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Martin Möller · Martin Ulirsch · Annette Werner

Realizability of tropical canonical divisors

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Abstract. We use recent results by Bainbridge–Chen–Gendron–Grushevsky–Möller on compactifications of strata of abelian differentials to give a comprehensive solution to the realizability problem for effective tropical canonical divisors in equicharacteristic zero. Given a pair (Γ, D) consisting of a stable tropical curve Γ and a divisor D in the canonical linear system on Γ, we give a purely combinatorial condition to decide whether there is a smooth curve X over a non-Archimedean field whose stable reduction has Γ as its dual tropical curve together with an effective canonical divisor K_X that specializes to D.

Keywords. Tropical geometry, moduli spaces, Hodge bundle, abelian differentials, canonical divisors, flat surfaces, Berkovich spaces

Contents

Introduction

The *realizability problem* in tropical geometry is a metaproblem that underlies many of the successful applications of tropical geometry to other areas of mathematics. It asks whether for a given synthetically defined tropical object, there exists an analogous algebraic geometric object whose tropicalization is precisely the given tropical object.

The realizability problem for divisors is, in general, notoriously difficult: see e.g. [BJ16, Section 10]. In this article we solve it for effective canonical divisors using recent results on the compactification of strata of abelian differentials in [BC$^+$18b].

M. Möller, M. Ulirsch, A. Werner: Institut für Mathematik, Goethe-Universität Frankfurt, Robert-Mayer-Str. 6-8, 60325 Frankfurt am Main, Germany;
e-mail: moeller@math.uni-frankfurt.de, ulirsch@math.uni-frankfurt.de, werner@math.uni-frankfurt.de

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Realizability of tropical canonical divisors

Let (Γ, D) be a tuple consisting of an (abstract) stable tropical curve Γ and a divisor D in the canonical linear system $|K_\Gamma|$ on Γ. Does there exist a smooth curve X together with a stable degeneration $\mathcal{X}$ as well as an effective canonical divisor K_X on X such that the following two conditions hold:

- the tropical curve given by the metrized weighted dual graph of the irreducible components in the special fiber of $\mathcal{X}$ is Γ; and
- the specialization of K_X, i.e. the multidegree of the special fiber of the closure of K_X in a suitably chosen semistable model of $\mathcal{X}$, is equal to D?

If that is the case, we say the pair (Γ, D) is *realizable*.

Our main theorem gives an exhaustive answer to this question over an algebraically closed field of characteristic 0. To state it, recall that an element in the tropical canonical series differs from the distinguished element K_Γ by the divisor of a piecewise affine function f on Γ with integral slopes. We declare the support of $\mathrm{div}(f)$ to be vertices of Γ and add to the graph Γ legs at the support of $\mathrm{div}(f)$, according to the local multiplicity. Now we simply use the value of such a function f to define an order among the vertices of Γ (making it into a *level graph*). Finally, we provide each half-edge of Γ with an enhancement consisting of the (outgoing) slope of f. The resulting object is called the *enhanced level graph* $\Gamma^+(f)$ associated with f.

Section 5 explains the algebro-geometric origin of this notion. The correspondence between rational functions and decorations on Γ is explained in Section 6. In particular, we introduce the notion of an *inconvenient vertex* v. A vertex $v \in \Gamma^+(f)$ of genus 0 is inconvenient if it has, roughly speaking, an edge with a 'large' positive decoration. For example, trivalent vertices where two edges have decoration less than -1 are always inconvenient.

Theorem 1. *Given a tropical curve* Γ *and an element* $D = K_\Gamma + \mathrm{div}(f)$ *in the tropical canonical linear series in* Γ*, the pair* (Γ, D) *is realizable if and only if the following two conditions hold:*

(i) *For every inconvenient vertex* v *of* $\Gamma^+(f)$ *there is a simple cycle* $\gamma \subset \Gamma$ *based at* v *that does not pass through any node on a level lower than* $f(v)$*.*
(ii) *For every horizontal edge* e *there is a simple cycle* $\gamma \subset \Gamma$ *passing though* e *which does not pass through any node on a level lower than* $f(e)$*.*

This theorem implies in particular that the canonical divisor K_Γ on Γ is in general not realizable (see Example 6.4 below). Note that K_Γ is always the tropicalization of some (non-effective) canonical divisor by [Bak08, Remark 4.21].

The realizability locus in the tropical Hodge bundle

In [LU17] Lin and the second author of this article synthetically constructed a tropical analogue $\mathbb{P}\Omega\mathcal{M}_g^{\mathrm{trop}}$ of the projective Hodge bundle. Set-theoretically it parametrizes isomorphism classes of pairs (Γ, D) where Γ is a stable tropical curve of genus g and D is an element of the canonical linear system on Γ. By [LU17, Theorem 1.2] it canonically carries the structure of a generalized (rational polyhedral) cone complex. We denote by $\mathbb{P}\Omega\mathcal{M}_g^{\mathrm{an}}$ the Berkovich analytic space associated to $\mathbb{P}\Omega\mathcal{M}_g$ in the sense of [Uli17]. There is a natural tropicalization map

$$\mathrm{trop}_\Omega : \mathbb{P}\Omega\mathcal{M}_g^{\mathrm{an}} \to \mathbb{P}\Omega\mathcal{M}_g^{\mathrm{trop}}$$

that sends an element in $\mathbb{P}\Omega\mathcal{M}_g^{\mathrm{an}}$, represented by a pair (X, K_X) consisting of a smooth algebraic curve X over a non-Archimedean extension of the base field and a canonical divisor K_X on X, to the point consisting of the dual tropical curve Γ of a stable reduction $\mathcal{X}$ of X together with specialization of K_X to Γ (see Section 4 below for details).

Theorem 1 thus gives a complete characterization of the elements in the so-called *realizability locus*, the image of trop_Ω in $\mathbb{P}\Omega\mathcal{M}_g^{\mathrm{trop}}$.

In general, by the classical Bieri–Groves Theorem (see [BG84, Theorem A] and [EKL06, Theorem 2.2.3]) the tropicalization of a subvariety of a split algebraic torus is a rational polyhedral complex of the same dimension. The Hodge bundle does not embed into an algebraic torus but rather in a suitably defined toroidal compactification (in the sense of [KK+73]). Consequently, we know in this case a priori only that the dimension of the realizability locus is bounded above by $4g - 4$ by [Uli15a, Theorem 1.1]. Our methods allow us to prove the following stronger result.

Theorem 2. *The realizability locus in $\mathbb{P}\Omega\mathcal{M}_g^{\mathrm{trop}}$ admits the structure of a generalized rational polyhedral cone complex of pure dimension $4g - 4$.*

The main ingredient in the proofs of both Theorems 1 and 2 is the description of compactifications of strata of abelian differentials in [BC+18b], which is achieved using the method of plumbing and gluing. So our proof only works in equicharacteristic zero. It would be highly interesting to find a purely algebraic-geometric proof of these results (and the ones in [BC+18b]) that generalizes to all characteristics.

Realizability locus for strata

The Hodge bundle has a natural stratification by locally closed subsets

$$\mathbb{P}\Omega\mathcal{M}_g = \bigcup_{m_1,\ldots,m_n} \mathbb{P}\Omega\mathcal{M}_g(m_1, \ldots, m_n)$$

where the strata parametrize canonical divisors whose support has multiplicity profile $(m_1, \ldots, m_n)$ for non-negative integers m_i with $m_1 + \cdots + m_n = 2g - 2$. In our proof we construct a realization of a tropical canonical divisor by an element in the open stratum $\mathbb{P}\Omega\mathcal{M}_g(1, \ldots, 1)^{\mathrm{an}}$ of $\mathbb{P}\Omega\mathcal{M}_g^{\mathrm{an}}$. However, our criterion works exactly the same way for the realizability by an element in a fixed stratum $\mathbb{P}\Omega\mathcal{M}_g(m_1, \ldots, m_n)^{\mathrm{an}}$. In Section 6.3 we discuss this criterion in detail and show, in an example, how it can be applied to study the realizability problem for Weierstrass points in genus 2.

Related works

The realizability problem for divisors (or divisor classes) of a certain fixed rank on tropical curves is central to many applications of the tropical approach to limit linear series and has recently received a significant amount of attention (see [BJ16, Section 10] and the references therein). It is a crucial element in the tropical approach to the maximal rank conjecture for quadrics [JP16], which is based on a realizability result coming from [CJP15], as well as in the recent works on the Brill–Noether varieties for curves of fixed gonality [Pfl17, JR17]. We also highlight [Car15], in which the author shows that this realizability problem fulfills a version of Murphy's Law in the sense of Mnëv universality, and [He19], which connects the classical smoothing problem for limit linear series with the divisor theory on metrized curve complexes of Amini and Baker [AB15].

In [BN16], Baker and Nicaise study the geometry of the Kontsevich–Soibelman weight function on a non-Archimedean curve X^{an} (see [KS06, MN15] for details). Its restriction to the skeleton Γ_X of X^{an} is precisely the specialization of effective (pluri-) canonical divisors from algebraic to tropical curves. In [BN16, Theorem 3.2.3] Baker and Nicaise show in particular that the specialization of an effective canonical divisor on X is, in fact, an effective canonical divisor on Γ_X (which is our Proposition 4.2). Moreover, in [BN16, Section 3.4] they found that the union of Kontsevich–Soibelman skeletons associated to all effective canonical divisors (and not to all pluri-canonical divisors) excluded precisely the bridges on Γ_X, which foreshadows the special role played by bridges in our realizability result (see Example 6.4). We also refer the reader to [Tem16] for a more general perspective on Kontsevich–Soibelman weight functions from the point of view of metrization of differential pluriforms.

1. Compactifying the moduli space of algebraic divisors

Fix an algebraically closed field k. Let X be a scheme over k. Recall that a Cartier divisor D on X is *effective* if it admits a representation (U_i, f_i) such that $f_i \in \Gamma(U_i, \mathcal{O}_X)$ and f_i is a non-zero divisor. We may think of D as the closed subscheme of X whose ideal sheaf $I(D) = \mathcal{O}(-D)$ is invertible and generated by f_i on U_i.

Given a morphism $X \to S$ of schemes over k, we say that an effective divisor D on X is a *relative effective Cartier divisor* if it is flat over S when regarded as a subscheme of X. If S is connected, the function $s \mapsto \deg D_s$ is constant and will be referred to as the *degree* of D.

Inspired by both [Cap94] and [Has03], we define the following moduli space.

Definition 1.1. Let $g \geq 2$ and $d \geq 0$. Define $\overline{\mathcal{D}iv}_{g,d}$ to be the fibered category over $\overline{\mathcal{M}}_g$ whose fiber over a family $\pi : X \to S$ of stable curves is the set of pairs (X', D) consisting of a semistable model X' of X, i.e. a semistable curve over S with stabilization X, and a relative effective Cartier divisor D on X' such that

(i) the support of D does not meet the nodes in each fiber X'_s of $\pi' \colon X' \to S$; and
(ii) the twisted canonical divisor $K_{X'} + D$ is relatively ample.

Denote by $\mathcal{D}iv_{g,d}$ the preimage of the locus $\mathcal{M}_g$ of smooth curves.

Theorem 1.2. *The fibered category $\overline{\mathcal{D}iv}_{g,d}$ is a Deligne–Mumford stack of dimension $N = 3g - 3 + d$ that is smooth and proper over k. Its coarse moduli space $\overline{\mathrm{Div}}_{g,d}$ is projective. The complement of $\mathcal{D}iv_{g,d}$ in $\overline{\mathcal{D}iv}_{g,d}$ is a divisor with (stack-theoretically) normal crossings, and the forgetful morphism $\overline{\mathcal{D}iv}_{g,d} \to \overline{\mathcal{M}}_g$ is toroidal.*

Given a smooth curve X over the field k, it is well-known that the restriction of $\mathcal{D}iv_{g,d}$ to the point $[X]$ in $\mathcal{M}_g$ is representable by the d-th symmetric product $\mathrm{Sym}^d X$ of X (see e.g. [Mil86, Theorem 3.1.3]). One can generalize this result to all of $\overline{\mathcal{D}iv}_{g,d}$ using an instance of Hassett's moduli space of weighted stable curves [Has03] that automatically provides us with a compactification of $\mathcal{D}iv_{g,d}$ with favorable properties. The precise statement is as follows:

Let $g \geq 2$, $d \geq 0$, and $\epsilon^d = (\epsilon, \ldots, \epsilon) \in \mathbb{Q}^n$ for $\epsilon > 0$ such that $d \cdot \epsilon \leq 1$, e.g. $\epsilon = 1/d$. The moduli space $\overline{\mathcal{M}}_{g,\epsilon^d}$ of ϵ^d-stable curves (in the sense of [Has03]) parametrizes semistable curves $(X, s_1, \ldots, s_d)$ with d sections such that each rational component has at least two nodes and contains a marked point whenever it has precisely two nodes. Then our moduli stack $\overline{\mathcal{D}iv}_{g,d}$ is the relative coarse moduli space of the stack quotient $[\overline{\mathcal{M}}_{g,\epsilon^d}/S_d]$ over $\overline{\mathcal{M}}_g$ in the sense of [AOV11, Theorem 3.1]. So, in particlar, every fiber of $\overline{\mathcal{D}iv}_{g,d} \to \overline{\mathcal{M}}_g$ is the coarse moduli space of the corresponding fiber of $[\overline{\mathcal{M}}_{g,\epsilon^d}/S_d] \to \overline{\mathcal{M}}_g$. Over $\mathcal{M}_g$ this specializes to an isomorphism between $\mathcal{D}iv_{g,d}$ and the relative symmetric product

$$\mathrm{Sym}^d \mathcal{X}_g = \mathcal{X}_g \times_{\mathcal{M}_g} \cdots \times_{\mathcal{M}_g} \mathcal{X}_g / S_d$$

of the universal curve $\mathcal{X}_g$ over $\mathcal{M}_g$.

Using these observations, the proof of Theorem 1.2 now consists of an adaption of well-known techniques that is left to the avid reader.

Remark 1.3. After making our preprint available, we learned that the moduli space $\overline{\mathcal{D}iv}_{g,d}$ is a special instance of the moduli space of stable quotients (see [MOP11, Section 4.1] for more details).

Remark 1.4. In [JR17, proof of Theorem 4.6] Jensen and Ranganathan work with the symmetric product $\mathrm{Sym}^d \overline{\mathcal{X}}_g$ of the compactified universal curve $\overline{\mathcal{X}}_g$ over $\overline{\mathcal{M}}_g$ as a compactification of $\mathcal{D}iv_{g,d}$. This compactification is not smooth and not even toroidal, in general; the authors get around this by working with the stacky symmetric product

$$[\mathrm{Sym}^d \overline{\mathcal{X}}_g] = [\mathcal{X}_g^d / S_d]$$

whose boundary admits (stack-theoretically) toroidal singularities. Our compactification is a relative coarse moduli space of a toroidal resolution of the singularities of $[\mathrm{Sym}^d \overline{\mathcal{X}}_g]$.

2. Tropical divisors and their moduli

Let us first introduce tropical curves (see e.g. [Mik06]). A *metric graph* is an equivalence class of tuples $(G, |\cdot|)$ consisting of a connected finite graph $G = (V, E)$ together with an edge length function $|\cdot| : E(G) \to \mathbb{R}_{>0}$. Two such tuples $(G, |\cdot|)$ and $(G', |\cdot|')$ are *equivalent* if there is a common length preserving refinement. We implicitly identify a metric graph, represented by $(G, |\cdot|)$, with its realization as a metric space, by gluing in an interval of length $|e|$ for every edge e according to the incidences in G.

A *tropical curve* Γ is a metric graph together with a function $h : \Gamma \to \mathbb{Z}_{\geq 0}$ with finite support. We refer to a tuple $(G, |\cdot|)$ as a *model* of Γ if it represents Γ as a metric graph, and if h is supported on the vertices of G. The *genus* of a tropical curve is defined to be

$$g(\Gamma) = b_1(\Gamma) + \sum_{p \in \Gamma} h(p). \tag{1}$$

A model G of a tropical curve is said to be *semistable* if for every vertex v of G we have $2h(v) - 2 + |v| \geq 0$, where $|v|$ denotes the valency of the vertex v. It is called *stable* if the above inequality is strict, i.e. $2h(v) - 2 + |v| > 0$ for all vertices v of G. Notice that when a tropical curve Γ admits a semistable model, its minimal model is necessarily stable. In this case, we call Γ *stable*.

Later we will also use the notion of a *tropical curve* Γ *with legs* for a tropical curve Γ decorated with a collection L of infinite half-edges, called legs, emanating from the vertices of G. In this case, we modify the definition of *stability* by also counting the legs, when determining the valency of a vertex. Whenever it is clear from the context, we refer to a tropical curve with legs simply as a *tropical curve*.

Let $g \geq 2$. The moduli space M_g^{trop} of stable tropical curves is defined to be the set of isomorphism classes of stable tropical curves (without legs) of genus g. By [ACP15] it has the structure of a *generalized* (*rational polyhedral*) *cone complex*, i.e. it arises as a colimit of a diagram of (not necessarily proper) face morphisms of rational polyhedral cones.

The goal of this section is to construct a moduli space $\mathrm{Div}_{g,d}^{\mathrm{trop}}$ of tropical divisors of degree d over M_g^{trop}.

2.1. *Divisors on tropical curves*

A *divisor* on a tropical curve Γ is a finite formal sum $D = \sum a_i p_i$ of points $p_i \in \Gamma$ with integral coefficients a_i. We let $\deg(D) = \sum a_i$ be the degree of the divisor and write $D(p)$ for the coefficient of a point $p \in \Gamma$. A divisor D is said to be *effective*, denoted by $D \geq 0$, if $D(p) \geq 0$ for all points p of Γ. Given a tropical curve Γ, we denote by $\mathrm{Div}(\Gamma)$ the group of divisors on Γ.

A *rational function* on Γ is a continuous function $f : \Gamma \to \mathbb{R}$ whose restriction to every edge Γ (thought of as an interval $[0, |e|]$) is a piecewise linear function whose slopes are integral. We write $\mathrm{Rat}(\Gamma)$ for the abelian group of rational functions on Γ. Given a rational function f on Γ and $P \in \Gamma$, we define the *order* $\mathrm{ord}_p(f)$ of f at p to be the sum of the outgoing slopes of f over all edges emanating from p. This defines a map

$$\mathrm{div} : \mathrm{Rat}(\Gamma) \to \mathrm{Div}(\Gamma), \qquad f \mapsto \sum_{p \in \Gamma} \mathrm{ord}_p(f) \cdot p,$$

that assigns to any rational function its divisor. The image of the map div is the subgroup $\mathrm{PDiv}(\Gamma) \subset \mathrm{Div}(\Gamma)$ of *principal divisors*. Divisors D and D' are called *equivalent* (denoted by $D \sim D'$) if $D - D' \in \mathrm{PDiv}(\Gamma)$.

We can now define the *linear system* of a divisor D to be

$$|D| = \{D' \in \mathrm{Div}(\Gamma) : D \geq 0 \text{ and } D \sim D'\}.$$

It is convenient to also introduce the tropical analogue

$$R(D) = \{f \in \mathrm{Rat}(\Gamma) : D + \mathrm{div}(f) \geq 0\}$$

of the global sections of $\mathcal{O}(D)$. Note that we can shift any element in $R(D)$ by adding a real number and that $|D| = R(D)/\mathbb{R}$.

For any divisor D the space $R(D)$ has the structure of a polyhedral complex (see e.g. [GK08, Lemma 1.9], [MZ08], and [LU17, Proposition 3.2]). However, this polyhedral complex is not equidimensional in general, as we will see in the case of the canonical linear system in Section 6.

2.2. *Moduli of tropical divisors*

Definition 2.1. Let $g \geq 2$. The moduli space $\mathrm{Div}_{g,d}^{\mathrm{trop}}$ is the set of isomorphism classes of tuples (Γ, D) consisting of a stable tropical curve of genus g and an effective divisor D on Γ of degree d.

Proposition 2.2. *The moduli space* $\mathrm{Div}_{g,d}^{\mathrm{trop}}$ *naturally has the structure of a generalized cone complex of dimension* $3g - 3 + d$.

Consider a pair (G', D) consisting of a finite semistable vertex-weighted graph G' of genus g and an effective divisor D on G' of degree d supported on the vertices of G'. We say that the pair (G', D) is *stable* if for every vertex v of G' we have $2h(v) - 2 + |v| + D(v) > 0$.

Proof of Proposition 2.2. Denote by $J_{g,d}$ the category of stable pairs (G', D) where G' is of genus g and D has degree d. The morphisms in $J_{g,d}$ are generated by

- automorphisms ϕ of the weighted graph G' such that $\phi^* D = D$; and
- weighted edge contractions $\pi : (G'_1, D_1) \to (G'_2, D_2)$ (i.e. edge contractions for which $g(\pi^{-1}(v)) = h(v)$ for all vertices v of G'_2) that fulfill $\pi_* D_1 = D_2$.

There is a natural functor from $J_{g,d}$ to the category $\mathbf{RPC}^{\text{face}}$ of (rational polyhedral) cones with (not necessarily proper) face morphisms, given by $(G', D) \mapsto \sigma_{G'} = \mathbb{R}_{\geq 0}^{E(G')}$. Recall that a *face morphism* is a morphism of rational polyhedral cones $\sigma \to \sigma'$ that induces an isomorphism between σ and a (not necessarily proper) face of σ'; the class of face morphisms includes in particular all automorphisms.

For a fixed (G', D), the open cone $\mathring{\sigma}_{G'} = \mathbb{R}_{>0}^{E(G')}$ parametrizes the space of triples consisting of

- a tropical curve Γ in M_g^{trop},
- an effective divisor D on Γ of degree d, and
- an isomorphism between G' and the unique minimal semistable model of Γ whose vertices contain the support of D.

The automorphism group $\text{Aut}(G', D)$ acts on $\sigma_{G'} = \mathbb{R}_{\geq 0}^{E(G')}$ by permuting the entries of the vectors accordingly and the natural map $\mathring{\sigma}_{G'} \to \text{Div}_{g,d}^{\text{trop}}$ factors through the injection $\mathring{\sigma}_{G'}/\text{Aut}(G', D) \hookrightarrow \text{Div}_{g,d}^{\text{trop}}$. Thus the set $\text{Div}_{g,d}^{\text{trop}}$ arises as a colimit of the diagram $J_{g,d} \to \mathbf{RPC}^{\text{face}}$ and therefore carries the structure of a generalized cone complex.

Finally, a maximally degenerate object (G', D) in $J_{g,d}$ with all vertex weights equal to zero has precisely $3g - 3 + d$ finite edges. Therefore the dimension of every maximal cone in $\text{Div}_{g,d}^{\text{trop}}$ is $3g - 3 + d$. □

Remark 2.3. Let Γ be a stable tropical curve. The set $\text{Div}_d^+(\Gamma)$ of effective divisors of degree d on Γ admits a natural rational polyhedral subdivision given by subdividing $\text{Sym}^d \Gamma$ along the folds that arise when taking the quotient of Γ^d by the S_d-operation. The moduli space $\text{Div}_{g,d}$ naturally recovers this polyhedral decomposition in the fiber over a tropical curve $[\Gamma] \in M_g^{\text{trop}}$ (see [BU18, Section 1] for details).

Remark 2.4. Using the language of tropical moduli stacks developed in [CC+20], we may consider the natural moduli functor $\mathcal{D}iv_{g,d}^{\text{trop}}$ that associates to a rational polyhedral cone σ the groupoid of semistable pairs (G', D) together with non-zero edge lengths on G' in the dual monoid S_σ. The proof of Proposition 2.2 actually shows that $\mathcal{D}iv_{g,d}^{\text{trop}}$ is representable by a *cone stack* in the sense of [CC+20] (see [CC+20, Section 3.4] for an analogous argument for the moduli stack $\mathcal{M}_{g,n}^{\text{trop}}$ of n-marked stable tropical curves of genus g).

A stable tropical curve Γ of genus g (with real edge lengths) then corresponds to a morphism $\mathbb{R}_{\geq 0} \to \mathcal{M}_g^{\text{trop}}$. Expanding on [CC+20, Section 4], one can show that the pullback $\mathbb{R}_{\geq 0} \times_{\mathcal{M}_g^{\text{trop}}} \mathcal{D}iv_{g,d}^{\text{trop}}$ is representable by a *cone space* and its fiber over $1 \in \mathbb{R}_{\geq 0}$ is exactly the polyhedral decomposition of $\text{Div}_d^+(\Gamma)$ we have considered in Remark 2.3.

3. Specialization versus tropicalization

Let k be an algebraically closed field endowed with the trivial absolute value. Denote by $\mathcal{D}iv_{g,d}^{\text{an}}$ the non-Archimedean analytification of the moduli space of smooth curves together with an effective divisor of degree d. In this section we define a natural tropicalization map

$$\mathrm{trop}_{g,d}\colon \mathcal{D}iv_{g,d}^{\mathrm{an}} \to \mathrm{Div}_{g,d}^{\mathrm{trop}}$$

from the Berkovich analytic space $\mathcal{D}iv_{g,d}^{\mathrm{an}}$ to the tropical moduli space $\mathrm{Div}_{g,d}^{\mathrm{trop}}$ and show that this map can be identified with a natural strong deformation retraction onto the non-Archimedean skeleton of $\mathcal{D}iv_{g,d}^{\mathrm{an}}$ in the sense of [Thu07].

Note that for every algebraic stack $\mathcal{X}$ which is locally of finite type over k, there is an associated analytic stack $\mathcal{X}^{\mathrm{an}}$ defined by pullback with respect to the usual analytification functor on k-schemes [Uli17, Definition 2.18]. We usually abuse notation and denote by $\mathcal{X}^{\mathrm{an}}$ the associated topological space as defined in [Uli17, Definition 3.3]. Hence, if $\mathcal{X}$ is a separated algebraic Deligne–Mumford stack, by [Uli17, Proposition 3.8], the space $\mathcal{X}^{\mathrm{an}}$ can be identified with the Berkovich analytification of the coarse moduli space associated to $\mathcal{X}$.

3.1. Tropicalization of $\mathcal{D}iv_{g,d}$

We begin by defining the tropicalization map

$$\mathrm{trop}_{g,d}\colon \mathcal{D}iv_{g,d}^{\mathrm{an}} \to \mathrm{Div}_{g,d}^{\mathrm{trop}}.$$

A point in $\mathcal{D}iv_{g,d}^{\mathrm{an}}$ is represented by a proper, smooth algebraic curve X of genus g over a field K that is a non-Archimedean extension of k together with an effective divisor D on X of degree d. Possibly after replacing K by a finite extension, there is a semistable model $\mathcal{X}/S$ of X over the spectrum S of the valuation ring R of K together with a relative effective divisor $\mathcal{D}$ on $\mathcal{X}$ that does not meet the singularities in the special fiber $\mathcal{X}_s$ and makes the divisor $K_{\mathcal{X}} + \mathcal{D}$ relatively ample. Here we use the fact that the moduli stack $\overline{\mathcal{D}iv}_{g,d}$ is proper. Its special fiber $(\mathcal{X}_s, \mathcal{D}_s)$ (as a Cartier divisor) is an element in $\overline{\mathcal{D}iv}_{g,d}(\widetilde{K})$, where $\widetilde{K}$ denotes the residue field of R. At the level of points, the tropicalization map

$$\mathrm{trop}_{g,d}: \mathcal{D}iv_{g,d}^{\mathrm{an}} \to \mathcal{D}iv_{g,d}^{\mathrm{trop}}$$

associates to the pair (X, D) the *dual tropical curve* Γ of $\mathcal{X}_s$ together with the *specialization* of D to Γ, an effective divisor of degree d on Γ. More precisely, the weighted dual graph G' is the incidence graph of $\mathcal{X}_s$ together with the vertex weights $h(v)$ given by the genus of (the normalization of) the corresponding irreducible component of $\mathcal{X}_s$. The *dual tropical curve* of $\mathcal{X}$ is the tropical curve with semistable model G' for which the *length* $|e|$ of the edge $e \in E(\Gamma)$ is defined to be $\mathrm{val}_R(f)$, where $xy = f$ is the local equation of the node corresponding to e, and where val_R denotes the valuation. For $v \in V(\Gamma)$ we denote the normalization of the component C_v of the special fiber $\mathcal{X}_s$ by $\widetilde{C}_v$. The *specialization of D to* Γ is then defined as the multidegree

$$\mathrm{mdeg}(\mathcal{D}_s) = \sum_{v\in V(\Gamma)} \deg(\mathcal{D}_s|_{\widetilde{C}_v}) \cdot [v] \tag{2}$$

of the special fiber $\mathcal{D}_s$ of $\mathcal{D}$, thought of as a divisor on Γ (with support contained in the vertices of G').

The independence of the choices made in this construction is checked in [Viv13] for the moduli space of stable curves. It also follows a posteriori from Theorem 3.2 below.

Remark 3.1. Instead of working with $(\mathcal{X}, \mathcal{D})$ as above, we can also work (at least in the case that S is the spectrum of a discrete valuation ring) with any semistable model $\widetilde{\mathcal{X}}/S$ in which $\mathcal{D}$ does not meet the singularities of $\mathcal{X}_s$. The dual graph of the special fiber, metrized as above, is *equivalent* to that of $\mathcal{X}/S$ in that it results from a subdivision of edges by a finite number of 2-valent genus zero vertices.

3.2. Baker's specialization map

Let X be a smooth curve over a non-Archimedean extension K of k, and let D be an effective divisor on X. Then there is a minimal skeleton Γ associated to X^{an} [Ber90, Section 4.3], which is a deformation retract of X^{an}, that is, there exists a continuous retraction map $\tau : X^{\mathrm{an}} \to \Gamma$. Denote by $\tau : X(\overline{K}) \to \Gamma$ its restriction to $X(\overline{K}) \subseteq X^{\mathrm{an}}$. By linear extension, and using the fact that $\mathrm{Div}_d(X_{\overline{K}}) = \mathrm{Div}_d(X(\overline{K}))$, we may define a homomorphism

$$\tau_* : \mathrm{Div}_d(X_{\overline{K}}) \to \mathrm{Div}_d(\Gamma), \qquad \sum_i a_i p_i \mapsto \sum_i a_i \tau(p_i).$$

We also refer the reader to [Bak08, Section 2C] and in particular to [BJ16, Section 6.3] for details of this construction.

Suppose now that D is effective. Since $\overline{\mathcal{D}iv}_{g,d}$ is proper, we may find a semistable model $\mathcal{X}$ of X over R (possibly after replacing R by a finite extension, and X and D by their base changes) such that the closure $\mathcal{D}$ of D on $\mathcal{X}$ does not meet the singularities of the special fiber $\mathcal{X}_s$ of $\mathcal{X}$ (and so that $K_{\mathcal{X}} + \mathcal{D}$ is relatively ample). If the support of D is K-rational and R is a discrete valuation ring, then the image $\tau_*(D)$ coincides with the multidegree of $\mathcal{D}_s$ on $\mathcal{X}_s$, thought of as a divisor on Γ. In other words,

$$\mathrm{trop}_{g,d}([X], D) = (\mathrm{trop}_g([X]), \tau_*(D)).$$

This observation has originally appeared in [Bak08, Remark 2.12] (see also [BJ16, Section 6.3]).

3.3. The retraction to the skeleton

Let $X_0 \hookrightarrow X$ be a *toroidal embedding*, i.e. an open immersion of normal schemes locally of finite type over k that étale locally on X admits an étale morphism $\gamma : X \to Z$ into a T-toric variety Z such that $\gamma^{-1}(T) = X_0$. Moreover, suppose for notational simplicity that X is proper over k.

In [Thu07] Thuillier has constructed a strong deformation retraction

$$\mathbf{p}_{X_0 \hookrightarrow X} : X_0^{\mathrm{an}} \to X_0^{\mathrm{an}}$$

onto a closed subset $\mathfrak{S}(X_0 \hookrightarrow X)$ of X_0^{an} with the structure of a generalized cone complex, the *non-Archimedean skeleton* of X_0 (defined with respect to the toroidal compactification $X_0 \hookrightarrow X$). We refer the reader to [ACP15, Section 6] for a generalization of this construction to separated toroidal Deligne–Mumford stacks.

In fact, Thuillier's construction in [Thu07] extends to the analytification of the toroidal compactification X^{an} and we obtain a strong deformation retraction to a compactified skeleton $\overline{\mathfrak{S}}_{X_0\hookrightarrow X}$ of X^{an}. The strong deformation retraction is proper and closed as a map $X^{\mathrm{an}} \to \overline{\mathfrak{S}}_{X_0\hookrightarrow X}$, since $\overline{\mathfrak{S}}_{X_0\hookrightarrow X}$ is compact. Therefore $\mathbf{p}_{X_0\hookrightarrow X}$ is proper and closed as well.

By Theorem 1.2, the open immersion $\mathcal{D}iv_{g,d} \hookrightarrow \overline{\mathcal{D}iv}_{g,d}$ is toroidal and hence there is a natural strong deformation retraction

$$\mathbf{p}_{g,d} : \mathcal{D}iv_{g,d}^{\mathrm{an}} \to \mathcal{D}iv_{g,d}^{\mathrm{an}}$$

onto the skeleton $\mathfrak{S}_{g,d}$ of $\mathcal{D}iv_{g,d}^{\mathrm{an}}$. The following Theorem 3.2 shows that the tropicalization map $\mathrm{trop}_{g,d}$ defined above is precisely equal to the retraction $\mathbf{p}_{g,d}$. This in particular implies that $\mathrm{trop}_{g,d}$ is well-defined, continuous, proper, and closed.

Theorem 3.2. *There is a natural isomorphism* $\Phi_{g,d} : \mathrm{Div}_{g,d}^{\mathrm{trop}} \xrightarrow{\sim} \mathfrak{S}_{g,d}$ *that makes the diagram*

$$\begin{tikzcd} \mathcal{D}iv_{g,d}^{\mathrm{an}} \arrow[rr, "\mathrm{trop}_{g,d}"] \arrow[dr, "\mathbf{p}_{g,d}"'] & & \mathrm{Div}_{g,d}^{\mathrm{trop}} \arrow[dl, "\Phi_{g,d}", "\cong"'] \\ & \mathfrak{S}_{g,d} & \end{tikzcd}$$

commute.

The proof of Theorem 3.2 is an adaption of the well-known methods of [ACP15] to our situation. Its central observation is that there is a natural one-to-one correspondence between the boundary strata of $\overline{\mathcal{D}iv}_{g,d}$ and stable pairs (G', D) as in Proposition 2.2. We leave the technical details of this argument to the avid reader.

Remark 3.3. Let $0 < \epsilon \leq 1/d$. At the level of underlying topological spaces, the two analytic stacks $[\mathcal{M}_{g,\epsilon^d}^{\mathrm{an}}/S_d]$ and $\mathcal{D}iv_{g,d}^{\mathrm{an}}$ are homeomorphic, since they have the same coarse moduli space (using [Uli17, Proposition 3.9]). Moreover, one may observe that $\mathrm{Div}_{g,d}^{\mathrm{trop}}$ is naturally homeomorphic to the quotient $M_{g,\epsilon^d}^{\mathrm{trop}}/S_d$ of the tropical analogue $M_{g,\epsilon^d}^{\mathrm{trop}}$ of Hassett's moduli space of weighted stable curves (as introduced in [CH$^+$16, Uli15b]) by the S_d-operation that permutes the markings of the legs. By [Uli15b, Theorem 1.2] we have a natural identification of the non-Archimedean skeleton of $\mathcal{M}_{g,\epsilon^d}^{\mathrm{an}}$ with the tropical moduli space $M_{g,\epsilon^d}^{\mathrm{trop}}$. Therefore, since this identification is invariant under the S_d-operations on both sides, we may also deduce from this earlier result that $\mathrm{Div}_{g,d}^{\mathrm{trop}}$ is homeomorphic to the skeleton of $\mathcal{D}iv_{g,d}^{\mathrm{an}}$.

Remark 3.4. Let X be a smooth curve over a non-Archimedean and algebraically closed extension K of k and denote by Γ its dual tropical curve. For a stable model $\mathcal{X}$ of X over R, corresponding to a morphism $S \to \overline{\mathcal{M}}_g$ with $S = \mathrm{Spec}\, R$, the fiber product $S \times_{\overline{\mathcal{M}}_g} \overline{\mathcal{D}iv}_{g,d}$ defines a polystable model of $\mathrm{Div}_d^+(X)$. In [BU18], the authors show that the non-Archimedean skeleton of $\mathrm{Div}_d^+(X)^{\mathrm{an}}$ in the sense of Berkovich [Ber99, Section 5] is isomorphic to $\mathrm{Div}_d^+(\Gamma)$ with the natural polyhedral structure described in Remark 2.3.

4. Tropicalizing the Hodge bundle

From now one we specialize from general divisors to canonical divisors and the Hodge bundle. In contrast to the case of algebraic curves, the canonical linear system on a tropical curve Γ without legs comes with a distinguished element

$$K_\Gamma = \sum_{v \in V(\Gamma)} (2h(v) + |v| - 2) \cdot v$$

with support at the vertices of Γ. We denote by $|K_\Gamma|$ the canonical linear series.

In [LU17] Lin and the second author introduce a tropical analogue of the Hodge bundle $\Omega\mathcal{M}_g$ and of its projectivization $\mathbb{P}\Omega\mathcal{M}_g$ on the moduli space M_g. As a set, the *tropical Hodge bundle* $\Omega\mathcal{M}_g^{\mathrm{trop}}$ is defined to be the set of isomorphism classes of pairs (Γ, f) consisting of a stable tropical curve Γ of genus g and a rational function $f \in \mathrm{Rat}(\Gamma)$ with $K_\Gamma + \mathrm{div}(f) \geq 0$. Its projectivization[1] $\mathbb{P}\Omega\mathcal{M}_g^{\mathrm{trop}}$ parametrizes pairs (Γ, D) consisting of a stable tropical curve Γ of genus g and an effective divisor $D = K_\Gamma + \mathrm{div}(f)$ in $|K_\Gamma|$. Both spaces come with a natural forgetful map to $\mathcal{M}_g^{\mathrm{trop}}$.

Proposition 4.1 ([LU17, Theorem 1]). *The tropical Hodge bundle $\mathbb{P}\Omega\mathcal{M}_g^{\mathrm{trop}}$ is a closed subset of* $\mathrm{Div}_{g,2g-2}^{\mathrm{trop}}$ *that canonically carries the structure of a generalized cone complex of (maximal) dimension* $5g - 5$.

The Hodge bundle is not equidimensional: see Example 6.7 below. Proposition 4.1 shows in particular that $\mathbb{P}\Omega\mathcal{M}_g^{\mathrm{trop}}$ is a closed subset of $\mathrm{Div}_{g,2g-2}^{\mathrm{trop}}$ that is a subcomplex of a subdivision of $\mathrm{Div}_{g,2g-2}^{\mathrm{trop}}$.

Proof of Proposition 4.1. Proposition 4.1 has already been proved as part of [LU17, Theorem 1] (and building upon the polyhedral description of tropical linear systems from [GK08, MZ08]). We rephrase the main insights of this proof using the language developed in Section 2.

Let G' be a semistable finite vertex-weighted graph of genus g and an effective divisor $D \in \mathrm{Div}_{g,2g-2}(G')$ making (G', D) into a stable pair. Write G for the stabilization of G'. We will show that the pullback of the tropical Hodge bundle to $\sigma_{G'} = \mathbb{R}_{\geq 0}^{E(G')}$ is given by a finite union of linear subspaces of $\sigma_{G'}$.

For simplicity choose an orientation on every edge of the graph G; the resulting structure will not depend on this choice. Consider a tropical curve Γ whose underlying graph is G'. In order to specify a rational function f on Γ such that $D = K_\Gamma + \mathrm{div}(f)$ (up to a global additive $\mathbb{R}$-operation) we need to specify a collection of integers $(m_e) \in \mathbb{Z}^{E(G)}$ (one for each edge of the stabilization G of G'), the initial slopes of f at the origin of the edge e, subject to the condition

$$2h(v) - 2 + |v| = \sum_{\text{outward edges at } v} m_e + \sum_{\text{inward edges at } v} -(\deg D|_e + m_e).$$

[1] This space was denoted $\mathcal{H}_g^{\mathrm{trop}}$ in [LU17], and $\Omega\mathcal{M}_g^{\mathrm{trop}}$ was denoted $\Lambda_g^{\mathrm{trop}}$ there. Here we mainly follow the notation conventions of [BC$^+$18b].

Notice that by [GK08, Lemma 1.8] there are, in fact, only finitely many choices for the initial slopes m_e.

In each of the finitely many cases where such an $f \in \mathrm{Rat}(\Gamma)$ exists, the continuity of f imposes a collection of linear conditions on the coordinates of $\sigma_{G'} = \mathbb{R}_{\geq 0}^{E(G')}$ (i.e. the edge lengths of G'). The intersection of $\sigma_{G'}$ with such a linear subspace is a cone in the generalized cone complex structure on $\mathbb{P}\Omega\mathcal{M}_g^{\mathrm{trop}}$. □

Let $\mathbb{P}\Omega\mathcal{M}_g^{\mathrm{an}}$ be the analytification of the projective Hodge bundle over an algebraically closed field k endowed with the trivial absolute value. In this section we recall in detail the construction of the tropicalization map on the Hodge bundle from [LU17, Proposition 6] and elaborate on its properties.

The moduli space $\mathbb{P}\Omega\mathcal{M}_g^{\mathrm{an}}$ parametrizes pairs $(X/K, K_X)$ consisting of a point $X/K \in \mathcal{M}_g^{\mathrm{an}}$ as recalled above, together with a divisor K_X that is equivalent to the canonical bundle $\omega_{X/K}$. We define a natural tropicalization map

$$\mathrm{trop}_\Omega : \mathbb{P}\Omega\mathcal{M}_g^{\mathrm{an}} \to \mathbb{P}\Omega\mathcal{M}_g^{\mathrm{trop}}$$

by setting

$$\mathrm{trop}_\Omega(X/K, K_X) = \mathrm{trop}_{g,2g-2}(X/K, K_X), \tag{3}$$

where $\mathrm{trop}_{g,2g-2}$ is the tropicalization map introduced in Section 3.1.

Proposition 4.2. *The tropicalization map* trop_Ω *is well-defined, continuous, proper, and closed.*

Proof. The fact that trop_Ω is well-defined can be shown using a moving lemma as in [Bak08, Lemma 4.20]. We will give an alternative proof of this fact in Section 6.1 in the framework of this article.

Let $\mathcal{X}$ be a semistable model of X over a discrete valuation ring R (or a finite extension thereof) extending k. We may assume that $\mathcal{X}$ is regular; otherwise we blow up accordingly. By a moving lemma, such as [Liu02, Proposition 9.1.11], we may find a (not necessarily effective) canonical divisor $K_{\mathcal{X}}$ on $\mathcal{X}$ that does not meet the singularities in the special fiber. It is well-known that the multidegree of $K_{\mathcal{X}}$ in the special fiber is equal to K_Γ (see e.g. [Bak08, Remark 4.18]). Any effective canonical divisor K_X on X is equivalent to the generic fiber of $K_{\mathcal{X}}$ and therefore the specialization of K_X to Γ is equivalent to K_Γ.

The discretely valued points in $\mathbb{P}\Omega\mathcal{M}_g^{\mathrm{an}}$ are dense, and since $\mathrm{trop}_{g,2g-2}$ is continuous and $\mathbb{P}\Omega\mathcal{M}_g^{\mathrm{trop}}$ is closed by Proposition 4.1 above, we find that $\mathrm{trop}_{g,2g-2}(x)$ is in $\mathbb{P}\Omega\mathcal{M}_g^{\mathrm{trop}}$ for every (not necessarily discretely valued) point $x \in \mathbb{P}\Omega\mathcal{M}_g^{\mathrm{an}}$. Since $\mathbb{P}\Omega\mathcal{M}_g^{\mathrm{trop}}$ is naturally a closed subset of $\mathrm{Div}_{g,2g-2}^{\mathrm{trop}}$, the properness and closedness of trop_Ω follow from the corresponding properties of $\mathrm{trop}_{g,2g-2}$. □

Definition 4.3. The *realizability locus* $\mathbb{P}\mathcal{R}_\Omega$ in $\mathbb{P}\Omega\mathcal{M}_g^{\mathrm{trop}}$ is the image

$$\mathbb{P}\mathcal{R}_\Omega = \mathrm{trop}_\Omega(\mathbb{P}\Omega\mathcal{M}_g^{\mathrm{an}})$$

of the tropicalization map.

The realizability locus $\mathbb{P}\mathcal{R}_\Omega$ is the locus of tuples (Γ, D) consisting of a stable tropical curve Γ and a canonical divisor D for which there is a stable family $\mathcal{X}$ of curves over a valuation ring R together with an effective canonical divisor K_X on the generic fiber X of $\mathcal{X}$ such that Γ is the dual tropical curve of $\mathcal{X}$ and D is the specialization of K_X.

5. Twisted differentials and the global residue condition

From now on we work over the field $\mathbb{C}$ of complex numbers. Let $X/\mathbb{C}$ be a smooth and proper algebraic curve, i.e. a compact Riemann surface. We let $\Omega\mathcal{M}_g \to \mathcal{M}_g$ be the space of pairs (X, ω) consisting of an algebraic curve together with a non-zero holomorphic one-form ω on X. This is the Hodge bundle over the complex-analytic moduli space of curves, deprived of the zero section. The multiplicities of the zeros of ω define a partition μ of $2g-2$ and the subspaces $\Omega\mathcal{M}_g(\mu)$ with fixed partition μ form a stratification of $\Omega\mathcal{M}_g$. Most of the time we will focus on the *principal stratum* corresponding to the partition $\mu = (1, \ldots, 1)$ and we usually put $n = |\mu|$.

In this section we recall from [BC$^+$18b] the description of a compactification of the strata of $\Omega\mathcal{M}_g$. More concretely, observe that there is a natural map $\varphi : \Omega\mathcal{M}_g(\mu) \to \mathcal{M}_{g,[\mu]}$ sending (X, ω) to the curve marked by the zeros of ω. Here $\mathcal{M}_{g,[\mu]}$ is the quotient of $\mathcal{M}_{g,n}$ by the symmetric group that permutes the entries of μ. The main theorem of [BC$^+$18b] is a characterization of the closure of φ in terms of twisted differentials that arise from rescaling degenerating one-parameter families of abelian differentials so that the limits on the components of the special fiber are non-zero. The version given here highlights the functions arising as scaling parameters. These scaling parameters will reflect the location of the support of the corresponding tropical divisors. We need to set up some notation to recall the theorem for the case of holomorphic abelian differentials.

Definition 5.1. *A type* is a tuple $\mu = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ such that $\sum m_i = 2g-2$ and $m_1 \geq \cdots \geq m_r > m_{r+1} = 0 = \cdots = m_{r+s} > m_{r+s+1} \geq \cdots \geq m_{r+s+p}$. We denote by p_1 the number of -1's occurring in this tuple.

We assign a type to any meromorphic differential via the multiplicities of its associated divisor.

The moduli space $\Omega\mathcal{M}_g(\mu)$ parametrizes meromorphic one-forms whose divisor is of type μ. We may view these spaces as strata of a twisted Hodge bundle (see [BC$^+$18b], but we do not need this viewpoint here).

5.1. *Level graphs*

Let $\Gamma = (V, E)$ be an (unmetrized) graph. A *full order* $\overline{\Gamma}$ on Γ is an order $\succcurlyeq$ on the vertices V that is reflexive, transitive, and such that for any $v_1, v_2 \in V$ at least one of the statements $v_1 \succcurlyeq v_2$ or $v_2 \succcurlyeq v_1$ holds. We call any function $\ell : V(\Gamma) \to \mathbb{Z}_{\leq 0}$ such that $\ell^{-1}(0) \neq \emptyset$ a *level function* on Γ. Note that a level function induces a full order on Γ by setting $v \preccurlyeq w$ if $\ell(v) \leq \ell(w)$. A *level graph* is a graph together with a choice of a level function (see e.g. Figure 2, where the level function is depicted as the height above Γ). Abusing notation, we use the symbol $\overline{\Gamma}$ also for level graphs.

For a given level L we call the subgraph of $\overline{\Gamma}$ that consists of all vertices v with $\ell(v) > L$ along with edges between them the *graph above level* L of $\overline{\Gamma}$, and denote it by $\overline{\Gamma}_{>L}$. We similarly define the graph $\overline{\Gamma}_{\geq L}$ *above or on level* L, and the graph $\overline{\Gamma}_{=L}$ *on level* L. An edge $e \in E(\overline{\Gamma})$ of a level graph $\overline{\Gamma}$ is called *horizontal* if it connects two vertices of the same level, and *vertical* otherwise. Given a vertical edge e, we denote by $v^+(e)$ (resp. $v^-(e)$) the vertex that is its endpoint of higher (resp. lower) level.

5.2. *Twisted differentials*

Let C be a nodal, in general non-smooth curve over the complex numbers. Let $\mu = (m_1, \dots, m_n)$ be a type. A *twisted differential of type* μ on a stable n-pointed curve $(C, \mathbf{s})$ is a collection of (possibly meromorphic) differentials η_v on the irreducible components C_v of C such that no η_v is identically zero with the following properties.

(0) **(Vanishing as prescribed)** Each differential η_v is holomorphic and non-zero outside of the nodes and marked points of C_v. Moreover, if a marked point s_i lies on C_v, then $\mathrm{ord}_{s_i} \eta_v = m_i$.
(1) **(Matching orders)** For any node of C that identifies $q_1 \in C_{v_1}$ with $q_2 \in C_{v_2}$, the vanishing orders satisfy $\mathrm{ord}_{q_1} \eta_{v_1} + \mathrm{ord}_{q_2} \eta_{v_2} = -2$.
(2) **(Matching residues at simple poles, MRC)** If $\mathrm{ord}_{q_1} \eta_{v_1} = \mathrm{ord}_{q_2} \eta_{v_2} = -1$ at a node of C that identifies $q_1 \in C_{v_1}$ with $q_2 \in C_{v_2}$, then $\mathrm{Res}_{q_1} \eta_{v_1} + \mathrm{Res}_{q_2} \eta_{v_2} = 0$.

Let Γ be the dual graph of C. Recall that the vertices v in Γ correspond to the irreducible components C_v of C. If ℓ is a level function on Γ, we write $C_{>L}$ for the subcurve of C containing only the components C_v with v of level strictly higher than L. Similarly, we define $C_{=L}$. If two components C_v and C_w with $\ell(v) < \ell(w)$ intersect in the point q, we denote by q^- the corresponding point on C_v, and we write $v = v^-(e)$ for the edge e in Γ connecting v and w.

Denote by $\overline{\Gamma}$ the full order on the dual graph Γ given by a level function. We say that a twisted differential η of type μ on C is *compatible with* $\overline{\Gamma}$ if in addition it also satisfies the following two conditions.

(3) **(Partial order)** If a node of C identifies $q_1 \in C_{v_1}$ with $q_2 \in C_{v_2}$, then $v_1 \succcurlyeq v_2$ if and only if $\mathrm{ord}_{q_1} \eta_{v_1} \geq -1$. Moreover, $v_1 \asymp v_2$ if and only if $\mathrm{ord}_{q_1} \eta_{v_1} = -1$.
(4) **(Global residue condition, GRC)** For every level L and every connected component Y of $C_{>L}$ the following condition holds: Let $q_1, \dots, q_b$ denote the set of all nodes where Y intersects $C_{=L}$. Then

$$\sum_{j=1}^{b} \mathrm{Res}_{q_j^-} \eta_{v^-(q_j)} = 0,$$

where we recall that $q_j^- \in C_{=L}$ and $v^-(q_j) \in \overline{\Gamma}_{=L}$.

5.3. The characterization of limit points

Suppose that S is the spectrum of a discrete valuation ring R with residue field $\mathbb{C}$, whose maximal ideal is generated by t. Let $\mathcal{X}/S$ be a family of semistable curves with smooth generic fiber X and special fiber C. Let ω be a section of $\omega_{\mathcal{X}/S}$ of type μ whose divisor is given by the sections $\mathbf{s} = (s_1, \dots, s_n)$ with multiplicity m_i. The triple $(\mathcal{X}/S, \mathbf{s}, \omega)$ is called a *pointed family of stable differentials* if moreover $(\mathcal{X}/S, \mathbf{s})$ is stable. Then we define the *scaling factor* $\ell(v)$ for the node v as the non-positive integer such that the restriction of the meromorphic differential $t^{-\ell(v)} \cdot \omega$ to the component C_v of the special fiber corresponding to v is a well-defined and generically non-zero differential η_v on C_v (see [BC+18b, Lemma 4.1]). The η_v are called the *scaling limits* of ω.

Theorem 5.2 ([BC+18b]). *If $(\mathcal{X}/S, \mathbf{s}, \omega)$ is as above, then the function $\ell(v)$ defines a full order on the dual graph Γ of the special fiber of $\mathcal{X}$ and the collection $\eta_v|_{X_v}$ is a twisted differential of type μ compatible with the level function ℓ.*

Conversely, suppose that C is a stable n-pointed curve with dual graph Γ and $\eta = \{\eta_v\}_{v \in V}$ is a twisted differential of type μ compatible with a full order $\overline{\Gamma}$ on Γ. Then for every level function $\ell : \Gamma \to \mathbb{Z}$ defining the full order $\overline{\Gamma}$ and for every assignment of integers n_e to horizontal edges there is a stable family $\mathcal{X}/S$ over $S = \operatorname{Spec}(\mathbb{C}[[t]])$ with smooth generic fiber and special fiber C that satisfies the following properties:

- (i) *There exists a global section ω of $\omega_{\mathcal{X}/S}$ whose horizontal divisor $\operatorname{div}_{\mathrm{hor}}(\omega) = \sum_{i=1}^{n} m_i \Sigma_i$ is of type μ and whose scaling limits are the collection $\{\eta_v\}_{v \in V}$.*
- (ii) *The intersections $\Sigma_i \cap C = \{s_i\}$ are smooth points of the special fiber and η has a zero of order m_i at s_i.*
- (iii) *There exists a positive integer N such that a local equation near every node corresponding to a horizontal edge e is $xy = t^{Nn_e}$, and it is $xy = t^{N(\ell(q^+(e)) - \ell(q^-(e)))}$ for every vertical edge e.*

Proof. The first statement is the necessity of [BC+18b, Theorem 1.3], proven in Section 4.1. Note that the arguments given in loc. cit. for this direction hold over any discrete valuation ring.

For the second statement one has to trace the proof of sufficiency of this theorem, given in Section 4.4 of loc. cit. As stated there (see equation (4.8) and the last two paragraphs of the proof of Addendum 4.8), there are no constraints on the plumbing fixtures to be used for plumbing horizontal nodes, whereas for the plumbing fixtures used for every vertical node, given by an edge e, the level function ℓ_0 used for plumbing has to satisfy the divisibility constraint

$$(\operatorname{ord}_{q^+(e)} \eta + 1) \mid (\ell_0(q^+(e)) - \ell_0(q^-(e)))\,.$$

Multiplying the prescribed function ℓ by a sufficiently divisible N, the resulting level function $\ell_0 = N \cdot \ell$ satisfies this divisibility property. □

5.4. Dimension and period coordinates

In preparation for the dimension statements in Section 6.2 we recall here two results about the geometry of strata of meromorphic differentials. Consider the neighborhood of a point

$(X, \omega) \in \Omega\mathcal{M}_g(\mu)$. We denote by Z the $r + s$ zeros and marked points of ω and let P be the p poles of ω. On such a neighborhood, integration of the meromorphic one-form against a basis of the relative cohomology group $H^1(X \setminus P, Z; \mathbb{Z})$ gives local coordinates, called *period coordinates*. See [BC+18a] for a proof of this statement (including the case of k-differentials) and for references to the history of this result. The fact that these functions are local coordinates also proves the following dimension statement.

Theorem 5.3. *The stratum $\Omega\mathcal{M}_g(\mu)$ has dimension $2g - 1 + n$ if the type μ is holomorphic (i.e. $p = 0$), and it has dimension $2g - 2 + n$ if the type μ is strictly meromorphic (i.e. $p > 0$).*

The following result is the special case for one-forms of a main result of [Che19].

Theorem 5.4 ([Che19]). *The projectivization of a stratum $\mathbb{P}\Omega\mathcal{M}_g(\mu)$ of strictly meromorphic type (i.e. with $p > 0$) does not contain a complete curve.*

5.5. The image of the residue map

Since the global residue condition imposes strong constraints on the residues, we need a criterion for which residues can actually be realized. Since a twisted differential is a collection of meromorphic differentials, rather than just holomorphic differentials, we have to deal more generally with types of meromorphic differentials. Recall from the beginning of Section 5 the conventions used to denote types of meromorphic differentials. In particular, $p_1 \leq p$ denotes the number of simple poles. For every type μ with $p \neq 0$ we let

$$\mathrm{Res}\colon \Omega\mathcal{M}_g(\mu) \to H$$

be the residue map, whose range is contained by the residue theorem in

$$H = \left\{ \mathbf{x} = (x_1, \ldots, x_p) \in \mathbb{C}^p : \sum_{i=1}^{p} x_i = 0 \right\}.$$

Moreover we define the 'non-zero set' $N \subseteq \mathbb{C}^p$ to consist of those $\mathbf{x}$ with $x_i \neq 0$ whenever $m_i = -1$. By definition of a stratum, the image of Res is obviously contained in $H \cap N$. This set is non-empty unless $p = p_1 = 1$, which we exclude from our discussion.

To illustrate the problem of determining the image of Res, consider differentials η of type $\mu = (a, -b, b - 2 - a)$ with $a \geq 0$, $b \geq 2$ and $b - a - 2 \leq -2$ on a projective line with coordinate z. We may assume that the zero of ω is at $z = 1$, while the poles are at $z = 0$ and $z = \infty$. Consequently, $\eta = C(z - 1)^a dz/z^b$ with $C \neq 0$. This implies that the residue

$$\mathrm{Res}_{z=0}(\eta) = C \cdot \mathrm{coeff}_{z^{-1}} \left(z^{-b} \sum_{i=0}^{a} (-1)^{a-i} \binom{a}{i} z^i \right) = C \cdot (-1)^{a-b+1} \binom{a}{b-1}$$

is non-zero since $a \geq b$ and $b - 1 \geq 0$. In fact $\mathrm{Im}(\mathrm{Res}) = H \setminus \{(0, 0)\} \subset H$. The image of the map Res in the general case was determined by Gendron and Tahar [GT17]. We restate a simplified version of their main result that is sufficient for our purposes.

Proposition 5.5. (i) *If $g \geq 1$ then* Res *is surjective onto $N \cap H$.*

(ii) *If $g = 0$ and $p > p_1 > 0$, or $p_1 = 0$ and there does not exist an index $1 \leq i \leq n$ with*
$$m_i > \Big(\sum_{j=r+s+1}^{r+s+p} -m_j \Big) - p - 1, \tag{4}$$
then Res *is surjective onto $N \cap H$.*

(iii) *If $g = 0$ and $p_1 = 0$ and there exists an index $1 \leq i \leq n$ with (4), then* Res *is surjective onto $N \cap H \setminus \{(0, \dots, 0)\}$.*

(iv) *If $g = 0$ and $p = p_1 = 2$, then* Res *is surjective onto $N \cap H$.*

(v) *If $g = 0$ and $p = p_1 > 2$, then the image of* Res *contains all tuples in $N \cap H$ consisting of p complex numbers that are not $\mathbb{R}$-collinear (i.e., whose $\mathbb{R}$-span is $\mathbb{C}$).*

Proof. All the conditions except for the case $g = 0$ and $p = p_1$ (only simply poles) are restatements of the case of abelian differentials in [GT17, Theorems 1.1–1.5]. The case $p = p_1 = 2$ follows directly from the preceding discussion. The case $p = p_1 > 2$ is stated for $r = 1$ in [GT17, Proposition 1.6]. For $r > 1$ it can be easily deduced from this by the procedure of splitting a zero, or it can be derived from the more involved formulation in [GT17, Proposition 1.7]. □

6. The realizability locus

In this section we prove Theorem 1 describing the realizability locus of tropical canonical divisors over an algebraically closed field of characteristic zero. For tropical curves with rational side lengths and canonical divisors with rational coordinates we use the complex-analytic techniques of the previous section to characterize the image of the tropicalization map. For general points the results follow from the general properties of trop_Ω.

6.1. From rational functions to enhanced level graphs

To formalize the correspondence between rational functions and level graphs, we introduce the following enhancement of the notion of a level graph. We consider vertex weighted graphs of the form $\Gamma = (V, E, L, h)$ *without edge lengths*, where (V, E) is a classical graph, L is a finite set of legs, i.e. infinite half-edges starting at a vertex, and h is a marking of the vertices with integers. After choosing a level function on the underlying graph (V, E), we call $\overline{\Gamma}$ a *level graph*. We divide every edge e into two half-edges and write Λ for the set containing all half-edges and all legs in Γ. Note that every $\lambda \in \Lambda$ is adjacent to a unique vertex. An *enhanced level graph* Γ^+ is a level graph $\overline{\Gamma}$ as above together with an assignment $k : \Lambda \to \mathbb{Z}$ such that the following compatibility conditions hold.

(i) If e^+ and e^- are two half-edges forming an edge e, then $k(e^+) + k(e^-) = -2$. An edge is horizontal if and only if $k(e^\pm) = -1$ for both its half-edges. Moreover, if e is a vertical edge consisting of the half-edges e^+ leading downwards and e^- leading upwards, then $k(e^+) > k(e^-)$.

(ii) For each vertex $v \in V$,

$$\sum k(\lambda) = 2h(v) - 2\,, \tag{5}$$

where the sum is over all $\lambda \in \Lambda$ which are adjacent to v.

Let v be a vertex in an enhanced level graph Γ^+. Then we define the type $\mu(v)$ as the ordered tuple consisting of all $k(\lambda)$, where λ is a half-edge adjacent to v. Note that $\mu(v)$ is a type in the sense of Definition 5.1 if we replace g by $h(v)$.

This definition is motivated by the notion of twisted differentials. In fact, given a stable curve X, let Γ be the associated graph consisting of the dual graph of X with legs attached for each marked point and h given by the genera of the irreducible components X_v for $v \in V$. Note that up to the choice of a metric this is the same construction as in Section 3. Every λ in Λ gives rise to a point z_λ in X_v, where v is the vertex adjacent to λ: If λ is a half-edge, we let z_λ be the node corresponding to the edge containing λ, and if λ is a leg we let z_λ be the associated smooth point. For any twisted differential η on X, we decorate all $\lambda \in \Lambda$ with the order of η at z_λ. This defines an enhanced level graph structure on Γ.

Lemma 6.1. *Let Γ be a tropical curve. To every element $D = K_\Gamma + \mathrm{div}(f) \in |K_\Gamma|$ we can associate a natural structure of an enhanced level graph $\Gamma^+ = \Gamma^+(f)$ on some realization of Γ.*

Proof. Let Γ_0 be the minimal realization of Γ subdivided with vertices at the places where f is not differentiable. We use the function f itself to give Γ_0 a full order, i.e. for nodes v, w of G_0 we declare $v \succcurlyeq w$ if and only if $f(v) \geq f(w)$.

We provide each vertex v of Γ_0 with $2h(v) - 2 + \sum_e (1 + s(e))$ legs, each given the decoration $k = 1$. Here the sum runs over all non-leg half-edges adjacent to v and $s(e)$ denotes the slope of f along e, oriented to be pointing away from v. The fact that $D \in |K_\Gamma|$ is equivalent to this number of legs being indeed non-negative for all vertices. We provide each half-edge e which is not a leg with $k = -s(e) - 1$, using the same orientation convention. The conditions for an enhanced level graph now follow immediately. □

We need two more notions to state our main theorem.

Definition 6.2. A vertex v of an enhanced level graph is called *inconvenient* if $h(v) = 0$ and if its type $\mu(v) = (m_1, \ldots, m_n)$ has the property that $p_1 = 0$ and there exists an index i such that (4) holds.

A cycle is called *simple* if it does not visit any vertex more than once. Recall the tropicalization map $\mathrm{trop}_\Omega : \mathbb{P}\Omega\mathcal{M}_g^{\mathrm{an}} \to \mathbb{P}\Lambda_g^{\mathrm{trop}}$ from Proposition 4.2.

Theorem 6.3. *Suppose that k is an algebraically closed field of characteristic zero. A pair (Γ, D) with $D = K_\Gamma + \mathrm{div}(f)$ in the tropical canonical linear series lies in the image of trop_Ω if and only if the following two conditions hold:*

(i) *For every inconvenient vertex v of $\Gamma^+(f)$ there is a simple cycle $\gamma \subset \Gamma$ based at v that does not pass through any node on a level lower than $f(v)$.*

(ii) *For every horizontal edge e there is a simple cycle $\gamma \subset \Gamma$ passing through e which does not pass through any node on a level lower than $f(e)$.*

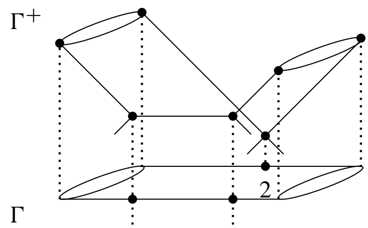

Fig. 1. Illustration of the edge condition.

Figure 1 illustrates the conditions of the theorem. The value of f is given by the height of the point in $\Gamma^+(f)$ over its image point in Γ. In this example there is a simple cycle through the horizontal edge in the foreground. However all the possible simple cycles through this edge pass through the vertex with two markings, which is on a lower level. Consequently, this graph is not realizable. Note also that realizability depends on the edge lengths here: If the edge containing the vertex with two markings were shorter (and all the other lengths remained the same), the corresponding vertex could be on a level above the horizontal edge and the corresponding divisor would be realizable.

Proof of Theorem 6.3. Proof that (i) and (ii) implies $D \in \mathrm{Im}(\mathrm{trop}_\Omega)$. First assume that $k = \mathbb{C}$. Conditions (i) and (ii) define a closed subset of $\mathbb{P}\Omega\mathcal{M}_g^{\mathrm{trop}}$. Tropical curves Γ with rational edge lengths and divisors $D = K_\Gamma + \mathrm{div}(f)$ associated to a function f are dense in this subset. Since trop_Ω is continuous and closed by Proposition 4.2, the image of trop_Ω is the closure of this locus. We may therefore assume that Γ has rational edge lengths. Moreover, if $(\Gamma, K_\Gamma + \mathrm{div}(f))$ is realizable, we may rescale the edge lengths by a global constant and still obtain a realizable object in $\mathbb{P}\Omega\mathcal{M}_g^{\mathrm{trop}}$. Therefore we may assume that Γ has integral edge lengths.

Suppose that the enhanced level graph $\Gamma^+(f)$ associated with D satisfies (i) and (ii) and the integrality hypothesis made above on Γ. We want to show that there is a twisted differential of type $\mu = (1, \ldots, 1)$ on a stable pointed curve C with dual graph Γ, compatible with the enhanced level structure $\Gamma^+(f)$, and then apply the 'converse' implication in Theorem 5.2. For every vertex v this amounts to finding a differential of type $\mu(v)$ on some smooth curve C_v. This is indeed the type of a meromorphic differential on C_v by property (ii) of an enhanced level graph. The matching order condition and the partial order condition of a twisted differential are built into the condition (i) of an enhanced level graph.

Hence the main point is to choose the curves C_v and the differential η_v so that the matching residue condition (MRC) and the GRC can be satisfied. For this purpose, we want to apply Proposition 5.5. By the following procedure we specify residues which on the one hand lie in $H \cap N$ at every node and satisfy both MRC and GRC, and on the other hand are non-zero at inconvenient nodes, and match the conditions of the last item in Proposition 5.5 at all nodes with only simple poles.

For each inconvenient vertex v_i choose a cycle γ_i as in condition (i) and let I_t be the index set for these γ_i. Similarly, for each horizontal edge e_j choose a cycle δ_j as in condition (ii) and collect the indices in the set J_h. We provide these cycles with some orientation. Let $\{\alpha_i, i \in I_t\} \cup \{\beta_j, j \in J_h\}$ be a collection of complex numbers such that no sum of a subset of the $\pm\alpha_i$ and $\pm\beta_j$ is real. (That is, the α_i and β_j lie in a complement of a finite union of real codimension 1 hyperplanes in $\mathbb{C}^{|I_t|+|J_h|}$.) Starting from residue zero at each edge we increase, for every $i \in I_t$, the prescribed residue at all the half-edges e with $f(e) = f(v_i)$ by α_i if the outward pointing orientation of e agrees with the orientation of the cycle, and by $-\alpha_i$ otherwise. For every $j \in J_h$ we increase the prescribed residue at all the half-edges e with $f(e) = f(e_j)$ by β_j. The collection of residues prescribed in this way is non-zero for every horizontal edge (since our choice of the cycles δ_j for $j \in J_h$ covers each such edge and the choice of the α_i and β_j avoids unintended cancellations), it is non-zero at every inconvenient node (by the choice of the cycles γ_i) and satisfies the residue theorem (since a cycle enters and exits any vertex the same number of times), i.e. the prescribed residues lie in the image of the residue map at each inconvenient node by Proposition 5.5. At each vertex with only simple poles, i.e. with $p = p_1$, the residues are non-zero and $\mathbb{R}$-linearly independent (if there are more than one pair of such poles, i.e. if $p_1 > 2$ at such a vertex). Consequently, by Proposition 5.5, the residues lie in the image of the residue map at each vertex. Finally, we check that the GRC continues to hold at each step of adding the contributions along a cycle γ_i or δ_j. We give the details for the first case, the second being the same, on replacing $f(v_i)$ by $f(e_j)$ everywhere. In fact, the addition procedure prescribes a zero total sum of residues to each component of $C_{>f(v_i)}$ the cycle passes through, so the GRC holds for $C_{\geq f(v_i)}$. Since the cycle does not pass through levels below $f(v_i)$, the GRC for those levels remains valid. If w is a vertex with level $f(w) > f(v_i)$ then all the edges up to level $f(v_i)$ are unseen in the GRC for $C_{\geq f(w)}$ and hence the GRC for these levels continues to hold, too.

Consequently, we can now use Theorem 5.2 with the level function $\ell = f$ as described in Lemma 6.1 and with $n_e = |e|$, the length in Γ for any horizontal edge. The conclusion of the theorem is precisely that the divisor $D + \operatorname{div}(f)$ is the specialization of an effective canonical divisor on a graph equivalent to Γ, with all the lengths rescaled by the integer N of Theorem 5.2(iii). Hence (Γ, D) lies in the image of $\operatorname{trop}_\Omega$ by definition of this map in (3).

Proof that $D \in \operatorname{Im}(\operatorname{trop}_\Omega)$ implies (i) and (ii). Points of the form (X, D), where X is the smooth generic fiber of a stable curve $\mathcal{X}$ over the valuation ring R of a finite extension of $\mathbb{C}(t)$, are dense in $\mathbb{P}\Omega\mathcal{M}_g^{\mathrm{an}}$. Since $\operatorname{trop}_\Omega$ is continuous by Proposition 4.2, it suffices to show that $\operatorname{trop}_\Omega(X, D)$ satisfies conditions (i) and (ii) in our claim. Denote by S the spectrum of R. Moreover, let ω be a stable differential on $\mathcal{X}$ such that the divisor of its generic fiber is D. We may assume, by the density of the principal stratum, that $\operatorname{div}(\omega)^{\mathrm{hor}} = \sum_{i=1}^{2g-2} s_i(S)$ consists of $2g-2$ images of sections.

Let ℓ be the level function on the dual graph Γ of the special fiber given by the scaling parameters of this family (cf. Section 5.3). Let η be the twisted differential on the special fiber C of $\mathcal{X}$, obtained as the scaling limit of ω, and let $\Gamma^+(\ell)$ be the enhanced level graph given by ℓ and the enhancement given by η, as described before Lemma 6.1. We want to show that $\operatorname{trop}_\Omega(X, D) = (\Gamma, K_\Gamma + \operatorname{div}(\ell))$ and that the enhanced level graph $\Gamma^+(\ell)$ satisfies (i) and (ii).

Concerning the first claim, we note that by definition of η exactly $2h(v) - 2 - \sum k(e)$ sections of ω lie in the irreducible component C_v of the special fiber associated to the vertex v, counted with multiplicity. Here the sum runs over all half-edges adjacent to v which are not legs at v. By definition of trop_Ω in (3) the first claim thus amounts to showing that the slope defined by ℓ of non-leg half-edges e is equal to $-k(e) - 1$. This in turn is a consequence of the way degenerating families are built by plumbing (cf. [BC+18b, Theorem 4.5]). The core observation is that at a node $xy = t^a$ the differential on the two ends of the node is $\big(x^k + t^{a(k+1)}\frac{r}{x}\big)dx$ and $-t^{a(k+1)} \cdot (y^{-k-2} + \frac{r}{y})dy$, where $t^{a(k+1)}r$ is the period of ω along the vanishing cycle. Consequently, the difference of the scaling parameters (or, equivalently, the values of the level function) is $a(k+1)$, proving the slope claim.

To show (i), we work with the complex topology. Note that there exists a disc Δ in $\mathbb{C}$ such that $\mathcal{X}(\mathbb{C})$ can be extended to a complex-analytic space $\mathcal{X}_\Delta(\mathbb{C})$ over Δ. At every inconvenient vertex v the restriction of η to v has non-zero residue at $q^-(e)$ for some edge e with $k(e) < 0$ by Proposition 5.5. To illustrate the idea of constructing the necessary cycle, let $e_1, \ldots, e_m$ be the edges adjacent to v with $k(e) < 0$. Choose a continuously varying family $\beta_j(s)$ for $s \in \Delta$ of simple closed curves in the fibers $\mathcal{X}_s$ belonging to the homotopy class which is pinched to the node e_j. If all the curves $\beta_j(s)$ are separating for one (hence every) $s \in \Delta$, the period of ω_s around α_0 is zero by Stokes' theorem. This gives a contradiction to the non-zero residue in the limit. Consequently, there is some non-trivial cycle γ passing through v.

To deal with the general case and to derive the claimed property of γ we revisit the proof of the GRC (compare [BC+18b, Section 4.1]. Let $A = \alpha_1(s) \cup \cdots \cup \alpha_m(s)$ be the union of simple closed curves which are pinched (when $s \to 0$) to the nodes joining a level $\geq \ell(v)$ to a level $< \ell(v)$. If $\beta_j(s)$ is a separating loop on the connected component of $X_s \setminus A$ that contains $\beta_j(s)$, we obtain the same contradiction from Stokes' theorem as before. In fact, let $I \subset \{0, \ldots, s\}$ be the index set of curves bounding a component of $X_s \setminus (A \cup \beta_j)$. Then

$$\int_{\beta_j} t^{\ell(v)}\omega(t) + \sum_{i\in I} \int_{\alpha_i} t^{\ell(v)}\omega(t) = 0$$

by Stokes. The first term of this sum tends to the residue we are interested in. The other terms tend to the residue of the limiting twisted differential on level $\geq \ell$ at a node corresponding to an edge to level $< \ell$, which is zero since the limiting differential is holomorphic there (see the 'partial order' condition in the definition of twisted differential in Section 5.2). Consequently, if some $\beta_j(s)$ does not separate its connected component in $X_s \setminus A$, there exists a cycle as claimed in (i). The argument for horizontal edges is the same and gives (ii).

Hence we have proved the theorem in the case $k = \mathbb{C}$. If $K \subset L$ is a field extension of two trivially valued algebraically closed fields of characteristic zero, we have a natural surjective projection map $(\mathbb{P}\Omega\mathcal{M}_{g,L})^{\mathrm{an}} \to (\mathbb{P}\Omega\mathcal{M}_{g,K})^{\mathrm{an}}$ which is compatible with the tropicalization map, i.e. the diagram

$$\begin{tikzcd}
(\mathbb{P}\Omega\mathcal{M}_{g,L})^{\mathrm{an}} \arrow[d] \arrow[dr, "\mathrm{trop}_\Omega"] & \\
(\mathbb{P}\Omega\mathcal{M}_{g,K})^{\mathrm{an}} \arrow[r, "\mathrm{trop}_\Omega"'] & \mathbb{P}\Omega\mathcal{M}_g^{\mathrm{trop}}
\end{tikzcd}$$

is commutative by the definition of trop_Ω in Section 3.1. Hence the realizability locus does not depend on the choice of the algebraically closed ground field of characteristic zero, which implies our claim. $\square$

Example 6.4. Figure 2 shows points in the realizability locus over the dumbbell graph in genus 2, i.e. all vertex genera zero. Those points in the realizability locus consist of two symmetrically placed marked points on either dumbbell end (left) or double point anywhere on the central edge (including the ends) of the dumbbell (right). A canonical divisor whose support consists of two different points on the central edge of the dumbbell (see Figure 3) is not in the realizability locus, since the edge between those two points is horizontal in the enhanced level graph and separating, thus violating condition (ii) in Theorem 6.3. These two figures are reinterpretations of the corresponding figures in [GK08] from our viewpoint of level graphs.

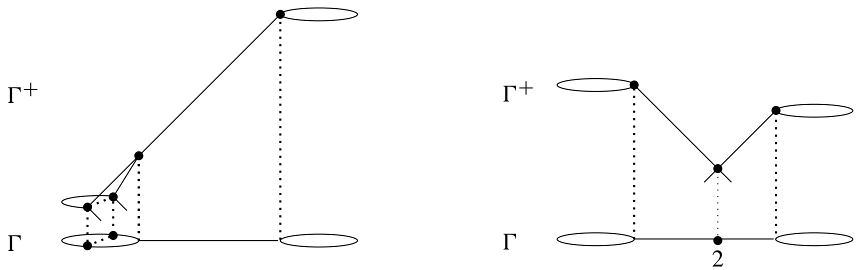

Fig. 2. Realizability locus over the dumbbell graph.

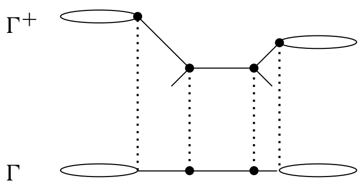

Fig. 3. A non-realizable configuration over the dumbbell graph.

Proof of Proposition 4.2, alternative proof that the image of trop_Ω *belongs to* $|K_\Gamma|$. We need to show that for any given graph Γ there exists a rational function $f \in \mathrm{Rat}(\Gamma)$ with $K_\Gamma + \mathrm{div}(f) \geq 0$ such that $\Gamma^+(f)$ satisfies conditions (i) and (ii) of the preceding theorem. We prove this by induction on the genus. Since adding marked points and increasing the vertex genus can only improve the situation concerning inconvenient vertices, it suffices to treat the case that all vertex genera $h(v)$ are zero.

For $g = 2$, there are two cases. For the graph with three nodes joining the two vertices (and in general, for any graph Γ without separating edges), the canonical divisor K_Γ is in the image of trop_Ω. For the dumbbell graph, we take a function f that is constant on the edges and has a global minimum on the separating edge (see Figure 2, right).

In the induction step, we consider a graph Γ of genus g and remove a non-separating edge e. Let $\Delta = \Gamma - e$ be the resulting graph. There are two cases to consider. First, suppose that the two ends of e are different nodes in Δ. Then Δ is semistable and we start with the f_0 given by induction on the stable graph equivalent to Δ. We complete this to a function f on Γ having slope ≥ -1 on each half-edge of e. (This is possible for all values of f_0 at the ends of e.) Together with the induction hypothesis this condition implies $K_\Gamma + \mathrm{div}(f) \geq 0$. Neither a horizontal separating edge nor a trivalent vertex with negative decorations has been added, hence conditions (i) and (ii) continue to hold.

Second, suppose that e is a cycle adjacent to some vertex v.

If $\Delta = \Gamma - e$ is semistable, we simply declare f to be constant on e. Otherwise, there is a separating edge e_s ending at v and $\Delta \setminus e_s$ is semistable. In this case we take f_0 from $\Delta \setminus e_s$ by induction and complete it to f constant on e and with slopes -1 on the two half-edges of e_s (i.e. $\mathrm{div}(f)$ contains twice the midpoint of e_s). Then conditions (i), (ii) and $K_\Gamma + \mathrm{div}(f) \geq 0$ follow from the construction. $\square$

Example 6.5. To give a more involved example we discuss the realizability locus over the complete graph K_4. We first claim that there are five types of maximal-dimensional cones, as given in Figures 4–6.

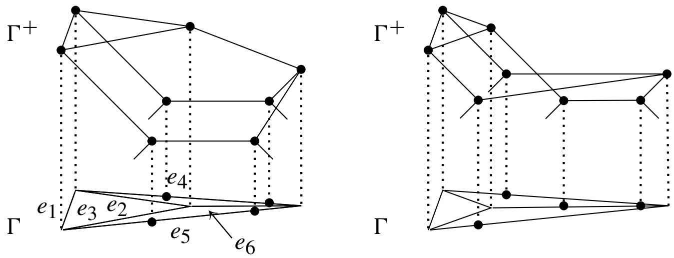

Fig. 4. Realizable configurations of maximal dimension on K_4: 3-cycles on top level, edges with two points.

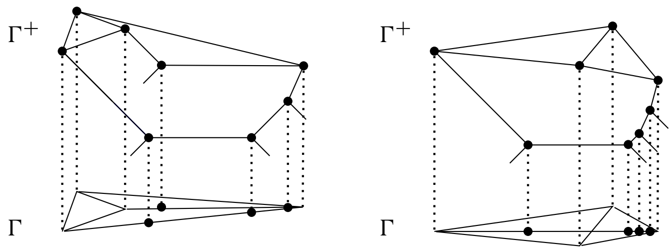

Fig. 5. Realizable configurations of maximal dimension on K_4: 3-cycles on top level, more than two points on some edge.

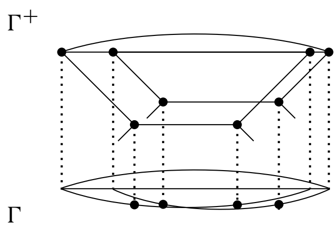

Fig. 6. Realizable configurations of maximal dimension on K_4: 4-cycle on top level.

To prove this claim, we establish some notation. Suppose that Γ^+ is the enhanced level graph corresponding to a canonical divisor in the realizability locus. Let $v_1, \ldots, v_4$ be the four vertices of the original K_4 and let $w_1, \ldots, w_n$ with $n \leq 4$ be the remaining vertices of Γ, each of them having at least one leg. Consider the vertices on the top level of Γ^+. A vertex w_i cannot be on the top level, since its two non-leg half-edges have $k = -1$, since it has at least one leg and since the sum of decorations is equal to -2. Suppose one of the v_i on the top level is decorated with a leg. This requires v_i to have (at least) three horizontal edges, otherwise the sum of decorations cannot be -2. Together with the previous argument this implies that all v_i lie on the top level, and that each of them is decorated with a single leg. By Lemma 6.8 below the cone with this configuration has dimension 6, strictly less than the maximal dimension 8.

Consequently, the top level consists of vertices v_i without legs, hence each of them is adjacent to precisely two horizontal edges. Thus the subgraph on the top level is a simple cycle. In K_4 the length of the cycle might be 3 or 4. In the case of a 4-cycle on top, the case of a single w_i with just one leg on one of the edges is ruled out by the sum of decorations being equal to -2. The configuration in Figure 6 remains and attains the maximal dimension.

Suppose the top level is a 3-cycle consisting of v_1, v_2, v_3 and that v_4 is on a lower level. Consider the edges e_i for $i = 4, 5, 6$ joining v_{i-3} to v_4. For each partition $(4, 0, 0)$, $(3, 1, 0)$, $(2, 2, 0)$ and $(2, 1, 1)$ of the four vertices w_i on these three edges there is a unique solution to the enhancement conditions, leading to the graphs in Figures 4 and 5, all of maximal dimension. Graphs with less than four w_i are degenerations thereof, and hence of strictly smaller dimension.

Note that for a given tropical curve Γ with underlying graph K_4 the preimage $\pi^{-1}([\Gamma])$ may not meet all of these maximal cones. For example the graph in Figure 6 is possible for any edge lengths, with the canonical divisor supported on an arbitrary pair of disjoint edges. However, the graph pictured in Figure 4 (left) with the canonical divisor supported on a pair of adjacent edges is possible if and only if $|e_6| < |e_4|, |e_6| < |e_5|$.

The realizability locus is connected in codimension 1 over the closure of the K_4-cone. To see this, note that contracting one of the horizontal edges on the top level 4-cycle in Figure 6 and reopening it as a vertical edge connects this cone to the cone in Figure 4 (left). This cone is connected to the cone in Figure 5 (left) by pushing one of the w_i adjacent to v_4 into v_4. This cone is connected to the cone in Figure 5 (right) by pushing the isolated w_i on e_6 through v_4 onto the edge e_5. Finally, the cone in Figure 5 (left) is also connected to the cone in Figure 4 (right) by pushing the vertex w_i on e_5 adjacent to v_4 through v_4.

It is an interesting combinatorial question whether in general the maximal cones of the realizability locus are connected in codimension 1.

6.2. Dimensions

The fundamental Theorem of Bieri–Groves [BG84, Theorem A] (see also [EKL06, Theorem 2.2.3]) shows that, given a closed subvariety of a split algebraic torus, its tropicalization admits the structure of a polyhedral complex of the same dimension. In our situation, the tropical Hodge bundle does not admit a natural embedding into a toric variety, but rather a toroidal embedding in the sense of [KK+73], the compactification $\overline{\mathcal{D}iv}_{g,2g-2}$ of $\mathcal{D}iv_{g,2g-2}$ over $\overline{\mathcal{M}}_g$. In this situation a weaker version of the Bieri–Groves Theorem holds (see [Uli15a, Theorem 1.1]) and we only know that the realizability locus (i.e. the tropicalization of $\mathbb{P}\Omega\mathcal{M}_g$) is a generalized cone complex of dimension $\leq 4g-4$. This result technically only applies when the boundary has no self-intersection, but the arguments immediately generalize to our situation. Our methods allow us to prove the following much stronger statement.

Theorem 6.6. *The realizability locus $\mathbb{P}\mathcal{R}_\Omega$ admits the structure of a generalized cone complex, all of whose maximal cones have dimension $4g-4$. The fiber in $\mathbb{P}\mathcal{R}_\Omega$ over a maximal-dimensional cone σ_G in M_g^{trop} (i.e. for a trivalent graph G with all vertex-weights $h(v)$ zero) is a generalized cone complex, all whose maximal cones have relative dimension $g-1$.*

Recall that in Figure 1 we have seen that the realizability locus is not a subcomplex of $\mathcal{D}iv^{\mathrm{trop}}_{g,2g-2}$.

Example 6.7. We revisit Example 6.4. The dumbbell graph is one of the two trivalent genus 2 graphs. For any edge lengths assigned to the dumbbell, the fiber of $\mathbb{P}\Omega\mathcal{M}_g^{\mathrm{trop}}$ over the corresponding tropical curve is the folded square with two ends pictured in Figure 7. The realizability locus corresponds to the thickened line segments, drawn horizontally.

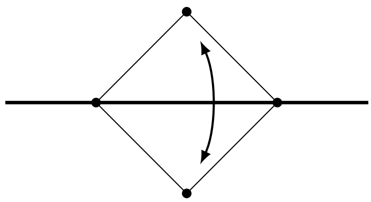

Fig. 7. The simplices over the dumbbell graph.

Notice that the canonical divisor K_Γ (which corresponds to the third corner in the triangle) is not in the realizability locus.

The dimension estimates are based on the following lemma. The contraction procedure in the lemma stems from the fact that the length information encoded in those genus zero nodes is not recorded when passing to the associated tropical curves with divisor. Note that for all enhanced level graphs that appear in Theorem 6.3, i.e. those resulting from Lemma 6.1, we have $\Gamma^+ = \Gamma_0^+$ in the following statement.

Lemma 6.8. *For every level graph Γ^+ let Γ_0^+ be the graph obtained by successively contracting edges in Γ^+ that have an $(n+1)$-valent genus zero node with $n \geq 1$ marked points at one of its ends. The dimension of a cone $\sigma(\Gamma^+)$ in the realizability locus with associated level graph Γ^+ is 1 less than the number of levels of $\mathrm{Lev}(\Gamma_0^+)$ plus the number of horizontal edges $E_H(\Gamma_0^+)$, i.e.*

$$\dim(\sigma(\Gamma^+)) = |\mathrm{Lev}(\Gamma_0^+)| - 1 + |E_H(\Gamma_0^+)|\,.$$

Proof. Assign a real number $d_i \leq 0$ ('depth') to each level $i \in \mathrm{Lev}(G, \ell)$ in such a way that $d_0 = 0$ and $d_i < d_j$ if $i < j$. Then endow any edge e joining the vertices v_1 and v_2 with $\ell(v_1) > \ell(v_2)$ with length $(d_{\ell(v_1)} - d_{\ell(v_2)})/(k(e^+) + 1)$ and endow horizontal edges with arbitrary lengths. By construction this tropical curve admits a unique continuous function f (up to addition of a global constant) that is linear of slope zero on horizontal edges and linear of slope $-k(e^+) - 1$ on each edge (as viewed from the top end). This implies that $\dim(\Gamma^+) \geq |\mathrm{Lev}(\Gamma_0^+)| - 1 + |E_H(\Gamma_0^+)|$.

On the other hand, every rational function f on a tropical curve with enhanced level graph Γ_0^+ determines uniquely a collection of real numbers d_i with $d_0 = 0$ and $d_{\ell(v_1)} - d_{\ell(v_2)} = |e|s(e)$ whenever $\ell(v_1) > \ell(v_2)$. This implies the converse estimate. □

Proof of Theorem 6.6. To prove the upper bound $4g - 4$, we compare with the complex dimension of the moduli space of twisted differentials (denoted by $\mathfrak{M}^{\mathrm{ab}}(\overline{\Gamma})$ in [BC+18a]) compatible with a level graph $\overline{\Gamma}$. (The dimension does not depend on the enhancement.) Each level contributes at least 1 to the dimension of $\mathfrak{M}^{\mathrm{ab}}(\overline{\Gamma})$, namely by rescaling the differentials on that level by a scalar, the dimension of $\mathfrak{M}^{\mathrm{ab}}(\overline{\Gamma})$ is the sum of the dimensions of the spaces of twisted differentials on each level. Consequently, the maximal dimension is bounded above by the number of horizontal edges plus $\dim_{\mathbb{C}} \mathfrak{M}^{\mathrm{ab}}(\overline{\Gamma}) - 1$. This sum is computed in [BC+18a, Theorem 6.1] to be equal to $\dim_{\mathbb{C}} \Omega\mathcal{M}_g(\mu) - 1$, where μ is the type of the twisted differential. This quantity is maximized for the principal stratum $\mu = (1, \ldots, 1)$ and gives $\dim_{\mathbb{C}} \Omega\mathcal{M}_g(\mu) - 1 = 2g - 2 + |\mu| = 4g - 4$ by Theorem 5.3 and thus the claimed upper bound.

To show that this upper bound is always attained we have to split vertices whose contribution to $\mathfrak{M}^{\mathrm{ab}}(\overline{\Gamma})$ is greater than 1. The claim follows from the more precise statement in the subsequent proposition. □

Proposition 6.9. *Maximal-dimensional cones of the realizability locus correspond precisely to the enhanced level graphs Γ^+ with the following properties.*

- (i) *All the vertices have vertex genus zero.*
- (ii) *Each vertex is either*
  - (ii.1) *n-valent ($n \geq 3$) with precisely two edges which are legs or edges to a lower level, or*
  - (ii.2) *n-valent ($n \geq 3$) with precisely one edge which is a leg or an edge to a lower level.*
- (iii) *Each level L contains either*
  - (iii.1) *precisely one vertex as in* (ii.1)*; all the edges of this vertex to a higher level disconnect the subgraph $\Gamma^+_{\geq L}$, or*
  - (iii.2) *only vertices as in* (ii.2)*; at each of these nodes v moreover $|v| - 2$ edges disconnect the subgraph $\Gamma^+_{\geq L}$ while the remaining edges of the nodes on level ℓ together with the connected components of the subgraph $\Gamma^+_{>L}$ form a simple cycle.*

In (iii.2) the valence $|v|$ refers to the valence of the subgraph $\Gamma^+_{\geq L}$, i.e. an edge to a lower level does not contribute. In the proof we will see that the conditions in the theorem can be explained using period coordinates (see Section 5.4).

Proof of Proposition 6.9. In order to show that these cones are maximal we need to show that the contribution of each level to the dimension of $\mathfrak{M}^{\mathrm{ab}}(\overline{\Gamma})$ is at most 1. Then we conclude using [BC+18a, Theorem 6.1], since then the number of levels has to be at least $4g-4-|E_H(\overline{\Gamma})|$.

We start by discussing the dimension contribution for the cones in the statement of the proposition. Recall from Theorem 5.3 that the space of differentials corresponding to an n-valent vertex of genus zero has dimension $n-2$.

If there are two marked points (or edges to a lower level) as in (ii.1), this space is parametrized by the relative period between the two marked points and $n-1$ residues. Moreover the condition in (iii.1) and the GRC imply that all the residues are zero and the resulting contribution of that level L is of dimension 1.

If there is only one marked point (or edges to a lower level), the space of differentials is parametrized by the $n-1$ residues with one constraint given by the residue theorem. The condition in (iii.2) and the GRC imply again that $n-3$ residues vanish. Hence each node contributes again individually 1 to the complex dimension of the space of differentials on that level. Moreover, the cycle constraint in (iii.2) implies that this residue is the same for each vertex on the given level. Consequently, the total contribution of that level to the space of twisted differentials is 1, as claimed.

To show that the cones listed in the proposition are the only cones of maximal dimension, we show that we can split the vertices in an enhanced level graph until the conditions of the proposition are met, while maintaining conditions (i) and (ii) of Theorem 6.3(i, ii). E.g. while some vertex genus is positive, we apply the splitting of Figure 8 where $a_i \geq 0$ and where $b_i \leq -1$.

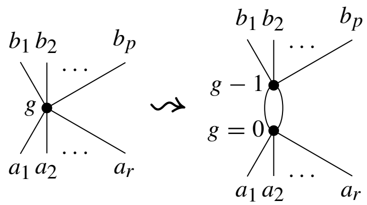

Fig. 8. Splitting positive genus.

We may thus assume from now on that all vertex genera are zero. In order to show that the nodes on a given level i can be split until their contribution to the dimension of $\mathfrak{M}^{\mathrm{ab}}(\overline{\Gamma})$ is 1 (in the sense at the beginning of the proof) we could argue combinatorially, but arguing geometrically as follows seems more enlightening. Consider the space of twisted differentials $\eta = \{\eta_v\}$ compatible with the level graph currently under consideration. Suppose that the subspace of twisted differentials with all η_v fixed except for those with $\ell(v) = i$ has dimension greater than 1. To put it differently, we assume that the projectivization of this subspace has positive dimension. This subspace is cut out inside $\mathbb{P}\Omega\mathcal{M}_g(\mu)$ by a collection of residue conditions. Since $\mathbb{P}\Omega\mathcal{M}_g(\mu)$ does not contain a projective curve by Theorem 5.4, a subspace defined by residue conditions does not contain such a curve either. Consequently, there is some way to degenerate the meromorphic differential, thus increasing either the number of levels or the number of horizontal

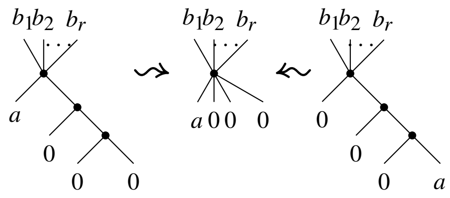

Fig. 9. Rearranging trees of marked points.

edges. This also increases the dimension of the corresponding cone in the realizability locus and we can repeat the process until each cone has dimension 1, with the caveat given in Lemma 6.8 that trees of marked points are contracted. If we replace each such tree as in Figure 9 then this is also a degeneration of the graph where all trees of marked points are contracted, the number of levels and horizontal edges is the same, and this graph satisfies the conditions of Theorem 6.3 if the graph prior to the replacement did. This concludes the proof of the existence of a splitting procedure. □

6.3. *The realizability locus for strata of abelian differentials*

Let μ be a partition of d. We say that an effective divisor D of degree d on a tropical curve Γ has *type* μ if the multiplicities at its support define the partition μ. Notice in particular that in complete analogy with the situation for the projective algebraic Hodge bundle $\mathbb{P}\Omega\mathcal{M}_g$, the tropical Hodge bundle $\mathbb{P}\Omega\mathcal{M}_g^{\mathrm{trop}}$ admits a stratification by strata $\mathbb{P}\Omega\mathcal{M}_g^{\mathrm{trop}}(\mu)$ that are indexed by partitions μ of $2g-2$.

Theorem 6.3 also contains a characterization of the *realizability locus* $\mathbb{P}\mathcal{R}_\Omega(\mu)$ *of the stratum of type* μ, defined as the image of the restriction of the tropicalization map to the corresponding stratum $\mathbb{P}\Omega\mathcal{M}_g(\mu)^{\mathrm{an}}$ of abelian differentials. In fact, the proof of our main theorem applies verbatim to give the following criterion.

Proposition 6.10. *An element $D = K_\Gamma + \mathrm{div}(f)$ in the tropical canonical linear series lies in $\mathbb{P}\mathcal{R}_\Omega(\mu)$ if and only if D is a divisor of type μ and for the enhanced level graph $\Gamma^+(f)$ conditions* (i) *and* (ii) *of Theorem 6.3 hold.*

Example 6.11. We give for example the realizability locus $\mathbb{P}\mathcal{R}_\Omega(2)$ if the underlying graph is the dumbbell graph, i.e. a subset of Example 6.4. Restricted to this graph, the tropical Hodge bundle is disconnected and consists of (isolated) double zeros at the midpoint of either of the dumbbell cycles and of a one-dimensional component with a double zero on the central edge of the dumbbell. The midpoints of the dumbbell cycles satisfy criteria (i) and (ii). A point of multiplicity 2 in the interior of the central edge (as in Figure 2, right) does not satisfy criterion (i), since a vertex with one zero (of order 2) and two poles (of order 2) is inconvenient. However, if the double point is located on the vertices of the dumbbell, the vertices are no longer inconvenient and the criteria are satisfied.

In conclusion, the realizability locus $\mathbb{P}\mathcal{R}_\Omega(2)$ on the dumbbell graph consists of four 'Weierstrass' points as in Figure 10.

Fig. 10. The realizability locus $\mathbb{P}\mathcal{R}_\Omega(2)$.

6.4. Algorithmic aspects

Our main theorem can be turned into an algorithm to compute the simplicial structure of the realizability locus.

(i) For each genus g construct the finitely many abstract graphs $G = (E, V, L)$ with a genus function $h : V \to \mathbb{Z}_{>0}$ of genus g (in the sense of (1)) and with $|L| = 2g - 2$ that are stable.

(ii) There are a finite number of partial orders on G such that any two vertices joined by an edge are comparable (equality permitted). For each of those partial orders there are a finite number of enhancements k with the properties that $k(e^\pm) = -1$ for both half-edges of e joining v_1 and v_2 with $v_1 \asymp v_2$ and such that whenever there is an edge e joining v_1 and v_2 with $v_1 \succ v_2$ then $k(e_1) \geq 0$, where e_1 is the half-edge of e adjacent to v_1.

To see this, we assign $k(e^\pm) = -1$ to all edges with $v_1 \asymp v_2$ and argue inductively top-down: for each vertex v such that all the upward pointing half-edges have already been assigned an enhancement, there are a finite number of possibilities to assign a non-negative enhancement to each of the downward pointing half-edges e^+ such that the genus formula (5) holds. We complete this enhancement on each of the complementary half-edges e^- using the condition $k(e^+) + k(e^-) = -2$ and proceed to another vertex.

(iii) For each of the partial orders there are a finite number of full orders that refine the partial order. We assume for notational convenience that the full order is given by the level function ℓ.

(iv) For each of the horizontal edges e (i.e. both half-edges are decorated with $k(e^\pm) = -1$) check if e disconnects the graph $G_{\geq \ell(e)}$ and discard the graph if this is the case. Here $\ell(e) := \ell(v)$ for any of the two vertices adjacent to e.

(v) Using the enhancement we can determine the set of inconvenient vertices $I \subset V$. For each vertex $v \in I$ check if v disconnects the graph $G_{\geq \ell(v)}$ and discard the graph if this is the case.

(vi) The realizability locus consists of a cone $\sigma = \sigma_{(G,h,k,\ell)}$ for each tuple (G, h, k, ℓ) not discarded. The cone $\sigma_{(G,h,k,\ell)}$ parametrizes the following tropical curves. Assign as in the proof of Lemma 6.8 a real number $d_i \leq 0$ ('depth') to each level $i \in \mathrm{Lev}(G, \ell)$ in such a way that $d_0 = 0$ and $d_i < d_j$ if $i < j$. Then endow any edge e joining the vertices v_1 and v_2 with $\ell(v_1) > \ell(v_2)$ with length

$$|e| = (d_{\ell(v_1)} - d_{\ell(v_2)})/(k(e^+) + 1)\,.$$

This algorithm is effective but not efficient, since most of the enhanced level graphs that are built in the process will be discarded in the end.

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