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The porous medium equation with measure data on negatively curved Riemannian manifolds

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Abstract. We investigate existence and uniqueness of weak solutions of the Cauchy problem for the porous medium equation on negatively curved Riemannian manifolds. We show existence of solutions taking as initial condition a finite Radon measure, not necessarily positive. We then establish uniqueness in the class of nonnegative solutions, under a quadratic lower bound on the Ricci curvature. On the other hand, we prove that any weak solution of the porous medium equation necessarily takes on as initial datum a finite Radon measure. In addition, we obtain some results in potential analysis on manifolds, concerning the validity of a modified version of the mean-value inequality for superharmonic functions, and properties of potentials of positive Radon measures. Those results are new and of independent interest, and are crucial for our approach.

Keywords. Porous medium equation, Sobolev inequalities, Green function, potential analysis, superharmonic functions, nonlinear diffusion equations, smoothing effect, asymptotics of solutions

1. Introduction

We are concerned with existence and uniqueness of weak solutions of Cauchy problems for the *porous medium equation* on Riemannian manifolds of the following type:

$$\begin{cases} u_t = \Delta(u^m) & \text{in } M \times (0, \infty), \\ u = \mu & \text{on } M \times \{0\}, \end{cases} \tag{1.1}$$

where M is an N-dimensional complete, simply connected Riemannian manifold with nonpositive sectional curvatures (a *Cartan–Hadamard manifold*), Δ is the Laplace–Beltrami operator on M, $m > 1$ and μ is a finite Radon measure on M. Note that when dealing with sign-changing solutions, as usual we set $u^m = |u|^{m-1}u$.

In the special case of the Euclidean space, problem (1.1) has been deeply investigated in [26]. In particular, existence and uniqueness results for nonnegative solutions have been

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established. More recently, similar results have been generalized to the *fractional* porous medium equation [32, 16, 15]. Furthermore, problem (1.1) with $M = \mathbb{H}^N$,

$$\begin{cases} u_t = \Delta(u^m) & \text{in } \mathbb{H}^N \times (0, \infty), \\ u = \mu & \text{on } \mathbb{H}^N \times \{0\}, \end{cases} \tag{1.2}$$

where $\mathbb{H}^N$ denotes the N-dimensional hyperbolic space, has lately been addressed in a number of papers. In fact, in [31] it has been studied for $m > 1$ and μ a Dirac delta, in [27, 28] for $m > 1$ and $\mu \in L^\infty(\mathbb{H}^N)$, and in [13] for $\mu \in L^p(\mathbb{H}^N)$ for any $p > p_0$ (for a certain $p_0(m, N)$) in a *fast diffusion* regime, i.e. $(N-2)/(N+2) < m < 1$. More precisely, in [31] a thorough analysis of the fundamental solution of the differential equation in (1.2), that is, the solution of (1.2) with $\mu = \delta$, is performed. That special solution is then used to study the large-time behaviour of nonnegative solutions to (1.2) with $\mu \in L^1(\mathbb{H}^N)$.

The aim of our paper is to investigate existence and uniqueness of weak solutions to problem (1.1) under the hypothesis that the sectional curvatures are nonpositive (this is enough for existence), and that the Ricci curvature is bounded from below by $-C(1 + \operatorname{dist}(x, o)^2)$ for some positive constant C and a fixed point $o \in M$ (this is required for uniqueness). Under our assumptions M is necessarily *nonparabolic* (see Section 3), hence the Green function $G(x, y)$ on M is finite for all $x \neq y$.

In particular, we show that for any given finite Radon measure μ (not necessarily positive) there exists a weak solution to problem (1.1) which takes on the initial condition in a suitable "dual" sense. Note that, in general, such a solution can change sign. On the other hand, we are able to prove uniqueness under the additional assumption that μ, and so the corresponding solution, is nonnegative. Furthermore, we show that any weak solution of the differential equation in problem (1.1) (i.e. without a prescribed initial condition) necessarily takes on, in a suitable weak sense, a finite Radon measure as $t \to 0^+$, which is uniquely determined (the initial trace). Observe that this property also justifies the fact that we consider a finite Radon measure as initial datum in problem (1.1). Let us stress that no result in the literature seems to be available as concerns *signed* measures, for which we can prove existence and trace results.

Let us mention that in order to prove that the initial condition is taken on in a suitable weak sense, we exploit some results from potential theory on Riemannian manifolds that we have established here precisely for this purpose, but which also have an independent interest. To be specific, we extend to Riemannian manifolds some results for potentials of nonnegative measures given in the monograph [21], and we obtain a suitable *mean-value inequality* for superharmonic (and subharmonic) functions, without assuming any sign condition and in particular dealing also with positive superharmonic functions. Note that, in contrast with the classical results in [22], where the standard mean value of nonnegative smooth subharmonic functions is considered, we deal with a modified mean value which takes into account the Green function of $-\Delta$ on M; this allows us to remove the nonnegativity assumption. This is essential for our purposes; in fact, since we deal with positive superharmonic functions, the results in [22] cannot be applied in that case. In addition, we work with lower semicontinuous functions with values in $(-\infty, \infty]$ which are superharmonic (or subharmonic with values in $[-\infty, \infty)$) in a distributional sense only: in fact, we shall apply such inequalities to potentials of Radon measures. In establishing such modified mean-value inequalities, we follow the arguments of [2] (see also [8] and references therein), where similar results are obtained in Euclidean space for general second-order elliptic operators.

Note that mean-value inequalities involving Green functions, in the context of general strongly nonparabolic Riemannian manifolds, have been first proved in [25]. However, such inequalities are established for smooth functions, although they can be shown to hold for Lipschitz functions (see [25, Remark 2.4]), a class of functions which is not sufficient for our purposes.

We remark that the above mentioned results in potential analysis will also be crucial in the proof of uniqueness. In fact, by adapting to the present setting the general "duality method" (see [26]), we consider the problem satisfied by the difference of the potentials of any two solutions taking on the same initial measure, and the corresponding dual one.

Let us also mention that, in a different framework, Green functions in connection with the porous medium equation have recently been used in [5] to obtain certain sharp a priori estimates.

From a general viewpoint, the fact that we are considering non-positively curved Riemannian manifolds implies substantial differences form the Euclidean case. In fact, in view of our hypotheses on sectional curvatures, we could have different properties for the Green function and for the growth of the volume of balls (which can be exponential in the radius, as in $\mathbb{H}^N$, or even faster). Therefore, we need to use more delicate cut-off arguments which exploit crucial integrability properties of the Green function. In addition, our assumption concerning the bound from below for the Ricci curvature (see (H)(ii) below) is essential since it ensures conservation of mass for the aforementioned dual problem, a key tool in the uniqueness proof. It is not surprising that such a bound on the Ricci curvature is essential for uniqueness, since it implies *stochastic completeness* of M, which is equivalent to uniqueness of bounded solutions in the linear case (i.e. for the heat equation); see [10].

The potential techniques we exploit allow us to establish an identity which expresses the Green function in terms of the time integral of the solution of problem (1.1) with $\mu = \delta_{x_0}$ for any $x_0 \in M$. The formula holds, indeed, on general Riemannian manifolds, without specific assumptions on its curvatures. In particular, it seems to be new, to our knowledge, even in the Euclidean framework. On the other hand, it extends to the nonlinear case a well-known formula, which relates the Green function to the heat kernel. This result implies in particular that a manifold is nonparabolic if and only if the Barenblatt solution is integrable in time. We are not aware of previous results connecting nonparabolicity of a manifold to properties of nonlinear evolutions of the kind studied here.

The paper is organized as follows. In Section 2 we state the main results and we give the precise definition of solution to problem (1.1). In Section 3 we recall some useful preliminaries in Riemannian geometry and basic facts concerning analysis on manifolds. Then in Section 4 we obtain some results in potential analysis on manifolds; they are mostly used in the subsequent sections, but they also have an independent interest. Existence of solutions is shown in Section 5, along with the integral identity involving the Green function. Finally, in Section 6 we prove both uniqueness of solutions and the results concerning the initial trace.

Remark 1.1. Our results are presented for simplicity in the case of Cartan–Hadamard manifolds of dimension $N \geq 3$. However, they hold with identical proofs under the following more general assumptions:

- M is nonparabolic, complete and noncompact. Moreover, it supports the Sobolev-type inequality $\|f\|_{2\sigma} \leq C \|\nabla f\|_2$ for some $\sigma > 1$, $C > 0$ and all $f \in C_c^\infty(M)$;
- $G(x, y) \to 0$ as $\operatorname{dist}(x, y) \to \infty$, uniformly in $x \in K$, given any compact set $K \subset M$;
- there exists $o \in M$ such that $x \mapsto \operatorname{dist}(x, o)$ is $C^2(M \setminus B)$ for some neighbourhood B of o and $|\Delta_x \operatorname{dist}(x, o)| \leq c \operatorname{dist}(x, o)$ for a suitable constant $c > 0$ and $\operatorname{dist}(x, o)$ large (not necessary for existence).

Note that the above properties are fulfilled if M is, for instance, a nonparabolic, complete and noncompact Riemannian manifold of dimension $N \geq 3$ possessing a pole o such that $\operatorname{cut}(o) = \emptyset$ (i.e. the *cut locus* at o is empty) and assumption (*H*)(ii) below holds, with nonpositive sectional curvatures outside a compact set.

2. Statements of the main results

We consider *Cartan–Hadamard manifolds*, i.e. complete, noncompact, simply connected Riemannian manifolds with nonpositive sectional curvatures. Observe that (see e.g. [10, 12]) on Cartan–Hadamard manifolds the *cut locus* of any point o is empty. So, for any $x \in M \setminus \{o\}$, one can define its *polar coordinates* with pole at o. Namely, for any point $x \in M \setminus \{o\}$ there exists a polar radius $\rho(x) := d(x, o)$ and a polar angle $\theta \in \mathbb{S}^{N-1}$ such that the geodesic from o to x starts at o with direction θ in the tangent space $T_o M$ (and has length ρ). Since we can identify $T_o M$ with $\mathbb{R}^N$, θ can be regarded as a point of $\mathbb{S}^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$.

The Riemannian metric in $M \setminus \{o\}$ in polar coordinates reads

$$ds^2 = d\rho^2 + A_{ij}(\rho, \theta) d\theta^i d\theta^j,$$

where $(\theta^1, \dots, \theta^{N-1})$ are coordinates in $\mathbb{S}^{N-1}$ and (A_{ij}) is a positive definite matrix.

Let

$$\mathcal{A} := \{f \in C^\infty((0, \infty)) \cap C^1([0, \infty)) : \ f'(0) = 1, \ f(0) = 0, \ f > 0 \text{ in } (0, \infty)\}.$$

We say that M is a *spherically symmetric manifold* or a *model manifold* if the Riemannian metric is given by

$$ds^2 = d\rho^2 + \psi(\rho)^2 d\theta^2,$$

where $d\theta^2$ is the standard metric on $\mathbb{S}^{N-1}$, and $\psi \in \mathcal{A}$. In this case, we write $M \equiv M_\psi$; furthermore, we have $\sqrt{A}(\rho, \theta) = \psi(\rho)^{N-1} \eta(\theta)$ (for a suitable function η).

Note that for $\psi(r) = r$, we have $M = \mathbb{R}^N$, while for $\psi(r) = \sinh r$, M is the N-dimensional hyperbolic space $\mathbb{H}^N$.

For most of our purposes, we shall assume that the following hypothesis is satisfied, where we denote by $\mathrm{Ric}_o(x)$ the *radial* Ricci curvature at x with a given pole $o \in M$ (see Section 3 for some more details):

$$\begin{cases} \text{(i) } M \text{ is a Cartan–Hadamard manifold of dimension } N \geq 3; \\ \text{(ii) } \mathrm{Ric}_o(x) \geq -C(1 + \mathrm{dist}(x, o)^2) \text{ for some } C \geq 0. \end{cases} \tag{H}$$

For instance, assumption (H) is satisfied if $M = \mathbb{H}^N$, and e.g. on Riemannian models (see Section 3 below) associated with functions ψ such that $\psi'' \geq 0$ and $\psi(r) = e^{r^\alpha}$ for any $r > 0$ large enough, for some $0 < \alpha \leq 2$.

Note that by (H) the Green function $G(x, y) > 0$ on M exists and is finite for all $x \neq y$ (see again Section 3), i.e. M is *nonparabolic*.

Let $\mathcal{M}^+(M)$ be the set of positive Radon measures on M, with $\mathcal{M}^+_F(M) := \{\mu \in \mathcal{M}^+(M) : \mu(M) < \infty\}$. We shall also denote by $\mathcal{M}_F(M)$ the space of *signed* finite measures on M, that is, measures that can be written as differences of two elements of $\mathcal{M}^+_F(M)$.

Definition 2.1. Given a measure $\mu \in \mathcal{M}_F(M)$, we say that a function u is a *weak solution* to problem (1.1) if

$$u \in L^\infty((0,\infty); L^1(M)) \cap L^\infty(M \times (\tau, \infty)) \quad \text{for all } \tau > 0, \tag{2.1}$$

$$\nabla(u^m) \in L^2((\tau,\infty); L^2(M)) \quad \text{for all } \tau > 0, \tag{2.2}$$

$$-\int_0^\infty \int_M u(x,t)\varphi_t(x,t)\, d\mathcal{V}(x)\, dt + \int_0^\infty \int_M \langle \nabla(u^m)(x,t), \nabla\varphi(x,t)\rangle\, d\mathcal{V}(x)\, dt = 0 \tag{2.3}$$

for any $\varphi \in C_c^\infty(M \times (0,\infty))$, and

$$\lim_{t\to 0} \int_M u(x,t)\phi(x)\, d\mathcal{V}(x) = \int_M \phi(x)\, d\mu(x) \quad \text{for any } \phi \in C_b(M) := C(M) \cap L^\infty(M). \tag{2.4}$$

In fact we shall prove (see Proposition 5.1 below) that weak solutions in the sense of Definition 2.1 are continuous curves in $L^1(M)$.

2.1. Existence and uniqueness results

Concerning existence of solutions starting from an initial finite (not necessarily positive) Radon measure, which are allowed to change sign, we prove the next result. The strategy of the proof is similar to the one of [16, Theorem 3.2], but new ideas are necessary, since the method of proof of [16, Theorem 3.2] works only in the case of positive Radon measures.

Theorem 2.2. *Let assumption* (H)(i) *be satisfied. Let* $\mu \in \mathcal{M}_F(M)$. *Then there exists a weak solution* u *to problem* (1.1) *which conserves the quantity*

$$\mu(M) = \int_M u(x,t)\, d\mathcal{V}(x) \quad \text{for all } t > 0 \tag{2.5}$$

and satisfies the smoothing effect

$$\|u(t)\|_\infty \le K t^{-\alpha} |\mu|(M)^\beta \quad \text{for all } t > 0, \tag{2.6}$$

where K *is a positive constant which only depends on* m, N, *and where*

$$\alpha := \frac{N}{(m-1)N+2}, \quad \beta := \frac{2}{(m-1)N+2}. \tag{2.7}$$

Note that this result can be extended, apart from the conservation of mass, to the case of the supercritical fast diffusion case $m \in ((N-2)/N, 1)$ (see Remark 5.3 below).

Concerning uniqueness of nonnegative solutions taking on the same initial positive finite measure, we show the following result. The ideas of the proof bear some similarities to the one given in [16, Section 5], being based on the duality method of Pierre [26], but substantial differences occur, mainly due to the very different properties of the heat semigroup and the Green function on M, related to our assumptions on sectional curvatures.

Theorem 2.3. *Let assumption* (H) *be satisfied. Let* u_1 *and* u_2 *be two nonnegative weak solutions to problem* (1.1). *Suppose that their initial datum, in the sense of* (2.4), *is the same* $\mu \in \mathcal{M}_F^+(M)$. *Then* $u_1 = u_2$.

Our final result concerns the existence and uniqueness of an initial trace for solutions to the differential equation in problem (1.1).

Theorem 2.4. *Let assumption* (H) *be satisfied. Let* u *be a weak solution of the differential equation in problem* (1.1), *in the sense that* u *satisfies* (2.1)–(2.3). *Then there exists* $\mu \in \mathcal{M}_F(M)$ *such that* (2.4) *is satisfied for any* $\phi \in C_c(M)$ *or for* ϕ *equal to a constant.*

Under the additional assumption that $u \ge 0$, *the conclusion holds for any* $\phi \in C_b(M)$, *for some* $\mu \in \mathcal{M}_F^+(M)$.

Remark 2.5. We point out that our existence and uniqueness results also hold in the linear case, i.e. for $m = 1$. To the best of our knowledge no results are available in the literature if the initial condition is a measure. Note that for the heat equation the explosion rate $-\operatorname{dist}(x, o)^2$ for the Ricci curvature is a sharp condition for uniqueness, as shown in [20]. For several other sharp results in the linear case see [23, 18, 19, 24]

2.2. Superharmonic functions and modified mean-value properties

In this section we establish a modified version of the mean-value inequality for distributional superharmonic functions. It should be stressed that these results, of independent interest, will be essential in the proofs of the potential-theoretic results of Section 4.2, which are in turn fundamental to the proof of uniqueness for solutions to problem (1.1).

Unless otherwise stated, we assume here that M is a nonparabolic manifold of dimension $N \geq 2$, with G being the minimal positive Green function of M.

Let $u : M \to (-\infty, \infty]$ be a lower semicontinuous (l.s.c.) function. For $r > 0$ we define

$$\mathfrak{m}_r[u](x) := \int_{\{y\in M\,:\,G(x,y)=1/r\}} u(y)|\nabla_y G(x, y)|\, dS(y) \quad \text{for all } x \in M, \tag{2.8}$$

where dS is the $(N-1)$-dimensional Hausdorff measure on M. Moreover, for any $\alpha > 0$, we set

$$\mathfrak{M}_r[u](x) := \frac{\alpha+1}{r^{\alpha+1}} \int_0^r \xi^\alpha \mathfrak{m}_\xi[u](x)\, d\xi \quad \text{for all } x \in M. \tag{2.9}$$

Let us recall the well-known smooth coarea formula (see e.g. [6, Exercise III.12]). Let $\phi : M \to \mathbb{R}$ be of class $C^\infty(M)$ with $|\nabla\phi| \in L^\infty(M)$, and let $f : M \to \mathbb{R}$ be either nonnegative or in $L^1(M)$. Then

$$\int_M f|\nabla\phi|\, d\mathcal{V} = \int_{\mathbb{R}} d\xi \int_{\{y\in\phi^{-1}(\xi)\}} f(y)\, dS(y).$$

By approximation it is not difficult to show that the formula is also true with the choices $\phi(y) = [G(x, y)]^{-1}$ and $f(y) = u(y)[G(x, y)]^{-\alpha}\chi_{\{y\in M\,:\,G(x,y)>r\}}$, for each fixed $x \in M$. So, one can rewrite (2.9) as

$$\mathfrak{M}_r[u](x) := \frac{\alpha+1}{r^{\alpha+1}} \int_{\{y\in M\,:\,G(x,y)>1/r\}} u(y)[G(x, y)]^{-\alpha-2}|\nabla_y G(x, y)|^2\, d\mathcal{V}(y) \tag{2.10}$$

for all $x \in M$.

Definition 2.6. We say that a l.s.c. function $u : M \to (-\infty, \infty]$ is $\mathfrak{m}$-*continuous* if

$$u(x) = \lim_{r\to 0} \mathfrak{m}_r[u](x) \quad \text{for all } x \in M.$$

Similarly, u is $\mathfrak{M}$-*continuous* if

$$u(x) = \lim_{r\to 0} \mathfrak{M}_r[u](x) \quad \text{for all } x \in M.$$

We point out that if u is continuous, then it is both $\mathfrak{m}$-continuous and $\mathfrak{M}$-continuous (see the proof of Lemma 4.1). Moreover, in general, if u is $\mathfrak{m}$-continuous, it is also $\mathfrak{M}$-continuous.

Definition 2.7. We say that $u \in L^1_{\text{loc}}(M)$ is *superharmonic* (resp. *subharmonic*) if

$$\int_M u(x)\Delta\phi(x)\, d\mathcal{V}(x) \leq (\geq)\, 0 \quad \text{for any } \phi \in C_c^\infty(M),\ \phi \geq 0.$$

Moreover, $u \in L^1_{\text{loc}}(M)$ is *harmonic* if it is both subharmonic and superharmonic.

Definition 2.8. We say that a l.s.c. function $u : M \to (-\infty, \infty]$ is $\mathfrak{m}$-*superharmonic* if

$$\mathfrak{m}_r[u](x) \leq u(x) \quad \text{for all } x \in M \text{ and a.e. } r > 0.$$

Similarly, u is $\mathfrak{M}$-*superharmonic* if

$$\mathfrak{M}_r[u](x) \leq u(x) \quad \text{for all } x \in M,\ r > 0.$$

Furthermore, we say that u is $\mathfrak{m}$-*subharmonic* if $-u$ is $\mathfrak{m}$-superharmonic, while u is $\mathfrak{M}$-*subharmonic* if $-u$ is $\mathfrak{M}$-superharmonic.

Finally, we say that u is $\mathfrak{m}$-*harmonic* if it is both $\mathfrak{m}$-subharmonic and $\mathfrak{m}$-superharmonic, while u is $\mathfrak{M}$-*harmonic* if it is both $\mathfrak{M}$-subharmonic and $\mathfrak{M}$-superharmonic.

We have the following result, which will be proved in Section 4.1.

Theorem 2.9. (i) *Let u be $\mathfrak{M}$-continuous, l.s.c. and superharmonic. Then u is $\mathfrak{M}$-superharmonic.*

(ii) *Let u be $\mathfrak{M}$-continuous, upper semicontinuous and subharmonic. Then u is $\mathfrak{M}$-subharmonic.*

Of course, the above theorem implies that if u is continuous and harmonic, then u is $\mathfrak{M}$-harmonic, in agreement with the results of [25], which are given in principle for more regular functions.

We stress again that the classical mean-value formula (for the Riemannian measure of a ball) need not be valid, and that in principle only a mean-value inequality for nonnegative subharmonic functions holds (see [22]).

By minor modifications in the proof of Theorem 2.9, a local version of such results on general Riemannian manifolds (possibly parabolic) can be obtained, without supposing that hypothesis (*H*) holds. In fact, we have the following.

Corollary 2.10. *Let $\Omega \subset M$ be an open bounded subset. Let u be $\mathfrak{M}$-continuous, l.s.c. and superharmonic in Ω. Then u is $\mathfrak{M}$-superharmonic in Ω. Similar statements hold for subharmonic and harmonic functions.*

Note that in Corollary 2.10, the function G in (2.8) is meant to be replaced by the Green function of $-\Delta$ in Ω' with homogeneous Dirichlet boundary conditions at $\partial\Omega'$, where Ω' is any open bounded domain with smooth boundary such that $\Omega \Subset \Omega'$.

We remark that, besides the previous ones, we expect that further results given in [2] can be extended to Riemannian manifolds. In particular, it should be true that if a function u is $\mathfrak{m}$-continuous, l.s.c. and superharmonic, then it is $\mathfrak{m}$-superharmonic. However, we limit ourselves to proving the results stated above, since they are the only ones we need in the study of existence and uniqueness for problem (1.1).

2.3. A connection between the Green function and the porous medium equation

In this section we state a nonlinear counterpart of a well-known result that relates the Green function to the heat kernel. In this case, the role of the heat kernel is taken over by the fundamental solution $\mathcal{B}_{x_0}$ of problem (1.1) with $\mu = \delta_{x_0}$, for each fixed $x_0 \in M$.

Suppose that hypothesis (H) is satisfied. Then by Theorem 2.2 the function $\mathcal{B}_{x_0}$ is well defined. If we drop that assumption, the method developed in Section 5.2 to construct $\mathcal{B}_{x_0}$ does not work. Nevertheless, the function $\mathcal{B}_{x_0}$ can always be defined as the monotone limit of approximate solutions to Dirichlet problems set in $B_R \times (0, \infty)$ (for the details, see the proof of Theorem 2.11 in Section 5.3). In general, we cannot in principle exclude that $\mathcal{B}_{x_0} = \infty$.

Theorem 2.11. *Let M be a complete noncompact Riemannian manifold of dimension $N \geq 2$. For any $x_0 \in M$, let $\mathcal{B}_{x_0}$ be the solution of problem (1.1) with $\mu = \delta_{x_0}$, in the sense described above. Then*

$$G(x_0, y) = \int_0^\infty \mathcal{B}_{x_0}^m(y, t)\, dt \quad \textit{for all } y \in M. \tag{2.11}$$

In particular, the time integral in (2.11) exists and is finite if and only if M is nonparabolic.

As a consequence of Theorem 2.11 and of symmetry of the Green function (see (3.12) below), we have the identity

$$\int_0^\infty \mathcal{B}_{x_0}^m(y, t)\, dt = \int_0^\infty \mathcal{B}_{y}^m(x_0, t)\, dt \quad \text{for all } x_0, y \in M.$$

Remark 2.12. Since sectional curvatures are by assumption nonpositive, Hessian comparison (see (3.5) below) shows that $B_0^E(\rho(x), t)$, where $B_0^E(|x|, t)$ is the Euclidean Barenblatt solution, is a supersolution of problem (1.1) with $\mu = \delta_0$. By the comparison principle in bounded domains, it is not difficult to show that, as a consequence, if u is a solution of (1.1) with $\mu \equiv u_0$ and $\operatorname{supp} u_0$ compact, then $\operatorname{supp} u(t)$ is also compact for all $t > 0$. For the details, we refer to the proof of Proposition 5.1.

Moreover, in view of the construction of $\mathcal{B}_0$, by the same arguments as above we have $\mathcal{B}_0 \leq B_0^E$ in $M \times (0, \infty)$. In particular, $\operatorname{supp} \mathcal{B}_0$ is compact.

3. Preliminaries on Riemannian geometry and analysis on manifolds

Let M be a complete noncompact Riemannian manifold. Let Δ denote the standard Laplace–Beltrami operator, ∇ the gradient (with respect to the metric of M) and $d\mathcal{V}$ the Riemannian volume element.

In [29] it is shown that $-\Delta$, defined on $C_c^\infty(M)$, is essentially self-adjoint in $L^2(M)$. In particular, this implies that if $f \in L^2(M)$ with $\Delta f \in L^2(M)$, then $\nabla f \in L^2(M)$, and there exists a sequence $\{f_j\} \subset C_c^\infty(M)$ such that

$$f_j \to f, \quad \nabla f_j \to \nabla f, \quad \Delta f_j \to \Delta f \quad \text{in } L^2(M).$$

In addition, for any $f, g \in L^2(M)$ with $\Delta f, \Delta g \in L^2(M)$ we have

$$\int_M f\, \Delta g\, d\mathcal{V} = -\int_M \langle \nabla f, \nabla g \rangle\, d\mathcal{V} = \int_M g \Delta f\, d\mathcal{V}.$$

It is seen directly that the Laplace–Beltrami operator in polar coordinates has the form

$$\Delta = \frac{\partial^2}{\partial \rho^2} + m(\rho, \theta)\frac{\partial}{\partial \rho} + \Delta_{S_\rho}, \tag{3.1}$$

where $m(\rho,\theta) := \frac{\partial}{\partial\rho}(\log\sqrt{A})$, $A := \det(A_{ij})$, Δ_{S_ρ} is the Laplace–Beltrami operator on the submanifold $S_\rho := \partial B(o,\rho) \setminus \mathrm{cut}(o)$ and $B(o,\rho)$ denotes the Riemannian ball of radius ρ centred at o ($B(\rho)$ for short). Furthermore, on model manifolds

$$\Delta = \frac{\partial^2}{\partial \rho^2} + (N-1)\frac{\psi'}{\psi}\frac{\partial}{\partial \rho} + \frac{1}{\psi^2}\Delta_{\mathbb{S}^{N-1}},$$

where $\Delta_{\mathbb{S}^{N-1}}$ is the Laplace–Beltrami operator on $\mathbb{S}^{N-1}$.

Let us recall comparison results for sectional and Ricci curvatures, which will be used in what follows. For any $x \in M \setminus \{o\}$, denote by $\mathrm{Ric}_o(x)$ the *Ricci curvature* at x in the direction $\partial/\partial\rho$. Let ω denote any pair of tangent vectors from T_xM having the form $(\partial/\partial\rho, X)$, where X is a unit vector orthogonal to $\partial/\partial\rho$. Denote by $\mathrm{K}_\omega(x)$ the *sectional curvature* at the point $x \in M$ of the 2-section determined by ω. By classical results (see e.g. [9], [10, Section 15]), if

$$\mathrm{K}_\omega(x) \le -\frac{\widetilde{\psi}''(\rho)}{\widetilde{\psi}(\rho)} \quad \text{for all } x \equiv (\rho,\theta) \in M \setminus \{o\}, \tag{3.2}$$

for some function $\widetilde{\psi} \in \mathcal{A}$, then

$$m(\rho,\theta) \ge (N-1)\frac{\widetilde{\psi}'(\rho)}{\widetilde{\psi}(\rho)} \quad \text{for all } \rho > 0,\ \theta \in \mathbb{S}^{N-1}.$$

Moreover (see e.g. [10, Section 3]),

$$\mathcal{V}(B_R) \ge \omega_N \int_0^R \widetilde{\psi}(\rho)^{N-1}\, d\rho, \tag{3.3}$$

where ω_N is the measure of the unit sphere $\mathbb{S}^{N-1}$.

On the other hand, if

$$\mathrm{Ric}_o(x) \ge -(N-1)\frac{\psi''(\rho)}{\psi(\rho)} \quad \text{for all } x \equiv (\rho,\theta) \in M \setminus \{o\},$$

for some function $\psi \in \mathcal{A}$, then

$$m(\rho,\theta) \le (N-1)\frac{\psi'(\rho)}{\psi(\rho)} \quad \text{for all } \rho > 0,\ \theta \in \mathbb{S}^{N-1}. \tag{3.4}$$

Note that if M_ψ is a model manifold, then for any $x \equiv (\rho,\theta) \in M_\psi \setminus \{o\}$ we have

$$\mathrm{K}_\omega(x) = -\frac{\psi''(\rho)}{\psi(\rho)} \quad \text{and} \quad \mathrm{Ric}_o(x) = -(N-1)\frac{\psi''(\rho)}{\psi(\rho)}.$$

Since in view of hypothesis (*H*) we have $\mathrm{K}_\omega(x) \le 0$, we infer that condition (3.2) is trivially satisfied with $\widetilde{\psi}(\rho) = \rho$. Therefore,

$$m(\rho,\theta) \ge \frac{N-1}{\rho} \quad \text{for all } x \equiv (\rho,\theta) \in M \setminus \{o\}. \tag{3.5}$$

Let $\mathrm{spec}(-\Delta)$ be the spectrum of $-\Delta$ in $L^2(M)$. Note that (see [10, Section 10])

$$\mathrm{spec}(-\Delta) \subseteq [0, \infty).$$

As a consequence of (H)(i), the *Sobolev inequality*

$$\|f\|_{\frac{2N}{N-2}} \le C_S \|\nabla f\|_2 \quad \text{for all } f \in C_c^\infty(M) \tag{3.6}$$

holds for some positive constant $C_S > 0$, which is equivalent to the *Faber–Krahn inequality*

$$\lambda_1(\Omega) \ge C_{FK} \mathcal{V}(\Omega)^{-2/N} \tag{3.7}$$

for some positive constant C_{FK}, for any bounded regular domain $\Omega \subset M$. Here $\lambda_1(\Omega)$ denotes the first eigenvalue for the operator $-\Delta$ in $L^2(\Omega)$ with homogeneous Dirichlet boundary conditions on $\partial\Omega$. Moreover, for some positive constant C_N one has

$$\mathcal{V}(B_R(x)) \ge C_N R^N \quad \text{for any } x \in M,\, R > 0. \tag{3.8}$$

Inequalities (3.6) and (3.7) and their connection are classical results, which follow e.g. from [12, Exercise 14.5, Corollary 14.23, Remark 14.24] or [17, Lemma 8.1, Theorem 8.3]. Furthermore, (3.8) is due to (H)(i) and (3.3) with $\widetilde{\psi}(\rho) = \rho$.

Let $G(x, y)$ be the *Green function* on M. Note that a priori (see [10]) either $G(x, y) = \infty$ for all $x, y \in M$ or $G(x, y) < \infty$ for all $x \neq y$.

Since M is by assumption a Cartan–Hadamard manifold and hence sectional curvatures are nonpositive, standard Hessian comparisons imply that

$$G(x, y) \le \widetilde{C}\, \mathrm{dist}(x, y)^{2-N} \quad \text{for all } x, y \in M, \tag{3.9}$$

for a suitable $\widetilde{C} > 0$ (we refer e.g. to [11, Theorem 4.2] and (3.15) below). In particular, the Green function $G(x, y)$ is finite for any $x \neq y$ and vanishes as $\mathrm{dist}(x, y) \to \infty$. Furthermore (see [10, Section 4]),

$$G(x, y) \sim \widetilde{C}\, \mathrm{dist}(x, y)^{2-N} \quad \text{as } \mathrm{dist}(x, y) \to 0 \text{ (for any fixed } y), \tag{3.10}$$

$$G(x, y) > 0 \quad \text{for all } x, y \in M, \tag{3.11}$$

$$G(x, y) = G(y, x) \quad \text{for all } x, y \in M. \tag{3.12}$$

In addition,

$$\begin{gathered} \text{for each fixed } y \in M,\ x \mapsto G(x, y) \text{ is of class } C^\infty(M \setminus \{y\}), \\ \Delta_x G(x, y) = 0 \quad \text{for any } x \in M \setminus \{y\}, \end{gathered} \tag{3.13}$$

and

$$\int_M G(x, y) \Delta\phi(x)\, d\mathcal{V}(x) = -\phi(y) \le 0 \tag{3.14}$$

for any $\phi \in C_c^\infty(M)$ with $\phi \ge 0$. Moreover, by Sard's theorem, for all $x \in M$ and a.e. (possibly depending on x) $a > 0$, one has $\nabla_y G(x, y) \neq 0$ on the level set $\{y \in M : G(x, y) = a\}$. In particular such level sets are smooth.

Let h be the *heat kernel* on M; we have the identity

$$G(x,y) = \int_0^\infty h(x,y,t)\,dt \quad \text{for all } x,y \in M \tag{3.15}$$

(see [10]). Moreover, let $\{T_t\}_{t\geq 0}$ denote the heat semigroup on M. The minimal positive solution of the Cauchy problem for the heat equation

$$\begin{cases} u_t = \Delta u & \text{in } M \times (0,\infty), \\ u = u_0 \in L^1(M),\ u_0 \geq 0 & \text{on } M \times \{0\}, \end{cases}$$

can be written as

$$T_t[u_0](x) = \int_M h(x,y,t) u_0(y)\,d\mathcal{V}(y) \quad \text{for all } x \in M,\ t \geq 0.$$

Note that

$$\|T_t\phi\|_p \leq \|\phi\|_p \quad \text{for all } t > 0,\ p \in [1,\infty],\ \phi \in L^p(M). \tag{3.16}$$

Furthermore, as a consequence of (3.6), we have

$$\|T_t\phi\|_\infty \leq \frac{C}{t^{N/2}}\|\phi\|_1 \quad \text{for any } t > 0,\ \phi \in L^1(M), \tag{3.17}$$

for some $C = C(N) > 0$ (see e.g. [7, Chapter 4]).

4. Auxiliary results in potential analysis on Riemannian manifolds

This section is devoted to establishing some crucial results for superharmonic functions and potentials of Radon measures, the latter being closely related to the former. Here, unless otherwise stated, M will always be assumed to be a nonparabolic Cartan–Hadamard manifold of dimension $N \geq 2$.

4.1. Proof of the modified mean-value inequality and properties of superharmonic functions

In order to show the modified mean-value inequality, we need a preliminary lemma.

Lemma 4.1. *For each fixed $y \in M$, the function $x \mapsto G(x,y)$ from M to $[0,\infty]$ is superharmonic. Moreover, it is both $\mathfrak{m}$- and $\mathfrak{M}$-continuous.*

Proof. In view of (3.10) and (3.14), the function $x \mapsto G(x,y)$ is superharmonic. Furthermore, an easy application of the divergence theorem yields, for any $x \in M$ and a.e. $r > 0$,

$$\begin{aligned} &-\int_{\{y\in M\,:\,G(x,y)>1/r\}} G(x,y)\Delta\phi(y)\,d\mathcal{V}(y) \\ &\qquad = -\mathfrak{m}_r[\phi](x) - \frac{1}{r}\int_{\{y\in M\,:\,G(x,y)>1/r\}} \Delta\phi(y)\,d\mathcal{V}(y) + \lim_{\rho\to 0}\mathfrak{m}_\rho[\phi](x) \end{aligned} \tag{4.1}$$

for any $\phi \in C^2(M)$. This can be shown exactly as in [2, formula (11.4)], upon noting that $\lim_{r\to 0} \mathfrak{m}_\rho[\phi](x)$ exists, as proved in formula (11.2) and just above (11.7) in [2].

Now, we choose $\phi = \xi$ with $\xi \in C_c^\infty(M)$, $\xi = 1$ in a neighbourhood of x, and $r > 0$ so large that supp $\xi \subset \{y \in M : G(x, y) > 1/r\}$. Hence, using (3.14), (4.1), an integration by parts, and the fact that $\mathfrak{m}_r[\phi](x) = 0$, we obtain

$$\lim_{\rho\to 0} \int_{\{y\in M\,:\,G(x,y)=1/\rho\}} |\nabla_y G(x, y)|\, dS(y) = 1. \tag{4.2}$$

From (4.2) it easily follows that any continuous function on M is automatically $\mathfrak{m}$- and so $\mathfrak{M}$-continuous. Therefore, for each $y \in M$, the function $x \mapsto G(x, y)$ is $\mathfrak{m}$-continuous at any $x \in M \setminus \{y\}$. It remains to show that it is $\mathfrak{m}$-continuous also at $x = y$. This is a straightforward consequence of the very definition of $\mathfrak{m}_r$ and (4.2):

$$\begin{aligned}\lim_{r\to 0} \mathfrak{m}_r[G(\cdot, y)](y) &= \lim_{r\to 0} \int_{\{z\in M\,:\,G(y,z)=1/r\}} G(y, z)|\nabla_z G(y, z)|\, dS(z)\\ &= \lim_{r\to 0} \frac{1}{r} \int_{\{z\in M\,:\,G(y,z)=1/r\}} |\nabla_z G(y, z)|\, dS(z) = \infty.\end{aligned}$$

Hence the function $x \mapsto G(x, y)$ is $\mathfrak{M}$-continuous, too. This completes the proof. □

Proof of Theorem 2.9. We shall prove that for every $x \in M$ the function $r \mapsto \mathfrak{M}_r[u](x)$ is nonincreasing in $(0, \infty)$. Note that this property combined with the fact that u is $\mathfrak{M}$-continuous easily gives the conclusion.

Now, let $\psi \in C^\infty([0, \infty))$ with $\psi \geq 0$, ψ constant in $[0, \varepsilon)$, $\psi = 0$ in $[R, \infty)$ for some $R > \varepsilon > 0$. Fix any $x_0 \in M$. Define

$$\phi(x) := \psi\left(\frac{1}{G(x_0, x)}\right) \quad \text{for all } x \in M, \tag{4.3}$$

with the obvious convention that $\phi(x_0) = \psi(0)$. In view of (3.13) and (3.9), we have $\phi \in C_c^\infty(M)$. Since u is superharmonic, Definition 2.7 implies

$$\int_M u\Delta\phi\, d\mathcal{V} \leq 0. \tag{4.4}$$

A straightforward computation yields

$$\Delta\phi(x) = \frac{|\nabla_x G(x_0, x)|^2}{G(x_0, x)^4}\left[\psi''\left(\frac{1}{G(x_0, x)}\right) + 2G(x_0, x)\psi'\left(\frac{1}{G(x_0, x)}\right)\right] \quad \text{for all } x \in M. \tag{4.5}$$

In view of (3.11), of the explicit form of $\Delta\phi(x)$ given above, and of the discussion after formula (3.14), we can apply the smooth coarea formula (see again [6, Exercise III.12]), (4.4) and (4.5) to get

$$
\begin{aligned}
0 &\geq \int_M u\Delta\phi \, d\mathcal{V} \\
&= \int_0^\infty \int_{\{x\in M\,:\,1/G(x_0,x)=t\}} u(x)\frac{|\nabla_x G(x_0,x)|}{G(x_0,x)^2} \\
&\quad \times \left[\psi''\left(\frac{1}{G(x_0,x)}\right) + 2G(x_0,x)\psi'\left(\frac{1}{G(x_0,x)}\right)\right] dt \\
&= \int_0^\infty t^2\left[\psi''(t) + \frac{2\psi'(t)}{t}\right] \int_{\{x\in M\,:\,1/G(x_0,x)=t\}} u(x)|\nabla_x G(x_0,x)|\, dS(x)\, dt \\
&= \int_0^\infty [t^2\psi''(t) + 2t\psi'(t)]\mathfrak{m}_t[u](x_0)\, dt = \int_0^\infty (t^2\psi'(t))'\mathfrak{m}_t[u](x_0)\, dt. \qquad (4.6)
\end{aligned}
$$

Given any $\eta \in C_c^\infty((0,\infty))$ with $\eta \geq 0$, we can pick

$$
\psi(t) := \int_t^\infty \frac{\eta(s)}{s^2}\, ds \qquad \text{for all } t \in [0,\infty).
$$

Using such ψ in (4.3) and (4.6) we obtain

$$
\int_0^\infty \eta'(t)\mathfrak{m}_t[u](x_0)\, dt \geq 0 \qquad \text{for all } \eta \in C_c^\infty((0,\infty)),\ \eta \geq 0. \qquad (4.7)
$$

By [3, Lemma 8.2.13], (4.7) implies that the function $r \mapsto \mathfrak{M}_r[u](x)$ is nonincreasing in $(0,\infty)$. This completes the proof. □

As a consequence of Lemma 4.1 and Theorem 2.9 we obtain the next result.

Corollary 4.2. *For each $y \in M$, the function $x \mapsto G(x,y)$ is $\mathfrak{M}$-superharmonic.*

We have two further lemmas, concerning superharmonic functions, which will be used in what follows.

Lemma 4.3. *Let u be an $\mathfrak{M}$-superharmonic function. Then u is $\mathfrak{M}$-continuous.*

Proof. Let $x \in M$. From Definition 2.8 we immediately deduce that

$$
u(x) \geq \limsup_{r\to 0} \mathfrak{M}_r[u](x). \qquad (4.8)
$$

Now, let $\varepsilon > 0$ and $u(x) < \infty$ (the proof for $u(x) = \infty$ is analogous). Since u is l.s.c. at x, there exists $\tilde{r}_\varepsilon > 0$ such that

$$
\inf_{B_{\tilde{r}_\varepsilon}(x)} u \geq u(x) - \varepsilon. \qquad (4.9)
$$

Due to (3.9), there exists $\bar{r}_\varepsilon > 0$ such that

$$
\{y \in M : G(x,y) = 1/\rho\} \subset B_{\tilde{r}_\varepsilon}(x) \qquad \text{for all } 0 < \rho \leq \bar{r}_\varepsilon. \qquad (4.10)
$$

Hence, in view of (4.9) and (4.10), we obtain

$$
\mathfrak{M}_r[u](x) \geq [u(x) - \varepsilon]\, \frac{\alpha+1}{r^{\alpha+1}} \int_0^r \rho^\alpha \int_{\{y\in M\,:\,G(x,y)=1/\rho\}} |\nabla_y G(x,y)|\, dS(y)\, d\rho \qquad (4.11)
$$

for all $0 < r \le \bar{r}_\varepsilon$. Due to (4.2), letting $r \to 0$ in (4.11) yields

$$\liminf_{r\to 0} \mathfrak{M}_r[u](x) \ge u(x) - \varepsilon. \tag{4.12}$$

The conclusion follows from (4.8) and (4.12), since ε is arbitrary. □

Lemma 4.4. *Let $\{u_n\}$ be a sequence of $\mathfrak{M}$-superharmonic functions. Then the function $x \mapsto \liminf_{n\to\infty} u_n(x)$ is $\mathfrak{M}$-superharmonic, provided it is l.s.c.*

Proof. Since each u_n is $\mathfrak{M}$-superhamonic, it satisfies

$$u_n(x) \ge \frac{\alpha+1}{r^{\alpha+1}} \int_0^r \rho^\alpha \int_{\{y\in M\,:\,G(x,y)=1/\rho\}} u_n(y)|\nabla_y G(x,y)|\, dS(y)\, d\rho \tag{4.13}$$

for all $x \in M$. By Fatou's Lemma applied to the right-hand side of (4.13),

$$\liminf_{n\to\infty} u_n(x) \ge \frac{\alpha+1}{r^{\alpha+1}} \int_0^r \rho^\alpha \int_{\{y\in M\,:\,G(x,y)=1/\rho\}} \liminf_{n\to\infty} u_n(y)|\nabla_y G(x,y)|\, dS(y)\, d\rho,$$

so $\liminf_{n\to\infty} u_n$ is $\mathfrak{M}$-superharmonic. □

4.2. Potentials of Radon measures and their properties

We start by recalling the definition of vague convergence for Radon measures.

Definition 4.5. Given a sequence $\{\mu_n\} \subset \mathcal{M}^+(M)$ and $\mu \in \mathcal{M}^+(M)$, we say that μ_n *converges vaguely* to μ, and we write

$$\mu_n \rightharpoonup \mu \quad \text{as } n \to \infty,$$

if

$$\int_M \phi\, d\mu_n \to \int_M \phi\, d\mu \quad \text{as } n \to \infty \text{ for all } \phi \in C_c(M). \tag{4.14}$$

The same definition holds for a sequence $\{\mu_n\} \subset \mathcal{M}_F(M)$ and $\mu \in \mathcal{M}_F(M)$. In the latter case the validity of (4.14) plus the condition $\sup_n |\mu_n|(M) < \infty$ is equivalent to the validity of (4.14) for all $\phi \in C_0(M) := \{\phi \in C(M) : \phi(x) \to 0 \text{ as } d(x,o) \to \infty\}$ (see e.g. [1, Definition 1.58]).

A well-known compactness result asserts that if $\sup_n |\mu_n|(M) < \infty$ then there exists $\mu \in \mathcal{M}_F(M)$ such that (4.14) holds for all $\phi \in C_0(M)$ along a subsequence [1, Theorem 1.59].

Furthermore, vague convergence implies a lower semicontinuity property:

$$|\mu|(M) \le \liminf_{n\to\infty} |\mu_n|(M).$$

For any $\mu \in \mathcal{M}^+(M)$ we define its *potential* as

$$\mathcal{G}^{\mu}(x) := \int_M G(x, y)\, d\mu(y) \quad \text{for all } x \in M.$$

Note that, in general, $\mathcal{G}^{\mu}$ is a function from M to $[0, \infty]$. When $d\mu(y) = f(y)\, d\mathcal{V}(y)$ for some measurable function $f \geq 0$, we shall use the simplified notation

$$\mathcal{G}^{f}(x) := \int_M G(x, y) f(y)\, d\mathcal{V}(y) \quad \text{for all } x \in M. \tag{4.15}$$

The same definition holds for any $\mu \in \mathcal{M}_F(M)$, namely $\mathcal{G}^{\mu} = \mathcal{G}^{\mu_+} - \mathcal{G}^{\mu_-}$. In this case $\mathcal{G}^{\mu}(x)$ only makes sense for almost every $x \in M$; by means of Tonelli's Theorem and estimate (3.9), it is straightforward to show that potentials of finite Radon measures are at least $L^1_{\text{loc}}(M)$ functions.

The main goal of this section is to prove the next result.

Proposition 4.6. *Let $\{\mu_n\} \subset \mathcal{M}^+(M)$ and $\mu \in \mathcal{M}^+(M)$, with $\mu_n \rightharpoonup \mu$. Suppose that for each compact subset $K \subset M$ and for any $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that*

$$\int_{B_R^c} \int_K G(x, y)\, d\mathcal{V}(y)\, d\mu_n(x) \leq \varepsilon \quad \textit{for any } R > R_\varepsilon,\ n \in \mathbb{N}. \tag{4.16}$$

Then

$$\mathcal{G}^{\mu}(x) = \liminf_{n\to\infty} \mathcal{G}^{\mu_n}(x) \quad \textit{for } \text{every } x \in M, \tag{4.17}$$

provided $x \mapsto \liminf \mathcal{G}^{\mu_n}(x)$ is l.s.c.

We point out that Proposition 4.6 will play a key role in the proof of Theorem 2.3. In particular, the fact that (4.17) holds *for every $x \in M$* will be fundamental.

The proof of Proposition 4.6 requires some preliminary tools.

Proposition 4.7 (Principle of descent). *Let $\{\mu_n\} \subset \mathcal{M}^+(M)$ and $\mu \in \mathcal{M}^+(M)$. Suppose that $\mu_n \rightharpoonup \mu$. Then*

$$\mathcal{G}^{\mu}(x) \leq \liminf_{n\to\infty} \mathcal{G}^{\mu_n}(x) \quad \textit{for all } x \in M. \tag{4.18}$$

Proof. Assume first that there exists a compact subset K such that $\operatorname{supp} \mu_n \subset K$ for any $n \in \mathbb{N}$ and $\operatorname{supp} \mu \subset K$. For each $\varepsilon > 0$ define

$$G_\varepsilon(x, y) := \phi_\varepsilon\left(\frac{1}{G(x, y)}\right) \quad \text{for all } x, y \in M,$$

where

$$\phi_\varepsilon(r) := \begin{cases} 1/\varepsilon, & r \leq \varepsilon, \\ 1/r, & r > \varepsilon. \end{cases}$$

Note that G_ε is continuous and bounded in $M \times M$; furthermore, for each $\varepsilon > 0$,

$$G_\varepsilon(x, y) \leq G(x, y) \quad \text{for all } x, y \in M, \tag{4.19}$$

and

$$G_\varepsilon(x, y) \to G(x, y) \quad \text{as } \varepsilon \to 0 \text{ for all } x, y \in M. \tag{4.20}$$

Hence, in view of (4.19) and of the fact that $\mu_n \rightharpoonup \mu$,

$$\int_M G_\varepsilon(x, y)\, d\mu(y) = \lim_{n\to\infty} \int_M G_\varepsilon(x, y)\, d\mu_n(y) \le \liminf_{n\to\infty} \int_M G(x, y)\, d\mu_n(y) \tag{4.21}$$

for all $x \in M$. As a consequence of (4.20), (4.21), and Fatou's Lemma, we obtain

$$\begin{aligned} \mathcal{G}^\mu(x) &= \int_M \lim_{\varepsilon\to 0} G_\varepsilon(x, y)\, d\mu(y) \le \liminf_{\varepsilon\to 0} \int_M G_\varepsilon(x, y)\, d\mu(y) \\ &\le \liminf_{n\to\infty} \int_M G(x, y)\, d\mu_n(y) \quad \text{for all } x \in M. \end{aligned}$$

In order to complete the proof, we have to get rid of the assumption $\operatorname{supp} \mu_n \subset K$ for any $n \in \mathbb{N}$ and $\operatorname{supp} \mu \subset K$. To this end, note that since μ is locally finite, the function $R \mapsto \mu(B_R)$ is locally bounded and nondecreasing, thus its jump set is countable. Therefore, we can select an increasing sequence $\{R_k\} \subset (0, \infty)$ such that $\mu(\partial B_k) = 0$ where $B_k := B_{R_k}$. This implies that $\mu_n^k := \mu_n \rfloor B_k \rightharpoonup \mu \rfloor B_k =: \mu^k$ as $n \to \infty$, for each $k \in \mathbb{N}$ (see [1, Proposition 1.62]). So,

$$\mathcal{G}^{\mu^k}(x) \le \liminf_{n\to\infty} \mathcal{G}^{\mu_n^k}(x) \le \liminf_{n\to\infty} \mathcal{G}^{\mu_n}(x) \quad \text{for all } x \in M.$$

Hence (4.18) follows by letting $k \to \infty$ in the above inequality, in view of the monotone convergence theorem. □

Lemma 4.8. *Let $\mu \in \mathcal{M}^+(M)$. Then $\mathcal{G}^\mu : M \to [0, \infty]$ is a l.s.c. function.*

Proof. Given $x_0 \in M$, take any sequence $\{x_n\} \subset M$ with $x_n \to x_0$. Due to Fatou's Lemma, the continuity of $y \mapsto G(x_0, y)$ in $M \setminus \{x_0\}$ for each $x_0 \in M$, and (3.10), we get

$$\mathcal{G}^\mu(x_0) = \int_M \lim_{n\to\infty} G(x_n, y)\, d\mu(y) \le \liminf_{n\to\infty} \int_M G(x_n, y)\, d\mu(y) = \liminf_{n\to\infty} \mathcal{G}^\mu(x_n).$$

This completes the proof. □

Lemma 4.9. *Let the assumptions of Proposition 4.6 be satisfied. Then*

$$\mathcal{G}^\mu(x) = \liminf_{n\to\infty} \mathcal{G}^{\mu_n}(x) \quad \textit{for } \mathcal{V}\textit{-a.e. } x \in M. \tag{4.22}$$

Proof. Towards a contradiction, suppose that for the set

$$E := \left\{ x \in M : \mathcal{G}^\mu(x) < \liminf_{n\to\infty} \mathcal{G}^{\mu_n}(x) \right\}$$

we have $\mathcal{V}(E) > 0$. We can therefore select a compact subset $K \subset E$ with $\mathcal{V}(K) > 0$. By Fatou's Lemma and the very definition of E, we have

$$\int_K \mathcal{G}^\mu\, d\mathcal{V} < \int_K \liminf_{n\to\infty} \mathcal{G}^{\mu_n}\, d\mathcal{V} \le \liminf_{n\to\infty} \int_K \mathcal{G}^{\mu_n}\, d\mathcal{V}. \tag{4.23}$$

Note that for any $\nu \in \mathcal{M}^+(M)$, by Tonelli's Theorem,

$$\int_K \mathcal{G}^\nu \, d\mathcal{V} = \int_M \phi_K \, d\nu,$$

where

$$\phi_K(x) := \int_K G(x, y) \, d\mathcal{V}(y) \quad \text{for all } x \in M.$$

Since $\phi_K = \mathcal{G}^{\chi_K}$, Lemma 4.14 below implies $\phi_K \in C(M) \cap L^\infty(M)$. For any $R > 0$ let ϕ_K^R be a continuous function on M with

$$\phi_K^R(x) = \begin{cases} \phi_K(x) & \text{for any } x \in B_R, \\ 0 & \text{for any } x \in B_{R+1}^c, \end{cases}$$

and

$$\phi_K^R \le \phi_K \quad \text{in } M.$$

We have

$$\begin{aligned} &\left| \int_M \phi_K \, d\mu - \int_M \phi_K \, d\mu_n \right| \\ &\quad \le \underbrace{\int_M (\phi_K - \phi_K^R) \, d\mu_n}_{I_1} + \underbrace{\int_M (\phi_K - \phi_K^R) \, d\mu}_{I_2} + \underbrace{\left| \int_M \phi_K^R \, d\mu_n - \int_M \phi_K^R \, d\mu \right|}_{I_3}. \end{aligned} \tag{4.24}$$

Thanks to (4.16), I_1 can be estimated as follows: for any $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that for all $R > R_\varepsilon, n \in \mathbb{N}$,

$$0 \le I_1 \le \int_{B_R^c} \phi_K \, d\mu_n \le \varepsilon. \tag{4.25}$$

Now, for any $R_2 > R_1 > 1$ let $\xi \in C(M)$ with

$$\xi_K^{R_1, R_2}(x) = \xi(x) = \begin{cases} \phi_K(x) & \text{for any } x \in B_{R_2} \setminus B_{R_1}, \\ 0 & \text{for any } x \in B_{R_2+1}^c \cup B_{R_1-1}, \end{cases}$$

and

$$\xi \le \phi_K \quad \text{in } M.$$

Since $\mu_n \rightharpoonup \mu$ as $n \to \infty$, property (4.16) and Fatou's Lemma imply

$$\int_{B_{R_1}^c} \phi_K \, d\mu \le \liminf_{R_2 \to \infty} \int_M \xi \, d\mu = \liminf_{R_2 \to \infty} \lim_{n \to \infty} \int_M \xi \, d\mu_n \le \limsup_{n \to \infty} \int_{B_{R_1-1}^c} \phi_K \, d\mu_n \le \varepsilon$$

provided $R_1 > R_\varepsilon + 1$. This yields

$$0 \le I_2 \le \varepsilon \quad \text{for all } R > R_\varepsilon + 1. \tag{4.26}$$

Moreover, $I_3 \to 0$ as $n \to \infty$ by the very definition of vague convergence. Hence, letting $n \to \infty$ in (4.24), choosing $R > R_\varepsilon + 1$, using (4.25) and (4.26), we deduce

$$\limsup_{n\to\infty} \left| \int_M \phi_K \, d\mu - \int_M \phi_K \, d\mu_n \right| \le 2\varepsilon, \tag{4.27}$$

which contradicts (4.23). Thus, (4.22) follows. □

Lemma 4.10. *Let $\mu \in \mathcal{M}^+(M)$. Then $\mathcal{G}^\mu$ is $\mathfrak{M}$-superharmonic.*

Proof. Let $x \in M$ and $r > 0$. Thanks to Tonelli's Theorem and Corollary 4.2, we have

$$\begin{aligned}
\mathcal{G}^\mu(x) &= \int_M G(x,y)\, d\mu(y) \ge \int_M \mathfrak{M}_r[G(\cdot,y)](x)\, d\mu(y) \\
&= \frac{\alpha+1}{r^{\alpha+1}} \int_M \int_0^r \rho^\alpha \mathfrak{m}_\rho[G(\cdot,y)](x)\, d\rho\, d\mu(y) \\
&= \frac{\alpha+1}{r^{\alpha+1}} \int_M \int_0^r \rho^\alpha \int_{\{z\in M\,:\,G(x,z)=1/\rho\}} G(y,z)|\nabla_z G(x,z)|\, dS(z)\, d\rho\, d\mu(y) \\
&= \frac{\alpha+1}{r^{\alpha+1}} \int_0^r \rho^\alpha \int_M \int_{\{z\in M\,:\,G(x,z)=1/\rho\}} G(y,z)|\nabla_z G(x,z)|\, dS(z)\, d\mu(y)\, d\rho \\
&= \frac{\alpha+1}{r^{\alpha+1}} \int_0^r \rho^\alpha \int_{\{z\in M\,:\,G(x,z)=1/\rho\}} \overbrace{\int_M G(z,y)\, d\mu(y)}^{\mathcal{G}^\mu(z)} |\nabla_z G(x,z)|\, dS(z)\, d\rho \\
&= \mathfrak{M}_r[\mathcal{G}^\mu](x),
\end{aligned}$$

and the proof is complete. □

Proof of Proposition 4.6. From Lemmas 4.4 and 4.10, both $\mathcal{G}^\mu$ and $\mathcal{L} := \liminf_{n\to\infty} \mathcal{G}^{\mu_n}$ are $\mathfrak{M}$-superharmonic. Hence, in view of Lemma 4.3 and (4.22), for *every* $x \in M$,

$$\begin{aligned}
\mathcal{L}(x) &= \lim_{r\to 0} \mathfrak{M}_r[\mathcal{L}](x) \\
&= \lim_{r\to 0} \frac{\alpha+1}{r^{\alpha+1}} \int_0^r \xi^\alpha \int_{\{y\in M\,:\,G(x,y)=1/\xi\}} \mathcal{L}(y)|\nabla_y G(x,y)|\, dS(y)\, d\xi \\
&= \lim_{r\to 0} \frac{\alpha+1}{r^{\alpha+1}} \int_0^r \xi^\alpha \int_{\{y\in M\,:\,G(x,y)=1/\xi\}} \mathcal{G}^\mu(y)|\nabla_y G(x,y)|\, dS(y)\, d\xi \\
&= \lim_{r\to 0} \mathfrak{M}_r[\mathcal{G}^\mu](x) = \mathcal{G}^\mu(x);
\end{aligned}$$

we point out that here we have used (2.10) in order to circumvent the fact that $\mathcal{G}^\mu$ and $\mathcal{L}$ coincide only $\mathcal{V}$-a.e. in M. □

Let us recall the following well-known result, which will be essential in the proof of Theorem 2.2, in the case of signed measures.

Lemma 4.11 (Jordan decomposition). *Let* $\mu \in \mathcal{M}_F(M)$. *There exists a* unique *couple* $(\mu_+, \mu_-) \in \mathcal{M}_F^+(M) \times \mathcal{M}_F^+(M)$ *such that* $\mu = \mu_+ - \mu_-$ *and*

$$\mu_{\mathcal{P}} \geq \mu_+, \qquad \mu_{\mathcal{N}} \geq \mu_- \tag{4.28}$$

for any other couple $(\mu_{\mathcal{P}}, \mu_{\mathcal{N}}) \in \mathcal{M}_F^+(M) \times \mathcal{M}_F^+(M)$ *such that*

$$\mu = \mu_{\mathcal{P}} - \mu_{\mathcal{N}}. \tag{4.29}$$

Moreover, (μ_+, μ_-) *is the* unique minimizer *of the functional*

$$(\mu_{\mathcal{P}}, \mu_{\mathcal{N}}) \mapsto \mu_{\mathcal{P}}(M) + \mu_{\mathcal{N}}(M) \quad \text{for all } \mu_{\mathcal{P}}, \mu_{\mathcal{N}} \in \mathcal{M}_F^+(M) \text{ subject to (4.29).}$$

The corresponding minimum is referred to as the total variation *of* μ*, and it is denoted by* $|\mu|(M)$*, the total mass of the positive finite Radon measure* $|\mu| = \mu_+ + \mu_-$.

Proof. This is a classical result in measure theory: see for instance [33, Theorem 10.8]. We point out that the last statement is just a consequence of (4.28). In fact, in view of the latter, given any decomposition $(\mu_{\mathcal{P}}, \mu_{\mathcal{N}}) \neq (\mu_+, \mu_-)$ there necessarily exists a Borel set $A \subset M$ such that either $\mu_{\mathcal{P}}(A) > \mu_+(A)$ or $\mu_{\mathcal{N}}(A) > \mu_-(A)$. In particular,

$$\begin{aligned} |\mu|(M) &= \mu_+(A) + \mu_+(M \setminus A) + \mu_-(A) + \mu_-(M \setminus A) \\ &< \mu_{\mathcal{P}}(A) + \mu_{\mathcal{P}}(M \setminus A) + \mu_{\mathcal{N}}(A) + \mu_{\mathcal{N}}(M \setminus A) = \mu_{\mathcal{P}}(M) + \mu_{\mathcal{N}}(M). \end{aligned}$$ □

Remark 4.12. If $d\mu(x) = f(x)\, d\mathcal{V}(x)$ for some $f \in L^1(M)$, then $d\mu_+(x) = f_+(x)\, d\mathcal{V}(x)$ and $d\mu_-(x) = f_-(x)\, d\mathcal{V}(x)$.

We now show a standard uniqueness result involving potentials of finite Radon measures.

Lemma 4.13. *Let* $\mu, \nu \in \mathcal{M}_F$*, and suppose that* $\mathcal{G}^\mu(x) = \mathcal{G}^\nu(x)$ *for almost every* $x \in M$. *Then* $\mu = \nu$. *In particular, if* μ *(or* ν*) is positive, then* $\mathcal{G}^\mu(x) = \mathcal{G}^\nu(x)$ *for* every $x \in M$.

Proof. Let $\phi \in C_c^\infty(M)$. In view of the assumptions, we have

$$\int_M \mathcal{G}^\mu(x) \Delta\phi(x)\, d\mathcal{V}(x) = \int_M \mathcal{G}^\nu(x) \Delta\phi(x)\, d\mathcal{V}(x). \tag{4.30}$$

By Fubini's Theorem (recall (3.9)), (4.30) is equivalent to

$$\int_M \int_M G(x, y) \Delta\phi(x)\, d\mathcal{V}(x)\, d\mu(y) = \int_M \int_M G(x, y) \Delta\phi(x)\, d\mathcal{V}(x)\, d\nu(y),$$

that is,

$$\int_M \phi(y)\, d\mu(y) = \int_M \phi(y)\, d\nu(y). \tag{4.31}$$

From (4.31) the conclusion follows thanks to density of $C_c^\infty(M)$ in $C_c(M)$. □

The following result, which is crucial for what follows, is concerned with integrability properties of potentials of functions in $L^1(M) \cap L^\infty(M)$.

Lemma 4.14. *Let $N \geq 3$ and $f \in L^1(M) \cap L^\infty(M)$. Then $\mathcal{G}^f \in C(M) \cap L^p(M)$ for all $p \in (N/(N-2), \infty]$, $\nabla \mathcal{G}^f \in [L^2(M)]^N$, and*

$$\int_M |\nabla \mathcal{G}^f|^2 \, d\mathcal{V} = \int_M f \, \mathcal{G}^f \, d\mathcal{V}. \tag{4.32}$$

Proof. We first show that $\mathcal{G}^f \in C(M)$. To this end, fix any $x_0 \in M$, $\varepsilon > 0$, and suppose that $x \in B_\varepsilon(x_0)$. We have

$$\begin{aligned} |\mathcal{G}^f(x) - \mathcal{G}^f(x_0)| &\leq \int_M |G(x,y) - G(x_0,y)| \, |f(y)| \, d\mathcal{V}(y) \\ &\leq \|f\|_\infty \int_{B_\varepsilon(x_0)} [G(x,y) + G(x_0,y)] \, d\mathcal{V}(y) \\ &\quad + \int_{B_\varepsilon^c(x_0)} |G(x,y) - G(x_0,y)| \, |f(y)| \, d\mathcal{V}(y). \end{aligned}$$

Due to (3.9), since $f \in L^1(M)$, by dominated convergence we get

$$\int_{B_\varepsilon^c(x_0)} |G(x,y) - G(x_0,y)| \, |f(y)| \, d\mathcal{V}(y) \to 0 \quad \text{as } x \to x_0.$$

On the other hand, since $y \mapsto G(x,y)$ is bounded e.g. in $L^{\frac{N-1}{N-2}}(B_\varepsilon(x_0))$ uniformly with respect to $x \in M$ (recall again (3.9) and the fact that the Riemannian measure $\mathcal{V}$ is locally Euclidean), and $G(x,y) \to G(x_0,y)$ as $x \to x_0$ for every $y \in M$, we see that $G(x,y)$ converges weakly to $G(x_0,y)$ in $L^{\frac{N-1}{N-2}}(B_\varepsilon(x_0))$, so that

$$\int_{B_\varepsilon(x_0)} G(x,y) \, d\mathcal{V}(y) \to \int_{B_\varepsilon(x_0)} G(x_0,y) \, d\mathcal{V}(y) \quad \text{as } x \to x_0.$$

Hence,

$$\limsup_{x \to x_0} |\mathcal{G}^f(x) - \mathcal{G}^f(x_0)| \leq 2 \, \|f\|_\infty \int_{B_\varepsilon(x_0)} G(x_0,y) \, d\mathcal{V}(y),$$

and the claim follows by letting $\varepsilon \to 0$, thanks to the local integrability of $y \mapsto G(x_0,y)$.

In order to prove that $\mathcal{G}^f \in L^p(M)$ for all $p \in (N/(N-2), \infty]$, it is convenient to use the representation formula (3.15) for the Green function. In fact, by means of (3.16), (3.17) and interpolation, it is straightforward to infer the following estimate:

$$\|T_t f\|_p \leq C t^{-\frac{N(p-1)}{2p}} \|f\|_1 \quad \forall p \in (1,\infty), \ \forall t > 0, \tag{4.33}$$

where C is a positive constant depending only on N, p. By (3.15),

$$\|\mathcal{G}^f\|_p \leq \int_0^\infty \|T_t f\|_p \, dt = \int_0^1 \|T_t f\|_p \, dt + \int_1^\infty \|T_t f\|_p \, dt \quad \forall p \in [1,\infty]. \tag{4.34}$$

By using (3.16) and $f \in L^1(M) \cap L^\infty(M)$, the first integral on the r.h.s. of (4.34) is finite for every $p \in [1,\infty]$. By means of (3.17) we can deduce that the second integral on the r.h.s. of (4.34) is finite for $p = \infty$; furthermore, thanks to (4.33), that integral is also finite for all $p \in (N/(N-2), \infty)$. We have thus shown that $\mathcal{G}^f \in L^p(M)$ for all $p \in (N/(N-2), \infty]$.

We are left with the proof of (4.32). We assume, with no loss of generality, that $f \geq 0$. For any $R > 0$, we denote by $\mathcal{G}_R^f$ the potential of f in B_R, that is, the unique $H_0^1(B_R)$ solution to

$$\begin{cases} -\Delta v = f & \text{in } B_R, \\ v = 0 & \text{on } \partial B_R. \end{cases}$$

Clearly,

$$\int_{B_R} |\nabla \mathcal{G}_R^f|^2 \, d\mathcal{V} = \int_{B_R} f \, \mathcal{G}_R^f \, d\mathcal{V}. \tag{4.35}$$

Because $\mathcal{G}_R^f$ converges monotonically from below to $\mathcal{G}^f$ and $f \in L^1(M) \cap L^\infty(M)$, and because from the first part of the proof we know that $\mathcal{G}^f \in L^p(M)$ for all $p \in (N/(N-2), \infty]$, we can pass to the limit in (4.35) as $R \to \infty$ to get

$$\int_M |\nabla \mathcal{G}^f|^2 \, d\mathcal{V} \leq \int_M f \, \mathcal{G}^f \, d\mathcal{V}$$

and

$$\nabla \mathcal{G}_R^f \rightharpoonup \nabla \mathcal{G}^f \quad \text{in } [L^2(M)]^N, \tag{4.36}$$

where $\nabla \mathcal{G}_R^f$ is set to be zero in B_R^c. Exploiting the fact that $\mathcal{G}_R^f = 0$ on ∂B_R and $-\Delta \mathcal{G}^f = f$ in M, we obtain

$$\int_{B_R} \langle \nabla \mathcal{G}_R^f, \nabla \mathcal{G}^f \rangle \, d\mathcal{V} = \int_{B_R} f \, \mathcal{G}_R^f \, d\mathcal{V}. \tag{4.37}$$

Identity (4.32) then follows by letting $R \to \infty$ in (4.37), on using (4.36) and the monotone convergence of $\mathcal{G}_R^f$ to $\mathcal{G}^f$. The case of signed functions follows by writing $f = f^+ - f^-$, and using the linearity of the potential operator. □

5. Existence of weak solutions: proofs

This section is devoted to the proofs of our main results concerning existence and fundamental properties of the weak solutions to (1.1) we construct.

5.1. *Consequences of the definition of weak solution*

The aim of this subsection is to prove the following result, which establishes some fundamental properties enjoyed by weak solutions in the sense of Definition 2.1.

Proposition 5.1. *Let assumption* (*H*)(i) *be satisfied. Let u be any function satisfying* (2.1)–(2.3). *Then*

$$u \in C((0, \infty); L^1(M)), \tag{5.1}$$

$$\int_M u(x, t_1) \, d\mathcal{V}(x) = \int_M u(x, t_2) \, d\mathcal{V}(x) \quad \textit{for all } t_2 > t_1 > 0, \tag{5.2}$$

$$\|u(t)\|_\infty \leq K t^{-\alpha} \|u\|^\beta_{L^\infty((0,\infty); L^1(M))} \quad \textit{for all } t > 0, \tag{5.3}$$

where K is a positive constant which only depends on m, N, and α, β are as in (2.7).

In order to prove Proposition 5.1 we need a preliminary lemma, which relies on results on the porous medium equation that are by now well known.

Lemma 5.2. *Let* $\mu \equiv u_0 \in L^1(M) \cap L^\infty(M)$*. Then there exists a unique weak solution* u *to problem* (1.1) *satisfying* (2.1)–(2.2) *down to* $\tau = 0$ *and*

$$-\int_0^\infty \int_M u(x,t)\varphi_t(x,t)\, d\mathcal{V}(x)\, dt + \int_0^\infty \int_M \langle \nabla(u^m)(x,t), \nabla\varphi(x,t)\rangle\, d\mathcal{V}(x)\, dt = \int_M u_0(x)\varphi(x,0)\, d\mathcal{V}(x) \tag{5.4}$$

for any $\varphi \in C_c^\infty(M \times [0,\infty))$*. Moreover,* $u \in C([0,\infty); L^1(M))$*, and if* v *is another weak solution to problem* (1.1) *with initial datum* $v_0 \in L^1(M) \cap L^\infty(M)$ *then*

$$\|u(t) - v(t)\|_1 \le \|u_0 - v_0\|_1 \quad \textit{for all } t > 0. \tag{5.5}$$

Proof. As $u_0 \in L^1(M) \cap L^\infty(M)$, existence of the so-called energy solutions, that is, solutions for which (2.1)–(2.2) hold down to $\tau = 0$ and (5.4) is satisfied, is rather standard (we refer e.g. to [30, Sections 5, 9] for the Euclidean case). The simplest way to construct them is e.g. by using approximate problems on balls, establishing suitable a priori estimates and then passing to the limit as the radius of the ball goes to infinity. A sketch of an analogous procedure is provided at the beginning of the proof of Theorem 2.2 below. Uniqueness in this class is due to a well-known theorem by Oleĭnik (see [30, Section 5.3]). The continuity of $u(t)$ as a curve in $L^1(M)$ is then a consequence of an alternative construction of the solution, which makes use of the Crandall–Liggett Theorem and proceeds by means of time discretization (see [30, Section 10]). Also the L^1-contractivity inequality (5.5) is a classical fact (see [30, Section 3]).

For similar issues involving existence, uniqueness and equivalence of different concepts of solution (in the framework of the fractional porous medium equation), we also refer to [15, Appendix A]. □

Proof of Proposition 5.1. Given almost every $t_0 > 0$, namely any Lebesgue point of $u(t)$ as a curve in $L^1(M)$, let $\{\theta_\varrho^{t_0}\}$ $(0 < \varrho < t_0)$ be a family of positive, smooth approximations of $\chi_{[t_0,\infty)}$ such that supp $\theta_\varrho^{t_0} \subset [t_0 - \varrho, \infty)$ and $(\theta_\varrho^{t_0})' \to \delta_{t_0}$ as $\varrho \to 0$. Let φ be any function in $C_c^\infty(M \times [t_0,\infty))$; we can assume that φ is the restriction to $M \times [t_0,\infty)$ of some function in $C_c^\infty(M \times (0,\infty))$. Hence, by plugging in (2.3) the test function

$$\varphi_\varrho(x,t) := \theta_\varrho^{t_0}(t)\varphi(x,t) \qquad \forall (x,t) \in M \times (0,\infty)$$

and letting $\varrho \to 0$, we end up with the identity

$$-\int_{t_0}^\infty \int_M u(x,t)\varphi_t(x,t)\, d\mathcal{V}(x)\, dt + \int_{t_0}^\infty \int_M \langle \nabla(u^m)(x,t), \nabla\varphi(x,t)\rangle\, d\mathcal{V}(x)\, dt = \int_M u(x,t_0)\varphi(x,t_0)\, d\mathcal{V}(x).$$

On the other hand, by Definition 2.1 it is apparent that (2.1)–(2.2) hold for $\tau = t_0$; we have therefore shown that $u\rfloor_{[t_0,\infty)}$ is a weak solution to (1.1) (with 0 replaced by t_0) in the sense of Lemma 5.2, starting from the initial datum $\mu \equiv u(t_0) \in L^1(M) \cap L^\infty(M)$. In particular $u \in C([t_0,\infty); L^1(M))$, whence (5.1) because t_0 can be arbitrarily small.

To establish (5.2), we use a reasoning similar to the one outlined in Remark 2.12. Indeed, by the same arguments, we know that the free-mass time-shifted Barenblatt functions

$$B_0^{E,D}(\rho(x),t) := (t+1)^{-\alpha}[D - k\,\rho(x)^2(t+1)^{-\beta}]_+^{1/(m-1)} \tag{5.6}$$

for all $(x,t) \in M \times (0,\infty)$, $\forall D > 0$ are (weak) supersolutions to (1.1) with initial datum $\mu \equiv B_0^{E,D}(\rho(x),0)$, where α, β are as in (2.7) and k is a positive constant depending only on m, N. Let us first prove (5.2) under the additional assumption that $u(t_1)$ is compactly supported. In this case, we can always choose D in (5.6) so large that $|u(x,t_1)| \le B_0^{E,D}(\rho(x),0)$. Hence, because $B_0^{E,D}(\rho(x),t)$ and $-B_0^{E,D}(\rho(x),t)$ are a supersolution and a subsolution, respectively, it follows that

$$-B_0^{E,D}(\rho(x),t-t_1) \le u(x,t) \le B_0^{E,D}(\rho(x),t-t_1) \qquad \text{for a.e. } (x,t) \in M \times (t_1,\infty). \tag{5.7}$$

Since $B_0^{E,D}$ is compactly supported for all times, estimate (5.7) implies that u is also compactly supported for all times. In particular, (5.2) holds. If $u(t_1)$ is not compactly supported, we can pick a sequence of initial data $u_{1,n} \in L^1(M) \cap L^\infty(M)$, with compact support, such that $\lim_{n\to\infty} u_{1,n} = u(t_1)$ in $L^1(M)$. If we denote by u_n the solution to (1.1) corresponding to $\mu \equiv u_{1,n}$, the above argument shows that (5.2) is satisfied with u replaced by $u_n(t-t_1)$: on the other hand, the L^1-contractivity inequality (5.5) ensures that the solution map is continuous in $L^1(M)$, so we can let $n \to \infty$ to get (5.2).

Let us finally deal with the smoothing effect (5.3). For initial data u_0 and corresponding solutions u as in Lemma 5.2, the estimate

$$\|u(t)\|_\infty \le K t^{-\alpha} \|u_0\|_{L^1(M)}^\beta \qquad \text{for all } t > 0 \tag{5.8}$$

is a consequence of the Sobolev inequality (3.6) (see e.g. [4, Theorem 4.1] or [14, Corollary 5.6]). Hence, by applying (5.8) to $u\rfloor_{[t_1,\infty)}$ we obtain

$$\|u(t)\|_\infty \le K t^{-\alpha} \|u(t_1)\|_{L^1(M)}^\beta \le K t^{-\alpha} \|u\|_{L^\infty((0,\infty);L^1(M))}^\beta \qquad \text{for all } t > t_1;$$

since $t_1 > 0$ is arbitrary, the conclusion follows. □

5.2. *Proof of the existence result*

Let us outline the main ideas behind the proof of Theorem 2.2. Suppose first that μ is a compactly supported measure. Take $\mu_\varepsilon \in L^1(M) \cap L^\infty(M)$ such that

$$\int_M \phi \mu_\varepsilon \, d\mathcal{V} \to \int_M \phi \, d\mu \qquad \text{as } \varepsilon \to 0 \text{ for any } \phi \in C_b(M), \tag{5.9}$$

and

$$\int_M |\mu_\varepsilon| \, d\mathcal{V} \to |\mu|(M) \quad \text{as } \varepsilon \to 0; \tag{5.10}$$

to this end it suffices, for instance, to mollify the image of μ on $\mathbb{R}^N$ and then go back to M through one of the regular bijections between M and $\mathbb{R}^N$. For any fixed $\varepsilon > 0$ and $R > 0$, consider then the following homogeneous Dirichlet problem:

$$\begin{cases} u_t = \Delta(u^m) & \text{in } B_R \times (0,\infty), \\ u = 0 & \text{on } \partial B_R \times (0,\infty), \\ u = \mu_\varepsilon \rfloor_{B_R} & \text{on } B_R \times \{0\}, \end{cases} \tag{5.11}$$

for which one can provide the same definition of weak solution as in Lemma 5.2 upon replacing M with B_R and requiring in addition that $u^m \in H^1_0(B_R)$. Existence, uniqueness and good properties of the weak (energy) solution to (5.11), which will be denoted by $u_{\varepsilon,R}$, can be shown by well-established methods (see again the proof of Lemma 5.2). Classical compactness arguments ensure that $\{u_{\varepsilon,R}\}$ converges (up to subsequences), as $R \to \infty$, to a function u_ε satisfying (2.1)–(2.3). A further passage to the limit as $\varepsilon \to 0$ yields a function u which still satisfies (2.1)–(2.3). The hardest point is to prove that u also fulfils (2.4), that is, its initial trace is precisely μ. To this end we have to adapt to our framework some potential techniques first introduced by M. Pierre [26] and then recently developed in [32, 16] in the nonlocal Euclidean context. Finally, we handle general finite measures (not necessarily compactly supported) by an additional approximation.

Proof of Theorem 2.2. By standard arguments one can infer that the weak (energy solution) $u_{\varepsilon,R}$ to (5.11) satisfies the nonexpansivity of the L^1 norms

$$\|u_{\varepsilon,R}(t)\|_{L^1(B_R)} \le \|\mu_\varepsilon\|_{L^1(B_R)} \quad \text{for all } t > 0, \tag{5.12}$$

the L^1-L^∞ smoothing effect

$$\|u_{\varepsilon,R}(t)\|_{L^\infty(B_R)} \le K t^{-\alpha} \|\mu_\varepsilon\|^{\beta}_{L^1(B_R)} \quad \text{for all } t > 0, \tag{5.13}$$

and the energy estimates

$$\int_{t_1}^{t_2} \int_{B_R} |\nabla(u^m_{\varepsilon,R})(x,t)|^2 \, d\mathcal{V}(x)\, dt + \int_{B_R} |u_{\varepsilon,R}(x,t_2)|^{m+1} \, d\mathcal{V}(x) \le K^m \, t_1^{-\alpha m} \|\mu_\varepsilon\|^{1+\beta m}_{L^1(B_R)}, \tag{5.14}$$

$$\int_{t_1}^{t_2} \int_{B_R} |(z_{\varepsilon,R})_t(x,t)|^2 \, d\mathcal{V}(x)\, dt \le \widetilde{C} \, t_1^{-\alpha m} \|\mu_\varepsilon\|^{1+\beta m}_{L^1(B_R)}, \tag{5.15}$$

for all $t_2 > t_1 > 0$, where $z_{\varepsilon,R} := u_{\varepsilon,R}^{(m+1)/2}$ and $\widetilde{C}$ is a positive constant that depends on N, m, t_1, t_2 but is independent of ε, R. In the Euclidean context estimates (5.12), (5.14), (5.15) are by now classical: see again [30], in particular Section 5 there. The fact that here B_R is a ball on a Riemannian manifold is inessential. The smoothing effect (5.13) is again a direct consequence of the Sobolev inequality (3.6).

Let $\mathcal{G}_{\varepsilon,R}$ be the potential of $u_{\varepsilon,R}$, that is,

$$\mathcal{G}_{\varepsilon,R}(x,t) := \int_M G_R(x,y)u_{\varepsilon,R}(y,t)\,d\mathcal{V}(y) \quad \text{for all } x \in M,\ t > 0,$$

where G_R is the Green function of the Dirichlet Laplacian in B_R. We claim that $\mathcal{G}_{\varepsilon,R}$ solves

$$(\mathcal{G}_{\varepsilon,R})_t = -u^m_{\varepsilon,R} \quad \text{in } B_R \times (0,\infty),$$

in the sense that

$$\int_{B_R} \mathcal{G}_{\varepsilon,R}(x,t_2)\phi(x)\,d\mathcal{V}(x) - \int_{B_R} \mathcal{G}_{\varepsilon,R}(x,t_1)\phi(x)\,d\mathcal{V}(x) = -\int_{t_1}^{t_2}\int_{B_R} u^m_{\varepsilon,R}(x,t)\phi(x)\,d\mathcal{V}(x)\,dt \tag{5.16}$$

for all $t_2 > t_1 > 0$ and any $\phi \in C_c^\infty(B_R)$. Indeed, by standard elliptic regularity,

$$\mathcal{G}^\phi_R(x) := \int_{B_R} G_R(x,y)\phi(y)\,d\mathcal{V}(y) \in C_0^\infty(B_R). \tag{5.17}$$

Hence, we are allowed to pick the test function $\varphi(x,t) = \mathcal{G}^\phi_R(x)\,[\theta^{t_1}_\varrho(t) - \theta^{t_2}_\varrho(t)]$ in the weak formulation of (5.11), with $\theta^{t_\cdot}_\varrho$ defined as in the proof of Proposition 5.1. By using the fact that $(-\Delta)\mathcal{G}^\phi_R = \phi$ in B_R, integrating by parts and letting $\varrho \to 0$, we get

$$\int_{B_R} u_{\varepsilon,R}(x,t_2)\mathcal{G}^\phi_R(x)\,d\mathcal{V}(x) - \int_{B_R} u_{\varepsilon,R}(x,t_1)\mathcal{G}^\phi_R(x)\,d\mathcal{V}(x) = -\int_{t_1}^{t_2}\int_{B_R} u^m_{\varepsilon,R}(x,t)\phi(x)\,d\mathcal{V}(x)\,dt, \tag{5.18}$$

which is (5.16) up to an application of Fubini's Theorem on the left-hand side. By letting $t_1 \to 0$ in (5.18), we obtain

$$\int_{B_R} u_{\varepsilon,R}(x,t_2)\mathcal{G}^\phi_R(x)\,d\mathcal{V}(x) - \int_{B_R} \mathcal{G}^\phi_R(x)\,\mu_\varepsilon(x)\,d\mathcal{V}(x) = -\int_0^{t_2}\int_{B_R} u^m_{\varepsilon,R}(x,t)\phi(x)\,d\mathcal{V}(x)\,dt. \tag{5.19}$$

We now let $R \to \infty$. Thanks to (5.12)–(5.15), routine compactness and lower-semi-continuity arguments ensure that $\{u_{\varepsilon,R}\}$, set to be zero outside B_R, converges almost everywhere (up to subsequences) to some function u_ε which satisfies (2.1)–(2.3) (with u replaced by u_ε) and the analogues of (5.12)–(5.15):

$$\|u_\varepsilon(t)\|_1 \le \|\mu_\varepsilon\|_1 \quad \text{for all } t > 0, \tag{5.20}$$

$$\|u_\varepsilon(t)\|_\infty \le K t^{-\alpha}\|\mu_\varepsilon\|_1^\beta \quad \text{for all } t > 0, \tag{5.21}$$

$$\int_{t_1}^{t_2}\int_M |\nabla(u_\varepsilon^m)(x,t)|^2\,d\mathcal{V}(x)\,dt + \int_M |u_\varepsilon(x,t_2)|^{m+1}\,d\mathcal{V}(x) \\ \leq K^m\,t_1^{-\alpha m}\,\|\mu_\varepsilon\|_1^{1+\beta m}, \tag{5.22}$$

$$\int_{t_1}^{t_2}\int_M |(z_\varepsilon)_t(x,t)|^2\,d\mathcal{V}(x)\,dt \leq \widetilde{C}\,t_1^{-\alpha m}\,\|\mu_\varepsilon\|_1^{1+\beta m}, \tag{5.23}$$

for all $t_2 > t_1 > 0$, where $z_\varepsilon := u_\varepsilon^{(m+1)/2}$. Moreover, since

$$\lim_{R\to\infty}\mathcal{G}_R^\phi(x) = \mathcal{G}^\phi(x) \quad \forall x\in M, \qquad \mathcal{G}^\phi \in C_0(M), \qquad |\mathcal{G}_R^\phi| \leq \mathcal{G}^{|\phi|}$$

(consequences of definition (4.15) plus (3.9), (3.10), (3.13)), by exploiting estimates (5.12)–(5.13) we can pass to the limit in (5.19) to get

$$\int_M u_\varepsilon(x,t_2)\mathcal{G}^\phi(x)\,d\mathcal{V}(x) - \int_M \mathcal{G}^\phi(x)\,\mu_\varepsilon(x)\,d\mathcal{V}(x) = -\int_0^{t_2}\int_M u_\varepsilon^m(x,t)\phi(x)\,d\mathcal{V}(x)\,dt.$$

As a final step, we let $\varepsilon\to 0$. In view of (5.20)–(5.23) and (5.9)–(5.10), proceeding as above we deduce that $\{u_\varepsilon\}$ converges almost everywhere (up to subsequences) to some function u which satisfies (2.1)–(2.3),

$$\|u(t)\|_1 \leq |\mu|(M) \qquad \text{for all } t>0, \tag{5.24}$$

$$\|u(t)\|_\infty \leq K t^{-\alpha}|\mu|(M)^\beta \qquad \text{for all } t>0 \tag{5.25}$$

and

$$\int_M u(x,t_2)\mathcal{G}^\phi(x)\,d\mathcal{V}(x) - \int_M \mathcal{G}^\phi(x)\,d\mu(x) = -\int_0^{t_2}\int_M u^m(x,t)\phi(x)\,d\mathcal{V}(x)\,dt,$$

that is,

$$\int_M \mathcal{G}(x,t_2)\phi(x)\,d\mathcal{V}(x) - \int_M \mathcal{G}^\mu(x)\phi(x)\,d\mathcal{V}(x) \\ = -\int_0^{t_2}\int_M u^m(x,t)\phi(x)\,d\mathcal{V}(x)\,dt \tag{5.26}$$

up to an application of Fubini's Theorem, where we denote by $\mathcal{G}(t)$ the potential of $u(t)$. In particular, by combining (5.24)–(5.25) and (5.26) we deduce the estimate

$$\left|\int_M \mathcal{G}(x,t_2)\phi(x)\,d\mathcal{V}(x) - \int_M \mathcal{G}^\mu(x)\phi(x)\,d\mathcal{V}(x)\right| \\ \leq \|\phi\|_\infty\,K^{m-1}\,|\mu|(M)^{1+\beta(m-1)}\int_0^{t_2} t^{-\alpha(m-1)}\,dt. \tag{5.27}$$

By compactness results in measure spaces (recall Definition 4.5), from (5.24) it follows that every sequence $t_n\to 0$ has a subsequence $\{t_{n_k}\}$ such that $\{u(t_{n_k})\}$ converges vaguely to a certain finite Radon measure ν. On the other hand, as noted above, $\mathcal{G}^{\phi}(x)$ is a continuous function that vanishes as $\operatorname{dist}(x,o) \to \infty$. We can therefore pass to the limit as $t_2 \to 0$ in (5.27) (by using Fubini's Theorem again): because ϕ is arbitrary, it follows that $\mathcal{G}^{\mu} = \mathcal{G}^{\nu}$ almost everywhere in M; so, thanks to Lemma 4.13, we have $\nu = \mu$ and the limit measure does not depend on the particular subsequence. We have thus proved that

$$\lim_{t\to 0}\int_M u(x,t)\phi(x)\,d\mathcal{V}(x) = \int_M \phi(x)\,d\mu(x) \quad \text{for any } \phi \in C_0(M). \tag{5.28}$$

In particular, given the lower semicontinuity of the total variation with respect to the vague topology,

$$|\mu|(M) \le \liminf_{t\to 0}\|u(t)\|_1, \tag{5.29}$$

so that by gathering (5.24) and (5.29) we obtain

$$\lim_{t\to 0}\|u(t)\|_1 = |\mu|(M). \tag{5.30}$$

We are then left with proving that (5.28) holds for any $\phi \in C_b(M)$. To this end, we exploit Lemma 4.11. In fact, by (5.24) and [1, Theorem 1.59], every sequence $t_n \to 0$ has a subsequence $\{t_{n_k}\}$ such that $\{u_+(t_{n_k})\}$ and $\{u_-(t_{n_k})\}$ converge vaguely to some positive finite Radon measures $\mu_{\mathcal{P}}$ and $\mu_{\mathcal{N}}$, respectively. Thanks to (5.28) it follows that $\mu = \mu_{\mathcal{P}} - \mu_{\mathcal{N}}$. Moreover, by (5.30) and the lower semicontinuity of the total variation with respect to the vague topology, we have

$$\begin{aligned}\mu_{\mathcal{P}}(M) + \mu_{\mathcal{N}}(M) &\le \liminf_{k\to\infty}\|u_+(t_{n_k})\|_1 + \liminf_{k\to\infty}\|u_-(t_{n_k})\|_1 \\ &\le \liminf_{k\to\infty}\|u(t_{n_k})\|_1 = |\mu|(M).\end{aligned} \tag{5.31}$$

By Lemma 4.11, (5.31) implies $\mu_{\mathcal{P}} = \mu_+$ and $\mu_{\mathcal{N}} = \mu_-$, so that

$$\lim_{k\to\infty}\|u_+(t_{n_k})\|_1 = \mu_+(M), \qquad \lim_{k\to\infty}\|u_-(t_{n_k})\|_1 = \mu_-(M). \tag{5.32}$$

From (5.32) and [1, Proposition 1.80] we then infer that

$$\lim_{k\to\infty}\int_M u_\pm(x,t_{n_k})\phi(x)\,d\mathcal{V}(x) = \int_M \phi(x)\,d\mu_\pm(x)$$

for all $\phi \in C_b(M)$. Since the same argument can be performed along any sequence, (2.4) follows. Note that the conservation of "mass" (2.5) is an immediate consequence of (5.2) and (2.4) with $\phi = 1$.

Finally, in order to handle a general finite Radon measure μ (not necessarily compactly supported), it is enough to approximate μ by the sequence $\{\mu\rfloor_{B_n}\}$ as $n \to \infty$, and proceed as above. □

Remark 5.3 (The case $(N-2)/N < m < 1$). By using the same techniques as in the proof of Theorem 2.2, we can establish existence of weak solutions to problem (1.1) also for m below 1, in the supercritical fast-diffusion range $(N-2)/N < m < 1$. Indeed, well-posedness of the approximate problems (5.11) still holds, as do the key estimates (5.12)–(5.15); the assumption $m > (N-2)/N$ plays a crucial role in the validity of the smoothing effect (5.13) (see [4, Theorem 4.1]). The only difference is that, since $m < 1$, the r.h.s. of (5.26) has to be bounded as follows:

$$\left| \int_0^{t_2} \int_M u^m(x,t)\phi(x)\, d\mathcal{V}(x)\, dt \right| \le \|\phi\|_\infty \mathcal{V}(\operatorname{supp}\phi)^{1-m} |\mu|(M)^m\, t_2.$$

Actually, the only point that we are not able to recover in Theorem 2.2 is the conservation of "mass" (2.5). The problem is that, for $m < 1$, the analogues of the Euclidean Barenblatt profiles (5.6) we exploit in the proof of Proposition 5.1 are no more compactly supported, and their decay rate at infinity is too slow compared to the possible volume growth of the Riemannian manifolds we are interested in. On the other hand, in general mass conservation fails: for instance, on Riemannian manifolds supporting the Poincaré/gap inequality $\|f\|_2 \le \|\nabla f\|_2$ for all $f \in C_c^\infty(M)$ (like those whose sectional curvatures are bounded from above by a negative constant), the $L^1(M)$ norm of the solution vanishes in finite time [4, Theorem 6.1].

5.3. Connection between the Green function and the porous medium equation: proof

Let us consider again the solutions $u_{\varepsilon,R}$ to the approximate problems (5.11). If $\mu \in \mathcal{M}_F^+(M)$, such solutions are by construction nonnegative; hence, by the standard comparison principle, for all $0 < R_1 < R_2$,

$$u_{\varepsilon,R_1}(x,t) \le u_{\varepsilon,R_2}(x,t) \quad \text{for a.e. } (x,t) \in B_{R_1} \times (0,\infty). \tag{5.33}$$

For any fixed $R > 0$, if we let $\varepsilon \to 0$ we obtain, by the techniques of the proof of Theorem 2.2, a nonnegative weak solution u_R to (5.11) with μ_ε replaced by μ. By letting $\varepsilon \to 0$ in (5.33) we also deduce that order is preserved:

$$u_{R_1}(x,t) \le u_{R_2}(x,t) \quad \text{for a.e. } (x,t) \in B_{R_1} \times (0,\infty),$$

so $\{u_R\}$ is nondecreasing in R. As a consequence, the pointwise limit u as $R \to \infty$ exists regardless of the validity of hypothesis (H); in such a general framework, this is precisely what we mean by a "solution" to (1.1) when μ is a positive measure.

Proof of Theorem 2.11. Let u_R be the solution of problem (5.11) with $\mu_\varepsilon \equiv \delta_{x_0}$, where R is so large that $x_0 \in B_R$. Let $\mathcal{G}_R$ be the potential of u_R, and $\mathcal{G}_R^\phi$ the potential of any $\phi \in C_c^\infty(B_R)$ (recall (5.17)). Given any $t_2 > t_1 > 0$, by plugging the test function $\varphi(x,t) = \mathcal{G}_R^\phi(x)[\theta_\varrho^{t_1}(t) - \theta_\varrho^{t_2}(t)]$ in the definition of weak solution ($\theta_\varrho^{t.}$ is as in the proof of Proposition 5.1), letting $\varrho \to 0$ and invoking Tonelli's Theorem, we end up with the identity

$$\int_{B_R} \mathcal{G}_R(x,t_2)\phi(x)\,d\mathcal{V}(x) - \int_{B_R} \mathcal{G}_R(x,t_1)\phi(x)\,d\mathcal{V}(x) = -\int_{t_1}^{t_2}\int_{B_R} u_R^m(x,t)\phi(x)\,d\mathcal{V}(x)\,dt. \tag{5.34}$$

From (5.34) we deduce that the map $t \mapsto \mathcal{G}_R(x,t)$ is nonincreasing (recall that u_R is nonnegative). Hence, $\mathcal{G}_R(t)$ has a pointwise limit as $t \to \infty$. The limit is necessarily zero: this is a straightforward consequence, for instance, of the smoothing estimate (2.6), which clearly holds for (5.11) as well. Passing to the limit in (5.34) as $t_2 \to \infty$ we get

$$\int_{B_R} \mathcal{G}_R(x,t_1)\phi(x)\,d\mathcal{V}(x) = \int_{t_1}^{\infty}\int_{B_R} u_R^m(x,t)\phi(x)\,d\mathcal{V}(x)\,dt. \tag{5.35}$$

Letting $t_1 \to 0$ in (5.35), recalling the initial condition and using again Tonelli's Theorem we infer that

$$\int_{B_R} G_R(x_0,x)\phi(x)\,d\mathcal{V}(x) = \int_{B_R} \phi(x)\int_0^{\infty} u_R^m(x,t)\,dt\,d\mathcal{V}(x). \tag{5.36}$$

Now we point out that both $x \mapsto \int_0^\infty u_R^m(x,t)\,dt$ and $x \mapsto G_R(x_0,x)$ are $\mathfrak{M}$-superharmonic functions belonging to $L^1(B_R)$. Indeed, in view of standard results concerning the porous medium equation on bounded domains (see [30]), it is well known that $\|u_R(t)\|_{L^\infty(B_R)}$ behaves at most like $t^{-N/[2+N(m-1)]}$ as $t \to 0$ and at most like $t^{-1/(m-1)}$ as $t \to \infty$. This immediately implies that $\int_0^\infty u_R^m(t)\,dt \in L^1(B_R)$. Moreover, by classical results, $u(x,t)$ is continuous in $B_R \times [t_1,\infty)$ for all $t_1 > 0$. In particular, by dominated convergence, $\int_{t_1}^\infty u_R^m(t)\,dt$ is also continuous for all $t_1 > 0$. As a consequence of the differential equation solved by u_R and of the fact that $\|u_R(t)\|_{L^\infty(B_R)}$ vanishes as $t \to \infty$, we deduce that $\int_{t_1}^\infty u_R^m(t)\,dt$ is superharmonic. Hence, thanks to Theorem 2.9, $\int_{t_1}^\infty u_R^m(t)\,dt$ is $\mathfrak{M}$-superharmonic; then, in view of Lemma 4.4, so is $\int_0^\infty u_R^m(t)\,dt$. The fact that $x \mapsto G_R(x_0,x)$ is $\mathfrak{M}$-superharmonic follows from Corollary 4.2; in addition, it belongs to $L^1(B_R)$ since B_R is bounded (see e.g. [12]). In view of the above remarks, (5.36) and Lemma 4.3, we have

$$G_R(x_0,x) = \int_0^\infty u_R^m(x,t)\,dt \qquad \text{for all } x \in B_R. \tag{5.37}$$

The conclusion then follows from (5.37) by monotone convergence and the fact that $G_R \uparrow G$ as $R \to \infty$ everywhere. □

6. Proof of the uniqueness result

We begin this section with a key lemma, which will be useful in what follows and which is essentially based on the potential-theoretic results given in Sections 2.2 and 4. For our purposes it is crucial that the limit in (6.3) below is valid for *every* x, which follows from Proposition 4.6.

Lemma 6.1. *Let u be a nonnegative weak solution of problem* (1.1) *with $\mu \in \mathcal{M}_F^+(M)$. Then the potential $\mathcal{G}(t)$ of $u(t)$ satisfies*

$$\mathcal{G}_t = -u^m \quad \text{in } M \times (0, \infty), \tag{6.1}$$

in the sense that

$$\int_M \mathcal{G}(x, t_2)\phi(x)\, d\mathcal{V}(x) - \int_M \mathcal{G}(x, t_1)\phi(x)\, d\mathcal{V}(x) = -\int_{t_1}^{t_2}\int_M u^m(x, t)\phi(x)\, d\mathcal{V}(x)\, dt \tag{6.2}$$

for all $t_2 > t_1 > 0$ and for any $\phi \in C_c^\infty(M)$. In particular, $\mathcal{G}(t)$ admits an absolutely continuous version on $(0, \infty)$ in $L^p(M)$ for all $p \in (N/(N-2), \infty)$, which is nonincreasing in t. Moreover,

$$\lim_{t\to 0} \mathcal{G}(x, t) = \mathcal{G}^\mu(x) \quad \text{for all } x \in M. \tag{6.3}$$

Proof. Consider a cut-off function $\xi \in C^\infty([0, \infty))$ with

$$\xi = \begin{cases} 1 & \text{in } [0, 1], \\ 0 & \text{in } [2, \infty), \end{cases} \qquad 0 \le \xi \le 1 \quad \text{in } [0, \infty).$$

Set $\rho(x) := d(x, o)$ for all $x \in M$. For every $R \ge 1$, define

$$\xi_R(x) := \xi(\rho(x)/R) \qquad \text{for all } x \in M.$$

Let $C := \max\{\sup_{[0,\infty)} |\xi'|, \sup_{[0,\infty)} |\xi''|\}$. In view of (3.1) we have

$$\Delta\xi_R(x) = \frac{1}{R^2}\,\xi''\left(\frac{\rho(x)}{R}\right) + \frac{m(\rho, \theta)}{R}\,\xi'\left(\frac{\rho(x)}{R}\right) \qquad \text{for all } x \in M. \tag{6.4}$$

Clearly

$$\nabla\xi_R = 0 \quad \text{and} \quad \Delta\xi_R = 0 \quad \text{in } B_R \cup B_{2R}^c; \tag{6.5}$$

moreover,

$$|\nabla\xi_R(x)| \le C/R \qquad \text{for all } x \in B_{2R} \setminus B_R$$

since $|\nabla\rho(x)| = 1$. Furthermore, thanks to assumption (*H*)(ii), it is not difficult to check that there exists a positive constant $\hat{C}$ such that (3.4) is fulfilled by a suitable ψ satisfying $\psi(\rho) = e^{\hat{C}\rho^2}$ for all ρ large enough. As a consequence, by exploiting also (3.5), from (6.4) we infer that

$$\begin{aligned} |\Delta\xi_R(x)| &\le \frac{1}{R^2}\left|\xi''\left(\frac{\rho(x)}{R}\right)\right| + \frac{N-1}{R}\left|\frac{\psi'(\rho(x))}{\psi(\rho(x))}\xi'\left(\frac{\rho(x)}{R}\right)\right| \\ &\le \frac{C}{R^2} + 2C\hat{C}(N-1) \le \bar{C} \qquad \forall x \in B_{2R} \setminus B_R \end{aligned} \tag{6.6}$$

provided R is large enough, for another positive constant $\bar{C}$ that depends only on N, C and $\hat{C}$.

In view of (3.9), (3.10) and (3.13), the potential

$$\mathcal{G}^{\phi}(x) := \int_M G(x,y)\phi(y)\,d\mathcal{V}(y) \quad \text{for all } x \in M$$

is a regular function belonging to $C_0(M)$. For every $R \geq 1$ and $\varrho > 0$, we are therefore allowed to pick the test function

$$\varphi(x,t) := \xi_R(x)\,\mathcal{G}^{\phi}(x)[\theta_{\varrho}^{t_1}(t) - \theta_{\varrho}^{t_2}(t)] \quad \text{for all } x \in M,\ t \geq 0$$

in (2.3), where $\theta_{\varrho}^{t.}$ is as in the proof of Proposition 5.1. By letting $\varrho \to 0$ we get

$$\begin{aligned} \int_M u(x,t_2)\xi_R(x)\mathcal{G}^{\phi}(x)\,d\mathcal{V}(x) - \int_M u(x,t_1)\xi_R(x)\mathcal{G}^{\phi}(x)\,d\mathcal{V}(x) \\ = \int_{t_1}^{t_2}\int_M u^m(x,t)\Delta(\xi_R\,\mathcal{G}^{\phi})(x)\,d\mathcal{V}(x)\,dt; \end{aligned} \tag{6.7}$$

the r.h.s. of (6.7) reads (we use the fact that $-\Delta\mathcal{G}^{\phi} = \phi$ in M)

$$\begin{aligned} -\int_{t_1}^{t_2}\int_M u^m(x,t)\xi_R(x)\phi(x)\,d\mathcal{V}(x)\,dt + \underbrace{\int_{t_1}^{t_2}\int_M u^m(x,t)\Delta\xi_R(x)\mathcal{G}^{\phi}(x)\,d\mathcal{V}(x)\,dt}_{I_1} \\ + \underbrace{2\int_{t_1}^{t_2}\int_M u^m(x,t)\langle\nabla\xi_R,\,\nabla\mathcal{G}^{\phi}\rangle(x)\,d\mathcal{V}(x)\,dt}_{I_2}\,. \end{aligned} \tag{6.8}$$

In view of (6.5)–(6.6), we can estimate the last two integrals of (6.8) as follows:

$$|I_1| \leq \bar{C}\|\mathcal{G}^{\phi}\|_{\infty}\int_{t_1}^{t_2}\int_{B_{2R}\setminus B_R} u^m(x,t)\,d\mathcal{V}(x)\,dt, \tag{6.9}$$

$$\begin{aligned} |I_2| \leq {} & \frac{C(t_2-t_1)^{1/2}}{R} \\ & \times \left(\int_{t_1}^{t_2}\int_{B_{2R}\setminus B_R} u^{2m}(x,t)\,d\mathcal{V}(x)\,dt\right)^{1/2}\left(\int_{B_{2R}\setminus B_R}|\nabla\mathcal{G}^{\phi}(x)|^2\,d\mathcal{V}(x)\right)^{1/2}. \end{aligned} \tag{6.10}$$

Since $u \in L^p(M \times (\tau,\infty))$ for all $\tau > 0$, $p \in [1,\infty]$, and since $\nabla\mathcal{G}^{\phi} \in [L^2(M)]^N$ (recall Lemma 4.14), by letting $R \to \infty$ we deduce that I_1 and I_2 vanish, so that passing to the limit in (6.7) yields

$$\int_M u(x,t_2)\mathcal{G}^{\phi}(x)\,d\mathcal{V}(x) - \int_M u(x,t_1)\mathcal{G}^{\phi}(x)\,d\mathcal{V}(x) = -\int_{t_1}^{t_2}\int_M u^m(x,t)\phi(x)\,d\mathcal{V}(x)\,dt,$$

which is (6.2) up to an application of Tonelli's Theorem. The absolute continuity of $\mathcal{G}(t)$ as a curve in $L^p(M)$ for any $p \in (N/(N-2),\infty)$ is then a consequence of (6.2) and

Lemma 4.14 (we use the fact that $u(t) \in L^1(M) \cap L^\infty(M)$). Since $u \geq 0$, still by (6.2) and Lemma 4.14 we deduce that for *every* $x \in M$ the function $t \mapsto \mathcal{G}(x,t)$ is nonincreasing.

In order to establish (6.3), pick a sequence $\{t_n\} \subset (0,\infty)$ such that $t_n \to 0$ as $n \to \infty$. From (3.9) and the fact that $u \in L^\infty((0,\infty); L^1(M))$ we infer that for each compact subset $K \subset M$ and for any $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that

$$\int_{B_R^c} \int_K G(x,y)\, d\mathcal{V}(y)\, u(x,t_n)\, d\mathcal{V}(x) \leq \varepsilon \qquad \text{for all } R > R_\varepsilon,\ n \in \mathbb{N}.$$

Furthermore, by Definition 2.1, $\{u(t_n)\}$ converges vaguely to μ. By monotonicity, the function $x \mapsto \liminf \mathcal{G}(x,t_n)$ is l.s.c., hence we can apply Proposition 4.6 to deduce that

$$\mathcal{G}^\mu(x) = \liminf_{n\to\infty} \mathcal{G}(x,t_n) \qquad \text{for every } x \in M.$$

This implies (6.3), due to the just mentioned monotonicity property of $t \mapsto \mathcal{G}(x,t)$. □

6.1. *Formal strategy of proof*

Our method of proof is modelled after the one given in [26] in the Euclidean context (see also [16, proof of Theorem 3.4]). We sketch it below.

Let u_1, u_2 be two weak solutions of problem (1.1) which take on the same initial measure $\mu \in \mathcal{M}_F^+(M)$. Let $\mathcal{G}_1(t)$ and $\mathcal{G}_2(t)$ be the corresponding potentials. Given any $h > 0$, define

$$W(x,t) := \mathcal{G}_2(x,t+h) - \mathcal{G}_1(x,t) \qquad \text{for all } x \in M,\ t > 0. \tag{6.11}$$

In view of Lemma 6.1,

$$W_t(x,t) = a(x,t)\, \Delta W(x,t) \qquad \text{in } M \times (0,\infty), \tag{6.12}$$

where

$$a(x,t) := \begin{cases} \frac{u_1^m(x,t) - u_2^m(x,t+h)}{u_1(x,t) - u_2(x,t+h)} > 0 & \text{if } u_1(x,t) \neq u_2(x,t+h), \\ 0 & \text{elsewhere.} \end{cases} \tag{6.13}$$

Still Lemma 6.1 yields $W(x,0) \leq 0$ in M. The conclusion would follow if we could show that $W \leq 0$ in $M \times (0,\infty)$, since this would imply, by interchanging the roles of u_1 and u_2, that $W = 0$, and hence, by letting $h \to 0$, that $u_1 = u_2$. In order to prove that $W \leq 0$ one considers solutions of the *dual* problem

$$\begin{cases} \varphi_t = -\Delta(a\varphi) & \text{in } M \times (0,T], \\ \varphi = \psi & \text{on } M \times \{T\}, \end{cases} \tag{6.14}$$

for any $\psi \in C_c^\infty(M)$ with $\psi \geq 0$ and $T > 0$.

Using the solutions of the dual problem as test functions in the weak formulation of (6.12) one formally gets

$$\int_M W(x,T)\psi(x)\, d\mathcal{V}(x) = \int_M W(x,0)\varphi(x,0)\, d\mathcal{V}(x) \leq 0.$$

The claim follows since φ is by construction nonnegative. In fact, the procedure must be carefully justified by means of suitable approximations of problem (6.14).

6.2. Existence and basic properties of the approximate solutions $\varphi_{\varepsilon,n}$

For every $n \in \mathbb{N}$ and $\varepsilon > 0$ we consider nonnegative solutions $\varphi_{n,\varepsilon}$ of the problem

$$\begin{cases} (\varphi_{n,\varepsilon})_t = -\Delta[(a_n+\varepsilon)\varphi_{n,\varepsilon}] & \text{in } M\times(0,T], \\ \varphi_{n,\varepsilon} = \psi & \text{on } M\times\{T\}, \end{cases} \tag{6.15}$$

where the sequence $\{a_n\}$ is a suitable approximation of the function a defined by (6.13). The functions $\varphi_{\varepsilon,n}$ are constructed by making use of linear semigroup theory; in particular, we take advantage of the fact that $-\Delta$ is a positive self-adjoint operator generating a Markov semigroup on $L^2(M)$ (see [10]).

The arguments one can exploit in the proof of the forthcoming lemma closely resemble those used to establish [16, Lemma 5.3], hence we skip it.

Lemma 6.2. *Let $\{a_n\}$ be a sequence of nonnegative functions converging a.e. to the function a defined in (6.13) such that:*

- *for any $n \in \mathbb{N}$ and $t > 0$, $x \mapsto a_n(x,t)$ is a regular function;*
- *for any $n \in \mathbb{N}$ and $x \in M$, $t \mapsto a_n(x,t)$ is a piecewise constant function, which is constant on each time interval $(T-(k+1)T/n,\ T-kT/n]$, $k \in \{0,\dots,n-1\}$;*
- *$\{\|a_n\|_{L^\infty(M\times(\tau,\infty))}\}$ is uniformly bounded with respect to $n \in \mathbb{N}$ for any $\tau > 0$.*

Then, for any $\varepsilon > 0$ and any $\psi \in C_c^\infty(M)$ with $\psi \geq 0$, there exists a nonnegative solution $\varphi_{n,\varepsilon}$ to problem (6.15), in the sense that $\varphi_{n,\varepsilon}(t)$ is a continuous curve in $L^p(M)$ (for all $1 < p < \infty$) satisfying $\varphi_{n,\varepsilon}(T) = \psi$ and it is absolutely continuous on $(T-(k+1)T/n,\ T-kT/n)$ for each $k \in \{0,\dots,n-1\}$, so that the identity

$$\varphi_{n,\varepsilon}(t_2) - \varphi_{n,\varepsilon}(t_1) = -\int_{t_1}^{t_2} \Delta[(a_n+\varepsilon)(\tau)\,\varphi_{n,\varepsilon}(\tau)]\,d\tau \tag{6.16}$$

holds in $L^p(M)$ (for all $1 < p < \infty$) for any $t_1, t_2 \in (T-(k+1)T/n,\ T-kT/n)$ and any $k \in \{0,\dots,n-1\}$. Moreover,

$$\begin{aligned} &\varphi_{n,\varepsilon} \in L^\infty((0,T);L^p(M)) && \text{for all } p \in [1,\infty], \\ &\|\varphi_{n,\varepsilon}(t)\|_1 \leq \|\psi\|_1 && \text{for all } t \in [0,T]. \end{aligned} \tag{6.17}$$

In the proofs of the next lemmas, even if we follow the general strategy used to show analogous results in [16], there are some additional difficulties to overcome. They are related to the fact that an analogue of [16, Proposition B.1] is not available in the present framework, because of a possible different growth of the volume of balls. Thus, more delicate cut-off arguments are required.

We now prove some crucial identities involving the functions $\varphi_{n,\varepsilon}$ and W.

Lemma 6.3. *Let W be defined as in (6.11), a as in (6.13), and a_n, $\varphi_{n,\varepsilon}$, ψ as in Lemma 6.2. Then for all $t \in (0,T)$,*

$$\begin{aligned} &\int_M W(x,T)\psi(x)\,d\mathcal{V}(x) - \int_M W(x,t)\varphi_{n,\varepsilon}(x,t)\,d\mathcal{V}(x) \\ &\quad = -\int_t^T \int_M [a_n(x,\tau)+\varepsilon-a(x,\tau)]\Delta W(x,\tau)\varphi_{n,\varepsilon}(x,\tau)\,d\mathcal{V}(x)\,d\tau. \end{aligned} \tag{6.18}$$

Proof. Let us set

$$t_k := T(n-k)/n \qquad \text{for all } k \in \{0, \dots, n\}.$$

Thanks to Lemma 6.1, we know that $W(t)$ is an absolutely continuous curve in $L^p(M)$ for all $p \in (N/(N-2), \infty)$, satisfying (6.12). On the other hand, Lemma 6.2 ensures that $\varphi_{n,\varepsilon}(t)$ is a continuous curve in $L^p(M)$ for all $p \in (1, \infty)$ on $(0, T]$, absolutely continuous on (t_{k+1}, t_k) for each $k \in \{0, \dots, n-1\}$ and satisfying the differential equation in (6.15) on such intervals. Hence, the function

$$t \mapsto \int_M \xi_R(x) W(x,t) \varphi_{\varepsilon,n}(x,t)\, d\mathcal{V}(x),$$

where $\{\xi_R\}_{R>0}$ is a cut-off family as in the proof of Lemma 6.1, is continuous on $(0, T]$, absolutely continuous on (t_{k+1}, t_k) and satisfies

$$\begin{aligned} &\frac{d}{dt} \int_M \xi_R(x) W(x,t) \varphi_{n,\varepsilon}(x,t)\, d\mathcal{V}(x) \\ &= \int_M \{\xi_R(x) a(x,t) \Delta W(x,t) \varphi_{n,\varepsilon}(x,t) - \xi_R(x) W(x,t) \Delta[(a_n + \varepsilon)\varphi_{n,\varepsilon}](x,t)\}\, d\mathcal{V}(x) \end{aligned} \tag{6.19}$$

on (t_{k+1}, t_k). By standard elliptic regularity, $W(t) \in W^{2,p}_{\text{loc}}(M)$ for all $p \in (1, \infty)$. We can therefore integrate by parts in the last term on the r.h.s. of (6.19) to get

$$\begin{aligned} &\int_M \xi_R(x) W(x,t) \Delta[(a_n + \varepsilon)\varphi_{n,\varepsilon}](x,t)\, d\mathcal{V}(x) \\ &\qquad = \int_M \xi_R(x) \Delta W(x,t) [a_n(x,t) + \varepsilon] \varphi_{n,\varepsilon}(x,t)\, d\mathcal{V}(x) \\ &\qquad\quad + \underbrace{\int_M \Delta\xi_R(x) W(x,t) [a_n(x,t) + \varepsilon] \varphi_{n,\varepsilon}(x,t)\, d\mathcal{V}(x)}_{I_1(t)} \\ &\qquad\quad + 2 \underbrace{\int_M \langle \nabla\xi_R(x), \nabla W(x,t) \rangle\, [a_n(x,t) + \varepsilon] \varphi_{n,\varepsilon}(x,t)\, d\mathcal{V}(x)}_{I_2(t)}. \end{aligned} \tag{6.20}$$

By reasoning similarly to the proof of Lemma 6.1 and exploiting Lemma 4.14 with $f = u_2(t+h) - u_1(t)$ and (6.5)–(6.6), we obtain

$$|I_1(t)| \le \bar{C}\, \|W(t)\|_\infty\, \|a_n(t) + \varepsilon\|_\infty \int_{B_{2R} \setminus B_R} \varphi_{n,\varepsilon}(x,t)\, d\mathcal{V}(x), \tag{6.21}$$

$$\begin{aligned} |I_2(t)| \le{}& \frac{C \|a_n(t) + \varepsilon\|_\infty}{2R} \\ &\times \left(\int_{B_{2R} \setminus B_R} |\nabla W(x,t)|^2\, d\mathcal{V}(x) + \int_{B_{2R} \setminus B_R} \varphi_{n,\varepsilon}(x,t)^2\, d\mathcal{V}(x) \right). \end{aligned} \tag{6.22}$$

Integrating (6.19)–(6.22) between any $t_{k+1} < t_* < t^* < t_k$, noting that $\nabla W \in [L^2(M \times (t_*, t^*))]^N$ and $\varphi_{n,\varepsilon} \in L^2(M \times (t_*, t^*))$, and letting $R \to \infty$, we end up with

$$\int_M W(x,t^*)\varphi_{n,\varepsilon}(x,t^*)\,d\mathcal{V}(x) - \int_M W(x,t_*)\varphi_{n,\varepsilon}(x,t_*)\,d\mathcal{V}(x)$$
$$= -\int_{t_*}^{t^*}\int_M [a_n(x,\tau)+\varepsilon - a(x,\tau)]\Delta W(x,\tau)\varphi_{n,\varepsilon}(x,\tau)\,d\mathcal{V}(x)\,d\tau. \tag{6.23}$$

Now (6.18) just follows from (6.23), since the r.h.s. of (6.23) is in $L^1((\tau,T))$ (e.g. as a function of t^*) for all $\tau \in (0,T)$. □

Lemma 6.4. *Let a_n, $\varphi_{n,\varepsilon}$, ψ be as in Lemma 6.2. Then*

$$\int_M \varphi_{n,\varepsilon}(x,t)\phi(x)\,d\mathcal{V}(x) - \int_M \psi(x)\phi(x)\,d\mathcal{V}(x)$$
$$= \int_M \Delta\phi(x)\left[\int_t^T (a_n(x,\tau)+\varepsilon)\varphi_{n,\varepsilon}(x,\tau)\,d\tau\right]d\mathcal{V}(x) \tag{6.24}$$

for all $t \in (0,T)$, $\phi \in C_c^\infty(M)$. In particular,

$$\int_M \varphi_{n,\varepsilon}(x,t)\,d\mathcal{V}(x) = \int_M \psi(x)\,d\mathcal{V}(x) \quad \text{for all } t \in (0,T). \tag{6.25}$$

Proof. The validity of (6.24) just a consequence of (6.16) plus the continuity of $\varphi_{n,\varepsilon}(t)$ as a curve in $L^2(M)$ (for instance).

In order to establish (6.25), let us plug $\phi = \xi_R$ in (6.24), with ξ_R still defined as in the proof of Lemma 6.1. Thanks to (6.5) and (6.6), we obtain

$$\left|\int_M \varphi_{n,\varepsilon}(x,t)\xi_R(x)\,d\mathcal{V}(x) - \int_M \psi(x)\xi_R(x)\,d\mathcal{V}(x)\right|$$
$$\le \bar{C}\,\|a_n+\varepsilon\|_{L^\infty(M\times(t,T))}\int_{B_{2R}\setminus B_R}\int_t^T \varphi_{n,\varepsilon}(x,\tau)\,d\mathcal{V}(x)\,d\tau. \tag{6.26}$$

Since (6.17) trivially implies $\varphi_{n,\varepsilon} \in L^1(M\times(0,T))$, by letting $R\to\infty$ in (6.26) we deduce (6.25). □

Lemma 6.5. *Let a_n, $\varphi_{n,\varepsilon}$, ψ be as in Lemma 6.2. Denote by $\Phi_{n,\varepsilon}(t)$ the potential of $\varphi_{n,\varepsilon}(t)$, that is,*

$$\Phi_{n,\varepsilon}(x,t) := \mathcal{G}^{\varphi_{n,\varepsilon}(t)}(x).$$

Then $\nabla\Phi_{n,\varepsilon}(t) \in [L^2(M)]^N$ and

$$\|\nabla\mathcal{G}^\psi\|_2^2 = \|\nabla\Phi_{n,\varepsilon}(t)\|_2^2 + 2\int_t^T\int_M [a_n(x,\tau)+\varepsilon]\varphi_{n,\varepsilon}(x,\tau)^2\,d\mathcal{V}(x)\,d\tau \tag{6.27}$$

for all $t \in (0,T]$.

Proof. Since $\Phi_{n,\varepsilon}(t)$ is the potential of $\varphi_{n,\varepsilon}(t)$, which belongs to $L^1(M)\cap L^\infty(M)$ (recall (6.17)), thanks to Lemma 4.14 we have $\Phi_{n,\varepsilon}(t) \in L^p(M)$ for all $p \in (N/(N-2),\infty]$, $\nabla\Phi_{n,\varepsilon}(t) \in [L^2(M)]^N$ and

$$\|\nabla\Phi_{n,\varepsilon}(t)\|_2^2 = \int_M \Phi_{n,\varepsilon}(x,t)\varphi_{n,\varepsilon}(x,t)\,d\mathcal{V}(x). \tag{6.28}$$

Furthermore, one can show that $\Phi_{n,\varepsilon}(t)$ is an absolutely continuous curve in $L^p(M)$ for all $p \in (N/(N-2), \infty)$, satisfying the differential equation

$$(\Phi_{n,\varepsilon})_t(x,t) = [a_n(x,t) + \varepsilon]\varphi_{n,\varepsilon}(x,t) \quad \text{for a.e. } (x,t) \in M \times (0,T). \tag{6.29}$$

This can be established exactly as for (6.1). Taking advantage of (6.15), (6.28) and (6.29), we then deduce that

$$\begin{aligned} \frac{d}{dt}\|\nabla\Phi_{n,\varepsilon}(t)\|_2^2 = &\int_M [a_n(x,t)+\varepsilon]\varphi_{n,\varepsilon}(x,t)^2\, d\mathcal{V}(x) \\ &- \int_M \Phi_{n,\varepsilon}(x,t)\Delta[(a_n+\varepsilon)\varphi_{n,\varepsilon}](x,t)\, d\mathcal{V}(x) \end{aligned} \tag{6.30}$$

for a.e. $t \in (0,T)$. In view of the integrability properties of $\Phi_{n,\varepsilon}$, $\nabla\Phi_{n,\varepsilon}$, $\varphi_{n,\varepsilon}$ and $\Delta[(a_n+\varepsilon)\varphi_{n,\varepsilon}]$, the last term in the r.h.s. of (6.30) can be integrated by parts (through the same cut-off techniques we used in the proof of Lemma 6.3), which yields

$$\frac{d}{dt}\|\nabla\Phi_{n,\varepsilon}(t)\|_2^2 = 2\int_M [a_n(x,t)+\varepsilon]\varphi_{n,\varepsilon}(x,t)^2\, d\mathcal{V}(x) \quad \text{for a.e. } t \in (0,T). \tag{6.31}$$

Since the r.h.s. of (6.31) is in $L^1((\tau,T))$ for each $\tau \in (0,T)$, it turns out that $t \mapsto \|\nabla\Phi_{n,\varepsilon}(t)\|_2^2$ is continuous on $(0,T]$ and absolutely continuous on every (t_{k+1},t_k), and the conclusion follows by integrating (6.31) over (t,T). □

6.3. Taking the limit of $\varphi_{n,\varepsilon}$ as $n \to \infty$

This section is devoted to showing that, for each fixed $\varepsilon > 0$, the sequence $\{\varphi_{n,\varepsilon}\}$ converges in a suitable sense to a limit function φ_ε as $n \to \infty$. Moreover, φ_ε inherits some fundamental integrability properties from $\{\varphi_{n,\varepsilon}\}$.

Lemma 6.6. *Let u_1, u_2 be any two solutions of problem (1.1), taking on the same initial datum $\mu \in \mathcal{M}_F^+(M)$. Let W be defined as in (6.11), a as in (6.13), and $\varphi_{n,\varepsilon}$, ψ as in Lemma 6.2. Then, up to subsequences, $\{\varphi_{n,\varepsilon}\}$ converges weakly in $L^2(M \times (\tau,T))$ (for each $\tau \in (0,T)$), as $n \to \infty$, to a suitable nonnegative function φ_ε. Moreover,*

$$\begin{aligned} &\int_M \varphi_\varepsilon(x,t)\phi(x)\, d\mathcal{V}(x) - \int_M \psi(x)\phi(x)\, d\mathcal{V}(x) \\ &\quad = \int_M \Delta\phi(x)\left[\int_t^T (a(x,\tau)+\varepsilon)\varphi_\varepsilon(x,\tau)\, d\tau\right] d\mathcal{V}(x) \end{aligned} \tag{6.32}$$

for a.e. $t \in (0,T)$ and any $\phi \in C_c^\infty(M)$,

$$\int_M \varphi_\varepsilon(x,t)\, d\mathcal{V}(x) = \int_M \psi(x)\, d\mathcal{V}(x) \quad \text{for a.e. } t \in (0,T), \tag{6.33}$$

and

$$\begin{aligned} &\left|\int_M W(x,T)\psi(x)\, d\mathcal{V}(x) - \int_M W(x,t)\varphi_\varepsilon(x,t)\, d\mathcal{V}(x)\right| \\ &\quad \le \varepsilon(T-t)\|\psi\|_1 \|u_2(\cdot+h) - u_1(\cdot)\|_{L^\infty(M\times(t,T))} \quad \text{for a.e. } t \in (0,T). \end{aligned} \tag{6.34}$$

Proof. From (6.27) we infer that $\{\varphi_{n,\varepsilon}\}$ converges weakly (up to subsequences) in $L^2(M \times (\tau, T))$ (for each $\tau \in (0, T)$) to some φ_ε. Moreover, thanks to (6.17), for every $t \in (0, T)$ there exists a subsequence (which a priori depends on t) such that

$$\int_M \varphi_{n,\varepsilon}(t)\phi(x)\, d\mathcal{V}(x) \to \int_M \phi(x)\, d\nu_\varepsilon^t(x) \quad \text{for any } \phi \in C_c(M), \tag{6.35}$$

for some $\nu_\varepsilon^t \in \mathcal{M}_F^+(M)$. In fact we have $d\nu_\varepsilon^t = \varphi_\varepsilon(t)\, d\mathcal{V}$ for a.e. $t \in (0, T)$. In order to show that, let $t \in (0, T)$ be a Lebesgue point for $\varphi_\varepsilon(t)$ as a curve in $L^1((\tau, T); L^2(M))$. Take any $\phi \in C_c^\infty(M)$. Since for each $\tau \in (0, T)$ the sequence $\{\|a_n + \varepsilon\|_{L^\infty(M\times(\tau,T))}\}$ is bounded, in view of (6.17) and (6.24) we obtain

$$\begin{aligned} &\left| \int_t^{t+\delta} \int_M \varphi_{n,\varepsilon}(x,\tau)\phi(x)\, d\mathcal{V}(x)\, d\tau - \int_t^{t+\delta} \int_M \varphi_{n,\varepsilon}(x,t)\phi(x)\, d\mathcal{V}(x)\, d\tau \right| \\ &\qquad \le \int_t^{t+\delta} C(\tau - t)\|\psi\|_1 \|\Delta\phi\|_\infty\, d\tau = \frac{\delta^2 C}{2}\|\psi\|_1\|\Delta\phi\|_\infty \end{aligned} \tag{6.36}$$

for all $0 < \delta < T - t$, for some positive constant C independent of n, δ. By letting $n \to \infty$ in (6.36) (up to subsequences) we get

$$\left| \int_t^{t+\delta} \int_M \varphi_\varepsilon(x,\tau)\phi(x)\, d\mathcal{V}(x)\, d\tau - \delta \int_M \phi(x)\, d\nu_\varepsilon^t(x) \right| \le \frac{\delta^2 C}{2}\|\psi\|_1\|\Delta\phi\|_\infty. \tag{6.37}$$

Upon dividing (6.37) by δ and then letting $\delta \to 0^+$ we deduce that

$$\int_M \varphi_\varepsilon(x,t)\phi(x)\, d\mathcal{V}(x) = \int_M \phi(x)\, d\nu_\varepsilon^t(x),$$

so $\varphi_\varepsilon(t)\, d\mathcal{V} = d\nu_\varepsilon^t$. Therefore, (6.32) easily follows by passing to the limit in (6.24) as $n \to \infty$, also in view of the convergence properties of $\{a_n\}$.

Identity (6.33) and estimate (6.34) can be obtained along the lines of [16, proof of Lemma 5.7]; we only mention that (6.33) follows by passing to the limit in (6.26), and (6.34) follows by passing to the limit in (6.18), which is feasible since $W(t) \in C_b(M)$ and, thanks to (6.33), (6.35) also holds for any $\phi \in C_b(M)$. □

6.4. Taking the limit of φ_ε as $\varepsilon \to 0$ and proof of Theorem 2.3

In order to prove Theorem 2.3, we need to exploit the properties of the functions φ_ε provided by Lemma 6.6, and then let $\varepsilon \to 0$.

Proof of Theorem 2.3. Let $\Phi_\varepsilon(t)$ be the potential of $\varphi_\varepsilon(t)$, that is, $\Phi_\varepsilon(x,t) = \mathcal{G}^{\varphi_\varepsilon(t)}(x)$. In view of (6.32),

$$\begin{aligned} &\mathcal{G}^\psi(x) - \Phi_\varepsilon(x,t) \\ &\qquad = \int_t^T [a(x,\tau) + \varepsilon]\varphi_\varepsilon(x,\tau)\, d\tau \ge 0 \quad \text{for a.e. } (x,t) \in M \times (0,T). \end{aligned} \tag{6.38}$$

This can be established as we did in the proof of (6.2): it is enough to plug $\xi_R \mathcal{G}^\phi$ in (6.32) and let $R \to \infty$, exploiting the fact that $\mathcal{G}^\phi \in L^\infty(M)$, $\nabla \mathcal{G}^\phi \in [L^2(M)]^N$, $a \in L^\infty(M \times (t,T))$ and $\varphi_\varepsilon \in L^1(M \times (t,T)) \cap L^2(M \times (t,T))$.

So, in particular,

$$0 \le \Phi_\varepsilon(x,t_1) \le \Phi_\varepsilon(x,t_2) \le \mathcal{G}^\psi(x) \quad \text{for a.e. } x \in M,\ 0 < t_1 < t_2 < T. \tag{6.39}$$

We now let $\varepsilon \to 0$. In view of (6.39) it follows that $\{\Phi_\varepsilon\}$ is bounded in $L^p(M \times (0,T))$ for any $p \in (N/(N-2), \infty]$. In particular, there exists a sequence $\{\Phi_{\varepsilon_n}\}$ that converges weakly in $L^p(M \times (0,T))$ to some $\Phi \in L^p(M \times (0,T))$. As a consequence, in view of (6.38) and (6.39), by arguments similar to those used at the beginning of the proof of Lemma 6.6, we can deduce that $\{\Phi_{\varepsilon_n}(t)\}$ converges weakly in $L^p(M)$ to $\Phi(t)$ for a.e. $t \in (0,T)$. Thanks to the boundedness of $\{\varphi_{\varepsilon_n}(t)\}$ in $L^1(M)$ (recall (6.33)), for a.e. $t \in (0,T)$ there exists a subsequence $\{\varepsilon_{n_k}\}$ (a priori depending on t) such that

$$\int_M \varphi_{\varepsilon_{n_k}}(x,t)\phi(x)\, d\mathcal{V}(x) \to \int_M \phi(x)\, d\nu^t(x) \quad \text{as } k \to \infty \text{ for any } \phi \in C_0(M), \tag{6.40}$$

for some $\nu^t \in \mathcal{M}_F^+(M)$. Hence, for a.e. $t \in (0,T)$ and any $\phi \in C_c(M)$,

$$\begin{aligned}\int_M \mathcal{G}^{\nu^t}\phi(x)\, d\mathcal{V}(x) &= \lim_{k\to\infty}\int_M \varphi_{\varepsilon_{n_k}}(x,t)\mathcal{G}^\phi(x)\, d\mathcal{V}(x) = \lim_{k\to\infty}\int_M \Phi_{\varepsilon_{n_k}}(x,t)\phi(x)\, d\mathcal{V}(x)\\ &= \int_M \Phi(x,t)\phi(x)\, d\mathcal{V}(x).\end{aligned} \tag{6.41}$$

From (6.41) and Lemma 4.13, we infer that ν^t is independent of the particular subsequence, so that (6.40) holds along the whole sequence $\{\varepsilon_n\}$ and $\Phi(t)$ is the potential of ν^t. Moreover, in view of (6.39) and of the convergence properties of $\{\Phi_{\varepsilon_n}\}$, we get

$$0 \le \Phi(x,t_1) \le \Phi(x,t_2) \le \mathcal{G}^\psi(x) \quad \text{for a.e. } x \in M,\ 0 < t_1 < t_2 < T. \tag{6.42}$$

We now aim at proving that (6.40) holds for any $\phi \in C_b(M)$. To this end, note that since (6.33) holds and $a \in L^\infty(M \times (\tau,T))$ for each $\tau \in (0,T)$, we see that, up to subsequences,

$$\int_M \phi(x)\left\{\int_t^T [a(x,\tau)+\varepsilon_n]\varphi_{\varepsilon_n}(x,\tau)\, d\tau\right\} d\mathcal{V}(x) \to \int_M \phi(x)\, d\sigma^{t,T}(x) \quad \text{as } n \to \infty,$$

for a.e. $t \in (0,T)$ and any $\phi \in C_c(M)$, where $\sigma^{t,T}$ is a suitable element of $\mathcal{M}_F^+(M)$. We can therefore pass to the limit as $n \to \infty$ in (6.32) (with $\varepsilon = \varepsilon_n$) to get

$$\int_M \phi(x)\, d\nu^t(x) - \int_M \psi(x)\phi(x)\, d\mathcal{V}(x) = \int_M \Delta\phi(x)\, d\sigma^{t,T}(x) \tag{6.43}$$

for a.e. $t \in (0,T)$ and any $\phi \in C_c^\infty(M)$. Now plug $\phi = \xi_R$ in (6.43), with ξ_R defined as in the proof of Lemma 6.1. Thanks to (6.5) and (6.6), we obtain

$$\left|\int_M \xi_R(x)\, d\nu^t(x) - \int_M \psi(x)\xi_R(x)\, d\mathcal{V}(x)\right| \le \bar{C}\int_{B_{2R}\setminus B_R} d\sigma^{t,T}(x) \quad \text{for a.e. } t \in (0,T). \tag{6.44}$$

Since $\sigma^{t,T}$ is a positive finite measure, by letting $R \to \infty$ in (6.44) we get

$$\int_M d\nu^t(x) = \int_M \psi(x)\, d\mathcal{V}(x). \tag{6.45}$$

From (6.33), (6.40), (6.45) and [1, Proposition 1.80] we then deduce that

$$\int_M \varphi_{\varepsilon_n}(x,t)\phi(x)\, d\mathcal{V}(x) \to \int_M \phi(x)\, d\nu^t(x) \quad \text{as } n \to \infty \text{ for any } \phi \in C_b(M). \tag{6.46}$$

As a consequence,

$$\int_M W(x,T)\psi(x)\, d\mathcal{V}(x) = \int_M W(x,t)\, d\nu^t(x) \quad \text{for a.e. } t \in (0,T), \tag{6.47}$$

by passing to the limit as $\varepsilon = \varepsilon_n \to 0$ in (6.34) (recall that, from Lemma 4.14, $W(t) \in C_b(M)$).

Since for a.e. $0 < t_* < t^* < T$ we have $\Phi(x,t_*) \le \Phi(x,t^*)$ for a.e. $x \in M$ (see (6.42)), it is direct to show that the curve ν^t can be extended to *every* $t \in (0,T]$ so that it still satisfies (6.42), (6.45) and (6.47). Hence, in view of Lemma 6.1 and (6.47),

$$\int_M W(x,T)\psi(x)\, d\mathcal{V}(x) \le \int_M [\mathcal{G}_2(x,h) - \mathcal{G}_1(x,t_0)]\, d\nu^t(x) \quad \text{for all } 0 < t < t_0 < T. \tag{6.48}$$

Thanks to (6.42) and (6.45), it is straightforward to check that there exists $\nu_0 \in \mathcal{M}_F^+(M)$ such that

$$\lim_{t\to 0} \int_M \phi(x)\, d\nu^t(x) = \int_M \phi(x)\, d\nu_0 \quad \text{for any } \phi \in C_c(M)$$

and

$$\mathcal{G}^{\nu_0}(x) = \lim_{t\to 0} \Phi(x,t) := \Phi_0(x) \quad \text{for a.e. } x \in M.$$

Thus, by dominated convergence and Tonelli's Theorem,

$$\begin{aligned}\lim_{t\to 0} \int_M \mathcal{G}_1(x,t_0)\, d\nu^t(x) &= \lim_{t\to 0} \int_M u_1(x,t_0)\Phi(x,t)\, d\mathcal{V}(x) = \int_M u_1(x,t_0)\Phi_0(x)\, d\mathcal{V}(x) \\ &= \int_M \mathcal{G}_1(x,t_0)\, d\nu_0(x).\end{aligned}$$

We can similarly prove that

$$\lim_{t\to 0} \int_M \mathcal{G}_2(x,h)\, d\nu^t(x) = \int_M \mathcal{G}_2(x,h)\, d\nu_0(x).$$

Hence, passing to the limit as $t \to 0$ in (6.48) we infer that

$$\int_M W(x,T)\psi(x)\, d\mathcal{V}(x) \le \int_M [\mathcal{G}_2(x,h) - \mathcal{G}_1(x,t_0)]\, d\nu_0(x) \quad \text{for all } 0 < t_0 < T. \tag{6.49}$$

Letting $t_0 \to 0$ in (6.49), by monotone convergence and in view of Lemma 6.1 we find

$$\int_M W(x,T)\psi(x)\, d\mathcal{V}(x) \le \int_M [\mathcal{G}_2(x,h) - \mathcal{G}^\mu]\, d\nu_0(x) \le 0. \tag{6.50}$$

Since $h >, T > 0$ and $\psi \in C_c^\infty(M)$ (with $\psi \geq 0$) are arbitrary, (6.50) implies $\mathcal{G}_1 \geq \mathcal{G}_2$. Interchanging the roles of u_1 and u_2, we also get $\mathcal{G}_1 \leq \mathcal{G}_2$, so that $\mathcal{G}_1 = \mathcal{G}_2$ and $u_1 = u_2$ in view of Lemma 4.13. □

Remark 6.7. As a consequence of the above method of proof, Theorem 2.3 still holds under the weaker assumption that (2.4) is satisfied for any $\phi \in C_c(M)$.

However, in contrast with the existence counterpart, in order to prove Theorem 2.3 we require some additional hypotheses. First of all, we assume (*H*)(ii): this is essential to provide a cut-off family ξ_R satisfying (6.6), which is the main tool we exploit to justify all the integration by parts, as well as the conservation of mass (6.45). The positivity of the initial datum, $\mu \in M_F^+(M)$, is crucial for the validity of (6.3) for *every* $x \in M$ and for the monotonicity of $\mathcal{G}$ as a function of t; both properties are deeply exploited in the final part of the proof of Theorem 2.3.

Finally, in Remark 5.3 we explained how to recover existence in the range $(N-2)/N < m < 1$. As for uniqueness, there are two main issues. A priori nothing guarantees that $u^m(t) \in L^1(M)$ for positive times: this prevents us from proving that the remainder integrals I_1 and I_2 in the proof of Lemma 6.1 vanish as $R \to \infty$. On the other hand, the function a in (6.13) is no more bounded for positive times, a crucial property that we exploit throughout the proof of Theorem 2.3.

6.5. *Proof of existence and uniqueness of the initial trace*

First note that, under the assumptions of Theorem 2.4, the proof of Lemma 6.1 works without further issues down to the proof of identity (6.2). Moreover, by combining the latter with the smoothing effect (5.3) and proceeding as in the proof of Theorem 2.2, we end up with

$$\left| \int_M \mathcal{G}(x,t_2)\phi(x)\,d\mathcal{V}(x) - \int_M \mathcal{G}(x,t_1)\phi(x)\,d\mathcal{V}(x) \right| \leq \|\phi\|_\infty \, K^{m-1} \|u\|^{1+\beta(m-1)}_{L^\infty((0,\infty);L^1(M))} \int_{t_1}^{t_2} t^{-\alpha(m-1)}\,dt, \tag{6.51}$$

where again $\mathcal{G}(t)$ is the potential of $u(t)$. Furthermore, given any $t_2 > t_1 > 0$ and any $R \geq 1$, by plugging in (2.3) the test function

$$\varphi(x,t) = \xi_R(x)[\theta_\varrho^{t_1}(t) - \theta_\varrho^{t_2}(t)] \quad \text{for all } x \in M, \ t > 0$$

($\theta_\varrho^{t.}$ is defined as in the proof of Proposition 5.1 and ξ_R as in the proof of Lemma 6.1), integrating by parts, letting $\varrho \to 0$ and using (6.6), we obtain

$$\left| \int_M u(x,t_2)\xi_R(x)\,d\mathcal{V}(x) - \int_M u(x,t_1)\xi_R(x)\,d\mathcal{V}(x) \right| \leq \bar{C} \int_{t_1}^{t_2} \int_{B_{2R}\setminus B_R} |u(x,t)|^m \,d\mathcal{V}(x)\,dt. \tag{6.52}$$

In addition, (2.1) and the smoothing effect (5.3) ensure that

$$\int_0^{t_2} \int_M |u(x,t)|^m \, d\mathcal{V}(x) \, dt \le \frac{K^{m-1}}{1-\alpha(m-1)} \|u\|_{L^\infty((0,\infty);L^1(M))}^{1+\beta(m-1)} t_2^{1-\alpha(m-1)} < \infty. \tag{6.53}$$

Having established (6.51)–(6.53) for general weak solutions to the differential equation in (1.1), we are in a position to prove Theorem 2.4.

Proof of Theorem 2.4. In view of (6.51) we infer that the family $\{\mathcal{G}(t)\}$ is Cauchy in $L^1_{\mathrm{loc}}(M)$ as $t \to 0$, hence there exists $\mathcal{G}_0 \in L^1_{\mathrm{loc}}(M)$ such that $\mathcal{G}(t) \to \mathcal{G}_0$ as $t \to 0$ in $L^1_{\mathrm{loc}}(M)$. Moreover, the fact that $u \in L^\infty((0,\infty); L^1(M))$ implies that for every sequence $t_n \to 0$ there exists $\mu \in \mathcal{M}_F(M)$ such that $\{u(t_n)\}$ converges vaguely to μ as $n \to \infty$, up to a subsequence (recall Definition 4.5). On the other hand, the convergence of $\{\mathcal{G}(t_n)\}$ to $\mathcal{G}_0$ in $L^1_{\mathrm{loc}}(M)$ implies $\mathcal{G}^\mu = \mathcal{G}_0$, so that by Lemma 4.13 the measure μ does not depend on the sequence $\{t_n\}$, and (2.4) holds for all $\phi \in C_c(M)$. In order to prove that (2.4) also holds for constant functions, we can exploit (6.52): by letting $t_1 \to 0$ and using the vague convergence of $\{u(t_1)\}$ to μ we end up with

$$\left| \int_M u(x,t_2)\xi_R(x) \, d\mathcal{V}(x) - \int_M \xi_R(x) \, d\mu(x) \right| \le \bar{C} \int_0^{t_2} \int_{B_{2R} \setminus B_R} |u(x,t)|^m \, d\mathcal{V}(x) \, dt.$$

We then let $R \to \infty$: thanks to (6.53) we obtain

$$\left| \int_M u(x,t_2) \, d\mathcal{V}(x) - \int_M d\mu(x) \right| \le 0,$$

which is the conservation of mass, or equivalently the fact that (2.4) holds for ϕ equal to any constant. If $u \ge 0$, the last assertion of the theorem is just a consequence of [1, Proposition 1.80], since for positive measures vague convergence plus convergence of measures is equivalent to convergence in the dual space of $C_b(M)$. □

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