-ize several objects (such as the field, the various groups and the various characters) before one is in a position to apply Theorem 1.1. Moreover, if we set $W_F = \mathrm{Hom}_F(N_F, \mathbb{G}_a)$ and fix a nontrivial character ψ_F of F, then composition with ψ_F identifies the F-vector space W_F with the set of unitary characters of N_F. Thus χ_F is an element of W_F and we are requiring in the corollary that its M_F-orbit is Zariski open in W_F.

Using the globalization of supercuspidal representations as given by Theorem 1.1 and Corollary 1.2, the second author has completed the Langlands–Shahidi theory in posi-tive characteristic [L1, L2], following Shahidi's work [Sh2] in characteristic zero. In par-ticular, one has a characterization of the Langlands–Shahidi gamma factors for generic representations of quasi-split groups over function fields by the usual properties: multi-plicativity, compatibility with class field theory in the case of tori and global functional equations. Special cases of this characterization over function fields were shown in [HL1, HL2, HL3, GL]. We refer the reader to [L2] for the general results.

When one combines Theorem 1.1 with the Langlands–Shahidi theory and the recent work [La] of V. Lafforgue on the global Langlands correspondence over function fields, one can obtain further applications. Let us highlight some of these here:

- In Theorem 5.1, we show the stability of general Langlands–Shahidi gamma factors in positive characteristic.
- In Theorem 6.1, we express the Plancherel measure (associated with parabolic induc-tion) in terms of Galois-theoretic gamma factors.
- Building upon these results, and appealing to the work of V. Lafforgue, L. Lafforgue, Deligne and others, we can attach local L-parameters to supercuspidal representations of quasi-split classical groups under a working hypothesis (see §7.5 and Theorem 7.5).
- Our results on Plancherel measure (together with a result of Silberger) also allow us to verify the basic assumption (BA) in the work of Mœglin–Tadić [MT] on the clas-sification of discrete series representations of quasi-split classical groups in terms of supercuspidal ones. As a consequence, we can extend the local Langlands correspon-dence for supercuspidal representations obtained above to all discrete series represen-tations, and then to all irreducible smooth representations by Langlands classification (see Theorem 7.13).

We will discuss these various applications in §5, §6 and §7. In §8, we discuss another approach to extending the local Langlands correspondence of classical groups from su-

percuspidal representations to discrete series representations, using a (conjectural) simple form of the Arthur–Selberg trace formula. The main point is to globalize a discrete series representation (in the style of our main theorem), but the Poincaré series argument does not apply, which necessitates the use of the trace formula. We hope that this application will provide some impetus for the systematic development of the local theory of invariant harmonic analysis and the global theory of the trace formula in positive characteristic.

The applications above are all obtained via a global-to-local argument, using a globalization result of the type in the theorem. As we mentioned above, in such globalization, one often loses control at some place of k. In characteristic 0, one sacrifices the archimedean places, and so one needs to have already established the desired theorem at archimedean places by purely local means. The local proof of the archimedean theorem could be highly nontrivial but is thankfully more attainable than the nonarchimedean one. In the context of the Langlands–Shahidi theory in characteristic 0, this archimedean input was provided by Shahidi [Sh1]. The main stumbling block preventing the development of the Langlands–Shahidi theory in positive characteristic was the constraint that one cannot sacrifice any local place, since the desired result is not known at any place. Hence, it would appear that this situation is one of the few instances where having some archimedean places is a blessing instead of a curse, which is quite contrary to the general principle that function fields are easier to handle than number fields because of a rich underlying geometry. Another such instance is the state of the Arthur–Selberg trace formula. With Theorem 1.1, however, we remove the previous constraint and there is no longer a need to sacrifice any place over a function field. So the globalization of supercuspidal representations over function fields turns out to be easier to handle than for number fields.

Finally, we show a variant of Theorem 1.1, which is a refinement in positive characteristic of a theorem of D. Prasad and R. Schulze-Pillot [PSP, Theorem 4.1] on globalizing supercuspidal representations that are distinguished with respect to a given closed algebraic subgroup (which is not necessarily unipotent):

Theorem 1.3. *Suppose we are given the following data:*

- $k = \mathbb{F}_q(Y)$ *is the global function field of an absolutely irreducible smooth projective curve Y over a finite field $\mathbb{F}_q$, with associated ring of adeles $\mathbb{A}$;*
- S_0 *is a nonempty finite set of places of k;*
- H *is a connected reductive group over k, with Z the identity component of its center;*
- $R \subset H$ *is a closed algebraic k-subgroup containing Z and such that R/Z has no nontrivial k-rational characters;*
- $\chi = \prod_v \chi_v : R(\mathbb{A}) \to \mathbb{C}^\times$ *is a (possibly trivial) unitary character trivial on $R(k)$;*
- *for each $v_0 \in S_0$, π_{v_0} is a supercuspidal representation of $H(k_{v_0})$ which is $(R(k_{v_0}), \chi_{v_0})$-distinguished, i.e. $\mathrm{Hom}_{R(k_{v_0})}(\pi_{v_0}, \chi_{v_0}) \neq 0$.*

We make the following two technical assumptions:

(a) *there is a semisimple algebraic representation $\iota : H \to \mathrm{GL}(V)$ defined over k such that R is the stabilizer of a vector $x_0 \in V(k)$ and $\mathrm{Lie}(R)$ is the infinitesimal stabilizer of x_0;*

(b) *for all places $v \notin S_0$, there exists an Iwahori subgroup I_v^{der} of $H^{\mathrm{der}}(k_v)$ with pro-p radical J_v^{der} such that χ_v is trivial on $R(k_v) \cap J_v^{\mathrm{der}}$.*

Then there exists a cuspidal representation Π of $H(\mathbb{A})$ satisfying:

(i) *for all $v_0 \in S_0$, $\Pi_{v_0} \cong \pi_{v_0}$;*

(ii) *for all $v \notin S_0$, Π_v belongs to a principal series representation induced from a minimal parabolic subgroup and has depth 0 when restricted to H_v^{der};*

(iii) *Π has nonzero automorphic (R, χ)-period.*

We make a couple of remarks about the technical conditions (a) and (b). By a well-known theorem of Chevalley [B, Theorem 5.1 and §5.5], given any closed algebraic subgroup R of H as in Theorem 1.3, there is an algebraic representation $\iota : H \to \mathrm{GL}(V)$ such that R is the stabilizer of a vector $x_0 \in V(k)$ and $\mathrm{Lie}(R)$ is the infinitesimal stabilizer of x_0 (here we are using the hypothesis that R/Z has no nontrivial rational characters). As R. Beuzart-Plessis explained to us, this implies that $R \backslash H$ is quasi-affine and hence $Z(\mathbb{A})R(k) \backslash R(\mathbb{A})$ is a closed subset of $Z(\mathbb{A})H(k) \backslash H(\mathbb{A})$. Since cusp forms on $H(k) \backslash H(\mathbb{A})$ are compactly supported modulo $Z(\mathbb{A})$, the automorphic (R, χ)-period is absolutely convergent and it makes sense to consider it. However, in the above, there is no guarantee that ι is a semisimple representation, and this semisimplicity is of course an issue in characteristic $p > 0$. For our proof of Theorem 1.3, we need ι to be semisimple and this explains the technical condition (a). For (b), note that it is satisfied automatically for almost all places v, and is satisfied for all $v \notin S_0$ if χ is the trivial character.

2. Proof of Theorem 1.1

In this section, we give the proof of Theorem 1.1. We first assume that H is semisimple. Even though a uniform argument can be given, we shall first deal with the case when H is split, as it is notationally cleaner and conceptually simpler. Throughout, let $C_N = \prod_v C_{N,v} \subset N(\mathbb{A})$ be a compact subgroup which projects surjectively onto $N(k) \backslash N(\mathbb{A})$. Note that C_N exists because $N(k) \backslash N(\mathbb{A})$ is compact and the totally disconnected group $N(\mathbb{A})$ contains arbitrarily large open compact subgroups.

2.1. Split semisimple case

With H a split semisimple group, choose an inclusion

$$\iota : H \to \mathrm{SL}_n \subset \mathrm{GL}_n$$

over k, which allows us to identify H as a closed subgroup of GL_n. Then we have the pullback of the n^2 coordinate functions x_{ij} on H. Without loss of generality, we may assume that the intersection of H with the upper (respectively lower) triangular Borel subgroup of GL_n is a Borel subgroup $B = T \cdot U$ (respectively $\overline{B} = T \cdot \overline{U}$) of H, and that $N \subset U$. Indeed, since N is unipotent, we may choose a Borel subgroup $B = T \cdot U$ of H such that $N \subset U$. Then $\iota(B) \subset \mathrm{SL}_n$ is a connected solvable subgroup and one may conjugate the pair $\iota(T)$ and $\iota(B)$ to lie inside the diagonal torus and the standard Borel subgroup of upper triangular matrices in SL_n respectively, from which it follows

that $\iota(\overline{U})$ is conjugated into the subgroup of lower triangular unipotent matrices. As an affine space, we may write $U = N \times N'$ with N and N' affine subspaces.

Let $\mathcal{O}_{S_0}$ denote the ring of S_0-integers (i.e. the subring of elements of k which have no poles outside S_0). Then the "natural" $\mathcal{O}_{S_0}$ integral structure on GL_n induces one on H and N. Now let S be a finite set of places of k disjoint from S_0 such that for all $v \notin S \cup S_0$,

- the groups H, B, T and U are smooth over $\mathcal{O}_v$ and $H(\mathcal{O}_v) = H(k_v) \cap \mathrm{GL}_n(\mathcal{O}_v)$ is a hyperspecial maximal compact subgroup;
- the intersections of $H(k_v)$ with the upper triangular and lower triangular Iwahori subgroups of $\mathrm{GL}_n(k_v)$ give Iwahori subgroups of $H(k_v)$; we denote these by I_v^+ and I_v^- respectively;
- the decomposition $U = N \times N'$ is defined over $\mathcal{O}_v$, with N and N' smooth over $\mathcal{O}_v$;
- $C_{N,v} = N(\mathcal{O}_v)$ and χ_v is trivial when restricted to $N(\mathcal{O}_v)$.

Note that the above conditions can be achieved when S is large enough. For the first condition, see [T, §3.9]. For the second condition, suppose that v is a place with associated residue field κ_v such that the first condition holds. Then one has a commutative diagram induced by the projection map $\mathcal{O}_v \to \kappa_v$:

$$
\begin{array}{ccccc}
B(\mathcal{O}_v) & \longrightarrow & H(\mathcal{O}_v) & \longrightarrow & \mathrm{GL}_n(\mathcal{O}_v) \\
\downarrow & & \downarrow & & \downarrow \\
B(\kappa_v) & \longrightarrow & H(\kappa_v) & \longrightarrow & \mathrm{GL}_n(\kappa_v)
\end{array}
$$

where $B(\kappa_v)$ is a Borel subgroup of $H(\kappa_v)$ and is the intersection of $H(\kappa_v)$ with the standard (upper triangular) Borel subgroup in $\mathrm{GL}_n(\kappa_v)$. Now recall from [T, §3.7] that the preimage in $\mathrm{GL}_n(\mathcal{O}_v)$ of the standard Borel subgroup of $\mathrm{GL}_n(\kappa_v)$ is the standard Iwahori subgroup of $\mathrm{GL}_n(k_v)$ and the preimage in $H(\mathcal{O}_v)$ of $B(\kappa_v)$ is an Iwahori subgroup of $H(k_v)$. It follows from this that the second condition holds.

Now fix an open compact subset C_{S_0} of $H(k_{S_0}) = \prod_{v \in S_0} H(k_v)$ and some nonempty finite set of places $S_1 \cup S_2$ of k disjoint from $S \cup S_0$. We are going to define an open compact subset $C = \prod_v C_v$ as follows:

- we take
$$\prod_{v \in S_0} C_v = C_{S_0};$$

- for $v \in S$, let C_v be an Iwahori subgroup I_v of $H(k_v)$ such that the character χ_v restricted to $N(k_v) \cap I_v$ is trivial;
- for $v \in S_1$, let C_v be the pro-p radical J_v^+ of I_v^+;
- for $v \in S_2$, let C_v be the Iwahori subgroup I_v^-;
- for all other places v, let $C_v = K_v = H(\mathcal{O}_v)$.

Note that for the second condition above, the desired Iwahori subgroup always exists. Indeed, suppose one starts with any Iwahori subgroup I_v' stabilizing a chamber in the apartment associated to T in the Bruhat–Tits building of H. For $t \in T(k_v)$, the compact open subgroup $t I_v' t^{-1} \cap N(k_v)$ can be made arbitrarily small by taking t sufficiently

deep into the positive Weyl chamber, i.e. by ensuring that $|\alpha(t)|_v$ is sufficiently small for all positive roots of H with respect to (T, B). Since χ_v is smooth, it will be trivial on $t I'_v t^{-1} \cap N(k_v)$ when the latter is sufficiently small, and one can take $I_v = t I'_v t^{-1}$.

The following is a key lemma:

Lemma 2.1. *If S_1 and S_2 are sufficiently large, then*

$$H(k) \cap C \cdot C_N \subset N(k),$$

with the intersection occurring in $H(\mathbb{A})$. Indeed, one may take S_1 and S_2 to be singleton sets, each containing a place whose residue field is sufficiently large.

Proof. We regard $\gamma \in H(k)$ as an element in $\mathrm{GL}_n(k)$, so that γ is determined by its coordinates $x_{ij}(\gamma) \in k$. Consider $\gamma \in H(k) \cap C \cdot C_N$. Away from the set $T = S \cup S_0 \cup S_1 \cup S_2$ of places,

$$C^T \cdot C_N^T = \prod_{v \notin T} K_v,$$

and so $x_{ij}(\gamma) \in \mathcal{O}_v$ for $v \notin T$. Hence $x_{ij}(\gamma)$ has no poles at the closed points of Y outside T. We now consider the places in T:

- At places $v \in S_0 \cup S$, $x_{ij}(\gamma)$ has bounded orders of poles (determined by the compact sets $C_v \cdot C_{N,v}$). More precisely, there exists a positive integer M (depending only on $C_{N,v}$, χ_v and C_v for $v \in S \cup S_0$) such that for all i, j, the order of poles of $x_{ij}(\gamma)$ is at most M at all $v \in S_0 \cup S$.
- At places $v_1 \in S_1$, the condition $\gamma \in C_{v_1} \cdot C_{N,v_1}$ implies that $x_{ij}(\gamma)$ vanishes at v_1 for all $i > j$, and $x_{ii}(\gamma) - 1$ vanishes at v_1 for all i.
- At places $v_2 \in S_2$, the condition $\gamma \in C_{v_2} \cdot C_{N,v_2}$ implies at least that $x_{ij}(\gamma)$ has no poles at v_2.

Since a principal divisor on Y has degree 0 (i.e. by the product formula), it is clear that if S_1 is sufficiently large, then $x_{ij}(\gamma) = 0$ for all $i > j$ and $x_{ii}(\gamma) = 1$ for all i. In particular, any $\gamma \in H(k) \cap C \cdot C_N$ is strictly upper triangular and hence lies in $U(k)$. In fact, one could take S_1 to contain only one place v_1 whose residue field (and hence the Galois orbit of points on $Y(\overline{\mathbb{F}}_q)$ associated to v_1) is sufficiently large. This is possible since $Y(\overline{\mathbb{F}}_q)$ is infinite whereas $Y(\mathbb{F}_{q^n})$ is finite for each n.

Now we claim that if S_2 is sufficiently large, then $\gamma \in C_{v_2} \cdot C_{N,v_2}$ implies $\gamma \in N(k)$. Indeed, one may change coordinates on U so that with respect to the new coordinates y_{ij} $(i < j)$, the subspace N is defined by the vanishing of a subset Σ of the y_{ij}'s. For $v \in S_2$, $U = N \times N'$ is defined over $\mathcal{O}_v$. For $\gamma \in U(k) \cap C_{v_2} \cdot C_{N,v_2}$, one sees that $y_{ij}(\gamma) \in \mathcal{O}_{v_2}$ for all $i < j$, and $y_{ij}(\gamma)$ vanishes at v_2 for $(i, j) \in \Sigma$. Hence if S_2 is sufficiently large, then $y_{ij}(\gamma) = 0$ for all $(i, j) \in \Sigma$, and we conclude that $\gamma \in N(k)$. As above, one could also have taken S_2 to consist of a single place v_2 whose residue field is sufficiently large. $\square$

We shall now define a test function $f = f_{S_1, S_2} = \prod_v f_v \in C_c^\infty(H(\mathbb{A}))$ as follows: For $v_0 \in S_0$, take

$$f_{v_0}(h) = \langle w_{v_0}^\vee, h \cdot w_{v_0} \rangle$$

to be the (compactly supported) matrix coefficient of π_{v_0} formed using nonzero vectors $w_{v_0} \in \pi_{v_0}$ and $w_{v_0}^\vee \in \pi_{v_0}^\vee$ such that

$$\int_{N(k_{v_0})} \chi(u)^{-1} \cdot \langle w_{v_0}^\vee, u \cdot w_{v_0} \rangle \, du \neq 0.$$

This is always possible since π_{v_0} is supercuspidal and $(N(k_{v_0}), \chi_{v_0})$-distinguished.

More precisely, let $\ell \in \operatorname{Hom}_{N(k_{v_0})}(\pi_{v_0}, \chi_{v_0})$ be a nonzero element. For any nonzero vector $w_1 \in \pi_{v_0}$, one has $\pi_{v_0}(C_c^\infty(H(k_{v_0}))) \cdot w_1 = \pi_{v_0}$ and so there exists $\varphi \in C_c^\infty(H(k_{v_0}))$ such that $\ell(\pi_{v_0}(\varphi) \cdot w_1) \neq 0$. On the other hand, the map $\phi \mapsto \pi_{v_0}(\phi)$ is an $H(k_{v_0}) \times H(k_{v_0})$-equivariant projection

$$C_c^\infty(H(k_{v_0})) \to \operatorname{End}^\infty(\pi_{v_0}) = \pi_{v_0} \otimes \pi_{v_0}^\vee$$

onto the maximal $\pi_{v_0} \otimes \pi_{v_0}^\vee$-isotypic quotient of $C_c^\infty(H(k_{v_0}))$. Since π_{v_0} is supercuspidal, this quotient in fact occurs as a submodule and

$$C_c^\infty(H(k_{v_0})) \cong (\pi_{v_0} \otimes \pi_{v_0}^\vee) \oplus \mathcal{C}'$$

where $\mathcal{C}'$ does not contain $\pi_{v_0} \otimes \pi_{v_0}^\vee$ as a subquotient. Moreover, the submodule $\pi_{v_0} \otimes \pi_{v_0}^\vee$ is realized by the formation of matrix coefficients of $\pi_{v_0}^\vee$. Hence, since $\pi_{v_0}(\varphi) \neq 0$, we may assume that

$$\varphi(h) = \langle w, h \cdot w^\vee \rangle$$

is a matrix coefficient of $\pi_{v_0}^\vee$. Then

$$0 \neq \ell(\pi_{v_0}(\varphi) \cdot w_1) = \int_{H(k_{v_0})} \varphi(h) \cdot \ell(h \cdot w_1) \, dh$$

$$= \int_{N(k_{v_0}) \backslash H(k_{v_0})} \ell(h \cdot w_1) \cdot \left(\int_{N(k_{v_0})} \chi(u) \cdot \varphi(uh) \, du \right) dh.$$

Thus, for some h, the inner integral is nonzero, as desired.

Now let C_{S_0} be the support of $f_{S_0} = \prod_{v \in S_0} f_{v_0}$ and define $C = \prod_v C_v$ as above. For $v \notin S_0$, let f_v be the characteristic function of C_v. Then $C = \prod_v C_v$ is the support of f.

Now we consider the Poincaré series associated to f:

$$P(f)(h) = \sum_{\gamma \in H(k)} f(\gamma h),$$

so that $P(f) \in C^\infty(H(k) \backslash H(\mathbb{A}))$. The fact that π_{v_0} is supercuspidal implies that $P(f)$ is a cuspidal function (i.e. all its constant terms vanish). Since $P(f)$ is smooth, it follows by [BJ, Prop. 5.9] that $P(f)$ is a cusp form and hence has compact support on $H(k) \backslash H(\mathbb{A})$ by [BJ, Prop. 5.2]. In particular, $P(f) \in L^2(H(k) \backslash H(\mathbb{A}))$.

To prove Theorem 1.1, we need to show that $P(f)$ is globally (N, χ)-distinguished. We have

$$W_{N,\chi}(P(f)) = \int_{N(k) \backslash N(\mathbb{A})} \chi(u)^{-1} \cdot P(f)(u) \, du = \int_{N(k) \backslash N(\mathbb{A})} \chi(u)^{-1} \cdot \sum_{\gamma \in H(k)} f(\gamma u) \, du.$$

It suffices to sum over those $\gamma \in H(k)$ such that

$$\gamma \in H(k) \cap C \cdot C_N.$$

Hence, when S_1 and S_2 are sufficiently large, Lemma 2.1 implies that

$$H(k) \cap C \cdot C_N \subset N(k).$$

Thus,

$$W_{N,\chi}(P(f))(1) = \int_{N(k)\backslash N(\mathbb{A})} \chi(u)^{-1} \cdot \sum_{\gamma \in N(k)} f(\gamma u)\, du = \int_{N(\mathbb{A})} \chi(u)^{-1} \cdot f(u)\, du$$

$$= \prod_v W_v(f_v)$$

where

$$W_v(f_v) = \int_{N(k_v)} \chi_v(u)^{-1} \cdot f_v(u)\, du.$$

Moreover, it follows by construction that for all v,

$$W_v(f_v) \neq 0,$$

and for almost all v, one has $W_v(f_v) = 1$. Thus, we have shown that $W_{N,\chi}(P(f)) \neq 0$, so that $P(f)$ is globally (N, χ)-distinguished. The spectral decomposition of $P(f)$ in $L^2(H(k)\backslash H(\mathbb{A}))$ then gives a cuspidal representation Π such that $\Pi_{v_0} \cong \pi_{v_0}$ for $v_0 \in S_0$, and for all $v \notin S_0$, Π_v has nonzero fixed vectors under a pro-p Sylow subgroup of an Iwahori subgroup of $H(k_v)$. It follows from results of Morris [Mo] and Moy–Prasad [MP, Prop. 6.7 and Theorem 6.11] that for all $v \notin S_0$, Π_v is a constituent of a depth zero principal series representation induced from a Borel subgroup. This proves the theorem in the split semisimple case.

2.2. General semisimple case

We may now consider the case when H is a general semisimple group over k. Let T be a maximal k-torus of H containing a maximal k-split torus. Let E/k be the splitting field of T, so that $H_E = H \times_k E$ is split over E. It is important to note that E is a separable extension of k, since all tori over k are split over a separable closure of k. Choose an embedding

$$H_E \hookrightarrow \mathrm{GL}_n(E) \quad \text{over } E$$

as in the split case; it induces a k-embedding

$$\iota : H \hookrightarrow \mathrm{Res}_{E/k} H_E \hookrightarrow \mathrm{Res}_{E/k} \mathrm{GL}_n.$$

In particular, the intersection of H_E with the upper triangular Borel subgroup of $\mathrm{GL}_n(E)$ is a Borel subgroup $T_E \cdot U_E$ of H_E, and $N_E = N \times_k E \subset U_E$. As in the split case, we may write $U_E = N_E \times N'_E$ as the product of two affine subspaces. Moreover, the $\mathcal{O}_{E,S_0}$-integral structure of $\mathrm{GL}_n(E)$ induces one on H_E and an $\mathcal{O}_{S_0}$-integral structure on H. For any $\gamma \in H(k)$, we may regard γ as a matrix $(x_{ij}(\gamma))$ with $x_{ij}(\gamma) \in E$.

Now let S be a finite set of places of k such that for all $v \notin S \cup S_0$,

- the groups $H \hookrightarrow \mathrm{Res}_{E/k} H_E \hookrightarrow \mathrm{Res}_{E/k} \mathrm{GL}_n(E)$ are smooth reductive groups over $\mathcal{O}_v$, so that their groups of $\mathcal{O}_v$-points are hyperspecial maximal compact subgroups;

- $N \hookrightarrow \mathrm{Res}_{E/k} N_E \hookrightarrow \mathrm{Res}_{E/k} U_E$ are closed immersions of smooth unipotent group schemes over $\mathcal{O}_v$;
- the intersections of $H(E_v)$ with the upper triangular and lower triangular Iwahori subgroups of $\mathrm{GL}_n(E_v)$ are Iwahori subgroups of $H(E_v)$ (where $E_v = E \otimes_k k_v$); for each w lying over v, we denote the Iwahori subgroups of $H(E_w)$ by I_w^+ and I_w^- respectively;
- $C_{N,v} = N(\mathcal{O}_v)$ and χ_v is trivial when restricted to $N(\mathcal{O}_v)$.

The existence of such a finite set S follows from the same argument as in the split semisimple case considered above. Now fix nonempty finite sets of places S_1 and S_2 of k disjoint from $S \cup S_0$ such that E splits completely at any $v \in S_1 \cup S_2$. We fix a test function $f = \bigotimes_v f_v \in C_c^\infty(H(\mathbb{A}))$ as follows:

- For places $v \in S_0 \cup S$, we let f_v be as in the split case.
- For places $v \in S_1$, suppose that the places of E over v are w_i, $i = 1, \dots, d$. Then one has natural isomorphisms

$$E \otimes_k k_v \cong \prod_i E_{w_i} \cong k_v \times \cdots \times k_v \quad (d \text{ times})$$

inducing an isomorphism

$$H_v \times_{k_v} E_v \cong \prod_i H_{E,w_i} \cong H_v \times \cdots \times H_v.$$

The natural embedding

$$\rho : H_v \hookrightarrow H_v \times_{k_v} E_v \cong \prod_i H_{E,w_i}$$

is diagonal, in the sense that the projection onto any factor of the product $\prod_i H_{E,w_i}$ is an isomorphism over k_v. Moreover, by our choice of S, the preimage under ρ of $\prod_i H(\mathcal{O}_{w_i})$ in $H(k_v)$ is the hyperspecial maximal compact subgroup $H(\mathcal{O}_v)$. In particular,

$$H(k_v) \cap \rho^{-1}\Big(I_{w_1}^+ \times \prod_{i>1} H(\mathcal{O}_{w_i})\Big)$$

is an Iwahori subgroup I_v^+ of $H(k_v)$. We then take f_v to be the characteristic function of the pro-p radical J_v^+ of I_v^+.
- For places $v \in S_2$, the analogous discussion as for S_1 defines an Iwahori subgroup I_v^- of $H(k_v)$, and we let f_v be the characteristic function of I_v^-.
- For all other places v, let f_v be the characteristic function of $K_v = H(\mathcal{O}_v)$.

Let $C_f = \prod_v C_{f,v}$ be the support of f. Then we claim that Lemma 2.1 continues to hold, i.e.

$$H(k) \cap C_f \cdot C_N \subset N(k).$$

To see this, note that:

- At places w of E lying above places of k outside $S \cup S_0 \cup S_1 \cup S_2$, $x_{ij}(\gamma) \in \mathcal{O}_w$.
- At each place w of E lying above $S \cup S_0$, the maximal order of poles of $x_{ij}(\gamma)$ (for all i, j) is at most some integer M.

- At places w of E lying over $S_1 \cup S_2$, $x_{ij}(\gamma)$ lies in $\mathcal{O}_w$. Moreover, for each $v_1 \in S_1$, there is at least one place w_1 lying over v_1 such that $x_{ij}(\gamma)$ vanishes at w_1 for all $i > j$ and $x_{ii}(\gamma) - 1$ vanishes at w_1 for all i. Together with the above, we see that if S_1 is sufficiently large, then $x_{ij}(\gamma) = 0$ for all $i > j$ and $x_{ii} = 1$ for all i.
- We have thus shown that

$$H(k) \cap C_f \cdot C_N \subset U_E(E) \subset H(E).$$

To show the desired statement, it remains to show that

$$H(k) \cap C_f \cdot C_N \subset N(E).$$

This follows by the same argument as in the split case (if S_2 is sufficiently large).

With the key lemma in hand, we may now form the Poincaré series $P(f)$ and show the nonvanishing of $W_{N,\chi}(P(f))$ by the same argument as in the split case. Hence Theorem 1.1 is proved when H is semisimple.

2.3. Reductive case

We now deal with the general reductive case. Consider the semisimple group $\bar{H} := H/Z$ over k and let

$$r : H \to \bar{H} = H/Z$$

be the natural projection map. In the semisimple case, we have constructed an open compact subset of $\bar{H}(\mathbb{A})$ of the form

$$\bar{C} = r(C_{S_0}) \times \prod_{v \in S} \bar{I}_v \times \prod_{v \in S_1} \bar{J}_v^+ \times \prod_{v \in S_2} \bar{I}_v^- \times \prod_{\text{other } v} \bar{K}_v$$

where

- C_{S_0} is the support of an appropriate matrix coefficient of the given supercuspidal representation π_{S_0} of $H(k_{S_0})$, which is compact modulo $Z(k_{S_0})$;
- S is some finite set of places and $\bar{I}_v$ is some Iwahori subgroup $\bar{I}_v$ of $\bar{H}(k_v)$ for $v \in S$;
- S_1 is some finite set of places and $\bar{J}_v^+$ is the pro-p unipotent radical of some Iwahori subgroup I_v^+ of $\bar{H}(k_v)$ for $v \in S_1$;
- S_2 is some finite set of places and $\bar{I}_v^-$ is some Iwahori subgroup of $\bar{H}(k_v)$ for $v \in S_2$;
- $\bar{K}_v$ is a hyperspecial maximal compact subgroup of $\bar{H}(k_v)$ for all other v's.

The key property of $\bar{C}$ is that expressed in Lemma 2.1:

$$\bar{H}(k) \cap \bar{C} \cdot r(C_N) \subset r(N(k)).$$

In the construction of $\bar{C}$, we may further assume that S is taken to be so large that, in addition to the conditions satisfied in the construction for $\bar{H}$, the groups H and Z and the character ω of Z are unramified outside S.

Note that for any Iwahori subgroup $\bar{I}_v$ of $\bar{H}(k_v)$, there is a unique Iwahori subgroup I_v of $H(k_v)$ such that

$$r^{-1}(\bar{I}_v) = Z(k_v) \cdot I_v.$$

Indeed, the Bruhat–Tits building $\mathcal{B}(H(k_v))$ of $H(k_v)$ projects onto the building $\mathcal{B}(\bar{H}(k_v))$ of $\bar{H}(k_v)$; if $\bar{I}_v$ is associated with a chamber $\bar{\mathcal{C}}_v \subset \mathcal{B}(\bar{H}(k_v))$, then I_v is associated with the unique chamber $\mathcal{C}_v \subset \mathcal{B}(H(k_v))$ projecting onto $\bar{\mathcal{C}}_v$. Likewise, there is a unique hyperspecial maximal compact subgroup K_v such that $r^{-1}(\bar{K}_v) = Z(k_v) \cdot K_v$. The preimage of $\bar{C}$ in $H(\mathbb{A})$ is

$$r^{-1}(\bar{C}) = C_{S_0} \times \prod_{v \in S} r^{-1}(\bar{I}_v) \times \prod_{v \in S_1} r^{-1}(\bar{J}_v^+) \times \prod_{v \in S_2} r^{-1}(\bar{I}_v^-) \times \prod_{\text{other } v} r^{-1}(\bar{K}_v).$$

We shall modify the subgroup $r^{-1}(\bar{I}_v) = Z(k_v) \cdot I_v$ at the places $v \in S$ slightly. For $v \in S$, let J_v be the pro-p radical of I_v and consider

$$J_v^{\text{der}} = H^{\text{der}}(k_v) \cap J_v,$$

which is the pro-p radical of the Iwahori subgroup $I_v^{\text{der}} = H^{\text{der}}(k_v) \cap I_v$ of the derived group $H^{\text{der}}(k_v)$. The subgroup $Z(k_v) \cdot J_v^{\text{der}}$ of $H(k_v)$ is compact modulo $Z(k_v)$. Moreover, observe that

$$Z(k_v) \cap J_v^{\text{der}} = \{1\}.$$

Indeed, $Z(k_v) \cap J_v^{\text{der}} \subset Z(k_v) \cap H^{\text{der}}(k_v)$ is a finite p-group. However, in $Z(k_v)$, there are no nontrivial elements of finite p-power order, because k has characteristic p. Hence

$$Z(k_v) \cdot J_v^{\text{der}} = Z(k_v) \times J_v^{\text{der}}.$$

Now we set

$$C = C_{S_0} \times \prod_{v \in S} Z(k_v) \cdot J_v^{\text{der}} \times \prod_{v \in S_1} r^{-1}(\bar{J}_v^+) \times \prod_{v \in S_2} r^{-1}(\bar{I}_v^-) \times \prod_{\text{other } v} r^{-1}(\bar{K}_v),$$

and note that $C \subset r^{-1}(\bar{C})$. We may define the following test function $f = \prod_v f_v$:

- For the places $v_0 \in S_0$, let f_{v_0} be a matrix coefficient of π_{v_0} as in the semisimple case.
- For all other places v, let f_v be supported on C_v, equivariant under $Z(k_v)$ with respect to ω_v and equal to 1 on the relevant compact subgroups J_v^{der}, J_v^+, I_v^- or K_v. This is possible because, by construction, ω_v is trivial on the intersection of $Z(k_v)$ with the relevant compact subgroup. In particular, our discussion above says that at places in S, there is no compatibility to check; this is the main reason for using J_v^{der} for $v \in S$.

Now f is supported on C and equivariant with respect to ω under $Z(\mathbb{A})$, so that f is left $Z(k)$-invariant. Define the Poincaré series

$$P(f)(h) = \sum_{\gamma \in Z(k) \backslash H(k)} f(\gamma h).$$

Then $P(f)$ is a cuspidal automorphic function on $H(k) \backslash H(\mathbb{A})$ with central character ω under $Z(\mathbb{A})$, and thus belongs to the space $L_\omega^2(Z(\mathbb{A}) H(k) \backslash H(\mathbb{A}))$ of functions which are $(Z(\mathbb{A}), \omega)$-equivariant and square-integrable on $Z(\mathbb{A}) H(k) \backslash H(\mathbb{A})$. Moreover, observe that the projection r induces an injection

$$r : Z(k) \backslash (H(k) \cap C \cdot C_N) \hookrightarrow \bar{H}(k) \cap \bar{C} \cdot r(C_N) \subset r(N(k)).$$

Hence, we deduce that

$$H(k) \cap C \cdot C_N \subset Z(k) \cdot N(k).$$

Then

$$W_{N,\chi}(P(f)) = \int_{N(\mathbb{A})} f(n) \cdot \chi(n)^{-1} \, dn = \prod_v W_v(f_v) \neq 0$$

as before. This completes the proof of Theorem 1.1.

3. Proof of Corollary 1.2.

In this section, we give the proof of Corollary 1.2 and will use the notations in the corollary. The corollary requires us to globalize a number of objects and we need to deal with each in turn before we are in a position to apply Theorem 1.1.

3.1. Globalizing the field

Consider the local field $F \cong \mathbb{F}_q((t))$. In many applications, it suffices to simply take $k_0 = \mathbb{F}_q(t)$ to be the function field of $\mathbb{P}_1$ over $\mathbb{F}_q$ and a place v_0 of k_0 such that $k_{0,v_0} \cong F$. However, we shall also need the following well-known fact; we briefly explain how it can be achieved by Krasner's lemma (see also [H, Lemma 3.6]).

Lemma 3.1. *Given a finite Galois extension E/F of local fields, one can find a finite Galois extension k_1/k of global fields with $[k_1 : k] = [E : F]$ and a place v of k such that $k_v \cong F$ and $k_1 \otimes_k k_v \cong E$. In particular, the natural map $\mathrm{Gal}(E/F) \hookrightarrow \mathrm{Gal}(k_1/k)$ is an isomorphism.*

Proof. Suppose that $E = F(\alpha)$ (by separability) and let $f(x) \in F[x]$ be the (irreducible) minimal polynomial of α. Then E is the splitting field of f. Let $f_0 \in k_0[x]$ be v_0-adically sufficiently close to f coefficientwise, so that f_0 is also irreducible over F. By Krasner's lemma, there is a root α_0 of f_0 which is close to α such that $E = F(\alpha_0)$. Thus, the global field $k_0(\alpha_0)$ satisfies

$$k_0(\alpha_0) \otimes_{k_0} F \cong k_0(\alpha_0) \cdot F = E.$$

However, the extension $k_0(\alpha_0)/k_0$ may not be Galois.

Since E/F is Galois, E contains all the roots of f_0. Let k_1 be the Galois closure (in E) of $k_0(\alpha_0)$ over k_0. Then for any place v_1 of k_1 lying over v_0, we have

$$k_{1,v_1} = k_1 \cdot F = E,$$

so that the associated decomposition group at v_1 is isomorphic to $\mathrm{Gal}(E/F)$. Thus, if we let k be the fixed field of this decomposition group, we obtain an extension k_1/k with v the unique place of k lying below v_1 and inert in k_1. Then we have $k_v \cong F$ and $k_1 \otimes_k k_v \cong E$, so that $\mathrm{Gal}(E/F) \cong \mathrm{Gal}(k_1/k)$. $\qquad\square$

3.2. Globalizing the groups

Next we consider the question of globalizing the pair $P_F \subset H_F$.

Lemma 3.2. *Given a parabolic F-subgroup $P_F \subset H_F$, one can find:*

- *a global function field k with a place v_0 such that $k_{v_0} \cong F$;*
- *a pair $P \subset H$ over k such that P is a parabolic k-subgroup of a connected reductive group H over k with $H_{v_0} \cong H_F$ and $P_{v_0} \cong P_F$.*

Moreover, if Z is the identity component of the center of H, so that $Z_{v_0} \cong Z_F$, then one can ensure that the k-rank of Z is equal to the F-rank of Z_F.

Proof. Assume first that H_F is a quasi-split group. Let H_s be the split form of H_F which is a Chevalley group defined over $\mathbb{Z}$. Fix a Borel subgroup B_s of H_s containing a maximal split torus T_s. Then H_s determines a based root datum

$$\Psi = \Psi(H_s, T_s, B_s) = (X(T_s), \Delta(T_s, B_s), Y(T_s), \Delta(T_s, B_s)^\vee)$$

where $X(T_s)$ and $Y(T_s)$ denote the character and cocharacter groups of T_s respectively, whereas $\Delta(T_s, B_s)$ and $\Delta(T_s, B_s)^\vee$ denote the set of simple roots and simple coroots respectively. The outer automorphism group $\mathrm{Out}(H_s)$ of H_s is a constant group scheme (defined over $\mathbb{Z}$) which is naturally isomorphic to $\mathrm{Aut}(\Psi)$.

Now H_F corresponds to an element in the Galois cohomology set $H^1(F, \mathrm{Out}(H_s))$. In fact, if H_s is split by the finite Galois extension E of F, then H_F determines an element in $H^1(\mathrm{Gal}(E/F), \mathrm{Out}(H_s))$. We pick a 1-cocycle $c : \mathrm{Gal}(E/F) \to \mathrm{Out}(H_s)$ representing the element H_F; it is simply a group homomorphism. This induces an action of $\mathrm{Gal}(E/F)$ on the based root datum Ψ. Much of the structure of H_F is controlled by the Galois module Ψ. For example, the F-rank of Z_F is the dimension of the $\mathrm{Gal}(E/F)$-fixed space in $Y(Z_F) \otimes_\mathbb{Z} \mathbb{Q} \subset Y(T_F) \otimes_\mathbb{Z} \mathbb{Q}$. Moreover, conjugacy classes of parabolic F-subgroups of H_F are in bijection with subsets of $\mathrm{Gal}(E/F)$-orbits on $\Delta(T_s, B_s)$.

By Lemma 3.1, we can find a finite Galois extension k_1 of k and a place v_0 of k such that $k_1 \otimes_k k_{v_0} = k_1 \otimes_k F \cong E$ and $\mathrm{Gal}(k_1/k)$ is naturally isomorphic to $\mathrm{Gal}(E/F)$. By composition with this isomorphism, c gives rise to a 1-cycle $\mathrm{Gal}(k_1/k) \to \mathrm{Out}(H_s)$. This in turn gives rise to a quasi-split group H over k, containing a pair $T \subset B$ of maximal torus contained in a Borel k-subgroup, which globalizes H_F, T_F and B_F. Moreover, the corresponding action of $\mathrm{Gal}(k_1/k)$ on the based root datum Ψ is the same as that of $\mathrm{Gal}(E/F)$ (under the isomorphism of the two Galois groups). Thus, the k-rank of Z is the same as the F-rank of Z_F. Finally, since parabolic subgroups of a quasi-split group are in bijection with subsets of Galois orbits on the set of simple roots in Ψ, there is a parabolic subgroup P of H whose localization at v_0 is P_F. This proves the lemma for H_F a quasi-split group.

Now suppose that H_F is an inner form of a quasi-split group H_F', so that H_F gives rise to an $\mathrm{Aut}(H_F')$-orbit in $H^1(F, H_{F,\mathrm{ad}}')$, where $H_{F,\mathrm{ad}}'$ is the adjoint group of H_F'. The quasi-split group H_F' will contain a parabolic subgroup P_F' which is a form of P_F. By what we showed above, we can find a global field k with a place v_0 with $k_{v_0} \cong F$ such that we may globalize the pair $P_F' \subset H_F'$ to $P' \subset H'$ as in the lemma.

It is known that the natural map

$$H^1(k, H'_{\mathrm{ad}}) \to H^1(F, H'_{F,\mathrm{ad}})$$

is surjective. In characteristic 0, this is a result of Borel–Harder [BH, Theorem 1.7], which has been extended to positive characteristic by N. Q. Thǎńg and N. D. Tân [TT, Theorem 3.8.1]. This shows that one can globalize H_F to a k-group H. However, we need to be more careful if we want to globalize the parabolic subgroup P_F as well.

For this, let $\mathrm{Inn}(H'_F, P'_F)$ denote the inner automorphism group of the pair $P'_F \subset H'_F$. Since parabolic subgroups are self-normalizing, $\mathrm{Inn}(H'_F, P'_F) = P'_{F,\mathrm{ad}}$ (the image of P'_F in $H'_{F,\mathrm{ad}}$). Over k, one similarly has $\mathrm{Inn}(H', P') = P'_{\mathrm{ad}}$. Then we need to show that the map

$$H^1(k, P'_{\mathrm{ad}}) \to H^1(F, P'_{F,\mathrm{ad}})$$

is surjective. But if $M'_{F,\mathrm{ad}}$ is the Levi factor of $P'_{F,\mathrm{ad}}$, then

$$H^1(F, P'_{F,\mathrm{ad}}) \cong H^1(F, M'_{F,\mathrm{ad}})$$

and likewise over k. Hence we need to show the surjectivity of

$$H^1(k, M'_{\mathrm{ad}}) \to H^1(F, M'_{F,\mathrm{ad}}).$$

Let $M'_{F,\mathrm{ad,der}}$ be the derived group of $M'_{F,\mathrm{ad}}$, so that $M'_{F,\mathrm{ad,der}}$ is semisimple and $A_F := M'_{F,\mathrm{ad}}/M'_{F,\mathrm{ad,der}}$ is a split torus. One has the analogous objects over k. Then the long exact sequence in Galois cohomology gives rise to a commutative diagram

$$
\begin{array}{ccccc}
H^1(k, M'_{\mathrm{ad,der}}) & \longrightarrow & H^1(k, M'_{\mathrm{ad}}) & \longrightarrow & H^1(k, A) = 0 \\
\downarrow & & \downarrow & & \\
H^1(F, M'_{F,\mathrm{ad,der}}) & \longrightarrow & H^1(F, M'_{F,\mathrm{ad}}) & \longrightarrow & H^1(F, A_F) = 0
\end{array}
$$

The first vertical arrow is surjective by the result of Thǎńg–Tân [TT, Theorem 3.8.1] alluded to above. It follows that the second vertical arrow is also surjective, so that $P_F \subset H_F$ can be globalized to $P \subset H$. Moreover, since inner automorphisms act as identity on the center of H'_F or H', it is clear that the F-rank of Z_F is the same as the k-rank of Z. This proves the lemma. $\qquad\square$

3.3. Globalizing the character χ_F

By the above lemma, we now have a pair $P = M \cdot N \subset H_0$ over k globalizing $P_F \subset H_F$ over F, with N the unipotent radical of P. Set $W = \mathrm{Hom}(N, \mathbb{G}_a)$, which is a vector group. If we fix a nontrivial character $\psi : k\backslash\mathbb{A} \to \mathbb{C}^\times$, then composition with $\psi = \prod_v \psi_v$ gives an identification

$$W_k = \mathrm{Hom}_k(N, \mathbb{G}_a) \cong \{\text{unitary characters of } N(k)\backslash N(\mathbb{A})\}.$$

Similarly, composition with ψ_{v_0} gives an identification

$$W_F = \mathrm{Hom}_F(N_F, \mathbb{G}_a) \cong \{\text{unitary characters of } N_F\}$$

so that extracting the v_0-component of an automorphic unitary character of N corresponds to the natural inclusion $W_k \subset W_F$. Since the M_F-orbit of χ_F is open (in the Zariski topology of V and hence in the v_0-adic topology of V_F), and W_k is dense in W_F, the M_F-orbit of χ_F contains an element of W_k. Thus, there is an automorphic unitary character χ of N whose local component at v_0 is in the same M_F-orbit as χ_F.

3.4. Globalizing central character

Finally, we need to globalize the central character ω_F. Recall that we have globalized H_F to H over k so that the k-rank of the connected center Z of H is the same as the F-rank of Z_F.

Lemma 3.3. *There exists an automorphic character ω of Z satisfying:*

- $\omega_{v_0} = \omega_F$;
- ω *is trivial on the compact group*

$$\prod_{v \in T} Z(k_v)^1 \times \prod_{v \notin T \cup \{v_0\}} Z(k_v)^0,$$

where T is some nonempty finite set of places of k, $Z(k_v)^0$ is the maximal compact subgroup of $Z(k_v)$, and $Z(k_v)^1$ is its pro-p radical.

Proof. The proof is an elaboration of that of [P, Lemma 3]. To construct ω_0, consider the natural map

$$i : \prod_{v \in T} Z(k_v)^1 \times \prod_{v \notin T} Z(k_v)^0 \to Z(k) \backslash Z(\mathbb{A}).$$

The kernel $\mathrm{Ker}(i)$ is a finite group and we shall show that it is trivial. Choose a splitting field E of Z and regard

$$Z(k) \hookrightarrow Z(E) \cong (E^\times)^r,$$

so that each element $z \in Z(k)$ is determined by r coordinates $z_j \in E^\times$. If $z \in \mathrm{Ker}(i)$, then z lies in the maximal compact subgroup of $Z(k_v)$ for all v and hence lies in $(\mathcal{O}_{E,w}^\times)^r$ for all places w of E. The coordinates z_j of z are thus constant functions on the smooth projective curve $\tilde{Y}$ with function field E. However, at places w lying over $v \in T$, $z_j \in 1 + \varpi_w \mathcal{O}_{E,w}$, so that z_j takes value 1 at such w. This implies that $z_j = 1$, so that $\mathrm{Ker}(i)$ is trivial, as desired.

Since i is injective and its image is compact and hence a closed subgroup of $Z(k) \backslash Z(\mathbb{A})$, we can find a character ω' of $Z(k) \backslash Z(\mathbb{A})$ whose restriction to the image of i is

$$\omega_F|_{Z(k_{v_0})^0} \otimes \left(\text{the trivial character of } \prod_{v \in T} Z(k_v)^1 \times \prod_{v \notin T \cup \{v_0\}} Z(k_v)^0\right).$$

Let $q : Z \to \mathbb{G}_{\mathrm{m}}^m$ be a surjective morphism of algebraic tori over k whose kernel is anisotropic over v_0 (possible since the split k-rank of Z is the same as its split F-rank). Then ω'_{v_0}/ω_F factors through

$$Z(k_{v_0}) \to (k_{v_0}^\times)^m \to \mathbb{Z}^m.$$

Twisting ω'_0 by the pullback to $Z(\mathbb{A})$ of a character of the form $|-|_{\mathbb{A}}^{s_1} \times \cdots \times |-|_{\mathbb{A}}^{s_m}$, we find a character ω of $Z(k)\backslash Z(\mathbb{A})$ satisfying $\omega_{v_0} = \omega_F$, as desired. $\square$

Remark. (i) It is necessary to know that the split k-rank of the torus Z is the same as the split F-rank of Z_F above. Consider the case when $T = \mathrm{U}(1)$ is an anisotropic torus of dimension 1 over k and suppose that v_0 is a place when T splits so that $T(k_{v_0}) \cong k_{v_0}^\times \cong \mathcal{O}_{v_0}^\times \times \mathbb{Z}$. The irreducible representations of $T(k_v)$ are classified by a discrete set of parameters (giving a character of the compact group $\mathcal{O}_v^\times$) and a continuous one (giving the image of $1 \in \mathbb{Z}$). On the other hand, since $T(k)\backslash T(\mathbb{A})$ is compact, its characters are classified by a discrete set of parameters. There are simply too many degrees of freedom at the place v_0 for every character of $T(k_{v_0})$ to be globalizable to a character of $T(k)\backslash T(\mathbb{A})$.

(ii) In the proof of Lemma 3.3, instead of insisting that ω is trivial on $\prod_{v \in T} Z(k_v)^1$, we could have stipulated that ω restricts to any given character of $\prod_{v \in T} Z(k_v)^1$. For example, one may require ω to be highly ramified at places in T. Then the proof of Lemma 3.3 shows that one can globalize ω_F to an automorphic character which is highly ramified at places in T but unramified outside $\{v_0\} \cup T$.

3.5. Proof of the corollary

We are now ready to complete the proof of Corollary 1.2. Let k' be a finite Galois extension of k which splits completely at v_0; suppose that v_0 splits into a different places $w_1, \ldots, w_a$ of k'. We may then base change the data (H, Z, N, χ, ω) to k'. This puts us in a position to apply Theorem 1.1, and the proof of Corollary 1.2 is complete.

4. Proof of Theorem 1.3

In this section, we give the proof of Theorem 1.3. The proof is a nontrivial refinement of that of [PSP, Theorem 4.1], but instead of appealing to the relative trace formula [PSP, Theorem 4.5] as a blackbox, we simply use the Poincaré series argument in the proof of Theorem 1.1. Indeed, the relative trace formula argument is simply a Poincaré series argument, and our treatment makes the argument in [PSP] somewhat more transparent.

By the technical assumption (a), one can find a semisimple algebraic representation (over k)

$$\iota : H \to \mathrm{GL}(V)$$

such that R is the stabilizer of a vector $x_0 \in V(k)$ and $\mathrm{Lie}(R)$ is the infinitesimal stabilizer of x_0. Let $\mathfrak{X} \cong H/R$ be the H-orbit of x_0, so that $\mathfrak{X}$ is a locally closed subvariety of V [B, Prop. 6.7 and Theorem 6.8]. Let E be a splitting field of H, so that ι induces

$$\iota_E : H_E \to \mathrm{GL}(V_E).$$

Now we note the following lemma, which is the only place where the semisimplicity of ι is used.

Lemma 4.1. *There is an E-basis of V_E consisting of vectors, each of which is fixed by some maximal unipotent subgroup of H_E.*

Proof. One easily reduces the proof to the case when V_E is irreducible. Let $v \in V_E$ be a highest weight vector with respect to a maximal unipotent subgroup $U_E \subset H_E$. Since the set $\{h \cdot v : h \in H(E)\}$ is a spanning set of V, a subset of it is a basis, and the vector $h \cdot v$ is fixed by the maximal unipotent subgroup $h \cdot U_E \cdot h^{-1}$. $\qquad\square$

We fix an E-basis $\mathcal{B}^* = \{e_1^*, \ldots, e_n^*\}$ of V_E^* as in the lemma, and let $\mathcal{B} = \{e_1, \ldots, e_n\}$ be the dual basis for V_E. The basis $\mathcal{B}$ gives an isomorphism $\mathrm{GL}(V_E) \cong \mathrm{GL}_n(E)$, and defines an $\mathcal{O}_w$-structure for $\mathrm{GL}(V_E)$ and H_E for each place w of E, as well as an $\mathcal{O}_v$-structure for H for each place v of k. As in §2.1 (using [T, §3.9]), let S be a finite set of places of k such that for all $v \notin S \cup S_0$,

- $x_0 \in V(\mathcal{O}_w)$ for all w lying over v;
- for each i, there is a maximal unipotent subgroup U_{e_i} fixing e_i, and smooth over $\mathcal{O}_w$ for all w lying over v;
- the natural map $H \hookrightarrow \mathrm{Res}_{E/k} H_E \to \mathrm{GL}(V_E)$ is a map of smooth reductive group schemes over $\mathcal{O}_v$;
- the representation ι induces a rational representation ι_v over the residue field κ_v;
- R is smooth over $\mathcal{O}_v$ and $R \hookrightarrow H$ is defined over $\mathcal{O}_v$;
- χ_v is trivial on $R(\mathcal{O}_v)$.

For each $e \in \mathcal{B}$, fix a maximal unipotent subgroup U_e of H_E fixing e^*. We choose a finite set S_e of places v of k which split completely in E, such that S_e is disjoint from $S \cup S_0$. We also ensure that the S_e's are pairwise disjoint as e ranges over $\mathcal{B}$. For $v \in S_e$, one has a commutative diagram

$$
\begin{array}{ccc}
H(\mathcal{O}_v) & \longrightarrow & \prod_{w|v} H_E(\mathcal{O}_w) \cong H(\mathcal{O}_v)^r \\
\downarrow & & \downarrow \\
H(\kappa_v) & \longrightarrow & \prod_{w|v} H_E(\kappa_w) \cong H(\kappa_v)^r
\end{array}
$$

where κ_v and κ_w denote the residue field at the places v and w respectively. The preimage of $U_e(\kappa_{w_1}) \times \prod_{w \neq w_1} H(\kappa_w)$ in $H(\mathcal{O}_v)$ is thus the pro-p radical J_v of an Iwahori subgroup of $H(k_v)$.

We will now define a test function $f = \prod_v f_v \in C_c^\infty(H(\mathbb{A}), \chi|_Z)$ as follows:

- For $v \in S_0$, let f_v be a matrix coefficient of π_v such that

$$
\int_{Z(k_v) \backslash R(k_v)} \chi_v(r)^{-1} \cdot f_v(r)\, dr \neq 0.
$$

- For $v \in S$, we shall make use of the technical assumption (b) in Theorem 1.3 and choose an Iwahori subgroup I_v^{der} of $H^{\mathrm{der}}(k_v)$ with pro-p radical J_v^{der} such that χ_v is trivial on $R(k_v) \cap J_v^{\mathrm{der}}$. Then we let f_v be supported on $Z(k_v) \cdot J_v = Z(k_v) \times J_v^{\mathrm{der}}$ and trivial on J_v^{der}.

- For each $e \in \mathcal{B}$ and $v \in S_e$, let f_v be supported on $Z(k_v) \cdot J_v$ and equal to 1 on J_v.
- For all other v, let f_v be supported on $Z(k_v) \cdot H(\mathcal{O}_v)$ and equal to 1 on $H(\mathcal{O}_v)$.

Form the Poincaré series

$$P(f)(h) = \sum_{\gamma \in Z(k) \backslash H(k)} f(\gamma h).$$

Computing its (R, χ)-period, one has

$$
\begin{aligned}
W_{R,\chi}(P(f)) &= \int_{Z(\mathbb{A})R(k) \backslash R(\mathbb{A})} P(f)(r) \cdot \chi(r)^{-1} \, dr \\
&= \sum_{\gamma \in H(k)/R(k)} \int_{Z(\mathbb{A})R(k) \backslash R(\mathbb{A})} \sum_{\delta \in R(k)} f(\gamma \cdot \delta \cdot r) \cdot \chi(r)^{-1} \, dr \\
&= \sum_{\gamma \in H(k)/R(k)} \int_{Z(\mathbb{A}) \backslash R(\mathbb{A})} f(\gamma r) \cdot \chi(r)^{-1} \, dr.
\end{aligned}
$$

Set

$$\phi_f(h) = \int_{Z(\mathbb{A}) \backslash R(\mathbb{A})} f(hr) \cdot \chi(r)^{-1} \, dr,$$

so that ϕ_f is a compactly supported function on $H(\mathbb{A})/R(\mathbb{A}) = H(\mathbb{A}) \cdot x_0 \subset \mathfrak{X}(\mathbb{A})$, and

$$\phi_f(x_0) \neq 0.$$

Now

$$W_{R,\chi}(P(f)) = \sum_{x \in H(k) \cdot x_0} \phi_f(x).$$

In this sum, it suffices to consider $x \in H(k) \cdot x_0 \cap \operatorname{supp}(\phi_f) \subset \mathfrak{X}(k)$. We have the key lemma:

Lemma 4.2. *When the S_e's are sufficiently large (for all $e \in \mathcal{B}$),*

$$H(k) \cdot x_0 \cap \operatorname{supp}(\phi_f) = \{x_0\}.$$

Proof. Suppose that $x \in V(k)$ lies in the intersection. To show that $x = x_0$, it suffices to show that for each $e \in \mathcal{B}$,

$$\alpha_e := \langle e^*, x - x_0 \rangle = 0 \in E.$$

There is an integer M such that for all $v \in S_0 \cup S$, and all w lying over v, the elements α_e have order of poles at most M at each w. On the other hand, for places w lying over $v \notin S \cup S_0$, α_e is in $\mathcal{O}_w$ and thus has no pole at w. For places $v \in S_e$, however, observe that

$$x = u \cdot x_0 \bmod \varpi_{w_1} \quad \text{for some } u \in U_e(\kappa_{w_1}).$$

Hence

$$\alpha_e = \langle e^*, u \cdot x_0 - x_0 \rangle = \langle u^{-1} \cdot e^* - e^*, x_0 \rangle = 0 \bmod \varpi_{w_1}$$

since $u^{-1} \cdot e^* = e^*$. Thus, for each $v \in s_e$, α_e vanishes at some w lying over v. It is now clear that if S_e is sufficiently large, one must have $\alpha_e = 0 \in E$. This proves the lemma. $\qquad \square$

By the lemma, we deduce that

$$W_{R,\chi}(P(f)) = \sum_{x \in H(k)\cdot x_0} \phi_f(x) = \phi_f(x_0) \neq 0.$$

Moreover, $P(f)$ is fixed (at least) by the pro-p unipotent radical of some Iwahori subgroup at each $v \notin S_0$. Thus, in considering the spectral decomposition of $P(f)$, we obtain a cuspidal representation Π which globalizes the given π_i's at S_0, is globally (R, χ)-distinguished, and whose local components Π_v at $v \notin S_0$ belong to principal series representations induced from minimal parabolic subgroups and have depth 0 when restricted to H_v^{der}.

This completes the proof of Theorem 1.3.

5. Stability of LS gamma factors

In this and subsequent sections, we give some applications of the globalization theorem by combining it with the recent spectacular work of V. Lafforgue [La]. We begin by summarizing the results of V. Lafforgue that we need.

5.1. Results of V. Lafforgue

For a prime number $l \neq p$, fix henceforth an isomorphism

$$\iota_l : \overline{\mathbb{Q}}_l \to \mathbb{C}$$

which allows one to compare $\overline{\mathbb{Q}}_l$-valued functions with $\mathbb{C}$-valued ones. Given an algebraic group H over a global function field k, the isomorphism ι_l also induces an isomorphism

$$\iota_l : {}^L H(\overline{\mathbb{Q}}_l) \cong {}^L H(\mathbb{C}),$$

Now let Π be a cuspidal automorphic representation of $H(\mathbb{A})$. By V. Lafforgue [La], one can associate to Π a continuous global Galois representation

$$\rho = \rho_{\Pi,l} : \mathrm{Gal}(k^{\mathrm{sep}}/k) \to {}^L H(\overline{\mathbb{Q}}_l).$$

We list some of its properties which will be relevant:

(a) For almost all places v of k, the (Frobenius-semisimplification of the) local Galois representation ρ_v is unramified and the image of the geometric Frobenius element Frob_v is equal to the Satake parameter of Π_v (after composing with ι_l).

(b) The embedding $Z(H)^0 \hookrightarrow H$ induces a morphism

$$\rho_Z : {}^L H \to {}^L H / {}^L H_{\mathrm{der}}^0 \cong {}^L Z(H)^0,$$

and the central character ω_Π of Π corresponds to the map $\rho_Z \circ \rho_\Pi$ under the global Langlands correspondence for tori (again, after composing with ι_l). Indeed, by (a), the two characters correspond at almost all places, and so correspond by Chebotarev's density theorem.

For each place v of k, one obtains a continuous (Frobenius-semisimplified) l-adic local Galois representation

$$\rho_{\Pi,l,v} : \mathrm{Gal}(k_v^{\mathrm{sep}}/k_v) \to {}^L H(\overline{\mathbb{Q}}_l).$$

By a well-known construction due to Grothendieck for GL(n) (see [R]) and [GR, §2.1] in general, such an l-adic local Galois representation corresponds to a (Frobenius-semi-simple) representation of the Weil–Deligne group

$$W_{k_v} \times \mathrm{SL}_2(\overline{\mathbb{Q}}_l) \to {}^L H(\overline{\mathbb{Q}}_l).$$

Using the isomorphism ι_l, we get a (Frobenius-semisimple) Weil–Deligne representation

$$\rho_{\Pi,v} : WD_{k_v} = W_{k_v} \times \mathrm{SL}_2(\mathbb{C}) \to {}^L H(\mathbb{C}).$$

In the rest of this paper, we shall pass between the (Frobenius-semisimplified) l-adic representation $\rho_{\Pi,l,v}$ and the local L-parameter $\rho_{\Pi,v}$ without further comment. We will also drop the adjective "Frobenius-semisimple" henceforth, as all our local representations or parameters will be assumed to be Frobenius-semisimplified. Note that Frobenius-semisimplification does not change local Artin L-factors or gamma factors.

5.2. Langlands–Shahidi gamma factors

We can now introduce the Langlands–Shahidi (LS) gamma factors. Let F be a local field of characteristic $p > 0$, and let $P_F = H_F \cdot N_F$ be a maximal parabolic F-subgroup of a connected reductive quasi-split group G_F, with Levi factor H_F and unipotent radical N_F. One has a natural inclusion of Langlands L-groups

$$ {}^L H_F \to {}^L G_F.$$

Suppose that the adjoint action of ${}^L H_F$ on $\mathrm{Lie}(N_F)$ decomposes as

$$\mathrm{Lie}(N_F) = \bigoplus_i r_i$$

for irreducible representations r_i, $1 \le i \le m_r$, ordered as in [Sh2] according to nilpotency class. The second author has extended the Langlands–Shahidi theory to the case of function fields [L1, L2]. In particular, to an irreducible generic representation π of H_F, one can attach a local gamma factor

$$\gamma(\pi, r_i, \psi)$$

for each r_i, where ψ is a nontrivial additive character of F. These LS gamma factors are $\mathbb{C}$-valued meromorphic functions in π (as we shall explain shortly) and satisfy some natural properties which characterize them uniquely [L2].

Since H_F is a maximal Levi subgroup of G_F, it follows that the quotient $H_F/Z(G_F)^0$ of H_F by the connected center $Z(G_F)^0$ of G_F has 1-dimensional split center, so that $\mathrm{Hom}_F(H_F/Z(G_F)^0, \mathbb{G}_m)$ is a free $\mathbb{Z}$-module of rank 1. Let $\delta \in \mathrm{Hom}_F(H_F/Z(G_F)^0, \mathbb{G}_m)$ be the generator such that the modulus character $\det(\mathrm{Ad}_{H_F}|\mathrm{Lie}(N_F))$ is a positive multiple of δ. For any character $\chi : F^\times \to \mathbb{C}^\times$, the composite $\chi \circ \delta$ is a 1-dimensional character of H_F which is trivial on $Z(G_F)$. In particular, for an irreducible representation π

of H_F, one may consider the twist $\pi \otimes (\chi \circ \delta)$. We shall denote this twisted representation simply by $\pi \otimes \chi$, so that we have $\gamma(\pi \otimes \chi, r_i, \psi)$. The character δ corresponds to a morphism

$$\phi_\delta : \mathbb{C}^\times \to {}^L H_F,$$

taking values in the center of $({}^L H_F)^0$. For any character $\chi : F^\times \to \mathbb{C}^\times$, one then has an induced map

$$W_F \to F^\times \xrightarrow{\chi} \mathbb{C}^\times \xrightarrow{\phi_\delta} {}^L H_F,$$

which is the L-parameter for the character $\chi \circ \delta$ of H_F. For simplicity, we shall write χ for this map as well.

Recall that the set of characters of $F^\times$ is the countable disjoint union of 1-dimensional complex manifolds. As χ varies over the characters of $F^\times$, the function $\chi \mapsto \gamma(\pi \otimes \chi, r_i, \psi)$ is meromorphic. To be more precise, if $\chi = |-|_F^s$ for $s \in \mathbb{C}$, then the function

$$\gamma(\pi \otimes |-|_F^s, r_i, \psi) \in \mathbb{C}(q_F^{-s})$$

is a rational function in q_F^{-s}.

5.3. Stability

We now turn to an important stability property of LS γ-factors. This is an open problem in characteristic zero, but in [CPSS] important cases are established. The proof here in positive characteristic extends the cases of symmetric and exterior square γ-factors studied in [HL1].

Theorem 5.1. *Let $R = r_i$ for some i (in the notation of §5.2). Let π_1 and π_2 be irreducible generic representations of H_F with the same central character. For all sufficiently highly ramified characters χ of $F^\times$,*

$$\gamma(\pi_1 \otimes \chi, R, \psi) = \gamma(\pi_2 \otimes \chi, R, \psi).$$

Here, the set of characters χ with a fixed conductor is a complex manifold of dimension 1 under twisting by unramified characters, and the identity in the theorem is interpreted as an identity of meromorphic functions on this complex manifold.

Proof of Theorem 5.1. We first consider a generic supercuspidal representation π of H_F. Let χ be any character of $F^\times$. By Corollary 1.2, we may find a global function field k (indeed $k = \mathbb{F}_q(t)$ will do) with $k_{v_0} = F$ for some place v_0 of k and globalize the data $(H_F \cdot N_F \subset G_F, \pi)$ to $(H \cdot N \subset G, \Pi)$ where $H \subset G$ are quasi-split and Π is a globally generic cuspidal representation as in Corollary 1.2.

Without loss of generality, we may assume that the character ψ can be globalized to a character Ψ of $k \backslash \mathbb{A}$. Indeed, if we fix a nontrivial character $\Psi = \prod_v \Psi_v$ of $k \backslash \mathbb{A}$, then the nontrivial characters of $k \backslash \mathbb{A}$ are of the form $\Psi_a(x) = \Psi(ax)$ for $a \in k^\times$, and the nontrivial characters of F are of the form $\Psi_{v_0,a}(x) = \Psi_{v_0}(ax)$ for $a \in F^\times$. This shows that a dense subset of nontrivial characters of F can be globalized to characters of $k \backslash \mathbb{A}$. On the other hand, for $i = 1$ or 2, one has

$$\gamma(\pi_i \otimes \chi, R, \psi_a) = \alpha(a) \cdot \gamma(\pi_i \otimes \chi, R, \psi)$$

for some character α of $F^\times$ depending only on R, χ and the central character of π_i. This shows that if the identity to be shown in the theorem holds for one nontrivial ψ, then it holds for all nontrivial ψ.

By V. Lafforgue [La], one has a continuous semisimple l-adic global Galois representation

$$\rho = \rho_\Pi : \mathrm{Gal}(k^{\mathrm{sep}}/k) \to {}^L H(\overline{\mathbb{Q}}_l)$$

associated to Π. Let S be a nonempty finite set of places of k not containing the distinguished place v_0 such that for all $v \notin S$ different from v_0,

- H_v, Π_v and Ψ_v are unramified;
- ρ_v is unramified and $\rho_v(\mathrm{Frob}_v)$ is the Satake parameter of Π_v.

For $v \in S$, Π_v is nonetheless contained in a principal series representation induced from a Borel subgroup. We may globalize χ to a Hecke character $\mathcal{X}$ which is unramified outside $S \cup \{v_0\}$ and highly ramified for places in S (by part (ii) of the remark following Lemma 3.3). Then the global functional equation from Langlands–Shahidi theory gives

$$\prod_{v \in S \cup \{v_0\}} \gamma(s, \Pi_v \otimes \mathcal{X}_v, R, \Psi_v) = \frac{L^{S \cup \{v_0\}}(s, \Pi \otimes \mathcal{X}_v, R)}{L^{S \cup \{v_0\}}(1 - s, \Pi \otimes \mathcal{X}_v, R^\vee)}, \qquad (5.2)$$

where for the purpose of this proof, we have written

$$\gamma(s, \Pi_v \otimes \mathcal{X}_v, R, \Psi_v) := \gamma(\Pi_v \otimes \mathcal{X}_v |-|_v^s, R, \Psi_v)$$

and likewise for the L-functions on the right-hand side.

Now consider the representation

$$R \circ (\rho \otimes \mathcal{X}) : \mathrm{Gal}(k^{\mathrm{sep}}/k) \to \mathrm{GL}(V_R),$$

where we regard $\mathcal{X}$ as a 1-dimensional representation of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ by global class field theory. After composing with the isomorphism $\iota : \overline{\mathbb{Q}}_l \cong \mathbb{C}$, one may form the global L-function of Artin type

$$L(s, R \circ (\rho \otimes \mathcal{X})) = \prod_v L(s, R \circ (\rho_v \otimes \mathcal{X}_v))$$

which converges for $\mathrm{Re}(s)$ sufficiently large. By the work of Grothendieck and others, this L-function is in fact a rational function in q^{-s} (and hence admits meromorphic continuation to $\mathbb{C}$) and satisfies a functional equation of the form

$$L(s, R \circ (\rho \otimes \mathcal{X})) = \epsilon(s, R \circ (\rho \otimes \mathcal{X})) \cdot L(1 - s, (R \circ (\rho \otimes \mathcal{X}))^\vee)$$

for some global epsilon factor $\epsilon(s, R \circ \rho \otimes \mathcal{X})$. It is known by work of Laumon [Lau1, Theorems 3.1.5.4 and 3.2.1.1] (see also [D2]), that the epsilon factor admits an Euler product

$$\epsilon(s, R \circ (\rho \otimes \mathcal{X})) = \prod_v \epsilon(s, R \circ (\rho_v \otimes \mathcal{X}_v), \Psi_v) \quad \text{(a finite product)}$$

for any character $\Psi = \prod_v \Psi_v$ of $k \backslash \mathbb{A}$. In particular, one may define the local Galois-theoretic gamma factors

$$\gamma(s, R \circ (\rho_v \otimes \mathcal{X}_v), \psi_v) := \epsilon(s, R \circ (\rho_v \otimes \mathcal{X}_v,)\Psi_v) \cdot \frac{L(1-s, R^\vee \circ (\rho_v \otimes \mathcal{X}_v))}{L(s, R \circ (\rho_v \otimes \mathcal{X}_v))},$$

and the global functional equation can be expressed as

$$\prod_{v \in S \cup \{v_0\}} \gamma(s, R \circ (\rho_v \otimes \mathcal{X}_v), \Psi_v) = \frac{L^{S \cup \{v_0\}}(s, R \circ (\rho_v \otimes \mathcal{X}))}{L^{S \cup \{v_0\}}(1-s, R^\vee \circ (\rho_v \otimes \mathcal{X}))}. \quad (5.3)$$

Comparing (5.2) and (5.3) and using the compatibility of ρ_v and Π_v outside of S, we deduce that

$$\prod_{v \in S \cup \{v_0\}} \gamma(s, \Pi_v \otimes \mathcal{X}_v, R, \Psi_v) = \prod_{v \in S \cup \{v_0\}} \gamma(s, R \circ (\rho_v \otimes \mathcal{X}_v), \Psi_v). \quad (5.4)$$

In particular,

$$\frac{\gamma(s, \pi \otimes \chi, R, \psi)}{\gamma(s, R \circ (\rho_{v_0} \otimes \chi), \psi)} = \prod_{v \in S} \frac{\gamma(s, R \circ (\rho_v \otimes \mathcal{X}_v), \Psi_v)}{\gamma(s, \Pi_v \otimes \mathcal{X}_v, R, \Psi_v)}.$$

For $v \in S$, the representation Π_v is a constituent of a principal series representation $I_v(\mu_v)$ induced from the Borel subgroup. Hence, by multiplicativity of LS gamma factors and their compatibility with class field theory in the case of tori [L2], one has

$$\gamma(s, \Pi_v \otimes \mathcal{X}_v, R, \Psi_v) = \gamma(s, R \circ (\phi_v \otimes \mathcal{X}_v,)\Psi_v)$$

where ϕ_v is the composite

$$\phi_v : W_k \rightarrow {}^L T_v \rightarrow {}^L H_v$$

with the first map corresponding to the inducing data μ_v on $T(k_v)$. In particular,

$$\frac{\gamma(s, \pi \otimes \chi, R, \psi)}{\gamma(s, R \circ (\rho_{v_0} \otimes \chi), \psi)} = \prod_{v \in S} \frac{\gamma(s, R \circ (\rho_v \otimes \mathcal{X}_v), \Psi_v)}{\gamma(s, R \circ (\phi_v \otimes \mathcal{X}_v), \Psi_v)}.$$

Now we know that

$$\rho_Z \circ \rho_v = \rho_Z \circ \phi_v,$$

since both these characters correspond to the central character of Π_v. Moreover, the map

$$\det \circ R : {}^L H \rightarrow \bar{\mathbb{Q}}_l^\times \cong \mathbb{C}^\times$$

factors through ρ_Z. Thus we deduce that

$$\det R \circ \rho_v = \det R \circ \phi_v.$$

Since $\mathcal{X}_v$ can be as highly ramified as we wish for $v \in S$, it follows by the stability of Galois-theoretic gamma factors that for a suitable choice of $\mathcal{X}_v$ for all $v \in S$, we have

$$\frac{\gamma(s, R \circ (\rho_v \otimes \mathcal{X}_v), \Psi_v)}{\gamma(s, R \circ (\phi_v \otimes \mathcal{X}_v,)\Psi_v)} = 1.$$

Hence, we conclude that for every character χ of $F^\times$, one has

$$\gamma(s, \pi \otimes \chi, R, \psi) = \gamma(s, R \circ (\rho_{v_0} \otimes \chi), \psi) \tag{5.5}$$

for a supercuspidal representation π. The same then holds for general generic π by induction, using the multiplicativity of LS gamma factors. We have thus expressed arbitrary LS gamma factors in terms of (some) Galois-theoretic ones.

We can now complete the proof of the theorem. Given two irreducible generic representations π and π' of H_F with the same central character, we may apply the above discussion to each of them in turn. In particular, we have (5.5) for π and π', with ρ_{v_0} and ρ'_{v_0} satisfying

$$\det R \circ \rho_{v_0} = \det R \circ \rho'_{v_0}.$$

Thus if χ is sufficiently highly ramified, then

$$\gamma(s, R \circ (\rho_{v_0} \otimes \chi), \psi) = \gamma(s, R \circ (\rho'_{v_0} \otimes \chi), \psi),$$

and one obtains the desired stability of LS gamma factors. $\qquad\square$

We record here a corollary.

Corollary 5.6. *Let $\Pi = \bigotimes_v \Pi_v$ be a globally generic cuspidal representation of $H(\mathbb{A})$ (where H is a connected reductive group over the global function field k) and let ρ_Π be the l-adic global Galois representation associated to Π by V. Lafforgue. Let $\mathcal{X} = \bigotimes_v \mathcal{X}_v$ be a Hecke character and $\Psi = \prod_v \Psi_v$ a nontrivial character of $\mathbb{A}/k$. Then for all places v of k,*

$$\gamma(\Pi_v \otimes \mathcal{X}_v |-|_v^s, R, \Psi_v) = \gamma(R \circ (\rho_{\Pi,v} \otimes \mathcal{X}_v |-|_v^s), \Psi_v).$$

Proof. Fix a place v_0 of k. As in the proof of Theorem 5.1, one uses the global functional equations to obtain the identity (5.4), with v_0 the place of interest. In the notation of (5.4), for each $v \in S$, equation (5.5) gives

$$\gamma(s, \Pi_v, \mathcal{X}_v, R, \Psi_v) = \gamma(s, R \circ (\phi_v \otimes \mathcal{X}_v), \Psi_v)$$

for some $\phi_v : WD_{k_v} \to {}^L H(\mathbb{C})$ such that $\rho_Z \circ \phi_v$ corresponds to the central character of Π_v (under the local Langlands correspondence for the torus Z). Since $\rho_Z \circ \rho_{\Pi,v}$ also corresponds to the central character of Π_v, we deduce that

$$\det R \circ \phi_v = \det R \circ \rho_{\Pi,v}.$$

In particular, by multiplying $\mathcal{X}$ by a character $\mathcal{X}'$ which is trivial at the place v_0 of interest, highly ramified at all $v \in S$ and unramified outside $S \cup \{v_0\}$, we can appeal to the stability of Galois-theoretic gamma factors to deduce that

$$\gamma(s, R \circ (\phi_v \otimes \mathcal{X}_v \mathcal{X}'_v), \Psi_v) = \gamma(s, R \circ (\rho_{\Pi,v} \otimes \mathcal{X}_v \mathcal{X}'_v), \Psi_v)$$

for all $v \in S$. Now the desired result follows from (5.4) with $\mathcal{X}$ replaced by $\mathcal{X} \cdot \mathcal{X}'$. $\qquad\square$

6. Plancherel measures

We continue with the setup of the previous section, but now we assume that π is any irreducible representation of H_F, not necessarily generic. In this case, one can consider the Plancherel measure associated to the induced representation $\mathrm{Ind}_{P_F}^{G_F} \pi \otimes \chi$. More precisely, one has a standard intertwining operator [W, §IV]

$$M(\pi \otimes \chi, P_F, \bar{P}_F, \psi) : \mathrm{Ind}_{P_F}^{G_F} \pi \otimes \chi \to \mathrm{Ind}_{\bar{P}_F}^{G_F} \pi \otimes \chi$$

defined by the usual integral when $|\chi| = |-|_F^s$ with $\mathrm{Re}(s)$ sufficiently large and admits a meromorphic continuation to all χ. The composite

$$M(\pi \otimes \chi, \bar{P}_F, P_F, \psi) \circ M(\pi \otimes \chi, P_F, \bar{P}_F, \psi) = \mu(\pi \otimes \chi, \psi)^{-1}$$

is a scalar-valued meromorphic function (in χ) known as the Plancherel measure. Indeed, the function $\mu(\pi \otimes \chi |-|_F^s, \psi)$ is a rational function of q_F^{-s}.

Since the definition of the intertwining operators depends on the choice of Haar measures on N_F and $\bar{N}_F$, $\mu(s, \pi \otimes \chi)$ is a priori only well-defined up to scaling by a positive real number. For a precise normalization, see [GI, Appendix B]. In particular, there is a unique normalization such that when π is a constituent of a principal series representation induced from a Borel subgroup, one has

$$\mu(\pi, \psi) = \prod_i \gamma(r_i \circ \phi_\pi, \psi) \cdot \gamma(r_i^\vee \circ \phi_\pi, \overline{\psi})$$

with

$$\phi_\pi : W_F \to {}^L T \to {}^L G,$$

where the first map is the one attached to the inducing data for π.

Suppose now that π is supercuspidal and suppose we have globalized the data $(F, P_F = H_F \cdot N_F \subset G_F, \pi)$ to $(k, P = H \cdot N \subset G, \Pi)$ as in the previous section, with $H \subset G$ quasi-split and Π a cuspidal representation such that $\Pi_{v_0} = \pi$ and Π_v is a constituent of a principal series representation induced from a Borel subgroup for all other v. Let ρ be the Galois representation associated to Π by V. Lafforgue [La]. The goal of this subsection is to show:

Theorem 6.1. *With the above notations,*

$$\mu(\pi \otimes |-|_F^s, \psi) = \prod_i \gamma(r_i \circ (\rho_{v_0} \otimes |-|_F^s), \psi) \cdot \gamma(r_i^\vee \circ (\rho_{v_0} \otimes |-|_F^s), \overline{\psi}).$$

Proof. Over the global field k, one has the global analog of the discussion before the statement of the theorem. When the Haar measures on $N(\mathbb{A})$ and $\bar{N}(\mathbb{A})$ are taken to be the Tamagawa measures, then one has the global functional equation [MW, Theorem IV.1.10]

$$M(\Pi \otimes \mathcal{X}, \bar{P}, P) \circ M(\Pi \otimes \mathcal{X}, P, \bar{P}) = 1,$$

where $\mathcal{X}$ is a global Hecke character with $\mathcal{X}_{v_0} = 1$ and $\mathcal{X}_v$ is unramified outside a finite set S.

Comparing this with the global functional equation of the Galois-theoretic gamma factors, one sees that

$$\prod_{v \in S \cup \{v_0\}} \mu(\Pi_v \otimes \mathcal{X}|-|_v^s, \Psi_v)$$

$$= \prod_{v \in S \cup \{v_0\}} \prod_i \gamma(r_i \circ (\rho_v \otimes \mathcal{X}_v|-|_v^s), \Psi_v) \cdot \gamma(r_i^\vee \circ (\rho_v \otimes \mathcal{X}_v|-|_v^s), \overline{\Psi}_v). \quad (6.2)$$

It follows that

$$\frac{\mu(\pi \otimes |-|_F^s, \psi)}{\prod_i \gamma(r_i \circ (\rho_{v_0} \otimes |-|_F^s), \psi) \cdot \gamma(r_i^\vee \circ (\rho_{v_0} \otimes |-|_F^s), \overline{\psi})}$$

$$= \prod_{v \in S} \prod_i \frac{\gamma(r_i \circ (\rho_v \otimes \mathcal{X}_v|-|_v^s), \Psi_v) \cdot \gamma(r_i^\vee \circ (\rho_v \otimes \mathcal{X}_v|-|_v^s), \overline{\Psi}_v)}{\gamma(r_i \circ (\phi_{\Pi_v} \otimes \mathcal{X}_v|-|_v^s), \Psi_v) \cdot \gamma(r_i^\vee \circ (\phi_{\Pi_v} \otimes \mathcal{X}_v|-|_v^s), \overline{\Psi}_v)}. \quad (6.3)$$

Now suppose that $\mathcal{X}_v$ is sufficiently highly ramified for $v \in S$ and $\mathcal{X}_v$ is unramified for all other v's. Since $\det r_i \circ \rho_v = \det r_i \circ \phi_{\Pi_v}$ and $\mathcal{X}_v$ is sufficiently highly ramified for $v \in S$, we conclude that

$$\mu(\pi \otimes |-|_F^s, \psi) = \prod_i \gamma(r_i \circ (\rho_{v_0} \otimes |-|_F^s), \psi) \cdot \gamma(r_i^\vee \circ (\rho_{v_0} \otimes |-|_F^s), \overline{\psi}). \quad \square$$

Corollary 6.4. *Let π be any irreducible representation of H_F (where F is a local field). Then there is a local Galois representation $\phi : \mathrm{Gal}(F^{\mathrm{sep}}/F) \to {}^L H_F$ such that the (connected) central character ω_π of π corresponds to the character $\rho_Z \circ \phi$ under the local Langlands correspondence for tori and*

$$\mu(\pi \otimes |-|_F^s, \psi) = \prod_i \gamma(r_i \circ (\phi \otimes |-|_F^s), \psi) \cdot \gamma(r_i^\vee \circ (\phi \otimes |-|_F^s), \overline{\psi}).$$

Proof. This follows from the supercuspidal case (as demonstrated in Theorem 6.1) by multiplicativity of Plancherel measures (see [GI, Appendix B]). $\quad \square$

Corollary 6.5. *Let π_1 and π_2 be irreducible representations of H_F with the same central character. For all sufficiently highly ramified characters χ of $F^\times$,*

$$\mu(\pi_1 \otimes \chi, \psi) = \mu(\pi_2 \otimes \chi, \psi).$$

Proof. Since we have expressed the Plancherel measures in terms of Galois-theoretic gamma factors in Corollary 6.4 , the stability under twisting by highly ramified characters follows by that of the Galois-theoretic ones. $\quad \square$

Corollary 6.6. *Let Π be any cuspidal automorphic representation of $H(\mathbb{A})$ and let ρ be the Galois representation associated to Π by V. Lafforgue. For any v,*

$$\mu(\Pi_v \otimes |-|_F^s, \Psi_v) = \prod_i \gamma(r_i \circ (\rho_v \otimes |-|_v^s), \Psi_v) \cdot \gamma(r_i^\vee \circ (\rho_v \otimes |-|_v^s), \overline{\Psi}_v).$$

Proof. We start with a comment on the difference between this corollary and Theorem 6.1. In Theorem 6.1, we started with a local supercuspidal representation and globalized it to Π according to Theorem 1.1, so that we have control at all places outside v_0, which

allows us to deduce the identity in the corollary at v_0 (the identity being known at all other places). What the corollary asserts is that for any cuspidal Π, the same conclusion continues to hold at all places for Π.

The proof is via the same argument as in the proof of Corollary 5.6. To be more precise, fix a place v_0 of k which is the place of interest. The global functional equation for intertwining operators gives equation (6.2) in the current context. Now using Corollary 6.4, one obtains the analog of equation (6.3) for some local Galois representation ϕ_v (in place of ϕ_{Π_v} in (6.3)). Now one obtains the desired conclusion by appealing to the stability of Galois-theoretic gamma factors as in the proof of Corollary 5.6 to isolate the place v_0. $\qquad\square$

7. Local Langlands correspondence

In this section, we specialize the discussion of the previous section to the case when H_F is a quasi-split classical group.

7.1. Classical groups

Thus, let E be equal to F or a quadratic field extension, with $\mathrm{Aut}(E/F) = \langle c \rangle$ and let V be a finite-dimensional vector space over E equipped with a nondegenerate sesquilinear form $\langle -, - \rangle$. Then $H_F = \mathrm{Aut}(V, \langle -, - \rangle)^0$, and the various possibilities are

$$H_F = \mathrm{SO}_{2n+1} \quad \text{or} \quad \mathrm{Sp}_{2n} \quad \text{or} \quad \mathrm{SO}_{2n} \quad \text{or} \quad \mathrm{U}_n.$$

The Langlands dual group of H_F is

$$H_F^\vee = \mathrm{Sp}_{2n}(\mathbb{C}) \quad \text{or} \quad \mathrm{SO}_{2n+1}(\mathbb{C}) \quad \text{or} \quad \mathrm{SO}_{2n}(\mathbb{C}) \quad \text{or} \quad \mathrm{GL}_n(\mathbb{C}).$$

Following [GGP], an L-parameter $\phi : WD_F \to {}^L H_F$ for H_F gives rise to an equivalence class of self-dual or conjugate-self-dual representations

$$\phi : WD_E \to \begin{cases} \mathrm{Sp}_{2n}(\mathbb{C}) \\ \mathrm{SO}_{2n+1}(\mathbb{C}) \\ \mathrm{O}_{2n}(\mathbb{C}) \\ \mathrm{GL}_n(\mathbb{C}) \end{cases}$$

of appropriate sign $\epsilon = \pm 1$ in each case. Note that if $H_F = \mathrm{SO}_{2n}$, we are considering the L-parameters up to equivalence under conjugacy by $\mathrm{O}_{2n}(\mathbb{C})$ and not just by $\mathrm{SO}_{2n}(\mathbb{C})$. To be precise, a representation ϕ of WD_E is conjugate-self-dual if $\phi^c \cong \phi^\vee$, and the L-parameter of H_F gives rise to a conjugate-self-dual representation of WD_E with

$$\text{dimension} = \begin{cases} 2n \\ 2n+1 \\ 2n \\ n \end{cases} \quad \text{and} \quad \text{sign} = \epsilon(H_F) := \begin{cases} -1 & \text{if } H_F = \mathrm{SO}_{2n+1}, \\ +1 & \text{if } H_F = \mathrm{Sp}_{2n}, \\ +1 & \text{if } H_F = \mathrm{SO}_{2n}, \\ (-1)^{n-1} & \text{if } H_F = \mathrm{U}_n. \end{cases}$$

We shall frequently identify an L-parameter ϕ of H_F with its associated conjugate-self-dual representation of WD_E. Likewise, for a representation τ of $\mathrm{GL}_r(E)$, we will write τ^c for the associated c-conjugate representation.

7.2. Langlands–Shahidi factors

The group $H_F \times \mathrm{GL}_r(E)$ is the Levi factor of a maximal parabolic subgroup $P_{r,F}$ of a classical group G_F of the same type. The generator $\delta \in \mathrm{Hom}_F((H_F \times \mathrm{GL}_r(E))/Z(G_F)^0, \mathbb{G}_{\mathrm{m}})$ is simply the rational character $N_{E/F} \circ \det_{\mathrm{GL}_r(E)}$. We have the associated Plancherel measure $\mu(\sigma, \psi)$ if $\sigma = \pi \otimes \tau$ is a representation of $H_F \times \mathrm{GL}_r(E)$ and the associated LS gamma factors $\gamma(\sigma, r_i, \psi)$ if σ is generic. Furthermore, one can specify each r_i in this case. The dual group of the Levi factor is $H_F^\vee \times \mathrm{GL}_r(\mathbb{C})$ and we have

$$r_1 = \mathrm{std}_{H_F^\vee}^\vee \otimes \mathrm{std}_{\mathrm{GL}_n(\mathbb{C})}$$

where std stands for the standard representation of the relevant group. It is convenient and customary to write

$$\gamma(s, \pi \times \tau, \psi) := \gamma(\pi^\vee \otimes \tau |\det|_E^s, r_1, \psi). \qquad (7.1)$$

The representation r_2 is given by

$$R = \begin{cases} \mathrm{Sym}^2 & \text{if } H_F = \mathrm{SO}_{2n+1}, \\ \wedge^2 & \text{if } H_F = \mathrm{Sp}_{2n} \text{ or } \mathrm{SO}_{2n}, \\ \mathrm{Asai}^{(-1)^n} & \text{if } H_F = \mathrm{U}_n. \end{cases}$$

In fact, this second γ-factor depends only on τ and we write

$$\gamma(s, \tau, R, \psi) := \gamma(\sigma \otimes |\det|_E^{s/2}, r_2, \psi).$$

Hence, for $i = 1$ or 2, we have

$$\gamma(is, \pi \otimes \tau, r_i, \psi) = \gamma(\pi \otimes \tau |\det|_E^s, r_i, \psi).$$

7.3. The problem

The problem we shall consider in this section is the following. Starting with a supercuspidal representation π of H_F, we may globalize it to Π using Corollary 1.2 and then use V. Lafforgue's work [La] to obtain a global Galois representation ρ valued in $^L H_F(\overline{\mathbb{Q}}_l) \cong {}^L H_F(\mathbb{C})$. In particular, one obtains a local L-parameter ρ_F at the place v_0 where $k_{v_0} = F$. It is a natural question to ask if ρ_F depends on the choice of the globalization. It would, in general, since a supercuspidal representation can belong to two different Arthur packets and the resulting global Galois representations will be quite different; for example, one could be pure and the other not. The first goal of this section is to show that, despite this, one can attach a discrete series L-parameter to a supercuspidal representation of a classical group using the work of V. Lafforgue and others.

7.4. Generic case

We first consider the case of generic supercuspidal representations with respect to a fixed Whittaker datum. In [L1, L2], the second author has shown the Langlands functorial lifting from classical groups to GL_N using the converse theorem of Cogdell–Piatetski-Shapiro and the Langlands–Shahidi method (following [CK$^+$1, CK$^+$2] in the characteristic 0 case). As a consequence of this and the local Langlands correspondence for GL_N [LRS], one has a map

$$\{\text{generic supercuspidal representations of } H_F\} \to \{\text{elliptic } L\text{-parameters } W_F \to {}^L H_F\}$$
(7.2)

with the following property: for any irreducible generic representation τ of $GL_r(E)$ (for any r) with associated L-parameter ϕ_τ,

$$\gamma(s, \pi \times \tau, \psi) = \gamma(s, \phi_\pi \otimes \phi_\tau, \psi),$$

where we are using the LS gamma factor defined in (7.1) on the LHS. Moreover, one also knows by [HL1, HL2] that

$$\gamma(s, \tau, R, \psi) = \gamma(s, \pi \otimes \tau, r_2, \psi) = \gamma(s, R \circ \phi_\tau, \psi)$$

for any generic representation τ of $GL_r(E)$.

   In the characteristic 0 case, using the theory of local descent of Ginzburg–Rallis–Soudry, one can show that the map (7.2) is a surjection. The theory of local descent should continue to work over a local function field F. However, it is presently not written up in this generality in the literature. While we could have taken the surjectivity of (7.2) as a working hypothesis, we prefer to use a weaker one to be described below.

7.5. A working hypothesis

Suppose that H_F' is the split SO_{2n+1} or the quasi-split unramified SO_{2n}. As described in [S1], one can construct a tamely ramified regular elliptic L-parameter

$$\phi_1 : W_F \to Sp_{2n}(\mathbb{C}) \quad \text{or} \quad O_{2n}(\mathbb{C})$$

which is irreducible as a representation of W_F. If $H_F' = Sp_{2n}$ so that $H_F'^\vee = SO_{2n+1}(\mathbb{C})$, one still has an "almost irreducible" elliptic tamely ramified L-parameter

$$\phi_1 : W_F \to SO_{2n+1}(\mathbb{C})$$

of the form

$$\phi_1 = \chi_1 + \phi_1'$$

with χ_1 1-dimensional and $\phi_1' : W_F \to O_{2n}(\mathbb{C})$ irreducible. We would like to know that such a ϕ_1 is in the image of the map (7.2).

   In characteristic 0, this was shown by Savin [S1] for the symplectic and orthogonal groups by using the DeBacker–Reeder construction of depth 0 supercuspidal

L-packets [DR]. Unfortunately, both [DR] and [S1] are written in the context of characteristic 0 local fields. However, it seems likely that the parts which are relevant for us carry over to the characteristic $p > 0$ setting (at least if $p \neq 2$). In what follows, we shall take this as a working hypothesis:

Working Hypothesis I. H'_F is the split SO_{2n+1} or the quasi-split unramified SO_{2n}. The tamely ramified parameter ϕ_1 described here lies in the image of (7.2).

Let π_1 be the generic supercuspidal representation of H'_F which is mapped to ϕ_1 under (7.2) in the symplectic and orthogonal cases. In the unitary case, we shall let $H'_F = GL_n$ and ϕ_1 be any irreducible representation of W_F. In particular, we do not need this working hypothesis for unitary groups.

7.6. Globalization

Now, appealing to Corollary 1.2, given any supercuspidal representation π_F of H_F, there exist:

- a function field k with $k_{v_0} \cong F$ at a place v_0;
- a quasi-split group H_k over k such that $H_{k,v_0} \cong H_F$ and H_{k,v_1} is an unramified group as described in §7.5;
- a cuspidal representation Π of $H_k(\mathbb{A})$ such that $\Pi_{v_0} = \pi_F$, $\Pi_{v_1} = \pi_1$ (with π_1 defined in §7.5) and Π_v is a constituent of a principal series representation induced from a Borel subgroup for all other v.

Let

$$\rho_\Pi : \mathrm{Gal}(k^{\mathrm{sep}}/k) \to {}^L H_k(\overline{\mathbb{Q}}_l)$$

be the semisimple Galois representation associated to Π by V. Lafforgue. The following proposition describes some key properties of ρ_Π.

Proposition 7.3. (i) *The local representation ρ_{Π,v_1} corresponds to the Weil–Deligne representation ϕ_1. In particular, the global representation ρ_Π is either irreducible or the sum of a quadratic character and an irreducible self-dual representation.*
 (ii) *The global representation ρ_Π is ι_l-pure of weight 0 (in the sense of [D1, §1.2.6]).*
(iii) *The local representation ρ_{Π,v_0} corresponds to a discrete series L-parameter for H_F.*

Proof. (i) It follows from Corollary 6.6 that, with $\rho = \rho_\Pi$,

$$\mu(s, \pi_1 \times \tau, \psi) = \gamma(s, \rho_{v_1}^\vee \otimes \phi_\tau, \psi) \cdot \gamma(-s, \rho_{v_1} \otimes \phi_\tau^\vee, \overline{\psi})$$
$$\cdot \gamma(2s, R \circ \phi_\tau, \psi) \cdot \gamma(-2s, R^\vee \circ \phi_\tau, \overline{\psi}).$$

On the other hand, by Langlands–Shahidi theory [Sh2, L2] and the properties of the map (7.2), one has

$$\mu(s, \pi_1 \times \tau, \psi) = \gamma(s, \phi_1^\vee \otimes \phi_\tau, \psi) \cdot \gamma(-s, \phi_1 \otimes \phi_\tau^\vee, \overline{\psi})$$
$$\cdot \gamma(2s, R \circ \phi_\tau, \psi) \cdot \gamma(-2s, R^\vee \circ \phi_\tau, \overline{\psi}).$$

Thus,

$$\gamma(s, \rho_{v_1}^\vee \otimes \phi_\tau, \psi) \cdot \gamma(-s, \rho_{v_1} \otimes \phi_\tau^\vee, \overline{\psi}) = \gamma(s, \phi_1^\vee \otimes \phi_\tau, \psi) \cdot \gamma(-s, \phi_1 \otimes \phi_\tau^\vee, \overline{\psi}).$$

In general, such an identity is not sufficient to force $\rho_{v_1} = \phi_1$. However, in our case, ϕ_1 is almost an irreducible representation of W_F, and this additional property will give this identity.

More precisely, if ϕ_1 is irreducible as a representation of W_F, then for any ϕ_τ of dimension $< \dim \phi_1$, the RHS of the above equation has no zeros or poles. On the other hand, if ρ_{v_1} is not an irreducible representation of W_F, then the LHS of the above equation will have a zero for some ϕ_τ of dimension $< \dim \phi_1$. Thus, ρ_{v_1} is irreducible as a representation of W_F as well. Then, taking $\phi_\tau = \phi_1$ in the above equation, we deduce that the RHS is zero and thus so is the LHS, which implies that $\rho_{v_1} = \phi_1$.

If ϕ_1 is not irreducible, then $\phi_1 = \chi + \phi_1'$ with ϕ_1' irreducible of dimension $2n$. In this case, a similar argument gives $\rho_{v_1} = \phi_1$; we leave the details to the reader.

(ii) This follows from (i) and [LL, Theorem VII.6] (proving a conjecture of Deligne [D1, Conjecture 1.2.10]).

(iii) It follows from (ii) and a theorem of Deligne [D1, Theorem 1.8.4] that ρ_{v_0} is a tempered L-parameter. Moreover, by Corollary 6.6,

$$\mu(s, \pi \times \tau, \psi) = \gamma(s, \rho_{v_0}^\vee \otimes \phi_\tau, \psi) \cdot \gamma(-s, \rho_{v_0} \otimes \phi_\tau^\vee, \overline{\psi})$$
$$\cdot \gamma(2s, R \circ \phi_\tau, \psi) \cdot \gamma(-2s, R^\vee \circ \phi_\tau, \overline{\psi})$$

for any ϕ_τ. Now if ϕ_τ is not conjugate-self-dual, then it follows from [W, Prop. IV.2.2(ii)] that the LHS is nonzero, and hence so is the RHS. This implies that ρ_{v_0} does not contain any non-conjugate-self-dual summand. Further, it follows from [W, Corollary IV.1.2(ii)] that the LHS has a zero of order at most 2, which implies that ρ_{v_0} is multiplicity-free. Thus, ρ_{v_0} is the multiplicity-free sum of conjugate-self-dual representations of WD_F (of sign $\epsilon(H_F)$), and thus is a discrete series parameter for H_F. $\square$

7.7. Independence

The following proposition shows that the discrete series L-parameter obtained in (iii) above is independent of the various choices used in its construction.

Proposition 7.4. *Suppose that*

- *k and k' are global function fields such that $k_{v_0} \cong k'_{v_0'} \cong F$;*
- *H_k and $H_{k'}$ are algebraic groups over k and k' respectively with $H_{k,v_0} \cong H_{k',v_0'} \cong H_F$;*
- *Π and Π' are cuspidal representations such that $\Pi_{v_0} \cong \Pi'_{v_0'} \cong \pi_F$;*
- *the associated Galois representations $\rho_{\Pi,l}$ and $\rho_{\Pi',l'}$ are both pure of weight 0 (where l and l' are any two prime numbers different from p).*

Then the local representations $\rho_{\Pi,v_0,l}$ and $\rho_{\Pi',v_0',l'}$ are equivalent as L-parameters of H_F.

Proof. By the previous proposition, $\rho_{\Pi,v_0,l}$ and $\rho_{\Pi',v_0',l'}$ are both discrete series L-parameters of H_F. Moreover, by Corollary 6.6, we have

$$\gamma(-s, \rho_{\Pi,v_0,l} \otimes \phi_\tau^\vee, \overline{\psi}) \cdot \gamma(s, \rho_{\Pi,v_0,l}^\vee \otimes \phi_\tau, \psi)$$
$$= \gamma(-s, \rho_{\Pi',v_0',l'} \otimes \phi_\tau^\vee, \overline{\psi}) \cdot \gamma(s, \rho_{\Pi',v_0',l'}^\vee \otimes \phi_\tau, \psi)$$

for any irreducible representation ϕ_τ of W_E. By [GS, Lemma 12.3], this implies that $\rho_{\Pi,v_0,l}$ and $\rho_{\Pi',v_0',l'}$ are equivalent as L-parameters of H_F. $\square$

7.8. *L-parameters of supercuspidal representations*

To summarize, we have shown:

Theorem 7.5. *Assume Working Hypothesis I. Let τ be a supercuspidal representation of* $\mathrm{GL}_r(F)$ *(for any r) with associated L-parameter ϕ_τ.*

(i) *For each prime number $l \neq p$, there is a map*

$$\mathcal{L}_l : \{\text{supercuspidal representations of } H_F\} \to \{\text{elliptic L-parameters } WD_F \to {}^L H_F\}.$$

Write $\phi_\pi = \mathcal{L}_l(\pi)$ for the corresponding Langlands parameter of a representation π.

(ii) *Suppose π is a supercuspidal generic representation of H_F. The map $\mathcal{L}_l$ has the property that*

$$L(s, \pi \times \tau) = L(s, \phi_\pi \otimes \phi_\tau) \quad \text{and} \quad \varepsilon(s, \pi \times \tau, \psi) = \varepsilon(s, \phi_\pi \otimes \phi_\tau, \psi).$$

(iii) *The map $\mathcal{L}_l$ has the property that*

$$\mu(s, \pi \times \tau, \psi) = \gamma(s, \phi_\pi^\vee \otimes \phi_\tau, \psi) \cdot \gamma(-s, \phi_\pi \otimes \phi_\tau^\vee, \overline{\psi})$$
$$\cdot \gamma(2s, R \circ \phi_\tau, \psi) \cdot \gamma(-2s, R^\vee \circ \phi_\tau, \overline{\psi}).$$

(iv) *Moreover, $\mathcal{L}_l$ is characterized by properties* (ii) *and* (iii) *and is independent of l (so that we simply denote it by $\mathcal{L}$).*

It is natural to ask if the map $\mathcal{L}$ defined in Theorem 7.5 can be extended to all irreducible representations of H_F. The key step is to extend it to the set of irreducible discrete series representations. If one can do this, then an application of the Langlands classification theorem would extend it to all irreducible representations. To do so it is necessary to use another deep result, namely the classification of discrete series representations of classical groups in terms of supercuspidal ones due to Mœglin–Tadić [M, MT]. The results of [M, MT] were obtained under a basic assumption (BA). In the next theorem, we shall verify the assumption (BA) of [MT].

7.9. Reducibility of generalized principal series

Using the above results, we can obtain the reducibility points of the generalized principal series representations of quasi-split classical groups induced from supercuspidal representations of maximal parabolic subgroups. More precisely, let $P = M \cdot N \subset G$ be a maximal parabolic subgroup of a classical group G over F, and suppose that its Levi factor M is isomorphic to $\mathrm{GL}_r(E) \times H_F$. Let $\tau \otimes \pi$ be a unitary supercuspidal representation of M and consider the generalized principal series representation

$$\mathrm{I}(s, \tau \otimes \pi) = \mathrm{Ind}_P^G \, \tau \, |\mathrm{det}|_E^s \otimes \pi$$

for $s \in \mathbb{R}$, with its associated Plancherel measure $\mu(s, \tau \otimes \pi, \psi)$. We shall make use of the following well-known properties of the Plancherel measure for supercuspidal inducing data due to Harish-Chandra and Silberger (see [Si1], p. 296, Remark 2 and Lemma 5.4.2.4] and [Si2, Lemmas 1.1 and 1.2]; see also [Ca, S2, W]).

Lemma 7.6. (a) *If $\tau^\vee \not\cong \tau^c$, then $\mathrm{I}(s, \tau \otimes \pi)$ is irreducible for all $s \in \mathbb{R}$.*
(b) *If $\tau^\vee \cong \tau^c$, then $\mathrm{I}(0, \tau \otimes \pi)$ is reducible if and only if $\mu(0, \tau \otimes \pi, \psi)$ is nonzero, in which case, $\mathrm{I}(s, \tau \otimes \pi)$ is irreducible for all real numbers $s \neq 0$ and $\mu(s, \tau \otimes \pi, \psi)$ is holomorphic at all $s \in \mathbb{R}$.*
(c) *If $\tau^\vee \cong \tau^c$, but $\mu(0, \tau \otimes \pi, \psi) = 0$ so that $\mathrm{I}(0, \tau \otimes \pi)$ is irreducible, then $\mathrm{I}(s_0, \tau \otimes \pi)$ is reducible for $s_0 > 0$ if and only if $\mu(s, \tau \otimes \pi, \psi)$ has a pole at $s = s_0$. Moreover, there is a unique $s_0 > 0$ such that reducibility occurs, and at this point of reducibility, the pole of $\mu(s, \tau \otimes \pi, \psi)$ is simple. Further, $\mu(s, \tau \otimes \pi, \psi)$ is nonzero for any real $s \neq 0$.*
In particular, if $\tau^\vee \cong \tau^c$, there is a unique $s_0 \geq 0$ such that $\mathrm{I}(s_0, \tau \otimes \pi)$ is reducible.

The above properties of the Plancherel measure imply the following proposition:

Proposition 7.7. *Let $\phi_\pi = \mathcal{L}(\pi)$ be the L-parameter of the supercuspidal representation π supplied by Theorem 7.5. Then ϕ_π is "sans trou" (without holes) in the sense of Mœglin [M]. More precisely, for any irreducible representation ρ of W_E such that $\mathrm{det}\,\rho$ is unitary, let*

$$\mathrm{Jord}_\rho(\phi_\pi) = \{a \in \mathbb{N} : \rho \otimes S_a \subset \phi_\pi\},$$

where S_a denotes the a-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$. Then the set $\mathrm{Jord}_\rho(\phi_\pi)$ can be nonempty only if ρ is conjugate-self-dual of some sign $\epsilon(\rho)$, in which case all elements in $\mathrm{Jord}_\rho(\phi_\pi)$ are of the same parity: $a \in \mathrm{Jord}_\rho(\phi_\pi)$ is odd if and only if $\epsilon(\rho) = \epsilon(H_F)$. Then for all ρ such that $\mathrm{Jord}_\rho(\phi_\pi)$ is nonempty and any integer $a > 2$,

$$a \in \mathrm{Jord}_\rho(\phi_\pi) \;\Rightarrow\; a - 2 \in \mathrm{Jord}_\rho(\phi_\pi).$$

Proof. Suppose that ρ is conjugate-self-dual and $\mathrm{Jord}_\rho(\phi_\pi)$ is nonempty. Let τ_ρ be the supercuspidal representation of $\mathrm{GL}_r(E)$ (with $r = \dim \rho$) with L-parameter ρ, and consider the family of induced representations $\mathrm{I}(s, \tau_\rho \otimes \pi)$. Recall that

$$\mu(s, \tau_\rho \otimes \pi, \psi) = \gamma(s, \rho \otimes \phi_\pi^\vee, \psi) \cdot \gamma(-s, \rho^\vee \otimes \phi_\pi, \overline{\psi})$$
$$\cdot \gamma(2s, R \circ \rho, \psi) \cdot \gamma(-2s, R^\vee \circ \rho, \overline{\psi}).$$

The RHS is essentially a ratio of products of local L-functions and epsilon factors, and the part which could contribute poles or zeros in $s \geq 0$ is

$$\left(\prod_{a \in \mathrm{Jord}_\rho(\phi_\pi)} \frac{L\left(\frac{a+1}{2} - s, \rho^\vee \otimes \rho\right)}{L\left(\frac{a-1}{2} - s, \rho^\vee \otimes \rho\right)} \right) \cdot \left(\frac{L(1 - 2s, R^\vee \circ \rho) \cdot L(1 + 2s, R \circ \rho)}{L(2s, R \circ \rho) \cdot L(-2s, R^\vee \circ \rho)} \right).$$

From this, we see that the poles and zeros of $\mu(s, \tau_\rho \otimes \pi, \psi)$ occur at the following points:

- for $\epsilon(\rho) = \epsilon(H_F)$:

$$\text{poles at } \frac{a+1}{2} \text{ with } a \in \mathrm{Jord}_\rho(\phi_\pi) \text{ but } a + 2 \notin \mathrm{Jord}_\rho(\phi_\pi),$$

$$\text{zeros at } \frac{a-1}{2} \text{ with } a \in \mathrm{Jord}_\rho(\phi_\pi) \text{ but } a - 2 \notin \mathrm{Jord}_\rho(\phi_\pi);$$

- for $\epsilon(\rho) = -\epsilon(H_F)$:

$$\text{poles at } \frac{a+1}{2} \text{ with } a = 0 \text{ or } a \in \mathrm{Jord}_\rho(\phi_\pi), \text{ but } a + 2 \notin \mathrm{Jord}_\rho(\phi_\pi),$$

$$\text{zeros at } 0 \text{ and } \frac{a-1}{2} \text{ with } a \in \mathrm{Jord}_\rho(\phi_\pi) \text{ but } a - 2 \notin \mathrm{Jord}_\rho(\phi_\pi).$$

Hence, if we set

$$a_\rho(\pi) = \max \mathrm{Jord}_\rho(\phi_\pi),$$

then $\mu(s, \tau_\rho \otimes \pi, \psi)$ has a pole at $s = (a_\rho(\pi) + 1)/2 \geq 1$, so that $s_0 = (a_\rho(\pi) + 1)/2$ must be the unique reducibility point of $\mathrm{I}(s, \tau_\rho \otimes \pi)$ with $s \geq 0$. In particular, by Lemma 7.6, we must have $\mu(0, \tau_\rho \otimes \pi, \psi) = 0$ but $\mu(s, \tau_\rho \otimes \pi, \psi) \neq 0$ for any $s > 0$. Hence, we conclude that in both cases above, for all $a > 2$,

$$a \in \mathrm{Jord}_\rho(\phi_\pi) \implies a - 2 \in \mathrm{Jord}_\rho(\phi_\pi). \qquad \square$$

Now we have the following theorem which establishes the basic assumption (BA) of [MT].

Theorem 7.8. *Let $\phi_\pi = \mathcal{L}(\pi)$ be the L-parameter of π supplied by Theorem 7.5, and let ϕ_τ be the L-parameter of a unitary supercuspidal representation τ of $\mathrm{GL}_r(E)$. The representation $\mathrm{I}(s_0, \tau \otimes \pi)$ is reducible if and only if $\tau^\vee \cong \tau^c$ and one of the following holds:*

(i) *$s_0 = (a_\tau(\pi) + 1)/2 \geq 1$, with $a_\tau(\pi) = \max \mathrm{Jord}_{\phi_\tau}(\phi_\pi)$, if $\mathrm{Jord}_{\phi_\tau}(\phi_\pi)$ is nonempty;*
(ii) *$s_0 = 1/2$ if $\mathrm{Jord}_{\phi_\tau}(\phi_\pi)$ is empty and $L(2s, R \circ \phi_\tau)$ has a pole at $s = 0$ (i.e. $\epsilon(\phi_\tau) = -\epsilon(H_F)$);*
(iii) *$s_0 = 0$ if $\mathrm{Jord}_{\phi_\tau}(\phi_\pi)$ is empty and $L(2s, R \circ \phi_\tau)$ is holomorphic at $s = 0$ (i.e. $\epsilon(\phi_\tau) = \epsilon(H_F)$).*

Proof. This follows immediately from Lemma 7.6 and the proof of Proposition 7.7. $\square$

We note that such a theorem was first shown by Shahidi [Sh2] for general quasi-split groups and generic supercuspidal inducing data, in which case the reducibility points are at 0, 1/2 or 1.

7.10. Results of Mœglin–Tadić

Theorem 7.8 renders the results of Mœglin–Tadić [M, MT] unconditional. Thus we can extend the map $\mathcal{L}$ in Theorem 7.5 from supercuspidal representations to discrete series representations. The extension procedure (due to Mœglin–Tadić) has been explained in great detail and clarity in [CK$^+$2, §7]. Let us give a brief description here, following [CK$^+$2, §7] closely.

Mœglin–Tadić showed that any nonsupercuspidal discrete series representation π can be uniquely expressed as a subquotient of an induced representation of the form

$$\text{Ind}_{P_F}^{H_F}\Big(\bigotimes_{i \in S} \delta_i\Big) \otimes \Big(\bigotimes_{j \in T} \delta_j'\Big) \otimes \pi_0, \tag{7.9}$$

where:

- π_0 is a supercuspidal representation of a smaller classical group of the same type as H_F.
- For $i \in S$, δ_i is the generalized Steinberg representation of GL_{k_i} contained in the induced representation

$$\tau_i|-|^{-(b_i-1)/2} \times \cdots \times \tau_i|-|^{(a_i-1)/2},$$

  where $a_i > b_i > 0$ are integers of the same parity and τ_i is a supercuspidal representation which is conjugate-self-dual with sign $\epsilon(H_F) \cdot (-1)^{a_i-1}$.
- For $j \in T$, δ_j' is the generalized Steinberg representation of $\text{GL}_{k_j'}$ contained in the induced representation

$$\tau_j'|-|^{(c_j'+1)/2} \times \cdots \times \tau_j'|-|^{(a_j'-1)/2}$$

where τ_j' is a conjugate-self-dual supercuspidal representation and $c_j' \in \{1, 2\}$ has the same parity as a_j' with $a_j' \geq c_j' + 2$. Moreover, the τ_j''s are pairwise distinct and

$$a_j' \text{ odd} \quad \Rightarrow \quad L(s, \phi_{\tau_j'}^\vee \otimes \phi_{\pi_0}) \text{ has a pole at } s = 0,$$

$$a_j' \text{ even} \quad \Rightarrow \quad L(s, r_2 \circ \phi_{\tau_j'}) \text{ has a pole at } s = 0.$$

In particular, if a_j' is odd, then $\phi_{\tau_j'}$ is a summand in ϕ_{π_0} and

$$\phi_{\pi_0} - \bigoplus_{j \in T : a_j' \text{ odd}} \phi_{\tau_j'}$$

is an elliptic L-parameter for a smaller classical group of the same type as H_F.

Given this, one can define the L-parameter of π by

$$\mathcal{L}(\pi) = \Big(\bigoplus_{i \in S} \phi_{\tau_i} \otimes (S_{a_i} \oplus S_{b_i})\Big) \oplus \Big(\bigoplus_{j \in T : a_j' \text{ even}} \phi_{\tau_j'} \otimes S_{a_j'}\Big)$$

$$\oplus \Big(\bigoplus_{j \in T : a_j' \text{ odd}} \phi_{\tau_j'} \otimes S_{a_j'}\Big) \oplus \Big(\phi_{\pi_0} - \bigoplus_{j \in T : a_j' \text{ odd}} \phi_{\tau_j'}\Big). \tag{7.10}$$

It was shown in [M] and [MT] that this is a discrete series L-parameter for H_F, i.e., it is multiplicity-free.

To see that the Plancherel measure $\mu(s, \pi \times \tau, \psi)$ can be expressed in terms of $\mathcal{L}(\pi)$ as in Theorem 7.5(iii), we note that by the multiplicativity property of Plancherel measures, $\mu(s, \pi \times \tau, \psi)$ depends only on the supercuspidal support of $\pi \otimes \tau$. Consider the representation ϕ of W_F associated to the supercuspidal support of the induced representation (7.9). Setting

$$\rho_a = \bigoplus_{i=0}^{a-1} |-|^{(a-1)/2-i}$$

to be the L-parameter of the trivial representation of $\mathrm{GL}_a(F)$ for simplicity, we see that

$$\phi = \Big(\bigoplus_{i \in S} \phi_{\tau_i} \otimes (\rho_{a_i} \oplus \rho_{b_i}) \Big) \oplus \Big(\bigoplus_{j \in T : a'_j \text{ even}} \phi_{\tau'_j} \otimes \rho_{a'_j} \Big)$$

$$\oplus \Big(\bigoplus_{j \in T : a'_j \text{ odd}} \phi_{\tau'_j} \otimes \rho_{a'_j} \Big) \oplus \Big(\phi_{\pi_0} - \bigoplus_{j \in T : a'_j \text{ odd}} \phi_{\tau'_j} \Big). \tag{7.11}$$

By multiplicativity and Theorem 7.5(iii) (for supercuspidal representations),

$$\mu(s, \pi \times \tau, \psi) = \gamma(s, \phi^\vee \otimes \phi_\tau, \psi) \cdot \gamma(-s, \phi \otimes \phi_\tau^\vee, \overline{\psi})$$

$$\cdot \gamma(2s, R \circ \phi_\tau, \psi) \cdot \gamma(-2s, R^\vee \circ \phi_\tau, \overline{\psi}). \tag{7.12}$$

Comparing (7.10) and (7.11), and noting that

$$\gamma(s, \Sigma \otimes S_a, \psi) = \gamma(s, \Sigma \otimes \rho_a, \psi)$$

for any representation Σ of WD_F, we deduce that the RHS of (7.12) is equal to the same expression with ϕ replaced by $\mathcal{L}(\pi)$, as desired.

7.11. The LLC

In view of the above discussion and using the Langlands classification theorem, we obtain

Theorem 7.13. *Assume Working Hypothesis I* (*which is not needed when H_F is a unitary group*). *There is a map*

$$\mathcal{L} : \{\text{irreducible smooth representations of } H_F\} \to \{L\text{-parameters } WD_F \to {}^L H_F\}$$

with the following properties:

(i) *Write $\phi_\pi = \mathcal{L}(\pi)$ for the corresponding Langlands parameter of a representation π. Then*

$$\pi \text{ is a discrete series representation} \iff \phi_\pi \text{ is a discrete series } L\text{-parameter,}$$
$$\pi \text{ is a tempered representation} \iff \phi_\pi \text{ is a tempered } L\text{-parameter.}$$

(ii) *The map $\mathcal{L}$ is compatible with the Langlands classification theorem. More precisely, suppose π is the unique Langlands quotient of a standard module $\mathrm{Ind}_{P_F}^{H_F} \tau$ where $P_F = M_F N_F$ is a parabolic subgroup and τ is an essentially tempered representation of the Levi factor M_F. Then ϕ_π is given by the composite*

$$WD_F \xrightarrow{\phi_\tau} {}^L M_F \to {}^L H_F$$

   where the first arrow is the L-parameter ϕ_τ of τ.

(iii) *Suppose π is an irreducible generic representation of H_F. Then*

$$L(s, \pi \times \tau) = L(s, \phi_\pi \otimes \phi_\tau) \quad and \quad \varepsilon(s, \pi \times \tau, \psi) = \varepsilon(s, \phi_\pi \otimes \phi_\tau, \psi)$$

   where τ is any irreducible representation of $\mathrm{GL}_r(F)$ (for any r) with L-parameter ϕ_τ.

(iv) *For any π,*

$$\mu(s, \pi \times \tau, \psi) = \gamma(s, \phi_\pi^\vee \otimes \phi_\tau, \psi) \cdot \gamma(-s, \phi_\pi \otimes \phi_\tau^\vee, \overline{\psi})$$
$$\cdot \gamma(2s, R \circ \phi_\tau, \psi) \cdot \gamma(-2s, R^\vee \circ \phi_\tau, \overline{\psi}).$$

Moreover, $\mathcal{L}$ is characterized by properties (i)–(iv).

7.12. Some questions

Naturally, we are led to ask the following questions:

- Is the map $\mathcal{L}$ surjective?
- Are the fibers of $\mathcal{L}$ finite?
- If ϕ is a tempered L-parameter, is there a generic representation in its fiber under $\mathcal{L}$? This is the tempered L-packet conjecture of Shahidi.
- Is there a refined parametrization of the fibers of $\mathcal{L}$ in terms of characters of a certain component group?

Note that if one has the local descent results in positive characteristic, Working Hypothesis I would not be needed in Theorem 7.13 and the surjectivity of $\mathcal{L}$ would also follow.

   We remark that in [GV], R. Ganapathy and S. Varma have used the Deligne–Kazhdan theory of close local fields to deduce the local Langlands correspondence for split classical groups in characteristic $p > 0$ from the case of characteristic 0. Their map satisfies the properties of the above theorem and thus agrees with our map $\mathcal{L}$; moreover, the above questions all have affirmative answers. We should also mention that in an ongoing work, A. Genestier and V. Lafforgue are trying to establish the local Langlands correspondence by a local analog of [La], and in particular to obtain the map $\mathcal{L}$ as in the theorem, for a general reductive group G in characteristic $p > 0$. Their more geometric methods should complement and perhaps go further than those of this paper.

8. Application of the trace formula

We continue to assume that H_F is a quasi-spit classical group over the local function field F. In this section, we consider an alternative way of extending the map $\mathcal{L}$ of Theorem 7.5 from supercuspidal representations to discrete series representations. Instead of

appealing to the deep results of Mœglin–Tadić, we shall use a global-to-local argument similar to the construction of $\mathcal{L}$ in the supercuspidal case.

The construction of the map $\mathcal{L}$ for supercuspidal representations would also apply to discrete series representations if one can globalize discrete series representations. The Poincaré series argument used in our proof of Theorem 1.1 only works for supercuspidal representations. However, in the characteristic 0 situation, one can use the Arthur–Selberg trace formula to globalize discrete series representations. Indeed, it suffices to have a weak version of "limit multiplicity formula", such as that shown by Clozel [C]. Unfortunately, in positive characteristic, the local theory of invariant harmonic analysis and the global theory of the trace formula are not as fully developed as in the characteristic 0 case. Therefore, we shall make an additional working hypothesis (a simple trace formula) which we shall describe in a moment.

8.1. Pseudo-coefficients

In order to detect nonsupercuspidal discrete series representations using the trace formula, we need the notion of pseudo-coefficients. It has been shown by Henniart–Lemaire [HLe] that any irreducible discrete series representation π of H_F has a pseudo-coefficient f_π. More precisely, $f_\pi \in C_c^\infty(H_F)$ has the property that

$$\operatorname{Tr} \sigma(f_\pi) = \begin{cases} 1 & \text{if } \sigma \cong \pi, \\ 0 & \text{for any irreducible tempered representation } \sigma \not\cong \pi. \end{cases}$$

It follows that such an f_π satisfies the following additional properties:

- the orbital integral of f_π vanishes on all nonelliptic regular semisimple elements;
- $f_\pi(1)$ is equal to the formal degree of π (with respect to the same Haar measure used in the definition of $\pi(f_\pi)$) and thus is nonzero;
- for any standard module $I_P(\tau) = \operatorname{Ind}_{P(F)}^{H_F(F)} \tau$ with $P \neq H_F$ a proper parabolic subgroup, $\operatorname{Tr}(I_P(\tau)(f_\pi)) = 0$.

If π is supercuspidal, one can simply take f_π to be a matrix coefficient of π with $f_\pi(1) \neq 0$. Then such a pseudo-coefficient is a very cuspidal function in the sense of [Lau2, p. 133, Definition 5.1.4]. More precisely, for any proper parabolic subgroup $P = MN \subset H_F$ and a special maximal compact subgroup K in good relative position to P,

$$f_\pi^P(m) := \delta_P(m)^{1/2} \cdot \int_{N(F)} \int_K f_\pi(k^{-1}mnk)\, dk\, dn = 0$$

as a function on $M(F)$.

If π is the Steinberg representation, an explicit pseudo-coefficient f_{EP} has been constructed in characteristic 0 using the Bruhat–Tits building by Kottwitz [Ko]; he called them *Euler–Poincaré functions*. One can consider the same functions in positive characteristic, and this has been done by Laumon [Lau2, Chapter 5] for the group $\mathrm{GL}_n(F)$, who also showed that there exists a very cuspidal Euler–Poincaré function. These Euler–Poincaré functions f_{EP} have the additional property that their orbital integrals are nonvanishing only on elliptic semisimple elements.

We also note the following lemma:

Lemma 8.1. *Let π be a discrete series representation of H_F with pseudo-coefficient f_π. If σ is an irreducible representation of H_F such that $\operatorname{Tr} \sigma(f_\pi) \neq 0$, then σ and π have the same supercuspidal support. In particular, for any irreducible representation τ of $\operatorname{GL}_n(F)$, one has an equality of Plancherel measures*

$$\mu(s, \pi \times \tau, \psi) = \mu(s, \sigma \times \tau, \psi).$$

Proof. If $\sigma \not\cong \pi$, then σ is nontempered and can be written as a finite $\mathbb{Z}$-linear combination of standard modules, all of whose irreducible subquotients have the same supercuspidal support. Since the trace of f_π vanishes on any standard module induced from a proper parabolic subgroup as well as on any tempered representation different from π, one of the standard modules which intervene in the above linear combination must be π. Thus σ has the same supercuspidal support as π. $\square$

For the Euler–Poincaré functions f_{EP} of Kottwitz, one knows from [Ko] that the only irreducible unitary representations σ such that $\operatorname{Tr} \sigma(f_{\mathrm{EP}}) \neq 0$ are the Steinberg representation and the trivial representation.

8.2. *Another working hypothesis*

Now let k be a global function field and H_k a connected semisimple group over k (for simplicity). We shall formulate another working hypothesis which is basically a simple trace formula.

Working Hypothesis II. Let T and T' be nonempty finite sets of places of k. Suppose that $f = \prod_v f_v \in C_c^\infty(H_k(\mathbb{A}))$ is such that for $v \in T$, f_v is a matrix coefficient of a supercuspidal representation π_v with $f_v(1) = 1$, and for $v \in T'$, f_v is an Euler–Poincaré function à la Kottwitz. For such a test function f, consider the kernel function

$$K_f(x, y) = \sum_{\gamma \in H_k(k)} f(x^{-1} \gamma y)$$

for the right translation action $R(f)$ on $L^2(H(k)\backslash H(\mathbb{A}))$. Then assuming (as in [Gr]) that the characteristic of k does not divide the order of $Z(G)$, $K_f(x, y)$ is integrable on the diagonal and (at least when T and T' are sufficiently large) one has a spectral and geometric expansion

$$\sum_{\text{cuspidal } \Pi} \operatorname{Tr} \Pi(f) = \int_{H(k)\backslash H(\mathbb{A})} K_f(x, x)\, dx = \sum_{\{\gamma\}} a_\gamma\, O_\gamma(f)$$

where the sum over γ runs over conjugacy classes of elliptic semisimple elements in $H_k(k)$, $a_\gamma \neq 0$ are some nonzero constants and $O_\gamma(f)$ is the orbital integral of f over the conjugacy class of γ.

In characteristic 0, the hypothesis follows from the work of Arthur. For global function fields, the hypothesis was established by Laumon for $H = \operatorname{GL}_n$ [Lau3, Chapters 9 and 10]

and a variant was used for general H by Gross [Gr, §5]. One certainly hopes that Laumon's proof would extend to general groups H. This is not the right place to verify this, but let us make a few comments. The proof of the integrality of $K_f(x, x)$ is given in [Lau3, Theorem 10.2 and §10.4]; one would imagine that essentially the same proof should work for general groups H. The fact that only cuspidal representations intervene on the spectral side is because we have used the matrix coefficient of a supercuspidal representation at places in T. The main difficulty, due to the nonperfectness of k, is the geometric expansion which is dealt with in [Lau3, §§10.6–10.9]; for example, one would need the important [Lau3, Theorem 10.7.6]. The details of this geometric expansion need to be verified for general H.

8.3. Globalization of discrete series

Using the above Working Hypothesis II, we can demonstrate the following result, which is a weak version of a result of Clozel [C] in characteristic 0 (itself a weak version of the so-called "limit multiplicity formula"):

Proposition 8.2. *Let k be a global function field and H_k a connected semisimple group over k (for simplicity). Let $S \cup T$ be a disjoint union of finite sets of places of k with S nonempty and T sufficiently large. Suppose we are given discrete series representations π_v of $H_k(k_v)$ for each $v \in S$ and a supercuspidal representation π_{v_1} of $H_k(k_{v_1})$ for each $v_1 \in T$. Then there exists a cuspidal representation Π of $H_k(\mathbb{A})$ such that*

- *for all $v_1 \in T$, $\Pi_{v_1} \cong \pi_{v_1}$;*
- *for all $v \in S$,*

$$\mathrm{Tr}\, \Pi_v(f_{\pi_v}) \neq 0,$$

where f_π is a pseudo-coefficient for π_v; hence, Π_v and π_v have the same supercuspidal support.

Observe that for $v \in S$, we do not assert, nor do we know, that $\Pi_v \cong \pi_v$. Thus, Π is not exactly a globalization of $\bigotimes_{v \in S} \pi_v$. However, Lemma 8.1 implies that Π_v and π_v have the same supercuspidal support for $v \in S$, so one might call Π a "pseudo-globalization" of $\bigotimes_{v \in S} \pi_v$ (obtained as a consequence of using a pseudo-coefficient). Moreover, we do not care about the local components of Π outside the set $S \cup T$ (because we have the stability of Plancherel measures as in Corollary 6.5).

Proof of Proposition 8.2. To apply the trace formula supplied by Working Hypothesis II, we first fix a nonempty (sufficiently large) auxiliary set T' of places of k disjoint from $S \cup T$ and specify a test function $f = \prod_v f_v \in C_c^\infty(H_k(\mathbb{A}))$ as follows:

- for $v_1 \in T$, we take f_{v_1} to be a matrix coefficient of π_{v_1} with $f_{v_1}(1) = 1$;
- for $v' \in T'$, we take $f_{v'}$ to be an Euler–Poincaré function;
- for $v \in S$, we take f_v to be a pseudo-coefficient f_{π_v} for the discrete series representation π_v;
- for some fixed $v_2 \notin S \cup T \cup T'$, we take f_{v_2} to be the characteristic function of an open compact subgroup $J \subset H_k(k_{v_2})$;

- for all other v, we take f_v to be the characteristic function of a (hyper)special maximal compact subgroup.

Now we apply the trace formula in Working Hypothesis II to this test function f. On the geometric side, the sum of elliptic semisimple orbital integrals

$$\sum_{\{\gamma\}} a_\gamma \, O_\gamma(f)$$

is a finite sum. Thus, if we shrink the open compact subgroup $J \subset H_k(k_{v_2})$ to a sufficiently small neighborhood of 1, we see that the only term which contributes to the geometric side of the trace formula is the one given by $\gamma = 1$. Then the geometric side is equal to

$$a_1 \cdot f(1) \neq 0.$$

Thus, invoking the spectral side, we conclude that for this particular f, there exists a cuspidal representation Π such that

$$\operatorname{Tr} \Pi(f) = \prod_v \operatorname{Tr} \Pi_v(f_v) \neq 0.$$

By the properties of the local test functions f_v, this Π will satisfy the requirements of the proposition. □

We have assumed that H_k is semisimple for simplicity. The case of reductive H_k with anisotropic center is similarly handled by working with a fixed central character, with some care needed in globalizing the central character, as discussed in §3; we omit the details.

8.4. Definition of $\mathcal{L}$

Now we can define the extension of the map $\mathcal{L}$ to all discrete series representations. Given the local field F and a classical group H_F over F, choose a global field k such that $k_{v_0} \cong F$ and a classical group H_k over k such that $H_{k,v_0} \cong H_F$. We consider a finite set $S \cup T$ of places of k with $S = \{v_0\}$ and T sufficiently large. Given a discrete series representation π of H_F, we apply the proposition with

- $\pi_{v_0} = \pi$;
- for all $v_1 \in T$, $\pi_{v_1} = $ the supercuspidal representation π_1 with L-parameter $\mathcal{L}(\pi_1)$ equal to the L-parameter ϕ_1 in Working Hypothesis I.

Then the proposition provides a cuspidal Π such that

$$\Pi_{v_1} = \pi_1 \quad \text{for all } v_1 \in T, \quad \text{and} \quad \operatorname{Tr} \Pi_{v_0}(f_\pi) \neq 0.$$

Note that we do not know whether Π_{v_0} is isomorphic to π. However, if we believe in various standard conjectures in the theory of automorphic forms, we would expect that Π_{v_0} is tempered, and thus isomorphic to π.

We now consider the Galois representation $\rho_{\Pi,l}$ associated to Π by V. Lafforgue, as well as the Frobenius-semisimplification of its local component $\rho_{\Pi,v_0,l}$. We have:

- In view of Lemma 8.1 and the properties of Π, the statement and proof of Proposition 7.3 continue to apply to $\rho_{\Pi,l}$. The main point is that

$$\mu(s, \pi \times \tau, \psi) = \mu(s, \Pi_{v_0} \times \tau, \psi)$$
$$= \gamma(s, \rho_{\Pi,v_0,l} \times \phi_\tau, \psi) \cdot \gamma(-s, \rho^\vee_{\Pi,v_0,l} \times \phi^\vee_\tau, \psi)$$
$$\cdot \gamma(2s, r_2 \circ \phi_\tau, \psi) \cdot \gamma(-2s, r_2^\vee \circ \phi_\tau, \psi).$$

Here the first equality follows from Lemma 8.1, and the second from Corollary 6.6. Thus, the proof of Proposition 7.3 shows that $\rho_{\Pi,v_0,l}$ corresponds to a discrete series L-parameter of H_F.

- Thanks to Lemma 8.1 again, we deduce from the proof of Proposition 7.4 that $\rho_{\Pi,v_0,l}$ is independent of the choice of the prime l or the globalization Π used (as long as $\rho_{\Pi,l}$ is pure of weight 0, and Π_{v_0} has the same supercuspidal support as π).

In view of the above, we may set

$$\mathcal{L}(\pi) = \text{the Frobenius-semisimplification of } \rho_{\Pi,v_0,l}$$

where Π is a cuspidal representation constructed as above. In this way, we have extended the map $\mathcal{L}$ of Theorem 7.5 (except for property (ii)) to all discrete series representations. Applying the Langlands classification theorem, we then recover Theorem 7.13 (except for property (iii)), albeit under the additional Working Hypothesis II. We do not get property (iii) (for generic discrete series representations) this way because in using the trace formula to globalize, we could not ensure that the globalization of a generic representation is globally generic.

We hope that the application discussed in this section will provide some impetus for a systematic development of the local theory of invariant harmonic analysis and the global theory of the Arthur–Selberg trace formula for general reductive groups over function fields of characteristic $p > 0$.

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