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Metric-measure boundary and geodesic flow on Alexandrov spaces

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Abstract. We relate the existence of many infinite geodesics on Alexandrov spaces to a statement about the average growth of volumes of balls. We deduce that the geodesic flow exists and preserves the Liouville measure in several important cases. The analytic tools we develop have close ties to integral geometry.

Keywords. Alexandrov spaces, convex hypersurfaces, geodesic flow, Liouville measure

1. Introduction

1.1. Motivation and application

The following question in the theory of Alexandrov spaces was formulated in a slightly different way in [PP96] and remains open.

- Are there "many" infinite geodesics on any Alexandrov space without boundary?

We address this question and obtain an affirmative answer in several cases. The main new tool is the investigation of the Taylor expansion of the average volume growth. The central results relate the first coefficient of this expansion to the geodesic flow and show how to control the Taylor expansion. This tool might be interesting in its own right, beyond the realm of Alexandrov geometry.

In particular, we prove the existence of such infinite geodesics in the most classical examples of non-smooth Alexandrov spaces:

Theorem 1.1. *Let X be the boundary of a convex body in $\mathbb{R}^{n+1}$. Then almost every direction in the tangent bundle TX is the starting direction of a unique infinite geodesic on X. Moreover, the geodesic flow is defined almost everywhere and preserves the Liouville measure.*

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Apparently, the existence of a *single* infinite geodesic has not been known, even in the two-dimensional case [Zam92]. Our result might appear somewhat surprising since on *most* convex surfaces *most points in the sense of Baire category* are not inner points of any geodesic [Zam82].

1.2. Metric-measure boundary

On a smooth manifold with boundary the geodesic flow is not defined for all times. The amount of geodesics terminating at the boundary in a given time depends on the size of this boundary, due to Santaló's integral formula.

We are going to capture the size of the boundary by estimating the average volumes of small balls and their deviations from the corresponding volumes in the Euclidean space.

Let (X, d) be a locally compact separable metric space, and μ a Radon measure on X which takes finite values on bounded subsets. For $x \in X$ and $r > 0$ denote by $B(x, r)$ the open metric ball of radius r around the point x. Consider the volume growth function $b_r\colon X \to [0, \infty)$,

$$b_r(x) := \mu(B(x, r)). \tag{1.1}$$

For a natural number $n > 0$, let ω_n be the volume of the n-dimensional unit Euclidean ball. The deviation function

$$v_r(x) = 1 - \frac{b_r(x)}{\omega_n r^n}$$

measures in a very rough sense the deviation of the metric measure space (X, d, μ) from $\mathbb{R}^n$. Moreover, one can expect the behavior of v_r at the origin $r = 0$ to reflect some curvature-like properties of the space X, as in the following fundamental example.

Example 1.2. Let X^n be a smooth Riemannian manifold with Riemannian volume μ. Then $v_r(x) = \frac{1}{6(n+2)} \cdot \mathit{scal} \cdot r^2$ up to terms of higher order in r. Here *scal* denotes the scalar curvature of X.

In this paper we are interested in the order of vanishing of v_r at $r = 0$ and the first non-vanishing coefficient; in particular we assume that v_r converges to zero in some integral sense. In most interesting metric spaces (X, d), at least in the cases investigated here, the only reasonable choice of the measure μ for which v_r is "sufficiently small" in r is the n-dimensional Hausdorff measure $\mathcal{H}^n$:

Example 1.3. Let X be a countably n-rectifiable metric space. Assume that the Radon measure μ is non-zero on open subsets of X. If $\mu = \mathcal{H}^n$ then the functions v_r converge $\mathcal{H}^n$-almost everywhere to 0. Moreover, $\mu = \mathcal{H}^n$ is the only measure with this property [AK00, Theorem 5.4].

Therefore, in the following, the number n will always be the Hausdorff dimension of X and μ will be the n-dimensional Hausdorff measure.

As seen in Example 1.3, most points in reasonably nice spaces are rather regular. It is conceivable that by averaging the deviation functions v_r we will smooth out the "wildest singularities". The resulting objects will experience better behavior at $r = 0$ and tell us more about the regularity of the space.

Thus, instead of looking at the pointwise behavior of v_r at $r = 0$ we define the *deviation measure* $\mathcal{V}_r$ of X as the signed Radon measure

$$\mathcal{V}_r = v_r \mu, \tag{1.2}$$

absolutely continuous with respect to μ.

The vector space $\mathrm{M}(X)$ of *signed Radon measures on X is dual to the topological vector* space $\mathcal{C}_c(X)$ of compactly supported continuous functions. We consider the space $\mathrm{M}(X)$ with the topology of weak convergence. Recall that a subset $\mathcal{F} \subset \mathrm{M}(X)$ is relatively compact if and only if it is *uniformly bounded*: for any compact subset $K \subset X$ the values $\nu(K)$, $\nu \in \mathcal{F}$, are uniformly bounded.

The next example, fundamental for this paper, can be obtained by computations in local coordinates. Since it is formally not needed later, we omit the details, but a rigorous proof can be extracted from the proof of Theorem 1.7 in Section 7.

Example 1.4. Let X be a smooth n-dimensional Riemannian manifold with boundary ∂X. Then, for $r \to 0$, the measures $\mathcal{V}_r/r$ converge in $\mathrm{M}(X)$ to $c_n \mathcal{H}^{n-1}_{\partial X}$ for some constant $c_n > 0$ depending only on n.

This example suggests viewing the first Taylor coefficient of $\mathcal{V}_r$ as the "boundary" of the metric measure space (X, d, μ). It motivates the following definition.

Definition 1.5. Let (X, d, μ) be a metric measure space as above. Let $\mathcal{V}_r$ be the deviation measure of X, as in (1.2). We say that X *has locally finite metric-measure boundary*, abbreviated as mm-boundary, if the family of signed Radon measures

$$\{\mathcal{V}_r/r\, ;\ 0 < r \le 1\}$$

is uniformly bounded. If $\lim_{r\to 0} \mathcal{V}_r/r = \nu$ in $\mathrm{M}(X)$, we call ν the *mm-boundary* of X. If $\nu = 0$ we say that X *has vanishing mm-boundary*.

We refer to Subsection 1.8 and Section 8 for a discussion of examples and questions, and now we state our central result connecting mm-boundaries to the existence of infinite geodesics in Alexandrov spaces:

Theorem 1.6. *Let X be an Alexandrov space. If X has vanishing mm-boundary, then almost each direction of the tangent bundle TX is the starting direction of an infinite geodesic. Moreover, the geodesic flow preserves the Liouville measure on TX.*

In different settings, geodesic flows on singular spaces have been investigated in [BB95] and [Bam17].

1.3. Size of the mm-boundary in Alexandrov spaces

The next theorem shows that, similarly to Example 1.4, the topological boundary is closely related to the mm-boundary in Alexandrov spaces.

Theorem 1.7. *Let X^n be an n-dimensional Alexandrov space. Then X has locally finite mm-boundary. If $\nu = \lim \mathcal{V}_{s_j}/s_j$ for a sequence $s_j \to 0$, then ν is a Radon measure and the following hold:*

(1) *There is a Borel set A_0 with $\mathcal{H}^n(X \setminus A_0) = \nu(A_0) = 0$.*
(2) *If the topological boundary ∂X is non-empty then $\nu \geq c\mathcal{H}^{n-1}_{\partial X}$ for a positive constant c depending only on n.*
(3) *If the topological boundary ∂X is empty then $\nu(A) = 0$ for any Borel subset $A \subset X$ with $\mathcal{H}^{n-1}(A) < \infty$.*

We believe that an Alexandrov space with empty topological boundary ∂X has vanishing mm-boundary, which would solve the question of existence of infinite geodesics. This conjecture will be proved in two cases.

Theorem 1.8. *Let X^n be a convex hypersurface in $\mathbb{R}^{n+1}$ or let X be a two-dimensional Alexandrov space without boundary. Then X has vanishing mm-boundary.*

In combination with Theorem 1.6, this proves Theorem 1.1. The two-dimensional case could be derived from the statement about convex hypersurfaces and Alexandrov's embedding theorems. Another proof follows from a much stronger result discussed in the next subsection.

1.4. Metric-measure curvature

Motivated by Example 1.2, one can naively hope that the second Taylor coefficient at 0 of the map $r \mapsto \mathcal{V}_r \in \mathrm{M}(X)$ describes the scalar curvature of the space.

Definition 1.9. Let X, $\mathcal{V}_r$ be as in Definition 1.5. If the family $\mathcal{V}_r/r^2$, $r \leq 1$, is uniformly bounded then we say that X has *locally finite mm-curvature*. If the measures $\mathcal{V}_r/r^2$ converge to a measure ν, we call ν the *mm-curvature* of X.

Clearly, local finiteness of mm-curvature as defined above implies that the mm-boundary vanishes. Thus, the following result proves Theorem 1.8 in the 2-dimensional case.

Theorem 1.10. *Let X be a 2-dimensional Alexandrov space without boundary. Then X has locally finite mm-curvature.*

This finiteness result holds true in the much greater generality of surfaces with bounded integral curvature in the sense of Alexandrov–Zalgaller–Reshetnyak [Res93], [AZ67] (see Section 4).

Note, however, that the mm-curvature in Theorem 1.10 need not coincide with the "curvature measure" as defined in [AZ67], even in the case of a cone: see Example 1.14. In particular, this shows that the mm-curvatures in 2-dimensional Alexandrov spaces are not stable under Gromov–Hausdorff convergence.

Remark 1.11. Nina Lebedeva and the third named author [LP17] have found a "scalar curvature measure" on all smoothable Alexandrov spaces. There is hope, supported by our proof of Theorem 1.10, that a better understanding of this "stable curvature measure" will lead to some control of the mm-boundary and mm-curvature discussed here.

1.5. Relation to the Lipschitz–Killing curvatures

Let M be a compact smooth submanifold in $\mathbb{R}^n$. Given $r > 0$, consider the volume $w(r) = \mathcal{H}^n(B(M,r))$ of the distance tube $B(M,r)$ around M. The function $r \mapsto w(r)$ is a polynomial, at least for small positive r. The coefficients of $w(r)$, called the *Lipschitz–Killing curvatures* of M, are given as integrals of some intrinsically defined curvature terms. Moreover, these coefficients can be localized and considered as measures on M. We refer to [Ale18] for a short account of the theory, connection to [LP17] and further hypothetical relations to the theory of Alexandrov spaces.

To make the formal similarity to our approach to mm-boundary and mm-curvature more transparent, we observe that (at least for a smooth n-dimensional manifold M) the number $\int_M \mathcal{H}^n(B(x,r))\, d\mathcal{H}^n(x)$ can be interpreted as the $\mathcal{H}^{2n}$-measure of the distance tube $B(\Delta, r/\sqrt{2})$ around the diagonal Δ in the Cartesian product $M \times M$.

1.6. Idea of the proof of Theorem 1.6

The interpretation of the tangent bundle of M as the normal bundle of the diagonal Δ in $M \times M$ gives a connection between the measure-theoretical properties of tubes around Δ and the dynamical properties of the geodesic flow.

We clarify this abstract statement by explaining the main idea of our proof of Theorem 1.6 in the case of a complete smooth Riemannian manifold $X = M$. In this case the existence of geodesics is trivial. Thus, we just sketch a new proof of the classical fact that the geodesic flow ϕ preserves the Liouville measure $\mathcal{M}$ on TM. This proof is sufficiently stable to be transferred to the singular situation,

Denote by $\pi\colon TM \to M$ the tangent bundle of M. Let $\phi_t\colon TM \to TM$ be the geodesic flow for time t. Define $E\colon TM \to M \times M$ by

$$E(v) = (\pi(v), \pi(\phi_1(v))).$$

By construction, $E(-\phi_1(v)) = J(E(v))$, where J is the involution of $M \times M$ which switches the coordinates. Since J preserves the measure $\mathcal{H}^{2n}$ on $M \times M$ and $v \mapsto -v$ preserves the Liouville measure $\mathcal{M}$ on TM, the statement that ϕ is measure preserving hinges upon the smallness of measure-distortion of $E\colon (TM, \mathcal{M}) \to (M \times M, \mathcal{H}^{2n})$ close to the 0-section.

In the present case of a Riemannian manifold, this property of ϕ_1 is expressed by the fact that the differential of E is the identity (after suitable identifications). Similarly, in the general case of Alexandrov spaces, we observe that the "infinitesimal" deviation (via the canonical map E) of ϕ_1 being measure preserving (which is what we want to show) from J being measure preserving (which we know) is expressed as the triviality of the mm-boundary.

1.7. Stability and relation to quasi-geodesics

Many Alexandrov spaces, for instance all convex hypersurfaces, appear naturally as Gromov–Hausdorff limits of smooth Riemannian manifolds. However, the properties of the geodesic flow, mm-boundaries and mm-curvature are unstable under limit operations; see also the discussion at the end of Subsection 1.4. Thus, there is no hope to deduce Theorem 1.10, Theorem 1.8 or Theorem 1.1 by a direct limiting argument.

For instance, being a geodesic is a local notion, not preserved under limits. However, any limit of geodesics in a non-collapsed limit of Alexandrov spaces is a curve sharing many properties with geodesics. These properties are used to define the so called *quasi-geodesics*; see [PP96], [Pet07] and the references therein. It has been shown that any direction is the starting direction of an infinite quasi-geodesic. One motivation for the present paper was an attempt to prove Liouville's theorem for the "quasi-geodesic flow" (see Subsection 3.6).

1.8. Examples

The estimates of the mm-boundary and mm-curvature are quite involved even in quite simple situations. The following examples are not needed later and we omit the somewhat tedious computations. Examples 1.14–1.16 should be compared with [Ber03] and [Ber02] revealing further natural connections to the theory of Lipschitz–Killing curvature on singular subsets of the Euclidean space.

Example 1.12. Let X be a Riemannian manifold with a Lipschitz continuous metric. Then X has vanishing mm-boundary.

Example 1.13. If X is a manifold with two-sided bounded curvature in the sense of Alexandrov then its mm-curvature is well-defined and absolutely continuous with respect to the Hausdorff measure.

Example 1.14. Let X be the Euclidean cone over the circle S_ρ of length ρ. The curvature measure and the mm-curvature are Dirac measures concentrated at the tip of the cone. The mass of the curvature measure is $\alpha = 2\pi - \rho$. From Example 1.2 one would expect the mass of the mm-curvature to be $m(\alpha) = \alpha/12$. However, a straightforward calculation shows that $m(\alpha) = \alpha/12 + f(\alpha)$, where $f(\alpha) = O(\alpha^2)$ is a non-zero function.

Example 1.15. Let X be a finite n-dimensional simplicial complex with an intrinsic metric d. Assume that the restriction of d to each simplex is given by a smooth Riemannian metric. Then X has a finite mm-boundary ν with support on the $(n-1)$-skeleton X^{n-1}.

Example 1.16. Assume that X as in the last example is a pseudo-manifold. Then X has finite mm-curvature. If all simplices are flat then the mm-curvature is concentrated on the $(n-2)$-skeleton.

1.9. *Structure of the paper*

After preliminaries collected in Section 2, we prove Theorem 1.6 in Section 3 along the lines sketched above. In Sections 4, 5 and 7 we prove Theorems 1.10, 1.8 and 1.7 respectively. Their proofs all rely on a decomposition of the space into a regular and a singular part, with a quantitative estimate of the size of the singular part. Finally, on the regular part we estimate the mm-curvature and mm-boundary by comparing them to other natural measures on these spaces.

In the case of surfaces, the comparison measure is the classical curvature measure; in the case of convex hypersurfaces, it is the mean curvature. Finally, in the case of a general Alexandrov space, the comparison is given by the derivative of the metric tensor expressed in DC-coordinates [Per95].

The needed control of the ball growth in terms of these measures is given by a theorem of Mario Bonk and Urs Lang in the case of surfaces, and follows from classical convex geometry in the case of hypersurfaces. The analytical comparison result needed for Alexandrov spaces is established in Section 6.

In the final Section 8 we collect a number of comments and open questions which naturally arose during the work on this paper.

2. Preliminaries

2.1. *Metric spaces*

We refer to [BBI01] for basics on metric spaces. The distance between points x, y in a metric space X will be denoted by $d(x, y)$. By $B(x, r)$ we will denote the open metric ball of radius r around a point x. For $A \subset X$ we denote by $B(A, r)$ the open r-neighborhood $B(A, r) = \bigcup_{x \in A} B(x, r)$.

A *minimizing geodesic* γ in a metric space X is a map $\gamma\colon \mathbb{I} \to X$ defined on an interval $\mathbb{I}$ such that for some number $\lambda \geq 0$ and all $t, s \in \mathbb{I}$,

$$d(\gamma(t), \gamma(s)) = \lambda |t - s|.$$

In particular, we allow γ to have any constant velocity $\lambda \geq 0$. A *geodesic* is a curve $\gamma\colon \mathbb{I} \to X$ whose restriction to a small neighborhood of any point in $\mathbb{I}$ is a minimizing geodesic. Note that a geodesic is a curve of constant velocity.

2.2. *Metric measure spaces*

We refer to [Fed69] and [EG15] for basics on measure theory.

Let X be a locally compact separable metric space. A *Radon measure* on X is a measure on X for which all compact subsets are measurable and have finite measure. Any Radon measure defines an element of $\mathrm{M}(X)$, the dual space to the topological vector space $\mathcal{C}_c(X)$ of compactly supported continuous functions on X. All elements in $\mathrm{M}(X)$ are called *signed Radon measures*. Any $\mu \in \mathrm{M}(X)$ can be uniquely written as

$\mu_+ - \mu_-$, where $\mu_\pm$ are Radon measures concentrated on disjoint subsets. The measure $|\mu| = \mu_+ + \mu_-$ is called the *total variation* of μ.

A family $\mathcal{F}$ of signed Radon measures on X is *uniformly bounded* if for any compact subset $K \subset X$ there exists a constant $C(K) > 0$ such that $|\mu|(K) \leq C(K)$ for any $\mu \in \mathcal{F}$. Any uniformly bounded sequence of signed measures μ_i has a convergent subsequence.

The following lemma will be repeatedly used.

Lemma 2.1. *Let X be a metric space with two Radon measures μ and ν. Let $r > 0$ be arbitrary and let $A \subset X$ be a Borel subset. Then*

$$\int_A \mu(B(x,r))\, d\nu(x) \leq \int_{B(A,r)} \nu(B(x,r))\, d\mu(x).$$

Proof. By Fubini's theorem the left hand side is the volume of

$$S = \{(y,x) \in X \times X\,;\; x \in A,\; d(y,x) < r\}$$

with respect to the product measure $\nu \otimes \mu$. And the right hand side is the volume of the larger set

$$T = \{(y,x) \in X \times X\,;\; x \in B(A,r),\; d(y,x) < r\}$$

with respect to the same measure. □

2.3. Alexandrov spaces

We assume that the reader is familiar with the basic theory of Alexandrov spaces and refer to [BGP92] for an introduction to the subject. In this paper, an *Alexandrov space* is a complete, locally compact, geodesic metric space of finite Hausdorff dimension and of curvature bounded from below by some $\kappa \in \mathbb{R}$. For Alexandrov spaces, an upper index will indicate the Hausdorff dimension; that is, X^n denotes an n-dimensional Alexandrov space, equipped with the n-dimensional Hausdorff measure $\mathcal{H}^n$.

The set of starting directions of geodesics starting at a given point $x \in X^n$ carries a natural metric, whose completion is the *tangent space* $T_x = T_x X$ of X at the point x. It is an n-dimensional Alexandrov space of non-negative curvature. Moreover, it is the Euclidean cone over the space Σ_x of unit directions. The Euclidean cone structure defines multiplication by positive scalars $\lambda \geq 0$ on $T_x X$. The origin of the cone $T_x X$ is denoted by $0 = 0_x$. Elements of $T_x X$ are called *tangent vectors* at x, despite the fact that $T_x X$ is not a vector space in general. For $v \in T_x X$ the *norm* $|v|$ of v is the distance from v to the origin 0_x.

Geodesics in X do not branch; moreover, any two geodesics with identical starting vectors coincide. For $x \in X$ the *exponential map* $\exp_x$ is defined as follows. Let D_x denote the set of all vectors $v \in T_x X$ for which there exists an (always unique) *minimizing geodesic* $\gamma_v \colon [0,1] \to X$ with starting direction v. Then $\exp_x$ is defined on D_x as

$$\exp_x(v) = \gamma_v(1).$$

For any $r > 0$, the map $\exp_x$ sends $D_x \cap B(0_x, r) \subset T_x X$ surjectively onto $B(x, r) \subset X$. Moreover, for a constant $C = C(\kappa) \geq 0$ and all $r < 1/C$ the map $\exp_x \colon D_x \cap B(0_x, r) \to B(x, r)$ is $(1 + Cr^2)$-Lipschitz continuous.

By the Bishop–Gromov theorem, the volume $b_r(x) = \mathcal{H}^n(B(x, r))$ is bounded from above by the corresponding volume in the space of constant curvature κ. In particular, $b_r(x) \leq \omega_n r^n + Cr^{n+2}$ for all $r \leq 1/C$, where the constant C can be chosen as before. Thus, the deviation measures $\mathcal{V}_r$ from (1.2) satisfy

$$\mathcal{V}_r \geq -Cr^2 \mathcal{H}^n$$

for all sufficiently small r. Here and above we can set $C = 0$ if $\kappa \geq 0$.

Denote by X_{reg} the set of all points $x \in X$ with $T_x X$ isometric to the Euclidean space. The set X_{reg} has full $\mathcal{H}^n$-measure in X. Any inner point of any geodesic starting on X_{reg} is contained in X_{reg} [Pet98].

The topological boundary ∂X of X can be defined as the closure of the set of all points $x \in X$ with $T_x X$ isometric to a Euclidean half-space. Up to a subset of Hausdorff dimension $n - 2$, ∂X is an $(n - 1)$-dimensional Lipschitz manifold.

2.4. *Volume and bi-Lipschitz maps*

Let $\mu = \mathcal{H}^n$ be a Radon measure on the metric space X. Let $U \subset X$ and $V \subset \mathbb{R}^n$ be open and assume that there is a surjective $(1 + \delta)$-bi-Lipschitz map $f \colon U \to V$, that is,

$$\frac{1}{1 + \delta} \leq \frac{|f(x) - f(y)|}{d(x, y)} \leq 1 + \delta$$

for any distinct points $x, y \in U$.

Let $A \subset U$ satisfy $B(A, (1 + \delta)r) \subset U$ and $B(f(A), (1 + \delta)r) \subset V$. Then, for all $x \in A$,

$$(1 + \delta)^{-2n} \leq \frac{b_r(x)}{\omega_n r^n} \leq (1 + \delta)^{2n}. \tag{2.1}$$

Therefore, if δ is sufficiently small, then $|\mathcal{V}_r|(A) \leq 3n\delta \mathcal{H}^n(A)$.

3. Liouville measure and geodesics

3.1. *Tangent bundle and Liouville measure*

Let X be an n-dimensional Alexandrov space. Denote by TX the disjoint union of the tangent spaces at all points,

$$TX = \bigsqcup_{x \in X} T_x X.$$

Let $\pi \colon TX \to X$ be the footpoint projection, so $\pi(T_x X) = \{x\}$ for any $x \in X$. For a subset $K \subset X$ denote by TK the inverse image $\pi^{-1}(K) = \bigcup_{x \in K} T_x X$. Given $r > 0$, denote by $T^r K$ the set of all vectors in TK of norm smaller than r.

The Riemannian structure on the set of regular points discussed in [OS94] (see also [KMS01], [Per95]) provides TX_{reg} with the structure of a Euclidean vector bundle over X_{reg}. In this topology, for any sequence of geodesics γ_i in X_{reg} converging to a geodesic γ, the starting directions of γ_i converge to the starting direction of γ.

On the Euclidean vector bundle TX_{reg} over X_{reg} we have a natural choice of measure, which locally coincides with the product measure of $\mathcal{H}^n_X$ and the Lebesgue measures on the fibers. More precisely, it is the unique Borel measure $\mathcal{M}$ on TX_{reg} such that for any Borel set $A \subset TX_{\mathrm{reg}}$,

$$\mathcal{M}(A) = \int_X \mathcal{H}^n(A \cap T_x X)\, d\mathcal{H}^n(x).$$

We extend $\mathcal{M}$ to a measure on TX by setting $\mathcal{M}(TX \setminus TX_{\mathrm{reg}})$ to be 0.

By definition, a subset $A \subset TX$ is *$\mathcal{M}$-measurable* if there exists a Borel subset $A' \subset A \cap TX_{\mathrm{reg}}$ such that for $\mathcal{H}^n$-almost all $x \in X$ the intersection $(A \setminus A') \cap T_x X$ has $\mathcal{H}^n$-measure zero in $T_x X$.

For any $\lambda > 0$, we have $\mathcal{M}(\lambda A) = \lambda^n \mathcal{M}(A)$ for any measurable set $A \subset TX$. The involution $I : TX_{\mathrm{reg}} \to TX_{\mathrm{reg}}$ defined by $I(v) = -v$ preserves $\mathcal{M}$ since it preserves the Lebesgue measure in each tangent space.

3.2. *Geodesic flow*

Let us define the geodesic flow ϕ on a maximal subset $\mathcal{F}$ of $TX \times \mathbb{R}$.

For any $v \in T_x X$ we set $\phi_0(v) = v$. If no geodesic starts in the direction of v, the value $\phi_t(v)$ will not be defined for $t \neq 0$. If such a geodesic γ_v exists, then γ_v can be uniquely extended to a maximal half-open interval $\gamma_v : [0, a) \to X$. For $t \geq a$ the value $\phi_t(v)$ will not be defined. For $0 < t < a$ we set $\phi_t(v)$ to be $\gamma_v^+(t) \in T_{\gamma_v(t)} X$, the starting direction of $\gamma_v : [t, a) \to X$ at $\gamma_v(t)$.

If the geodesic $\gamma_v : [0, a) \to X$ extends to an (again uniquely defined, maximal) geodesic $\gamma_v : (b, a) \to X$ for some $b < 0$ then we define $\phi_t(v)$ for $b < t < 0$ to be $\gamma_v^+(t)$ as above.

We denote by $\mathcal{F}$ the set of all pairs $(v, t) \in TX \times \mathbb{R}$ for which $\phi_t(v)$ is defined.

For $\lambda > 0$, for $t, s \in \mathbb{R}$ and $v \in T_x X$ we have

$$\phi_t(\lambda v) = \lambda \phi_{\lambda t}(v) \quad \text{and} \quad \phi_{t+s}(v) = \phi_t(\phi_s(v)),$$

whenever the right hand sides are defined.

The partial flow ϕ preserves the norm of tangent vectors. Since inner points of geodesics starting in X_{reg} are contained in X_{reg}, the set TX_{reg} is invariant under the flow ϕ.

By construction, the domain of the definition of the geodesic flow almost includes the domain of the definition of the exponential map. More precisely, consider the set

$$D = \bigcup_{x \in X} D_x \subset TX,$$

that is, the set of all vectors $v \in TX$ for which $\exp_{\pi(v)}(v)$ is defined. Note that $\lambda D \subset D$ for any $0 \leq \lambda \leq 1$. Moreover, for all $v \in D$ and all $0 \leq \lambda < 1$ the geodesic flow $\phi_1(\lambda v)$ is defined (equivalently $(\lambda v, 1) \in \mathcal{F}$) and

$$\pi(\phi_1(\lambda v)) = \exp_{\pi(v)}(\lambda v).$$

Thus, for $\mathcal{M}$-almost all $v \in D$ we have the following:

- $v \in TX_{\text{reg}}$;
- $\phi_1(v) \in TX_{\text{reg}}$ is defined, hence $(v, 1) \in \mathcal{F}$;
- $w = -\phi_1(v) \in D$ and

$$(\pi(w), \exp(w)) = (\exp(v), \pi(v)) \in X \times X. \tag{3.1}$$

3.3. *Measurability*

In order to use measure-theoretic arguments we will need the following lemma (see also Subsection 3.6).

Lemma 3.1. *The set $\mathcal{F} \subset TX \times \mathbb{R}$ is measurable with respect to the product of the Liouville measure $\mathcal{M}$ on TX and the Lebesgue measure on $\mathbb{R}$. Moreover the map $\phi \colon \mathcal{F} \to TX$ is measurable.*

Proof. Fix $(v, \tau) \in \mathcal{F}$ and set $\gamma(\tau t) = \phi_t(v)$, $t \in [0, 1]$. Note that there exists some $k > 0$ such that the restriction of γ to any subinterval of length $1/k$ is a (minimizing) geodesic. We will call such a γ a *k-geodesic* and write $(v, \tau) \in \mathcal{F}_k$.

The limit of any converging sequence of k-geodesics is a k-geodesic. It follows that $\mathcal{F}'_k = \mathcal{F}_k \cap (TX_{\text{reg}} \times \mathbb{R})$ is a closed set in $TX_{\text{reg}} \times \mathbb{R}$. Therefore, $\mathcal{F} \cap (TX_{\text{reg}} \times \mathbb{R})$ is a countable union of closed subsets $\mathcal{F}'_k$, hence is measurable. Moreover, the restriction $\phi \colon \mathcal{F}'_k \to TX_{\text{reg}}$ is continuous, and therefore $\phi \colon \mathcal{F} \cap (TX_{\text{reg}} \times \mathbb{R}) \to TX_{\text{reg}}$ is a Borel-measurable map.

Since $\mathcal{M}(X \backslash X_{\text{reg}}) = 0$, the statement follows. $\square$

3.4. *Liouville property*

Denote by $\mathcal{G}$ the set of all vectors $v \in TX$ such that $\phi_t(v)$ is defined for all $t \in \mathbb{R}$. Note that $\mathcal{G}$ contains the 0-section; it is invariant under multiplication by any $\lambda > 0$ and under the geodesic flow ϕ. Moreover, $\mathcal{G} \cap TX_{\text{reg}}$ is invariant under the involution $I(v) = -v$.

Definition 3.2. We say that an Alexandrov space X has the *Liouville property* if $\mathcal{M}(TX \setminus \mathcal{G}) = 0$ and for any $t \in \mathbb{R}$ the geodesic flow $\phi_t \colon \mathcal{G} \to \mathcal{G}$ preserves the Liouville measure.

The Liouville property can be checked infinitesimally using the following lemma.

Lemma 3.3. *An Alexandrov space* X does not *have the Liouville property if and only if there is a compact subset* $K \subset X$*, a positive number* ε *and a sequence of positive numbers* $r_m \to 0$ *with the following property. For every* m*, there exists a Borel subset* $A_m \subset T^{r_m} K$ *such that*

$$\varepsilon r_m^{n+1} \leq \mathcal{M}(A_m) - \mathcal{M}(\phi_1(A_m)). \tag{3.2}$$

Here $\phi_1(A_m)$ *is the set of all* $\phi_1(v)$, $v \in A_m$*, for which* $\phi_1(v)$ *is defined.*

Proof. If at least one r_m with the above property exists, then X does not have the Liouville property by definition.

Assume that X does not have the Liouville property. Then, by homogeneity of the geodesic flow, either ϕ_1 is undefined on a subset of TX with positive measure, or ϕ_1 does not preserve the measure $\mathcal{M}$. In both cases we can find a compact subset $K_1 \subset X_{\text{reg}}$, a Borel subset $A \subset T^1 K_1$ and $\varepsilon > 0$ such that

$$\varepsilon < \mathcal{M}(A) - \mathcal{M}(\phi_1(A)).$$

Since

$$\phi_1(A) = 2\phi_2\big(\tfrac{1}{2}A\big) = 2\phi_1 \circ \phi_1\big(\tfrac{1}{2}A\big),$$

we deduce

$$\begin{aligned}\frac{\varepsilon}{2^n} \leq\ & \mathcal{M}\big(\tfrac{1}{2}A\big) - \mathcal{M}\big(\phi_1\big(\tfrac{1}{2}A\big)\big) \\ &+ \mathcal{M}\big(\phi_1\big(\tfrac{1}{2}A\big)\big) - \mathcal{M}\big(\phi_1\big(\phi_1\big(\tfrac{1}{2}A\big)\big)\big).\end{aligned}$$

Thus, taking either $A_{1/2} := \frac{1}{2}A$ or $A_{1/2} := \phi_1(\frac{1}{2}A)$ we infer

$$\frac{\varepsilon}{2^{n+1}} < \mathcal{M}(A_{1/2}) - \mathcal{M}(\phi_1(A_{1/2})).$$

The set $A_{1/2}$ constructed above is contained in $T^{1/2}K_{1/2}$, where $K_{1/2} = B(K_1, 1/2)$.

Iterating the above procedure we obtain, for $r_m = 1/2^m$, a subset $A_{r_m} \subset T^{r_m} K_m$ with $K_m = B(K_{m-1}, r_m)$ such that (3.2) holds true.

The claim follows since all K_m are contained in the set $B(K_1, 1)$, whose closure is compact, by completeness of X. □

Remark 3.4. The completeness of the space X is used in the proof of Theorem 1.6 only once, namely in the last line of the above proof.

3.5. Relation to the mm-boundary

Let us interpret the deviation measures $\mathcal{V}_r$ from (1.2) in suitable geometric terms.

Let $K \subset X$ be measurable and let $r > 0$ be arbitrary. Since $\mathcal{H}^n(X \setminus X_{\text{reg}}) = 0$, we have

$$\mathcal{M}(T^r K) = \omega_n r^n \mathcal{H}^n(K).$$

Denote now by $U^r(K)$ the set of all pairs $(x, y) \in X \times X$ with $x \in K$ and $d(x, y) < r$. By Fubini's theorem the set $U^r(K)$ is $\mathcal{H}^n \otimes \mathcal{H}^n = \mathcal{H}^{2n}$ measurable and we have

$$\mathcal{H}^{2n}(U^r(K)) = \int_K b_r(x)\, d\mathcal{H}^n(x).$$

Taking both equations together, we see that the signed measure $\mathcal{V}_r$ expresses the difference between $\mathcal{H}^{2n}$ and $\mathcal{M}$. More precisely,

$$\mathcal{V}_r(K) = \frac{1}{\omega_n r^n}\big(\mathcal{M}(T^r K) - \mathcal{H}^{2n}(U^r(K))\big). \tag{3.3}$$

The following statement is a reformulation of Theorem 1.6.

Theorem 3.5. *If an Alexandrov space X has vanishing mm-boundary then it has the Liouville property.*

Proof. Assume that X does not have the Liouville property. Consider the compact subset $K \subset X$, the positive numbers ε, r_m and the Borel subsets $A_m \subset T^{r_m} K$ provided by Lemma 3.3.

Let Y be the closure of $B(K, 1)$. Recall that $D \subset TX$ is the set of all vectors at which the exponential map is defined. For $r > 0$, denote by D^r the intersection of D with $T^r Y$ and consider the "total exponential map" $E\colon D^r \to X \times X$ given by

$$E(v) = (\pi(v), \pi(\exp(v))).$$

As above, let $U^r = U^r(Y)$ be the set of all pairs $(y, x) \in X \times X$ with $y \in Y$ and $d(x, y) < r$. Note that

$$E(D^r) = U^r. \tag{3.4}$$

Moreover, for any fixed $x \in Y$, the restriction of E to $D_x \cap D^r$ is a $(1 + Cr^2)$-Lipschitz continuous map from $D_x \subset T_x X$ onto the set $U^r \cap (\{x\} \times X)$ (see Subsection 2.3). Thus, for all sufficiently small r, and any Borel subset $S \subset D_x \cap D^r$, we have

$$\mathcal{H}^n(E(S)) \le (1 + 4nCr^2)\mathcal{H}^n(S).$$

Using the definition of the Liouville measure $\mathcal{M}$ and Fubini's formula for the product measure $\mathcal{H}^{2n} = \mathcal{H}^n \otimes \mathcal{H}^n$ on $X \times X$ we obtain, for any $\mathcal{M}$-measurable subset S of D^r,

$$\mathcal{H}^{2n}(E(S)) \le (1 + 4nCr^2)\mathcal{M}(S). \tag{3.5}$$

Due to (3.3), the vanishing of the mm-boundary of X implies

$$\lim_{r\to 0} \frac{1}{r^{n+1}} |\mathcal{M}(T^r Y) - \mathcal{H}^{2n}(U^r)| = 0. \tag{3.6}$$

Thus, up to terms of order higher than r^{n+1}, the map E does not increase the measure of subsets, but the total mass of the image coincides with the total mass of the target. Therefore, E is measure preserving up to terms of order higher than r^{n+1} on all subsets of $T^r Y$. More precisely, for every $\delta > 0$, there is $s > 0$ with the following property. For all $0 < r < s$ and all measurable subsets $S \subset D^r$ we have

$$0 \leq \mathcal{M}(T^r Y) - \mathcal{M}(D^r) < \delta r^{n+1}, \quad |\mathcal{H}^{2n}(E(S)) - \mathcal{M}(S)| < \delta r^{n+1}. \tag{3.7}$$

Indeed, violation of the first inequality would imply by (3.5) an upper bound on $\mathcal{H}^{2n}(U^r) = \mathcal{H}^{2n}(E(D^r))$, which would contradict (3.6). Similarly, (3.5) provides the right upper bound for $\mathcal{M}(S)$ in the second inequality. On the other hand, (3.5) applied to $T^r Y \setminus S$ together with (3.6) imply the right lower bound for $\mathcal{M}(S)$.

For any measurable subset $S \subset D^r \cap TK$ we now claim that

$$|\mathcal{H}^{2n}(E(S)) - \mathcal{M}(\phi_1(S))| < 2\delta r^{n+1}. \tag{3.8}$$

In order to prove (3.8), let S^+ be the subset of all vectors $v \in S$ for which $\phi_1(v)$ exists and is contained in TX_{reg}. For all $v \in S^+$, we have $-\phi_1(v) \in D^r$ and, due to (3.1),

$$E(-\phi_1(v)) = J(E(v)).$$

The involution $I(v) = -v$ is $\mathcal{M}$-preserving on TX_{reg}. And the involution $J\colon X \times X \to X \times X$ given by $J(x, y) = (y, x)$ preserves $\mathcal{H}^{2n}$. Therefore, from (3.7) we deduce

$$|\mathcal{H}^{2n}(E(S^+)) - \mathcal{M}(\phi_1(S^+))| < \delta r^{n+1}. \tag{3.9}$$

On the other hand, by construction,

$$\mathcal{M}(S \setminus S^+) = 0 \quad \text{and} \quad \phi_1(S \setminus S^+) \cap TX_{\text{reg}} = \emptyset.$$

Hence, applying (3.7), we see that

$$|\mathcal{H}^{2n}(E(S)) - \mathcal{H}^{2n}(E(S^+))| < \delta r^{n+1} \quad \text{and} \quad \mathcal{M}(\phi_1(S \setminus S^+)) = 0.$$

Together with (3.9) this finishes the proof of (3.8).

Coming back to our subsets $A_m \subset T^{r_m} K$, we have

$$\varepsilon r_m^{n+1} \leq \mathcal{M}(A_m) - \mathcal{M}(\phi_1(A_m)) \leq \mathcal{M}(A_m) - \mathcal{M}(\phi_1(A_m \cap D^{r_m})).$$

Setting $S_m = A_m \cap D^{r_m}$ we estimate the right hand side as the sum of the following three terms:

$$|\mathcal{M}(A_m) - \mathcal{M}(S_m)|,$$
$$|\mathcal{M}(S_m) - \mathcal{H}^{2n}(E(S_m))|,$$
$$|\mathcal{H}^{2n}(E(S_m)) - \mathcal{M}(\phi_1(S_m)|.$$

In view of (3.7) and (3.8) this sum is bounded above by $4\delta r_m^{n+1}$ for all large m.

Therefore

$$\varepsilon r_m^{n+1} < 4\delta r_m^{n+1}$$

for all large m. Since δ is an arbitrary positive number, this is a contradiction. □

3.6. *Quasi-geodesic flow*

Finally, we discuss some relations to quasi-geodesics, referring the reader to [Pet07] for the basic properties of such curves. Recall that whenever a unit speed minimizing geodesic $\gamma_v\colon [0,a] \to X$ start at a point x in the direction v then this is the unique quasi-geodesic defined on the interval $[0,a]$ [PP96, p. 8], thus the same statement is also true for (local) geodesics γ_v.

Using this and the fact that a limit of quasi-geodesics is a quasi-geodesic, it is not difficult to conclude that the partial geodesic flow $\phi\colon \mathcal{F} \cap TX_{\mathrm{reg}} \to TX_{\mathrm{reg}}$ defined above is continuous. The latter statement slightly strengthens Lemma 3.1.

As in Subsection 3.1, we have a canonical measure $\mathcal{M}_1$ on the unit tangent bundle $\Sigma X \subset TX$ of X, which we also call the Liouville measure. Whenever X has the Liouville property, the geodesic flow is defined $\mathcal{M}_1 \otimes \mathcal{H}^1$-almost everywhere on $\Sigma X \times \mathbb{R}$ and preserves $\mathcal{M}_1$. In this case for $\mathcal{M}_1$-almost each unit direction there exists exactly one quasi-geodesic starting in this direction.

Let now X be an Alexandrov space with topological boundary ∂X and let Z be the doubling $X \sqcup_{\partial X} X$, which is an Alexandrov space without boundary [Per91]. Quasi-geodesics in X are exactly the projections of the quasi-geodesics in Z under the folding $f\colon Z \to X$. From this we deduce that if Z has the Liouville property, then $\mathcal{M}_1$-almost each direction $v \in \Sigma X$ is the starting direction of a unique infinite quasi-geodesic in X. Moreover, in this case, the corresponding quasi-geodesic flow preserves $\mathcal{M}_1$.

Finally, as an application of Theorems 1.6 and 1.7 we see that the above assumptions are fulfilled whenever the complement $X \setminus \partial X$ has vanishing mm-boundary. Indeed, in this case the mm-boundary of Z must be concentrated on $\partial X \subset Z$, hence it must be trivial by Theorem 1.7(3).

4. Surfaces with bounded integral curvature in the sense of Alexandrov

4.1. Preparations

We assume that the reader is familiar with the theory of surfaces with bounded integral curvature (see [AZ67] and [Res93]).

Let X be a surface with bounded integral curvature; it is a locally geodesic metric space, homeomorphic to a two-dimensional surface. It has Hausdorff dimension 2 and the Hausdorff measure $\mathcal{H}^2$ is a Radon measure on X. There is another signed Radon measure on X, called the *curvature measure*, which will be denoted Ω [Res93, Section 8]. We will not assume that X is complete.

We will derive Theorem 1.10 as a consequence of the following weak local version of a theorem of Mario Bonk and Urs Lang [BL03], which relates the curvature measure to the volume of balls.

Lemma 4.1. *There exists some $\delta_0 > 0$ with the following property. Let X be a surface with bounded integral curvature and let $\Omega \in \mathrm{M}(X)$ be its curvature measure. Assume X is homeomorphic to a plane and $|\Omega|(X) < \delta_0$. Then for any point $x \in X$ and $r > 0$ such that $\bar{B}(x,r)$ is compact we have*

$$\left| 1 - \frac{b_r(x)}{\pi r^2} \right| \leq 3|\Omega|(B(x,r)).$$

Proof. Set $\delta = |\Omega|(B(x,r))$. By continuity, it is sufficient to prove that $\left|1 - \frac{b_s(x)}{\pi s^2}\right| \leq 3\delta$ for any $s < r$. Using approximations of the metric on X by polyhedral metrics [Res93, Theorems 8.4.3, 8.1.9], we assume from now on that X is polyhedral and homeomorphic to $\mathbb{R}^2$.

Claim. *There exists a complete polyhedral surface $\hat{X}$ homeomorphic to a plane such that X contains a copy of $B(x,s)$ and the curvature measure $\hat{\Omega}$ of $\hat{X}$ satisfies $|\hat{\Omega}|(\hat{X}) < 3\delta$.*

Once the claim is proven, [BL03] provides a bi-Lipschitz map $f\colon \hat{X} \to \mathbb{R}^2$ with constant $L \leq 1 + \frac{3\delta}{2\pi - 3\delta}$. Since δ is small, an application of (2.1) finishes the proof of the lemma.

It remains to prove the Claim, certainly well-known to experts. Take some $r > t > s$ and consider the compact metric ball $\bar{B}(x,t) \subset B(x,r)$. We may assume that the boundary S_t of $\bar{B}(x,t)$ does not contain singular points of X. By [Res93, Theorems 9.1, 9.3], the boundary S_t is a (piecewise smooth) Jordan curve once $\delta_0 < 2\pi$, moreover, the negative part κ^- of the geodesic curvature κ of S_t satisfies $|\kappa^-|(S_t) \leq \delta$. Since X is homeomorphic to a plane, this implies that $\bar{B}(x,t)$ is homeomorphic to a closed disk $\bar{D}^2$ in $\mathbb{R}^2$.

We find a polygonal Jordan curve Γ in $B(x,t)$ approximating S_t such that the negative part of the geodesic curvature of Γ is smaller than 2δ. Consider the closed Jordan domain Y bounded by Γ, which can be assumed to contain $B(x,s)$. Now we glue to Y, along any edge of Γ, a flat half-strip. The boundary of the arising polyhedral surface consists of pairs of rays $\gamma_i^\pm$ emanating from the vertices $V_1, \dots, V_k$ of Γ. The rays $\gamma_i^\pm$ enclose an angle equal to $2\pi - \alpha_i$, where $\pi - \alpha_i$ is the angle of Γ at V_i measured in Y. In order to finish the construction of $\hat{X}$ we glue a flat sector of angle α_i between $\gamma_i^\pm$ if $\alpha_i > 0$, and we glue $\gamma_i^\pm$ together if $\alpha_i \leq 0$. Since Y was a polyhedral disk, the arising space $\hat{X}$ is a complete polyhedral plane. All of the singularities of $\hat{X}$ are contained in $B(x,s) \cup \{V_1, \dots, V_k\}$. Moreover, by construction, the curvature measure $\hat{\Omega}$ of $\hat{X}$ satisfies

$$\hat{\Omega}(V_i) = \min\{0, \alpha_i\}.$$

We deduce

$$|\hat{\Omega}|(\hat{X}) = |\Omega(B(x,s))| + |\hat{\Omega}|(\Gamma) \leq \delta + |\kappa^-|(\Gamma) < 3\delta.$$

This finishes the proof of the Claim and of Lemma 4.1. □

4.2. Local finiteness of mm-curvature

Now we are ready to prove the following generalization of Theorem 1.10.

Theorem 4.2. *Let X be an Alexandrov surface with integral curvature bounds. Then, equipped with the Hausdorff measure $\mathcal{H}^2$, the space X has locally finite mm-curvature.*

Proof. Let again Ω denote the curvature measure of X. Let $\delta_0 > 0$ be sufficiently small and satisfy the conclusion of Lemma 4.1. The statement of Theorem 4.2 is local, so we need to prove it only in a small neighbourhood of any point. Thus we may (and will) assume that there is a point $x_0 \in X$ such that $|\Omega|(X \backslash \{x_0\}) < \delta_0$ and that X is homeomorphic to a plane.

Let $A \subset X$ be compact. Choose some $\varepsilon > 0$ such that the closure of $B(A, 2\varepsilon)$ in X is compact and $\mathcal{H}^2(B(x_0, 3r)) < \frac{1}{\varepsilon} r^2$ for any $0 < 2r < \varepsilon$ [Res93, Lemma 8.1.1].

Let $r < \varepsilon$ be arbitrary. For any $x \in B(x_0, 2r)$ we have

$$b_r(x) = \mathcal{H}^2(B(x, r)) \leq \mathcal{H}^2(B(x_0, 3r)) \leq \tfrac{1}{\varepsilon} r^2.$$

For any $x \notin B(x_0, r)$ we have $|\Omega|(B(x, r)) < \delta_0$. Thus, by Lemma 4.1,

$$\left| 1 - \frac{b_r(x)}{\pi r^2} \right| \leq 3|\Omega|(B(x, r)).$$

For the deviation measures $\mathcal{V}_r$ from (1.2) we estimate

$$|\mathcal{V}_r|(A \cap B(x_0, 2r)) \leq |\mathcal{V}_r|(B(x_0, 2r)) \leq \left(1 + \tfrac{1}{\varepsilon}\right)\mathcal{H}^2(B(x_0, 2r)) \leq \left(1 + \tfrac{1}{\varepsilon}\right)\tfrac{1}{\varepsilon} r^2.$$

On the other hand,

$$\begin{aligned} |\mathcal{V}_r|(A \setminus B(x_0, 2r)) &\leq \int_{A \setminus B(x_0, 2r)} 3|\Omega|(B(x, r))\, d\mathcal{H}^2(x) \\ &\leq 3 \int_{B(A, r) \setminus B(x_0, 2r)} \mathcal{H}^2(B(x, r))\, d|\Omega|(x), \end{aligned}$$

where we have used Lemma 2.1 in the last step. For any x in the domain of integration of the last integral, we have $\mathcal{H}^2(B(x, r)) = b_r(x) \leq 2\pi r^2$, by Lemma 4.1, once δ_0 has been chosen to be sufficiently small. We deduce $|\mathcal{V}_r|(A \setminus B(x_0, 2r)) \leq 6\pi \delta_0 r^2$.

Thus, for some constant $C = C(\varepsilon)$ and all $r < \varepsilon$, we obtain

$$|\mathcal{V}_r|(A) = |\mathcal{V}_r|(A \setminus B(x_0, 2r)) + |\mathcal{V}_r|(A \cap B(x_0, 2r)) \leq C r^2.$$

This finishes the proof of the theorem. □

5. Convex hypersurfaces

In this section we are going to prove Theorem 1.1.

The proof will follow from Theorem 1.6 by comparing the mm-boundary with the mean curvature measure on convex hypersurfaces.

It is possible to deduce the theorem without a reference to Theorem 1.7, from Lemma 5.2 alone, but the use of Theorem 1.7 shortens the proof.

All results in this section are local, but for simplicity we consider only closed convex hypersurfaces. The hypersurfaces will always be equipped with the induced intrinsic metric.

We assume that the reader is familiar with the basics of the theory of convex functions and convex geometry.

5.1. Mean curvature

Let X be a convex hypersurface in $\mathbb{R}^{n+1}$. Recall that there exists a Radon measure $\mathcal{K}$ on X, called the *mean curvature measure* (see [Sch93], [Fed59]).

The measure $\mathcal{K}$ has the following properties. For smooth hypersurfaces X, we have $\mathcal{K} = \kappa\mathcal{H}^n$, where κ is the usual mean curvature function of X. The mean curvature measure is stable under Hausdorff convergence of convex hypersurfaces in $\mathbb{R}^{n+1}$. If the hypersurface is rescaled by λ, the mean curvature $\mathcal{K}$ is rescaled by λ^{n-1}.

A point x in the convex hypersurface X is called *smooth* if there is a unique supporting hyperplane of X at this point. For any smooth point $x \in X$, any sequence $x_j \in X$ converging to x and any sequence of positive numbers t_j converging to 0, the sequence of convex hypersurfaces X_j obtained from X by the dilatation by the factor $1/t_j$ centered at the point x_j converges to the tangent hyperplane of X at x.

The stability of the mean curvature measure $\mathcal{K}$, vanishing of $\mathcal{K}$ on flat hyperplanes and the behavior of $\mathcal{K}$ under rescalings give

Lemma 5.1. *Let X be a convex hypersurface in $\mathbb{R}^{n+1}$. Let A be a compact set of smooth points in X and $\delta > 0$. Then there exists some $t > 0$ such that*

$$\mathcal{K}(B(y,r)) \leq \delta r^{n-1}$$

for any $y \in B(A,t)$ and any $0 < r < t$.

Thus, the following lemma applies to all small balls in a neighborhood of any smooth point.

Lemma 5.2. *There exist numbers $\delta_0, C > 0$ depending only on n with the following property. Let X be a convex hypersurface in $\mathbb{R}^{n+1}$. Let $x \in X$ be a point and $r > 0$ be such that the mean curvature $\mathcal{K}$ satisfies $\mathcal{K}(B(x,6r)) < \delta r^{n-1}$ with $\delta < \delta_0$. Then*

$$\left|1 - \frac{b_r(x)}{\omega_n r^n}\right| < C\delta\mathcal{K}(B(x,6r))r^{1-n}. \tag{5.1}$$

Proof. By rescaling, it suffices to prove the existence of $\delta_0, C > 0$ such that the lemma holds for $r = 1$. By approximation, it is sufficient to prove the result for smooth convex hypersurfaces.

Fix a sufficiently small $\varepsilon_0 > 0$. The mean curvature vanishes on $B(x,6)$ if and only if $B(x,6)$ is contained in a flat hyperplane. Due to the stability of $\mathcal{K}$ under convergence, if δ_0 is small, then the ball $U = B(x,5) \subset X$ is close to a flat hyperplane in $\mathbb{R}^{n+1}$. Thus, we may assume that the tangent hyperplanes to points in U are ε_0-close to the tangent space $W = T_xX \subset \mathbb{R}^{n+1}$. Therefore, U is the graph $\{(x, f(x))\}$ of a convex function $f : V \to \mathbb{R}$ defined on an open subset $V \subset W$. Moreover, V contains the ball of radius 4 in W around x. Denote by $B(x,2)_W$ the ball of radius 2 in W around x. Set

$$a := \sup\{|\nabla f(y)|\,;\; y \in B(x,2)_W\}.$$

If δ_0 is small, then $a < \varepsilon_0$. The orthogonal projection $P: U \to V$ is 1-Lipschitz and the restriction of the inverse P^{-1} to $B(x, 2)_W$ has Lipschitz constant

$$\sqrt{1+a^2} \leq 1 + a^2 \leq 1 + \varepsilon_0^2.$$

Applying (2.1) we only need to prove that $a < C\delta$ for a constant C.

Denote by $|D^2 f|$ the largest eigenvalue of the Hessian $D^2 f$. Since f is convex and ε_0 is small, the mean curvature $\kappa(x)$ at the point $(x, f(x))$ of the graph U of f satisfies $\kappa(x) \geq \frac{n}{2}|D^2 f|$. Hence, the conclusion follows from the following statement.

Claim. *Let $f : B \to \mathbb{R}$ be a smooth convex function on the open ball $B = B(0, 4) \subset \mathbb{R}^n$. If $f(0) = |\nabla f(0)| = 0$ then, for some $C = C(n) > 0$,*

$$\sup_{y \in B(0,2)} |\nabla f(y)| \leq C \int_B |D^2 f|.$$

By convexity, it is sufficient to find some $C = C(n) > 0$ with

$$\sup_{y \in B(0,3)} |f(y)| \leq C \int_B |D^2 f| \tag{5.2}$$

(see also [EG15, Theorem 6.7]).

First note that $f(z) \geq 0$ for all z since $f(0) = |\nabla f(0)| = 0$ and f is convex.

In order to verify (5.2), we can multiply f by a constant and assume that f takes its maximum on the closed ball $\bar{B}(0, 3)$ at the point y_0 and $f(y_0) = 1$. Convexity of f implies that $|y_0| = 3$. Since $f(0) = 0$ and f is convex, we must have $f(y) \leq 1/3$ for all $y \in B(0, 1)$.

By convexity and the choice of y_0, the restriction of f to the supporting hyperplane H of $\bar{B}(0, 3)$ at y_0 is bounded from below by 1. Consider the ball S of radius $1/2$ in H around y_0. For any point $z \in S$ consider the restriction

$$f_z(t) = f\big(z - \tfrac{t}{3} y_0\big), \quad t \in [0, 6],$$

to the segment of length 6 starting at z orthogonal to H. Then

$$f_z(0) \geq 1, \quad f_z(3) \leq 1/3, \quad f_z(6) \geq 0.$$

Thus for some $t \in (0, 3)$ we have $f_z'(t) \leq -2/9$ and for some $t \in (3, 6)$ we have $f_z'(t) \geq -1/9$. Therefore

$$\int_0^6 f_z''(t)\, dt \geq \frac{1}{9}.$$

Integrating over S we obtain by Fubini's theorem a uniform positive lower bound on $\int_B |D^2 f|$. This finishes the proof of (5.2). Hence the Claim and the lemma follow. □

5.2. The proof

The next theorem is the first part of Theorem 1.8; the second part follows from Theorem 4.2. In combination with Theorem 1.6, it also finishes the proof of Theorem 1.1.

Theorem 5.3. *Let X be a convex hypersurface in $\mathbb{R}^{n+1}$. Then it has vanishing mm-boundary.*

Proof. Since X has locally finite mm-boundary by Theorem 1.7, it suffices to prove that any partial limit measure ν of a sequence $\frac{1}{r_j}\mathcal{V}_{r_j}$ for $r_j \to 0$ must be the zero measure.

Fix a partial limit measure ν. By Theorem 1.7, $\nu(A) = 0$ for any Borel subset $A \subset X$ with $\mathcal{H}^{n-1}(A) < \infty$. Let $Y \subset X$ be the set of smooth points of X. The complement $X \setminus Y$ is a countable union of subsets with finite $(n-1)$-dimensional Hausdorff measure (see [Zaj79] and [Sch93, Theorem 1.4]), so $\nu(X \setminus Y) = 0$. Therefore, it is sufficient to prove $\nu(A) = 0$ for any compact subset $A \subset Y$.

Fix a compact subset $A \subset Y$ and let $\delta > 0$ be an arbitrary sufficiently small number. Consider $1 > t > 0$ provided by Lemma 5.1. Let U be the open set $B(A, t)$.

Assume $0 < r < t$. Applying Lemma 5.2, for $x \in U$ we get

$$\begin{aligned}|\mathcal{V}_r|(U) &\leq \int_U C\delta r^{1-n}\mathcal{K}(B(y,6r))\,d\mathcal{H}^n(y)\\ &\leq C\delta r^{1-n}\int_{B(A,7t)}\mathcal{H}^n(B(y,6r))\,d\mathcal{K}(y)\\ &\leq C\delta r^{1-n}(6r)^n\mathcal{K}(B(A,7t));\end{aligned}$$

we have used Lemma 2.1 in the second inequality and the Bishop–Gromov inequality in the last inequality. Hence

$$|\nu|(A) \leq |\nu|(U) \leq C\delta 6^n\mathcal{K}(B(A,7)).$$

Since δ can be chosen arbitrarily small, we obtain $|\nu|(A) = 0$.

This finishes the proof of the claim, and therefore of Theorem 5.3. □

6. An integral inequality for Riemannian metrics

6.1. The smooth case

We start by estimating from above the deviation measure $\mathcal{V}_r$ on a smooth Riemannian manifold in terms of the first derivatives of the metric. We do not know how to prove a similar estimate from below (see Problem 8.3). However, for the applications to Alexandrov spaces discussed in the next section, the estimate from below is a consequence of the theorem of Bishop–Gromov.

For a smooth Riemannian metric g defined on an open subset $U \subset \mathbb{R}^n$ we denote by $|g'|\colon U \to [0,\infty)$ the sum $\sum_{i,j,k}\left|\frac{\partial}{\partial x_k}g_{ij}\right|$.

Proposition 6.1. *There exists a constant $C = C(n) > 1$ with the following property. Let $U \subset \mathbb{R}^n$ be an open subset with a smooth Riemannian metric g which is $(1 + 1/C)$-bi-Lipschitz to the background Euclidean metric. Let $A \subset U$ be a Borel subset. Let $r > 0$ be such that $B(A, 2r)$ is relatively compact in U. Then*

$$\mathcal{V}_r(A) \leq Cr \int_{B(A,2r)} |g'|.$$

Proof. We will denote by C_i various (explicit) constants which depend only on n.

We will use the following notations. We denote by $|\cdot|$ and $\mathcal{L}^n$ respectively the norm and the Lebesgue measure on $\mathbb{R}^n$. For $x \in U$ we denote by g_x the Riemannian tensor at x and by $|\cdot|_x$ the corresponding norm. The Hausdorff measure of the Riemannian metric g has the form $u\mathcal{L}^n$ with $u = \sqrt{\det(g_{ij})}$.

For $x \in \mathbb{R}^n$, we consider the function $K : U \to [0, \infty)$ given by

$$K(x) = \sup_{|v|_x=1} \frac{d}{dt}\bigg|_{t=0} |v|_{x+tv}.$$

By smoothness of the determinant and the square root, we find a constant C_1 such that for all $x \in U$ we have

$$|u'(x)| \leq C_1|g'(x)| \quad \text{and} \quad K(x) \leq C_1|g'(x)|. \tag{6.1}$$

We fix $A \subset X$ and $r > 0$ as in the formulation of the proposition. For $x \in U$ denote by B_x the metric ball $B(x, r)$ in U, and B^x the metric ball of radius r in the Euclidean norm $|\cdot|_x$. In this Euclidean metric the ball B^x has measure

$$\omega_n r^n = u(x) \int_{B^x} d\mathcal{L}^n.$$

Thus, in order to estimate the deviation measure $\mathcal{V}_r$, we only need to control the summands on the right of the following inequality:

$$\omega_n r^n - b_r(x) \leq u(x)\mathcal{L}^n(B^x \setminus B_x) + \int_{B_x} |u(x) - u(y)|\, d\mathcal{L}^n(y). \tag{6.2}$$

We may assume that the bi-Lipschitz constant $1 + 1/C$ is close to 1, so that $1/2 < u < 2$. Moreover, we may assume B_x and B^x are contained in the ball of radius $\frac{4}{3}r$ around x with respect to the Euclidean metric.

In order to bound the first summand, for $x \in A$ and $|v|_x = 1$, we set l_v^x to be the length of the segment $[x, x + v]$ in the Riemannian metric g. Then we compute

$$\begin{aligned} l_v^x - r &= \int_0^r |v|_{x+tv}\, dt - \int_0^r |v|_x\, dt \leq \int_0^r \left(\int_0^t K(x + sv)\, ds \right) dt \\ &\leq \int_0^r \left(\int_0^r K(x + sv)\, ds \right) dt = r \int_0^r K(x + sv)\, ds. \end{aligned}$$

Observe now that the intersection of $B^x \setminus B_x$ with the ray starting at x in the direction of v has $\mathcal{H}^1$-measure (with respect to the norm $|\cdot|_x$) at most $2(l_v^x - r)$, once the bi-Lipschitz constant $1 + 1/C$ is close to 1. Integrating in polar coordinates over the ball $B^x \subset (\mathbb{R}^n, |\cdot|_x)$ we infer that

$$\begin{aligned} u(x)\mathcal{L}^n(B^x \setminus B_x) &\leq r^{n-1} \int_{v \in S_x^{n-1}} \left(2r \int_0^r K(x+sv)\, ds\right) d\mathcal{H}^{n-1}(v) \\ &= 2r^n \int_{B^x} K(y)|y-x|^{n-1} u(x)\, d\mathcal{L}^n(y), \end{aligned}$$

where S_x^{n-1} is the unit sphere in $(\mathbb{R}^n, |\cdot|_x)$.

To get a similar estimate of the other summand in (6.2), we only need to recall the following inequality from [EG15, Lemma 4.1], valid for any $\mathcal{C}^1$ function u on a Euclidean ball:

$$\int_{|x-y|<r} |u(y) - u(x)|\, d\mathcal{L}^n(y) \leq C_2 r^n \int_{|x-y|<r} |u'(y)| \cdot |y-x|^{1-n}\, d\mathcal{L}^n(y).$$

Taking both estimates together with (6.1), embedding B_x and B^x in slightly larger Euclidean balls and using $1/2 < u < 2$, we conclude that

$$\omega_n r^n - b_r(x) \leq C_3 r^n \int_{|x-y|<\frac{4}{3}r} |g'(y)| \cdot |y-x|^{1-n}\, d\mathcal{L}^n(y).$$

We divide both sides by $\omega_n r^n$ and integrate over A. Since the bi-Lipschitz constant is close to 1, we see that

$$\begin{aligned} \mathcal{V}_r(A) &\leq C_4 \int_A \left(\int_{|x-y|<\frac{4}{3}r} |g'(y)| \cdot |y-x|^{1-n}\, d\mathcal{L}^n(y) \right) d\mathcal{L}^n(x) \\ &\leq C_4 \int_{B(A,2r)} \left(\int_{|x-y|<\frac{4}{3}r} |g'(y)| \cdot |y-x|^{1-n}\, d\mathcal{L}^n(x) \right) d\mathcal{L}^n(y) \\ &= \frac{4}{3} C_4 \int_{B(A,2r)} |g'(y)| r\, d\mathcal{L}^n(y), \end{aligned}$$

where we have used Lemma 2.1 in the second inequality. This finishes the proof of Proposition 6.1. □

6.2. *Functions of bounded variation*

Let U be an open subset of $\mathbb{R}^n$. A function $f \in L^1(U)$ is of class BV (bounded variation) if its first partial derivatives, $\frac{\partial f}{\partial x^i}$ (here and below always in the sense of distributions) are signed Radon measures with finite mass $\left|\frac{\partial f}{\partial x^i}\right|(U)$. We denote by $[Df]$ the Radon measure $\sum_i \left|\frac{\partial f}{\partial x^i}\right|$ on U. If $f: U \to \mathbb{R}$ is a BV function which is continuous on a subset $R \subset U$ with $\mathcal{H}^{n-1}(U \setminus R) = 0$ then the Radon measure $[Df]$ vanishes on all Borel subsets $A \subset U$ with $\mathcal{H}^{n-1}(A) < \infty$ [GL80].

Let $f: U \to \mathbb{R}$ be of class BV. Then for $\mathcal{H}^n$-almost every point $x \in U$ there exists an affine function $\hat{f}_x: \mathbb{R}^n \to \mathbb{R}$ such that for the BV function $h_x = f - \hat{f}_x$ we have

$$\lim_{r \to 0} \frac{1}{r^{n+1}} \int_{B(x,r)} |h_x| = 0 \quad \text{and} \quad \lim_{r \to 0} \frac{1}{r^n} [Dh_x](B(x,r)) = 0; \tag{6.3}$$

see [EG15, Theorem 6.1(2), (3)] for the second equality, and use the Hölder inequality and [EG15, Theorem 6.1(1)] for the first equality.

6.3. Almost Riemannian metric spaces

The following definition provides a suitable description of a large part of any Alexandrov space (see Section 7).

Let $C = C(n)$ be the constant determined in Proposition 6.1. We will call a locally geodesic metric space X an *almost Riemannian metric space* if it has the following properties (see [AB15] for a careful discussion of such DC_0-Riemannian manifolds in the language of [AB15] and [Per95]):

(1) There is a Borel subset $R \subset X$, called the subset of *regular points*, satisfying $\mathcal{H}^{n-1}(X \setminus R) = 0$ and with the following properties.
(2) Any minimizing geodesic γ in X can be approximated by curves γ_i in R such that the lengths of γ_i converge to the length of γ.
(3) For any $x \in X$, there is a neighborhood U of x, called a *regular chart*, and a bi-Lipschitz map $\phi: U \to O$ onto an open subset $O \subset \mathbb{R}^n$, with bi-Lipschitz constant less than $1 + 1/C$ and with the following properties.
(4) There is a continuous Riemannian tensor g_{ij} on $\phi(U \cap R)$ such that g_{ij} is a function of bounded variation on O for each $1 \le i, j \le n$.
(5) The length of any curve $\gamma \subset R$ can be computed as the length of $\phi(\gamma)$ via this Riemannian tensor g.

For any regular chart U as above, we set $\mathcal{N}_0(U)$ to be the Radon measure $[g']$ on U given as the sum of the Radon measures $[Dg_{ij}]$ over the coordinates g_{ij} of the metric tensor g. For an almost Riemannian metric space X, we define an outer measure $\mathcal{N}$ on X in the following way. For a subset $A \subset X$, we consider all coverings $A \subset \bigcup U_i$ by countably many regular charts U_i and let $\mathcal{N}(A)$ be the infimum of the sums $\sum_i \mathcal{N}_0(U_i)$ over all such coverings. This is indeed an outer measure, which takes finite values on compact subsets. Since $\mathcal{N}$ satisfies the Carathéodory criterion [EG15, Theorem 1.9], it is indeed a Radon measure. We will call $\mathcal{N}$ the *minimal metric derivative measure* on the almost Riemannian metric space X.

Lemma 6.2. *Let X^n be an almost Riemannian metric space and let $\mathcal{N}$ be its minimal metric derivative measure. Then $\mathcal{N}(A) = 0$ for any Borel subset $A \subset X$ with $\mathcal{H}^{n-1}(A) < \infty$. There exists a Borel subset $C \subset X$ of full $\mathcal{H}^n$-measure in X with $\mathcal{N}(C) = 0$, thus $\mathcal{N}$ is absolutely singular with respect to $\mathcal{H}^n$.*

Proof. Clearly, both claims are local. Hence we need to verify them only in a regular chart U, which we identify with its image $\phi(U) \subset \mathbb{R}^n$. The first statement follows directly from the continuity of the metric tensor g on $U \cap R$ and the result of [GL80] cited above.

In order to verify the second claim we only need to show the following statement (see also [EG15, Section 1.6]). For almost all $x \in U$ there is another regular chart $V \ni x$ such that the derivative measure $[h']$ of the Riemannian tensor h in the chart V has n-dimensional density 0 at x, thus

$$\lim_{r\to 0} \frac{1}{r^n}[h'](B(x,r)) = 0. \tag{6.4}$$

Here and below, the ball $B(x,r)$ over which we integrate can be equally considered with respect to the Euclidean metric or to the original metric on U, since they are bi-Lipschitz equivalent. In order to prove (6.4), we follow [Per95, Section 4.2] and consider the Riemannian tensor g of the original chart U. Applying (6.3) to the coordinates of g, we find for $\mathcal{H}^n$-almost all $x \in U$ a smooth symmetric 2-tensor $\hat{g} = \hat{g}_x$ on U such that for $u = g - \hat{g}$ we have

$$\lim_{r\to 0} \frac{1}{r^{n+1}} \int_{B(x,r)} \|u\| = 0 \quad \text{and} \quad \lim_{r\to 0} \frac{1}{r^n}[Du](B(x,r)) = 0. \tag{6.5}$$

The first statement implies that $\hat{g}$ is indeed a Riemannian metric in a neighborhood U_0 of x.

Fix such a point x, a neighborhood U_0 and $\hat{g}$. Consider a small neighborhood W of 0 in $\mathbb{R}^n$ and let $\xi\colon W \to U$ be the exponential map with respect to the metric $\hat{g}$. Then $\xi(0) = x$, $D\xi(0) = \mathrm{Id}$ and the pull-back Riemannian metric $\hat{h} = \xi^*(\hat{g})$ has zero derivative at 0. Since $D\xi$ is the identity, the bi-Lipschitz constant of the restriction $F = \xi^{-1} \circ \phi$ to a sufficiently small neighborhood V of x is still less than $1 + 1/C$. Hence, $F\colon V \to \mathbb{R}^n$ is a regular chart.

The Riemannian tensor h in this chart equals $\hat{h} + \xi^*(g - \hat{g})$. Now, $D\hat{h}(0) = 0$, thus (6.4) holds for $\hat{h}$ instead of h. For the other summand $\xi^*(u)$, the density estimate (6.4) follows from (6.5) and the fact that ξ is a $\mathcal{C}^2$-diffeomorphism if W is sufficiently small. This finishes the proof of Lemma 6.2. □

6.4. *The upper bound on the deviation measures*

Continuing to denote by $C = C(n)$ the constant from Proposition 6.1 we show

Corollary 6.3. *Let U be a regular chart of an almost Riemannian metric space X. Identifying U with its image $O = \phi(U)$, let g be the metric tensor and let $\mathcal{N}_0 = [Dg]$ be the derivative of the metric tensor. For any Borel subset $A \subset U$ and any r such that $B(A, 3r)$ is relatively compact in U we have*

$$\mathcal{V}_r(A) \leq 2C \cdot \mathcal{N}_0(B(A, 3r)).$$

Proof. Consider a relatively compact open subset $V \subset U$ which contains $B(A, 2r)$. Apply (coordinatewise) the standard mollifying construction to the Riemannian tensor g. For all small positive ε, we thus obtain smooth metrics g_ε on V with the following properties. The total derivatives $|g'_\varepsilon|$, considered as measures, satisfy $|g'_\varepsilon| \leq \mathcal{N}_0$ on V [Zie89, Theorem 5.3.1]. Since g is pointwise $1/C$-close to the Euclidean inner product, the same is true for g_ε. For all sufficiently small ε the $2r$-tubular neighborhood of A with respect to g_ε is contained in the $3r$-tubular neighborhood around A with respect to the original distance in X. Moreover, g_ε converges to g pointwise at all points of R [Zie89, Theorem 1.6.1].

Denote by d_ε the distance function induced by g_ε. From the last statement and the properties (2), (5) in the definition of an almost Riemannian metric space we deduce that

$$\lim_{\varepsilon \to 0} \sup \{|d_\varepsilon(x, y) - d(x, y)|\, ;\ x, y \in V,\ d(x, y) < r\} = 0.$$

Finally, the Hausdorff measures of the Riemannian metrics g_ε converge on V to the Hausdorff measure of V with respect to the original metric.

Now the result follows directly from Proposition 6.1 applied to the metrics g_ε, by letting ε go to 0. □

As a consequence of Corollary 6.3, the minimal metric derivative measure bounds from above the deviation measure $\mathcal{V}_r$ on any almost Riemannian metric space:

Lemma 6.4. *Let X be an almost Riemannian metric space with metric derivative measure $\mathcal{N}$. Then for any compact subset $A \subset X$, there exists some $r_0 > 0$ such that for all $r < r_0$ we have*

$$\mathcal{V}_r(A) \leq 2r(n+2)C\mathcal{N}(A).$$

Proof. Cover A by finitely many regular charts U_i such that $\sum \mathcal{N}_0(U_i)$ is sufficiently close to $\mathcal{N}(A)$. Since the covering dimension of X is n, we find a finite covering V_j of A which refines the first covering but has intersection multiplicity less than $n + 2$. Considering each V_j as a subchart of the corresponding chart U_i we see that

$$\sum \mathcal{N}_0(V_j) \leq (n+2) \sum \mathcal{N}(A).$$

Consider $r_0 > 0$ such that for any $x \in A$ the ball $B(x, 4r_0)$ is contained in one of the sets V_j. Denote by A_j the set of all such x. Then $\mathcal{V}_r(A) \leq \sum \mathcal{V}_r(A_j)$ and, due to Corollary 6.3, $\mathcal{V}_r(A_j) \leq 2C \cdot r \cdot N_0(V_j)$. Combining these inequalities finishes the proof. □

7. Alexandrov spaces

7.1. Strained points

Strainers and strainer maps are basic tools for Alexandrov spaces (see [BGP92], [OS94], [KMS01]) and will play an important role in the proof of Theorem 1.7.

Let us list the main properties of the subsets of strained points. We fix a natural number n. Then for all sufficiently large A and any $0 < r, \delta \leq 1/A^2$ the following properties hold true for all n-dimensional Alexandrov spaces X of curvature ≥ -1:

(1) The set $X_{r,\delta}$ of points in X which have an Ar-long (n,δ)-strainer is open in X. For $s < r$, we have $X_{r,\delta} \subset X_{s,\delta}$ [BGP92, 9.7].
(2) Assume a sequence (X_i^n, x_i) of Alexandrov spaces of curvature ≥ -1 converges to an n-dimensional Alexandrov space (X, x) in the pointed Gromov–Hausdorff topology. If $x \in X_{r,\delta}$ then, for all large i, the point x_i has an Ar-long (n,δ)-strainer in X_i.
(3) Rescaling X with a constant $\lambda \geq 1$ sends the subset $X_{r,\delta}$ to a subset $(\lambda X)_{\lambda r,\delta}$ of the rescaled Alexandrov space λX.
(4) The union $X_\delta := \bigcup_{r>0} X_{r,\delta}$ contains the set X_{reg} of all regular points of X. The Hausdorff dimension of the set $X \setminus (X_\delta \cup \partial X)$ is at most $n-2$ [BGP92, 10.6, 10.6.1, 12.8].
(5) For any point $x \in X_{r,\delta}$ there are natural distance coordinates $\phi\colon B(x,3r) \to \mathbb{R}^n$ which are $(1+\varepsilon)$-bi-Lipschitz onto an open subset $O \subset \mathbb{R}^n$. Here, $\varepsilon \to 0$ as $A \to \infty$ [BGP92, 9.4].
(6) The chart ϕ can be smoothed to satisfy the following property [OS94, Theorem B]. There exists a continuous Riemannian metric g on $\phi(X_{\text{reg}} \cap B(x,3r)) \subset O$ such that for any curve $\gamma \subset X_{\text{reg}} \cap B(x,3r)$ its length coincides with the length of $\phi(\gamma)$ with respect to the Riemannian metric g.
(7) The metric tensor g on a chart O defined above is of bounded variation on O [Per95, 4.2] (see also [AB15]).

The last three statements in the above list together with the density and convexity of the set X_{reg} of regular points imply the following.

Corollary 7.1. *In the above notations, the subset $X_\delta \subset X$ is an almost Riemannian metric space if A is sufficiently large.*

In fact, the arguments in [Per95, 4.2] provide a slightly more precise version of (7) in the above list:

Lemma 7.2. *In the notations above, the constant A can be chosen so large that the following holds true. The derivative measure $[g']$ of the Riemannian tensor g in the canonical distance chart O satisfies $[g'](\hat{O}) \leq Ar^{n-1}$, where $\hat{O}$ is the image $\phi(B(x,2r)) \subset O = \phi(B(x,3r))$.*

Proof. We only sketch the proof, referring to [Per95] for details. First we fix $r = 1/A^2$.

The fact that g has bounded variation in the chart O follows in [Per95, Section 4.2] by writing the coordinates of g as a universal smooth map $\Phi(f_1, \dots, f_\alpha)$ of a finite number of distance functions f_j on X and their partial derivatives, both expressed in the chart ϕ. It is shown in [Per95, Section 3] that any such distance function f_j is expressed in the chart O as a difference of two L-Lipschitz and λ-concave functions, where L, λ depend only on the semiconcavity of the corresponding distance functions in X. Since we have fixed $r > 0$, the numbers λ, L can be chosen independently of X. Thus, f_j can be written in the chart O as the difference of two convex functions with universal Lipschitz constants L'. Therefore, for any unit vector $v \in \mathbb{R}^n$, we have a uniform bound on the total mass of the Radon measure $\left[\frac{\partial^2 f_j}{\partial^2 v}\right](\hat{O})$. This implies that all partial second derivatives of f have uniformly bounded mass on $\hat{O}$ (see also [EG15, Theorem 6.8]).

From this we deduce a uniform bound A' on the total mass $[g'](\hat{O})$ for the fixed value of $r_0 = 1/A^2$.

For any $r < r_0$ we rescale the space by r_0/r. The total mass of the Riemannian tensor g is then rescaled by $(r_0/r)^{n-1}$. Thus,

$$[g'](\hat{O}) \leq A' r_0^{1-n} r^{n-1}.$$

We finish the proof by replacing A by $\max(A, A' r_0^{1-n})$. □

Now we use Corollary 6.3 to deduce

Proposition 7.3. *Let $C = C(n)$ and $A = A(n)$ be the constants from Proposition 6.1 and Lemma 7.2. For any point $x \in X_{r,\delta}$, any $s < r$ and any Borel subset $K \subset B(x, r)$ the deviation measure $\mathcal{V}_s$ satisfies $\mathcal{V}_s(K) \leq 2CAr^{n-1}$.*

7.2. Decomposition in good balls

Let the constant A be as above. A ball $B(x, r)$ in X^n will be called *good* if $x \in X_{r,\delta}$, and *bad* otherwise. In this subsection we give a controlled covering result (see also Problem 8.10 and its resolution in [LiN19] strengthening the next result).

Proposition 7.4. *Let X^n be an n-dimensional Alexandrov space without boundary. For every compact $W \subset X$ and every $\alpha > n - 2$ there exists $q = q(W, \alpha) > 0$ with the following property. For every $x \in W$ and every $s < 1$ there exists a countable collection of good balls $B_m = B(x_m, r_m) \subset X$ such that*

(1) *$r_m < s$ for all m;*
(2) *$\mathcal{H}^n(B(x, s) \setminus \bigcup_m B_m) = 0$;*
(3) *$\sum_m r_m^\alpha < q s^\alpha$.*

The proof will be obtained by a recursive application of the following lemma.

Lemma 7.5. *There is an integer $N = N(W, \alpha)$ with the following property. For any $p \in W$ and $\rho < 1$ the ball $B(p, \rho)$ can be covered by at most N balls $B_i = B(x_i, r_i)$ such that $r_i < \rho$ for all i, and*

$$\sum_{i \in \mathrm{BAD}} r_i^\alpha < \tfrac{1}{2}\rho^\alpha,$$

where $i \in$ BAD means that B_i is a bad ball.

Proof. Assume the contrary. Thus we can find a sequence of balls $K_l = B(p_l, \rho_l)$ such that $p_l \in W$, $\rho_l < 1$ and one needs at least l balls to cover K_l so that the conditions in the lemma are fulfilled.

Taking a subsequence we may assume that the following limit exists in the pointed Gromov–Hausdorff metric:

$$\left(\frac{1}{\rho_m} X, p_m\right) \xrightarrow{\text{GH}} (Y, p).$$

Since the points p_m range over a compact subset of X and $\rho_m < 1$, the sequence is non-collapsing, i.e. Y is an n-dimensional Alexandrov space. By Perelman's stability theorem, ∂Y is empty. Therefore, $S := (Y \setminus Y_\delta) \cap \bar{B}(p, 2)$ is a compact set of Hausdorff dimension $\leq n - 2$.

By the definition of Hausdroff dimension, we can cover S by a finite number of balls $B_i = B(x_i, r_i)$ such that

$$\sum_i r_i^\alpha < (1/2)^\alpha.$$

Any point in the remaining compact set $K \setminus \bigcup_i B_i$ is contained in Y_δ. Therefore a small ball centered at any point of this set is good. By compactness, we can cover $K \setminus \bigcup_i B_i$ by a finite number of good balls. Let N be the total number of balls in the resulting covering of K.

Lifting the constructed covering to K_l, for all large l, we cover the ball K_l by at most N balls satisfying the conditions of the lemma. This contradiction to our assumption finishes the proof of the lemma. □

Proof of Proposition 7.4. Cover $B(x, s)$ by N balls as in Lemma 7.5 and call this covering $\mathcal{F}_1$. Now cover every bad ball from $\mathcal{F}_1$ by at most N balls provided by Lemma 7.5. Together with the good balls from $\mathcal{F}_1$ the new balls define a covering $\mathcal{F}_2$ of $B(x, s)$. Continuing in this way, define for each natural number k a covering $\mathcal{F}_k$ of $B(x, s)$.

Denote by g_l^+ and g_l^- the sum of r_i^α over good, respectively bad, balls $B(x_i, r_i)$ in the covering $\mathcal{F}_l$. Then, by construction, $g_{l+1}^- < \frac{1}{2} g_l^-$ and $g_{l+1}^+ \leq g_l^+ + N g_l^-$. Therefore, $g_l^- \leq 2^{-l} g_1^-$ and g_l^+ is uniformly bounded from above. The volume of the union of bad balls in $\mathcal{F}_l$ is at most g_l^- and converges to 0 as $l \to \infty$.

Let $\mathcal{F}$ be the set of all good balls $B_j = B(x_j, r_j)$ from all the coverings $\mathcal{F}_l$. Then $\mathcal{H}^n(B(x, s) \setminus \bigcup_{\mathcal{F}} B_j \leq \lim_{l \to \infty} g_l^- = 0$. On the other hand, by construction,

$$\sum_{B_j \in \mathcal{F}} r_j^\alpha = \lim_{l \to \infty} g_l^+ \leq 3 N s^\alpha.$$

Setting $q = 3N$ finishes the proof. □

7.3. *Final step*

Now we can provide

Proof of Theorem 1.7. Let X be a fixed n-dimensional Alexandrov space. By the inequality of Bishop–Gromov, the deviation measures $\mathcal{V}_r$ are uniformly bounded from below by a quadratic term in r. Thus in order to control the mm-boundary we only need to bound $\mathcal{V}_r$ from above on balls in X.

Let the constants A, C be as above, so that Proposition 7.3 can be applied.

First assume that ∂X is empty. Let $W \subset X$ be an arbitrary compact subset. Fix $\alpha = n - 3/2$ and choose the constant q as in Proposition 7.4. For any $x \in W$ and $s < 1$ consider the good balls $B_i = B(x_i, r_i)$ provided by Proposition 7.4 and set $K' = \bigcup_i B_i$. Let $r < 1/A^2$ be sufficiently small. Since $\mathcal{H}^n(K \setminus K') = 0$, we have $\mathcal{V}_r(K) = \mathcal{V}_r(K')$.

For all m with $r < r_m$, we apply Proposition 7.3 to infer that

$$\mathcal{V}_r(B_m \cap K) \leq 2CArr_m^{n-1}.$$

On the other hand, for $r_m < r$, we have

$$\mathcal{V}_r(B_m \cap K) \leq \mathcal{H}^n(B_m) \leq 2\omega_n r_m^n < 2\omega_n r r_m^{n-1}.$$

Summing and using $r_m^{n-1} < r_m^\alpha$ we obtain

$$\mathcal{V}_r(K) \leq \sum_m \mathcal{V}_r(B_m \cap K) \leq (2CA + 2\omega_n) q r s^\alpha. \tag{7.1}$$

This proves that X has locally finite mm-boundary. As already mentioned and used above, any signed Radon measure ν obtained as a limit of a sequence $\mathcal{V}_{r_j}/r_j$ for some $r_j \to 0$ must be non-negative, hence a Radon measure. We fix such a ν.

Inequality (7.1) implies that ν has finite α-dimensional density at every point of X, in particular, ν vanishes on subsets of Hausdorff dimension $\leq n-2$. Thus $\nu(X \setminus X_\delta) = 0$.

Recall that X_δ is an almost Riemannian space. Denote by $\mathcal{N}$ its minimal metric derivative measure. We extend it to a measure on all of X (still denoted by $\mathcal{N}$) by setting it to be 0 on $X \setminus X_\delta$. By Lemma 6.4 the Radon measure ν is absolutely continuous with respect to $\mathcal{N}$ on compact subsets of X_δ. Now (1) and (3) of Theorem 1.7 follow from Lemma 6.2. This finishes the proof in the case $\partial X = \emptyset$.

Assume now that $\partial X \neq \emptyset$ and consider the doubling $Y = X \sqcup_{\partial X} X$ of X. Consider X as a convex subset of Y and let $K \subset X$ be compact. We find a constant $L > 0$ such that for all sufficiently small $r > 0$, we have $\mathcal{H}^n(K \cap B(\partial X, 2r)) \leq Lr$. (This follows, for example, by the coarea formula using the Lipschitz properties of the gradient flow of the distance function $d(\cdot, \partial X)$ which is semiconcave.) On the other hand, for $x \in K \setminus B(\partial X, r)$, the volumes of the r-ball in X and in Y coincide. Since Y has locally finite mm-boundary, we deduce that $\mathcal{V}_r(K)$ (computed in the space X) is bounded from above by $Lr + \tilde{\mathcal{V}}_r(K)$, where $\tilde{\mathcal{V}}_r(K)$ is the deviation measure of K considered as a subset of Y. This implies that $\mathcal{V}_r/r$ is uniformly bounded for $r \to 0$. Thus X has locally finite mm-boundary as well.

Any limit of a sequence $\mathcal{V}_{r_j}/r_j$ for some $r_j \to 0$ must again be non-negative, hence a Radon measure. Outside of ∂X, ν coincides with the restriction of the corresponding measure defined on Y. From the corresponding statement about Y we deduce that ν is singular with respect to $\mathcal{H}^n$. Moreover, ν vanishes on all subsets $S \subset X \setminus \partial X$ with $\mathcal{H}^{n-1}(S)$ finite.

It remains to prove (2), i.e. to show that the restriction of ν to ∂X is at least $c\mathcal{H}^{n-1}$ for a universal constant $c = c(n)$. This statement is local on ∂X and needs to be verified only in small neighborhoods of points x whose tangent $T_x X$ is isometric to a flat half-space.

We fix such a point $x \in \partial X$. We further fix a sufficiently small $\varepsilon > 0$ and find a small neighborhood U of x in X which is $(1+\varepsilon)$-bi-Lipschitz to a half-ball in the Euclidean space. Choose an arbitrary $s > 0$ such that $B(x, 2s) \subset U$. Let $K = \bar{B}(x, s) \cap \partial X$ be the closed ball of radius s in ∂X with respect to the ambient metric. Due to [EG15, Section 1.6], it is sufficient to prove that $\nu(K) \geq c_0 s^{n-1}$ for a universal constant c_0 depending only on the dimension.

In order to prove this inequality, we consider any open neighborhood V of K in X. For all small $r > 0$, the neighborhood V contains $B(K, 2r)$. Once ε has been chosen sufficiently small, for any point $z \in B\big(K, \frac{1}{10}r\big)$ the ball $B(z,r)$ in X has volume at most $(1-k_1)\omega_n r^n$. Here $k_1 = k_1(n) > 0$ is a universal constant. Moreover, the set $B\big(K, \frac{1}{10}r\big)$ has volume at least $\frac{1}{20} r\omega_{n-1} s^{n-1}$. Integrating over V (and using the inequality of Bishop–Gromov on the complement of $B\big(K, \frac{1}{10}r\big)$) we deduce that

$$\mathcal{V}_r(V) \geq k_1 \tfrac{1}{20}\omega_{n-1} r s^{n-1} - k_3 r^2$$

for some k_3 depending only on the volume of V and independent of r. Dividing by r and letting it go to 0 we obtain $\nu(V) \geq k_4 s^{n-1}$ for a universal constant $k_4 > 0$. Since the neighborhood V of K was arbitrary, we infer the same inequality for K instead of V, finishing the proof. □

8. Questions and comments

8.1. Manifolds

The notions of mm-boundary and mm-curvature are very easy to define but difficult to control. For instance, the examples mentioned in the introduction require a fair amount of computations and estimates. On the other hand, interesting examples seem to be difficult to construct as well. The first question in this direction is:

Problem 8.1. *Construct a closed manifold with a continuous Riemannian metric that does not have finite mm-boundary.*

The following problem is motivated by our approach to Theorem 1.7 in Sections 6 and 7.

Problem 8.2. *Let X be an almost Riemannian space. Can the minimal metric derivative measure be non-zero?*

In the language of DC-calculus discussed in [AB15], this question can be reformulated as follows. Given a compact subset K on any DC_0-Riemannian manifold and any $\varepsilon > 0$, can one cover K by charts such that the total mass of the derivative of the metric tensor in these coordinates is bounded by ε? Note that the minimal metric derivative measure must vanish if the metric can be locally defined by a Riemannian tensor of class $W^{1,1}$, since the metric derivative measure is singular with respect to the Hausdorff measure by Lemma 6.2.

The following question is motivated by Lemma 6.4 and potential applications to geodesic flows of spaces with curvature bounded from above (see also Problem 8.12).

Problem 8.3. *Let X be an almost Riemannian space. Can one use the minimal metric derivative measure in order to control the deviation measures $\mathcal{V}_r$ from below?*

8.2. Surfaces and hypersurfaces

The answer to the following question is non-trivial in view of Example 1.14.

Problem 8.4. *Can one express the mm-curvature of an Alexandrov surface in terms of its curvature measure?*

In view of Theorem 1.10 it is reasonable to expect an affirmative answer to the following question.

Problem 8.5. *Do convex hypersurfaces of* $\mathbb{R}^n$ *have locally finite mm-curvature?*

A natural approach to this question is related to the following conjectural generalization of the Bonk–Lang theorem [BL03]:

Problem 8.6. *Let X be a convex hypersurface sufficiently close to a flat hyperplane. Can we bound the optimal bi-Lipschitz constant for maps into the Euclidean space in terms of the total scalar curvature?*

Some natural generalizations of our Theorem 5.3 are possible. Probably, slightly refined arguments can be used to prove that any DC-submanifold of a Euclidean space has vanishing mm-boundary. Using the embedding theorem of Nash, this would also provide an easy generalization of Theorems 5.3 and 1.1 to convex hypersurfaces of smooth Riemannian manifolds.

8.3. Alexandrov geometry and beyond

As the next generalization of Theorem 1.1, one should study the case of smoothable Alexandrov spaces.

Problem 8.7. *Does the mm-boundary vanish in smoothable Alexandrov spaces? Are there relations to scalar curvature measures defined in* [LP17]*?*

Due to the observation after Problem 8.2, the vanishing of mm-boundary would follow from the existence of slightly smoother coordinates than the ones provided by Perelman's DC-structure.

Problem 8.8. *Let X be an Alexandrov space. Can one introduce coordinates on a neighborhood of the set of regular points such that the metric is locally given by a Riemannian tensor of class* $W^{1,1}$*?*

In the two-dimensional case, the answer to this question is "yes" by the work of Reshetnyak [Res93] (see also [AB16]).

Due to Theorem 1.6, an affirmative answer to the following question should be expected. A partial answer has been announced by Jérôme Bertrand.

Problem 8.9. *Are there further connections between the size of the mm-boundary of an Alexandrov space X, the existence of the geodesic flow and the "average size" of the cut loci of points in X?*

Should one have a chance to go beyond mm-boundary and towards mm-curvature, one would definitely need to improve the decomposition statement of Proposition 7.4, which provides geometric control of the size of the set of singular points of an Alexandrov space.

Problem 8.10. *Can one replace $\alpha > n-2$ by $\alpha = n-2$ in the statement of Proposition 7.4?*

In the meantime, an affirmative answer has been provided in [LiN19].

It is interesting to understand if our results provide a quantitative version of bi-Lipschitz closeness of small balls to Euclidean balls. It is known [BGP92] that there exist $\kappa(n,\delta) \to 0$ as $\delta \to 0$ such that if $X = X^n$ is an Alexandrov space of curvature ≥ -1 and $x \in X$ with $\omega_n r^n - \mathcal{H}^n(B(x,r)) \leq \delta r^n$ then $B(x, r/4)$ is $(1+\kappa(n,\delta))$-bi-Lipschitz to a Euclidean ball.

Problem 8.11. *Can $\kappa(n,\delta)$ above be chosen of the form $C(n)\delta$?*

It is also natural to look at what happens for spaces with curvature bounded above:

Problem 8.12. *Can one obtain similar estimates and applications for the mm-boundary in geodesically complete spaces with upper curvature bounds? Note that such spaces share many properties with Alexandrov spaces* [LN19].

Finally, it seems reasonable to expect some generalizations to spaces with Ricci curvature bounds, for instance:

Problem 8.13. *Can one control the mm-boundary of non-collapsed limits of Riemannian manifolds with Ricci curvature bounded below? Can one expect something like a geodesic flow in this setting?*

From the work of Jeff Cheeger and Aaron Naber [CN15] it should follow that on any non-collapsed limit of manifolds with both-sided Ricci curvature bounds, the mm-curvature is locally finite and the mm-boundary is zero. Vanishing of the mm-boundary should then imply that the geodesic flow is defined almost everywhere and preserves the Liouville measure by the same argument as in the proof of Theorem 1.6.

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