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Excursion theory for Brownian motion indexed by the Brownian tree

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Abstract. We develop an excursion theory for Brownian motion indexed by the Brownian tree, which in many respects is analogous to the classical Itô theory for linear Brownian motion. Each excursion is associated with a connected component of the complement of the zero set of the tree-indexed Brownian motion. Each such connected component is itself a continuous tree, and we introduce a quantity measuring the length of its boundary. The collection of boundary lengths coincides with the collection of jumps of a continuous-state branching process with branching mechanism $\psi(u) = \sqrt{8/3}\, u^{3/2}$. Furthermore, conditionally on the boundary lengths, the different excursions are independent, and we determine their conditional distribution in terms of an excursion measure $\mathbb{M}_0$ which is the analog of the Itô measure of Brownian excursions. We provide various descriptions of $\mathbb{M}_0$, and we also determine several explicit distributions, such as the joint distribution of the boundary length and the mass of an excursion under $\mathbb{M}_0$. We use the Brownian snake as a convenient tool for defining and analysing the excursions of our tree-indexed Brownian motion.

Keywords. Excursion theory, tree-indexed Brownian motion, continuum random tree, Brownian snake, exit measure, continuous-state branching process

1. Introduction

The concept of Brownian motion indexed by a Brownian tree has appeared in various settings in the last 25 years. The Brownian tree of interest here is the so-called CRT (Brownian Continuum Random Tree) introduced by Aldous [1, 2], or more conveniently a scaled version of the CRT with a random "total mass". The CRT is a universal model for a continuous random tree, in the sense that it appears as the scaling limit of many different classes of discrete random trees (see in particular [2, 14, 37]), and of other discrete random structures (see the recent papers [8, 35]). At least informally, the meaning of Brownian motion indexed by the Brownian tree should be clear: Labels, also called spatial positions, are assigned to the vertices of the tree, in such a way that the root has label 0 and labels evolve like linear Brownian motion when moving away from the root along a geodesic segment of the tree, and of course the increments of the labels along disjoint segments are independent. Combining the branching structure of the CRT with Brownian displacements led Aldous to introduce the Integrated Super-Brownian Excursion or

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ISE [3], which is closely related to the canonical measures of super-Brownian motion. On the other hand, the desire to get a better understanding of the historical paths of superprocesses motivated the definition of the so-called Brownian snake [19], which is a Markov process taking values in the space of all finite paths. Roughly speaking, the value of the Brownian snake at time s is the path recording the spatial positions along the ancestral line of the vertex visited at the same time s in the contour exploration of the Brownian tree. One may view the Brownian snake as a convenient representation of Brownian motion indexed by the Brownian tree, avoiding the technical difficulty of dealing with a random process indexed by a random set.

The preceding concepts have found many applications. The Brownian snake has proved a powerful tool in the study of sample path properties of super-Brownian motion and of its connections with semilinear partial differential equations [20, 21]. ISE, and more generally Brownian motion indexed by the Brownian tree and its variants, also appear in the scaling limits of various models of statistical mechanics above the critical dimension, including lattice trees [12], percolation [16] or oriented percolation [17]. More recently, scaling limits of large random planar maps have been described by the so-called Brownian map [23, 31], which is constructed as a quotient space of the CRT by an equivalence relation defined in terms of Brownian labels assigned to the vertices of the CRT.

Our main goal in this work is to show that a very satisfactory excursion theory can be developed for Brownian motion indexed by the Brownian tree, or equivalently for the Brownian snake, which in many aspects resembles the classical excursion theory for linear Brownian motion due to Itô [18]. We also expect the associated excursion measure to be an interesting probabilistic object, which hopefully will have significant applications in related fields.

Let us give an informal description of the main results of our study. The underlying Brownian tree that we consider is denoted by $\mathcal{T}_\zeta$, for the tree coded by a Brownian excursion $(\zeta_s)_{s\geq 0}$ under the classical Itô excursion measure (see Section 2.1 for more details about this coding, and note that the Itô excursion measure is a σ-finite measure). The tree $\mathcal{T}_\zeta$ may be viewed as a scaled version of the CRT, for which $(\zeta_s)_{s\geq 0}$ would be a Brownian excursion with duration 1. This tree is rooted at a particular vertex ρ. We write V_u for the Brownian label assigned to the vertex u of $\mathcal{T}_\zeta$. As explained above, the collection $(V_u)_{u\in\mathcal{T}_\zeta}$ should be interpreted as Brownian motion indexed by $\mathcal{T}_\zeta$, starting from 0 at the root ρ. Similarly to the case of linear Brownian motion, we may then consider the connected components of the open set

$$\{u \in \mathcal{T}_\zeta : V_u \neq 0\},$$

which we denote by $(\mathcal{C}_i)_{i\in I}$. Of course these connected components are not intervals as in the classical case, but they are connected subsets of the tree $\mathcal{T}_\zeta$, and thus subtrees of this tree. One then considers, for each component $\mathcal{C}_i$, the restriction $(V_u)_{u\in\mathcal{C}_i}$ of the labels to $\mathcal{C}_i$, and this restriction again yields a random process indexed by a continuous random tree, which we call the excursion E_i. Our main results completely determine the "law" of the collection $(E_i)_{i\in I}$ (we speak about the law of this collection though we are working under an infinite measure). A first important ingredient of this description is an infinite excursion measure $\mathbb{M}_0$, which plays a similar role to the Itô excursion measure in the classical setting, in the sense that $\mathbb{M}_0$ describes the distribution of a typical excursion E_i (this is a little informal as $\mathbb{M}_0$ is an infinite measure).

We can then completely describe the law of the collection $(E_i)_{i\in I}$ using the measure $\mathbb{M}_0$ and an independence property analogous to the classical setting. For this description, we first need to introduce a quantity $\mathcal{Z}_i$, called the exit measure of E_i, that measures the size of the boundary of $\mathcal{C}_i$: Note that in the classical setting the boundary of an excursion interval just consists of two points, but here of course the boundary of $\mathcal{C}_i$ is much more complicated. Furthermore, one can define, for every $z \geq 0$, a conditional probability measure $\mathbb{M}_0(\cdot \mid \mathcal{Z} = z)$ which corresponds to the law of an excursion conditioned to have boundary size z (this is somehow the analog of the Itô measure conditioned to have a fixed duration in the classical setting). Finally, we introduce a "local time exit process" $(\mathcal{X}_t)_{t\geq 0}$ such that, for every $t > 0$, $\mathcal{X}_t$ measures the number of vertices u of the tree $\mathcal{T}_\zeta$ with label 0 and such that the total accumulated local time at 0 of the label process along the geodesic segment between ρ and u is equal to t. The distribution of $(\mathcal{X}_t)_{t>0}$ is known explicitly and can be interpreted as an excursion measure for the continuous-state branching process with stable branching mechanism $\psi(\lambda) = \sqrt{8/3}\,\lambda^{3/2}$.

With all these ingredients at hand, we can complete our description of the distribution of the collection of excursions: Excursions E_i are in one-to-one correspondence with jumps of the local time exit process $(\mathcal{X}_t)_{t\geq 0}$, in such a way that, for every $i \in I$, the boundary length $\mathcal{Z}_i$ of E_i is equal to the size z_i of the corresponding jump, and furthermore, conditionally on the process $(\mathcal{X}_t)_{t\geq 0}$, the excursions E_i, $i \in I$, are independent, and, for every fixed j, E_j is distributed according to $\mathbb{M}_0(\cdot \mid \mathcal{Z} = z_j)$. There is a striking analogy with the classical setting (see e.g. [36, Chapter XII]), where excursions of linear Brownian excursion are in one-to-one correspondence with jumps of the inverse local time process, and the distribution of an excursion corresponding to a jump of size ℓ is the Itô measure conditioned to have duration ℓ.

The preceding discussion is somewhat informal, in particular because we did not give a mathematically precise definition of the excursions E_i. It would be possible to view these excursions as random elements of the space of all "spatial trees" in the terminology of [13] (compact $\mathbb{R}$-trees $\mathcal{T}$ equipped with a continuous mapping $\phi : \mathcal{T} \to \mathbb{R}$) but for technical reasons we prefer to use the Brownian snake approach. We now describe this approach in order to give a more precise formulation of our results. Let $\mathcal{W}$ stand for the set of all finite real paths. Here a finite real path is just a continuous function $\mathrm{w} : [0, \zeta] \to \mathbb{R}$, where $\zeta = \zeta_{(\mathrm{w})} \geq 0$ depends on w and is called the lifetime of w, and, for every $\mathrm{w} \in \mathcal{W}$, we write $\hat{\mathrm{w}} = \mathrm{w}(\zeta_{(\mathrm{w})})$ for the endpoint of w. The topology on $\mathcal{W}$ is induced by a distance whose definition is recalled at the beginning of Section 2.2.

The Brownian snake is a continuous Markov process $(W_s)_{s\geq 0}$ with values in $\mathcal{W}$ whose distribution is characterized as follows:

(i) The lifetime process $(\zeta_{(W_s)})_{s\geq 0}$ is a reflected Brownian motion on $\mathbb{R}_+$.
(ii) Conditionally on $(\zeta_{(W_s)})_{s\geq 0}$, $(W_s)_{s\geq 0}$ is time-inhomogeneous Markov, with transition kernels specified as follows: for $0 \leq s < s'$,

- $W_{s'}(t) = W_s(t)$ for every $0 \leq t \leq m(s, s') := \min\{\zeta_{(W_r)} : s \leq r \leq s'\}$;

- conditionally on W_s, $(W_{s'}(m(s,s')+t), 0 \le t \le \zeta_{(W_{s'})} - m(s,s'))$ is a linear Brownian motion started from $W_s(m(s,s'))$, on the time interval $[0, \zeta_{(W_{s'})} - m(s,s')]$.

We will write $\zeta_s = \zeta_{(W_s)}$ to simplify notation. Informally, the value W_s of the Brownian snake at time s is a random path with lifetime ζ_s evolving like reflected Brownian motion on $\mathbb{R}_+$. When ζ_s decreases, the path is erased from its tip, and when ζ_s increases, the path is extended by adding "little pieces" of Brownian paths at its tip.

For the sake of simplicity, in this introduction, we may and will assume that $(W_s)_{s\ge 0}$ is the canonical process on the space $C(\mathbb{R}_+, \mathcal{W})$ of all continuous mappings from $\mathbb{R}_+$ into $\mathcal{W}$. Later, it will be more convenient to define this process on a suitable canonical space of "snake trajectories" (see Section 2.2 below).

The trivial path with initial point 0 and zero lifetime is a regular recurrent point for the process $(W_s)_{s\ge 0}$, and thus we can introduce the associated excursion measure $\mathbb{N}_0$, which is called the Brownian snake excursion measure (from 0). This is a σ-finite measure on the space $C(\mathbb{R}_+, \mathcal{W})$—as mentioned earlier, we will later view $\mathbb{N}_0$ as a measure on the smaller space of snake trajectories. The measure $\mathbb{N}_0$ can be described via properties analogous to (i) and (ii), with the difference that in (i) the law of reflecting Brownian motion is replaced by the Itô measure of positive excursions of linear Brownian motion. In particular, under $\mathbb{N}_0$, the tree $\mathcal{T}_\zeta$ coded by $(\zeta_s)_{s\ge 0}$ has the distribution prescribed in the informal discussion at the beginning of this introduction—this distribution is a σ-finite measure on the space of trees. Recall that the coding of $\mathcal{T}_\zeta$ involves a canonical projection $p_\zeta : [0,\sigma] \to \mathcal{T}_\zeta$, where $\sigma = \sup\{s \ge 0 : \zeta_s > 0\}$ (see [27, Section 3.2] or Section 2.1 below). Notice that the definition of σ, as well as the definition of the tree $\mathcal{T}_\zeta$, are relevant under $\mathbb{N}_0$. Then the Brownian labels $(V_u)_{u\in\mathcal{T}_\zeta}$ are generated by taking $V_u = \hat{W}_s$, where $s \in [0,\sigma]$ is any instant such that $p_\zeta(s) = u$. Furthermore, the whole path W_s records the values of labels along the geodesic segment from the root ρ to u, and we sometimes say that W_s is the historical path of u.

From now on, we use the Brownian snake construction and argue under the excursion measure $\mathbb{N}_0$. This construction allows us to give a convenient representation for the excursions $(E_i)_{i\in I}$ discussed above. We observe that, $\mathbb{N}_0$-a.e., the connected components $(\mathcal{C}_i)_{i\in I}$ of $\{u \in \mathcal{T}_\zeta : V_u \ne 0\}$ are in one-to-one correspondence with the (countable) collection $(u_i)_{i\in I}$ of all vertices u of $\mathcal{T}_\zeta$ such that

(a) $V_u = 0$;
(b) u has a strict descendant v such that labels along the geodesic segment from u to v do not vanish except at u.

The correspondence is made explicit by saying that $\mathcal{C}_i$ consists of all strict descendants v of u_i such that property (b) holds, with $u = u_i$ (it is not hard to verify that, $\mathbb{N}_0$-a.e., no branching point of $\mathcal{T}_\zeta$ can satisfy property (b), and we discard the event of zero $\mathbb{N}_0$-measure where this might happen). Then, for every $i \in I$, there are exactly two times $0 < a_i < b_i < \sigma$ such that $p_\zeta(a_i) = p_\zeta(b_i) = u_i$. The paths W_s for $s \in [a_i, b_i]$ are the historical paths of the descendants of u_i. This leads us to define, for every $s \ge 0$, a random finite path $W_s^{(u_i)}$, with lifetime $\zeta_s^{(u_i)} = \zeta_{(a_i+s)\wedge b_i} - \zeta_{a_i}$, by setting

$$W_s^{(u_i)}(t) = W_{(a_i+s)\wedge b_i}(\zeta_{a_i} + t), \qquad 0 \le t \le \zeta_s^{(u_i)}.$$

If $0 < s < b_i - a_i$, , the path $W_s^{(u_i)}$ starts from 0 (note that $W_s^{(u_i)}(0) = W_{(a_i+s)\wedge b_i}(\zeta_{a_i}) = W_{a_i}(\zeta_{a_i}) = V_{u_i}$), and then stays positive during some time interval $(0, \eta)$, $\eta > 0$. Of course if $s = 0$ or $s \geq b_i - a_i$, then $W_s^{(u_i)}$ is just the trivial path with initial point 0.

The endpoints $\hat{W}_s^{(u_i)}$ of the paths $W_s^{(u_i)}$ correspond to the labels of all descendants of u_i in $\mathcal{T}_\zeta$. In fact, we are only interested in those descendants of u_i that belong to $\mathcal{C}_i$, and for this reason we introduce the time change

$$\tilde{W}_s^{(u_i)} = W^{(u_i)}_{\pi_s^{(u_i)}}$$

where, for every $s \geq 0$,

$$\pi_s^{(u_i)} := \inf\Big\{r \geq 0 : \int_0^r \mathrm{d}t\, \mathbf{1}_{\{\tau_0^*(W_t^{(u_i)}) \geq \zeta_t^{(u_i)}\}} > s\Big\},$$

with the notation $\tau_0^*(\mathrm{w}) := \inf\{t > 0 : \mathrm{w}(t) = 0\}$ for $\mathrm{w} \in \mathcal{W}$. The effect of this time change is to eliminate the paths $W_s^{(u_i)}$ that return to 0 and survive for some period after the return time.

Then, for every $i \in I$, the collection $(\tilde{W}_s^{(u_i)})_{s\geq 0}$, which we view as a random element of the space $C(\mathbb{R}_+, \mathcal{W})$, provides a mathematically precise representation of the excursion E_i—in fact the tree $\mathcal{C}_i$ (or rather its closure in $\mathcal{T}_\zeta$) is just the tree coded by the lifetime process $(\tilde{\zeta}_s^{(u_i)})_{s\geq 0}$ of $(\tilde{W}_s^{(u_i)})_{s\geq 0}$, and the labels on $\mathcal{C}_i$ correspond in this identification to the endpoints of the paths $\tilde{W}_s^{(u_i)}$.

In order to state our first theorem, we need one more piece of notation. For every $i \in I$, we let ℓ_i be the total local time at 0 of the historical path W_{a_i} of u_i.

Theorem 1. *There exists a σ-finite measure $\mathbb{M}_0$ on $C(\mathbb{R}_+, \mathcal{W})$ such that, for any non-negative measurable function Φ on $\mathbb{R}_+ \times C(\mathbb{R}_+, \mathcal{W})$, we have*

$$\mathbb{N}_0\Big(\sum_{i\in I} \Phi(\ell_i, \tilde{W}^{(u_i)})\Big) = \int_0^\infty \mathrm{d}\ell\, \mathbb{M}_0(\Phi(\ell, \cdot)).$$

The reason for considering a function depending on local times should be clear from the formula of the theorem: if $\Phi(\ell, \omega)$ does not depend on ℓ, the right-hand side will be either 0 or ∞. We may write $\mathbb{M}_0$ in the form

$$\mathbb{M}_0 = \tfrac{1}{2}(\mathbb{N}_0^* + \check{\mathbb{N}}_0^*)$$

where $\mathbb{N}_0^*$ is supported on positive excursions and $\check{\mathbb{N}}_0^*$ is the image of $\mathbb{N}_0^*$ under $\omega \mapsto -\omega$. Then, for every $\delta > 0$, $\mathbb{N}_0^*$ gives a finite mass to "excursions" ω that hit δ, and more precisely,

$$\mathbb{N}_0^*(\{\omega : \sup\{\hat{W}_s(\omega) : s \geq 0\} > \delta) = c_0\delta^{-3}$$

where c_0 is an explicit constant (see Lemma 25).

In a way similar to the classical setting, one can give various representations of the measure $\mathbb{N}_0^*$. For $\varepsilon > 0$, let $\mathbb{N}_\varepsilon$ be the Brownian snake excursion measure from ε (this is just the image of $\mathbb{N}_0$ under the shift $\omega \mapsto \varepsilon + \omega$). Consider under $\mathbb{N}_\varepsilon$ the time-changed process $\tilde{W}$ obtained by removing those paths W_s that hit 0 and then survive for some period (this is analogous to the time change we have used above to define $\tilde{W}^{(u_i)}$ from $W^{(u_i)}$). Then $\mathbb{N}_0^*$ may be obtained as the limit as $\varepsilon \to 0$ of ε^{-1} times the law of $\tilde{W}$ under $\mathbb{N}_\varepsilon$. See Theorem 23 and Corollary 26 for precise statements. This result is analogous to the classical result that the Itô measure of positive excursions is the limit (in a suitable sense) of $(2\varepsilon)^{-1}$ times the law of linear Brownian motion started from ε and stopped upon hitting 0.

Similarly, one can give a description of $\mathbb{N}_0^*$ analogous to the well-known Bismut decomposition for the Itô measure [36, Theorem XII.4.7]. Under $\mathbb{N}_0^*$, pick a vertex of the tree coded by $(\zeta_s)_{s\geq 0}$ according to the volume measure on this tree, re-root the tree at that vertex and shift all labels so that the label of the new root is again 0. This construction yields a new measure on $C(\mathbb{R}_+, \mathcal{W})$, which turns out to be the same (up to a simple density) as the measure obtained by picking $x \leq 0$ according to Lebesgue measure on $(-\infty, 0)$ and then, under the measure $\mathbb{N}_0$ restricted to the event where one of the paths W_s hits $-x$, removing all paths W_s that go below level x. See Theorem 28 below for a more precise statement.

We now introduce exit measures under $\mathbb{M}_0$.

Proposition 2. *One can choose a sequence $(\alpha_n)_{n\geq 1}$ of positive reals converging to 0 so that, $\mathbb{M}_0$-a.e., the limit*

$$Z_0^* := \lim_{n\to\infty} \alpha_n^{-2} \int_0^\infty \mathbf{1}_{\{0<|\hat{W}_s|<\alpha_n\}}\,\mathrm{d}s$$

exists and defines a positive random variable. Furthermore, this limit does not depend on the choice of the sequence $(\alpha_n)_{n\geq 1}$.

Remark. At this point, a comment about our terminology is in order. Frequently in this article, we will argue on σ-finite measure spaces, and measurable functions defined on these spaces will still be called "random variables", as in the preceding proposition. Similarly we will speak about the "law" or the "distribution" of these random variables, though these laws will be infinite (not necessarily σ-finite) measures.

Theorem 1 and Proposition 2 allow us to make sense of the quantity $Z_0^*(\tilde{W}^{(u_i)})$ for every $i \in I$. Informally, $Z_0^*(\tilde{W}^{(u_i)})$ counts the number of paths $\tilde{W}^{(u_i)}$ that return to 0, and thus measures the size of the boundary of $\mathcal{C}_i$. On the other hand, the quantity $\sigma(\tilde{W}^{(u_i)})$ corresponds to the volume of $\mathcal{C}_i$. Quite remarkably, one can obtain an explicit formula for the joint distribution of the pair (Z_0^*, σ) under $\mathbb{M}_0$. This distribution has density

$$f(z,s) = \frac{\sqrt{3}}{2\pi}\sqrt{z}\,s^{-5/2}\exp\left(-\frac{z^2}{2s}\right)$$

with respect to Lebesgue measure on $\mathbb{R}_+ \times \mathbb{R}_+$ (Proposition 31).

Using scaling arguments, one can then canonically define, for every $z > 0$, the conditional probability measure $\mathbb{M}_0(\cdot \mid Z_0^* = z)$, which will play an important role in our description of the distribution of the collection $(W^{(u_i)})_{i\in I}$. Before stating our theorem identifying this distribution, we need a last ingredient. For every $s \geq 0$ and $t \in [0, \zeta_s]$, write $L^0_t(W_s)$ for the local time at level 0 and at time t of the path W_s (this makes sense under the measure $\mathbb{N}_0$). We observe that, under the measure $\mathbb{N}_0$, the process

$$\mathbf{W}_s := (W_s, L^0(W_s)) = (W_s(t), L^0_t(W_s))_{0\le t\le\zeta_s}$$

can be viewed as the Brownian snake (under its excursion measure from (0, 0)) associated with a spatial motion which is now the pair consisting of a linear Brownian motion and its local time at 0 (the Brownian snake associated with a Markov process is defined by properties analogous to (i) and (ii) above, with the only difference that in (ii) linear Brownian motion is replaced by the Markov process under consideration). See [21], and notice that the spatial motion used to define the Brownian snake needs to satisfy certain continuity properties which hold in the present situation. Following [21, Chapter V], we can then define, for every $r > 0$, the exit measure of $\mathbf{W}$ from the open set $O_r = \mathbb{R}\times[0, r)$, and we denote this exit measure by $\mathcal{X}_r$—to be precise the exit measure is a measure on ∂O_r, but here it is easily seen to be concentrated on the singleton $\{0\}\times\{r\}$, and $\mathcal{X}_r$ denotes its total mass. Informally, $\mathcal{X}_r$ measures the quantity of paths W_s whose endpoint is 0 and which have accumulated a total local time at 0 equal to r.

One can explicitly determine the "law" of the exit measure process $(\mathcal{X}_r)_{r>0}$ under $\mathbb{N}_0$, using on the one hand Lévy's famous theorem relating the law of the local time process of a linear Brownian motion B to that of the supremum process of B, and on the other hand known results about exit measures from intervals. This process is Markovian, with the transition mechanism of the continuous-state branching process with stable branching mechanism $\psi(\lambda) = \sqrt{8/3}\,\lambda^{3/2}$. In particular the process $(\mathcal{X}_r)_{r>0}$ has a càdlàg modification, which we consider from now on.

Recall that, for every $i\in I$, ℓ_i denotes the local time at 0 of the historical path of u_i.

Proposition 3. *The numbers ℓ_i, $i\in I$ are exactly the jump times of the process $(\mathcal{X}_r)_{r>0}$. Furthermore, for every $i\in I$, the size $Z^*_0(\tilde W^{(u_i)})$ of the boundary of $\mathcal{C}_i$ is equal to the jump $\Delta\mathcal{X}_{\ell_i}$.*

We can now state the main result of this introduction.

Theorem 4. *Under $\mathbb{N}_0$, conditionally on the local time exit process $(\mathcal{X}_r)_{r>0}$, the excursions $(\tilde W^{(u_i)})_{i\in I}$ are independent and, for every $j\in I$, the conditional distribution of $\tilde W^{(u_j)}$ is $\mathbb{M}_0(\cdot \mid Z^*_0 = \Delta\mathcal{X}_{\ell_j})$.*

In the classical theory, the collection of excursions of linear Brownian motion is described in terms of a Poisson point process. Such a representation is also possible here and the relevant Poisson point process is linked with the Poisson process of jumps of the Lévy process that corresponds to the continuous-state branching process $\mathcal{X}$ via Lamperti's transformation. We refrain from explaining this representation in this introduction because the formulation is somewhat more intricate than in the classical case (see however Proposition 38) and requires adding extra randomness to get a complete construction of the Poisson point process.

Let us make a few remarks. First, although we state our main results under the infinite measure $\mathbb{N}_0$, one can give equivalent statements in the more familiar setting of probability measures, for instance by conditioning $\mathbb{N}_0$ on specific events with finite mass (such as the event where at least one of the paths W_s has accumulated a total local time at 0 greater than δ, for some fixed $\delta > 0$) or by dealing with a Poisson measure with intensity $\mathbb{N}_0$—such Poisson measures are in fact needed when one studies the connections between the Brownian snake and superprocesses. The second remark is that we could have considered excursions away from $a \neq 0$ instead of the particular case $a = 0$. There is a minor difference, due to the special connected component of $\{u \in \mathcal{T}_\zeta : V_u \neq a\}$ that contains the root. The study of the connected components other than the special one can be reduced to the case $a = 0$ by an application of the so-called special Markov property (see Section 2.4). As a last and important remark, most of the following proofs and statements deal with excursions "above the minimum" (see Section 3 for the definition) and not with excursions away from 0 that we considered in this introduction. However the results about excursions away from 0 can then be derived using the already mentioned theorem of Lévy, and we explain this derivation in detail in Section 8. The reason for considering first excursions above the minimum comes from the fact that certain technical details become significantly simpler. In particular, the local time exit process is replaced by the more familiar process of exit measures from intervals.

An important motivation for the present work comes from the construction of the Brownian map as a quotient space of the CRT for an equivalence relation defined in terms of Brownian motion indexed by the CRT (see e.g. [23, Section 2.5]). The recent paper [9] discusses the infinite volume version of the Brownian map called the Brownian plane. In a way similar to the Brownian map, the Brownian plane is obtained as a quotient space of an infinite Brownian tree equipped with nonnegative Brownian labels, in such a way that these labels correspond to distances from the root in the Brownian plane. The main goal of [9] is to study the process of hulls, where, for every $r > 0$, the hull of radius r is obtained by filling in the bounded holes in the ball of radius r centered at the root vertex of the Brownian plane. It turns out (see [9, formula (16)]) that discontinuities of the process of hulls correspond to excursions above the minimum for the process of labels, which is a tree-indexed Brownian motion under a special conditioning. Such a discontinuity appears when the hull of radius r "swallows" a connected component of the complement of the ball of radius r, and this connected component consists of (the equivalence classes of) the vertices belonging to the associated excursion above the minimum at level r. This relation explains why several formulas and calculations below are reminiscent of those in [9]. In particular the conditional distribution of the mass σ of an excursion given the boundary length $\mathcal{Z}_0^*$ (see Proposition 31) appears in [9, Theorem 1.3], as well as in the companion paper [10], where this distribution is interpreted as the limiting law of the number of faces of a Boltzmann triangulation with a boundary of fixed size tending to infinity.

In the same direction, there are close relations between the present article and the recent work of Miller and Sheffield [32, 33, 34] aiming at proving the equivalence of the Brownian map and Liouville quantum gravity with parameter $\gamma = \sqrt{8/3}$. In particular, the paper [32] uses what we call Brownian snake excursions above the minimum to define the notion of a Brownian disk, corresponding to bubbles appearing in the exploration of the Brownian map: see the definition of μ_{DISK}^L in [32, Proposition 4.4 and its proof]. A key idea of [32] is that one can use such Brownian disks to reconstruct the Brownian map by filling in the holes of the so-called "Lévy net", which itself corresponds to the union of the boundaries of hulls centered at the root (to be precise, the definition of hulls here requires that there is a marked vertex in addition to the root of the Brownian map). Interestingly, Bettinelli and Miermont [4] have developed a different method, based on an approximation by large planar maps with a boundary, to define the notion of a Brownian disk. The forthcoming paper [26] uses the excursion measure $\mathbb{N}_0^*$ introduced in the present work to unify these different approaches and derive new properties of Brownian disks.

An obvious question is whether the excursion theory developed here can be extended to more general tree-indexed processes. As a first remark, many of our arguments rely on the special Markov property (Proposition 13 below), which has been stated and proved rigorously only for processes indexed by the Brownian tree. It is likely that some version of the special Markov property holds for processes indexed by Lévy trees [13, 38], which are random $\mathbb{R}$-trees characterized by a branching property analogous to the one that holds for discrete Galton–Watson trees, but this has not been proven yet. One may then ask whether Brownian motion can be replaced by another Markov process indexed by the Brownian tree. The recent paper [25] shows that the special Markov property still holds provided the underlying Markov process satisfies certain strong continuity assumptions. These assumptions are satisfied by a "nice" diffusion process on the real line, and one may expect that analogs of our results will then hold in that more general setting. Proving this would however require a different approach, since we can no longer use the Lévy theorem mentioned above.

The present paper is organized as follows. Section 2 below presents a number of preliminary observations. In contrast with the text above where we consider the canonical space $C(\mathbb{R}_+, \mathcal{W})$, we have chosen to define the measure $\mathbb{N}_0$ on a smaller canonical space, the space of "snake trajectories" (see Section 2.2). The reason for this choice is that several transformations, such as the re-rooting operation, or the truncation operation allowing us to eliminate paths W_s hitting a certain level, are more conveniently defined and analysed on this smaller space. Snake trajectories are in one-to-one correspondence with tree-like paths (also defined in Section 2.2) via a homeomorphism theorem of Marckert and Mokkadem [29], and this bijection is useful to simplify certain convergence arguments. Section 2.4 gives a precise statement of the special Markov property which later plays an important role.

Section 3 provides a construction of the measure $\mathbb{N}_0^*$, by proving the analog of Theorem 1 for excursions above the minimum. As a by-product, this proof also yields the above-mentioned approximation of $\mathbb{N}_0^*$ in terms of the Brownian snake under $\mathbb{N}_\varepsilon$, truncated at level 0. Section 4 gives our analog of the Bismut decomposition theorem for the measure $\mathbb{N}_0^*$. The proof is based on a re-rooting invariance property of the Brownian snake which can be found in [28]. Then Section 5 describes an almost sure version of the approximation given in Section 3, which is useful in further developments.

Section 6 contains the definition of the exit measure Z_0^* under $\mathbb{N}_0^*$, and the derivation of the joint distribution of the pair (Z_0^*, σ). As an important technical ingredient of the proof of our main results, we also verify that the approximation of the measure $\mathbb{N}_0^*$ by a truncated Brownian snake under $\mathbb{N}_\varepsilon$ can be stated jointly with the convergence of the corresponding exit measures (Proposition 32). Section 7 contains the proof of the results analogous to Proposition 3 and Theorem 4 in the slightly different setting of excursions above the minimum. In a way very similar to the classical theory, we introduce an auxiliary Poisson point process with intensity $\mathrm{d}t \otimes \mathbb{N}_0^*(\mathrm{d}\omega)$, such that all excursions above the minimum can be recovered from the atoms of this process—but as mentioned earlier, the construction of this Poisson point process is somewhat more delicate than in the classical case. Finally, Section 8 explains how the results of the present introduction can be derived from those concerning excursions above the minimum.

Warning. As already mentioned, we define the Brownian snake below on a smaller canonical space than $C(\mathbb{R}_+, \mathcal{W})$, namely on the space $\mathcal{S}$ of all snake trajectories introduced in Definition 6. In particular, $(W_s)_{s\geq 0}$ will be the canonical process on $\mathcal{S}$, and $\mathbb{N}_0$ and $\mathbb{N}_0^*$ will be viewed as σ-finite measures on $\mathcal{S}$ rather than on $C(\mathbb{R}_+, \mathcal{W})$. The notation used below is therefore slightly different from the one in the introduction, but this should cause no confusion.

Main notation

- $\mathcal{T}_h$ the tree coded by a function h (Section 2.1)
- $\prec$ the genealogical order on $\mathcal{T}_\zeta$ (Section 2.1)
- p_h the canonical projection from $\mathbb{R}_+$ onto $\mathcal{T}_h$ (Section 2.1)
- $\mathcal{W}$ the set of all finite paths, $\mathcal{W}_x$ the set of all finite paths started at x (Section 2.2)
- $\zeta_{(\mathrm{w})}$ the lifetime of $\mathrm{w} \in \mathcal{W}$ (Section 2.2)
- $\hat{\mathrm{w}} = \mathrm{w}_{\zeta_{(\mathrm{w})}}$ for $\mathrm{w} \in \mathcal{W}$ (Section 2.2)
- $\underline{\mathrm{w}} = \min\{\mathrm{w}(t) : 0 \leq t \leq \zeta_{(\mathrm{w})}\}$ for $\mathrm{w} \in \mathcal{W}$ (Section 2.2)
- $\tau_y(\mathrm{w}) = \inf\{t \in [0, \zeta_{(\mathrm{w})}] : \mathrm{w}(t) = y\}$, $\tau_y^*(\mathrm{w}) = \inf\{t \in (0, \zeta_{(\mathrm{w})}] : \mathrm{w}(t) = y\}$ (Section 2.2)
- $\mathcal{S}$ the set of all snake trajectories, $\mathcal{S}_x$ the set of all snake trajectories with initial point x (Section 2.2)
- $(W_s)_{s\geq 0}$ the canonical process on $\mathcal{S}$ (Section 2.2)
- $\zeta_s(\omega) = \zeta_{(W_s(\omega))}$ the lifetime process on $\mathcal{S}$ (Section 2.2)
- $\sigma(\omega)$ the duration of the snake trajectory $\omega \in \mathcal{S}$ (Section 2.2)
- $\|\omega\| = \sup\{|\omega_s(t)| : 0 \leq s \leq \sigma(\omega),\ 0 \leq t \leq \zeta_{(\omega_s)}\}$ for $\omega \in \mathcal{S}$ (Section 2.2)
- $\mathcal{S}^{(\delta)} = \{\omega \in \mathcal{S} : \|\omega\| > \delta\}$ (Section 3)
- $M(\omega) = \sup\{\omega_s(t) : 0 \leq s \leq \sigma(\omega), 0 \leq t \leq \zeta_{(\omega_s)}\}$ for $\omega \in \mathcal{S}$ (Section 2.2)
- $\mathbb{T}$ the set of all tree-like paths, $\mathbb{T}_x$ the set of all tree-like paths with initial point x (Section 2.2)
- $\omega \mapsto \kappa_a(\omega)$ the shift on snake trajectories (Section 2.2)
- $\omega \mapsto R_s(\omega)$ re-rooting on snake trajectories (Section 2.2)
- $\mathrm{tr}_y(\omega)$ the truncation of $\omega \in \mathcal{S}$ at y (Section 2.2)
- $\mathbb{N}_x$ the Brownian snake excursion measure from x (Section 2.3)
- W_* the minimum of the Brownian snake (Section 2.3)
- $\mathcal{E}_x^U$ the σ-field generated by the Brownian snake paths under $\mathbb{N}_x$ before they exit U (Section 2.4)
- $\mathcal{Z}^U$ the exit measure of the Brownian snake from U (Section 2.4)
- $\mathcal{Z}_y = \langle \mathcal{Z}^{(y,\infty)}, 1\rangle$, $Z_a = \mathcal{Z}_{-a}$ (Section 2.5)
- $V_u = \hat{W}_s$ if $u = p_\zeta(s)$, the label of $u \in \mathcal{T}_\zeta$ (Section 3)

- D the set of all excursion debuts (Section 3)
- $C_u = \{w \in \mathcal{T}_\zeta : u \prec w \text{ and } V_v > V_u,\ \forall v \in]\!]u, w]\!]\}$ for $u \in D$ (Section 3)
- $M_u = \sup\{V_v - V_u : v \in C_u\}$ the height of the excursion debut u (Section 3)
- D_δ the set of all excursion debuts with height greater than δ (Section 3)
- $W^{(u)}$ the snake trajectory describing the labels of descendants of $u \in D$, shifted so that $W^{(u)} \in \mathcal{S}_0$ (Section 3)
- $\tilde{W}^{(u)} = \mathrm{tr}_0(W^{(u)})$ the truncation of $W^{(u)}$ at 0, for $u \in D$ (Section 3)
- $\tilde{W} = \mathrm{tr}_0(W)$ the truncation at 0 of the canonical process W (Section 3)
- $\tilde{M} = M(\tilde{W})$ (Section 3)
- $\mathbb{N}_0^*$ the Brownian snake excursion measure "above the minimum" (Section 3)
- $\mathcal{N}_k^\varepsilon(\omega) = \sum_{i \in I_k^\varepsilon} \delta_{\omega_i^{k,\varepsilon}}$ the point measure of excursions of $\omega \in \mathcal{S}_0$ outside $(-k\varepsilon, \infty)$ (Section 3)
- $\tilde{\omega}_i^{k,\varepsilon} = \mathrm{tr}_0 \circ \kappa_{(k+1)\varepsilon}(\omega_i^{k,\varepsilon})$ the truncation at 0 of the excursion $\omega_i^{k,\varepsilon}$ shifted so that its initial point is ε (Section 3)
- θ_λ the scaling operator on $\mathcal{S}$ (Section 3)
- $W^{[s]}(\omega) = \kappa_{-\hat{W}_s(\omega)} \circ R_s(\omega)$ the snake trajectory ω re-rooted at s and shifted so that the spatial position of the root is 0 (Section 5)
- Z_0^* the exit measure at 0 under $\mathbb{N}_0^*$ (Section 6)
- $\mathbb{N}_0^{*,z} = \mathbb{N}_0^*(\cdot \mid Z_0^* = z)$ (Section 6)
- $\mathcal{Y}_b = \int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{\tau_{-b}(W_s)=\infty\}}$ for $b \geq 0$ (Section 6)
- $\mathbb{N}_0^{(\beta)} = \mathbb{N}_0(\cdot \mid W_* < -\beta)$ (Section 7)

2. Preliminaries

2.1. *Coding a real tree by a function*

In this subsection, we recall without proof a number of simple properties of the coding of compact $\mathbb{R}$-trees by functions. We refer to [13] and [27] for additional details.

Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a nonnegative continuous function on $\mathbb{R}_+$ such that $h(0) = 0$. We assume that h has compact support, so that

$$\sigma_h := \sup\{t \geq 0 : h(t) > 0\} < \infty.$$

Throughout we make the convention that $\sup \emptyset = 0$.

For every $s, t \in \mathbb{R}_+$, we set

$$d_h(s,t) := h(s) + h(t) - 2 \min_{s \wedge t \leq r \leq s \vee t} h(r).$$

Then d_h is a pseudo-distance on $\mathbb{R}_+$. We introduce the associated equivalence relation on $\mathbb{R}_+$, defined by setting $s \sim_h t$ if and only if $d_h(s,t) = 0$, or equivalently

$$h(s) = h(t) = \min_{s \wedge t \leq r \leq s \vee t} h(r).$$

Then d_h induces a distance on the quotient space $\mathcal{T}_h := \mathbb{R}_+ / \sim_h$. The canonical projection from $\mathbb{R}_+$ onto $\mathcal{T}_h$ is denoted by p_h.

Lemma 5. *The quotient space $\mathcal{T}_h := \mathbb{R}_+/\sim_h$ equipped with the distance d_h is a compact $\mathbb{R}$-tree called the tree coded by h.*

See e.g. [13, Theorem 2.1] for a proof of this lemma as well as for the definition of $\mathbb{R}$-trees. For every $u, v \in \mathcal{T}_h$, the segment $[\![u, v]\!]$ is defined as the range of the (unique) geodesic from u to v in $(\mathcal{T}_h, d_h)$. The notations $]\!]u, v[\![$ or $]\!]u, v]\!]$ have the obvious meaning.

Write ρ for the equivalence class of 0 in $\mathbb{R}_+/\sim_h$, and note that $d_h(\rho, p_h(s)) = h(s)$ for every $s \geq 0$. We call ρ the *root* of $\mathcal{T}_h$, and the *ancestral line* of a point $u \in \mathcal{T}_h$ is the geodesic segment $[\![\rho, u]\!]$. We can then define a genealogical relation on $\mathcal{T}_h$ by saying that u is an *ancestor* of v (or v is a *descendant* of u), written $u \prec v$, if u belongs to $[\![\rho, v]\!]$. If $s, t \geq 0$, then $p_h(s) \prec p_h(t)$ if and only if

$$h(s) = \min_{s\wedge t \leq r \leq s\vee t} h(r).$$

If $u, v \in \mathcal{T}_h$, the last common ancestor of u and v is the unique point, denoted by $u \wedge v$, such that

$$[\![\rho, u]\!] \cap [\![\rho, v]\!] = [\![\rho, u \wedge v]\!].$$

If $u = p_h(s)$ and $v = p_h(t)$ then $u \wedge v = p_h(r)$, where r is any time in $[s \wedge t, s \vee t]$ such that $h(r) = \min\{h(r') : r' \in [s \wedge t, s \vee t]\}$.

We define a *leaf* of $\mathcal{T}_h$ to be any point $u \in \mathcal{T}_h$ which has no descendant other than itself. We let $\mathrm{Sk}(\mathcal{T}_h)$, the *skeleton* of $\mathcal{T}_h$, be the set of all points of $\mathcal{T}_h$ that are not leaves. The *multiplicity* of a point $u \in \mathcal{T}_h$ is the number of connected components of $\mathcal{T}_h \backslash \{u\}$. A point $u \neq \rho$ is a leaf if and only if its multiplicity is 1.

Suppose in addition that h satisfies the following properties:

(i) h does not vanish on $(0, \sigma_h)$;
(ii) h is not constant on any nontrivial subinterval of $(0, \sigma_h)$;
(iii) the local minima of h on $(0, \sigma_h)$ are distinct.

All these properties hold in the applications developed below, where h is a Brownian excursion away from 0. Then the multiplicity of any point of $\mathcal{T}_h$ is at most 3. Furthermore, a point u has multiplicity 3 if and only if u is the form $u = p_h(r)$ where r is a time of local minimum of h on $(0, \sigma_h)$. In that case there are exactly three values of s such that $p_h(s) = u$, namely $s = \sup\{t < r : h(t) > h(r)\}$, $s = r$ and $s = \inf\{t > r : h(t) \leq h(r)\}$. Points of multiplicity 3 will be called *branching points* of $\mathcal{T}_h$. If u and v are two points of $\mathcal{T}_h$, and if $u \wedge v \neq u$ and $u \wedge v \neq v$, then $u \wedge v$ is a branching point. Finally, if u is a point of $\mathrm{Sk}(\mathcal{T}_h)$ which is not a branching point, then there are exactly two times $0 \leq s_1 < s_2 \leq \sigma_h$ such that $p_h(s_1) = p_h(s_2) = u$, and the descendants of u are the points $p_h(s)$ when s varies over $[s_1, s_2]$.

2.2. *Canonical spaces for the Brownian snake*

Before we recall the basic facts that we need about the Brownian snake, we start by discussing the canonical space on which this random process will be defined (for technical reasons, we choose a canonical space suitable for the definition of the Brownian snake excursion measures, which would not be appropriate for the Brownian snake starting from an arbitrary initial value as considered above in the introduction).

Recall the notion of a finite path from the introduction. We let $\mathcal{W}$ denote the space of all finite paths in $\mathbb{R}$, and write $\zeta_{(\mathrm{w})}$ for the lifetime of a finite path $\mathrm{w} \in \mathcal{W}$. The set $\mathcal{W}$ is a Polish space when equipped with the distance

$$d_{\mathcal{W}}(\mathrm{w}, \mathrm{w}') = |\zeta_{(\mathrm{w})} - \zeta_{(\mathrm{w}')}| + \sup_{t \geq 0} |\mathrm{w}(t \wedge \zeta_{(\mathrm{w})}) - \mathrm{w}'(t \wedge \zeta_{(\mathrm{w}')})|.$$

The endpoint or tip of the path w is denoted by $\hat{\mathrm{w}} = \mathrm{w}(\zeta_{(\mathrm{w})})$. For every $x \in \mathbb{R}$, we set $\mathcal{W}_x = \{\mathrm{w} \in \mathcal{W} : \mathrm{w}(0) = x\}$. The trivial element of $\mathcal{W}_x$ with zero lifetime is identified with the point x—in this way we view $\mathbb{R}$ as the subset of $\mathcal{W}$ consisting of all finite paths with zero lifetime. We will also use the notation $\underline{\mathrm{w}} = \min\{\mathrm{w}(t) : 0 \leq t \leq \zeta_{(\mathrm{w})}\}$.

We next turn to snake trajectories.

Definition 6. Let $x \in \mathbb{R}$. A *snake trajectory* with initial point x is a continuous mapping

$$\omega : \mathbb{R}_+ \to \mathcal{W}_x, \quad s \mapsto \omega_s,$$

which satisfies the following two properties:

(i) We have $\omega_0 = x$ and $\sup\{s \geq 0 : \omega_s \neq x\} < \infty$.
(ii) For every $0 \leq s \leq s'$, we have

$$\omega_s(t) = \omega_{s'}(t) \quad \text{for every } 0 \leq t \leq \min_{s \leq r \leq s'} \zeta_{(\omega_r)}.$$

We write $\mathcal{S}_x$ for the set of all snake trajectories with initial point x, and

$$\mathcal{S} := \bigcup_{x \in \mathbb{R}} \mathcal{S}_x$$

for the set of all snake trajectories.

If $\omega \in \mathcal{S}_x$, we write

$$\sigma(\omega) = \sup\{s \geq 0 : \omega_s \neq x\}$$

and call $\sigma(\omega)$ the *duration* of the snake trajectory ω. For $\omega \in \mathcal{S}$, we will also use the notation $W_s(\omega) = \omega_s$ and $\zeta_s(\omega) = \zeta_{(\omega_s)}$ for every $s \geq 0$, so that in particular $(W_s)_{s \geq 0}$ is the canonical process on $\mathcal{S}$.

Remark. Property (ii) is called the *snake property*. It is not hard to verify that, for any mapping $\omega : \mathbb{R}_+ \to \mathcal{W}_x$ such that both the lifetime function $s \mapsto \zeta_s(\omega)$ and the tip function $s \mapsto \hat{\omega}_s = \hat{W}_s(\omega)$ are continuous, the snake property (ii) implies that ω is continuous.

The set $\mathcal{S}$ is equipped with the distance

$$d_{\mathcal{S}}(\omega, \omega') = |\sigma(\omega) - \sigma(\omega')| + \sup_{s \geq 0} d_{\mathcal{W}}(W_s(\omega), W_s(\omega')).$$

Note that $\mathcal{S}$ is a measurable subset of the space $C(\mathbb{R}_+, \mathcal{W})$, which is equipped as usual with the Borel σ-field associated with the topology of uniform convergence on every compact interval.

We will use the notation

$$\|\omega\| = \sup\{|\omega_s(t)| : s \geq 0,\ 0 \leq t \leq \zeta_s(\omega)\} = \sup\{|\hat{\omega}_s| : s \geq 0\},$$
$$M(\omega) = \sup\{\omega_s(t) : s \geq 0,\ 0 \leq t \leq \zeta_s(\omega)\} = \sup\{\hat{\omega}_s : s \geq 0\},$$

for $\omega \in \mathcal{S}$. The fact that the two suprema in the definition of $\|\omega\|$ (or in the definition of $M(\omega)$) are equal is a simple consequence of the snake property, which implies that

$$\{\omega_s(t) : s \geq 0,\ 0 \leq t \leq \zeta_s(\omega)\} = \{\hat{\omega}_s : s \geq 0\}.$$

One easily checks that a snake trajectory ω is completely determined by the two functions $s \mapsto \zeta_s(\omega)$ and $s \mapsto \hat{W}_s(\omega)$. We will state this in a more precise form, but for this we first need to introduce tree-like paths.

Definition 7. A *tree-like path* is a pair (h, f) where $h : \mathbb{R}_+ \to \mathbb{R}_+$ and $f : \mathbb{R}_+ \to \mathbb{R}$ are continuous functions that satisfy the following properties:

(i) $h(0) = 0$ and $\sigma_h := \sup\{s \geq 0 : h(s) \neq 0\} < \infty$.
(ii) For every $0 \leq s \leq s'$,

$$h(s) = h(s') = \min_{s \leq r \leq s'} h(r) \quad \text{implies} \quad f(s) = f(s').$$

The set of all tree-like paths is denoted by $\mathbb{T}$, and, for every $x \in \mathbb{R}$, $\mathbb{T}_x := \{(h, f) \in \mathbb{T} : f(0) = x\}$ denotes the set of all tree-like paths with initial point x.

Remark. Our terminology is inspired by the work of Hambly and Lyons, who give a slightly different definition of a tree-like path in a more general setting (see [15, Definition 1.2]).

It follows from property (ii) that if $(h, f) \in \mathbb{T}_x$, we have $f(s) = x$ for every $s \geq \sigma_h$. The set $\mathbb{T}$ is equipped with the distance

$$d_{\mathbb{T}}((h, f), (h', f')) = |\sigma_h - \sigma_{h'}| + \sup_{s \geq 0}\big(|h(s) - h'(s)| + |f(s) - f'(s)|\big).$$

If (h, f) is a tree-like path, h satisfies the assumptions required in Section 2.1 to define the tree $\mathcal{T}_h$. Then property (ii) just says that, for every $s \geq 0$, $f(s)$ only depends on $p_h(s)$, and thus f can as well be viewed as a function on the tree $\mathcal{T}_h$. Furthermore the function induced by f on $\mathcal{T}_h$ is also continuous. For $u \in \mathcal{T}_h$, we then interpret $f(u)$ as a spatial position, or a label, assigned to the point u.

Proposition 8. *The mapping $\Delta : \mathcal{S} \to \mathbb{T}$ defined by $\Delta(\omega) = (h, f)$, where $h(s) = \zeta_s(\omega)$ and $f(s) = \hat{W}_s(\omega)$, is a homeomorphism from $\mathcal{S}$ onto $\mathbb{T}$.*

This is essentially the homeomorphism theorem of Marckert and Mokkadem [29, Theorem 2.1]. Marckert and Mokkadem impose the extra condition $\sigma = 1$ for snake trajectories, and the similar condition for tree-like paths, but the proof is the same without this condition. We mention that $\sigma(\omega) = \sigma_h$ if $(h, f) = \Delta(\omega)$.

Let us briefly explain why Proposition 8 is relevant to our purposes. Much of what follows is devoted to studying the convergence of certain (random) snake trajectories. By Proposition 8, this convergence is equivalent to that of the associated tree-like paths, which is often easier to establish.

Remark. Let (h, f) be a tree-like path, and let ω be the associated snake trajectory. We have already noticed that f can be viewed as a continuous function on the tree $\mathcal{T}_h$ coded by ζ. The same holds for the mapping $s \mapsto \omega_s$. More precisely, for every $s \geq 0$, and every $t \leq \zeta_s(\omega) = h(s)$, $\omega_s(t)$ is the value of f at the unique ancestor of $p_h(s)$ at distance t from the root (recall that $d_h(\rho, p_h(s)) = h(s)$). Thus the finite path $\omega_s = (\omega_s(t))_{0\leq t\leq \zeta_s(\omega)}$ provides the values of f along the ancestral line of $p_h(s)$. We say that ω_s is the *historical* path of $p_h(s)$.

Lemma 9. *Let ω be a snake trajectory and $(h, f) = \Delta(\omega)$. Let $0 < s < s' < \sigma(\omega)$ be such that*

$$h(s) = h(s') = \min_{s\leq r\leq s'} h(r).$$

Set, for every $r \geq 0$,

$$h'(r) = h((s + r) \wedge s') - h(s), \qquad f'(r) = f((s + r) \wedge s').$$

Then, (h', f') is a tree-like path and the corresponding snake trajectory $\omega' = \Delta^{-1}(h', f')$ is called the subtrajectory of ω associated with the interval $[s, s']$.

We omit the easy proof. The assumption of the lemma is equivalent to saying that $p_h(s) = p_h(s')$. Suppose in addition that $\{r \geq 0 : p_h(r) = p_h(s)\} = \{s, s'\}$. Then $u := p_h(s)$ is a point of multiplicity 2 of $\mathrm{Sk}(\mathcal{T}_h)$, and the subtree of descendants of u is coded by f'. Furthermore the snake trajectory ω' describes the spatial positions of the descendants of u.

Let us finally introduce three useful operations on snake trajectories. The first one is just the obvious translation. If $a \in \mathbb{R}$ and $\omega \in \mathcal{S}$, then $\kappa_a(\omega)$ is obtained by adding a to all paths ω_s: in other words, $\zeta_s(\kappa_a(\omega)) = \zeta_s(\omega)$ and $\hat{W}_s(\kappa_a(\omega)) = \hat{W}_s(\omega) + a$ for every $s \geq 0$.

The second operation is the *re-rooting operation*. Let ω be a snake trajectory and let (h, f) be the associated tree-like path. Fix $s \in [0, \sigma(\omega)]$. We will define a new snake trajectory $R_s(\omega)$, which is more conveniently described in terms of its associated tree-like path $(h^{[s]}, f^{[s]}) = \Delta(R_s(\omega))$. Roughly speaking, $h^{[s]}$ is the coding function for the tree $\mathcal{T}_h$ re-rooted at $p_h(s)$ (this is informal since the coding function of a tree is not unique) and $f^{[s]}$ describes the "same function" as f but viewed on the re-rooted tree. To make this more precise we set, for every $r \in [0, \sigma(\omega)]$,

$$h^{[s]}(r) = h(s \oplus r) + h(s) - 2 \min_{s\wedge(s\oplus r)\leq t\leq s\vee(s\oplus r)} h(t),$$

where $s \oplus r = s + r$ if $s + r \leq \sigma(\omega)$ and $s \oplus r = s + r - \sigma(\omega)$ otherwise. We also set $h^{[s]}(r) = 0$ if $r > \sigma(\omega)$. Furthermore we set $f^{[s]}(r) = f(s \oplus r)$ if $r \in [0, \sigma(\omega)]$ and $f^{[s]}(r) = f(s)$ if $r > \sigma(\omega)$. See [13, Lemma 2.2] for the fact that the mapping $[0, \sigma(\omega)] \ni r \mapsto s \oplus r$ induces an isometry from the tree $\mathcal{T}_{h^{[s]}}$ onto the tree $\mathcal{T}_h$ (this in particular implies that $(h^{[s]}, f^{[s]})$ is a tree-like path), and [28, Section 2.3] for more details about this re-rooting operation.

The third and last operation is the *truncation* of snake trajectories, which will be important in this work. Roughly speaking, if $\omega \in \mathcal{S}_x$ and $y \neq x$, the truncation of ω at y is the new snake trajectory ω' such that the values ω'_s are exactly all values ω_s for s such that ω_s does not hit y, or hits y for the first time at its lifetime. Let us give a more precise definition. First, for any $\mathrm{w} \in \mathcal{W}$ and $y \in \mathbb{R}$, we set

$$\tau_y(\mathrm{w}) := \inf\{t \in [0, \zeta_{(\mathrm{w})}] : \mathrm{w}(t) = y\}, \qquad \tau^*_y(\mathrm{w}) := \inf\{t \in (0, \zeta_{(\mathrm{w})}] : \mathrm{w}(t) = y\},$$

with the usual convention $\inf \emptyset = \infty$. Note that $\tau^*_y(\mathrm{w})$ may be different from $\tau_y(\mathrm{w})$ only if $\mathrm{w}(0) = y$, but this case will be important in what follows.

Proposition 10. *Let $x, y \in \mathbb{R}$. Let $\omega \in \mathcal{S}_x$, and for every $s \geq 0$, set*

$$A_s(\omega) = \int_0^s \mathrm{d}r \, \mathbf{1}_{\{\zeta_r(\omega) \leq \tau^*_y(\omega_r)\}}, \qquad \eta_s(\omega) = \inf\{r \geq 0 : A_r(\omega) > s\}.$$

Then setting $\omega'_s = \omega_{\eta_s(\omega)}$ for every $s \geq 0$ defines an element of $\mathcal{S}_x$, which will be denoted by $\omega' = \mathrm{tr}_y(\omega)$ and called the truncation of ω at y.

Proof. First note that, by property (i) of the definition of a snake trajectory, we have $A_s(\omega) \to \infty$ as $s \to \infty$ (because $\zeta_r(\omega) \leq \tau^*_y(\omega_r)$ if $r \geq \sigma(\omega)$), and therefore $\eta_s(\omega) < \infty$ for every $s \geq 0$, so that the definition of ω' makes sense.

We need to verify that $\omega' \in \mathcal{S}_x$. To this end, we observe that the mapping $s \mapsto \eta_s(\omega)$ is right-continuous with left limits given by

$$\eta_{s-}(\omega) = \inf\{r \geq 0 : A_r(\omega) = s\}, \qquad \forall s > 0.$$

To simplify notation, we write $\eta_s = \eta_s(\omega)$, $\eta_{s-} = \eta_{s-}(\omega)$, $A_s = A_s(\omega)$ and $\zeta_s = \zeta_s(\omega)$ in what follows.

We first verify the continuity of the mapping $s \mapsto \omega'_s$. Let $s \geq 0$ be such that $\zeta_{\eta_s} > 0$. By the definition of η_s there are values of $r > \eta_s$ arbitrarily close to η_s such that $\zeta_r \leq \tau^*_y(\omega_r)$. Using the snake property, it then follows that the path $(\omega_{\eta_s}(t))_{0<t\leq\zeta_{\eta_s}}$ does not hit y, or hits y only at time ζ_{η_s} (notice that we have excluded the value $t = 0$ because of the particular case $y = x$, since trivially $\omega_{\eta_s}(0) = y$ in that case). Similarly, for every $s > 0$ such that $\zeta_{\eta_{s-}} > 0$, the path $(\omega_{\eta_{s-}}(t))_{0<t\leq\zeta_{\eta_{s-}}}$ does not hit y, or hits y only at time $\zeta_{\eta_{s-}}$.

Let $s > 0$ be such that $\eta_{s-} < \eta_s$. The key observation is to note that

$$\zeta_r \geq \zeta_{\eta_{s-}} = \zeta_{\eta_s}, \qquad \forall r \in [\eta_{s-}, \eta_s]. \tag{1}$$

In fact, suppose that (1) fails, so that certain values of ζ on the time interval (η_{s-}, η_s) are strictly smaller that $\zeta_{\eta_s} \vee \zeta_{\eta_{s-}}$. Suppose for definiteness that $\zeta_{\eta_{s-}} \leq \zeta_{\eta_s}$ (the other case

$\zeta_{\eta_{s-}} \geq \zeta_{\eta_s}$ is treated similarly). Then we can find $r \in (\eta_{s-}, \eta_s)$ such that $0 < \zeta_r < \zeta_{\eta_s}$ and $\zeta_r = \min\{\zeta_u : u \in [r, \eta_s]\}$. By the snake property this means that ω_r is the restriction of ω_{η_s} to $[0, \zeta_r]$, and since we know that $(\omega_{\eta_s}(t))_{0<t<\zeta_{\eta_s}}$ does not hit y, it follows that $\tau_y^*(\omega_r) = \infty$. Hence we also have $\tau_y^*(\omega_{r'}) = \infty$ for all r' sufficiently close to r, and therefore $A_{\eta_s} > A_{\eta_{s-}}$, which is a contradiction.

The mapping $s \mapsto \omega_{\eta_s}$ is right-continuous and its left limit at $s > 0$ is $\omega_{\eta_{s-}}$. Property (1) and the snake property show that, for every s such that $\eta_{s-} < \eta_s$, we have $\omega_{\eta_{s-}} = \omega_{\eta_s}$, so that the mapping $s \mapsto \omega_{\eta_s} = \omega'_s$ is continuous.

Furthermore, it also follows from (1) that, for every $s \leq s'$,

$$\min_{r\in[s,s']} \zeta_{\eta_r} = \min_{r\in[\eta_s,\eta_{s'}]} \zeta_r,$$

and the snake property for ω' is a consequence of the same property for ω.

We also need to verify that $\omega'_0 = x$. This is immediate if $y \neq x$ (because clearly $\eta_0 = 0$ in that case) but an argument is required in the case $y = x$, which we consider now. It suffices to verify that $\zeta_{\eta_0} = 0$. To reach a contradiction, assume that $\zeta_{\eta_0} > 0$, which implies that $\eta_0 > 0$. By previous observations, the path ω_{η_0} does not hit x during the time interval $(0, \zeta_{\eta_0})$. However, by the snake property again, this implies that there is a set of values of $r \in (0, \eta_0)$ of positive Lebesgue measure such that $\tau_x^*(\omega_r) = \infty$, which contradicts the definition of η_0.

We finally notice that, for $s \geq \int_0^{\sigma(\omega)} \mathrm{d}r\, \mathbf{1}_{\{\zeta_r(\omega)\leq\tau_y^*(\omega)\}}$, we have $\eta_s(\omega) \geq \sigma(\omega)$ and thus $\omega'_s = x$. This completes the proof of the property $\omega' \in \mathcal{S}_x$. □

Remark. If $s > 0$ is such that $\eta_{s-} < \eta_s$, and furthermore $\zeta_{\eta_s} > 0$, then $\tau_y^*(\omega_{\eta_s}) = \zeta_s$. Indeed, since $A_{\eta_s} = A_{\eta_{s-}} = s$, there exist values of $r < \eta_s$ arbitrarily close to η_s such that $\tau_y^*(\omega_r) < \zeta_r$, and by the snake property it follows that $\hat{\omega}_{\eta_s} = y$. Since we saw in the previous proof that $(\omega_{\eta_s}(t))_{0<t<\zeta_{\eta_s}}$ does not hit y, we conclude that $\tau_y^*(\omega_{\eta_s}) = \zeta_s$.

The truncation operation tr_y is a measurable mapping from $\mathcal{S}_x$ into $\mathcal{S}_x$. If $y \neq x$, and if $\omega' = \mathrm{tr}_y(\omega)$ is the truncation of a snake trajectory $\omega \in \mathcal{S}_x$, then the paths ω'_s stay in $[y, \infty)$ (if $y < x$) or in $(-\infty, y]$ (if $y > x$) and can only hit y at their lifetime.

The following lemma gives a simple continuity property of the truncation operations.

Lemma 11. *Let $\omega \in \mathcal{S}_0$ and $b < 0$. Suppose that*

$$\int_0^{\sigma(\omega)} \mathrm{d}s\, \mathbf{1}_{\{\tau_b(\omega_s)=\zeta_s(\omega)\}} = 0.$$

Then, for any sequence $(b_n)_{n\geq1}$ such that $b_n \downarrow b$ as $n \to \infty$, we have $\mathrm{tr}_{b_n}(\omega) \to \mathrm{tr}_b(\omega)$ in $\mathcal{S}$ as $n \to \infty$.

We omit the easy proof of this lemma. We conclude this subsection with another lemma that will be useful in the proof of one of our main results. The proof is somewhat technical and may be omitted at first reading. Recall the notation $\underline{\mathrm{w}} = \min\{\mathrm{w}(t) : 0 \leq t \leq \zeta_{(\mathrm{w})}\}$ for $\mathrm{w} \in \mathcal{W}$.

Lemma 12. *Let $\omega \in \mathcal{S}$, and let ω' be a subtrajectory of ω associated with the interval $[a, b]$. Assume that $\omega' \in \mathcal{S}_0$ and, for every $n \geq 1$, let $\omega^{(n)}$ be a subtrajectory of ω associated with the interval $[a_n, b_n]$ such that $[a, b] \subset [a_n, b_n]$ for every $n \geq 1$ and $a_n \to a$, $b_n \to b$ as $n \to \infty$. Assume furthermore that the following properties hold:*

(i) *$\omega_a(t) \geq 0$ for every $0 \leq t \leq \zeta_{(\omega_a)}$;*
(ii) *for every $s \in (0, b-a)$, $\tau_0^*(\omega'_s) \wedge \zeta_{(\omega'_s)} > 0$ and $\omega'_s(t) \geq 0$ for $0 \leq t \leq \tau_0^*(\omega_s) \wedge \zeta_{(\omega'_s)}$;*
(iii) *for every $s \in (0, b-a)$ such that $\zeta_{(\omega'_s)} > \tau_0^*(\omega'_s)$, we have $\underline{\omega}'_s < 0$.*

Then, if $(\delta_n)_{n\geq 1}$ is any sequence of negative real numbers converging to 0, we have $\mathrm{tr}_{\delta_n}(\omega^{(n)}) \to \mathrm{tr}_0(\omega')$ in $\mathcal{S}$ as $n \to \infty$.

Proof. The first step is to verify that $\omega^{(n)}$ converges to ω' in $\mathcal{S}$. To this end, let (h, f) be the tree-like path associated with ω, and notice that the tree-like path associated with $\omega^{(n)}$ is $(h^{(n)}, f^{(n)})$, with $h^{(n)}(r) = h((a_n + r) \wedge b_n) - h(a_n)$ and $f^{(n)}(r) = f((a_n + r) \wedge b_n)$. From the convergences $a_n \to a$, $b_n \to b$, it immediately follows that the pair $(h^{(n)}, f^{(n)})$ converges to the tree-like path (h', f') associated with ω', and Proposition 8 implies that $\omega^{(n)}$ converges to ω'.

We also note that, for every $n \geq 1$, we have $f^{(n)}(0) = f(a_n) = \omega_{a_n}(h(a_n)) = \omega_a(h(a_n))$, where the last equality holds because $p_h(a_n)$ is an ancestor of $p_h(a)$. Using (i), we get $f^{(n)}(0) \geq 0$. By preceding remarks, we know that the paths of $\mathrm{tr}_{\delta_n}(\omega^{(n)})$ stay in $[\delta_n, \infty)$.

Set $\tilde{\omega}^{(n)} = \mathrm{tr}_{\delta_n}(\omega^{(n)})$ and $\tilde{\omega}' = \mathrm{tr}_0(\omega')$ to simplify notation. Then set, for every $s \geq 0$,

$$A_s^{(n)} := \int_0^s \mathrm{d}r\, \mathbf{1}_{\{h^{(n)}(r) \leq \tau^*_{\delta_n}(\omega_r^{(n)})\}}, \qquad A'_s := \int_0^s \mathrm{d}r\, \mathbf{1}_{\{h'(r) \leq \tau_0^*(\omega'_r)\}},$$
$$\eta_s^{(n)} := \inf\{r \geq 0 : A_r^{(n)} > s\}, \qquad \eta'_s := \inf\{r \geq 0 : A'_r > s\},$$

so that $\tilde{\omega}_s^{(n)} = \omega^{(n)}_{\eta_s^{(n)}}$ and $\tilde{\omega}'_s = \omega'_{\eta'_s}$ by the definition of truncations. We observe that, for every $s \geq 0$,

$$A_s^{(n)} \xrightarrow[n\to\infty]{} A'_s. \tag{2}$$

To see this, note that, for $r \in [a, b]$, the paths ω_r are the same as ω_a up to time $h(a) = \zeta_a(\omega)$, and thus stay nonnegative on the time interval $[0, h(a)]$ by (i). From our definitions, it follows that the paths $\omega^{(n)}_{a-a_n+r}$, for $0 \leq r \leq b-a$, stay nonnegative up to time $h(a) - h(a_n) \geq 0$. Then, for $r \in [0, b-a]$, we have $\omega'_r(\cdot) = \omega^{(n)}_{a-a_n+r}(h(a) - h(a_n) + \cdot)$, and by (ii) we see that if $h'(r) \leq \tau_0^*(\omega'_r)$, the path $\omega^{(n)}_{a-a_n+r}$ does not hit $\delta_n < 0$ between times $h(a) - h(a_n)$ and $h^{(n)}(a - a_n + r)$. Hence, for every $r \in [0, b-a]$,

$$\mathbf{1}_{\{h'(r) \leq \tau_0^*(\omega'_r)\}} \leq \mathbf{1}_{\{h^{(n)}(a-a_n+r) \leq \tau^*_{\delta_n}(\omega^{(n)}_{a-a_n+r})\}}.$$

It follows that $A'_s \leq A^{(n)}_{a-a_n+s} \leq A_s^{(n)} + (a - a_n)$, which implies

$$\liminf_{n\to\infty} A_s^{(n)} \geq A'_s$$

for every $s \geq 0$. Conversely, we claim that, for every $r \in (0, b-a)$,

$$\limsup_{n\to\infty} \mathbf{1}_{\{h^{(n)}(r) \leq \tau^*_{\delta_n}(\omega_r^{(n)})\}} \leq \mathbf{1}_{\{h'(r) \leq \tau_0^*(\omega'_r)\}}.$$

Indeed, if $\tau_0^*(\omega'_r) < h'(r)$, then assumption (iii) implies that ω'_r takes negative values before its lifetime. By the convergence of $\omega_r^{(n)}$ to ω'_r, we must have $\tau^*_{\delta_n}(\omega_r^{(n)}) < h^{(n)}(r)$ for n large, proving our claim. The claim now gives

$$\limsup_{n\to\infty} A_s^{(n)} \leq A'_s,$$

completing the proof of (2). Notice that (2) also implies that $A^{(n)}_{b_n-a_n} \to A'_{b-a}$, from which one gets $\sigma(\tilde{\omega}^{(n)}) \to \sigma(\tilde{\omega}')$, because $\sigma(\tilde{\omega}') = A'_{b-a}$ as a consequence of (ii) (if $0 < s < A'_{b-a}$, then $\tilde{\omega}'_s = \omega'_{\eta'_s}$ is not a trivial path by (ii) and the fact that $0 < \eta'_s < b-a$).

It follows from (2) that $\eta_s^{(n)} \to \eta'_s$, and consequently $\tilde{\omega}_s^{(n)} \to \tilde{\omega}'_s$, as $n \to \infty$, for every $s \geq 0$ such that $\eta'_s = \eta'_{s-}$. We now prove by contradiction that this implies the uniform convergence of $\tilde{\omega}_s^{(n)}$ toward $\tilde{\omega}'$ (which will complete the proof of the proposition). If the latter convergence does not hold, then by taking a subsequence of $(\tilde{\omega}^{(n)})_{n\geq 1}$ we may suppose that there exist a sequence $(s_n)_{n\geq 1}$ and a real $\xi > 0$ such that, for every n,

$$d_{\mathcal{S}}(\tilde{\omega}^{(n)}_{s_n}, \tilde{\omega}'_{s_n}) > \xi. \tag{3}$$

Since both $\tilde{\omega}_r^{(n)}$ and $\tilde{\omega}'_r$ are constant (and equal to a trivial path) when $r \geq \sigma(\omega)$, we can assume that $s_n \in [0, \sigma(\omega)]$ for every n and then, modulo taking a subsequence, that $s_n \to s_\infty$ as $n \to \infty$. We must then have $\eta'_{s_\infty-} < \eta'_{s_\infty}$ because otherwise (2) would imply $\eta^{(n)}_{s_n} \to \eta_{s_\infty}$ and therefore $\tilde{\omega}^{(n)}_{s_n} \to \tilde{\omega}'_{s_\infty}$, contradicting (3). We can also assume that $0 < s_\infty < \sigma(\tilde{\omega}')$, and therefore $0 < \eta'_{s_\infty} < b-a$, since it follows from assumption (ii) that η' is continuous at $\sigma(\tilde{\omega}') = A'_{b-a}$ (if $0 < s < b-a$, property (ii) and the snake property imply that the interval $[s, b-a]$ contains a set of positive Lebesgue measure of values of r such that $\tau_0^*(\omega(r)) = \infty$, and this is what we need to get the latter continuity property). Also notice that (ii) implies $h'(r) > 0$ for $0 < r < b-a$, and consequently $h'(\eta'_r) > 0$ for $0 < r < \sigma(\tilde{\omega}')$.

From (2), we see that any accumulation point of the sequence $(\eta^{(n)}_{s_n})_{n\geq 1}$ must lie in the interval $[\eta'_{s_\infty-}, \eta'_{s_\infty}]$. We claim that for any such accumulation point r we have $\omega'_r = \omega'_{\eta'_{s_\infty}}$. This implies that $\tilde{\omega}^{(n)}_{s_n} = \omega^{(n)}_{\eta^{(n)}_{s_n}}$ converges to $\omega'_{\eta'_{s_\infty}} = \tilde{\omega}'_{s_\infty}$ and contradicts (3).

To verify our claim, let $r \in [\eta'_{s_\infty-}, \eta'_{s_\infty}]$ be an accumulation point of $(\eta^{(n)}_{s_n})_{n\geq 1}$. By property (1) in the proof of Proposition 10, the path ω'_r coincides with $\omega'_{\eta'_{s_\infty}}$ up to $h'(\eta'_{s_\infty}) = \tau_0^*(\omega'_{\eta'_{s_\infty}})$ (the last equality by the remark following Proposition 10). However, $h'(r) > h'(\eta'_{s_\infty})$ is impossible since assumption (iii) would imply that ω'_r takes negative values and cannot be an accumulation point of $\tilde{\omega}^{(n)}_{s_n}$ (because $\tilde{\omega}^{(n)}_{s_n}$ takes values in $[\delta_n, \infty)$ and δ_n tends to 0 as $n \to \infty$). Therefore $h'(r) = h'(\eta'_{s_\infty})$, meaning that $\omega'_r = \omega'_{\eta'_{s_\infty}}$ as desired. This completes the proof. $\square$

2.3. *The Brownian snake*

In this section we discuss the (one-dimensional) Brownian snake excursion measures. We avoid defining the Brownian snake starting from a general initial value (which is briefly presented in the introduction above), as this definition is not required in what follows, except in the proof of one technical lemma (Lemma 16) which the reader can skip at first reading.

Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ satisfy the assumptions of Section 2.1 (including assumptions (i)–(iii) from the end of that subsection) and also assume that h is Hölder continuous with exponent δ for some $\delta > 0$. Let $(G^h_s)_{s\geq 0}$ be the centered real Gaussian process with covariance

$$\mathrm{cov}(G^h_s, G^h_t) = \min_{s\wedge t\leq r\leq s\vee t} h(r) \tag{4}$$

for all $s, t \geq 0$. We leave it as an exercise to verify that the right-hand side of (4) is a covariance function (see [27, Lemma 4.1]). Note that we then have

$$E[(G^h_s - G^h_t)^2] = d_h(s,t). \tag{5}$$

An application of the classical Kolmogorov lemma shows that $(G^h_s)_{s\geq 0}$ has a continuous modification, which we consider from now on. Then property (5) entails that, for every fixed $0 \leq s \leq t$ such that $d_h(s,t) = 0$, we have $P(G^h_s = G^h_t) = 1$. A continuity argument, using the assumptions satisfied by h, then shows that, a.s., for every $0 \leq s \leq t$, the property $d_h(s,t) = 0$ implies $G^h_s = G^h_t$. This means that outside a set of probability zero which we may discard, the pair (h, G^h) is a (random) tree-like path in the sense of the preceding subsection.

The (one-dimensional) Brownian snake driven by h is the random snake trajectory $W^h = (W^h_s)_{s\geq 0}$ associated with the tree-like path (h, G^h). We write $\mathbf{P}_h(\mathrm{d}\omega)$ for the law of W^h on the space $\mathcal{S}_0$.

We next randomize h: We let $\mathbf{n}(\mathrm{d}h)$ stand for Itô's excursion measure of positive excursions of linear Brownian motion (see e.g. [36, Chapter XII]) normalized so that, for every $\varepsilon > 0$,

$$\mathbf{n}\Big(\max_{s\geq 0} h(s) > \varepsilon\Big) = \frac{1}{2\varepsilon}.$$

Notice that $\mathbf{n}$ is supported on functions h that satisfy the assumptions required above to define W^h and the probability measure $\mathbf{P}_h(\mathrm{d}\omega)$. The *Brownian snake excursion measure* $\mathbb{N}_0$ is then the σ-finite measure on $\mathcal{S}_0$ defined by

$$\mathbb{N}_0(\mathrm{d}\omega) = \int \mathbf{n}(\mathrm{d}h)\,\mathbf{P}_h(\mathrm{d}\omega).$$

In other words, the "lifetime process" $(\zeta_s)_{s\geq 0}$ is distributed under $\mathbb{N}_0(\mathrm{d}\omega)$ according to Itô's measure $\mathbf{n}(\mathrm{d}h)$, and, conditionally on $(\zeta_s)_{s\geq 0}$, $(W_s)_{s\geq 0}$ is distributed as the Brownian snake driven by $(\zeta_s)_{s\geq 0}$. The reader will easily check that the preceding definition of $\mathbb{N}_0$ is consistent with the slightly different presentation given in the introduction above (see [21] for more details about the Brownian snake). For every $x \in \mathbb{R}$, we also define $\mathbb{N}_x$ as the measure on $\mathcal{S}_x$ which is the image of $\mathbb{N}_0$ under the translation κ_x.

Let us recall the first-moment formula for the Brownian snake [21, Section IV.2]. For every nonnegative measurable function ϕ on $\mathcal{W}$,

$$\mathbb{N}_x\bigg(\int_0^\sigma \mathrm{d}s\,\phi(W_s)\bigg) = \mathbb{E}_x\bigg[\int_0^\infty \mathrm{d}t\,\phi((B_r)_{0\leq r\leq t})\bigg], \tag{6}$$

where $B = (B_r)_{r\geq 0}$ stands for a linear Brownian motion starting from x under the probability measure $\mathbb{P}_x$. Here we recall that $\mathbb{N}_x$ is a measure on $\mathcal{S}_x$, and so the duration σ is well-defined under $\mathbb{N}_x$ as in Definition 6.

We define the range $\mathcal{R}$ by

$$\mathcal{R} := \{\hat{W}_s : s \geq 0\} = \{W_s(t) : s \geq 0,\ 0 \leq t \leq \zeta_s\},$$

and we set

$$W_* := \min \mathcal{R}.$$

Then, if $x, y \in \mathbb{R}$ and $y < x$, we have

$$\mathbb{N}_x(W_* \leq y) = \frac{3}{2(x-y)^2}. \tag{7}$$

See e.g. [21, Section VI.1].

2.4. *Exit measures and the special Markov property*

In this section, we briefly describe a key result of [20] that plays a crucial role in the present work. Let U be a nonempty open interval of $\mathbb{R}$ such that $U \neq \mathbb{R}$. For any $\mathrm{w} \in \mathcal{W}$, set

$$\tau^U(\mathrm{w}) := \inf\{t \in [0, \zeta_{(\mathrm{w})}] : \mathrm{w}(t) \notin U\}.$$

If $x \in U$, the limit

$$\langle \mathcal{Z}^U, \phi \rangle = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{\tau^U(W_s) < \zeta_s < \tau^U(W_s)+\varepsilon\}}\, \phi(W_s(\tau^U(W_s))) \tag{8}$$

exists $\mathbb{N}_x$-a.e. for any function ϕ on ∂U and defines a finite random measure $\mathcal{Z}^U$ supported on ∂U (see [21, Chapter V]). Notice that here ∂U has at most two points, but the preceding definition holds in the same form for the Brownian snake in higher dimensions with an arbitrary open set U. Informally, the measure $\mathcal{Z}^U$ "counts" the exit points of the paths W_s from U, for those values of s such that W_s exits U. In particular, $\mathcal{Z}^U = 0$ if none of the paths W_s exits U.

Exit measures are needed to state the so-called special Markov property. Before stating it, we introduce the excursions outside U of a snake trajectory. We fix $x \in U$ and let $\omega \in \mathcal{S}_x$. We observe that the set

$$\{s \geq 0 : \tau^U(\omega_s) < \zeta_s\}$$

is open and can therefore be written as a union of disjoint open intervals (a_i, b_i), $i \in I$, where I may be empty. From the fact that ω is a snake trajectory, it is not hard to verify that we must have $p_\zeta(a_i) = p_\zeta(b_i)$ for every $i \in I$, where p_ζ is the canonical projection from $\mathbb{R}_+$ onto the tree $\mathcal{T}_\zeta$ coded by $(\zeta_s(\omega))_{s\geq 0}$. Furthermore the path $\omega_{a_i} = \omega_{b_i}$ exits U exactly at its lifetime $\zeta_{a_i} = \zeta_{b_i}$. We can then define the excursion ω_i, for every $i \in I$, as the subtrajectory of ω associated with the interval $[a_i, b_i]$ (equivalently $W_s(\omega_i)$ is the finite path $(\omega_{(a_i+s)\wedge b_i}(\zeta_{a_i} + t))_{0\leq t\leq \zeta^i(s)}$ with lifetime $\zeta^i(s) = \zeta_{(a_i+s)\wedge b_i} - \zeta_{a_i}$, for every $s \geq 0$). The ω_i's are the "excursions" of the snake trajectory ω outside U—the word "outside" is a little misleading here, because although these excursions start from ∂U, they will typically come back inside U. We define the point measure of excursions of ω outside U by

$$\mathcal{P}^U(\omega) := \sum_{i \in I} \delta_{\omega_i}.$$

We also need to define the σ-field on $\mathcal{S}_x$ containing the information given by the paths ω_s before they exit U. To this end we slightly generalize the definition of truncations in Section 2.2. If $\omega \in \mathcal{S}_x$, we set

$$\mathrm{tr}^U(\omega)_s := \omega_{\eta^U_s}$$

where

$$\eta^U_s := \inf\left\{ r \geq 0 : \int_0^r \mathrm{d}t\, \mathbf{1}_{\{\zeta_t(\omega) \leq \tau^U(\omega_t)\}} > s \right\}.$$

Just as in Proposition 10, we can verify that this defines a measurable mapping from $\mathcal{S}_x$ into $\mathcal{S}_x$. We define the σ-field $\mathcal{E}^U_x$ on $\mathcal{S}_x$ as the σ-field generated by this mapping and completed by the measurable sets of $\mathcal{S}_x$ of $\mathbb{N}_x$-measure 0.

We can now state the special Markov property.

Proposition 13. *Let $x \in U$. The random measure $\mathcal{Z}^U$ is $\mathcal{E}^U_x$-measurable. Furthermore, under the probability measure $\mathbb{N}_x(\cdot \mid \mathcal{R} \cap U^c \neq \emptyset)$, conditionally on $\mathcal{E}^U_x$, the point measure $\mathcal{P}^U$ is Poisson with intensity*

$$\int \mathcal{Z}^U(\mathrm{d}y)\, \mathbb{N}_y(\cdot).$$

See [20, Theorem 2.4] for a proof in a much more general setting. Note that, on the event $\{\mathcal{R} \cap U^c = \emptyset\}$, there are no excursions outside U, and this is the reason why we restrict our attention to the event $\{\mathcal{R} \cap U^c \neq \emptyset\}$, which has finite $\mathbb{N}_x$-measure by (7) (in fact, since $\mathcal{Z}^U = 0$ on $\{\mathcal{R} \cap U^c = \emptyset\}$, we could as well give a statement similar to Proposition 13 without conditioning).

2.5. *The exit measure process*

We now specialize the discussion of the previous subsection to the case $U = (y, \infty)$ and $x > y$. The exit measure $\mathcal{Z}^{(y,\infty)}$ is then a random multiple of the Dirac mass at y, and is determined by its total mass, which will be denoted by $\mathcal{Z}_y = \langle \mathcal{Z}^{(y,\infty)}, 1 \rangle$. We have

$$\{\mathcal{Z}_y > 0\} = \{W_* < y\} = \{W_* \leq y\}, \qquad \mathbb{N}_x\text{-a.e.}$$

Note that the identity $\{W_* < y\} = \{W_* \leq y\}$, $\mathbb{N}_x$-a.e., follows from the fact that the right-hand side of (7) is a continuous function of y. The fact that $\{\mathcal{Z}_y > 0\} = \{W_* < y\}$, $\mathbb{N}_x$-a.e., can then be deduced from the special Markov property (Proposition 13).

The Laplace transform of $\mathcal{Z}_y$ under $\mathbb{N}_x$ can be computed from the connections between exit measures and semilinear partial differential equations [21, Chapter V]. For every $\lambda > 0$,

$$\mathbb{N}_x(1 - \exp(-\lambda \mathcal{Z}_y)) = \left(\lambda^{-1/2} + \sqrt{2/3}\,(x - y)\right)^{-2}. \tag{9}$$

See [9, formula (6)] for a brief justification. Note that letting $\lambda \to \infty$ in (9) is consistent with (7). A consequence of (9) is that

$$\mathbb{N}_x(\mathcal{Z}_y) = 1. \tag{10}$$

Let us discuss Markovian properties of the process of exit measures. If $y' < y < x$, an application of the special Markov property combined with formula (9) gives, on the event $\{W_* \leq y\}$, for every $\lambda > 0$,

$$\begin{aligned}\mathbb{N}_x(\exp(-\lambda \mathcal{Z}_{y'}) \mid \mathcal{E}_x^{(y,\infty)}) &= \exp\big(-\mathcal{Z}_y \mathbb{N}_y(1-\exp(-\lambda \mathcal{Z}_{y'}))\big)\\ &= \exp\big(-\mathcal{Z}_y\big(\lambda^{-1/2}+\sqrt{2/3}\,(y-y')\big)^{-2}\big).\end{aligned}$$

It follows that the process $(\mathcal{Z}_{x-a})_{a>0}$ is Markovian under $\mathbb{N}_x$, with the transition kernels of the continuous-state branching process with branching mechanism $\psi(\lambda) = \sqrt{8/3}\,\lambda^{3/2}$ (see e.g. [9, Section 2.1] for the definition and properties of this process). Although $\mathbb{N}_x$ is an infinite measure, the previous statement makes sense by arguing on the event $\{W_* \leq x-\delta\}$, which has finite $\mathbb{N}_x$-measure for any $\delta > 0$, and considering $(\mathcal{Z}_{x-\delta-a})_{a\geq 0}$.

We will use an approximation of $\mathcal{Z}_y$ by $\mathcal{E}_x^{(y,\infty)}$-measurable random variables (notice that this is not the case for (8)). Recall our notation $\tau_y(\mathrm{w}) := \inf\{t \in [0, \zeta_{(\mathrm{w})}] : \mathrm{w}(t) = y\}$ for $\mathrm{w} \in \mathcal{W}$.

Lemma 14. *Let $y < x$. We have*

$$\varepsilon^{-2}\int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{\zeta_s \leq \tau_y(W_s),\, \hat{W}_s < y+\varepsilon\}} \xrightarrow[\varepsilon\to 0]{} \mathcal{Z}_y$$

where the convergence holds in probability under $\mathbb{N}_x(\cdot \mid W_* \leq y)$.

Proof. This follows from arguments similar to [9, Section 4.1, proof of Proposition 1.1], and we only sketch the proof. For every $\varepsilon > 0$, set

$$\Lambda_\varepsilon = \int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{\zeta_s \leq \tau_y(W_s),\, \hat{W}_s < y+\varepsilon\}}.$$

If $\varepsilon \in (0, x-y)$, the special Markov property applied to the domain $(y+\varepsilon, \infty)$ shows that the conditional distribution of Λ_ε, under $\mathbb{N}_x(\cdot \mid W_* \leq y+\varepsilon)$ and knowing $\mathcal{E}^{(y+\varepsilon,\infty)}$, is the law of $S_\varepsilon(\mathcal{Z}_{y+\varepsilon})$, where $(S_\varepsilon(t))_{t\geq 0}$ is a subordinator whose Lévy measure is the law of Λ_ε under $\mathbb{N}_{y+\varepsilon}$ (recall the comments following Proposition 2 about laws of random variables under σ-finite measures), and S_ε is assumed to be independent of $\mathcal{Z}_{y+\varepsilon}$. The first-moment formula for the Brownian snake (6) gives $\mathbb{N}_{y+\varepsilon}(\Lambda_\varepsilon) = \varepsilon^2$, so that $S_\varepsilon(t)$ has mean $\varepsilon^2 t$. On the other hand, scaling arguments entail that $(S_\varepsilon(t))_{t\geq 0}$ has the same distribution as $(\varepsilon^4 S_1(\varepsilon^{-2}t))_{t\geq 0}$. Hence, under $\mathbb{N}_x(\cdot \mid W_* \leq y+\varepsilon)$ and conditionally on $\mathcal{E}^{(y+\varepsilon,\infty)}$, $\varepsilon^{-2}\Lambda_\varepsilon$ has the law of $\varepsilon^2 S_1(\varepsilon^{-2}\mathcal{Z}_{y+\varepsilon})$, and the latter random variable is close in probability to $\mathcal{Z}_{y+\varepsilon}$ by the law of large numbers ($t^{-1}S_1(t)$ converges in probability to 1 as $t \to \infty$). The result of the lemma follows since $\mathcal{Z}_{y+\varepsilon}$ converges to $\mathcal{Z}_y$ in probability when $\varepsilon \to 0$. ◻

We note that the quantities $\int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{\zeta_s \leq \tau_y(W_s),\, \hat{W}_s < y+\varepsilon\}}$ are functions of the truncation $\mathrm{tr}_y(\omega)$, and therefore $\mathcal{E}_x^{(y,\infty)}$-measurable. As a consequence of Lemma 14, we can fix a sequence $(\alpha_n)_{n\geq 1}$ of positive reals converging to 0 such that

$$\mathcal{Z}_y = \lim_{n\to\infty} \alpha_n^{-2}\int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{\zeta_s \leq \tau_y(W_s),\, \hat{W}_s < y+\alpha_n\}}, \qquad \mathbb{N}_x\text{-a.e.} \tag{11}$$

and we can even choose the sequence $(\alpha_n)_{n\geq 1}$ independently of the pair (x, y) such that $y < x$ (observe that if (11) holds for $y = x - \delta$, then an application of the special Markov property (Proposition 13) shows that it holds for every $y \in (-\infty, x - \delta])$. It will be convenient to define $\mathcal{Z}_y(\omega)$ for every $\omega \in \mathcal{S}_x$, by setting

$$\mathcal{Z}_y(\omega) = \liminf_{n\to\infty} \alpha_n^{-2} \int_0^{\sigma(\omega)} \mathrm{d}s\, \mathbf{1}_{\{\zeta_s(\omega)\leq\tau_y(W_s(\omega)),\, \hat{W}_s(\omega)<y+\alpha_n\}}.$$

By the previous considerations, this definition is consistent with (8) up to an $\mathbb{N}_x$-negligible set. Furthermore, we have $\mathcal{Z}_y(\omega) = \mathcal{Z}_y(\mathrm{tr}_y(\omega))$ for every $\omega \in \mathcal{S}_x$.

In much of what follows, we will argue under the measure $\mathbb{N}_0$, and we simply write $\mathcal{E}^{(y,\infty)}$ instead of $\mathcal{E}_0^{(y,\infty)}$, for every $y < 0$. For $\omega \in \mathcal{S}_0$, we use the notation

$$Z_a(\omega) = \mathcal{Z}_{-a}(\omega)$$

for every $a > 0$. Because continuous-state branching processes are Feller processes, we know that the process $(Z_a)_{a>0}$ has a càdlàg modification under $\mathbb{N}_0$, and we will always consider this modification. We call $(Z_a)_{a>0}$ the *exit measure process*.

We will need some bounds on the moments of Z_a. By (10), we already know that $\mathbb{N}_0(Z_a) = 1$ for every $a > 0$. Moreover, an application of the special Markov property shows that the process $(Z_{\delta+a})_{a\geq 0}$ is a martingale under $\mathbb{N}_0(\cdot \mid W_* \leq -\delta)$ for every $\delta > 0$ (this also follows from the fact that $\psi(\lambda) = \sqrt{8/3}\,\lambda^{3/2}$ is a critical branching mechanism).

Lemma 15. *Let $p \in (1, 3/2)$. For every $0 < b \leq a$, we have $\mathbb{N}_0((Z_b)^p) \leq \mathbb{N}_0((Z_a)^p) < \infty$.*

Proof. Write $\mathbb{N}_0^{(a)} := \mathbb{N}_0(\cdot \mid W_* \leq -a)$ to simplify notation. As a consequence of (9) and (7), we see that, for every $\lambda > 0$,

$$\mathbb{N}_0^{(a)}(e^{-\lambda Z_a}) = 1 - \left(1 + a^{-1}\sqrt{\frac{3}{2\lambda}}\right)^{-2},$$

and also $\mathbb{N}_0^{(a)}(Z_a) = 2a^2/3$. From a Taylor expansion, we get

$$\mathbb{N}_0^{(a)}(e^{-\lambda Z_a}) - (1 - \lambda\mathbb{N}_0^{(a)}(Z_a)) = 2(2/3)^{3/2}a^3\lambda^{3/2} + o(\lambda^{3/2})$$

as $\lambda \to 0$. By [6, Theorem 8.1.6], this implies the existence of a constant C such that $\mathbb{N}_0^{(a)}(Z_a > x) \leq Cx^{-3/2}$ for every $x > 0$. Thus, $\mathbb{N}_0^{(a)}((Z_a)^p) < \infty$ if $1 < p < 3/2$.

Finally, if $b \in (0, a)$, by using the martingale property of the exit measure process we get

$$\mathbb{N}_0((Z_b)^p) = \frac{3}{2b^2}\mathbb{N}_0^{(b)}((Z_b)^p) \leq \frac{3}{2b^2}\mathbb{N}_0^{(b)}((Z_a)^p) = \mathbb{N}_0((Z_a)^p) < \infty. \qquad \square$$

2.6. *A technical lemma*

We finally give a technical lemma concerning local minima of the process $\hat{W}$.

Lemma 16. $\mathbb{N}_0$-*a.e., there exists no value of* $s \in (0, \sigma)$ *such that:*

(i) s *is a time of local minimum of* $\hat{W}$, *in the sense that there exists* $\varepsilon > 0$ *such that* $\hat{W}_r \geq \hat{W}_s$ *for every* $r \in (s - \varepsilon, s + \varepsilon)$.
(ii) $\hat{W}_s = \underline{W}_s$ *and there exists* $t \in (0, \zeta_s)$ *such that* $W_s(t) = \underline{W}_s$.

Proof. The proof uses more involved properties of the Brownian snake, which we have not recalled but for which we refer the reader to [21]. We start by observing that, for any reals $y < x$, we have, $\mathbb{N}_x$-a.e.,

$$\inf\{s \geq 0 : \hat{W}_s < y\} = \inf\{s \geq 0 : \hat{W}_s \leq y\}. \tag{12}$$

In other words, when the Brownian snake hits y, it immediately hits values strictly smaller than y. See [21, proof of Theorem VI.9] for an argument in a more general setting.

Then, fix $\mathrm{w} \in \mathcal{W}_0$ and let $(W'_s)_{s\geq 0}$ be a Brownian snake that starts from w under the probability measure $\mathbb{P}_{\mathrm{w}}$ (we write W'_s and not W_s because $\mathbb{P}_{\mathrm{w}}$ is not defined on the space $\mathcal{S}$ of snake trajectories). We let $(\zeta'_s)_{s\geq 0}$ be the lifetime process of $(W'_s)_{s\geq 0}$. Suppose that there is a unique time $t_0 \in (0, \zeta_{(\mathrm{w})})$ such that $\mathrm{w}(t_0) = \underline{\mathrm{w}}$, and introduce the stopping time

$$\tau := \inf\{s \geq 0 : \zeta'_s \leq t_0\}.$$

Notice that the path W'_τ is equal to the restriction of w to $[0, t_0]$, and thus $\hat{W}'_\tau = \mathrm{w}(t_0) = \underline{\mathrm{w}}$. We then claim that, $\mathbb{P}_{\mathrm{w}}$-a.s. on the event where $\inf\{s > 0 : \hat{W}'_s \leq \underline{\mathrm{w}}\} < \tau$, we have

$$\inf\{s > 0 : \hat{W}'_s \leq \underline{\mathrm{w}}\} = \inf\{s > 0 : \hat{W}'_s < \underline{\mathrm{w}}\}.$$

This follows by using the subtree decomposition of the Brownian snake started at w (see [21, Lemma V.5]) together with property (12) above.

We can now combine the previous observations with the Markov property of the Brownian snake under $\mathbb{N}_0$. We find that $\mathbb{N}_0$-a.e. for every rational $r \in (0, \sigma)$ such that $t \mapsto W_r(t)$ attains its minimum at a (necessarily unique) time $t_0 \in (0, \zeta_r)$, the property

$$\inf\{s > r : \hat{W}_s \leq \underline{W}_r\} < \inf\{s \geq r : \zeta_s \leq t_0\}$$

implies

$$\inf\{s > r : \hat{W}_s < \underline{W}_r\} = \inf\{s > r : \hat{W}_s \leq \underline{W}_r\}. \tag{13}$$

Let us show that this implies the statement of the lemma. Towards a contradiction, assume that there is a value $s_0 \in (0, \sigma)$ such that (i) and (ii) hold for $s = s_0$. Write t_0 for the (unique) time in $(0, \zeta_{s_0})$ such that $W_{s_0}(t_0) = \underline{W}_{s_0}$ and choose $\delta > 0$ such that $t_0 < \zeta_{s_0} - \delta$. Then, using (i) for $s = s_0$ and the properties of the Brownian snake, we can find a rational $r < s_0$ sufficiently close to s_0 such that, for some $\chi > 0$,

(a) $\hat{W}_s \geq \hat{W}_{s_0}$ for every $s \in [r, s_0 + \chi]$;
(b) $\zeta_r + \delta/2 > \zeta_s > \zeta_r - \delta/2$ for every $s \in [r, s_0]$.

We note that W_r coincides with W_{s_0} at least up to time $\zeta_r - \delta/2 > \zeta_{s_0} - \delta > t_0$. In particular t_0 is also the unique time of the minimum of $t \mapsto W_r(t)$ on $(0, \zeta_r)$, and $\underline{W}_r = \underline{W}_{s_0} = \hat{W}_{s_0}$ (it already follows from (a) that $\underline{W}_r \geq \hat{W}_{s_0}$). Property (b) then gives

$$\inf\{s > r : \hat{W}_s \leq \underline{W}_r\} \leq s_0 < \inf\{s \geq r : \zeta_s \leq t_0\}.$$

This allows us to apply (13) to get

$$\inf\{s > r : \hat{W}_s < \underline{W}_r\} = \inf\{s > r : \hat{W}_s \leq \underline{W}_r\} \leq s_0.$$

Since $\underline{W}_r = \hat{W}_{s_0}$, this contradicts (a), which completes the proof. □

3. Construction of the excursion measure above the minimum

The main goal of this section is to construct the positive excursion measure $\mathbb{N}_0^*$. For this construction, we will be arguing under the measure $\mathbb{N}_0$. Several properties stated below hold only outside an $\mathbb{N}_0$-negligible set, but we will frequently omit the words $\mathbb{N}_0$-a.e. Recall the notation $\mathcal{T}_\zeta$ for the random real tree coded by $(\zeta_s)_{s\geq 0}$, and $\mathrm{Sk}(\mathcal{T}_\zeta)$ for the skeleton of $\mathcal{T}_\zeta$. If $u \in \mathcal{T}_\zeta$ and $s \geq 0$ is such that $p_\zeta(s) = u$, we have already noticed that W_s does not depend on the choice of s, and it will be convenient to write $V_u = \hat{W}_s$. Then V_u is interpreted as the label or spatial position of u.

Definition 17. A vertex $u \in \mathcal{T}_\zeta$ is an *excursion debut above the minimum* if:

(1) $u \in \mathrm{Sk}(\mathcal{T}_\zeta)$;
(2) $V_u = \min\{V_v : v \in [\![\rho, u]\!]\}$;
(3) u has a strict descendant w such that such that $V_v > V_u$ for all $v \in]\!]u, w]\!]$.

We write D for the set of all excursion debuts above the minimum. If $u \in D$, then V_u is called the *level* of the excursion debut u.

In what follows, except in Section 8, we will be interested only in excursions above the minimum, and for this reason we will say "excursion debut" instead of "excursion debut above the minimum". By definition, excursion debuts belong to the skeleton of $\mathcal{T}_\zeta$. Clearly, $\mathbb{N}_0$-a.e., the root ρ is not an excursion debut (it is easy to see that (3) fails for $u = \rho$) and we have $V_u < 0$ for every $u \in D$. Furthermore, the quantities $V_u, u \in D$, are pairwise distinct, $\mathbb{N}_0$-a.e., as a consequence of the fact that local minima of Brownian paths are a.s. distinct (this implies that two local minima of labels that correspond to disjoint segments of the tree $\mathcal{T}_\zeta$ must be distinct).

Lemma 18. $\mathbb{N}_0$*-a.e., no branching point is an excursion debut.*

Proof. Any branching point can be represented as $p_\zeta(r)$, where $r \in (s,t)$ and $\zeta_r = \min\{\zeta_{r'} : s \leq r' \leq t\}$, for rationals s and t such that $0 < s < t < \sigma$. Then, for any strict descendant w of $p_\zeta(r)$, the historical path of w coincides either with W_s or with W_t, up to a time (strictly) greater than ζ_r. Since, conditionally on the lifetime process ζ, W_s is just a Brownian path over the time interval $[0, \zeta_s]$, it must take values smaller than $W_s(\zeta_r)$ immediately after time ζ_r, a.s., and the same holds for W_t. We conclude that $p_\zeta(r)$ is a.s. not an excursion debut, and by varying s and t we get the desired result outside a countable union of negligible sets. □

Let u be an excursion debut. We set

$$C_u = \{w \in \mathcal{T}_\zeta : u \prec w \text{ and } V_v > V_u,\ \forall v \in \rrbracket u, w \llbracket\},$$

where we recall that the notation $v \prec w$ means that v is an ancestor of w. Note that $u \in C_u$ and that saying that u is an excursion debut implies that $C_u \neq \{u\}$. We clearly have $V_w \geq V_u$ for every $w \in C_u$. Also, if $w \in C_u$, then $w' \in C_u$ for every $w' \in \llbracket u, w \llbracket$.

Lemma 19. $\mathbb{N}_0$*-a.e., for every* $u \in D$*, the set* C_u *is a closed subset of* $\mathcal{T}_\zeta$*, and its interior is*

$$\mathrm{Int}(C_u) = \{w \in C_u : V_w > V_u\}. \tag{14}$$

Proof. The closedness is easy: if (w_n) is a sequence in C_u that converges to w for the metric of $\mathcal{T}_\zeta$, then $u \prec w$ and the "interval" $\rrbracket u, w \llbracket$ is contained in the union of the intervals $\rrbracket u, w_n \llbracket$.

To verify (14), first note that the set $\{w \in C_u : V_w > V_u\}$ is open (if w belongs to this set and if w' is sufficiently close to w, then w' is still a descendant of u and $V_v > V_u$ for all $v \in \rrbracket u, w' \rrbracket$).

We also need to check that if $w \in C_u$ and $V_w = V_u$, then w does not belong to the interior of C_u. Consider first the case $w = u$. Letting s_1 be the first time such that $p_\zeta(s) = u$, u belonging to the interior of C_u would imply that $\hat{W}_s \geq \hat{W}_{s_1} = V_u$ for all $s \geq s_1$ sufficiently close to s_1. But then s_1 would a point of (right) increase for both ζ and $\hat{W}$, and by [22, Lemma 2.2] we know that this cannot occur. Suppose then that $w \in C_u$, $V_w = V_u$ and $w \neq u$. Let $s \in (0, \sigma)$ be such that $p_\zeta(s) = w$. Then property (ii) of Lemma 16 holds, and thus property (i) of that lemma cannot hold. This shows that for any neighborhood $\mathcal{N}$ of w we can find $w' \in \mathcal{N}$ such that $V_{w'} < V_u$, and therefore $w' \notin C_u$. □

Proposition 20. $\mathbb{N}_0$*-a.e., the sets* $\mathrm{Int}(C_u)$*, when* u *varies in* D*, are exactly the connected components of the open set* $\{w \in \mathcal{T}_\zeta : V_w > \min\{V_v : v \in \llbracket \rho, w \rrbracket\}\}$.

Proof. If $w \in \mathcal{T}_\zeta$ is such that $V_w > \min\{V_v : v \in \llbracket \rho, w \rrbracket\}$, then $w \in \mathrm{Int}(C_u)$, where u is the (unique) ancestor of w such that $V_u = \min\{V_v : v \in \llbracket \rho, w \rrbracket\}$. This shows that $\{w \in \mathcal{T}_\zeta : V_w > \min\{V_v : v \in \llbracket \rho, w \rrbracket\}\}$ is the union of all sets $\mathrm{Int}(C_u)$ when u varies in D. Then, if $u \in D$ and w and w' are two vertices in $\mathrm{Int}(C_u)$, their last common ancestor also belongs to $\mathrm{Int}(C_u)$ (because u is not a branching point, by Lemma 18), and the whole interval $\llbracket w, w' \rrbracket$ is contained in $\mathrm{Int}(C_u)$. It follows that, for every $u \in D$, the set $\mathrm{Int}(C_u)$ is connected. Finally, if u and u' are two distinct vertices in D, the sets $\mathrm{Int}(C_u)$ and $\mathrm{Int}(C_{u'})$ are disjoint: We prove this by contradiction. Suppose that there exists $v \in \mathrm{Int}(C_u) \cap \mathrm{Int}(C_{u'})$. Then u and u' are both ancestors of v, hence u is an ancestor of u' (or u' is an ancestor of u). However, the properties $u \prec u' \prec v$ and $v \in \mathrm{Int}(C_u)$ imply that $V_{u'} > V_u$, which contradicts property (2) in the definition of an excursion debut. □

Remark. A minor modification of the end of the proof shows in fact that the sets C_u, $u \in D$, are pairwise disjoint, which is slightly stronger.

The last proposition implies that the set D is countable, which can also be seen directly.

Definition 21. If u is an excursion debut, we set

$$M_u := \sup\{V_v - V_u : v \in C_u\} > 0$$

and we call M_u the *height* of the excursion debut u. For every $\delta > 0$, we define $D_\delta := \{u \in D : M_u > \delta\}$.

Lemma 22. *Let $\delta > 0$. The set D_δ is finite $\mathbb{N}_0$-a.e.*

Proof. By a uniform continuity argument, there exists a (random) $\chi > 0$ such that, for any $v, v' \in \mathcal{T}_\zeta$, the condition $d_\zeta(v, v') \le \chi$ implies $|V_v - V_{v'}| \le \delta$. Then let $u \in D_\delta$, and let $v \in C_u$ be such that $V_v - V_u > \delta$. We claim that the ball of radius $\chi/2$ centered at v in $\mathcal{T}_\zeta$, which we denote by $B_{d_\zeta}(v, \chi/2)$, is contained in $\mathrm{Int}(C_u)$. If the claim holds, the result of the lemma follows since the sets $\mathrm{Int}(C_u)$ are disjoint when u varies (Proposition 20), and there can only be finitely many values of v such that the balls $B_{d_\zeta}(v, \chi/2)$ are disjoint.

To verify our claim, we first note that we must have $d_\zeta(u, v) > \chi$ by our choice of χ, and it follows that $B_{d_\zeta}(v, \chi/2)$ is contained in the set of descendants of u. Next, if $v' \in B_{d_\zeta}(v, \chi/2)$, then $V_w \ge V_v - \delta > V_u$ for every $w \in [\![v, v']\!]$, showing that $v' \in \mathrm{Int}(C_u)$ since $[\![u, v']\!] \subset [\![u, v]\!] \cup [\![v, v']\!]$. This gives our claim and completes the proof. □

Let u be an excursion debut. Since $u \in \mathrm{Sk}(\mathcal{T}_\zeta)$ and u is not a branching point, there are two uniquely defined times $0 < s_1 < s_2 < \sigma$ such that $p_\zeta(s_1) = p_\zeta(s_2) = u$. Note that $\hat{W}_{s_1} = \hat{W}_{s_2} = V_u$ and $\zeta_{s_1} = \zeta_{s_2} = d_\zeta(\rho, u)$. We then define a random snake trajectory $W^{(u)} \in \mathcal{S}_0$ as the image under the translation κ_{-V_u} of the subtrajectory of ω associated with the interval $[s_1, s_2]$ (recall that the latter subtrajectory corresponds to the spatial displacements of the descendants of u). Note that $W^{(u)}$ has duration $\sigma(W^{(u)}) = s_2 - s_1$. Alternatively, the tree-like path corresponding to $W^{(u)}$ is $(\zeta_{(s_1+s)\wedge s_2} - \zeta_{s_1}, \hat{W}_{(s_1+s)\wedge s_2} - V_u)_{s \ge 0}$. By the definition of D, each of the paths $W_s^{(u)}$, for $0 < s < s_2 - s_1$, stays strictly above 0 during a small interval $(0, \delta)$, for some $\delta > 0$. We are not in fact interested in the behavior of these paths after they return to 0 (if they do), and for this reason we introduce the *truncation of* $W^{(u)}$ *at* 0,

$$\tilde{W}^{(u)} := \mathrm{tr}_0(W^{(u)}),$$

with the notation introduced in Section 2.2. We also write $\tilde{\zeta}_s^{(u)}$ for the lifetime of $\tilde{W}_s^{(u)}$, for every $s \ge 0$. For every $s \in (0, \sigma(\tilde{W}^{(u)}))$, the path $\tilde{W}_s^{(u)}$ starts from 0, stays positive during the interval $(0, \tilde{\zeta}_s^{(u)})$, and may or may not return to 0 at time $\tilde{\zeta}_s^{(u)}$.

It follows from our definitions that the paths $\tilde{W}_s^{(u)}$, $0 \le s \le \sigma(\tilde{W}^{(u)})$, correspond to the historical paths after time $d_\zeta(\rho, u)$ of all vertices $v \in C_u$, provided these paths are shifted by $-V_u$ so that they start from 0. In particular, $M(\tilde{W}^{(u)}) = M_u$ is the height of the excursion debut u. We sometimes call $\tilde{W}^{(u)}$ the *excursion above the minimum starting from* u.

Before stating the main theorem of this section, we introduce one more piece of notation. On the canonical space $\mathcal{S}$, we let $\tilde{W} = \mathrm{tr}_0(W)$ stand for the truncation at 0 of the canonical process $(W_s)_{s\geq 0}$.

Theorem 23. *There exists a σ-finite measure denoted by $\mathbb{N}_0^*$ on the space $\mathcal{S}$, which is supported on $\mathcal{S}_0$, such that for every nonnegative measurable function Φ on $\mathbb{R}_+ \times \mathcal{S}$, we have*

$$\mathbb{N}_0\Big(\sum_{u\in D} \Phi(V_u, \tilde{W}^{(u)})\Big) = \int_{-\infty}^{0} d\ell \int \mathbb{N}_0^*(\mathrm{d}\omega)\, \Phi(\ell, \omega). \tag{15}$$

The measure $\mathbb{N}_0^*$ gives finite mass to the set $\mathcal{S}^{(\delta)} := \{\omega \in \mathcal{S} : \|\omega\| > \delta\}$, for every $\delta > 0$. Moreover, if G is a bounded continuous real function on $\mathcal{S}$, and if there exists $\delta > 0$ such that G vanishes on $\mathcal{S} \setminus \mathcal{S}^{(\delta)}$, we have

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\mathbb{N}_\varepsilon(G(\tilde{W})) = \mathbb{N}_0^*(G). \tag{16}$$

The proof of Theorem 23 relies on an important technical lemma, which we state after introducing some notation. We consider a fixed sequence $(\varepsilon_n)_{n\geq 1}$ of positive real numbers converging to 0. Let ε be an element of this sequence; then for every $\omega \in \mathcal{S}_0$ and every integer $k \geq 1$, we let $\mathcal{N}_k^\varepsilon(\omega)$ be the point measure of excursions of ω outside $(-k\varepsilon, \infty)$, and we write

$$\mathcal{N}_k^\varepsilon(\omega) = \sum_{i\in I_k^\varepsilon} \delta_{\omega_i^{k,\varepsilon}}.$$

By construction, for every $i \in I_k^\varepsilon$, $\omega_i^{k,\varepsilon}$ is a subtrajectory of ω, and we write $[r_i^{k,\varepsilon}, s_i^{k,\varepsilon}]$ for the corresponding interval. We will also use the notation $\tilde{\omega}_i^{k,\varepsilon}$ for $\omega_i^{k,\varepsilon}$ translated so that its starting point is ε and then truncated at level 0: with the notation of Section 2.2, $\tilde{\omega}_i^{k,\varepsilon} = \mathrm{tr}_0 \circ \kappa_{(k+1)\varepsilon}(\omega_i^{k,\varepsilon}) \in \mathcal{S}_\varepsilon$.

Recall our notation Z_a for the total mass of the exit measure from $(-a, \infty)$. By the special Markov property (Proposition 13), we know that the conditional distribution of $\mathcal{N}_k^\varepsilon$ under $\mathbb{N}_0(\cdot \mid Z_{k\varepsilon} \neq 0)$ and given $Z_{k\varepsilon}$ is that of a Poisson point measure with intensity

$$Z_{k\varepsilon}\mathbb{N}_{-k\varepsilon}(\cdot).$$

On the other hand, we have $\mathcal{N}_k^\varepsilon(\omega) = 0$, $\mathbb{N}_0$-a.e. on $\{Z_{k\varepsilon} = 0\}$.

Lemma 24. *The following properties hold $\mathbb{N}_0$-a.e. Let $u \in D$, and let $0 < s_1 < s_2 < \sigma$ be determined by $p_\zeta(s_1) = p_\zeta(s_2) = u$. Then, for every sufficiently small ε in the sequence $(\varepsilon_n)_{n\geq 1}$, if $k_{u,\varepsilon} \geq 1$ is the integer determined by $-(k_{u,\varepsilon} + 1)\varepsilon < V_u \leq -k_{u,\varepsilon}\varepsilon$, there exists a unique index $i_{u,\varepsilon} \in I_{k_{u,\varepsilon}}^\varepsilon$ such that*

$$(s_1, s_2) \subset (r_{i_{u,\varepsilon}}^{k_{u,\varepsilon},\varepsilon}, s_{i_{u,\varepsilon}}^{k_{u,\varepsilon},\varepsilon}),$$

and we have

$$\tilde{\omega}_{i_{u,\varepsilon}}^{k_{u,\varepsilon},\varepsilon} \to \tilde{W}^{(u)} \quad \text{as } \varepsilon \to 0 \text{ along the sequence } (\varepsilon_n)_{n\geq 1}.$$

Remark. The convergence in the last assertion of the lemma holds in $\mathcal{S}$, on noting that $\tilde{W}^{(u)} \in \mathcal{S}_0$ whereas $\tilde{\omega}^{k_{u,\varepsilon},\varepsilon}_{i_{u,\varepsilon}} \in \mathcal{S}_\varepsilon$.

Proof of Lemma 24. Note that a priori we could have $k_{u,\varepsilon} = 0$, but this does not occur for ε small enough since $V_u < 0$. Then the index $i_{u,\varepsilon}$ is determined by the fact that the excursion $\omega^{k_{u,\varepsilon},\varepsilon}_{i_{u,\varepsilon}}$ corresponds to the descendants of the first ancestor of u at spatial position $-k_{u,\varepsilon}\varepsilon$. More specifically, the index $i_{u,\varepsilon}$ is determined by

$$r^{k_{u,\varepsilon},\varepsilon}_{i_{u,\varepsilon}} = \sup\{s \le s_1 : \zeta_s \le \tau_{-k_{u,\varepsilon}\varepsilon}(W_{s_1})\} \tag{17}$$

where we recall the notation $\tau_a(\mathrm{w}) = \inf\{t \ge 0 : \mathrm{w}(t) = a\}$. Since the image under p_ζ of the interval $(r^{k_{u,\varepsilon},\varepsilon}_{i_{u,\varepsilon}}, s^{k_{u,\varepsilon},\varepsilon}_{i_{u,\varepsilon}})$ corresponds to descendants of an ancestor of u, the inclusion

$$(s_1, s_2) \subset (r^{k_{u,\varepsilon},\varepsilon}_{i_{u,\varepsilon}}, s^{k_{u,\varepsilon},\varepsilon}_{i_{u,\varepsilon}})$$

is immediate. For the last property of the lemma, we first verify that

$$r^{k_{u,\varepsilon},\varepsilon}_{i_{u,\varepsilon}} \to s_1, \qquad s^{k_{u,\varepsilon},\varepsilon}_{i_{u,\varepsilon}} \to s_2 \tag{18}$$

as $\varepsilon \to 0$ along the sequence $(\varepsilon_n)_{n\ge1}$.

To this end, let s be such that $0 < s < s_1$, and observe that then

$$\inf_{r\in[s,s_1]} \zeta_r < \zeta_{s_1}$$

(otherwise u would be a branching point). On the other hand, for any $\gamma > 0$, there exists $\chi > 0$ such that $W_{s_1}(t) \ge V_u + \chi$ if $0 \le t \le \zeta_{s_1} - \gamma$ (by property (2) of the definition of an excursion debut, and the fact that a Brownian path cannot have two local minima at the same level). It follows that $\tau_{-k_{u,\varepsilon}\varepsilon}(W_{s_1}) \to \zeta_{s_1}$ as $\varepsilon \to 0$, and together with (17) the preceding observations imply that $r^{k_{u,\varepsilon},\varepsilon}_{i_{u,\varepsilon}} > s$ for ε small enough, giving the desired convergence $r^{k_{u,\varepsilon},\varepsilon}_{i_{u,\varepsilon}} \to s_1$. The proof of the other convergence $s^{k_{u,\varepsilon},\varepsilon}_{i_{u,\varepsilon}} \to s_2$ is analogous.

Once we have obtained the convergences (18), we deduce the last assertion of the lemma from Lemma 12. With the notation of that lemma, we take $\omega' = W^{(u)}$ and $\omega^{(n)} = \kappa_{-V_u}(\omega^{k_n,\varepsilon_n}_{i_n})$, where we write $k_n = k_{u,\varepsilon_n}$ and $i_n = i_{u,\varepsilon_n}$ to simplify notation. We also take $\delta_n = -(k_n + 1)\varepsilon_n - V_u \in (-\varepsilon_n, 0)$. The conclusion of the lemma then implies that $\mathrm{tr}_{\delta_n}(\omega^{(n)})$ converges to $\mathrm{tr}_0(\omega') = \tilde{W}^{(u)}$. This is the result we need since one easily checks that $\mathrm{tr}_{\delta_n}(\omega^{(n)})$ coincides with $\tilde{\omega}^{k_n,\varepsilon_n}_{i_n}$ translated by δ_n. We still need to verify that assumptions (i)–(iii) of Lemma 12 hold with our choice of ω'. Assumptions (i) and (ii) hold by the definition of an excursion debut. Assumption (iii) holds because otherwise there would be two distinct local minimum times corresponding to the same local minimum of a path W_s, which is impossible. This completes the proof of the last assertion of the lemma. □

Proof of Theorem 23. In order to prove the first part of the theorem, it is enough to construct the σ-finite measure $\mathbb{N}_0^*$ such that the identity (15) holds whenever $\Phi(\ell, \omega) = g(\ell)G(\omega)$, where g and G are nonnegative measurable functions defined on $\mathbb{R}$ and on $\mathcal{S}$ respectively. We fix two such functions g and G, and, in a first step, we assume that both

g and G are bounded and continuous and take nonnegative values. Moreover, we assume that g is nontrivial and is supported on a compact subinterval of $(-\infty, 0)$, and that there exists $\delta > 0$ such that $G(\omega) = 0$ if $\omega \notin \mathcal{S}^{(\delta)}$. The functions G and g will be fixed until the last lines of the proof, where we explain how to get rid of the extra assumptions on G and g.

By our assumptions on G, the quantity $G(\tilde{W}^{(u)})$ is zero if $u \notin D_\delta$, and a fortiori if $u \notin D_{\delta/2}$. Since $D_{\delta/2}$ is a.e. finite (Lemma 22), using the notation and the conclusion of Lemma 24 we get

$$\sum_{u\in D} g(V_u)G(\tilde{W}^{(u)}) = \sum_{u\in D_{\delta/2}} g(V_u)G(\tilde{W}^{(u)}) = \lim_{\varepsilon\to 0} \sum_{u\in D_{\delta/2}} g(-\varepsilon k_{u,\varepsilon})G(\tilde{\omega}_{i_{u,\varepsilon}}^{k_{u,\varepsilon},\varepsilon}),$$

$\mathbb{N}_0$-a.e. (here and in the remaining part of the proof, we consider only values of ε in the sequence $(\varepsilon_n)_{n\geq 1}$, even if this is not mentioned explicitly). We next observe that

$$\sum_{u\in D_{\delta/2}} g(-\varepsilon k_{u,\varepsilon})G(\tilde{\omega}_{i_{u,\varepsilon}}^{k_{u,\varepsilon},\varepsilon}) = \sum_{k=1}^{\infty}\sum_{i\in I_k^\varepsilon} g(-\varepsilon k)G(\tilde{\omega}_i^{k,\varepsilon}) \tag{19}$$

for ε small enough, $\mathbb{N}_0$-a.e. To see this, suppose that $\varepsilon < \delta/2$, and fix $k \geq 1$ and $i \in I_k^\varepsilon$. If $G(\tilde{\omega}_i^{k,\varepsilon}) \neq 0$, there exists a real $s \geq 0$ such that the path $W_s(\tilde{\omega}_i^{k,\varepsilon})$ hits level δ. This also means that there exists a real $s' \geq 0$ such that the path $W_{s'}(\omega_i^{k,\varepsilon})$ hits $-(k+1)\varepsilon+\delta$ before hitting $-(k+1)\varepsilon$, and we can take the smallest such s'. Let s'' be such that $W_{s''}(\omega_i^{k,\varepsilon})$ coincides with $W_{s'}(\omega_i^{k,\varepsilon})$ truncated at the (unique) time where it reaches its minimum before hitting $-(k+1)\varepsilon+\delta$ (in the tree coded by $\zeta(\omega_i^{k,\varepsilon})$, s'' corresponds to the unique ancestor with minimal spatial position of the vertex s'). Then it follows from our definitions that $u := p_\zeta(r_i^{k,\varepsilon} + s'')$ is an excursion debut, with $k_{u,\varepsilon} = k$ and $i_{u,\varepsilon} = i$ by construction, and the height of u is at least $\delta - \varepsilon > \delta/2$, so that $u \in D_{\delta/2}$. Thus any (nonzero) term appearing in the right-hand side of (19) also appears, at least once, in the left-hand side. To complete the proof of (19), we must still verify that, for ε small enough, no (nonzero) term in the right-hand side appears twice in the left-hand side. But this follows from the fact that the values of V_u for $u \in D$ are all distinct: since $D_{\delta/2}$ is finite, for ε small enough, there cannot be two distinct elements u, u' of $D_{\delta/2}$ such that V_u and $V_{u'}$ lie in the same interval $(-(k+1)\varepsilon, -k\varepsilon)$.

From the preceding considerations, we get

$$\sum_{u\in D} g(V_u)G(\tilde{W}^{(u)}) = \lim_{\varepsilon\to 0}\sum_{k=1}^{\infty}\sum_{i\in I_k^\varepsilon} g(-\varepsilon k)G(\tilde{\omega}_i^{k,\varepsilon}),$$

$\mathbb{N}_0$-a.e. We then notice that we can fix $\chi > 0$ such that $g(x) = 0$ if $x \geq -\chi$, and restrict our attention to the set $\{W_* \leq -\chi\}$, which has finite $\mathbb{N}_0$-measure. The next step is to deduce from the preceding convergence that also

$$\mathbb{N}_0\Big(\sum_{u\in D} g(V_u)G(\tilde{W}^{(u)})\Big) = \lim_{\varepsilon\to 0}\mathbb{N}_0\Big(\sum_{k=1}^{\infty}\sum_{i\in I_k^\varepsilon} g(-\varepsilon k)G(\tilde{\omega}_i^{k,\varepsilon})\Big). \tag{20}$$

For this, some uniform integrability is needed. For every integer $k \geq 1$, set

$$n_k^\varepsilon := \sum_{i \in I_k^\varepsilon} \mathbf{1}_{\{\|\tilde{\omega}_i^{k,\varepsilon}\| > \delta\}}.$$

Recalling our assumptions on g and G, we see that in order to deduce (20) from the preceding convergence, it suffices to verify that, for $p \in (1, 3/2)$, and for every $A > \chi$,

$$\mathbb{N}_0\Big[\Big(\sum_{k=\lfloor \chi/\varepsilon \rfloor + 1}^{\lfloor A/\varepsilon \rfloor} n_k^\varepsilon\Big)^p\Big] \tag{21}$$

is bounded independently of ε. By the special Markov property (Proposition 13), conditionally on the σ-field $\mathcal{E}^{(-k\varepsilon,\infty)}$, n_k^ε is Poisson with intensity $c_{\varepsilon,\delta} Z_{k\varepsilon}$, where $c_{\varepsilon,\delta} = \mathbb{N}_\varepsilon(\|\tilde{W}\| > \delta)$. In particular,

$$\mathbb{N}_0((n_k^\varepsilon - c_{\varepsilon,\delta} Z_{k\varepsilon})^2 \mid \mathcal{E}^{(-k\varepsilon,\infty)}) = c_{\varepsilon,\delta} Z_{k\varepsilon}$$

and

$$\mathcal{M}_k^\varepsilon := \sum_{j=\lfloor \chi/\varepsilon \rfloor + 1}^{k} (n_j^\varepsilon - c_{\varepsilon,\delta} Z_{j\varepsilon}), \qquad k \geq \lfloor \chi/\varepsilon \rfloor,$$

is a martingale with respect to the filtration $(\mathcal{E}^{(-(k+1)\varepsilon,\infty)})_{k \geq \lfloor \chi/\varepsilon \rfloor}$—note that, by the construction of the truncated excursions $\tilde{\omega}_i^{k,\varepsilon}$, n_k^ε is $\mathcal{E}^{(-(k+1)\varepsilon,\infty)}$-measurable. The discrete Burkholder–Davis–Gundy inequalities (see e.g. [30, Théorème 5]) now give, for $p \in (1, 3/2)$ and for some constant $K_{(p)}$ depending only on p,

$$\begin{aligned}
\mathbb{N}_0(|\mathcal{M}_{\lfloor A/\varepsilon \rfloor}^\varepsilon|^p) &\leq K_{(p)} \mathbb{N}_0\Big[\Big(\sum_{j=\lfloor \chi/\varepsilon \rfloor + 1}^{\lfloor A/\varepsilon \rfloor} (\mathcal{M}_j^\varepsilon - \mathcal{M}_{j-1}^\varepsilon)^2\Big)^{p/2}\Big] \\
&\leq K_{(p)} \mathbb{N}_0(M_* < -\chi)^{1-p/2} \Big(\mathbb{N}_0\Big[\sum_{j=\lfloor \chi/\varepsilon \rfloor + 1}^{\lfloor A/\varepsilon \rfloor} (\mathcal{M}_j^\varepsilon - \mathcal{M}_{j-1}^\varepsilon)^2\Big]\Big)^{p/2} \\
&= K_{(p)} \mathbb{N}_0(M_* < -\chi)^{1-p/2} c_{\varepsilon,\delta}^{p/2} \Big(\mathbb{N}_0\Big[\sum_{j=\lfloor \chi/\varepsilon \rfloor + 1}^{\lfloor A/\varepsilon \rfloor} Z_{j\varepsilon}\Big]\Big)^{p/2} \\
&= K_{(p)} \mathbb{N}_0(M_* < -\chi)^{1-p/2} c_{\varepsilon,\delta}^{p/2} (\lfloor A/\varepsilon \rfloor - \lfloor \chi/\varepsilon \rfloor)^{p/2},
\end{aligned} \tag{22}$$

by using Jensen's inequality (with respect to the probability measure $\mathbb{N}_0(\cdot \mid W_* < -\chi)$) in the second line, and in the last line the fact that $\mathbb{N}_0(Z_r) = 1$ for every $r > 0$ (see (10)).

Then observe that

$$\begin{aligned}
c_{\varepsilon,\delta} &= \mathbb{N}_\varepsilon(\|\tilde{W}\| > \delta) = \mathbb{N}_\varepsilon(M(\tilde{W}) > \delta) = \mathbb{N}_\varepsilon(\langle \mathcal{Z}^{(0,\delta)}, \mathbf{1}_{\{\delta\}} \rangle > 0) \\
&= \mathbb{N}_0(\langle \mathcal{Z}^{(-\varepsilon,\delta-\varepsilon)}, \mathbf{1}_{\{\delta-\varepsilon\}} \rangle > 0),
\end{aligned}$$

where the third equality follows from the special Markov property (Proposition 13). It follows from [24, Section 4, formula (9)], together with a monotonicity argument, that

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} c_{\varepsilon,\delta} = c_0 \delta^{-3}, \tag{23}$$

where c_0 is a positive constant (made explicit in Lemma 25 below). In particular, there exists a constant $c_\delta < \infty$ such that $c_{\varepsilon,\delta} \le c_\delta \varepsilon$ for every $\varepsilon < \delta/2$. From (22), we then infer that the quantities $\mathbb{N}_0(|\mathcal{M}^\varepsilon_{\lfloor A/\varepsilon \rfloor}|^p)$ are uniformly bounded when $\varepsilon < \delta/2$. Finally, we write

$$\sum_{k=\lfloor \chi/\varepsilon \rfloor+1}^{\lfloor A/\varepsilon \rfloor} n^\varepsilon_k = \mathcal{M}^\varepsilon_{\lfloor A/\varepsilon \rfloor} + c_{\varepsilon,\delta} \sum_{k=\lfloor \chi/\varepsilon \rfloor+1}^{\lfloor A/\varepsilon \rfloor} Z_{k\varepsilon}$$

and we use again the bound $c_{\varepsilon,\delta} \le c_\delta \varepsilon$ together with the fact that the random variables Z_a, $0 < a \le A$, are bounded in $L^p(\mathbb{N}_0)$ when $1 < p < 3/2$ (Lemma 15). This gives us the desired bound for the quantities in (21), and justifies the passage to the limit under the integral in (20)—incidentally, this also shows that the left-hand side of (20) is finite.

We then use the special Markov property once again to obtain

$$\begin{aligned}\mathbb{N}_0\Big(\sum_{k=1}^\infty \sum_{i\in I^\varepsilon_k} g(-\varepsilon k) G(\tilde{\omega}^{k,\varepsilon}_i)\Big) &= \sum_{k=1}^\infty g(-\varepsilon k)\mathbb{N}_0(Z_{-k\varepsilon}\mathbb{N}_\varepsilon(G(\tilde{W})))\\ &= \Big(\sum_{k=1}^\infty g(-\varepsilon k)\Big)\mathbb{N}_\varepsilon(G(\tilde{W})),\end{aligned}$$

where the last equality holds because $\mathbb{N}_0(Z_{-k\varepsilon}) = 1$, by (10). Now note that

$$\varepsilon \sum_{k=1}^\infty g(-\varepsilon k) \underset{\varepsilon\to 0}{\longrightarrow} \int_{-\infty}^0 g(x)\,\mathrm{d}x,$$

and so we deduce from (20) and the preceding two displays that

$$\varepsilon^{-1}\mathbb{N}_\varepsilon(G(\tilde{W})) \underset{\varepsilon\to 0}{\longrightarrow} K_G$$

where the limit $K_G < \infty$ is such that

$$\mathbb{N}_0\Big(\sum_{u\in D} g(V_u) G(\tilde{W}^{(u)})\Big) = K_G \int_{-\infty}^0 g(x)\,\mathrm{d}x.$$

We now set, for every measurable subset F of $\mathcal{S}$,

$$\mathbb{N}^*_0(F) := \frac{\mathbb{N}_0(\sum_{u\in D} g(V_u)\,\mathbf{1}_F(\tilde{W}^{(u)}))}{\int_{-\infty}^0 g(x)\,\mathrm{d}x}. \tag{24}$$

This defines a positive measure on $\mathcal{S}$, which is supported on $\mathcal{S}_0$ since $W^{(u)} \in \mathcal{S}_0$ for every $u \in D$. Furthermore,

$$\mathbb{N}^*_0(G) = K_G < \infty.$$

Noting that the definition (24) of $\mathbb{N}^*_0$ does not involve the choice of G, this implies that the sets $\mathcal{S}^{(\delta)}$ have finite $\mathbb{N}^*_0$-measure. Since it is clear that $\mathbb{N}^*_0(\|\omega\| = 0) = 0$, we see that $\mathbb{N}^*_0$ is σ-finite. Furthermore,

$$\mathbb{N}^*_0(G) = \lim_{\varepsilon\to 0} \varepsilon^{-1}\mathbb{N}_\varepsilon(G(\tilde{W})),$$

which gives (16) for the function G we had fixed, and then also for any function G satisfying the same assumptions, since (24) does not depend on the choice of G (note that we have been considering a fixed sequence of values of ε, but the same would hold for any such sequence).

Finally, the last display shows that $\mathbb{N}_0^*$ does not depend on the choice of g, since a measure on $\mathcal{S}$ supported on $\mathcal{S}_0$ and which is finite on the sets $\mathcal{S}^{(\delta)}$ and puts no mass on $\{\omega : \|\omega\| = 0\}$ is determined by its values on functions G satisfying the assumptions of the beginning of the proof. By (24), formula (15) holds if $\Phi(\ell, \omega) = g(\ell)G(\omega)$ where G is an indicator function and the function g satisfies the previous assumptions. By standard monotone class arguments, it holds when $\Phi(\ell, \omega) = g(\ell)G(\omega)$ for any nonnegative measurable functions g and G. This completes the proof. □

Recall the notation $M(\omega) = \sup\{\omega_s(t) : s \geq 0,\ 0 \leq s \leq \zeta_s\}$. We also set

$$\tilde{M} = M(\tilde{W}) = \sup\{W_s(t) : s \geq 0,\ t \leq \zeta_s \wedge \tau_0^*(W_s)\}.$$

We can derive the distribution of M under $\mathbb{N}_0^*$.

Lemma 25. *For every $\delta > 0$, we have*

$$\mathbb{N}_0^*(M > \delta) = c_0\delta^{-3}, \quad \textit{where} \quad c_0 = 3\pi^{-3/2}\Gamma(1/3)^3\Gamma(7/6)^3.$$

Proof. By (23), we have

$$\lim_{\varepsilon \to 0} \varepsilon^{-1}\mathbb{N}_\varepsilon(\tilde{M} > \delta) = c_0\delta^{-3} \tag{25}$$

and the value of c_0 is determined in [24, Section 4]. On the other hand,

$$\mathbb{N}_0^*(G) = \lim_{\varepsilon \to 0} \varepsilon^{-1}\mathbb{N}_\varepsilon(G(\tilde{W}))$$

for any bounded continuous function G vanishing on the complement of $\mathcal{S}^{(\chi)}$ for some $\chi > 0$. Noting that the limit in (25) depends continuously on δ, we can approximate the indicator function of the set $\{M > \delta\}$ by such functions G, and obtain

$$\mathbb{N}_0^*(M > \delta) = \lim_{\varepsilon \to 0} \varepsilon^{-1}\mathbb{N}_\varepsilon(\tilde{M} > \delta) = c_0\delta^{-3}. \qquad \square$$

We may now restate the last assertion of Theorem 23 in a way more suitable for our applications.

Corollary 26. *Let $\delta > 0$. As $\varepsilon \to 0$, the law of $\tilde{W}$ under $\mathbb{N}_\varepsilon(\cdot \mid \tilde{M} > \delta)$ converges weakly to $\mathbb{N}_0^*(\cdot \mid M > \delta)$.*

Proof. Let G be bounded and continuous on $\mathcal{S}$ and such that $G(\omega) = 0$ if $\omega \notin \mathcal{S}^{(\delta)}$. Then, for $\varepsilon \in (0, \delta)$,

$$\mathbb{N}_\varepsilon(G(\tilde{W}) \mid \tilde{M} > \delta) = \frac{\mathbb{N}_\varepsilon(G(\tilde{W}))}{\mathbb{N}_\varepsilon(\tilde{M} > \delta)} \underset{\varepsilon \to 0}{\longrightarrow} \frac{\mathbb{N}_0^*(G)}{\mathbb{N}_0^*(M > \delta)} = \mathbb{N}_0^*(G \mid M > \delta),$$

by (16), (25), and Lemma 25. The desired result follows. □

We conclude this section by deriving a useful scaling property of $\mathbb{N}_0^*$. For $\lambda > 0$, for every $\omega \in \mathcal{S}$, we define $\theta_\lambda(\omega) \in \mathcal{S}$ by $\theta_\lambda(\omega) = \omega'$ with

$$\omega'_s(t) = \sqrt{\lambda}\,\omega_{s/\lambda^2}(t/\lambda) \quad \text{for } s \geq 0,\ 0 \leq t \leq \zeta'_s = \lambda\zeta_{s/\lambda^2}.$$

Note that $\theta_\lambda(\mathbb{N}_x) = \lambda \mathbb{N}_{x\sqrt{\lambda}}$ for every $x \geq 0$. The measure $\mathbb{N}_0^*$ enjoys a similar scaling property.

Lemma 27. *For every $\lambda > 0$, $\theta_\lambda(\mathbb{N}_0^*) = \lambda^{3/2}\mathbb{N}_0^*$.*

Proof. Let G be a function on $\mathcal{S}$ satisfying the conditions required for (16). Then

$$\begin{aligned}\mathbb{N}_0^*(G) &= \lim_{\varepsilon\to 0} \varepsilon^{-1}\mathbb{N}_\varepsilon(G(\tilde{W})) = \lim_{\varepsilon\to 0} \varepsilon^{-1}\lambda^{-1}\mathbb{N}_{\varepsilon/\sqrt{\lambda}}(G(\theta_\lambda(\tilde{W}))) \\ &= \lim_{\varepsilon\to 0} \lambda^{-3/2} \times (\varepsilon/\sqrt{\lambda})^{-1}\mathbb{N}_{\varepsilon/\sqrt{\lambda}}(G(\theta_\lambda(\tilde{W}))) = \lambda^{-3/2}\mathbb{N}_0^*(G\circ\theta_\lambda),\end{aligned}$$

giving the desired result. □

4. The re-rooting representation

In this section, we provide a formula connecting the measures $\mathbb{N}_0$ and $\mathbb{N}_0^*$ via a re-rooting technique. We first need to introduce some notation.

Recall the re-rooting operator R_s from Section 2.2. For every $\omega \in \mathcal{S}_0$, for every $s \in [0, \sigma(\omega)]$, we set

$$W^{[s]}(\omega) = \kappa_{-\hat{W}_s(\omega)} \circ R_s(\omega).$$

In other words, $W^{[s]}(\omega)$ is just ω re-rooted at s and then shifted so that the spatial position of the root is again 0. Note that we slightly abuse notation here because it would be more consistent with the notation of Section 2.2 to take $W^{[s]}(\omega) = R_s(\omega)$.

Theorem 28. *For every nonnegative measurable function G on $\mathcal{S}$, we have*

$$\mathbb{N}_0^*\bigg(\int_0^\sigma \mathrm{d}r\, G(W^{[r]})\bigg) = 2\mathbb{N}_0\bigg(\int_0^\infty \mathrm{d}b\, G(\mathrm{tr}_{-b}(W))Z_b\bigg),$$

where we recall that Z_b stands for the total mass of the exit measure outside $(-b, \infty)$.

Proof. We start from the re-rooting theorem in [28, Theorem 2.3]: for every nonnegative measurable function F on $\mathbb{R}_+ \times \mathcal{S}$,

$$\mathbb{N}_0\bigg(\int_0^\sigma \mathrm{d}s\, F(s, W^{[s]})\bigg) = \mathbb{N}_0\bigg(\int_0^\sigma \mathrm{d}s\, F(s, W)\bigg). \tag{26}$$

We apply this result to a function F of the form

$$F(s,\omega) = G(\mathrm{tr}_{\underline{\omega}_{\sigma-s}}(\omega))g(\underline{\omega}_{\sigma-s} - \hat{\omega}_{\sigma-s}),$$

where we recall the notation $\underline{\mathrm{w}} = \min\{\mathrm{w}(t) : 0 \leq t \leq \zeta_{(\mathrm{w})}\}$, and we suppose that G and g satisfy the assumptions stated at the beginning of the proof of Theorem 23, and the additional assumption that there exists a constant $K > 0$ such that $G(\omega) = 0$ if $\|\omega\| \geq K$. We note that our definitions give, under $\mathbb{N}_0$,

$$\hat{W}^{[s]}_{\sigma-s} = -\hat{W}_s, \quad \underline{W}^{[s]}_{\sigma-s} = \underline{W}_s - \hat{W}_s.$$

Consequently,

$$F(s, W^{[s]}) = G(\mathrm{tr}_{\underline{W}_s - \hat{W}_s}(W^{[s]}))g(\underline{W}_s).$$

We can then decompose the integral

$$\int_0^\sigma \mathrm{d}s\, F(s, W^{[s]})$$

as a sum over the sets $\{s \in [0,\sigma] : p_\zeta(s) \in C_u\}$ where u varies over D. These sets cover $[0,\sigma]$ (except for a Lebesgue negligible subset) and they are pairwise disjoint. Furthermore, if $u \in D$, it follows from our definitions that $\underline{W}_s = V_u$ for every $s \in [0,\sigma]$ such that $p_\zeta(s) \in C_u$, and

$$\int_{\{s\in[0,\sigma]:\, p_\zeta(s)\in C_u\}} \mathrm{d}s\, G(\mathrm{tr}_{\underline{W}_s - \hat{W}_s}(W^{[s]})) = H(\tilde{W}^{(u)}),$$

where

$$H(\omega) = \int_0^{\sigma(\omega)} \mathrm{d}r\, G(W^{[r]}(\omega)).$$

Summarizing, the left-hand side of (26) is equal to

$$\mathbb{N}_0\Big(\sum_{u\in D} g(V_u)H(\tilde{W}^{(u)})\Big) = \Big(\int_{-\infty}^0 g(x)\,\mathrm{d}x\Big)\mathbb{N}_0^*(H) \tag{27}$$

by Theorem 23.

On the other hand, the right-hand side of (26) is equal to

$$\mathbb{N}_0\Big(\int_0^\sigma \mathrm{d}s\, G(\mathrm{tr}_{\underline{W}_s}(W))g(\underline{W}_s - \hat{W}_s)\Big).$$

We can evaluate this quantity via a discrete approximation. Using Lemma 11, we see that, $\mathbb{N}_0$-a.e.,

$$\begin{aligned}\int_0^\sigma \mathrm{d}s\, &G(\mathrm{tr}_{\underline{W}_s}(W))g(\underline{W}_s - \hat{W}_s)\\ &= \lim_{n\to\infty}\int_0^\sigma \mathrm{d}s\, g(\underline{W}_s - \hat{W}_s)\sum_{k=1}^\infty \mathbf{1}_{\{\underline{W}_s\in(-(k+1)/n,-k/n]\}}G(\mathrm{tr}_{-k/n}(W)),\end{aligned}$$

and we note that, if g is supported on $[-A, 0]$, the quantities on the right-hand side are bounded independently of $n \geq 1$ by a constant times

$$\int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{\underline{W}_s\geq -K-1\}}\mathbf{1}_{\{\underline{W}_s-\hat{W}_s\geq -A\}}.$$

The point is that if $s \in [0,\sigma]$ is such that $\underline{W}_s < -K-1$, then the unique integer k such that $\underline{W}_s \in (-(k+1)/n, -k/n]$ also satisfies $-k/n < -K$, and we have $G(\mathrm{tr}_{-k/n}(W)) = 0$ by our assumption on G. The quantity in the last display is integrable under $\mathbb{N}_0$ by a simple application of the first-moment formula for the Brownian snake (6). This makes it possible to use dominated convergence to get

$$\mathbb{N}_0\bigg(\int_0^\sigma \mathrm{d}s\, G(\mathrm{tr}_{\underline{W}_s}(W))g(\underline{W}_s - \hat{W}_s)\bigg)$$
$$= \lim_{n\to\infty}\sum_{k=1}^\infty \mathbb{N}_0\bigg(\int_0^\sigma \mathrm{d}s\, g(\underline{W}_s - \hat{W}_s)\,\mathbf{1}_{\{\underline{W}_s\in(-(k+1)/n,-k/n]\}}G(\mathrm{tr}_{-k/n}(W))\bigg). \tag{28}$$

Then, for every integer $k \geq 1$, an application of the special Markov property (note that $G(\mathrm{tr}_{-k/n}(W))$ is $\mathcal{E}^{(-k/n,\infty)}$-measurable by the very definition of this σ-field) gives

$$\begin{aligned}
&\mathbb{N}_0\bigg(\int_0^\sigma \mathrm{d}s\, g(\underline{W}_s - \hat{W}_s)\,\mathbf{1}_{\{\underline{W}_s\in(-(k+1)/n,-k/n]\}}G(\mathrm{tr}_{-k/n}(W))\bigg)\\
&\quad= \mathbb{N}_0\bigg(Z_{k/n}G(\mathrm{tr}_{-k/n}(W))\,\mathbb{N}_{-k/n}\bigg(\int_0^\sigma \mathrm{d}s\,\mathbf{1}_{\{\underline{W}_s>-(k+1)/n\}}g(\underline{W}_s - \hat{W}_s)\bigg)\bigg)\\
&\quad= \mathbb{N}_0(Z_{k/n}G(\mathrm{tr}_{-k/n}(W)))\times\mathbb{N}_{-k/n}\bigg(\int_0^\sigma \mathrm{d}s\,\mathbf{1}_{\{\underline{W}_s>-(k+1)/n\}}g(\underline{W}_s - \hat{W}_s)\bigg)\\
&\quad= \mathbb{N}_0(Z_{k/n}G(\mathrm{tr}_{-k/n}(W)))\\
&\qquad\times\mathbb{E}_{-k/n}\bigg[\int_0^\infty \mathrm{d}t\,\mathbf{1}_{\{\min\{B_r:0\leq r\leq t\}>-(k+1)/n\}}g(\min\{B_r : 0\leq r\leq t\} - B_t)\bigg]\\
&\quad= \frac{2}{n}\bigg(\int_{-\infty}^0 \mathrm{d}x\, g(x)\bigg)\,\mathbb{N}_0(Z_{k/n}G(\mathrm{tr}_{-k/n}(W))),
\end{aligned}$$

by using again the first-moment formula for the Brownian snake (6) in the third equality, and in the last one the property

$$\mathbb{E}_0\bigg[\int_0^\infty \mathrm{d}t\,\mathbf{1}_{\{\min\{B_r:0\leq r\leq t\}>-\varepsilon\}}g(\min\{B_r : 0\leq r\leq t\} - B_t)\bigg] = 2\varepsilon\int_{-\infty}^0 \mathrm{d}x\, g(x),$$

which holds for every $\varepsilon > 0$, by direct calculations since the law of $(B_t, \min\{B_r : 0 \leq r \leq t\})$ is known explicitly (or via a simple application of standard excursion theory). From (28), we then deduce that

$$\begin{aligned}
&\mathbb{N}_0\bigg(\int_0^\sigma \mathrm{d}s\, G(\mathrm{tr}_{\underline{W}_s}(W))g(\underline{W}_s - \hat{W}_s)\bigg)\\
&\qquad= \lim_{n\to\infty}\frac{2}{n}\bigg(\int_{-\infty}^0 \mathrm{d}x\, g(x)\bigg)\sum_{k=1}^\infty \mathbb{N}_0(Z_{k/n}G(\mathrm{tr}_{-k/n}(W)))\\
&\qquad= 2\bigg(\int_{-\infty}^0 \mathrm{d}x\, g(x)\bigg)\mathbb{N}_0\bigg(\int_0^\infty \mathrm{d}b\, Z_b G(\mathrm{tr}_{-b}(W))\bigg),
\end{aligned}$$

where the last equality is justified by Lemma 11 together with our assumptions on G and the integrability properties of the exit measure process Z that were already used in the proof of Theorem 23.

Finally, the equality between the right-hand side of the last display and the right-hand side of (27) gives the identity of the theorem under our special assumptions on G. However, since both sides of this identity define σ-finite measures (which are finite on sets of the form $\{\delta < \|\omega\| < K\}$), the fact that these measures take the same values on the particular functions G considered in the proof implies that the measures are equal. □

5. An almost sure construction

In this section, we fix $\delta > 0$ and we give an almost sure construction of a snake trajectory distributed according to $\mathbb{N}^*_0(\cdot \mid M > \delta)$. This construction will be useful later when we discuss exit measures.

Let $0 < \varepsilon < \varepsilon' < \delta$, and let $W^{\delta,\varepsilon}$ be a random snake trajectory distributed according to $\mathbb{N}_\varepsilon(\cdot \mid \tilde{M} > \delta)$. Consider the excursions of $W^{\delta,\varepsilon}$ outside the interval $(0, \varepsilon')$. The conditioning on $\{\tilde{M} > \delta\}$ implies that there is at least one such excursion ω' starting from ε' and such that $\tilde{M}(\omega') > \delta$. Furthermore, if we pick uniformly at random one of the excursions ω' starting from ε' that satisfy $\tilde{M}(\omega') > \delta$, the special Markov property (Proposition 13) ensures that this excursion will be distributed according to $\mathbb{N}_{\varepsilon'}(\cdot \mid \tilde{M} > \delta)$. For $\omega \in \mathcal{S}_\varepsilon$ such that $\tilde{M}(\omega) > \delta$, let $\Theta_{\varepsilon,\varepsilon'}(\omega, \mathrm{d}\omega')$ be the probability measure on $\mathcal{S}_{\varepsilon'}$ defined as the law of an excursion of ω outside $(0, \varepsilon')$ chosen uniformly at random among those excursions that satisfy $\tilde{M} > \delta$. Then, the preceding considerations show that the second marginal of the probability measure $\Pi_{\varepsilon,\varepsilon'}$ defined on $\mathcal{S}_\varepsilon \times \mathcal{S}_{\varepsilon'}$ by

$$\Pi_{\varepsilon,\varepsilon'}(\mathrm{d}\omega\, \mathrm{d}\omega') = \mathbb{N}_\varepsilon(\mathrm{d}\omega \mid \tilde{M} > \delta)\Theta_{\varepsilon,\varepsilon'}(\omega, \mathrm{d}\omega')$$

is $\mathbb{N}_{\varepsilon'}(\cdot \mid \tilde{M} > \delta)$.

Now let $(\varepsilon_n)_{n\geq 1}$ be a sequence of positive reals in $(0, \delta)$ decreasing to 0. We claim that we can construct, on a suitable probability space, a sequence $(W^{\delta,\varepsilon_n})_{n\geq 1}$ of random variables with values in $\mathcal{S}$ such that the following holds:

(i) For every $n \geq 1$, W^{δ,ε_n} is distributed according to $\mathbb{N}_{\varepsilon_n}(\cdot \mid \tilde{M} > \delta)$.
(ii) For every $1 \leq n < m$, W^{δ,ε_n} is an excursion of W^{δ,ε_m} outside $(0, \varepsilon_n)$.

Indeed, we use the Kolmogorov extension theorem to construct the sequence $(W^{\delta,\varepsilon_n})_{n\geq 1}$ so that, for every $n \geq 1$, the law of $(W^{\delta,\varepsilon_n}, W^{\delta,\varepsilon_{n-1}}, \ldots, W^{\delta,\varepsilon_1})$ is

$$\mathbb{N}_{\varepsilon_n}(\mathrm{d}\omega_n \mid \tilde{M} > \delta)\Theta_{\varepsilon_n,\varepsilon_{n-1}}(\omega_n, \mathrm{d}\omega_{n-1})\Theta_{\varepsilon_{n-1},\varepsilon_{n-2}}(\omega_{n-1}, \mathrm{d}\omega_{n-2}) \ldots \Theta_{\varepsilon_2,\varepsilon_1}(\omega_2, \mathrm{d}\omega_1)$$

and properties (i) and (ii) hold by construction.

For every $n \geq 1$, set $\tilde{W}^{\delta,\varepsilon_n} = \mathrm{tr}_0(W^{\delta,\varepsilon_n})$, and let $\sigma_n = \sigma(\tilde{W}^{\delta,\varepsilon_n})$ be the duration of $\tilde{W}^{\delta,\varepsilon_n}$. Clearly, it is still true that, for $1 \leq n < m$, $\tilde{W}^{\delta,\varepsilon_n}$ is an excursion of $\tilde{W}^{\delta,\varepsilon_m}$ outside $(0, \varepsilon_n)$. Therefore, for every $1 \leq n < m$, $\tilde{W}^{\delta,\varepsilon_n}$ is a subtrajectory of $\tilde{W}^{\delta,\varepsilon_m}$ and we write $[a_{n,m}, b_{n,m}] \subset [0, \sigma_m]$ for the associated interval. Note that $b_{n,m} - a_{n,m} = \sigma_n$.

Furthermore, if $1 \le n < m < \ell$, we have $[a_{n,\ell}, b_{n,\ell}] \subset [a_{m,\ell}, b_{m,\ell}]$, and more precisely

$$a_{n,\ell} = a_{n,m} + a_{m,\ell}\,, \tag{29}$$

$$\sigma_\ell - b_{n,\ell} = (\sigma_m - b_{n,m}) + (\sigma_\ell - b_{m,\ell})\,. \tag{30}$$

In particular, for n fixed, the sequence $(a_{n,m})_{m>n}$ is increasing, and we denote its limit by $a_{n,\infty}$ (the fact that this limit is finite will be obtained at the beginning of the proof of the next proposition).

Proposition 29. *We have a.s.*

$$\tilde{W}^{\delta,\varepsilon_n} \xrightarrow[n\to\infty]{} W^{\delta,0} \quad \text{in } \mathcal{S}\,,$$

where the a.s. limit $W^{\delta,0}$ is distributed according to $\mathbb{N}_0^*(\cdot \mid M > \delta)$. Furthermore, $\tilde{W}^{\delta,\varepsilon_n}$ is a subtrajectory of $W^{\delta,0}$, for every $n \ge 1$, and $\sigma(\tilde{W}^{\delta,\varepsilon_n}) \uparrow \sigma(W^{\delta,0})$ as $n \to \infty$.

Proof. By Corollary 26, we already know that the sequence $(\tilde{W}^{\delta,\varepsilon_n})_{n\ge1}$ converges in distribution to $\mathbb{N}_0^*(\cdot \mid M > \delta)$, and in particular $\sigma_n = \sigma(\tilde{W}^{\delta,\varepsilon_n})$ converges in distribution to the law of σ under $\mathbb{N}_0^*(\cdot \mid M > \delta)$. On the other hand, the sequence $(\sigma_n)_{n\ge1}$ is increasing and thus has an a.s. limit σ_∞. We conclude that σ_∞ is distributed as σ under $\mathbb{N}_0^*(\cdot \mid M > \delta)$, and in particular $\sigma_\infty < \infty$ a.s.

Since $a_{n,m} \le \sigma_m - \sigma_n$ if $n < m$, we find that, for every n,

$$a_{n,\infty} \le \sigma_\infty - \sigma_n.$$

It follows that

$$\lim_{n\to\infty} a_{n,\infty} = 0, \qquad \text{a.s.} \tag{31}$$

Then, for every fixed n, $b_{n,m} = a_{n,m} + \sigma_n$ converges as $m \uparrow \infty$ to $b_{n,\infty} = a_{n,\infty} + \sigma_n$, and, by letting ℓ tend to ∞ in (29) and (30), we get, for $n < m$,

$$a_{n,\infty} = a_{n,m} + a_{m,\infty}, \qquad \sigma_\infty - b_{n,\infty} = (\sigma_\infty - b_{m,\infty}) + (\sigma_m - b_{n,m})\,. \tag{32}$$

Set $\tilde{\zeta}_s^{\delta,\varepsilon_n} = \zeta_s(\tilde{W}^{\delta,\varepsilon_n})$ to simplify notation. By the definition of subtrajectories we know that $\tilde{\zeta}_s^{\delta,\varepsilon_n} = \tilde{\zeta}^{\delta,\varepsilon_m}_{(a_{n,m}+s)\wedge b_{n,m}} - \tilde{\zeta}^{\delta,\varepsilon_m}_{a_{n,m}}$ if $n < m$. We claim that a.s.,

$$\lim_{n\to\infty}\Big(\sup_{m>n}\Big(\sup_{0\le s\le a_{n,m}} \tilde{\zeta}_s^{\delta,\varepsilon_m}\Big)\Big) = 0. \tag{33}$$

To verify this claim, first observe that if $n < n' < m$, we have

$$\sup_{0\le s\le a_{n',m}} \tilde{\zeta}_s^{\delta,\varepsilon_m} \le \sup_{0\le s\le a_{n,m}} \tilde{\zeta}_s^{\delta,\varepsilon_m}$$

because $a_{n',m} \le a_{n,m}$. It then follows that the supremum over $m > n$ in (33) is a decreasing function of n, and so the limit in the left-hand side of (33) exists a.s. as a decreasing limit. Call this limit L. Towards a contradiction, assume that $P(L > 0) > 0$. Then we choose $\xi > 0$ such that $P(L > \xi) > 0$, and we note that, on the event $\{L > \xi\}$, we can find a sequence $n_1 < m_1 < n_2 < m_2 < \cdots$ such that, for every $i = 1, 2, \ldots$,

$$\sup_{0\le s\le a_{n_i,m_i}} \tilde{\zeta}_s^{\delta,\varepsilon_{m_i}} > \xi.$$

It then follows that, on the same event $\{L > \xi\}$ of positive probability, for any integer $k \geq 1$, and for every large enough n, there exist k disjoint intervals $[r_1, s_1], \dots, [r_k, s_k]$ such that $\tilde{\zeta}^{\delta,\varepsilon_n}_{s_i} - \tilde{\zeta}^{\delta,\varepsilon_n}_{r_i} > \xi$ for every $1 \leq i \leq k$. The latter property contradicts the tightness of the sequence of the laws of $\tilde{W}^{\delta,\varepsilon_n}$ in $\mathcal{S}$, and this contradiction proves our claim (33).

By the same argument,

$$\lim_{n\to\infty}\Big(\sup_{m>n}\Big(\sup_{b_{n,m}\leq s\leq \sigma_m} \tilde{\zeta}^{\delta,\varepsilon_m}_s\Big)\Big) = 0. \tag{34}$$

We can now use (33) and (34) to verify that $(\tilde{\zeta}^{\delta,\varepsilon_n}_s)_{s\geq 0}$ converges uniformly as $n\to\infty$, a.s. To this end, we define

$$\zeta^{(n)}_s = \begin{cases} 0 & \text{if } s \leq a_{n,\infty}, \\ \tilde{\zeta}^{\delta,\varepsilon_n}_{s-a_{n,\infty}} & \text{if } a_{n,\infty} \leq s \leq b_{n,\infty}, \\ 0 & \text{if } s \geq b_{n,\infty}. \end{cases}$$

Recalling the formula $\tilde{\zeta}^{\delta,\varepsilon_n}_s = \tilde{\zeta}^{\delta,\varepsilon_m}_{(a_{n,m}+s)\wedge b_{n,m}} - \tilde{\zeta}^{\delta,\varepsilon_m}_{a_{n,m}}$, and using (32), we get, for $n < m$,

$$\sup_{s\geq 0} |\zeta^{(n)}_s - \zeta^{(m)}_s| \leq \sup_{0\leq s\leq a_{n,m}} \tilde{\zeta}^{\delta,\varepsilon_m}_s + \sup_{b_{n,m}\leq s\leq \sigma_m} \tilde{\zeta}^{\delta,\varepsilon_m}_s,$$

and the right-hand side tends to 0 a.s. as n and m tend to ∞ with $n < m$, by (33) and (34). This gives the a.s. uniform convergence of $(\zeta^{(n)}_s)_{s\geq 0}$ as $n \to \infty$. Write $(\zeta^{\delta,0}_s)_{s\geq 0}$ for the limit. The a.s. uniform convergence of $(\tilde{\zeta}^{\delta,\varepsilon_n}_s)_{s\geq 0}$ toward the same limit then follows by using now (31). Moreover, $\sup\{s \geq 0 : \tilde{\zeta}^{\delta,\varepsilon_n}_s > 0\} = \sigma_n \to \sigma_\infty = \sup\{s \geq 0 : \zeta^{\delta,0}_s > 0\}$ as $n \to \infty$.

Let $\Gamma^{\delta,\varepsilon_n}_s$ stand for the endpoint of the path $\tilde{W}^{\delta,\varepsilon_n}_s$. Very similar arguments show that the analogs of (33) and (34) with $\tilde{\zeta}^{\delta,\varepsilon_m}_s$ replaced by $\Gamma^{\delta,\varepsilon_m}_s$ hold, and it follows that $(\Gamma^{\delta,\varepsilon_n}_s)_{s\geq 0}$ also converges uniformly to a limit denoted by $(\Gamma^{\delta,0}_s)_{s\geq 0}$, a.s. The pair $(\zeta^{\delta,0}, \Gamma^{\delta,0})$ is then a random tree-like path, and letting $W^{\delta,0}$ be the associated snake trajectory, we have shown that $\tilde{W}^{\delta,\varepsilon_n}$ converges a.s. to $W^{\delta,0}$. Since $\tilde{W}^{\delta,\varepsilon_n}$ converges in distribution to $\mathbb{N}^*_0(\cdot \mid M > \delta)$, $W^{\delta,0}$ is distributed according to $\mathbb{N}^*_0(\cdot \mid M > \delta)$.

Finally, it follows from our construction that, for every $n \geq 1$, $\tilde{W}^{\delta,\varepsilon_n}$ is the subtrajectory of $W^{\delta,0}$ associated with the interval $[a_{n,\infty}, b_{n,\infty}]$, and the property $\sigma(\tilde{W}^{\delta,\varepsilon_n}) \uparrow \sigma(W^{\delta,0})$ is just the fact that $\sigma_n \uparrow \sigma_\infty$. This completes the proof. □

6. The exit measure

We now define the exit measure from $(0, \infty)$ under $\mathbb{N}^*_0$. Informally, this measure corresponds to the quantity of snake trajectories that return to 0.

Proposition 30. *The limit*

$$\lim_{\varepsilon\to 0} \varepsilon^{-2} \int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{\hat{W}_s<\varepsilon\}}$$

exists in probability under $\mathbb{N}^*_0(\cdot \mid \sigma > \chi)$*, for every* $\chi > 0$*, and defines a finite random variable denoted by* Z^*_0.

Proof. We rely on the re-rooting property of Section 4. Let $(\varepsilon_n)_{n\geq 1}$ be a sequence of positive reals converging to 0. Recalling Lemma 14 and the subsequent remarks, we can extract from $(\varepsilon_n)_{n\geq 1}$ a subsequence $(\beta_n)_{n\geq 1}$ such that, for every $b < 0$,

$$Z_b = \lim_{n\to\infty} \beta_n^{-2} \int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{\zeta_s \leq \tau_{-b}(W_s),\, \hat{W}_s < -b+\beta_n\}}, \qquad \mathbb{N}_0\text{-a.e.} \tag{35}$$

Then, for $\omega \in \mathcal{S}$, we set $G(\omega) = 0$ if the limit

$$\lim_{n\to\infty} \beta_n^{-2} \int_0^{\sigma(\omega)} \mathrm{d}s\, \mathbf{1}_{\{\hat{W}_s(\omega) < W_*(\omega)+\beta_n\}}$$

exists (and is finite), and $G(\omega) = 1$ otherwise. By (35), we have $G(\mathrm{tr}_{-b}(W)) = 0$, $\mathbb{N}_0$-a.e. on the event $\{W_* \leq -b\} = \{Z_b > 0\}$, for every $b > 0$. By Theorem 28, we then have

$$\mathbb{N}_0^*\left(\int_0^\sigma \mathrm{d}s\, G(W^{[s]})\right) = 0.$$

Consequently, $\mathbb{N}_0^*$-a.e., for Lebesgue a.e. $r \in [0, \sigma]$, we have $G(W^{[r]}) = 0$. By considering just one value of r for which $G(W^{[r]}) = 0$, this says that the convergence of the proposition holds $\mathbb{N}_0^*$-a.e. along $(\beta_n)_{n\geq 1}$. We have thus shown that from any sequence of positive real numbers converging to 0 we can extract a subsequence along which the convergence of the proposition holds $\mathbb{N}_0^*$-a.e. The statement of the proposition follows. □

Recall from Section 2.5 that we have fixed a sequence $(\alpha_n)_{n\geq 1}$ such that (11) holds. We then define $Z_0^*(\omega)$ for *every* $\omega \in \mathcal{S}$, by setting

$$Z_0^*(\omega) = \liminf_{n\to\infty} \alpha_n^{-2} \int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{\hat{W}_s(\omega) < W_*(\omega)+\alpha_n\}}. \tag{36}$$

By the argument we have just given in the proof of Proposition 30, the liminf is a limit $\mathbb{N}_0^*$-a.e. In what follows, we will be concerned with the values of $Z_0^*(\omega)$ under $\mathbb{N}_0^*$, and we note that the quantity $W_*(\omega)$ in (36) can be replaced by 0, $\mathbb{N}_0^*$-a.e., so that (36) is consistent with Proposition 30.

Our next goal is to compute the joint distribution of the pair (Z_0^*, σ) under $\mathbb{N}_0^*$.

Proposition 31. *The distribution of the pair (Z_0^*, σ) under $\mathbb{N}_0^*$ has a density f given for $z, s > 0$ by*

$$f(z,s) = \frac{\sqrt{3}}{2\pi}\sqrt{z}\, s^{-5/2} \exp\left(-\frac{z^2}{2s}\right).$$

In particular, the respective densities g of Z_0^* and h of σ under $\mathbb{N}_0^*$ are given by

$$g(z) = \sqrt{\frac{3}{2\pi}}\, z^{-5/2}, \qquad z > 0,$$
$$h(s) = \frac{\sqrt{3}}{2\pi}\, 2^{-1/4}\Gamma(3/4) s^{-7/4}, \qquad s > 0.$$

Proof. We fix $\lambda, \mu > 0$, and compute

$$\mathbb{N}_0^*(\sigma \exp(-\lambda Z_0^* - \mu\sigma)).$$

Recalling (36), and using Lemma 14, we get $Z_0^*(\mathrm{tr}_{-b}(W)) = Z_b$, $\mathbb{N}_0$-a.e. on $\{Z_b > 0\}$, for every $b > 0$. Hence, by applying Theorem 28 to $G(\omega) = \exp(-\lambda Z_0^*(\omega) - \mu\sigma(\omega))$, we obtain

$$\mathbb{N}_0^*(\sigma \exp(-\lambda Z_0^* - \mu\sigma)) = 2\int_0^\infty \mathrm{d}b\, \mathbb{N}_0(Z_b \exp(-\lambda Z_b - \mu\mathcal{Y}_b))$$

with the notation

$$\mathcal{Y}_b = \int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{\tau_{-b}(W_s)=\infty\}}$$

(note that $\mathcal{Y}_b = \sigma(\mathrm{tr}_{-b}(W))$, $\mathbb{N}_0$-a.e.). Set

$$u_{\lambda,\mu}(b) = \mathbb{N}_0(1 - \exp(-\lambda Z_b - \mu\mathcal{Y}_b)),$$

and note that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} u_{\lambda,\mu}(b) = \mathbb{N}_0(Z_b \exp(-\lambda Z_b - \mu\mathcal{Y}_b)).$$

The quantity $u_{\lambda,\mu}(b)$ is computed explicitly in [9, Lemma 4.5]: if $\lambda < \sqrt{\mu/2}$, then

$$u_{\lambda,\mu}(b) = \sqrt{\frac{\mu}{2}}\left(3\left(\tanh^2\left((2\mu)^{1/4}b + \tanh^{-1}\sqrt{\frac{2}{3} + \frac{1}{3}\sqrt{\frac{2}{\mu}}\lambda}\right)\right) - 2\right),$$

and a similar formula holds if $\lambda > \sqrt{\mu/2}$. From this explicit formula, for $\lambda < \sqrt{\mu/2}$ one gets

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\lambda} u_{\lambda,\mu}(b) &= K_{\lambda,\mu}^{-1} \tanh\left((2\mu)^{1/4}b + \tanh^{-1}\sqrt{\frac{2}{3} + \frac{1}{3}\sqrt{\frac{2}{\mu}}\lambda}\right)\\ &\quad \times \left(\cosh^2\left((2\mu)^{1/4}b + \tanh^{-1}\sqrt{\frac{2}{3} + \frac{1}{3}\sqrt{\frac{2}{\mu}}\lambda}\right)\right)^{-1}\end{aligned}$$

where

$$K_{\lambda,\mu} = \frac{1}{3}\left(1 - \sqrt{\frac{2}{\mu}}\lambda\right)\sqrt{\frac{2}{3} + \frac{1}{3}\sqrt{\frac{2}{\mu}}\lambda}.$$

By integrating the last formula between $b = 0$ and $b = \infty$, we arrive at

$$\int_0^\infty \mathrm{d}b\, \mathbb{N}_0(Z_b \exp(-\lambda Z_b - \mu\mathcal{Y}_b)) = \int_0^\infty \mathrm{d}b\, \frac{\mathrm{d}}{\mathrm{d}\lambda} u_{\lambda,\mu}(b) = \frac{1}{2}\sqrt{\frac{3}{2}}(\lambda + \sqrt{2\mu})^{-1/2}.$$

Similar calculations give the same result when $\lambda > \sqrt{\mu/2}$ (and also for $\lambda = \sqrt{\mu/2}$ by a suitable passage to the limit). Summarizing, we have proved that, for all $\lambda, \mu > 0$,

$$\mathbb{N}_0^*(\sigma \exp(-\lambda Z_0^* - \mu\sigma)) = \sqrt{3/2}\,(\lambda + \sqrt{2\mu})^{-1/2}.$$

At this stage, we only need to verify that, with the function f defined in the proposition, we also have

$$\int_0^\infty \int_0^\infty s \exp(-\lambda z - \mu s)\, f(z,s)\, \mathrm{d}z\, \mathrm{d}s = \sqrt{3/2}\,\big(\lambda + \sqrt{2\mu}\big)^{-1/2}.$$

To see this, first note that, for every $z > 0$,

$$z \int_0^\infty s^{-3/2} \exp\Big(-\frac{z^2}{2s} - \mu z\Big)\, \mathrm{d}s = \sqrt{2\pi}\, e^{-z\sqrt{2\mu}},$$

by the classical formula for the Laplace transform of hitting times of a standard linear Brownian motion. The desired result easily follows. □

We now state a technical result that will be important for our purposes. Fix $\delta > 0$, and, for every $\varepsilon \in (0,\delta)$, write $W^{\delta,\varepsilon}$ for a random snake trajectory distributed according to $\mathbb{N}_\varepsilon(\cdot \mid \tilde M > \delta)$, where we recall the notation $\tilde M = \sup\{W_s(t) : s \geq 0, t \leq \zeta_s \wedge \tau_0(W_s)\}$. As usual, write $\tilde W^{\delta,\varepsilon}$ for $W^{\delta,\varepsilon}$ truncated at level 0. By Corollary 26, the distribution of $\tilde W^{\delta,\varepsilon}$ converges to $\mathbb{N}_0^*(\cdot \mid M > \delta)$ as $\varepsilon \to 0$. The next proposition shows that this convergence holds jointly with that of the exit measures from $(0,\infty)$. Recall the notation $\mathcal{Z}_0(W^{\delta,\varepsilon})$ for the (total mass of the) exit measure of $W^{\delta,\varepsilon}$ from $(0,\infty)$.

Proposition 32. *As $\varepsilon \to 0$, the distribution of the pair $(\tilde W^{\delta,\varepsilon}, \mathcal{Z}_0(W^{\delta,\varepsilon}))$ converges weakly to that of the pair $(W^{\delta,0}, Z_0^*(W^{\delta,0}))$, where $W^{\delta,0}$ is distributed according to $\mathbb{N}_0^*(\cdot \mid M > \delta)$.*

Proof. We may argue along a sequence $(\varepsilon_n)_{n\geq 1}$ strictly decreasing to 0. To simplify notation, we set $W^n = W^{\delta,\varepsilon_n}$ and $\tilde W^n = \tilde W^{\delta,\varepsilon_n}$. From Proposition 29, we may construct (on a suitable probability space) the whole sequence $(W^n)_{n\geq 1}$ and the snake trajectory $W^{\delta,0}$ in such a way that W^n is an excursion of W^m outside $(0,\varepsilon_n)$ for every $n < m$, $\tilde W^n$ is a subtrajectory of $W^{\delta,0}$ for every $n \geq 1$, $\tilde W^n \to W^{\delta,0}$ in $\mathcal{S}$ as $n \to \infty$, a.s., and moreover $\sigma(\tilde W^n) \uparrow \sigma(W^{\delta,0})$ as $n \to \infty$. These properties imply that, for every $\gamma > 0$ and every $1 \leq n \leq m$, we have

$$\begin{aligned}\int_0^{\sigma(W^n)} \mathrm{d}s\, \mathbf{1}_{\{\zeta^n_s \leq \tau_0(W^n_s),\, \hat W^n_s < \gamma\}} &\leq \int_0^{\sigma(W^m)} \mathrm{d}s\, \mathbf{1}_{\{\zeta^m_s \leq \tau_0(W^m_s),\, \hat W^m_s < \gamma\}} \\ &\leq \int_0^{\sigma(W^{\delta,0})} \mathrm{d}s\, \mathbf{1}_{\{\hat W^{\delta,0}_s < \gamma\}}.\end{aligned}$$

If we multiply this inequality by γ^{-2} and let γ tend to 0, we find that, for every $1 \leq n \leq m$,

$$\mathcal{Z}_0(W^n) \leq \mathcal{Z}_0(W^m) \leq Z_0^*(W^{\delta,0}).$$

In particular the almost sure increasing limit

$$Z_0' := \lim_{n\to\infty} \uparrow \mathcal{Z}_0(W^n)$$

exists and we have $Z'_0 \leq Z^*_0(W^{\delta,0})$. The result of the proposition will follow if we can verify that indeed $Z'_0 = Z^*_0(W^{\delta,0})$ a.s. To this end, fix $\lambda, \mu > 0$. Write $E[\cdot]$ for the expectation on the probability space where the sequence $(W^n)_{n\geq 1}$ and $W^{\delta,0}$ are defined. We note that

$$E[\exp(-\lambda Z'_0)(1-\exp(-\mu\sigma(W^{\delta,0})))] \leq \liminf_{n\to\infty} E[\exp(-\lambda \mathcal{Z}_0(W^n))(1-\exp(-\mu\sigma(\tilde{W}^n)))] \tag{37}$$

by Fatou's lemma. We will verify that

$$\liminf_{n\to\infty} E[\exp(-\lambda \mathcal{Z}_0(W^n))(1-\exp(-\mu\sigma(\tilde{W}^n)))] \leq E[\exp(-\lambda Z^*_0(W^{\delta,0}))(1-\exp(-\mu\sigma(W^{\delta,0})))]. \tag{38}$$

If (38) holds, then by combining this with the previous display, we get

$$E[\exp(-\lambda Z'_0)(1-\exp(-\mu\sigma(W^{\delta,0})))] \leq E[\exp(-\lambda Z^*_0(W^{\delta,0}))(1-\exp(-\mu\sigma(W^{\delta,0})))],$$

and since we already know that $Z'_0 \leq Z^*_0(W^{\delta,0})$, this is only possible if $Z'_0 = Z^*_0(W^{\delta,0})$ a.s.

Let us prove (38). Since W^n is distributed according to $\mathbb{N}_{\varepsilon_n}(\cdot \mid \tilde{M} > \delta)$ and $W^{\delta,0}$ is distributed according to $\mathbb{N}^*_0(\cdot \mid \tilde{M} > \delta)$, and since $\mathbb{N}_\varepsilon(\tilde{M} > \delta) \sim \varepsilon\, \mathbb{N}^*_0(M > \delta)$ as $\varepsilon \to 0$ (see the proof of Lemma 25), we see that (38) is equivalent to

$$\liminf_{n\to\infty} \frac{1}{\varepsilon_n} \mathbb{N}_{\varepsilon_n}\big(\exp(-\lambda\mathcal{Z}_0)(1-\exp(-\mu\mathcal{Y}_0))\, \mathbf{1}_{\{\tilde{M}>\delta\}}\big) \leq \mathbb{N}^*_0\big(\exp(-\lambda Z^*_0)(1-\exp(-\mu\sigma))\, \mathbf{1}_{\{M>\delta\}}\big), \tag{39}$$

where we recall that

$$\mathcal{Y}_0 = \int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{\tau_0(W_s)=\infty\}}.$$

Observe that, for any choice of $\gamma \in (0,\delta)$, the argument leading to (37) (using also the fact that $M(\tilde{W}^n)$ converges a.s. to $M(W^{\delta,0})$) gives

$$\liminf_{n\to\infty} \frac{1}{\varepsilon_n} \mathbb{N}_{\varepsilon_n}\big(\exp(-\lambda\mathcal{Z}_0)(1-\exp(-\mu\mathcal{Y}_0))\, \mathbf{1}_{\{\gamma<\tilde{M}\leq\delta\}}\big) \geq \mathbb{N}^*_0\big(\exp(-\lambda Z^*_0)(1-\exp(-\mu\sigma))\, \mathbf{1}_{\{\gamma<M\leq\delta\}}\big), \tag{40}$$

and by letting γ tend to 0,

$$\liminf_{n\to\infty} \frac{1}{\varepsilon_n} \mathbb{N}_{\varepsilon_n}\big(\exp(-\lambda\mathcal{Z}_0)(1-\exp(-\mu\mathcal{Y}_0))\, \mathbf{1}_{\{\tilde{M}\leq\delta\}}\big) \geq \mathbb{N}^*_0\big(\exp(-\lambda Z^*_0)(1-\exp(-\mu\sigma))\, \mathbf{1}_{\{M\leq\delta\}}\big).$$

So if (39) fails, we get

$$\liminf_{n\to\infty} \frac{1}{\varepsilon_n} \mathbb{N}_{\varepsilon_n}\big(\exp(-\lambda\mathcal{Z}_0)(1-\exp(-\mu\mathcal{Y}_0))\big) > \mathbb{N}^*_0\big(\exp(-\lambda Z^*_0)(1-\exp(-\mu\sigma))\big).$$

We will prove that

$$\lim_{n\to\infty} \frac{1}{\varepsilon_n} \mathbb{N}_{\varepsilon_n}\big(\exp(-\lambda \mathcal{Z}_0)(1-\exp(-\mu \mathcal{Y}_0))\big) = \mathbb{N}_0^*\big(\exp(-\lambda Z_0^*)(1-\exp(-\mu\sigma))\big), \tag{41}$$

showing by contradiction that (39) and thus also (38) hold.

The right-hand side of (41) can be computed from the formula

$$\mathbb{N}_0^*(\sigma \exp(-\lambda Z_0^* - \mu\sigma)) = \sqrt{3/2}\,\big(\lambda + \sqrt{2\mu}\big)^{-1/2},$$

which was obtained in the proof of Proposition 31. We get

$$\begin{aligned}
\mathbb{N}_0^*\big(\exp(-\lambda Z_0^*)(1-\exp(-\mu\sigma))\big) &= \mathbb{N}_0^*\Big(\exp(-\lambda Z_0^*) \int_0^\mu \mathrm{d}\mu'\, \sigma \exp(-\mu'\sigma)\Big) \\
&= \int_0^\mu \mathrm{d}\mu' \sqrt{3/2}\,\big(\lambda + \sqrt{2\mu'}\big)^{-1/2} = \sqrt{3/2} \int_0^{\sqrt{2\mu}} \mathrm{d}x\, x(\lambda + x)^{-1/2} \\
&= \sqrt{2/3}\Big(\big(\lambda + \sqrt{2\mu}\big)^{3/2} - 3\lambda\big(\lambda + \sqrt{2\mu}\big)^{1/2} + 2\lambda^{3/2}\Big).
\end{aligned} \tag{42}$$

On the other hand, we have, for every $\varepsilon > 0$,

$$\begin{aligned}
&\mathbb{N}_\varepsilon\big(\exp(-\lambda\mathcal{Z}_0)(1-\exp(-\mu\mathcal{Y}_0))\big) \\
&= \mathbb{N}_\varepsilon(1-\exp(-\lambda\mathcal{Z}_0 - \mu\mathcal{Y}_0)) - \mathbb{N}_\varepsilon(1-\exp(-\lambda\mathcal{Z}_0)) = u_{\lambda,\mu}(\varepsilon) - \big(1/\sqrt{\lambda} + \varepsilon\sqrt{2/3}\big)^{-2},
\end{aligned}$$

recalling (9) and using the notation introduced in the proof of Proposition 31. Formula (26) in [9] gives

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}(u_{\lambda,\mu}(\varepsilon) - \lambda) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} u_{\lambda,\mu}(\varepsilon)_{|\varepsilon=0} = \sqrt{2/3}\big(\lambda + \sqrt{2\mu}\big)^{1/2}\big(\sqrt{2\mu} - 2\lambda\big).$$

It follows that

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{\varepsilon_n}\mathbb{N}_{\varepsilon_n}\big(\exp(-\lambda\mathcal{Z}_0)(1-\exp(-\mu\mathcal{Y}_0))\big) \\
&\qquad = \sqrt{2/3}\,\big(\lambda + \sqrt{2\mu}\big)^{1/2}\,\big(\sqrt{2\mu} - 2\lambda\big) + 2\sqrt{2/3}\,\lambda^{3/2},
\end{aligned}$$

and one immediately verifies that the right-hand side coincides with the right-hand side of (42). This completes the proof of (41) and of the proposition. □

In view of our applications, it will be important to define the measure $\mathbb{N}_0^*$ conditioned on a given value of the exit measure. This is the goal of the next proposition. Before that, we mention a useful scaling property. Recall the definition of the scaling operator θ_λ at the end of Section 3. Then for every $\lambda > 0$ and every $\omega \in \mathcal{S}$,

$$Z_0^* \circ \theta_\lambda(\omega) = \lambda Z_0^*(\omega). \tag{43}$$

The proof is easy: recall from (36) the definition of $Z_0^*(\omega)$ for all $\omega \in \mathcal{S}$ and write

$$\begin{aligned} Z_0^* \circ \theta_\lambda(\omega) &= \liminf_{n\to\infty} \alpha_n^{-2} \int_0^{\lambda^2\sigma(\omega)} \mathrm{d}s\, \mathbf{1}_{\{\sqrt{\lambda}\hat{W}_{s/\lambda^2}(\omega) < \sqrt{\lambda}W_* + \alpha_n\}} \\ &= \lambda \liminf_{n\to\infty} (\alpha_n/\sqrt{\lambda})^{-2} \int_0^{\sigma(\omega)} \mathrm{d}s\, \mathbf{1}_{\{\hat{W}_s < W_* + \alpha_n/\sqrt{\lambda}\}} = \lambda Z_0^*(\omega). \end{aligned}$$

Proposition 33. *There exists a unique collection* $(\mathbb{N}_0^{*,z})_{z>0}$ *of probability measures on* $\mathcal{S}$ *such that:*

(i) *We have*

$$\mathbb{N}_0^* = \sqrt{\frac{3}{2\pi}} \int_0^\infty \mathrm{d}z\, z^{-5/2} \mathbb{N}_0^{*,z}.$$

(ii) *For every* $z > 0$, $\mathbb{N}_0^{*,z}$ *is supported on* $\{Z_0^* = z\}$.

(iii) *For every* $z, z' > 0$, $\mathbb{N}_0^{*,z'} = \theta_{z'/z}(\mathbb{N}_0^{*,z})$.

We will write $\mathbb{N}_0^{*,z} = \mathbb{N}_0^*(\cdot \mid Z_0^* = z)$.

Proof. Recall from Proposition 31 that the “law” of Z_0^* under $\mathbb{N}_0^*$ is the measure $\mathbf{1}_{\{z>0\}} \sqrt{3/(2\pi)}\, z^{-5/2}\, \mathrm{d}z$, which we denote here by $\nu(\mathrm{d}z)$ to simplify notation. The existence of a collection of probability measures on $\mathcal{S}$ that satisfy both (i) and (ii) is a consequence of standard disintegration theorems (see e.g. [11, Chapter III, §§70–74]). Two such collections coincide up to a negligible set of values of z. We need to verify that we can choose this collection so that the additional scaling property (iii) also holds (which will imply the stronger uniqueness in the proposition).

We start with any measurable collection $(\mathbb{Q}_z)_{z>0}$ of probability measures on $\mathcal{S}$ such that (i) and (ii) hold with $(\mathbb{N}_0^{*,z})_{z>0}$ replaced by $(\mathbb{Q}_z)_{z>0}$. From Lemma 27, we find that, for every $\lambda > 0$,

$$\int \theta_\lambda(\mathbb{Q}_z)\, \nu(\mathrm{d}z) = \theta_\lambda(\mathbb{N}_0^*) = \lambda^{3/2}\mathbb{N}_0^* = \lambda^{3/2} \int \mathbb{Q}_z\, \nu(\mathrm{d}z).$$

From the change of variables $z = z'/\lambda$ in the first integral, we thus get

$$\int \theta_\lambda(\mathbb{Q}_{z/\lambda})\, \nu(\mathrm{d}z) = \int \mathbb{Q}_z\, \nu(\mathrm{d}z).$$

Using the scaling property (43), we see that the collection $(\theta_\lambda(\mathbb{Q}_{z/\lambda}))_{z>0}$ also satisfies (i) and (ii), and so we get, for every fixed $\lambda > 0$,

$$\theta_\lambda(\mathbb{Q}_{z/\lambda}) = \mathbb{Q}_z, \qquad \mathrm{d}z\text{-a.e.}$$

From Fubini's theorem, we then have $\theta_\lambda(\mathbb{Q}_{z/\lambda}) = \mathbb{Q}_z$, $\mathrm{d}\lambda$-a.e., $\mathrm{d}z$-a.e. At this stage, we can pick $z_0 > 0$ such that the equality $\theta_\lambda(\mathbb{Q}_{z_0/\lambda}) = \mathbb{Q}_{z_0}$ holds $\mathrm{d}\lambda$-a.e., and define $\mathbb{N}_0^{*,z} := \theta_{z/z_0}(\mathbb{Q}_{z_0})$ for every $z > 0$. Then $\mathbb{N}_0^{*,z} = \mathbb{Q}_z$, $\mathrm{d}z$-a.e., so that (i) holds for the collection $(\mathbb{N}_0^{*,z})_{z>0}$. Similarly (ii) holds because $\mathbb{Q}_{z_0}$ is supported on $\{Z_0^* = z_0\}$, and we use the scaling property (43). Property (iii) holds by construction.

To get uniqueness, observe that (iii) implies that the mapping $z \mapsto \mathbb{N}_0^{*,z}$ is continuous for the weak convergence of probability measures. The uniqueness is then a simple consequence of this continuous dependence and the fact that two collections that satisfy both (i) and (ii) must coincide up to a negligible set of values of z. □

7. The excursion process

For technical reasons in this section, it is preferable to argue under a probability measure rather than under $\mathbb{N}_0$. So we fix $\beta > 0$, and we argue under the conditional measure $\mathbb{N}_0^{(\beta)} := \mathbb{N}_0(\cdot \mid W_* < -\beta)$. We will then consider, under $\mathbb{N}_0^{(\beta)}$, the excursion debuts whose level is smaller than $-\beta$. For every $\delta > 0$, we write $u_1^\delta, \dots, u_{N_\delta}^\delta$ for the excursion debuts with height greater than δ whose level is smaller than $-\beta$, listed in decreasing order of the levels, so that

$$V_{u^\delta_{N_\delta}} < V_{u^\delta_{N_\delta-1}} < \cdots < V_{u^\delta_1} < -\beta.$$

Notice that N_δ and $u_1^\delta, \dots, u_{N_\delta}^\delta$ depend on the choice of β, which will remain fixed in the first three subsections below (although on a couple of occasions we mention the consequences that one derives by letting β tend to 0, which should cause no confusion). For every integer $i \geq 1$, we also set

$$T_i^\delta := \begin{cases} -V_{u_i^\delta} & \text{if } i \leq N_\delta, \\ \infty & \text{if } i > N_\delta. \end{cases}$$

It is easy to verify that, for every $a > 0$, the event $\{T_i^\delta < a\}$ belongs to the σ-field $\mathcal{E}^{(-a,\infty)}$ (the knowledge of $\mathcal{E}^{(-a,\infty)}$ gives enough information to recover the excursion debuts—and the corresponding heights—such that $V_u > -a$). Since $\{T_i^\delta = a\}$ is $\mathbb{N}_0$-negligible, it follows that T_i^δ is a stopping time of the filtration $(\mathcal{E}^{(-a,\infty)})_{a \geq 0}$, where, by convention, $\mathcal{E}^{(0,\infty)}$ is the σ-field generated by the $\mathbb{N}_0$-negligible sets. Finally, it will also be useful to write $N_\delta^\circ = \# D_\delta$ for the total number of excursion debuts with height greater than δ.

7.1. The excursions with height greater than δ

Recall the notation $\tilde{W}^{(u)}$ for the excursion starting at the excursion debut $u \in D$.

Proposition 34. *Let $j \geq 1$. Then, under the conditional probability measure $\mathbb{N}_0^{(\beta)}(\cdot \mid N_\delta \geq j)$, $\tilde{W}^{(u_j^\delta)}$ is independent of the σ-field generated by $(\tilde{W}^{(u_1^\delta)}, \dots, \tilde{W}^{(u_{j-1}^\delta)})$ and $\mathcal{E}^{(-\beta,\infty)}$, and is distributed according to $\mathbb{N}_0^*(\cdot \mid M > \delta)$.*

Important remark. In view of the analogous statement for linear Brownian motion, one might naively expect that $\tilde{W}^{(u_1^\delta)}, \dots, \tilde{W}^{(u_j^\delta)}$ are (independent and) identically distributed under $\mathbb{N}_0^{(\beta)}(\cdot \mid N_\delta \geq j)$. This is not true as soon as $j \geq 2$: the point is that the knowledge of the event $\{N_\delta \geq j\}$ influences the distribution of $(\tilde{W}^{(u_1^\delta)}, \dots, \tilde{W}^{(u_{j-1}^\delta)})$.

Proof of Proposition 34. The first step of the proof is to determine the law of $\tilde{W}^{(u_1^\delta)}$ under $\mathbb{N}_0^{(\beta)}(\cdot \mid N_\delta \geq 1)$. We fix two bounded nonnegative functions G and g defined respectively on $\mathcal{S}$ and on $\mathbb{R}$. We assume that G is bounded and continuous on the set $\{\omega : M(\omega) > \delta\}$, and vanishes outside this set. The function g is assumed to be continuous with compact support contained in $(-\infty, -\beta]$.

We retain much of the notation of the proof of Theorem 23. In particular, for any integers $n \geq 1$ and $k \geq 1$, we let $\mathcal{N}_k^{2^{-n}}$ be the point measure of excursions of the Brownian snake outside $(-k2^{-n}, \infty)$, and we write

$$\mathcal{N}_k^{2^{-n}} = \sum_{i \in I_k^{2^{-n}}} \delta_{\omega_i^{k,2^{-n}}}.$$

Recall that, for every atom $\omega_i^{k,2^{-n}}$, $\tilde{\omega}_i^{k,2^{-n}}$ stands for $\omega_i^{k,2^{-n}}$ translated so that its starting point is 2^{-n} and then truncated at level 0. Furthermore, we let $A_{n,k}$ stand for the event $\{T_1^\delta \geq k2^{-n}\} = \{V_{u_1^\delta} \leq -k2^{-n}\}$. Finally, we let $B \in \mathcal{E}^{(-\beta,\infty)}$.

We then claim that

$$\begin{aligned}
&\mathbb{N}_0^{(\beta)}\big(\mathbf{1}_B\, g(V_{u_1^\delta}) G(\tilde{W}^{(u_1^\delta)})\, \mathbf{1}_{\{N_\delta \geq 1\}}\big) \\
&\qquad = \lim_{n\to\infty} \mathbb{N}_0^{(\beta)}\Big(\mathbf{1}_B \sum_{k=1}^{\infty} \mathbf{1}_{A_{n,k}}\, g(-k2^{-n}) \sum_{i \in I_k^{2^{-n}}} G(\tilde{\omega}_i^{k,2^{-n}})\Big).
\end{aligned} \tag{44}$$

In order to verify our claim, we first observe that

$$\sum_{k=1}^{\infty} \mathbf{1}_{A_{n,k}}\, g(-k2^{-n}) \sum_{i \in I_k^{2^{-n}}} G(\tilde{\omega}_i^{k,2^{-n}}) \xrightarrow[n\to\infty]{} g(V_{u_1^\delta}) G(\tilde{W}^{(u_1^\delta)})\, \mathbf{1}_{\{N_\delta \geq 1\}}, \qquad \mathbb{N}_0\text{-a.e.} \tag{45}$$

To see this, note that if $N_\delta = 0$ then, for n large enough, all quantities $G(\tilde{\omega}_i^{k,2^{-n}})$ vanish (the point is that if $G(\tilde{\omega}_i^{k,2^{-n}}) > 0$, then the excursion $\omega_i^{k,2^{-n}}$ must "contain" an excursion debut with height greater than $\delta - 2^{-n}$, and no such excursion debut exists when n is large enough, under the condition $N_\delta = 0$). Then, if $N_\delta \geq 1$, similar arguments show that, for n large enough, the only nonzero term in the sum over k in the left-hand side of (45) corresponds to the integer $k_0 = k_0(n)$ such that $-(k_0 + 1)2^{-n} < V_{u_1^\delta} \leq -k_0 2^{-n}$. Indeed, we have $\mathbf{1}_{A_{n,k}} = 0$ if $k > k_0$, since $A_{n,k} = \{V_{u_1^\delta} \leq -k2^{-n}\}$. On the other hand, if n is large enough, then, for $k < k_0$, the quantities $G(\tilde{\omega}_i^{k,2^{-n}})$, $i \in I_k^{2^{-n}}$, vanish by the same argument as used above in the case $N_\delta = 0$—recall that G is zero outside the set $\{\omega : M(\omega) > \delta\}$,

Next, for $k = k_0$, the sum over $i \in I_k^{2^{-n}}$ reduces (for n large enough) to a single term, namely $i = i_0 = i_{u_\delta^1, 2^{-n}}$ with the notation of Lemma 24. The last assertion of Lemma 24 implies that $G(\tilde{\omega}_{i_0}^{k_0,2^{-n}})$ converges to $G(\tilde{W}^{(u_1^\delta)})$ as $n \to \infty$, and (45) follows.

To derive (44) from (45), we use exactly the same uniform integrability argument as in the proof of Theorem 23 to justify the convergence (20).

Next recall that $A_{n,k}$ is measurable with respect to the σ-field $\mathcal{E}^{(-k2^{-n},\infty)}$, and note that $g(-k2^{-n}) = 0$ if $k \le 2^n\beta$. By applying the special Markov property, we then get

$$\begin{aligned}
&\mathbb{N}_0^{(\beta)}\Big(\mathbf{1}_B \sum_{k\ge 2^n\beta} \mathbf{1}_{A_{n,k}} g(-k2^{-n}) \sum_{i\in I_k^{2^{-n}}} G(\tilde\omega_i^{k,2^{-n}})\Big)\\
&\quad= \sum_{k\ge 2^n\beta} g(-k2^{-n})\, \mathbb{N}_0^{(\beta)}\Big(\mathbf{1}_B \mathbf{1}_{A_{n,k}} \mathbb{N}_0^{(\beta)}\Big(\sum_{i\in I_k^{2^{-n}}} G(\tilde\omega_i^{k,2^{-n}}) \,\Big|\, \mathcal{E}^{(-k2^{-n},\infty)}\Big)\Big)\\
&\quad= \sum_{k\ge 2^n\beta} g(-k2^{-n})\, \mathbb{N}_0^{(\beta)}\big(\mathbf{1}_B \mathbf{1}_{A_{n,k}} Z_{k2^{-n}}\, \mathbb{N}_{2^{-n}}(G(\tilde W))\big)\\
&\quad= \Big(\sum_{k\ge 2^n\beta} g(-k2^{-n})\, \mathbb{N}_0^{(\beta)}(\mathbf{1}_B \mathbf{1}_{A_{n,k}} Z_{k2^{-n}})\Big) \times \mathbb{N}_{2^{-n}}(G(\tilde W)).
\end{aligned}$$

Recalling (44), we have thus obtained

$$\begin{aligned}
\lim_{n\to\infty}&\Big(\sum_{k\ge 2^n\beta} g(-k2^{-n})\mathbb{N}_0^{(\beta)}(\mathbf{1}_B \mathbf{1}_{A_{n,k}} Z_{k2^{-n}})\Big) \times \mathbb{N}_{2^{-n}}(G(\tilde W))\\
&= \mathbb{N}_0^{(\beta)}\big(\mathbf{1}_B g(V_{u_1^\delta}) G(\tilde W^{(u_1^\delta)})\, \mathbf{1}_{\{N_\delta\ge 1\}}\big).
\end{aligned} \tag{46}$$

In the particular case $G = \mathbf{1}_{\{M>\delta\}}$ this gives

$$\begin{aligned}
\lim_{n\to\infty}&\Big(\sum_{k\ge 2^n\beta} g(-k2^{-n})\mathbb{N}_0^{(\beta)}(\mathbf{1}_B \mathbf{1}_{A_{n,k}} Z_{k2^{-n}})\Big) \times \mathbb{N}_{2^{-n}}(\tilde M > \delta)\\
&= \mathbb{N}_0^{(\beta)}\big(\mathbf{1}_B g(V_{u_1^\delta})\, \mathbf{1}_{\{N_\delta\ge 1\}}\big),
\end{aligned} \tag{47}$$

since $M(\tilde W^{(u_1^\delta)}) > \delta$ by construction. It follows from (46) and (47) that

$$\begin{aligned}
\mathbb{N}_0^{(\beta)}\big(\mathbf{1}_B g(V_{u_1^\delta}) G(\tilde W^{(u_1^\delta)})\, \mathbf{1}_{\{N_\delta\ge 1\}}\big) &= \mathbb{N}_0^{(\beta)}\big(\mathbf{1}_B g(V_{u_1^\delta})\, \mathbf{1}_{\{N_\delta\ge 1\}}\big) \times \lim_{n\to\infty} \frac{\mathbb{N}_{2^{-n}}(G(\tilde W))}{\mathbb{N}_{2^{-n}}(\tilde M > \delta)}\\
&= \mathbb{N}_0^{(\beta)}\big(\mathbf{1}_B g(V_{u_1^\delta})\, \mathbf{1}_{\{N_\delta\ge 1\}}\big) \times \mathbb{N}_0^*(G \mid M > \delta),
\end{aligned}$$

by Corollary 26. The last display shows both that $\tilde W^{(u_1^\delta)}$ is distributed according to $\mathbb{N}_0^*(\cdot \mid M > \delta)$ under $\mathbb{N}_0^{(\beta)}(\cdot \mid N_\delta \ge 1)$ (take a sequence of functions g that increase to the indicator function of $(-\infty, -\beta)$) and that $\tilde W^{(u_1^\delta)}$ is independent of the σ-field generated by $V_{u_1^\delta}$ and $\mathcal{E}^{(\beta,\infty)}$, still under $\mathbb{N}_0^{(\beta)}(\cdot \mid N_\delta \ge 1)$.

We have shown that the law of the first excursion above the minimum with height greater than δ and level smaller than $-\beta$, under $\mathbb{N}_0^{(\beta)}(\cdot \mid N_\delta \ge 1)$, is $\mathbb{N}_0^*(\cdot \mid M > \delta)$. By letting β tend to 0, we deduce that the law of the first excursion above the minimum with height greater than δ, under $\mathbb{N}_0(\cdot \mid N_\delta^\circ \ge 1)$, is also $\mathbb{N}_0^*(\cdot \mid M > \delta)$—we recall our notation N_δ° for the total number of excursion debuts with height greater than δ. Moreover, the same passage to the limit shows that this first excursion is independent of the level at which it occurs. These remarks will be useful in the second part of the proof.

The general statement of the proposition can be deduced from the special case $j = 1$, via an induction argument using the special Markov property. Let us explain this argument in detail when $j = 2$ (the reader will be able to fill in the details needed for a general value of j). Let G_1 and G_2 be two nonnegative measurable functions on $\mathcal{S}$, and consider again $B \in \mathcal{E}^{(-\beta,\infty)}$. Recall that $T_1^\delta > \beta$ by definition. By monotone convergence, we have

$$\begin{aligned}&\mathbb{N}_0^{(\beta)}\big(\mathbf{1}_B G_1(\tilde{W}^{(u_1^\delta)})G_2(\tilde{W}^{(u_2^\delta)})\,\mathbf{1}_{\{N_\delta\geq 2\}}\big)\\&\quad=\lim_{n\to\infty}\sum_{k\geq 2^n\beta}\mathbb{N}_0^{(\beta)}\big(\mathbf{1}_B G_1(\tilde{W}^{(u_1^\delta)})G_2(\tilde{W}^{(u_2^\delta)})\,\mathbf{1}_{\{k2^{-n}\leq T_1^\delta<(k+1)2^{-n}\leq T_2^\delta<\infty\}}\big).\end{aligned}\tag{48}$$

Then, for every $k \geq 2^n\beta$, noting that $\mathbf{1}_B G_1(\tilde{W}^{(u_1^\delta)})\,\mathbf{1}_{\{T_1^\delta<(k+1)2^{-n}\}}$ is $\mathcal{E}^{(-(k+1)2^{-n},\infty)}$-measurable, we get

$$\begin{aligned}&\mathbb{N}_0^{(\beta)}\big(\mathbf{1}_B G_1(\tilde{W}^{(u_1^\delta)})G_2(\tilde{W}^{(u_2^\delta)})\,\mathbf{1}_{\{k2^{-n}\leq T_1^\delta<(k+1)2^{-n}\leq T_2^\delta<\infty\}}\big)\\&\qquad=\mathbb{N}_0^{(\beta)}\big(\mathbf{1}_B G_1(\tilde{W}^{(u_1^\delta)})\,\mathbf{1}_{\{k2^{-n}\leq T_1^\delta<(k+1)2^{-n}\leq T_2^\delta\}}\\&\qquad\qquad\times\mathbb{N}_0^{(\beta)}\big(G_2(\tilde{W}^{(u_2^\delta)})\,\mathbf{1}_{\{T_2^\delta<\infty\}}\;\big|\;\mathcal{E}^{(-(k+1)2^{-n},\infty)}\big)\big).\end{aligned}\tag{49}$$

Applying the special Markov property (Proposition 13) to the interval $(-(k+1)2^{-n},\infty)$ now gives, on the event $\{T_1^\delta < (k+1)2^{-n} \leq T_2^\delta\}$,

$$\begin{aligned}&\mathbb{N}_0^{(\beta)}\big(G_2(\tilde{W}^{(u_2^\delta)})\,\mathbf{1}_{\{T_2^\delta<\infty\}}\;\big|\;\mathcal{E}^{(-(k+1)2^{-n},\infty)}\big)\\&\qquad=\big(1-\exp(-Z_{(k+1)2^{-n}}\,\mathbb{N}_0(N_\delta\geq 1))\big)\,\mathbb{N}_0^*(G_2\mid M>\delta).\end{aligned}\tag{50}$$

Let us explain this. From the special Markov property, there is a Poisson number ν with parameter $Z_{(k+1)2^{-n}}\mathbb{N}_0(N_\delta^\circ \geq 1)$ of Brownian snake excursions outside $(-(k+1)2^{-n},\infty)$ that contain at least one excursion debut with height greater than δ, and these excursions are independent and distributed according to $\mathbb{N}_0(\cdot \mid N_\delta^\circ \geq 1)$, modulo the obvious translation by $(k+1)2^{-n}$. For each of these ν excursions, the first excursion above the minimum with height greater than δ is distributed according to $\mathbb{N}_0^*(\cdot \mid M > \delta)$, and is independent of the level at which it occurs (by the first part of the proof). On the event $\{T_1^\delta < (k+1)2^{-n} \leq T_2^\delta\}$, $\tilde{W}^{(u_2^\delta)}$ is well defined if $T_2^\delta < \infty$, which is equivalent to $\nu \geq 1$, and is obtained by taking, among those first excursions above the minimum with height greater than δ, the one that occurs at the highest level. Clearly it is also distributed according to $\mathbb{N}_0^*(\cdot \mid M > \delta)$.

Since we have $1-\exp(-Z_{(k+1)2^{-n}}\,\mathbb{N}_0(N_\delta^\circ \geq 1)) = \mathbb{N}_0^{(\beta)}(T_2^\delta < \infty \mid \mathcal{E}^{(-(k+1)2^{-n},\infty)})$ on the event $\{T_1^\delta < (k+1)2^{-n} \leq T_2^\delta\}$, we deduce from (49) and (50) that, for every $k \geq 2^n\beta$,

$$\begin{aligned}&\mathbb{N}_0^{(\beta)}\big(\mathbf{1}_B G_1(\tilde{W}^{(u_1^\delta)})G_2(\tilde{W}^{(u_2^\delta)})\,\mathbf{1}_{\{k2^{-n}\leq T_1^\delta<(k+1)2^{-n}\leq T_2^\delta<\infty\}}\big)\\&\quad=\mathbb{N}_0^{(\beta)}\big(\mathbf{1}_B G_1(\tilde{W}^{(u_1^\delta)})\,\mathbf{1}_{\{k2^{-n}\leq T_1^\delta<(k+1)2^{-n}\leq T_2^\delta\}}\mathbb{N}_0^{(\beta)}(T_2^\delta<\infty\mid\mathcal{E}^{(-(k+1)2^{-n},\infty)})\big)\\&\qquad\times\mathbb{N}_0^*(G_2\mid M>\delta)\\&\quad=\mathbb{N}_0^{(\beta)}\big(\mathbf{1}_B G_1(\tilde{W}^{(u_1^\delta)})\,\mathbf{1}_{\{k2^{-n}\leq T_1^\delta<(k+1)2^{-n}\leq T_2^\delta<\infty\}}\big)\times\mathbb{N}_0^*(G_2\mid M>\delta).\end{aligned}$$

Finally, returning to (48), by monotone convergence we obtain

$$\mathbb{N}_0^{(\beta)}\big(\mathbf{1}_B G_1(\tilde W^{(u_1^\delta)})G_2(\tilde W^{(u_2^\delta)})\,\mathbf{1}_{\{N_\delta\geq 2\}}\big) = \mathbb{N}_0^{(\beta)}\big(\mathbf{1}_B G_1(\tilde W^{(u_1^\delta)})\,\mathbf{1}_{\{N_\delta\geq 2\}}\big)\mathbb{N}_0^*(G_2 \mid M>\delta).$$

This gives the case $j=2$ of the proposition. □

Remark. We could have shortened the proof a little by using a strong version of the special Markov property (applying to a random interval $(-T,\infty)$) of the type discussed in [9].

The next lemma shows that the sequence $(\tilde W^{(u_1^\delta)},\dots,\tilde W^{(u_{N_\delta}^\delta)})$ can be viewed as the beginning of an i.i.d. sequence.

Lemma 35. *On an auxiliary probability space $(\Omega,\mathcal{F},\mathbb{P})$, consider a sequence $(\overline{W}^{\delta,1},\overline{W}^{\delta,2},\dots)$ of independent random variables distributed according to $\mathbb{N}_0^*(\cdot\mid M>\delta)$. Under the product probability measure $\mathbb{P}\otimes\mathbb{N}_0^{(\beta)}$, consider the sequence $(W^{\delta,1},W^{\delta,2},\dots)$ defined by*

$$W^{\delta,j}=\begin{cases}\tilde W^{(u_j^\delta)} & \text{if } 1\leq j\leq N_\delta,\\ \overline{W}^{\delta,j-N_\delta} & \text{if } j>N_\delta.\end{cases}$$

Then $(W^{\delta,1},W^{\delta,2},\dots)$ is a sequence of i.i.d. random variables distributed according to $\mathbb{N}_0^*(\cdot\mid M>\delta)$, and this sequence is independent of the σ-field $\mathcal{E}^{(-\beta,\infty)}$.

Proof. This follows from Proposition 34 by an argument which is valid in a much more general setting. Let us give a few details. Let $k\geq 2$, and let $\phi_1,\dots,\phi_k$ be bounded nonnegative measurable functions defined on $\mathcal{S}$. Also let $B\in\mathcal{E}^{(-\beta,\infty)}$. We need to verify that

$$E[\mathbf{1}_B\phi_1(W^{\delta,1})\phi_2(W^{\delta,2})\cdots\phi_k(W^{\delta,k})]=\mathbb{N}_0^{(\beta)}(B)\times\prod_{i=1}^k\mathbb{N}_0^*(\phi_i\mid M>\delta), \tag{51}$$

where $E[\cdot]$ stands for the expectation under $\mathbb{P}\otimes\mathbb{N}_0^{(\beta)}$. By dealing separately with the possible values of N_δ and using the independence of the $\overline{W}^{\delta,j}$'s, we immediately get

$$E[\mathbf{1}_{\{N_\delta<k\}}\mathbf{1}_B\phi_1(W^{\delta,1})\cdots\phi_k(W^{\delta,k})] = E[\mathbf{1}_{\{N_\delta<k\}}\mathbf{1}_B\phi_1(W^{\delta,1})\cdots\phi_{k-1}(W^{\delta,k-1})]\times\mathbb{N}_0^*(\phi_k\mid M>\delta).$$

On the other hand, Proposition 34 exactly says that

$$\begin{aligned}E[\mathbf{1}_{\{N_\delta\geq k\}}\mathbf{1}_B\phi_1(W^{\delta,1})\cdots\phi_k(W^{\delta,k})] &= E[\mathbf{1}_{\{N_\delta\geq k\}}\mathbf{1}_B\phi_1(\tilde W^{(u_1^\delta)})\cdots\phi_k(\tilde W^{(u_k^\delta)})]\\ &= E[\mathbf{1}_{\{N_\delta\geq k\}}\mathbf{1}_B\phi_1(W^{\delta,1})\cdots\phi_{k-1}(W^{\delta,k-1})]\times\mathbb{N}_0^*(\phi_k\mid M>\delta).\end{aligned}$$

By summing the last two displays, we get

$$E[\mathbf{1}_B\phi_1(W^{\delta,1})\cdots\phi_k(W^{\delta,k})]=E[\mathbf{1}_B\phi_1(W^{\delta,1})\cdots\phi_{k-1}(W^{\delta,k-1})]\times\mathbb{N}_0^*(\phi_k\mid M>\delta),$$

and the proof of (51) is completed by an induction argument. □

7.2. Excursion debuts and discontinuities of the exit measure process

We start with a first proposition that relates levels of excursion debuts to discontinuity times for the process $(Z_x)_{x>0}$.

Proposition 36. $\mathbb{N}_0$*-a.e., discontinuity times for the process* $(Z_x)_{x>0}$ *are exactly all reals of the form* $-V_u$ *for* $u \in D$.

Proof. Recall that for every $x \geq 0$ we have set

$$\mathcal{Y}_x = \int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{\tau_{-x}(W_s)=\infty\}}.$$

If (x_n) is an increasing sequence that converges to $x > 0$, then the indicator functions $\mathbf{1}_{\{\tau_{-x_n}(W_s)=\infty\}}$ converge to $\mathbf{1}_{\{\tau_{-x}(W_s)=\infty\}}$, and by dominated convergence it follows that $(\mathcal{Y}_x)_{x>0}$ has left-continuous sample paths. On the other hand, if (x_n) is a decreasing sequence that converges to $x > 0$, with $x_n > x$ for every n, one immediately gets

$$\int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{\tau_{-x_n}(W_s)=\infty\}} \xrightarrow[n\to\infty]{} \int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{\underline{W}_s \geq -x\}}.$$

It follows that $(\mathcal{Y}_x)_{x>0}$ also has right limits, and that x is a discontinuity point of $\mathcal{Y}$ if and only if

$$\int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{\underline{W}_s = -x\}} > 0.$$

The latter condition holds if and only if there exists $s \in [0, \sigma]$ such that $\hat{W}_s > -x$ and $\underline{W}_s = -x$ (we use the fact that $\mathbb{N}_0$-a.e. for every $y \in \mathbb{R}$, $\int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{\hat{W}_s = y\}} = 0$, which follows from the existence of local times for the tip process of the Brownian snake; see e.g. [7]). However, the existence of $s \in [0, \sigma]$ such that $\hat{W}_s > -x$ and $\underline{W}_s = -x$ implies that there is an excursion debut u with $V_u = -x$, and the converse is also true. Summarizing, we have shown that the discontinuity times for the process $(\mathcal{Y}_x)_{x>0}$ are exactly all reals of the form $-V_u$ for $u \in D$.

To complete the proof of the proposition, we use the fact that the discontinuity times for $(\mathcal{Y}_x)_{x>0}$ are the same as the discontinuity times for $(Z_x)_{x>0}$, as a consequence of [9, Corollary 4.9] which essentially identifies the joint distribution of this pair of processes. To be precise, the latter result is not concerned with the processes Z and $\mathcal{Y}$ under $\mathbb{N}_0$ but with superpositions of these processes corresponding to a Poisson measure with intensity $\mathbb{N}_0$. A simple argument however shows that this implies the result we need. □

We now identify the value of the jump of the process Z at time $-V_u$ when $u \in D$. For every $u \in D$, the exit measure $Z_0^*(\tilde{W}^{(u)})$ makes sense by (36), and can also be defined by the approximation in Proposition 30, by using Proposition 34 to relate properties of $\tilde{W}^{(u)}$ to those valid $\mathbb{N}_0^*$-a.e.

Proposition 37. $\mathbb{N}_0$*-a.e. for every* $u \in D$*, the jump of the process* Z *at time* $-V_u$ *is equal to* $Z_0^*(\tilde{W}^{(u)})$.

Proof. We fix $\delta > 0$; we will prove that the assertion of the proposition holds $\mathbb{N}_0^{(\beta)}$-a.e. when $u = u_1^\delta$, the first excursion debut with level smaller than $-\beta$ and height greater than δ, on the event $\{N_\delta \geq 1\}$. We then observe that, for any excursion debut u, there are choices of rationals β and δ that make u the first excursion debut with level smaller than $-\beta$ and height greater than δ. This gives the desired result for every $u \in D$.

So from now on we focus on the case $u = u_1^\delta$, and in what follows we restrict our attention to the event $\{N_\delta \geq 1\}$, so that u_1^δ is well-defined. Recall that for integers $n \geq 1$ and $k \geq 1$, $(\omega_i^{k,2^{-n}})_{i \in I_k^{2^{-n}}}$ is the collection of excursions of the Brownian snake outside $(-k2^{-n}, \infty)$, and we keep using the notation $\tilde{\omega}_i^{k,2^{-n}}$ for $\omega_i^{k,2^{-n}}$ translated so that its starting point is 2^{-n} and then truncated at level 0. Let n_0 be the first integer such that $2^{n_0}\beta \geq 1$. From now on we consider values of n such that $n \geq n_0$. We define $H_n = \lfloor -2^n V_{u_1^\delta} \rfloor \geq 1$, in such a way that

$$H_n \, 2^{-n} \leq -V_{u_1^\delta} < (H_n + 1)2^{-n}. \tag{52}$$

If we set for $\omega \in \mathcal{S}$,

$$O(\omega) = \sup\{\hat{W}_s(\omega) - \underline{W}_s(\omega) : 0 \leq s \leq \sigma\},$$

then H_n is the first integer $k \geq 1$ such that $O(\tilde{\omega}_i^{k,2^{-n}}) > \delta$ for some $i \in I_k^{2^{-n}}$. This i may not be unique, and for this reason we introduce the event $A_n \subset \{N_\delta \geq 1\}$ where the property $O(\tilde{\omega}_i^{k,2^{-n}}) > \delta$ holds for $k = H_n$ for exactly one $i = i_n \in I_{H_n}^{2^{-n}}$. On the event A_n, we let $\omega_{(n)} = \tilde{\omega}_{i_n}^{H_n,2^{-n}}$ be the corresponding excursion, and on the complement of A_n we let $\omega_{(n)}$ be the trivial snake path with duration 0 in $\mathcal{S}_0$. Notice that, on the event A_n, the excursion debut u_1^δ must then belong to (the subtree coded by the interval corresponding to) the excursion $\omega_{i_n}^{H_n,2^{-n}}$. We also note that the sequence $(A_n)_{n \geq n_0}$ is increasing, and $\mathbb{N}_0^{(\beta)}(A_n \mid N_\delta \geq 1)$ converges to 1 as $n \to \infty$ because there cannot be two excursion debuts at the same level (and therefore—recall Lemma 22—two excursion debuts with height greater than δ must be "macroscopically separated").

Furthermore, we claim that the distribution of $\omega_{(n)}$ under $\mathbb{N}_0^{(\beta)}(\cdot \mid A_n)$ is the law of $\tilde{W}$ under $\mathbb{N}_{2^{-n}}(\cdot \mid O(\tilde{W}) > \delta)$. This is basically a consequence of the special Markov property, but we will provide a few details. Let Φ be a nonnegative measurable function on $\mathcal{S}$ such that $\Phi(\omega) = 0$ if $O(\omega) \leq \delta$. For every $k \geq 1$, let $B_{n,k}$ be the event that there is a unique $i \in I_k^{2^{-n}}$ such that $O(\tilde{\omega}_i^{k,2^{-n}}) > \delta$. Then

$$\mathbb{N}_0^{(\beta)}(\mathbf{1}_{A_n} \mathbf{1}_{\{H_n = k\}} \Phi(\omega_{(n)})) = \mathbb{N}_0^{(\beta)}\Big(\mathbf{1}_{\{H_n \geq k\}} \mathbf{1}_{B_{n,k}} \sum_{i \in I_k^{2^{-n}}} \Phi(\tilde{\omega}_i^{k,2^{-n}})\Big).$$

We observe that the event $\{H_n \geq k\}$ is measurable with respect to the σ-field $\mathcal{E}^{(-k2^{-n},\infty)}$, because, if $j < k$, the property $O(\tilde{\omega}_i^{j,2^{-n}}) > \delta$ for some $i \in I_j^{2^{-n}}$ can be checked from the snake W truncated at level $-k2^{-n}$. Therefore we can apply the special Markov property, using the fact that if a Poisson measure with intensity μ is conditioned to have a single atom in a measurable set C of positive and finite μ-measure, the law of this atom is $\mu(\cdot \mid C)$. It follows that the quantities in the last display are equal to

$$\mathbb{N}_0^{(\beta)}\big(\mathbf{1}_{\{H_n\geq k\}}\mathbf{1}_{B_{n,k}}\mathbb{N}_{2^{-n}}(\Phi(\tilde{W}) \mid O(\tilde{W}) > \delta)\big) \\ = \mathbb{N}_0^{(\beta)}(\mathbf{1}_{A_n}\mathbf{1}_{\{H_n=k\}}) \times \mathbb{N}_{2^{-n}}(\Phi(\tilde{W}) \mid O(\tilde{W}) > \delta).$$

We then sum over $k \geq 1$ to get the desired claim.

We then note that, for every $n \geq n_0$, on the event A_n we have

$$Z_{(H_n+1)2^{-n}} = \sum_{i\in I_{H_n}^{2^{-n}}} \mathcal{Z}_0(\tilde{\omega}_i^{H_n,2^{-n}}) = \mathcal{Z}_0(\omega_{(n)}) + \sum_{i\in I_{H_n}^{2^{-n}},\, i\neq i_n} \mathcal{Z}_0(\tilde{\omega}_i^{H_n,2^{-n}}). \tag{53}$$

To simplify notation, we write $b = -V_{u_1^\delta}$. We claim that

$$\sum_{i\in I_{H_n}^{2^{-n}},\, i\neq i_n} \mathcal{Z}_0(\tilde{\omega}_i^{H_n,2^{-n}}) \xrightarrow[n\to\infty]{} Z_{b-}, \tag{54}$$

where the convergence holds in probability under $\mathbb{N}_0^{(\beta)}(\cdot \mid N_\delta \geq 1)$—the fact that i_n is only defined on A_n poses no problem here since $\mathbb{N}_0^{(\beta)}(A_n \mid N_\delta \geq 1)$ converges to 1.

Proof of (54). It will be convenient to introduce the point measure

$$\tilde{\mathcal{N}}_k^{2^{-n}} = \sum_{i\in I_k^{2^{-n}}} \delta_{\tilde{\omega}_i^{k,2^{-n}}},$$

for every $n \geq 1$ and $k \geq 1$. We first observe that, on the event A_n, we have the equality

$$\sum_{i\in I_{H_n}^{2^{-n}},\, i\neq i_n} \mathcal{Z}_0(\tilde{\omega}_i^{H_n,2^{-n}}) = \int_{\{O\leq\delta\}} \tilde{\mathcal{N}}_{H_n}^{2^{-n}}(\mathrm{d}\omega)\, \mathcal{Z}_0(\omega).$$

Since $\mathbb{N}_0^{(\beta)}(A_n \mid N_\delta \geq 1)$ converges to 1, the proof of (54) reduces to checking that

$$\int_{\{O\leq\delta\}} \tilde{\mathcal{N}}_{H_n}^{2^{-n}}(\mathrm{d}\omega)\, \mathcal{Z}_0(\omega) \xrightarrow[n\to\infty]{} Z_{b-}.$$

Since $2^{-n}H_n \uparrow -V_{u_1^\delta} = b$, we have $Z_{2^{-n}H_n} \to Z_{b-}$, a.e. under $\mathbb{N}_0^{(\beta)}(\cdot \mid N_\delta \geq 1)$, and so it is enough to prove that

$$\int_{\{O\leq\delta\}} \tilde{\mathcal{N}}_{H_n}^{2^{-n}}(\mathrm{d}\omega)\, \mathcal{Z}_0(\omega) - Z_{2^{-n}H_n} \xrightarrow[n\to\infty]{} 0.$$

Note that we may have $H_n = \lfloor -2^n V_{u_1^\delta}\rfloor < 2^n\beta$ although $-V_{u_1^\delta} \geq \beta$, but this occurs with $\mathbb{N}_0^{(\beta)}$-probability tending to 0. Thanks to this observation, the preceding convergence will hold provided that, for every $\varepsilon > 0$, the quantities in the next display tend to 0 as $n \to \infty$:

$$\begin{aligned}
&\mathbb{N}_0^{(\beta)}\bigg(\bigg\{\bigg|\int_{\{O\le\delta\}} \tilde{\mathcal{N}}_{H_n}^{2^{-n}}(\mathrm{d}\omega)\,\mathcal{Z}_0(\omega) - Z_{2^{-n}H_n}\bigg| > \varepsilon\bigg\} \cap \{N_\delta \ge 1\} \cap \{H_n \ge 2^n\beta\}\bigg)\\
&= \sum_{k\ge 2^n\beta} \mathbb{N}_0^{(\beta)}\bigg(\bigg\{\bigg|\int_{\{O\le\delta\}} \tilde{\mathcal{N}}_{k}^{2^{-n}}(\mathrm{d}\omega)\,\mathcal{Z}_0(\omega) - Z_{k2^{-n}}\bigg| > \varepsilon\bigg\} \cap \{N_\delta \ge 1\} \cap \{H_n = k\}\bigg)\\
&= \sum_{k\ge 2^n\beta} \mathbb{N}_0^{(\beta)}\bigg(\bigg\{\bigg|\int_{\{O\le\delta\}} \tilde{\mathcal{N}}_{k}^{2^{-n}}(\mathrm{d}\omega)\,\mathcal{Z}_0(\omega) - Z_{k2^{-n}}\bigg| > \varepsilon\bigg\}\\
&\qquad\qquad\qquad\qquad \cap \{\tilde{\mathcal{N}}_k^{2^{-n}}(O>\delta) \ge 1\} \cap \{H_n \ge k\}\bigg).
\end{aligned}$$

The last equality holds because the event $\{N_\delta \ge 1\} \cap \{H_n = k\}$ coincides with $\{\tilde{\mathcal{N}}_k^{2^{-n}}(O > \delta) \ge 1\} \cap \{H_n \ge k\}$. Next we recall that the event $\{H_n \ge k\}$ is $\mathcal{E}^{(-k2^{-n},\infty)}$-measurable and we notice that, under $\mathbb{N}_0^{(\beta)}$, conditionally on $\mathcal{E}^{(-k2^{-n},\infty)}$, $\tilde{\mathcal{N}}_k^{2^{-n}}$ is a Poisson measure whose intensity is $Z_{k2^{-n}}$ times the "law" of $\tilde{W}$ under $\mathbb{N}_{2^{-n}}$. It follows that the quantities in the last display are also equal to

$$\sum_{k\ge 2^n\beta} \mathbb{N}_0^{(\beta)}\big(\psi_\varepsilon^n(Z_{k2^{-n}})\,\mathbf{1}_{\{\tilde{\mathcal{N}}_k^{2^{-n}}(O>\delta)\ge 1\}\cap\{H_n\ge k\}}\big), \tag{55}$$

where, for every $a \ge 0$,

$$\psi_\varepsilon^n(a) = P\bigg(\bigg|\int_{\{O\le\delta\}} \mathcal{N}_{n,a}(\mathrm{d}\omega)\,\mathcal{Z}_0(\omega) - a\bigg| > \varepsilon\bigg),$$

and $\mathcal{N}_{n,a}$ denotes a Poisson measure whose intensity is a times the "law" of $\tilde{W}$ under $\mathbb{N}_{2^{-n}}$. It is easy to verify that $\psi_\varepsilon^n(a)$ tends to 0 as $n \to \infty$, for every fixed a. First note that we can remove the restriction to $\{O \le \delta\}$ since $P(\mathcal{N}_{n,a}(O > \delta) > 0)$ tends to 0. Then we just have to observe that $\int \mathcal{N}_{n,a}(\mathrm{d}\omega)\,\mathcal{Z}_0(\omega)$ converges in probability to a as $n \to \infty$, as a straightforward consequence of (9). Furthermore, a simple monotonicity argument shows that the convergence of $\psi_\varepsilon^n(a)$ to 0 holds uniformly when a varies over a compact subset of $\mathbb{R}_+$.

Finally, since again $\{\tilde{\mathcal{N}}_k^{2^{-n}}(O > \delta) \ge 1\} \cap \{H_n \ge k\} = \{N_\delta \ge 1\} \cap \{H_n = k\}$, we see that the quantity in (55) is bounded by

$$\mathbb{N}_0^{(\beta)}(\psi_\varepsilon^n(Z_{2^{-n}H_n})\,\mathbf{1}_{\{N_\delta\ge 1\}}),$$

and this tends to 0 as $n\to\infty$ by the previous observations and since $\sup\{Z_a : a \ge 0\} < \infty$, $\mathbb{N}_0$-a.e. This completes the proof of our claim (54). □

Let us complete the proof of the proposition. We already noticed that the distribution of $\omega_{(n)}$ under $\mathbb{N}_0^{(\beta)}(\cdot \mid A_n)$ is the law of $\tilde{W}$ under $\mathbb{N}_{2^{-n}}(\cdot \mid O(\tilde{W}) > \delta)$. We observe that, for every $\varepsilon > 0$, the following inclusions hold $\mathbb{N}_\varepsilon$-a.e.:

$$\{\tilde{M} > \delta + \varepsilon\} \subset \{O(\tilde{W}) > \delta\} \subset \{\tilde{M} > \delta\}$$

and moreover the ratio $\mathbb{N}_\varepsilon(\tilde{M} \geq \delta + \varepsilon)/\mathbb{N}_\varepsilon(\tilde{M} > \delta)$ tends to 1 as $\varepsilon \to 0$. It follows that the result of Proposition 32 remains valid if, in the definition of $W^{\delta,\varepsilon}$, the conditioning by $\{\tilde{M} > \delta\}$ is replaced by $\{O(\tilde{W}) > \delta\}$. Thanks to this simple observation, we can deduce from Proposition 32 that

$$(\omega_{(n)}, \mathcal{Z}_0(\omega_{(n)})) \xrightarrow[n\to\infty]{\text{(d)}} (W^{\delta,0}, Z_0^*(W^{\delta,0})), \tag{56}$$

where $W^{\delta,0}$ is distributed according to $\mathbb{N}_0^*(\cdot \mid M > \delta)$ and the convergence holds in distribution under $\mathbb{N}_0^{(\beta)}(\cdot \mid N_\delta \geq 1)$. Furthermore, from the last assertion of Lemma 24, and the fact that $\omega_{i_n}^{H_n,2^{-n}}$ is, on the event A_n, the excursion outside $(-H_n 2^{-n}, \infty)$ that "contains" u_1^δ, we deduce that $\omega_{(n)}$ converges to $\tilde{W}^{(u_1^\delta)}$, $\mathbb{N}_0^{(\beta)}$-a.e. on $\{N_\delta \geq 1\}$.

On the other hand, (52) and the right-continuity of sample paths of Z imply that

$$Z_{(H_n+1)2^{-n}} \xrightarrow[n\to\infty]{} Z_b, \tag{57}$$

$\mathbb{N}_0^{(\beta)}$-a.s. on $\{N_\delta \geq 1\}$. Then, using (53), (54) and (57), we immediately see that $\mathcal{Z}_0(\omega_{(n)})$ converges to the random variable $Z_b - Z_{b-}$, in probability under $\mathbb{N}_0^{(\beta)}(\cdot \mid N_\delta \geq 1)$. So we infer that the pair $(\omega_{(n)}, \mathcal{Z}_0(\omega_{(n)}))$ converges in probability to $(\tilde{W}^{(u_1^\delta)}, Z_b - Z_{b-})$ under $\mathbb{N}_0^{(\beta)}(\cdot \mid N_\delta \geq 1)$, and it follows from (56) that the law of $(\tilde{W}^{(u_1^\delta)}, Z_b - Z_{b-})$ under $\mathbb{N}_0^{(\beta)}(\cdot \mid N_\delta \geq 1)$ is the law of $(W^{\delta,0}, Z_0^*(W^{\delta,0}))$. This forces $Z_b - Z_{b-} = Z_0^*(\tilde{W}^{(u_1^\delta)})$, which completes the proof. □

7.3. The Poisson process of excursions

The following proposition is reminiscent of Itô's famous Poisson point process of excursions of linear Brownian motion. We recall that $\beta > 0$ is fixed and that $u_1^\delta, \ldots, u_{N_\delta}^\delta$ are the successive excursion debuts with height greater than δ and level smaller than $-\beta$.

Proposition 38. *There is an auxiliary probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that, on the product space $\Omega \times \mathcal{S}$ equipped with the probability measure $\mathbb{P} \otimes \mathbb{N}_0^{(\beta)}$, we can construct a Poisson measure $\mathcal{P}$ on $\mathbb{R}_+ \times \mathcal{S}$ with intensity $\mathrm{d}t \otimes \mathbb{N}_0^*(\mathrm{d}\omega)$ so that the following holds. For every $\delta > 0$, if $(t_1^\delta, \omega_1^\delta), (t_2^\delta, \omega_2^\delta), \ldots$ is the sequence of atoms of the measure $\mathcal{P}(\cdot \cap (\mathbb{R}_+ \times \{M > \delta\}))$, arranged so that $t_1^\delta < t_2^\delta < \cdots$, then $\tilde{W}^{(u_i^\delta)} = \omega_i^\delta$ for every $1 \leq i \leq N_\delta$. Furthermore, the Poisson measure $\mathcal{P}$ is independent of $\mathcal{E}^{(-\beta,\infty)}$.*

This proposition means that all excursions above the minimum (with level smaller than β) can be viewed as the atoms of a certain Poisson point process. In contrast with the classical Itô theorem of excursion theory for Brownian motion, we have enlarged the underlying probability space in order to construct the Poisson measure $\mathcal{P}$.

Proof of Proposition 38. We first explain how we can choose the auxiliary random variables $\overline{W}^{\delta,j}$ of Lemma 35 in a consistent way when δ varies. We set $\delta_k = 2^{-k}$ for every $k \geq 1$ and we restrict our attention to values of δ in the sequence $(\delta_k)_{k\geq 1}$. On an auxiliary probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $\overline{\mathcal{P}}$ be a Poisson measure on $\mathbb{R}_+ \times \mathcal{S}$ with intensity $\mathrm{d}t \otimes \mathbb{N}_0^*(\mathrm{d}\omega)$. For every $k \geq 1$, let $(\bar{t}^{k,j}, \overline{W}^{k,j})_{j\geq 1}$ be the sequence of atoms of $\overline{\mathcal{P}}$ that fall in $\mathbb{R}_+ \times \{M > \delta_k\}$ (ordered so that $\bar{t}^{k,1} < \bar{t}^{k,2} < \cdots$). Then, for every $k \geq 1$, $(\overline{W}^{k,1}, \overline{W}^{k,2}, \ldots)$ forms an i.i.d. sequence of variables distributed according to $\mathbb{N}_0^*(\cdot \mid M > \delta_k)$. By Lemma 35, under the product probability measure $\mathbb{P} \otimes \mathbb{N}_0^{(\beta)}$, the sequence $(W^{k,1}, W^{k,2}, \ldots)$ defined by

$$W^{k,j} = \begin{cases} \tilde{W}^{(u_j^{\delta_k})} & \text{if } 1 \leq j \leq N_\delta, \\ \overline{W}^{k,j-M_\delta} & \text{if } j > N_\delta, \end{cases}$$

is also a sequence of i.i.d. random variables distributed according to $\mathbb{N}_0^*(\cdot \mid M > \delta_k)$, and is independent of the σ-field $\mathcal{E}^{(-\beta,\infty)}$.

Obviously, if $k < k'$, the excursions $\tilde{W}^{(u_j^{\delta_k})}$, $1 \leq j \leq N_{\delta_k}$, are obtained by considering the elements of the finite sequence $\tilde{W}^{(u_j^{\delta_{k'}})}$, $1 \leq j \leq N_{\delta_{k'}}$, that belong to $\{M > \delta_k\}$, and similarly $(\overline{W}^{k,j})_{j\geq 1}$ consists of those terms of $(\overline{W}^{k',j})_{j\geq 1}$ that belong to $\{M > \delta_k\}$. It follows that, for every $k < k'$, the sequence $(W^{k,j})_{j\geq 1}$ is obtained by keeping only those terms of $(W^{k',j})_{j\geq 1}$ that belong to the set $\{M > \delta_k\}$. Note that the law of the collection

$$(W^{k,j})_{j,k\geq 1}$$

is then completely determined by this consistency property and the fact that, for every fixed $k \geq 1$, $(W^{k,j})_{j\geq 1}$ is a sequence of i.i.d. random variables distributed according to $\mathbb{N}_0^*(\cdot \mid M > \delta_k)$. In particular,

$$(W^{k,j})_{j,k\geq 1} \stackrel{\text{(d)}}{=} (\overline{W}^{k,j})_{j,k\geq 1}. \tag{58}$$

Also note that the collection $(W^{k,j})_{j,k\geq 1}$ is independent of the σ-field $\mathcal{E}^{(-\beta,\infty)}$.

It is a simple exercise in Poisson measures to verify that $\overline{\mathcal{P}}$ is equal a.s. to a measurable function of the collection $(\overline{W}^{k,j})_{j,k\geq 1}$. Indeed, it suffices to verify that the times $(\bar{t}^{k,j})_{j,k\geq 1}$ are (a.s.) measurable functions of this collection. Let us outline the argument in the case $k = j = 1$. If, for every $k \geq 1$, we write

$$m_k := \#\{j \geq 1 : \bar{t}^{k,j} < \bar{t}^{1,1}\}$$

then m_k is just the number of terms in the sequence $(\overline{W}^{k,j})_{j\geq 1}$ before the first term that belongs to $\{M > \delta_1\}$, and is thus a function of $(\overline{W}^{\ell,j})_{j,\ell\geq 1}$. Elementary arguments using Lemma 25 show that we have the almost sure convergence

$$\mathbb{N}_0^*(M > \delta_k)^{-1} m_k \xrightarrow[k\to\infty]{} \bar{t}^{1,1},$$

thus giving the desired measurability property.

So there exists a measurable function Φ such that a.s.,

$$\overline{\mathcal{P}} = \Phi((\overline{W}^{k,j})_{j,k\geq 1}).$$

Then we can just set

$$\mathcal{P} = \Phi((W^{k,j})_{j,k\geq 1}).$$

By (58), $\mathcal{P}$ has the same distribution as $\overline{\mathcal{P}}$. By construction, the properties stated in the proposition hold when $\delta = \delta_k$, for every $k \geq 1$. This implies that they hold for every $\delta > 0$. □

In what follows, we will use not only the statement of Proposition 38 but also the explicit construction of $\mathcal{P}$ that is given in the preceding proof (we have not included this construction in the statement of Proposition 38 for the sake of conciseness).

We now state an important lemma, which shows that the process $(Z_{\beta+r})_{r\geq 0}$ can be recovered from (Z_β and) the Poisson measure $\mathcal{P}$. To this end, we introduce the point measure $\mathcal{P}^\circ$ defined as the image of $\mathcal{P}$ under the mapping $(t,\omega) \mapsto (t, Z_0^*(\omega))$. From the form of the "law" of Z_0^* under $\mathbb{N}_0^*$ given in Proposition 31, $\mathcal{P}^\circ$ is (under $\mathbb{P}\otimes\mathbb{N}_0^{(\beta)}$) a Poisson measure on $\mathbb{R}_+ \times (0,\infty)$ with intensity

$$\mathrm{d}t \otimes \sqrt{\frac{3}{2\pi}}\, z^{-5/2}\,\mathrm{d}z.$$

We can associate with this point measure a centered Lévy process $U = (U_t)_{t\geq 0}$ (with no negative jumps) started from 0 such that

$$\sum_{t\in\mathcal{D}_U} \delta_{(t,\Delta U_t)} = \mathcal{P}^\circ,$$

where $\mathcal{D}_U$ is the set of discontinuity times of U. Note that the Laplace transform of U_t is

$$E[\exp(-\lambda U_t)] = \exp(t\psi(\lambda)),$$

where

$$\psi(\lambda) = \sqrt{\frac{3}{2\pi}} \int_0^\infty (e^{-\lambda z} - 1 + \lambda z) z^{-5/2}\,\mathrm{d}z = \sqrt{8/3}\,\lambda^{3/2}.$$

Notice that we get the same function $\psi(\lambda)$ as in Section 2.5.

Lemma 39. *Set $X_t = Z_\beta + U_t$ for every $t \geq 0$. Then, $\mathbb{P}\otimes\mathbb{N}_0^{(\beta)}$-a.s.,*

$$Z_{\beta+r} = X_{\inf\{t\geq 0:\, \int_0^t (X_s)^{-1}\mathrm{d}s > r\}} \quad \textit{for every } 0 \leq r < -W_* - \beta.$$

Remark. We have $Z_r = 0$ for every $r \geq -W_*$, so that the above formula indeed expresses $(Z_{\beta+r})_{r\geq 0}$ as a function of X, which is itself defined in terms of Z_β and the point measure $\mathcal{P}^\circ$.

Proof of Lemma 39. First notice that $(U_t)_{t\geq 0}$ is independent of Z_β because $\mathcal{P}$ is independent of $\mathcal{E}^{(-\beta,\infty)}$. Therefore, $(X_t)_{t\geq 0}$ is a Lévy process started from Z_β. On the other hand, $(Z_{\beta+r})_{r\geq 0}$ is under $\mathbb{N}_0^{(\beta)}$ a continuous-state branching process with branching mechanism ψ. By the classical Lamperti transformation (see e.g. [5]), if we set $T'_0 := \int_0^\infty Z_{\beta+t}\,\mathrm{d}t$ and, for every $0 \leq r < T'_0$,

$$X'_r := Z_{\beta+\inf\{s\geq 0:\,\int_0^s Z_{\beta+t}\,\mathrm{d}t>r\}}, \tag{59}$$

then the process $(X'_r)_{0\leq r<T'_0}$ has the same distribution as $(X_r)_{0\leq r<T_0}$, where $T_0 := \inf\{t \geq 0 : X_t = 0\}$. Furthermore, by inverting (59), we also have

$$Z_{\beta+r} = X'_{\inf\{t\geq 0:\,\int_0^t (X'_s)^{-1}\mathrm{d}s>r\}} \qquad \text{for every } 0 \leq r < T_0^Z, \tag{60}$$

where $T_0^Z = -W_* - \beta$ is the hitting time of 0 by Z.

Comparing (60) with the statement of the lemma, we see that we only need to verify the a.s. equality $(X_r)_{0\leq r<T_0} = (X'_r)_{0\leq r<T'_0}$. To this end, we first extend the definition of X'_t to values $t \geq T_0$. Recalling the Poisson measure $\overline{\mathcal{P}}$ in the proof of Proposition 38, we define $\overline{\mathcal{P}}^\circ$ as the image of $\overline{\mathcal{P}}$ under the mapping $(t,\omega) \mapsto (t, Z_0^*(\omega))$, and associate with $\overline{\mathcal{P}}^\circ$ a Lévy process $(\overline{U}_t)_{t\geq 0}$ having the same distribution as $(U_t)_{t\geq 0}$. We complete the definition of X' by setting, for every $t \geq 0$,

$$X'_{T'_0+t} = \overline{U}_t.$$

We then observe that X and X' are two Lévy processes with the same distribution and the same (random) initial value Z_β. Furthermore, a.s. for every $\alpha > 0$, the ordered sequence of jumps of size greater than α is the same for X' and for X. First note that the jumps of X' that occur before the hitting time of 0 are the same as the jumps of Z after time β, and, by Proposition 37, these are exactly the quantities $Z_0^*(\tilde{W}^{(u)})$ when u varies over the excursion debuts with level smaller than $-\beta$. Recalling our construction of X from the point measure $\mathcal{P}^\circ$, we find that, for every $\alpha > 0$, the ordered sequence of jumps of X' of size greater than α that occur before the hitting time of 0 will also appear as the first n_α jumps of X of size greater than α, for some random integer n_α depending on α. Then, the ordered sequence of jumps of X' of size greater than α that occur after the hitting time of 0 consists of the quantities $Z_0^*(\omega)$ where (t,ω) varies over the atoms of $\overline{\mathcal{P}}$ such that $Z_0^*(\omega) > \alpha$, and these quantities are ranked according to the values of t. Recalling the way $\mathcal{P}$ was defined, we see that the same sequence will appear as the sequence of jumps of X of size greater than α occurring after the n_α-th one.

Finally, once we know that, for every $\alpha > 0$, the ordered sequence of jumps of size greater than α is the same for X' and for X, the fact that X and X' are two Lévy processes with the same distribution and the same initial value implies that they are a.s. equal, which completes the proof. □

7.4. The main theorem

Our main result identifies the conditional distribution of excursions above the minimum given the exit measure process Z. We let $\mathcal{D}_Z$ stand for the set of all jump times of Z. Recall from Proposition 36 that there is a one-to-one correspondence between $\mathcal{D}_Z$ and excursions above the minimum. If u is an excursion debut, and $r = -V_u$ is the associated element of $\mathcal{D}_Z$, we write $\tilde{W}^{(r)} = \tilde{W}^{(u)}$ in the following statement. We let $\mathbb{D}(0,\infty)$ stand for the usual Skorokhod space of càdlàg functions from $(0,\infty)$ into $\mathbb{R}$.

Theorem 40. *Let F be a nonnegative measurable function on $\mathbb{D}(0,\infty)$, and let G be a nonnegative measurable function on $\mathbb{R}_+ \times \mathcal{S}$. Then*

$$\mathbb{N}_0\Big(F(Z)\exp\Big(-\sum_{r\in\mathcal{D}_Z} G(r,\tilde{W}^{(r)})\Big)\Big) = \mathbb{N}_0\Big(F(Z)\prod_{r\in\mathcal{D}_Z} \mathbb{N}_0^*(\exp(-G(r,\cdot))\,|\,Z_0^* = \Delta Z_r)\Big).$$

In other words, under $\mathbb{N}_0$ and conditionally on the exit measure process Z, the excursions above the minimum are independent, and, for every $r \in \mathcal{D}_Z$, the conditional law of the associated excursion is $\mathbb{N}_0^*(\cdot \mid Z_0^* = \Delta Z_r)$.

Proof. Let us start with simple reductions of the proof. First we may assume that $\mathbb{N}_0(F(Z)) < \infty$ since the general case will follow by monotone convergence. Then, we may assume that $G(r,\omega) = 0$ if $r \le \gamma$, for some $\gamma > 0$, and it is also sufficient to prove that the statement holds when $\mathbb{N}_0$ is replaced by $\mathbb{N}_0^{(\beta)}$ for some fixed $\beta > 0$. Finally, we may restrict the sum or the product over r to jump times such that $\Delta Z_r > \alpha$, for some fixed $\alpha > 0$.

In view of the preceding observations, we only need to verify that, for every $\alpha > 0$ and $\beta > 0$,

$$\mathbb{N}_0^{(\beta)}\Big(F(Z)\exp\Big(-\sum_{\substack{r\in\mathcal{D}_Z^{(\beta)}\\ \Delta Z_r>\alpha}} G(r,\tilde{W}^{(r)})\Big)\Big) = \mathbb{N}_0^{(\beta)}\Big(F(Z)\prod_{\substack{r\in\mathcal{D}_Z^{(\beta)}\\ \Delta Z_r>\alpha}} \mathbb{N}_0^*(\exp(-G(r,\cdot))\,|\,Z_0^* = \Delta Z_r)\Big),$$

where $\mathcal{D}_Z^{(\beta)} = \mathcal{D}_Z \cap (\beta,\infty)$.

From now on, we fix $\alpha, \beta > 0$. We will use the notation and definitions of the previous subsections, where $\beta > 0$ was fixed and we argued under $\mathbb{N}_0^{(\beta)}$. In particular it will be convenient to consider the product probability measure $\mathbb{P}\otimes\mathbb{N}_0^{(\beta)}$ as in Section 7.3. Recall the definition of the Poisson measure $\mathcal{P}$ and of the process X in Lemma 39 (these objects depend on the choice of β, which is fixed here), and the notation $T_0 = \inf\{t \ge 0 : X_t = 0\}$. Also recall that $\mathcal{P}^\circ$ is the image of $\mathcal{P}$ under the mapping $(t,\omega) \mapsto (t, Z_0^*(\omega))$.

The first step is to rewrite the quantity

$$\sum_{\substack{r\in\mathcal{D}_Z^{(\beta)}\\ \Delta Z_r>\alpha}} G(r,\tilde{W}^{(r)})$$

in a different form. Recall from Lemma 39 that every jump time r of Z after time β, hence every excursion debut u with level smaller than $-\beta$, corresponds to a jump time of X before time T_0, and is therefore associated with an atom (t,ω) of $\mathcal{P}$, with $t < T_0$, such that $\omega = \tilde{W}^{(u)}$ and $Z_0^*(\omega) = Z_0^*(\tilde{W}^{(u)}) = \Delta Z_r$, where the last equality is Proposition 37. Then, let $(t_1^\alpha, \omega_1^\alpha), (t_2^\alpha, \omega_2^\alpha), \dots$ be the time-ordered sequence of all atoms (t,ω) of $\mathcal{P}$ such that $Z_0^*(\omega) > \alpha$. Also set $n_\alpha = \max\{i \geq 1 : t_i^\alpha < T_0\}$. For every $1 \leq i \leq n_\alpha$, write $z_i^\alpha = Z_0^*(\omega_i^\alpha)$ and r_i^α for the jump time of Z corresponding to the jump z_i^α. We can rewrite

$$\sum_{\substack{r\in\mathcal{D}_Z^{(\beta)}\\ \Delta Z_r>\alpha}} G(r,\tilde{W}^{(r)}) = \sum_{i=1}^{n_\alpha} G(r_i^\alpha, \omega_i^\alpha).$$

Writing $E[\cdot]$ for the expectation under $\mathbb{P}\otimes\mathbb{N}_0^{(\beta)}$, we then have

$$\mathbb{N}_0^{(\beta)}\Big(F(Z)\exp\Big(-\sum_{\substack{r\in\mathcal{D}_Z^{(\beta)}\\ \Delta Z_r>\alpha}} G(r,\tilde{W}^{(r)})\Big)\Big) = E\Big[F(Z)\exp\Big(-\sum_{i=1}^{n_\alpha} G(r_i^\alpha,\omega_i^\alpha)\Big)\Big].$$

We evaluate the right-hand side by conditioning first with respect to the σ-field $\mathcal{H}$ generated by $\mathcal{E}^{(-\beta,\infty)}$ and the point measure $\mathcal{P}^\circ$. Notice that the process Z is measurable with respect to $\mathcal{H}$ (because U is obviously a measurable function of $\mathcal{P}^\circ$, and we can use Lemma 39). The finite sequence $r_1^\alpha,\dots,r_{n_\alpha}^\alpha$ is also measurable with respect to $\mathcal{H}$ as it is the sequence of jump times of Z (after time β) corresponding to jumps of size greater than α. In particular, n_α is measurable with respect to $\mathcal{H}$. Finally, the quantities $z_1^\alpha,\dots,z_{n_\alpha}^\alpha$ are the corresponding jumps and therefore are also measurable with respect to $\mathcal{H}$.

On the other hand, by standard properties of Poisson measures, we know that $\omega_1^\alpha,\omega_2^\alpha,\dots$ is a sequence of i.i.d. variables distributed according to $\mathbb{N}_0^*(\cdot\mid Z_0^*>\alpha)$. Recalling that $\mathcal{P}$ is independent of $\mathcal{E}^{(-\beta,\infty)}$, we see that conditioning this sequence on the σ-field $\mathcal{H}$ has the effect of conditioning on the values of $Z_0^*(\omega_1^\alpha), Z_0^*(\omega_2^\alpha),\dots$. In a more precise way, the conditional distribution of $\omega_1^\alpha,\omega_2^\alpha,\dots$ knowing $\mathcal{H}$ is the distribution of a sequence of independent variables distributed respectively according to $\mathbb{N}_0^*(\cdot\mid Z_0^*=z_1^\alpha)$, $\mathbb{N}_0^*(\cdot\mid Z_0^*=z_2^\alpha),\dots$, where these conditional measures are defined thanks to Proposition 33.

By combining the preceding considerations, we get

$$E\Big[F(Z)\exp\Big(-\sum_{i=1}^{n_\alpha} G(r_i^\alpha,\omega_i^\alpha)\Big)\Big] = E\Big[F(Z)\prod_{i=1}^{n_\alpha}\mathbb{N}_0^*(\exp(-G(r_i^\alpha,\cdot))\mid Z_0^*=z_i^\alpha)\Big].$$

Now note that, with our definitions,

$$\prod_{i=1}^{n_\alpha}\mathbb{N}_0^*(\exp(-G(r_i^\alpha,\cdot))\mid Z_0^*=z_i^\alpha) = \prod_{\substack{r\in\mathcal{D}_Z^{(\beta)}\\ \Delta Z_r>\alpha}}\mathbb{N}_0^*(\exp(-G(r,\cdot))\mid Z_0^*=\Delta Z_r),$$

and so we have obtained

$$\begin{aligned}\mathbb{N}_0^{(\beta)}\Big(F(Z)\exp\Big(-\sum_{\substack{r\in\mathcal{D}_Z^{(\beta)}\\ \Delta Z_r>\alpha}} G(r,\tilde{W}^{(r)})\Big)\Big)\\ &= E\Big[F(Z)\prod_{\substack{r\in\mathcal{D}_Z^{(\beta)}\\ \Delta Z_r>\alpha}} \mathbb{N}_0^*(\exp(-G(r,\cdot))\mid Z_0^*=\Delta Z_r)\Big]\\ &= \mathbb{N}_0^{(\beta)}\Big(F(Z)\prod_{\substack{r\in\mathcal{D}_Z^{(\beta)}\\ \Delta Z_r>\alpha}} \mathbb{N}_0^*(\exp(-G(r,\cdot))\mid Z_0^*=\Delta Z_r)\Big),\end{aligned}$$

which completes the proof of the theorem. □

8. Excursions away from a point

In this section, we briefly explain how we can derive the results stated in the introduction from our statements concerning excursions above the minimum. This derivation relies on the famous theorem of Lévy stating that if $(B_t)_{t\geq 0}$ is a linear Brownian motion starting from 0, and if $(L_t^0(B))_{t\geq 0}$ is its local time process at 0, then the pair of processes

$$\big(B_t-\min\{B_r:0\leq r\leq t\},-\min\{B_r:0\leq r\leq t\}\big)_{t\geq 0}$$

has the same distribution as $(|B_t|,L_t^0(B))_{t\geq 0}$. Notice that $L_t^0(B)$ can also be interpreted as the local time of $|B|$ at 0, provided we consider here the "symmetric local time", namely

$$L_t^0(|B|)=\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_0^t \mathbf{1}_{[-\varepsilon,\varepsilon]}(|B_r|)\,\mathrm{d}r.$$

Lévy's identity will show that (the absolute values of) excursions away from 0 for our tree-indexed process have the same distribution as excursions above the minimum, which is essentially what we need to derive the results stated in the introduction.

Let us explain this in greater detail. For any finite path $\mathrm{w}\in\mathcal{W}_0$, define two other finite paths $\mathrm{w}^\bullet$ and $\ell_{\mathrm{w}}^\bullet$ with the same lifetime as w by the formulas

$$\begin{aligned}\mathrm{w}^\bullet(t)&:=\mathrm{w}(t)-\min\{\mathrm{w}(r):0\leq r\leq t\},\\ \ell_{\mathrm{w}}^\bullet(t)&:=-\min\{\mathrm{w}(r):0\leq r\leq t\}.\end{aligned}$$

On our canonical space $\mathcal{S}_0$ of snake trajectories, we can then make sense of $W_s^\bullet$ and $\ell_{W_s}^\bullet$ for every $s\geq 0$, and we write $L_s^\bullet=\ell_{W_s}^\bullet$ to simplify notation. Then, under $\mathbb{N}_0$, the pair $(W_s^\bullet,L_s^\bullet)_{s\geq 0}$ defines a random element of the space of two-dimensional snake trajectories with initial point $(0,0)$ (the latter space is defined by an obvious extension of Definition 6). Thanks to Lévy's theorem recalled above, it is then a simple matter to verify that the "law" of the pair $(W_s^\bullet,L_s^\bullet)_{s\geq 0}$ under $\mathbb{N}_0$ is the excursion measure from the point $(0,0)$ of the Brownian snake whose spatial motion is the Markov process $(|B_t|,L_t^0(B))$. We refer to [21, Chapter IV] for the definition of the Brownian snake associated with a general spatial motion and of its excursion measures. In a way similar to the beginning of

Section 3, we then set

$$V_u^\bullet = \hat{W}_s^\bullet = \hat{W}_s - \min\{W_s(t) : 0 \le t \le \zeta_s\} = V_u - \min\{V_v : v \in [\![\rho, u]\!]\}$$

for every $u \in \mathcal{T}_\zeta$ and $s \ge 0$ such that $p_\zeta(s) = u$.

Say that $u \in \mathcal{T}_\zeta$ is an *excursion debut away from* 0 for $V^\bullet$ if

(i) $V_u^\bullet = 0$;
(ii) u has a strict descendant w such that $V_v^\bullet \neq 0$ for all $v \in]\!]\rho, w]\!]$.

It follows from our definitions that u is an excursion debut away from 0 for $V^\bullet$ if and only if u is an excursion debut above the minimum in the sense of Section 3, that is, if and only if $u \in D$. Then, Proposition 20 shows that the connected components of the open set $\{u \in \mathcal{T}_\zeta : V_u^\bullet > 0\}$ are exactly the sets $\mathrm{Int}(C_u)$, $u \in D$. Furthermore, for every $u \in D$, the values of $V^\bullet$ over C_u are described by the snake trajectory $\tilde{W}^{(u)}$ (which can thus be viewed as the excursion of $V^\bullet$ away from 0 corresponding to u).

In order to recover the setting of the introduction, we still need to assign signs to the excursions of $V^\bullet$ away from 0. To this end, we let $(v_1, v_2, \dots)$ be a measurable enumeration of D—formally we should rather enumerate times $s_1, s_2, \dots$ such that $p_\zeta(s_1) = v_1$, $p_\zeta(s_2) = v_2, \dots$. On an auxiliary probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we then consider a sequence $(\xi_1, \xi_2, \dots)$ of i.i.d. random variables such that

$$\mathbb{P}(\xi_i = 1) = \mathbb{P}(\xi_i = -1) = 1/2$$

for every $i \ge 1$. Under the product measure $\mathbb{P} \otimes \mathbb{N}_0$, we then set, for every $u \in \mathcal{T}_\zeta$,

$$V_u^* := \begin{cases} \xi_i \, V_u^\bullet & \text{if } u \in \mathrm{Int}(C_{v_i}) \text{ for some } i \ge 1, \\ 0 & \text{if } V_u^\bullet = 0. \end{cases}$$

The fact that $u \mapsto V_u^\bullet$ is continuous implies that $u \mapsto V_u^*$ is also continuous on $\mathcal{T}_\zeta$. Furthermore the pair $(V^*_{p_\zeta(s)}, \zeta_s)_{s \ge 0}$ is a tree-like path, and we denote the associated snake trajectory by $(W_s^*)_{s \ge 0}$. Then the "law" of $(W_s^*)_{s \ge 0}$ under $\mathbb{P} \otimes \mathbb{N}_0$ is just the excursion measure $\mathbb{N}_0$. This is a consequence of the fact that, starting from a process distributed as $(|B_t|)_{t \ge 0}$, one can reconstruct a linear Brownian motion started from 0 by assigning independently signs $+1$ or -1 with probability $1/2$ to excursions away from 0. We omit the details.

Since the law of $(W_s^*)_{s \ge 0}$ under $\mathbb{P} \otimes \mathbb{N}_0$ is $\mathbb{N}_0$, we may replace the process $(W_s)_{s \ge 0}$ under $\mathbb{N}_0$ by the process $(W_s^*)_{s \ge 0}$ under $\mathbb{P} \otimes \mathbb{N}_0$ in order to prove the various statements of the introduction. To this end, we first notice that the excursion debuts away from 0 for V^* (obviously defined by properties (i) and (ii) with $V^\bullet$ replaced by V^*) are the same as excursion debuts away from 0 for $V^\bullet$, and thus the same as excursion debuts above the minimum in the sense of Section 3. Moreover, for every $i = 1, 2, \dots$, the excursion of V^* corresponding to v_i is described by

$$\tilde{W}^{*(v_i)} = \begin{cases} \tilde{W}^{(v_i)} & \text{if } \xi_i = 1, \\ -\tilde{W}^{(v_i)} & \text{if } \xi_i = -1. \end{cases}$$

In addition, if a_i is such that $p_\zeta(a_i) = v_i$, the local time at 0 of the path $W^*_{a_i}$ is equal to the (symmetric) local time at 0 of $|W^*_{a_i}| = W^\bullet_{a_i}$,

$$\ell^*_i = \hat{L}^\bullet_{a_i} = -\underline{W}_{a_i} = -V_{v_i}.$$

From the preceding remarks, it is now easy to derive Theorem 1 from Theorem 23. Indeed, the left-hand side of the formula of Theorem 1 can be rewritten as

$$\mathbb{P} \otimes \mathbb{N}_0\Big(\sum_{i=1}^{\infty} \Phi(\ell^*_i, W^{*(v_i)})\Big)$$

and, by the previous observations, the last display is equal to

$$\begin{aligned}\mathbb{P} \otimes \mathbb{N}_0\Big(\sum_{i=1}^{\infty} \Phi(-V_{v_i}, \xi_i \tilde{W}^{(v_i)})\Big) &= \frac{1}{2}\mathbb{N}_0\Big(\sum_{i=1}^{\infty}\big(\Phi(-V_{v_i}, \tilde{W}^{(v_i)}) + \Phi(-V_{v_i}, -\tilde{W}^{(v_i)})\big)\Big)\\ &= \frac{1}{2}\int \mathbb{N}^*_0(\mathrm{d}\omega)\Big(\int_0^\infty \mathrm{d}x\, (\Phi(x,\omega) + \Phi(x,-\omega))\Big)\end{aligned}$$

where the last equality follows from Theorem 23. This shows that Theorem 1 holds with $\mathbb{M}_0 = \frac{1}{2}(\mathbb{N}^*_0 + \check{\mathbb{N}}^*_0)$, where $\check{\mathbb{N}}^*_0$ is the image of $\mathbb{N}^*_0$ under $\omega \mapsto -\omega$. Then Proposition 2 follows from Proposition 30.

In order to derive Proposition 3, we note that, for every $r > 0$, the (total mass of the) exit measure of the snake $(W^\bullet, L^\bullet)$ outside the open set $\Delta_r := \mathbb{R}_+ \times [0, r)$, which is denoted by $\mathcal{X}_r$, satisfies the following approximation $\mathbb{N}_0$-a.e.:

$$\mathcal{X}_r = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{\zeta_s - \varepsilon < \tau_{\Delta_r}(W^\bullet_s, L^\bullet_s) < \zeta_s\}},$$

where $\tau_{\Delta_r}(W^\bullet_s, L^\bullet_s)$ stands for the first exit time from Δ_r of the path $(W^\bullet_s(t), L^\bullet_s(t))_{0\le t\le\zeta_s}$. This is indeed the analog of the approximation result (8), which holds in a very general setting: see [21, Proposition V.1]. Coming back to the definition of $W^\bullet_s$ and $L^\bullet_s$ in terms of W_s, we see that

$$\mathcal{X}_r = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_0^\sigma \mathrm{d}s\, \mathbf{1}_{\{\zeta_s - \varepsilon < \tau_{-r}(W_s) < \zeta_s\}} = \mathcal{Z}_{-r},$$

where the last equality follows from (8). This simple remark allows us to identify the process $(\mathcal{X}_r)_{r>0}$ with the exit measure process $(Z_r)_{r>0}$, and justifies the observations preceding Proposition 3 in the introduction. Proposition 3 itself then follows from Propositions 36 and 37. Finally, Theorem 4 is a consequence of Theorem 40 and the fact that the excursions $\tilde{W}^{*(v_i)}$ can be written in the form $\xi_i\, \tilde{W}^{(v_i)}$ for $i = 1, 2, \ldots$.

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