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Rigidity of cones with bounded Ricci curvature

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Abstract. We show that the only metric measure space with the structure of an N-cone and with two-sided synthetic Ricci bounds is the Euclidean space $\mathbb{R}^{N+1}$ for N integer. This is based on a novel notion of Ricci curvature upper bounds for metric measure spaces given in terms of the short time asymptotic of the heat kernel in the L^2-transport distance. Moreover, we establish rigidity results of independent interest which characterize the N-dimensional standard sphere $\mathbb{S}^N$ as the unique minimizer e.g. of

$$\int_X \int_X \cos(d(x,y))\, m(\mathrm{d}y)\, m(\mathrm{d}x)$$

among all metric measure spaces with dimension bounded above by N and Ricci curvature bounded below by $N-1$.

Keywords. Metric measure space, synthetic Ricci bounds, rigidity

1. Introduction

The theory of synthetic curvature-dimension bounds for non-smooth space has been very active and successful in the last decades. It was initiated in the works of Bakry–Émery [5] from the point of view of abstract Markov semigroups and Lott–Villani [18] and Sturm [22] from the point of view of optimal transport and metric measure spaces. Generalized lower bounds on the Ricci curvature and upper bounds on the dimension lead to a large number of geometric and functional inequalities and powerful control on the underlying diffusion process. By now, many precise analytic and geometric results for metric measure spaces under curvature-dimension bounds have been established such as Li–Yau type estimates for the heat semigroup [11] and splitting and rigidity results [12, 16] and a clear picture of the fine structure of such spaces is emerging [19].

Recently, significant progress has been made in developing more detailed synthetic control on the Ricci curvature in a non-smooth context. Gigli [13] and Han [15] provide a definition of the full Ricci tensor on metric measure spaces, building upon a similar

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construction in the context of Γ-calculus by Sturm [23]. Naber [20] characterized two-sided bounds on the Ricci curvature in terms of functional inequalities in the path space; see also recent work of Cheng–Thalmaier [9] and of Wu [26].

A drawback of the previous approaches to detailed control on Ricci is that they do not see curvature concentrated in singular sets such as the tip of a cone. One goal of the present article is to analyze a different concept of synthetic upper Ricci bounds introduced recently by the second author [24] and to exhibit a remarkable rigidity: the only metric measure spaces with cone structure and with Ricci curvature bounded above and below are Euclidean spaces $\mathbb{R}^N$.

We will work in the setting of $\mathrm{RCD}^*(K', N')$ metric measure spaces (see Section 2 for definitions and references). In this setting an equivalent definition of lower Ricci bound K is the contraction estimate

$$W_2(\hat{P}_t\mu, \hat{P}_t\nu) \le e^{-Kt} W_2(\mu, \nu)$$

for the dual heat flow $\hat{P}_t$ in L^2-Wasserstein distance. The central object in [24] to define upper Ricci bounds is a reverse estimate asymptotically for short times. More precisely, for a mm-space (X, d, m) and $x, y \in X$ consider

$$\vartheta^+(x, y) := -\liminf_{t\to 0} \frac{1}{t} \log\left(\frac{W_2(\hat{P}_t\delta_x, \hat{P}_t\delta_y)}{d(x, y)}\right), \qquad \vartheta^*(x) := \limsup_{y,z\to x} \vartheta^+(y, z).$$

For smooth Riemannian mainfolds an upper bound $\mathrm{Ric} \le K$ is equivalent to requiring $\vartheta^*(x) \le K$ for all x. For an RCD space (X, d, m) we take the latter as a definition of $\mathrm{Ric} \le K$ (see Section 2.2 for more details).

Our first main result is the following rigidity theorem for cones (cf. Definition 2.6).

Theorem 1.1. *Let (Y, d_Y, m_Y) be a mm-space satisfying the curvature-dimension condition* $\mathrm{RCD}^*(K', N')$ *for some $K' \in R$ and $N' \in [0, \infty)$ and assume that it is the N-cone over a mm-space (X, d_X, m_X) for some $N \ge 1$, i.e. $Y = \mathrm{Con}_0^N(X)$. Then either*

(i) *$\vartheta^*(o) = +\infty$ for o the tip of the cone, or*
(ii) *N is an integer and (Y, d_Y, m_Y) is isomorphic to Euclidean space $\mathbb{R}^{N+1}$ with the Euclidean distance and a multiple of the Lebesgue measure.*

In particular, (up to isomorphism) the only N-cone with bounded Ricci curvature among all mm-spaces is the $N+1$-dimensional Euclidean space for N an integer.

Remark 1.2. If Y is not Euclidean space, we still might have special directions in which Ricci curvature is bounded above. For instance, if N is an integer and $X = \mathrm{Con}_1^{N-1}(X')$ with $X' \ne \mathbb{S}^{N-1}$, then Proposition 4.1 will show that $\vartheta^+(o, y) = 0$ for $y = (r, x_0)$ with x_0 being one of the poles of the suspension X.

An important ingredient to establish the rigidity of cones will be a novel class of rigidity results characterizing the standard sphere $\mathbb{S}^N$ which will be applied to characterize the base of the cone. They are of independent interest and form the second goal of this article.

Let $f : [0, \pi] \to \mathbb{R}$ be continuous and strictly increasing and put, for a mm-space (X, d, m) with $m(X) < \infty$ and $\mathrm{diam}(X) \le \pi$,

$$M_f(X) := \frac{1}{m(X)^2} \int_X \int_X f(d(x, y))\, m(\mathrm{d}x)\, m(\mathrm{d}y),$$
$$M^*_{f,N} := \int_0^\pi f(r) \sin(r)^{N-1}\, \mathrm{d}r \Big/ \int_0^\pi \sin(r)^{N-1}\, \mathrm{d}r.$$

Theorem 1.3. *Let* (X, d, m) *be an* $\mathrm{RCD}^*(N-1, N)$ *space with* $N \ge 1$, $\mathrm{diam}(X) \le \pi$. *Then* $M_f(X) \le M^*_{f,N}$. *Moreover, the following are equivalent:*

(i) $M_f(X) = M^*_{f,N}$,
(ii) *N is an integer and X is isomorphic to the sphere $\mathbb{S}^N$ with the round metric and a multiple of the volume measure.*

In particular, we see that for $N \in \mathbb{N}$ the standard sphere $\mathbb{S}^N$ is the unique maximizer of the expected distance between points and of the variance among $\mathrm{RCD}^*(N-1, N)$ spaces, choosing $f(r) = r$ or $f(r) = r^2$ respectively. We also establish a corresponding almost rigidity theorem (see Theorem 3.1). It is easy to see that the extremum of M_f among $\mathrm{RCD}^*(N-1, N)$ spaces is attained also for non-integer N. It would be an interesting question to characterize the extremizers in this case.

The proof of Theorem 1.3 will rely on the maximal diameter theorem obtained by Ketterer [16], which in turn stems from Gigli's non-smooth splitting theorem [12]. In fact, we will see that (i) will imply that every point in X will have a partner at the maximal distance π. Also, the other known rigidity results for $\mathrm{RCD}^*(K, N)$ spaces with $K > 0$, namely Ketterer's non-smooth Obata theorem [17] for spaces with extremal spectral gap and the rigidity of spaces saturating the Lévy–Gromov isoperimetric inequality [7], are based on the maximal diameter theorem.

An analogous statement (with $M_f(X) \ge M^*_{f,N}$ in place of $M_f(X) \le M^*_{f,N}$) holds for strictly decreasing f. Of particular interest is the case $f = \cos$, which leads to $M^*_{\cos,N} = 0$.

Corollary 1.4. *Let* (X, d, m) *be an* $\mathrm{RCD}^*(N-1, N)$ *space with* $N \ge 1$, $\mathrm{diam}(X) \le \pi$. *Then the following are equivalent:*

(i) $\int_X \int_X \cos(d(x, y))\, m(\mathrm{d}x)\, m(\mathrm{d}y) = 0$.
(ii) *N is an integer and X is isomorphic to the sphere $\mathbb{S}^N$ with the round metric and a multiple of the volume measure.*

Note the condition $\mathrm{diam}(X) \le \pi$ is only required in the case $N = 1$. For $N > 1$, it already follows from the $\mathrm{RCD}^*(N-1, N)$ condition.

In order to obtain Theorem 1.1 from this corollary, note that the distance on the cone Y is built from the distance on X via the law of cosines. We will show that as long as

$$a := \int_X \cos(d(x, y))\, m(\mathrm{d}y) > 0$$

for some point $x \in X$, we have for $p = (r, x)$ in the cone Y the estimate

$$W_2(\hat{P}_t\delta_o, \hat{P}_t\delta_p)^2 \leq d(o, p)^2 - ca\sqrt{t} + O(t)$$

for some constant $c > 0$, which implies that $\vartheta(o, p) = +\infty$.

Organization

The article is organized as follows. In Section 2 we recall definitions and results concerning synthetic curvature-dimension bounds for metric measure spaces, as well as the notion of upper bounds on the Ricci curvature considered here. The proof of Theorem 1.3 will be given in Section 3 together with the corresponding almost rigidity statements. In Section 4 we give the proof of Theorem 1.1.

2. Preliminaries

2.1. Synthetic Ricci bounds for metric measure spaces

We briefly recall the main definitions and results concerning synthetic curvature-dimension bounds for metric measure spaces that will be used in what follows.

A *metric measure space* (*mm-space* for short) is a triple (X, d, m) where (X, d) is a complete and seperable metric space and m is a locally finite Borel measure on X. In addition, we will always assume the integrability condition $\int_X \exp(-cd(x_0, x)^2)\, \mathrm{d}m(x) < \infty$ for some $c > 0$ and $x_0 \in X$. We denote by $\mathcal{P}_2(X)$ the space of Borel probability measures on X with finite second moment and by W_2 the L^2-Kantorovich–Wasserstein distance.

The *Boltzmann entropy* of $\mu \in \mathcal{P}(X)$ is defined by $\mathrm{Ent}(\mu) = \int \rho \log \rho \, \mathrm{d}m$ provided $\mu = \rho m$ is absolutely continuous with respect to m and $\int \rho(\log \rho)_+ \, \mathrm{d}m < \infty$; otherwise $\mathrm{Ent}(\mu) = +\infty$. The *Cheeger energy* of $f \in L^2(X, m)$ is defined by

$$\mathrm{Ch}(f) = \liminf_{\substack{g \to f \text{ in } L^2(X,m) \\ g \in \mathrm{Lip}(X,d)}} \frac{1}{2} \int |\nabla g|^2 \, \mathrm{d}m,$$

where $|\nabla g|$ denotes the local Lipschitz constant. A mm-space is called *infinitesimally Hilbertian* if Ch is quadratic. In this case, Ch gives rise to a strongly local Dirichlet form. The associated generator Δ is called the *Laplacian* and the associated Markov semigroup $(P_t)_{t\geq 0}$ on $L^2(X, m)$ is called the *heat flow* on (X, d, m) (see [2] for more details).

For $\kappa \in \mathbb{R}$ and $\theta \geq 0$ define the functions

$$\mathfrak{s}_\kappa(\theta) = \begin{cases} \frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa}\,\theta), & \kappa > 0, \\ \theta, & \kappa = 0, \\ \frac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa}\,\theta), & \kappa < 0, \end{cases}$$

and $\mathfrak{c}_\kappa(\theta) = \frac{d}{d\theta}\mathfrak{s}_\kappa(\theta)$. Moreover, for $t \in [0, 1]$ define the distortion coefficients

$$\sigma_\kappa^{(t)}(\theta) = \begin{cases} \frac{\mathfrak{s}_\kappa(t\theta)}{\mathfrak{s}_\kappa(\theta)}, & \kappa\theta^2 \neq 0 \text{ and } \kappa\theta^2 < \pi^2, \\ t, & \kappa\theta^2 = 0, \\ +\infty, & \kappa\theta^2 \geq \pi^2. \end{cases}$$

Definition 2.1. (i) A metric measure space satisfies the condition $\mathrm{CD}^*(K, N)$ with $K \in \mathbb{R}$ and $N \in [1, \infty)$ if for each pair $\mu_0 = \rho_0 m$ and $\mu_1 = \rho_1 m$ in $\mathcal{P}_2(X)$ there exists an optimal coupling q of μ_0, μ_1 and a geodesic $\mu_t = \rho_t m$ connecting them such that

$$\int \rho_t^{-1/N'} \rho_t \,\mathrm{d}m \geq \int [\sigma_{K/N'}^{(1-t)}(d(x_0, x_1))\rho_0^{-1/N'} + \sigma_{K/N'}^{(t)}(d(x_0, x_1))\rho_1^{-1/N'}] \,\mathrm{d}q(x_0, x_1)$$

for all $t \in [0, 1]$ and all $N' \geq N$ (see [4]).

(ii) A mm-space satisfies the condition $\mathrm{RCD}^*(K, N)$ for $K \in \mathbb{R}$ and $N \in [1, \infty)$ if it is infinitesimally Hilbertian and satisfies $\mathrm{CD}^*(K, N)$.

It has been shown in [10] that the $\mathrm{RCD}^*(K, N)$ condition can be formulated equivalently in terms of evolution variational inequalities. In particular, for each $\mu_0 \in \mathcal{P}_2(X)$ there exists a (unique) EVI gradient flow emanating from μ_0, denoted by $\hat{P}_t\mu_0$ and called the *heat flow* acting on measures. For $\mu_0 = fm$ with $f \in L^2(X, m)$ it coincides with the heat flow [2], i.e. $\hat{P}_t(fm) = (P_t f)m$. It has been shown ([3, Thm. 6.1], [1, Thm. 7.1]) that the RCD condition entails several regularization properties for P_t. For instance, $P_t f(x) = \int f \,\mathrm{d}\hat{P}_t\delta_x$ for m-a.e. x for every $f \in L^2(X, m)$. This representation of $P_t f$ has the strong Feller property, that is, $x \mapsto \int f \,\mathrm{d}\hat{P}_t\delta_x$ is bounded and continuous for any bounded $f \in L^2(X, m)$. In particular, we have the following estimate for the quadratic variation.

Lemma 2.2. *Let X be an* $\mathrm{RCD}^*(0, N)$ *space. Then for $\mu \in \mathcal{P}_2(X)$ and all $t > 0$,*

$$W_2(\hat{P}_t\mu, \mu)^2 \leq 2Nt.$$

Proof. Choosing $K = 0$, $\nu = \mu$ and $s = 0$ (or more precisely, considering the limit $s \searrow 0$) in [10, Thm. 4.1] yields the claim. □

The $\mathrm{CD}^*(K, N)$ condition is a priori slightly weaker than the original condition $\mathrm{CD}(K, N)$ given in [22], where the coefficients $\sigma_{K/N}^{(t)}(\theta)$ are replaced by $\tau_{K/N}^{(t)}(\theta) = t^{1/N}\sigma_{K/(N-1)}^{(t)}(\theta)^{1-1/N}$. Recently, however, Cavalletti and Milman [6] succeeded in showing that $\mathrm{CD}^*(K, N)$ is in fact equivalent to $\mathrm{CD}(K, N)$ provided (X, d, m) is non-branching, which in particular will be the case if it is infinitesimally Hilbertian. Thus in turn $\mathrm{RCD}^*(K, N)$ will imply the sharp Bonnet–Myers diameter and Bishop–Gromov volume comparison estimates (see also [8, 22] for an alternative argument). Given $x_0 \in \mathrm{supp}[m]$ and $r > 0$ we denote by $v(r) := m(\bar{B}_r(x_0))$ the volume of the closed ball of radius r around x_0 and by

$$s(r) := \limsup_{\delta \to 0} \frac{1}{\delta} m(\overline{B_{r+\delta}(x_0)} \setminus B_r(x_0))$$

the volume of the corresponding sphere.

Proposition 2.3. *Assume that (X, d, m) is non-branching and satisfies $\mathrm{CD}^*(K, N)$. Then each bounded closed subset of $\mathrm{supp}[m]$ is compact and has finite volume. For each $x_0 \in \mathrm{supp}[m]$ and $0 < r \le R \le \pi\sqrt{N/(K \wedge 0)}$ we have*

$$\frac{s(r)}{s(R)} \ge \left(\frac{\mathfrak{s}_{K/(N-1)}(r)}{\mathfrak{s}_{K/(N-1)}(R)}\right)^{N-1} \quad \text{and} \quad \frac{v(r)}{v(R)} \ge \frac{\int_0^r \mathfrak{s}_{K/(N-1)}(t)^{N-1}\,\mathrm{d}t}{\int_0^R \mathfrak{s}_{K/(N-1)}(t)^{N-1}\,\mathrm{d}t}.$$

Moreover, if $K > 0$ then $\mathrm{supp}[m]$ is compact and its diameter is bounded by $\pi\sqrt{N/K}$.

2.2. Upper Ricci bounds

Here, we briefly introduce the synthetic notion of upper Ricci curvature bounds considered in this paper. For more details we refer to [24]. Let us mention that also other approaches in terms of the behaviour of the entropy along Wasserstein geodesics and their relations are discussed there.

Let (X, d, m) be an $\mathrm{RCD}^*(K', N')$ mm-space and let $\hat{P}_t$ denote the dual heat flow acting on measures. For points $x, y \in X$ we set

$$\vartheta^+(x, y) := -\liminf_{t\to 0} \frac{1}{t} \log\left(\frac{W_2(\hat{P}_t\delta_x, \hat{P}_t\delta_y)}{d(x, y)}\right), \qquad \vartheta^*(x) := \limsup_{y,z\to x} \vartheta^+(y, z).$$

It is shown in [24, Thm. 2.10] that a lower bound $\vartheta^+(x, y) \ge K$ is equivalent to the $\mathrm{RCD}^*(K, \infty)$ condition and in particular to the following Wasserstein contraction estimate: $W_2(\hat{P}_t\mu, \hat{P}_t\nu) \le e^{-Kt} W_2(\mu, \nu)$ for all $\mu, \nu \in \mathcal{P}_2(X)$ and all $t > 0$.

If $(X, d, m) = (M, d, e^{-V}\mathrm{vol})$ is a smooth weighted Riemannian manifold we have the following precise estimate on ϑ^+ in terms of the Bakry–Émery Ricci curvature $\mathrm{Ric}_f = \mathrm{Ric} + \mathrm{Hess}\, f$.

Theorem 2.4 ([24, Thm. 3.1]). *For all pairs of non-conjugate points $x, y \in M$,*

$$\mathrm{Ric}_f(\gamma) \le \vartheta^+(x, y) \le \mathrm{Ric}_f(\gamma) + \sigma(\gamma)\tan(\sqrt{\sigma(\gamma)}d(x, y)/2)^2,$$

where $\gamma = (\gamma^a)_{a\in[0,1]}$ is the (unique) constant speed geodesic connecting x and y,

$$\mathrm{Ric}_f(\gamma) = \frac{1}{d(x, y)^2}\int_0^1 \mathrm{Ric}_f(\dot{\gamma}^a, \dot{\gamma}^a)\,\mathrm{d}a,$$

and $\sigma(\gamma)$ denotes the maximal modulus of the Riemann tensor along the geodesic γ.

In particular one sees that an upper bound $\mathrm{Ric}_f \le K$ for some $K \in R$ is equivalent to the estimate $\vartheta^*(x) \le K$ for all $x \in M$. This motivates the following definition.

Definition 2.5. We say that a number $K \in \mathbb{R}$ is a *synthetic upper Ricci bound* for the mm-space (X, d, m) if

$$\vartheta^*(x) \le K \quad \text{for all } x \in X.$$

2.3. Cones and suspensions

We recall the construction of cones for metric measure spaces.

Definition 2.6. For a metric measure space (X, d_X, m_X) and $K \geq 0$, $N \geq 1$ the (K, N)-*cone* $\mathrm{Con}_K^N(X) = (C, d_C, m_C)$ over (X, d_X, m_X) is defined by

$$C = \begin{cases} [0, \pi/\sqrt{K}] \times X/(\{0, \pi/\sqrt{K}\} \times X), & K > 0, \\ [0, \infty) \times X/(\{0\} \times X), & K = 0, \end{cases}$$

with $m_C(dr, dx) = \mathfrak{s}_K(r)^N dr\, m_X(dx)$ and d_C given for $(r, x), (s, y) \in C$ by

$$d_C((r, x), (s, y)) = \begin{cases} \mathfrak{c}_K^{-1}[\mathfrak{c}_K(r)\mathfrak{c}_K(s) + K\mathfrak{s}_K(r)\mathfrak{s}_K(s)\cos(d_X(x, y) \wedge \pi)], & K > 0, \\ \sqrt{r^2 + s^2 - 2rs\cos(d_X(x, y) \wedge \pi)}, & K = 0. \end{cases}$$

We refer to the $(0, N)$-cone simply as the N-*cone* of X and to the $(1, N)$-cone as the N-*spherical suspension* of X.

Curvature-dimension bounds for cones are intimately related to curvature-dimension bounds for the base space. We recall the following result by Ketterer [16].

Theorem 2.7. *Let (X, d_X, m_X) be a metric measure space and let $K \geq 0$ and $N \geq 1$. Then the (K, N)-cone $\mathrm{Con}_K^N(X)$ satisfies $\mathrm{RCD}^*(KN, N+1)$ if and only if X satisfies $\mathrm{RCD}^*(N-1, N)$ and $\mathrm{diam}(X) \leq \pi$.*

In fact, any curvature-dimension bound on the cone is sufficient to infer bounds on the base space, as we will show here. More precisely, the following generalization holds.

Theorem 2.8. *Let (X, d_X, m_X) be a metric measure space and let $N \geq 1$. Then the following statements are equivalent:*

(i) *The (K, N)-cone $\mathrm{Con}_K^N(X)$ satisfies $\mathrm{RCD}^*(K', N')$ for some $K' \in \mathbb{R}$ and $N' \geq N+1$.*
(ii) *X satisfies $\mathrm{RCD}^*(N-1, N)$ and $\mathrm{diam}(X) \leq \pi$.*

In this case $\mathrm{Con}_K^N(X)$ satisfies $\mathrm{RCD}^*(KN, N+1)$.

A close inspection of the proof in [16] reveals that, at least in the case of the Euclidean cone $K = 0$, the arguments there already show that $\mathrm{RCD}^*(0, N')$ on the cone implies $\mathrm{RCD}^*(N-1, N)$ on the base space, although this is not explicitly stated. Since the argument is quite technical and involved we sketch the main steps for the reader's convenience and highlight the modifications. See the proof of [16, Thm. 1.2] for more details. To obtain the statement in the case $K > 0$ and under the relaxed curvature bound K' we provide additional arguments.

Proof of Theorem 2.8. We only need to treat the implication (i)⇒(ii). We proceed in three steps.

Step 1: Let us first consider the case $K = 0$ and assume that $\mathrm{Con}_0^N(X)$ satisfies $\mathrm{RCD}^*(0, N')$.

(a) Following the argument of Bacher and Sturm [4] one finds that the $\mathrm{CD}^*(0, N')$ condition for $C = \mathrm{Con}_0^N(X)$ implies that $\mathrm{diam}(X) \leq \pi$ and hence C coincides with the warped product $[0, \infty) \times_{\mathrm{id}}^N X$. Corollary 5.15 in [16] shows that X is infinitesimally Hilbertian. Proposition 5.11 and Corollary 5.12 in [16] show that the Cheeger energy of C coincides with the skew product of the Dirichlet forms on $[0, \infty)$ and X and that the intrinsic distance of the latter coincides with d_C. Moreover, with $I = [0, \infty)$ one has $C_0^\infty(I) \otimes D(\Gamma_2^X) \subset D(\Gamma_2^C)$ and $1 \otimes D_+^{b,2}(L^X) \subset D_+^{b,2}(L^C)$. Here $D_+^{b,2}(L)$ denotes the set of non-negative bounded functions ϕ in the domain of L such that $L\phi$ is bounded. Finally, [16, Thm. 4.26] implies that the Bakry–Émery condition $\mathrm{BE}(0, N')$ holds for the Dirichlet form on C.

(b) Following the proof of [16, Thm. 3.23], using the explicit expression of the Γ_2-operator on C (see [16, (27)]):

$$\begin{aligned}\Gamma_2^C(u \otimes v) &= \left((u'')^2 + \frac{N}{r^2}(u')^2\right)v^2 + \frac{1}{r^4}u^2\Gamma_2^X(v) - \frac{N-1}{r^4}u^2\Gamma^X(v)\\ &\quad + \frac{2}{r^3}uu'L^X(v)v + \left(\frac{2}{r^2}(u')^2 - \frac{4}{r^3}uu' + \frac{2}{r^4}u^2\right)\Gamma^X(v),\end{aligned}$$

choosing in particular $u(r) = r$ locally, and using the Bochner inequality with parameters $(0, N')$ in C one arrives at the following integrated estimate for $v \in D(\Gamma_2^X)$ and a test function $\phi \in D_+^{b,2}(L^X)$:

$$\begin{aligned}&\int L^X\phi\Gamma^X(v)\,\mathrm{d}m_X - \int \Gamma^X(v, L^Xv)\phi\,\mathrm{d}m_X\\ &\qquad \geq (N-1)\int \Gamma^X(v)\phi\,\mathrm{d}m_X + \frac{1}{N'}\int (L^Xv + Nv)^2\phi\,\mathrm{d}m_X\\ &\qquad\quad - \int \phi(v^2N + 2vL^Xv)\,\mathrm{d}m_X\\ &\qquad = (N-1)\int \Gamma^X(v)\phi\,\mathrm{d}m_X + \frac{1}{N}\int (L^Xv + Nv)^2\phi\,\mathrm{d}m_X\\ &\qquad\quad - \int \phi(v^2N + 2vL^Xv)\,\mathrm{d}m_X - \frac{N'-N}{N'N}\int (L^Xv + Nv)^2\phi\,\mathrm{d}m_X\\ &\qquad = (N-1)\int \Gamma^X(v)\phi\,\mathrm{d}m_X + \frac{1}{N}\int (L^Xv)^2\phi\,\mathrm{d}m_X\\ &\qquad\quad - \frac{N'-N}{N'N}\int (L^Xv + Nv)^2\phi\,\mathrm{d}m_X. \qquad (2.1)\end{aligned}$$

(c) It remains to get rid of the last term in (2.1) in order to conclude that X satisfies $\mathrm{RCD}^*(N-1, N)$. For a given point x_0 one could simply replace v by $v - (1/N)L^Xv(x_0)$ in order to make the last term vanish at x_0 leaving all other terms invariant. However, since the Bochner inequality is an integrated estimate, more care is needed.

One deduces from (2.1) the gradient estimate

$$|\nabla P_t^X v|^2 + \frac{c(t)}{N}\left((L^X P_t^X v)^2 - \frac{N'-N}{N} P_t^X (L^X v + Nv)^2\right) \leq P_t^X |\nabla v|^2.$$

From here one can follow the argument in [16] to deduce the usual gradient estimate without the extra term $-\frac{N'-N}{N} P_t^X (L^X v + Nv)^2$, which in turn implies the $\mathrm{RCD}^*(N-1, N)$ condition.

Step 2: Let us still consider the case $K = 0$ but assume that $\mathrm{Con}_0^N(X)$ satisfies $\mathrm{RCD}^*(K', N')$ for some $K' \in \mathbb{R}$. For $\lambda > 0$ consider the homothety Φ_λ of $\mathrm{Con}_0^N(X)$ given by $\Phi_\lambda(s, y) = (\lambda s, y)$ and note that it maps geodesics to geodesics. Consequently, also the induced map from $\mathcal{P}(\mathrm{Con}_0^N(X))$ to itself acting by push-forward maps W_2-geodesics to W_2-geodesics. Let $(\mu_t)_{t\in[0,1]}$ be a W_2-geodesic and let $\mu_t^\lambda = (\Phi_\lambda)_\# \mu_t$. By the $\mathrm{RCD}^*(K', N')$ condition the entropy is (K', N')-convex along the geodesic μ_t^λ. One finds that $\mathrm{Ent}(\mu_t^\lambda) = \mathrm{Ent}(\mu_t) - (N+1)\log\lambda$ and that $W_2(\mu_0^\lambda, \mu_1^\lambda) = \lambda W_2(\mu_0, \mu_1)$. This implies $(K'\lambda^2, N')$-convexity of the entropy along the original geodesic (μ_t). Since (μ_t) was arbitrary, letting $\lambda \to 0$ shows that $\mathrm{Con}_0^N(X)$ satisfies $\mathrm{RCD}^*(0, N')$ and we apply the first step.

Step 3: Let us finally consider the case $K > 0$ and assume that $\mathrm{Con}_K^N(X)$ satisfies $\mathrm{RCD}^*(K', N')$. The result will follow from a simple blow-up argument. Note that the pointed rescaled spaces $(\mathrm{Con}_{K/n^2}^N(X), o)$ converge in pointed measured Gromov–Hausdorff sense to the pointed Euclidean cone $(\mathrm{Con}_0^N(X), o)$ and that they satisfy $\mathrm{RCD}^*(K'/n^2, N')$. By the stability of the conditions $\mathrm{CD}^*(K, N)$ and $\mathrm{RCD}^*(K, \infty)$ under pointed measured Gromov–Hausdorff convergence (see [25, Thm. 29.25] and [14, Thm. 7.2, Prop. 3.33]) we deduce that $\mathrm{Con}_0^N(X)$ satisfies $\mathrm{RCD}^*(0, N')$. From the first part of the proof we infer that X satisfies $\mathrm{RCD}^*(N-1, N)$. □

3. Rigidity of the standard sphere

Here, we give the proof of the rigidity theorem for the standard sphere, Theorem 1.3. Then, we formulate an almost rigidity statement.

Proof of Theorem 1.3. Without restriction we can assume that $m(X) = 1$. Possibly adding a constant to f we can assume that $f(0) = 0$.

Recall that the Bishop–Gromov volume comparison (Proposition 2.3) asserts that for any $x \in X$,

$$\frac{m_X(\bar{B}_r(x))}{m_X(\bar{B}_R(x))} \geq \frac{\int_0^r \sin(t)^{N-1}\,\mathrm{d}t}{\int_0^R \sin(t)^{N-1}\,\mathrm{d}t} =: \frac{V_r^*}{V_R^*}. \tag{3.1}$$

Fix $x \in X$ and put $g(y) = f(d_X(x,y))$. Using $m_X(X) = 1$ and $D := \max\{d(x,y) : y \in X\} \leq \pi$ we can estimate

$$\begin{aligned}\int_X g(y)\, m_X(\mathrm{d}y) &= \int_0^\infty m_X(\{g \geq s\})\,\mathrm{d}s = \int_0^{f(D)} m_X(\bar{B}^c_{f^{-1}(s)}(x))\,\mathrm{d}s \\ &= \int_0^{f(D)} (1 - m_X(\bar{B}_{f^{-1}(s)}(x)))\,\mathrm{d}s \leq \int_0^{f(\pi)} \left(1 - \frac{V^*_{f^{-1}(s)}}{V^*_\pi}\right)\mathrm{d}s \\ &= \int_0^\pi f(r)\sin(r)^{N-1}\,\mathrm{d}r \Big/ \int_0^\pi \sin(r)^{N-1}\,\mathrm{d}r = M^*_{f,N}.\end{aligned}$$

Integrating over x then yields the first statement.

Let us now prove the rigidity statement. From the above argument we also see that the equality $M_f(X) = M^*_{f,N}$ implies that for m_X-a.e. point x there must exist a point x' with $d_X(x,x') = \pi$. But then this must hold even for every $x \in X$. Indeed, given x we can find a sequence of points x_n with $d(x,x_n) \to 0$ and x'_n such that $d(x_n,x'_n) = \pi$. Since X is compact, we can assume that $d(x'_n,x') \to 0$ for some $x' \in X$. But then we have $d(x,x') = \lim_n d(x_n,x'_n) = \pi$.

Now, we can show that N is an integer and that X is isomorphic to $\mathbb{S}^N$ by iteratively applying the maximal diameter theorem [16, Thm. 1.4]. Indeed, recall that the existence of points x_1, x'_1 with $d_X(x_1,x'_1) = \pi$ implies that

(a) if $N \in [1,2)$ then either X is isomorphic to the interval $[0,\pi]$ or $N = 1$ and X is isomorphic to the circle $\mathbb{S}^1$ with normalized Hausdorff measure;
(b) if $N \geq 2$, then X is isomorphic to a spherical suspension $\mathrm{Con}_1^{N-1}(Y)$ for some $\mathrm{RCD}^*(N-2, N-1)$ space (Y, d_Y, m_Y) with $\mathrm{diam}(Y) \leq \pi$ and $m(Y) = 1$.

In case (a), we must have $N = 1$ and X isomorphic to $\mathbb{S}^1$ since otherwise there would be points that do not have a partner at distance π. In case (b) we pick $x_2 \in X$ of the form $x_2 = (\pi/2, y_2)$ and x'_2 such that $d_X(x_2,x'_2) = \pi$. Then we have $x'_2 = (\pi/2, y'_2)$ and $d_Y(y_2,y'_2) = \pi$. We then repeat the previous argument inductively. After $\lfloor N \rfloor$ steps we arrive at case (a). Thus, we conclude that N is an integer and that X is the $N-1$-fold spherical suspension over $\mathbb{S}^1$, i.e. X is isomorphic to $\mathbb{S}^N$. □

We have the following almost rigidity statement.

Theorem 3.1. *For all $\epsilon > 0$ and integer $N \geq 1$ there exists $\delta > 0$ depending only on ϵ and N such that the following holds: If X is an $\mathrm{RCD}^*(N-1-\delta, N+\delta)$ space with $m(X) = 1$ and $M^*_{f,N} - M_f(X) \leq \delta$, then $d_{\mathrm{mGH}}(X, \mathbb{S}^N) \leq \epsilon$, where $\mathbb{S}^N$ is the standard N-sphere with normalized volume.*

Proof. Assume on the contrary that there is $\epsilon_0 > 0$ and a sequence X_n of normalized $\mathrm{RCD}^*(N-1-1/n, N+1/n)$ spaces with $M^*_{f,N} - M_f(X_n) \leq 1/n$ and $d_{\mathrm{mGH}}(X_n, \mathbb{S}^N) \geq \epsilon_0$ for all n. By compactness of the class of $\mathrm{RCD}^*(K,N)$ spaces, there exists a normalized $\mathrm{RCD}^*(N-1,N)$ space X such that X_n converges to X in mGH-sense along a subsequence. Obviously, we still have $d_{\mathrm{mGH}}(X, \mathbb{S}^N) > \epsilon_0$. On the other hand, since M_f is readily checked to be continuous with respect to measured Gromov–Hausdorff convergence, $M_f(X) = \lim_n M_f(X_n) = 0$. But then, by the rigidity result of Theorem 1.3, X is isomorphic to $\mathbb{S}^N$, a contradiction. $\square$

Let us give an alternative proof of Theorem 1.3 in the special case $f = \cos$ that will yield the rigidity of cones with bounded Ricci curvature. In this case $M^*_{\cos,N} = 0$. The proof is based on a slightly different induction argument, noting that the condition $M_{\cos}(X) = 0$ directly implies $M_{\cos}(Y) = 0$ if X is a suspension over Y.

Proof of Theorem 1.3 for $f = \cos$. First note that by Bishop–Gromov volume comparison we have, for any $x_0 \in X$,

$$\int_X \cos(d_X(x_0, y))\, m_X(\mathrm{d}y) \geq 0.$$

Indeed, denote by $s(r)$ the volume of the sphere of radius r around x_0 in X. Since X satisfies $\mathrm{RCD}^*(N-1, N)$, the Bishop–Gromov volume comparison (Proposition 2.3) asserts that for all $0 < r \leq R \leq \pi$,

$$\frac{s(r)}{s(R)} \geq \left(\frac{\sin(r)}{\sin(R)}\right)^{N-1}. \tag{3.2}$$

Thus we obtain

$$\begin{aligned}
\int_X \cos(d_X(x_0, y))\, m_X(\mathrm{d}y) &= \int_0^\pi \cos(r)s(r)\,\mathrm{d}r \\
&= \int_0^{\pi/2} \cos(r)s(r)\,\mathrm{d}r + \int_{\pi/2}^{\pi} \cos(r)s(r)\,\mathrm{d}r \\
&= \int_0^{\pi/2} \cos(r)[s(r) - s(\pi - r)]dr \geq 0.
\end{aligned}\tag{3.3}$$

Here we have used $\cos(r) = -\cos(\pi - r)$ and $s(r) \geq s(\pi - r)$ for $r \leq \pi/2$ by (3.2).

The previous argument also shows that in order for $M_{\cos}(X)=0$ to hold, for a.e. $x \in X$ there must exist a point $x' \in X$ at maximal distance, i.e. with $d_X(x, x') = \pi$. But then again by compactness and continuity this must hold for every x. The maximal diameter theorem [16, Thm. 1.4] again shows that one of the two cases (a), (b) above must hold and that in case (a) we must have $N = 1$ and X isomorphic to $\mathbb{S}^1$.

In case (b), from the definition of distance and measure in the spherical suspension we have

$$\begin{aligned}
0 &= \int_X \int_X \cos(d_X(x, y))\, m_X(\mathrm{d}x)\, m_X(\mathrm{d}y) \\
&= \int_0^\pi \int_Y \int_0^\pi \int_Y [\cos(r)\cos(s) + \sin(r)\sin(s)\cos(d_Y(\theta, \phi))] \\
&\qquad\qquad \times \sin(s)^{N-1}\sin(r)^{N-1}\,\mathrm{d}s\,\mathrm{d}r\, m_Y(\mathrm{d}\theta)\, m_Y(\mathrm{d}\phi) \\
&= A^2 \int_Y \int_Y \cos(d_Y(\theta, \phi))\, m_Y(\mathrm{d}\theta)\, m_Y(\mathrm{d}\phi),
\end{aligned}$$

with

$$A = \int_0^\pi \sin(s)^N \, \mathrm{d}s > 0.$$

This implies that also $M_{\cos}(Y) = 0$ holds and we repeat the previous argument inductively. After $\lfloor N \rfloor$ steps we arrive at case (a) and conclude that N is an integer and that X is the $N-1$-fold spherical suspension over $\mathbb{S}^1$, i.e. X is isomorphic to $\mathbb{S}^N$. □

4. Rigidity of cones with bounded Ricci curvature

Here, we give the proof of the rigidity for cones with bounded Ricci curvature, Theorem 1.1. A crucial ingredient in the proof will be the relation between the vanishing of the integral

$$\int_X \cos(d(x,y))\, m(\mathrm{d}y) = 0 \tag{4.1}$$

and the asymptotic behaviour as $t \to 0$ of $W_2(\hat{P}_t\delta_o, \hat{P}_t\delta_q)$ for the vertex o of the cone and the points $q = (r, x)$. We will first prove the following pointwise equivalence which is somewhat stronger than what is needed in the proof of Theorem 1.1.

Proposition 4.1. *Let* (X, d_X, m_X) *be an* $\mathrm{RCD}^*(N-1, N)$ *space with* $N \geq 1$ *and* $\mathrm{diam}(X) \leq \pi$. *Then for any* $p_0 = (r_0, x_0) \in \mathrm{Con}_0^N(X)$ *and* o *the vertex, one of the following statements holds:*

(i) $\int_X \cos(d_X(x_0, y)) m_X(\mathrm{d}y) = 0$ *and* $\vartheta^+(o, p_0) = 0$.
(ii) $\int_X \cos(d_X(x_0, y)) m_X(\mathrm{d}y) > 0$ *and* $\vartheta^+(o, p_0) = +\infty$.

Proof. **Step 1:** Fix $p_0 = (r_0, x_0) \in C = \mathrm{Con}_0^N(X)$. Recall from (3.3) that

$$a := \frac{1}{m_X(X)} \int_X \cos(d_X(x_0, y))\, m_X(\mathrm{d}y) \geq 0.$$

Step 2: Assume first that $a > 0$. We claim that as $t \to 0$ we have

$$W_2(\hat{P}_t\delta_o, \hat{P}_t\delta_{p_0})^2 \leq d_C(o, p_0)^2 - O(\sqrt{t}), \tag{4.2}$$

which immediately implies that $-\partial_t^-|_{t=0} \log W_2(\hat{P}_t\delta_o, \hat{P}_t\delta_{p_0}) = +\infty$. To this end, denote by $\nu_p^t = \hat{P}_t\delta_p$ the heat kernel measure at time t centred at $p = (r, x) \in \mathrm{Con}_0^N(X)$. Denote by $\bar{\nu}_p^t$ its marginal in the radial component. Further we consider the desintegration $\nu_{p,s}^t$ of ν_p^t after $\bar{\nu}_p^t$, i.e. $\nu_{p,s}^t$ are measures on $\mathrm{Con}_0^N(X)$ such that

$$\nu_p^t = \int_0^\infty \nu_{p,s}^t \, \mathrm{d}\bar{\nu}_p^t(s).$$

Lemma 4.2 below shows that for $p = o$, using polar coordinates we have $\nu^t_{o,s}(\mathrm{d}s', \mathrm{d}y) = \delta_s(\mathrm{d}s')\bar{m}_X(\mathrm{d}y)$ with $\bar{m}_X = m_X(X)^{-1}m_X$. Let now $\pi = \nu^t_{p_0} \otimes \nu^t_o$ be the product coupling. We obtain

$$\begin{aligned}
W_2(\hat{P}_t\delta_o, \hat{P}_t\delta_{p_0})^2 &\leq \int d_C^2 \,\mathrm{d}\pi \\
&= \int [r^2 + s^2 - 2rs\cos(d_X(x, y))]\, \nu^t_{p_0}(\mathrm{d}r, \mathrm{d}x)\, \nu^t_o(\mathrm{d}s, \mathrm{d}y) \\
&= \int r^2\, \bar{\nu}^t_{p_0}(\mathrm{d}r) + \int s^2\, \bar{\nu}^t_o(\mathrm{d}s) \\
&\quad - 2\int rs\cos(d_X(x, y))\, \nu^t_{p_0}(\mathrm{d}r, \mathrm{d}x)\bar{\nu}^t_o(\mathrm{d}s)\, \bar{m}_X(\mathrm{d}y) \\
&= \int r^2\, \bar{\nu}^t_{p_0}(\mathrm{d}r) + \int s^2\, \bar{\nu}^t_o(\mathrm{d}s) - 2\int f\, \mathrm{d}\nu^t_{p_0}\Big(\int s\, \bar{\nu}^t_o(\mathrm{d}s)\Big),
\end{aligned}$$

where we have set $f(r, x) = r\int_X \cos(d_X(x, y))\, \bar{m}_X(\mathrm{d}y)$. Note that $f(p_0) = r_0 a$. By Lemma 4.3 (and Jensen's inequality) f is a 1-Lipschitz function on $\mathrm{Con}^N_0(X)$. From Lemma 2.2 we infer that

$$\left|\int f \mathrm{d}\nu^t_{p_0} - f(p_0)\right| \leq \sqrt{W_2(\nu^t_{p_0}, \delta_{p_0})} \leq \sqrt{2Nt}.$$

Thus, using the moment estimates from Lemma 4.2 we obtain

$$W_2(\hat{P}_t\delta_o, \hat{P}_t\delta_{p_0})^2 \leq r_0^2 + t \cdot 4(N + 1) - 2c\sqrt{t}(r_0 a - \sqrt{2Nt})$$

for a constant $c > 0$. This proves (4.2).

Step 3: Let us now assume that $\int_X \cos(d_X(x_0, y))\, m_X(\mathrm{d}y) = 0$. We claim that

$$W_2(\hat{P}_t\delta_o, \hat{P}_t\delta_{p_0}) \geq d_C(o, p_0) + O(t), \tag{4.3}$$

which immediately implies that $-\partial^-_t|_{t=0} \log W_2(\hat{P}_t\delta_o, \hat{P}_t\delta_{p_0}) \leq 0$. To this end, consider the function $\phi : \mathrm{Con}^N_0(X) \to \mathbb{R}$ given by $\phi(s, y) = s\cos(d_X(x_0, y))$. By Lemma 4.3, ϕ is 1-Lipschitz with respect to the cone distance. Hence, by Kantorovich–Rubinstein duality, we obtain

$$W_2(\hat{P}_t\delta_o, \hat{P}_t\delta_{p_0}) \geq W_1(\hat{P}_t\delta_o, \hat{P}_t\delta_{p_0}) \geq \int \phi\, \mathrm{d}(\nu^t_{p_0} - \nu^t_o) = \int \phi\, \mathrm{d}\nu^t_{p_0} =: g(t).$$

Using the definition of the cone distance and Lemma 4.2 we obtain

$$\begin{aligned}
2r_0 g(t) &= -\int d_C(p_0, \cdot)^2\, \mathrm{d}\nu^t_{p_0} + r_0^2 + \int s^2\, \bar{\nu}^t_{p_0}(\mathrm{d}s) \\
&= -\int d_C(p_0, \cdot)^2\, \mathrm{d}\nu^t_{p_0} + 2r_0^2 + t \cdot 2(N + 1).
\end{aligned}$$

From Lemma 2.2 we infer that $g(t) = r_0 + O(t)$, which yields (4.3).

Step 4: Finally, recall that the $\mathrm{RCD}^*(0, N+1)$ property of $\mathrm{Con}_0^N(X)$ implies the contraction estimate

$$W_2(\hat{P}_t\delta_p, \hat{P}_t\delta_q) \le d_C(p,q) \quad \forall p, q \in C,$$

which implies that $-\partial_t^-|_{t=0} \log W_2(\hat{P}_t\delta_o, \hat{P}_t\delta_{p_0}) \ge 0$. □

Proof of Theorem 1.1. If Y is isomorphic to $\mathbb{R}^{N+1}$, it satisfies $\mathrm{RCD}^*(0, N+1)$ and it is isomorphic to the N-cone $\mathrm{Con}_0^N(\mathbb{S}^N)$. Moreover,

$$W_2(\hat{P}_t\delta_p, \hat{P}_t\delta_q) = d_Y(p,q)$$

for all p, q and hence $\vartheta^+(o, y) = 0$ for all $y \in Y$.

Let us now assume that (i) does not hold, i.e. $\vartheta^*(o) = k < \infty$. In particular, this implies that $\limsup_{r\to 0} \vartheta^+(o,(r,x)) \le k$ for all $x \in X$. By Theorem 2.8, X satisfies $\mathrm{RCD}^*(N-1, N)$. Since, by Proposition 4.1 we have either $\vartheta^+(o, y) = 0$ or $\vartheta^+(o, y) = +\infty$, we infer that $\vartheta^+(o,(r,x)) = 0$ for all x and all r sufficiently small (in fact, this holds for all r, x by the scaling property of the heat flow on the cone). Thus, again by Proposition 4.1,

$$\int_X \cos(d_X(x,y))\, m_X(\mathrm{d}y) = 0 \quad \forall x \in X,$$

and in particular (4.1) holds. Theorem 1.3 with $f = \cos$ shows that N is an integer and X is isomorphic to $\mathbb{S}^N$ with the round metric and a multiple of the volume measure. Hence Y is isomorphic to $\mathbb{R}^{N+1}$ with Euclidean distance and a multiple of the Lebesgue measure, i.e. (ii) holds. □

Let (X, d_X, m_X) be an $\mathrm{RCD}^*(N-1, N)$ space and let $C = \mathrm{Con}_0^N(X)$. Further let B denote the $\mathrm{RCD}^*(0, N+1)$ space $([0,\infty), |\cdot|, r^N\,\mathrm{d}r)$, where $|\cdot|$ stands for the Euclidean distance. Denote by P_t^B, P_t^X, and P_t^C the heat semigroups on the spaces B, X, and C respectively and denote their adjoints acting on measures by $\hat{P}_t^B$, $\hat{P}_t^X$, and $\hat{P}_t^C$.

Lemma 4.2. *Let $\nu_p^t = \hat{P}_t^C\delta_p$ for $p = (r,x) \in C$ and denote by $\bar{\nu}_p^t$ its marginal in the radial component. Further, let $\nu_{p,s}^t \in \mathcal{P}(C)$ be the desintegration of ν_p^t with respect to $\bar{\nu}_p^t$, i.e.*

$$\nu_p^t(\mathrm{d}q) = \int_0^\infty \nu_{p,s}^t(\mathrm{d}q)\, \bar{\nu}_p^t(\mathrm{d}s).$$

Then $\bar{\nu}_p^t = \hat{P}_t^B\delta_r$ for any $p \in C$. For $p = o$ we have $\bar{\nu}_{o,s}^t(\mathrm{d}s', \mathrm{d}y) = \delta_s(\mathrm{d}s')\bar{m}_X(\mathrm{d}y)$ with $\bar{m}_X = m_X(X)^{-1} \cdot m_X$. Finally, consider the moments $m_\alpha(p,t) := \int s^\alpha\, \mathrm{d}\bar{\nu}_p^t(s)$ for $\alpha > 0$. Then

$$m_2(p,t) = r^2 + t \cdot 2(N+1), \qquad m_1(o,t) = \sqrt{t} \cdot m_1(o,1).$$

Proof. First, we recall from [16, Sec. 2.3] that the generator L^C of the semigroup P_t^C is given explicitly on functions $u \otimes v$ of product form with $u \in C_0^\infty(0,\infty))$ and $v \in D(L^X)$ by

$$(L^C u \otimes v)(r,x) = (L^B u)(r) + \frac{1}{r^2}(L^X v)(x),$$

where L^B and L^X denote the Laplacians on B and X respectively, i.e. the generators of P_t^B and P_t^X. Note that

$$(L^B u)(r) = u''(r) + \frac{N}{r}u'(r).$$

It follows that $P_t^C(u \otimes 1) = (P_t^B u) \otimes 1$. Thus for all suitable u,

$$\int u \, \mathrm{d}\bar{\nu}_p^t = \int u \otimes 1 \, \mathrm{d}(\hat{P}_t^C \delta_p) = P_t^C(u \otimes 1)(p) = P_t^B u(r) = \int u \, \mathrm{d}\hat{P}_t^B \delta_r,$$

and hence $\bar{\nu}_p^t = \hat{P}_t^B \delta_r$.

Let us prove the second statement. To this end, we first consider measures $\mu \in \mathcal{P}_2(C)$ of the form $\mu = (\rho \otimes 1)m_C$ with $\rho \in C_0^\infty((0,\infty))$. Its marginal in the radial component is $\bar{\mu}(\mathrm{d}s) := m_X(X)\rho(s)s^N \, \mathrm{d}s$. Then

$$\hat{P}_t^C \mu = [P_t^C(\rho \otimes 1)]m_C = [(P_t^B \rho) \otimes 1]m_C = \int_0^\infty (\delta_s \otimes \bar{m}_X) \, \hat{P}_t^B \bar{\mu}(\mathrm{d}s).$$

The claim of the lemma follows by approximating δ_o with such measures μ.

To calculate the moments of $\bar{\nu}_p^t$, we employ a stochastic argument. The stochastic process associated with the generator $\frac{1}{2}L^B$ is the $(N+1)$-dimensional Bessel process, which is the solution to the Itô stochastic differential equation

$$Y_t = Y_0 + \frac{N}{2}\int_0^t \frac{1}{Y_s} \, \mathrm{d}s + B_s,$$

with B_s a standard Brownian motion (see e.g. [21, Chap. XI] for details on Bessel processes; for N integer this process can be realized as the absolute value of an $(N+1)$-dimensional Brownian motion). In particular

$$\int u \, \mathrm{d}\bar{\nu}_p^t = \mathbb{E}[u(Y_{2t})],$$

where Y_t is the above process with $Y_0 = r$. We then obtain $m_\alpha(p,t)$ by choosing $u(r) = r^\alpha$. The second moment is given by $m_2(p,t) = r^2 + 2t(N+1)$. One way to see this is to observe that $Y_t^2 = Y_0^2 + 2\int_0^t Y_t \, \mathrm{d}B_s + t(N+1)$. To obtain $m_1(o,t)$ we use the fact that Y_t has the Brownian scaling property, i.e. if $Y_0 = 0$, then Y_t has the same law as $\sqrt{t}\, Y_1$. This yields $m_1(o,t) = \sqrt{t} \cdot m_1(o,1)$. □

Lemma 4.3. *Let* (X, d_X, m_X) *be a metric measure space with* $\mathrm{diam}(X) \le \pi$ *and* $x \in X$. *Then the function* $\phi : \mathrm{Con}_0^N(X) \to \mathbb{R}$ *given by*

$$(s, y) \mapsto s\cos(d_X(x, y))$$

is 1*-Lipschitz with respect to the cone distance.*

Proof. Let $(s, y), (s', y') \in C = \mathrm{Con}_0^N(X)$ and set $\alpha = d_X(x, y)$, $\alpha' = d_X(x, y')$ and $\beta = d_X(y, y')$. Note that $\alpha, \alpha', \beta \le \pi$ and $\beta \ge |\alpha - \alpha'|$. Let $p, p' \in \mathbb{R}^2$ be points at angle α and α' to the first coordinate axis respectively and $\|p\| = s$, $\|p'\| = s'$. Now, we have

$$\begin{aligned} d_C((s, y), (s', y'))^2 &= s^2 + (s')^2 - 2ss'\cos\beta \ge s^2 + (s')^2 - 2ss'\cos|\alpha - \alpha'| \\ &= \|p - p'\|^2. \end{aligned}$$

On the other hand,

$$|\phi(s, y) - \phi(s', y')| = |s\cos\alpha - s'\cos\alpha'| = \|q - q'\| \le \|p - p'\|,$$

where q and q' are the projections of p and p' respectively onto the first coordinate axis. □

Example 4.4. Consider the special case $X = \mathbb{S}^2(1/\sqrt{3}) \times \mathbb{S}^2(1/\sqrt{3})$ equipped with the Cartesian product of the standard Riemannian distances on the spheres $\mathbb{S}^2(1/\sqrt{3})$ with radius $1/\sqrt{3}$ and the normalized product measure, which is an $\mathrm{RCD}^*(3, 4)$ space. Hence, the 4-cone over $\mathbb{S}^2(1/\sqrt{3}) \times \mathbb{S}^2(1/\sqrt{3})$ is an $\mathrm{RCD}^*(0, 5)$ space with Ricci curvature $+\infty$ at the tip.

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References

[1] Ambrosio, L., Gigli, N., Mondino, A., Rajala, T.: Riemannian Ricci curvature lower bounds in metric measure spaces with σ-finite measure. Trans. Amer. Math. Soc. **367**, 4661–4701 (2015) Zbl 1317.53060 MR 3335397

[2] Ambrosio, L., Gigli, N., Savaré, G.: Calculus and heat flow in metric measure spaces and applications to spaces with Ricci bounds from below. Invent. Math. **195**, 289–391 (2014) Zbl 1312.53056 MR 3152751

[3] Ambrosio, L., Gigli, N., Savaré, G.: Metric measure spaces with Riemannian Ricci curvature bounded from below. Duke Math. J. **163**, 1405–1490 (2014) Zbl 1304.35310 MR 3205729

[4] Bacher, K., Sturm, K.-T.: Ricci bounds for Euclidean and spherical cones. In: Singular Phenomena and Scaling in Mathematical Models, Springer, Cham, 3–23 (2014) Zbl 1328.53050 MR 3205034

[5] Bakry, D., Émery, M.: Diffusions hypercontractives. In: Séminaire de probabilités, XIX, 1983/84, Lecture Notes in Math. 1123, Springer, Berlin, 177–206 (1985) Zbl 0561.60080 MR 0889476

[6] Cavalletti, F., Milman, E.: The globalization theorem for the curvature dimension condition. arXiv:1612.07623 (2016)

[7] Cavalletti, F., Mondino, A.: Sharp and rigid isoperimetric inequalities in metric-measure spaces with lower Ricci curvature bounds. Invent. Math. **208**, 803–849 (2017) Zbl 1375.53053 MR 3648975

[8] Cavalletti, F., Sturm, K.-T.: Local curvature-dimension condition implies measure-contraction property. J. Funct. Anal. **262**, 5110–5127 (2012) Zbl 1244.53050 MR 2916062

[9] Cheng, L.-J., Thalmaier, A.: Characterizations of pinched Ricci curvature by functional inequalities. J. Geom. Anal. **28**, 2312–2345 (2018) Zbl 1405.60121 MR 3833795

[10] Erbar, M., Kuwada, K., Sturm, K.-T.: On the equivalence of the entropic curvature-dimension condition and Bochner's inequality on metric measure spaces. Invent. Math. **201**, 993–1071 (2015) Zbl 1329.53059 MR 3385639

[11] Garofalo, N., Mondino, A.: Li–Yau and Harnack type inequalities in $\mathrm{RCD}^*(K,N)$ metric measure spaces. Nonlinear Anal. **95**, 721–734 (2014) Zbl 1286.58016 MR 3130557

[12] Gigli, N.: The splitting theorem in non-smooth context. arXiv:1302.5555 (2013)

[13] Gigli, N.: On the differential structure of metric measure spaces and applications. Mem. Amer. Math. Soc. **236**, no. 1113, vi+91 pp. (2015) Zbl 1325.53054 MR 3381131

[14] Gigli, N., Mondino, A., Savaré, G.: Convergence of pointed non-compact metric measure spaces and stability of Ricci curvature bounds and heat flows. Proc. London Math. Soc. (3) **111**, 1071–1129 (2015) Zbl 1398.53044 MR 3477230

[15] Han, B.: Ricci tensor on $\mathrm{RCD}^*(K,N)$ spaces. J. Geom. Anal. **28**, 1295–1314 (2018) Zbl 1395.53046 MR 3790501

[16] Ketterer, C.: Cones over metric measure spaces and the maximal diameter theorem. J. Math. Pures Appl. (9) **103**, 1228–1275 (2015) Zbl 1317.53064 MR 3333056

[17] Ketterer, C.: Obata's rigidity theorem for metric measure spaces. Anal. Geom. Metr. Spaces **3**, 278–295 (2015) Zbl 1327.53051 MR 3403434

[18] Lott, J., Villani, C.: Ricci curvature for metric-measure spaces via optimal transport. Ann. of Math. (2) **169**, 903–991 (2009) Zbl 1178.53038 MR 2480619

[19] Mondino, A., Naber, A.: Structure theory of metric measure spaces with lower Ricci curvature bounds. J. Eur. Math. Soc. **21**, 1809–1854 (2019) Zbl 07060417 MR 3945743

[20] Naber, A.: Characterizations of bounded Ricci curvature on smooth and nonsmooth spaces. arXiv:1306.6512 (2015)

[21] Revuz, D., Yor, M.: Continuous Martingales and Brownian Motion. Grundlehren Math. Wiss. 293, Springer, Berlin (1999) Zbl 1087.60040 MR 1725357

[22] Sturm, K.-T.: On the geometry of metric measure spaces. I, II. Acta Math. **196**, 65–131, 133–177 (2006) Zbl 1106.53032 MR 2237207

[23] Sturm, K.-T.: Ricci tensor for diffusion operators and curvature-dimension inequalities under conformal transformations and time changes. J. Funct. Anal. **275**, 793–829 (2018) Zbl 1419.58020 MR 3807777

[24] Sturm, K.-T.: Remarks about synthetic upper Ricci bounds for metric measure spaces. Tohoku Math. J. (to appear); arXiv:1711.01707

[25] Villani, C.: Optimal Transport: Old and New. Grundlehren Math. Wiss. 338, Springer, Berlin (2009) Zbl 1156.53003 MR 2459454

[26] Wu, B.: Characterizations of the upper bound of Bakry–Émery curvature. J. Geom. Anal. (online, 2019)