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A Simple Proof of Kwapien's Theorem

By

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The purpose of the present note is to prove in a self-contained way Kwapien's theorem that gives a criterion for Hilbertizability of normed spaces. Original proof [1], [2] has required much more complicated discussions. Many authors have investigated [3]~[6] relating problems, but their researches are based on mutual quotations, thus it is difficult to follow their studies for readers not so familiar in the topic. So the author believes that a self-contained proof has some meaning.

§1. Kwapien's Theorem

Theorem *If a real Banach space E is of type 2 and cotype 2, then E is isomorphic to a Hilbert space.*

Proof. The definition of type 2 and cotype 2 is:

$$^{\exists}a,\ b>0\ ^{\forall}n\ ^{\forall}x_1,\ x_2,\ \cdots,\ x_n\in E\ (\text{except } x_1=x_2=\cdots=x_n=0)$$

$$(1.1)\qquad a\sum_{i=1}^{n}||x_i||^2<\int||\sum_{i=1}^{n}\varepsilon_i x_i||^2\,dm(\varepsilon)<b\sum_{i=1}^{n}||x_i||^2$$

where $\varepsilon=(\varepsilon_i)_{i=1}^{n}$, $\varepsilon_i=\pm1$ and m is the product measure of $\frac{1}{2}(\delta_1+\delta_{-1})$, δ_1 being the Dirac measure placed at 1.

In (1.1), the first inequality is the definition of cotype 2, while the second inequality is the definition of type 2.

First step From (1.1) we can derive

$$(1.2)\qquad a\sum_{i=1}^{n}||x_i||^2<\int||\sum_{i=1}^{n}t_i x_i||^2dg(t)<b\sum_{i=1}^{n}||x_i||^2$$

where $t=(t_i)_{i=1}^{n}$ and g is the product measure of one-dimensional gaussian measures with variance 1.

$\boldsymbol{R}^n$ can be written as the product $\boldsymbol{R}^n=\boldsymbol{R}_+^n\times\{-1,\ 1\}^n$, and the measure g can be written as $g=g_+\times m$, where g_+ is the (normalized) gaussian measure on

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$R_+^n=\{(t_i)\,|\,t_i\geqq 0 \text{ for } {}^\forall i\}$. Therefore we get

$$\int\|\sum_{i=1}^n t_i x_i\|^2 dg(t)=\iint\|\sum_{i=1}^n |t_i|\,\varepsilon_i x_i\|^2 dg_+ dm\,. \tag{1.3}$$

Assuming (1.1) we have

$$(1.3)<b\int\sum_{i=1}^n t_i^2\|x_i\|^2 dg_+=b\sum_{i=1}^n\|x_i\|^2\int t_i^2 dg_+=b\sum_{i=1}^n\|x_i\|^2\,.$$

In a similar way, also we have $(1.3)>a\sum_{i=1}^n\|x_i\|^2$. Thus we have proved (1.2).

Second step Assume that $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n\in E$ and

$$\sum_{i=1}^n f(x_i)^2=\sum_{i=1}^n f(y_i)^2 \qquad \text{for } {}^\forall f\in E^*\,. \tag{1.4}$$

Then we have

$$\int\|\sum_{i=1}^n t_i x_i\|^2 dg(t)=\int\|\sum_{i=1}^n t_i y_i\|^2 dg(t)\,. \tag{1.5}$$

Denote with $g_{x_1,x_2,\cdots,x_n}$ the measure on E induced by the mapping $t=(t_i)\to\sum_{i=1}^n t_i x_i\in E$ from the measure g. Then the characteristic function of $g_{x_1,x_2,\cdots,x_n}$ is

$$\begin{aligned}\chi_{x_1,x_2,\cdots,x_n}(f)&=\int_E \exp[if(x)]dg_{x_1,x_2,\cdots,x_n}(x)=\int_{R^n}\exp[if(\sum_{i=1}^n t_i x_i)]dg(t)\\&=\int_{R^n}\exp[i\sum_{i=1}^n t_i f(x_i)]dg(t)=\exp[-\sum_{i=1}^n f(x_i)^2/2]\,.\end{aligned}$$

Therefore under the assumption of (1.4), we have $\chi_{x_1,x_2,\cdots,x_n}(f)=\chi_{y_1,y_2,\cdots,y_n}(f)$. From the one-to-one correspondence between the characteristic function and the measure, we see that $g_{x_1,x_2,\cdots,x_n}=g_{y_1,y_2,\cdots,y_n}$. Hence

$$\begin{aligned}\int_{R^n}\|\sum_{i=1}^n t_i x_i\|^2 dg(t)&=\int_E\|x\|^2 dg_{x_1,x_2,\cdots,x_n}(x)=\int_E\|x\|^2 dg_{y_1,y_2,\cdots y_n}(x)\\&=\int_{R^n}\|\sum_{i=1}^n t_i y_i\|^2 dg(t)\,,\end{aligned}$$

namely we get (1.5).

Third step Let $C(S^*)$ be the Banach space of weakly continuous functions on the unit sphere S^* of E^*. For every $x_1, x_2, \cdots, x_n\in E$, we shall define $\Phi_{x_1,x_2,\cdots,x_n}\in C(S^*)$ as follows:

$$\Phi_{x_1,x_2,\cdots,x_n}(f)=\sum_{i=1}^n f(x_i)^2\,. \tag{1.6}$$

Combining (1.2) with the result of Second step, we see that

$$(1.7)\qquad \Phi_{x_1,x_2,\cdots,x_n}=\Phi_{y_1,y_2,\cdots,y_m} \textit{ implies } a\sum_{i=1}^{n}||x_i||^2<b\sum_{i=1}^{m}||y_i||^2\,.$$

Even if $n>m$, we can add $y_{m+1}=y_{m+2}=\cdots=y_n=0$ to prove (1.7).

Fourth step Consider a subset A of $C(S^*)$ such that

$$(1.8)\qquad A=\{\Phi_{x_1,x_2,\cdots,x_n}\mid \sum_{i=1}^{n}||x_i||^2=1\}\,.$$

A is convex, because $\lambda\Phi_{x_1,x_2,\cdots,x_n}+(1-\lambda)\Phi_{y_1,y_2,\cdots,y_m}=\Phi_{\sqrt{\lambda}x_1,\cdots,\sqrt{\lambda}x_n,\sqrt{1-\lambda}\,y_1,\cdots,\sqrt{1-\lambda}\,y_m}$ and $\sum_{i=1}^{n}||\sqrt{\lambda}\,x_i||^2+\sum_{i=1}^{m}||\sqrt{1-\lambda}\,y_i||^2=\lambda\sum_{i=1}^{n}||x_i||^2+(1-\lambda)\sum_{i=1}^{m}||y_i||^2=1$.

(1.7) means $bA\cap aA=\phi$, because if $b\Phi_{x_1,x_2,\cdots,x_n}=a\Phi_{y_1,y_2,\cdots,y_m}$ with $\sum_{i=1}^{n}||x_i||^2=\sum_{i=1}^{m}||y_i||^2=1$, then we get $\Phi_{\sqrt{b}x_1,\sqrt{b}x_2,\cdots,\sqrt{b}x_n}=\Phi_{\sqrt{a}y_1,\sqrt{a}y_2,\cdots,\sqrt{a}y_m}$ with $\sum_{i=1}^{n}||\sqrt{b}x_i||^2=b$ and $\sum_{i=1}^{m}||\sqrt{a}y_i||^2=a$, which contradicts to (1.7).

Fifth step Since A is a convex set and $bA\cap aA=\phi$, by the corollary of the separation theorem (the proof is given later) there exists a linear function F which is positive on A and satisfies

$$(1.9)\qquad a\sup_{\Phi\in A}F(\Phi)\leqq b\inf_{\Phi\in A}F(\Phi)\,.$$

Using this F, we shall define $(x,y)_H=F(f(x)f(y))$. $(x,y)_H$ is bilinear and $(x,x)_H=F(\Phi_x)>0$, therefore $||x||_H=\sqrt{(x,x)_H}$ is a Hilbertian norm.

From (1.9) we get

$$(1.10)\qquad a\sup_{x\in S}||x||_H^2\leqq b\inf_{x\in S}||x||_H^2\,.$$

So, replacing $||\cdot||_H$ by its suitable constant multiple, we can suppose that $a\leqq||x||_H^2\leqq b$ on S, or equivalently

$$(1.11)\qquad a||x||^2\leqq||x||_H^2\leqq b||x||^2\qquad \text{for } {}^{\forall}x\in E\,.$$

This means that E is isomorphic to the Hilbert space whose norm is $||\cdot||_H$.

(q.e.d.)

§2. Separation Theorem

Theorem *Let N be a real normed space and U be its open unit ball. If A is a convex subset of N such that $A\cap U=\phi$, then there exists a linear function F on N such that*

$$(2.1)\qquad ||F||^*\leqq 1,\quad {}^{\forall}x\in A\quad F(x)\geqq 1\,.$$

Proof. First step If N is finite dimensional, then two disjoint compact convex sets A and B can be separated by a hyperplane.

Since N is isomorphic to $\boldsymbol{R}^n$, we shall adopt the Euclidean metric. There exists $x_0 \in A$ and $y_0 \in B$ such that $d(x_0, y_0) = d(A, B)$ (=the shortest distance between A and B). Let l be the line connecting x_0 and y_0. Then the hyperplane π, which is orthogonal to l and passes through $(x_0+y_0)/2$, separates evidently A and B.

Second step In general case, let B^* be the closed unit ball of N^*. B^* is weakly compact. For $x \in A$ and $\varepsilon > 0$, we shall put

$$K_{x,\varepsilon} = \{F \in B^* \mid F(x) \geqq 1-\varepsilon\} . \tag{2.2}$$

For every $x_1, x_2, \cdots, x_n \in A$ and $\varepsilon > 0$, we have

$$\bigcap_{k=1}^{n} K_{x_k,\varepsilon} \neq \phi , \tag{2.3}$$

because the convex hull of $\{x_1, x_2, \cdots, x_n\}$ is finite dimensional and disjoint with the closed ball of radius $1-\varepsilon$, so that First step is applicable. (A linear continuous function on a subspace of N can be extended on N without changing its norm, according to Hahn-Banach's theorem).

Since B^* is weakly compact, the finite intersection property implies the complete intersection property, therefore

$$\bigcap_{\varepsilon>0} \bigcap_{x \in A} K_{x,\varepsilon} \neq \phi . \tag{2.4}$$

Take an element F from this set, then evidently we have (2.1). (q.e.d.)

Corollary *Let E be a real vector space, A be a convex set and suppose that $A \cap aA = \phi$ for some $a > 1$. Then there exists a linear function F such that*

$$\sup_{x \in A} F(x) \leqq 1 \leqq a \inf_{x \in A} F(x) . \tag{2.5}$$

Proof We can assume that A spans E, because a linear function on a subspace of E can be extended on E.

Since A is convex, every element z of E can be written in the form:

$$z = \lambda x - \mu y, \quad x, y \in A, \lambda \geqq 0, \mu \geqq 0 . \tag{2.6}$$

Define $||z||$ by

$$||z|| = \inf \{\lambda + \mu \mid z = \lambda x - \mu y, x, y \in A, \lambda \geqq 0, \mu \geqq 0\} . \tag{2.7}$$

Then $||\cdot||$ becomes a semi-norm on E. (It may not be a norm, but the following discussions are kept valid considering the factor space $E/\{z \mid ||z||=0\}$).

Suppose that $z = ax \in aA$. If $ax = \lambda x' - \mu y'$ for some $x', y' \in A$ and $\lambda \geqq 0$,

$\mu \geqq 0$, we have $x' = \frac{a}{\lambda} x + \frac{\mu}{\lambda} y' \in \frac{a+\mu}{\lambda} A$. Since $A \cap aA = \phi$, this implies $\frac{a+\mu}{\lambda} < a$, so that $\lambda > 1$. Thus $z \in aA$ implies $||z|| \geqq 1$.

By the separation theorem just proved, there exists a linear function F such that $||F||^* \leqq 1$ and ${}^{\forall}z \in aA \quad F(z) \geqq 1$ or equivalently ${}^{\forall}x \in A \quad aF(x) \geqq 1$. Since $x \in A$ implies $||x|| \leqq 1$, we have evidently ${}^{\forall}x \in A \quad F(x) \leqq 1$. This completes the proof of (2.5). (q.e.d.)

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