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Tristan Rivière

Willmore minmax surfaces and the cost of the sphere eversion

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Abstract. We develop a general minmax procedure in Euclidian spaces for constructing Willmore surfaces of non-zero indices. We apply this procedure to the Willmore minmax sphere eversion in the 3-dimensional Euclidian space. We compute the cost of sphere eversion in terms of Willmore energies of the Willmore spheres in $\mathbb{R}^3$.

Keywords. Willmore surface, eversion, minmax, calculus of variations, mountain pass

I. Introduction

I.1. The search for Willmore minmax surfaces

Finding optimal shapes is a search probably as old as mathematics and whose motivation goes beyond the exclusive quest for beauty. It is often closely related to the understanding of deep mathematical structures and ultimately to natural phenomena happening to be governed by these pure ideas.

The existence of closed geodesics on arbitrary manifolds and its higher dimensional counterpart, the existence of minimal surfaces, belongs to this search and has been since the 19th century a very active area of research diffusing in other areas of mathematics and science in general much beyond the field of differential geometry.

The theory of Willmore surfaces, introduced by Wilhelm Blaschke around 1920, grew out of the attempt to merge *minimal surface theory* and *conformal invariance*.

For an arbitrary immersion $\vec{\Phi}$ of a given oriented abstract surface Σ into a Euclidian space $\mathbb{R}^m$ Wilhelm Blaschke introduced the Lagrangian

$$W(\vec{\Phi}) = \int_\Sigma |\vec{H}_{\vec{\Phi}}|^2 \, d\mathrm{vol}_{g_{\vec{\Phi}}}$$

where $\vec{H}_{\vec{\Phi}}$ and $d\mathrm{vol}_{g_{\vec{\Phi}}}$ are respectively the mean curvature vector and the volume form of the metric induced by the immersion. He proved in particular that for a closed surface Σ this Lagrangian is invariant under conformal transformations.

T. Rivière: Department of Mathematics, ETH Zentrum, CH-8093 Zürich, Switzerland; e-mail: riviere@math.ethz.ch

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The critical points to W are called “Willmore surfaces[1]”. The known set of critical points to the Willmore Lagrangian has been for a long time reduced to the minimal surfaces and their conformal transformations. This maybe explains why the study of its variations has been more or less stopped for several decades following the seminal work of Wilhelm Blaschke which was slowly sinking into oblivion.

After the reviving work of Tom Willmore the first main contribution to “Willmore surfaces” has been brought by Robert Bryant [11]. Using algebraic geometric techniques he succeeded in describing all the immersed “Willmore spheres” in $\mathbb{R}^3$ as being given exclusively by the images by inversions of simply connected complete non-compact minimal surfaces with planar ends. The Willmore energy of the immersed Willmore spheres was consequently proved to be equal to 4π times the number of planar ends. Due to the non-triviality of the space of holomorphic quartic forms on any other Riemann surface this approach has been restricted to the sphere exclusively. Other algebraic geometric approaches for studying critical points to the Willmore Lagrangian include “spectral curve methods” and integrable system theory, but these rather abstract methods addres issues which are mostly local and, until now, hardly translatable into “down to earth” results exhibiting new complete Willmore surfaces or characterizing the space of Willmore critical points in a decisive way.

Besides algebraic-geometric methods a natural strategy for producing new Willmore surfaces would consist in developing the fundamental principles of the calculus of variations applied to the Willmore Lagrangian. Since $\vec{H}_{\vec{\Phi}} = 2^{-1}\Delta_{g_{\vec{\Phi}}}\vec{\Phi}$, the Willmore Lagrangian is nothing but $1/4$ of the L^2 norm of the Laplacian of the immersion and shows in that sense its 4th order elliptic nature. This coercive structure gives some hope for the success of variational methods. In the pioneering work studying the variations of W, Leon Simon [45] proved the existence of a torus minimizing the Willmore energy. This existence result was also motivated by the conjecture formulated by Willmore [51] according to which the torus obtained by rotating around the vertical z-axis of the vertical circle of radius 1 centered at $(\sqrt{2}, 0, 0)$ and included in the plane $y = 0$ would be the unique minimizer modulo conformal transformations. This conjecture was finally proved some years ago by Fernando Codá Marques and André Neves [31]. In [4] Matthias Bauer and Ernst Kuwert succeeded in proving a succession of strict inequalities excluding possible degeneracies and the splitting of the underlying surface which had been left open in Leon Simon's argument for arbitrary genus. As a consequence the authors proved the existence of a minimizer of W for any closed orientable two-dimensional manifold Σ.

Leon Simon's approach to the minimization of the Willmore energy is based on energy comparison arguments and local bi-harmonic graph approximation procedures and in that sense is shaped for studying the ground states of index 0. This approach mostly considers the image of the immersion, $\vec{\Phi}(\Sigma)$, and not the immersion *per se* and can be called “ambient” for that reason. In [42] the present author gave an alternative proof to Leon Simon's existence result using an approach called “parametric”. In this approach the study of the variations of the immersion is made possible by local extraction of “Coulomb

[1] This terminology has spread, and is now generally used, after the work [51] which relaunched the study of these surfaces that Blaschke originally named “conformal-minimal surfaces”.

gauges" (isothermic parametrization) and the use of the conservation laws coming from the application of the Noether theorem combined with the *integrability by compensation theory* (see also a systematic presentation of this theory and its application in Willmore [40]). Since this approach does not make use of comparison arguments and since it is based on a weak formulation of the Willmore Euler–Lagrange equation discovered in [41], it gave the author hopes to apply it to more diversified calculus of variation arguments than strict minimizations. This is the main achievement of the present work. More precisely, the purpose of the paper is to present a minmax method for producing critical points to the Willmore energy of non-zero indices.

I.2. "Smoothers" based on the second fundamental form

As already mentioned, the Willmore energy is invariant under the action of the Möbius group of conformal transformations of $\mathbb{R}^m$, which is known to be non-compact. For that reason in particular it does not satisfy the Palais–Smale condition. This is an obstruction to applying minmax variational principles such as the *mountain pass lemma* directly. We shall therefore adopt a *viscosity approach* and add to the Willmore energy what we call a "smoother" times a small "viscosity parameter" σ^2,

$$\text{Full Energy}(\vec{\Phi}) := W(\vec{\Phi}) + \sigma^2 \text{Smoother}(\vec{\Phi}),$$

so that for the new energy the Palais–Smale condition is satisfied. One can then apply the mountain pass arguments to such energy and produce minmax critical points. In the second part of the procedure one lets σ tend to zero and one studies the process hopefully converging to a Willmore minmax surface.

In a first approach, following [44], one could think of adding to W a term of the form

$$\text{Smoother}(\vec{\Phi}) = \int_\Sigma (1 + |\vec{\mathbb{I}}_{\vec{\Phi}}|^2)^2 \, d\text{vol}_{g_{\vec{\Phi}}}$$

where $\vec{\mathbb{I}}_{\vec{\Phi}}$ is the second fundamental form of the immersion $\vec{\Phi}$. This will make the new Willmore relaxed energy satisfy the Palais–Smale condition (as proved in [24], see also [27]) but this will bring us to the study of p-harmonic systems which makes the analysis of the convergence rather involved—in particular the energy quantization—when the small viscosity parameter σ tends to zero. From that perspective of p-harmonic versus harmonic systems, as observed below and as also used in [9], replacing the full second fundamental form by its trace $\vec{H}_{\vec{\Phi}}$ and considering instead

$$\text{Smoother}(\vec{\Phi}) = \int_\Sigma (1 + |\vec{H}_{\vec{\Phi}}|^2)^2 \, d\text{vol}_{g_{\vec{\Phi}}}$$

has the surprising effect of making the highest order term in the Euler–Lagrange equation be Δ and not $\Delta_2 := \text{div}((1 + |H|^2)\nabla)$ if one makes use of the various conservation laws coming from the Noether theorem, following the main lines of [41]. The drawback however is that $\int_\Sigma (1 + |\vec{H}_{\vec{\Phi}}|^2)^2$ fails to satisfy the Palais–Smale condition and cannot be a smoother by itself and has to be "reinforced".

I.3. *Frame energies*

In the portfolio of surface energies, the author, in collaboration with Andrea Mondino, introduced in [35] the notion of *frame energy* for an arbitrary immersion of a torus, $\vec{\Phi} : T^2 \to \mathbb{R}^m$, equipped by an orthonormal tangent frame $\vec{e} \ : \ T^2 \to S^{m-1} \times S^{m-1}$ where $\vec{e}(x)$ provides an orthonormal basis of $\vec{\Phi}_* T_x T^2$. The frame energy is then simply given by

$$\mathcal{F}(\vec{\Phi}, \vec{e}) := \frac{1}{4} \int_{T^2} |d\vec{e}|^2_{g_{\vec{\Phi}}} \, d\mathrm{vol}_{g_{\vec{\Phi}}} \geq W(\vec{\Phi}). \tag{I.1}$$

If one considers the “Coulomb frame” associated to the conformal immersion of a fixed flat torus,[2] then the *frame energy* $F(\vec{\Phi}) := \inf_{\vec{e}} \mathcal{F}(\vec{\Phi}, \vec{e})$ defines an energy of the immersion $\vec{\Phi}$ that happens to be more coercive than the Willmore energy itself. One could ask whether it can be naturally extended to any other immersion of an arbitrary surface Σ. This happens to be indeed the case as we explain in the following.

Let $\vec{\Phi}$ be an arbitrary immersion of a closed surface Σ and denote by g_0 a constant scalar curvature metric of volume 1 on Σ for which there exists $\alpha : \Sigma \to \mathbb{R}$ with

$$g_{\vec{\Phi}} = e^{2\alpha} g_0. \tag{I.2}$$

For $\Sigma \neq S^2$ the function α is defined without ambiguity, whereas in the sphere case we have to deal with the action of a “gauge group”, the space $\mathcal{M}^+(S^2)$ of positive conformal transformations of S^2, and α is uniquely defined modulo the action of this gauge group. For the case of the torus one proves in this paper that

$$F(\vec{\Phi}) = W(\vec{\Phi}) + \frac{1}{2} \int_{T^2} |d\alpha|^2_{g_{\vec{\Phi}}} \, d\mathrm{vol}_{g_{\vec{\Phi}}}.$$

We generalize the frame energy for any surface of genus larger than 1 as follows: $F := W + \mathcal{O}$ where

$$\mathcal{O}(\vec{\Phi}) = \frac{1}{2} \int_{\Sigma} |d\alpha|^2_{g_{\vec{\Phi}}} \, d\mathrm{vol}_{g_{\vec{\Phi}}} + K_{g_0} \int_{\Sigma} \alpha e^{-2\alpha} \, d\mathrm{vol}_{g_{\vec{\Phi}}} - 2^{-1} K_{g_0} \log\left(\int_{\Sigma} d\mathrm{vol}_{g_{\vec{\Phi}}} \right), \tag{I.3}$$

where K_{g_0} is the constant scalar curvature metric of g_0. The reason for adding to the Dirichlet energy of α the term

$$K_{g_0} \int_{\Sigma} \alpha e^{-2\alpha} \, d\mathrm{vol}_{g_{\vec{\Phi}}}$$

comes from the fact that the first variation can be expressed miraculously by <u>local</u> quantities! (although the operation which to g_Φ assigns α is highly non-local—see for instance [50]). Finally, the reason for adding the third term is twofold: it makes the energy scaling invariant and non-negative as a direct application of Jensen’s inequality when $K_{g_0} < 0$.

Finally, when $\Sigma = S^2$ we define the *frame energy* to be

$$F(\vec{\Phi}) := W(\vec{\Phi}) + \frac{1}{2} \int_{S^2} |d\alpha|^2_{g_{\vec{\Phi}}} \, d\mathrm{vol}_{g_{\vec{\Phi}}} + 4\pi \int_{S^2} \alpha e^{-2\alpha} \, d\mathrm{vol}_{g_{\vec{\Phi}}} - 2\pi \log\left(\int_{S^2} d\mathrm{vol}_{g_{\vec{\Phi}}} \right).$$

Besides the fact that it naturally generalizes to S^2 the Dirichlet energy of *Coulomb frames* on tori, there are four main reasons for considering this special expression with these

[2] By the uniformization theorem such a parametrization always exists.

particular coefficients:

(i) $F - W$ is the well known *Onofri energy* of α (see [37] and a more recent presentation in [14]) and satisfies[3]

$$\mathcal{O}(\vec{\Phi}) := \frac{1}{2}\int_{S^2} |d\alpha|^2_{g_{\vec{\Phi}}}\, d\mathrm{vol}_{g_{\vec{\Phi}}} + 4\pi \int_{S^2} \alpha e^{-2\alpha}\, d\mathrm{vol}_{g_{\vec{\Phi}}} - 2\pi \log\left(\int_{S^2} d\mathrm{vol}_{g_{\vec{\Phi}}}\right) \geq 0. \tag{I.4}$$

Observe that the Lagrangian $\mathcal{O}(\vec{\Phi})$ viewed as a functional depending on $\vec{\Phi}$ is nothing but the main term in the *Polyakov–Alvarez* formula of the *zeta regularized determinant of the Laplacian* of the underlying Riemannian two-dimensional manifold. This formula is also named the *Polyakov–Alvarez conformal anomaly formula* in *conformal field theory* (see [39]).

(ii) The first variation of F is explicit and can be expressed using local quantities.[4]

(iii) The energy $\mathcal{O}(\vec{\Phi})$ is gauge invariant with respect to the action of the Möbius group on S^2 and is independent of the choice of α and g_0 satisfying (I.2) and depends only on $\vec{\Phi}$.

(iv) The F-energy is dilation invariant: $F(e^t\vec{\Phi}) = F(\vec{\Phi})$ for any $t \in \mathbb{R}$.

Open Problem 1. It would be interesting to study the existence of minimizers of the frame energy F in each regular homotopy class of immersions of spheres in $\mathbb{R}^4$. Since the work of Stephen Smale [46] we have known that there exist countably many of them given by $\pi_2(V_{4,2}(\mathbb{R})) = \mathbb{Z}$, the second homotopy class of the Stiefel manifold of 2-frames in $\mathbb{R}^4$. It would also be interesting to study the asymptotic dependence on the class of the infimum of the F-energy as the class goes to infinity (whether the dependence is linear or sublinear).

Finally, for a given immersion $\vec{\Phi}$ it would be interesting to study the minimal Dirichlet energy of any bundle map from $T\Sigma$ into $\mathbb{R}^3 \times \mathbb{R}^3$ which is an isometry from each fiber $(T_x\Sigma, g_{\vec{\Phi}})$ into $\vec{\Phi}_* T_x\Sigma \subset G_2(\mathbb{R}^3)$ and which projects onto $\vec{\Phi}$—the map $(\vec{\Phi}, d\vec{\Phi})$ is one of such maps of course. Starting from $(\vec{\Phi}, d\vec{\Phi})$ such a bundle map is just given by the choice of an S^1 rotation at each point.

In the case of $\Sigma = T^2$ this coincides with the Dirichlet energy of an optimal global frame and is equal to $F(\vec{\Phi})$.

I.4. A viscous approximation of Willmore energy

Inspired by the discussion above we propose to consider the following approximation of the Willmore energy:

$$F^\sigma(\vec{\Phi}) := W(\vec{\Phi}) + \sigma^2 \int_\Sigma (1 + |\vec{H}|^2)^2\, d\mathrm{vol}_{\vec{\Phi}} + \frac{1}{\log(1/\sigma)} \mathcal{O}(\vec{\Phi})$$

where $\mathcal{O}(\vec{\Phi})$ is given by (I.4) if $\Sigma = S^2$ or by the expression (I.3) otherwise.

[3] This inequality is not a direct consequence of the Jensen inequality when $K_{g_0} > 0$ and requires more elaborate arguments.

[4] This is a very striking fact which is going to make the analysis simpler in the following sections. It very much depends on the choice of the coefficient of each main term of the energy.

We prove in Section V that F^σ satisfies the Palais–Smale condition. One can then apply minmax arguments to F^σ for any admissible family. One of the main achievements of the present work is the proof of ε-regularity independent of σ (see Lemma IV.1). It makes use of the special choice we made of the logarithmic dependence on the viscosity parameter σ of the small coefficient of $\mathcal{O}$. This ε-regularity permits one to pass to the limit in the equation for well chosen sequences of minmax critical points of F^σ as σ goes to zero. The last main lemma in the paper is an energy quantization result when $\Sigma = S^2$ (see Lemma VI.1). It roughly says that no energy can be dissipated in neck regions.

We then have the main tools for performing minmax procedures for the Willmore energy of spheres. To that end we introduce the space of $W^{2,4}(S^2, \mathbb{R}^m)$ immersions of S^2 into $\mathbb{R}^m$, which we denote $\mathcal{E}_{\Sigma,2}(\mathbb{R}^m)$. This space is equipped with the $W^{2,4}$ topology. It is proved in [44] that this defines a Banach manifold with a Finsler structure. Before giving the statement of our main result we will recall the definition of admissible families.

Definition I.1. A family $\mathcal{A} \subset \mathcal{P}(\mathcal{M})$ of subsets of a Banach manifold $\mathcal{M}$ is called an *admissible family* if for every homeomorphism Ξ of $\mathcal{M}$ isotopic to the identity we have

$$\forall A \in \mathcal{A} \qquad \Xi(A) \in \mathcal{A}. \qquad \square$$

Our main result is the following.

Theorem I.1 (Willmore minmax procedures for spheres). *Let $m \geq 3$ and $k \geq 1$. Let $\mathcal{A}$ be an admissible family of $W^{2,4}$ immersions of the sphere S^2 into $\mathbb{R}^m$. Let*

$$\beta_0 := \inf_{A \in \mathcal{A}} \max_{\vec{\Phi} \in A} W(\vec{\Phi}).$$

Then there exist finitely many Willmore immersions $\vec{\xi}_1, \ldots, \vec{\xi}_n$ of S^2 minus finitely many points such that

$$\beta_0 = \sum_{i=1}^{n} W(\vec{\xi}_i) - 4\pi N \tag{I.5}$$

for some $N \in \mathbb{N}$. $\square$

Remark I.1. Theorem I.1 involves a "bubble tree convergence" of a sequence of minmax critical points of F^{σ^k} for some well chosen sequence σ^k tending to zero. This convergence produces asymptotically a "bubble tree" of Willmore spheres some of them being shrunk to zero. Among the ones which shrink to zero there might be non-compact (after asymptotic rescaling) simply connected Willmore surfaces with ends at infinity that we have to invert in order to make them Willmore spheres. This operation produces energy given by an integer multiple of 4π. That is why such a quantity is subtracted in (I.5). One of the hard parts in the whole proof is to show that between two successive asymptotic Willmore spheres in that "bubble tree" no energy is lost in the limit. This is the so called "no neck energy" property.

True Willmore surfaces. It has been recently proved in [33] by the author in collaboration with Alexis Michelat that the maps $\vec{\xi}_i$ obtained after inversions in the bubble tree define "true" Willmore immersions, possibly branched. By "true" we mean that the first residue[5] defined in [41],

$$\int_\Gamma (\partial_\nu \vec{H} - 3\pi_{\vec{n}}(\partial_\nu \vec{H}) + \star\, \partial_\tau \vec{n} \wedge \vec{H})\, dl,$$

is zero for any closed curve Γ avoiding the center of the inversion. The inversion of the catenoid is not a "true" Willmore surface in that sense whereas the inversion of the Enneper surface is a "true" Willmore sphere with a branch point of multiplicity 3 at the origin (see [7]).

Open Problem 2. Extend the previous result to general surfaces. The "only" obstruction comes from the energy quantization result which is missing when the conformal class of the minmax sequence of F^{σ^k} possibly degenerates. The recent progress made in [29] and [30] should be of help in overcoming this difficulty.

One consequence of the previous result is the following corollary. One considers the family $\mathcal{A}$ of loops into $\mathrm{Imm}(S^2,\mathbb{R}^3) \simeq_{\mathrm{hom}} \mathrm{SO}(3)\times\Omega^2(\mathrm{SO}(3))$ realizing a non-trivial element of $\pi_1(\mathrm{Imm}(S^2,\mathbb{R}^3)) \simeq \mathbb{Z}_2\times\mathbb{Z}$. It is proved in [3] that for instance the Froisart–Morin sphere eversion followed by the mirror image of the time reversal of the same eversion generates $\pi_1(\mathrm{Imm}(S^2,\mathbb{R}^3))$. In order to avoid uninteresting loops coming from the action of $\mathrm{Diff}(S^2)$ one should rather work modulo the action of reparametrization of the sphere and consider the infinite orbifold[6] $\mathrm{Imm}(S^2,\mathbb{R}^3)/\mathrm{Diff}(S^2)$ instead of $\mathrm{Imm}(S^2,\mathbb{R}^3)$ which is an open subspace of the Banach space $W^{2,4}(S^2,\mathbb{R}^3)$.

One can then take $\mathcal{A}$ to be the canonical projections onto $\mathrm{Imm}(S^2,\mathbb{R}^3)/\mathrm{Diff}(S^2)$ of paths from $[0,1]$ into $\mathrm{Imm}(S^2,\mathbb{R}^3)$ homotopic to a non-trivial element in the group $\pi_1(\mathrm{Imm}(S^2,\mathbb{R}^3)/\mathrm{Diff}(S^2)) = \mathbb{Z}$. The projection of the Froisart–Morin sphere eversion gives such a loop for instance.[7]

Corollary I.1 (The cost of sphere eversion). *Let Ω be the space of continuous paths of C^2 immersions into $\mathbb{R}^3$ joining the standard sphere S^2 with the two opposite orientations and homotopic to the Froisart–Morin sphere eversion. Define the "cost of sphere*

[5] In three dimension this residue is also a multiple of the one that can be deduced from the integration of the 1-form (4.5) in [25].

[6] While the quotient $\mathrm{Imm}(S^2,\mathbb{R}^3)/\mathrm{Diff}^+(S^2)$ of $\mathrm{Imm}(S^2,\mathbb{R}^3)$ by the group of positive diffeomorphisms of S^2 has a nice bundle structure due to the free action of $\mathrm{Diff}^+(S^2)$ on $\mathrm{Imm}(S^2,\mathbb{R}^3)$, this is not the case anymore for $\mathrm{Imm}(S^2,\mathbb{R}^3)/\mathrm{Diff}(S^2)$. The latter space is an infinite orbifold obtained by quotienting $\mathrm{Imm}(S^2,\mathbb{R}^3)/\mathrm{Diff}^+(S^2)$ by the map $x\mapsto -x$. This induces a 2-sheet covering away from the subspace of singular orbits which happens to have infinite codimension (see [12, Section 3]). Because of the smallness of the size of singular orbits, using transversality arguments one can compute homotopy groups of $\mathrm{Imm}(S^2,\mathbb{R}^3)/\mathrm{Diff}(S^2)$ as if this covering map were without singularities.

[7] The canonical projection π of $\mathrm{Imm}(S^2,\mathbb{R}^3)$ onto the infinite orbifold $\mathrm{Imm}(S^2,\mathbb{R}^3)/\mathrm{Diff}(S^2)$ induces a morphism $\pi_*\colon \pi_1(\mathrm{Imm}(S^2,\mathbb{R}^3)) = \mathbb{Z}_2\times\mathbb{Z} \to \pi_1(\mathrm{Imm}(S^2,\mathbb{R}^3)/\mathrm{Diff}(S^2)) = \mathbb{Z}$ equal to multiplication by 2.

eversion" by

$$\beta_0 := \inf_{\omega\in\Omega} \max_{\vec{\Phi}\in\omega} W(\vec{\Phi}).$$

Then there exist finitely many Willmore immersions $\vec{\xi}_1, \ldots, \vec{\xi}_n$ *of* S^2 *minus finitely many points such that*

$$\beta_0 = \sum_{i=1}^{n} W(\vec{\xi}_i) - 4\pi N$$

for some $N \in \mathbb{N}$.

Remark I.2. Performing the Willmore minmax sphere eversion has been originally proposed by Robert Kusner. □

Open Problem 3. Is it true that in Corollary I.1,

$$\beta_0 = 16\pi \ ?$$

A topological result [3] (see also the enlightening proof in [22]) asserts that any element in Ω has to contain at least one immersion with a point of self-intersection of order 4 (i.e. a quadruple point). Hence using Li–Yau's result we deduce that $\beta_0 \geq 16\pi$. In [18] a candidate for the realization of β_0 is proposed. It is the inversion of a simply connected complete minimal surface with four planar ends. Hence the Willmore energy of this candidate is 16π. Interesting computations reinforcing this conjecture are performed in that work. Establishing an upper bound of the lowest energy minmax sphere eversion is made difficult by the fact that producing concrete sphere eversions is highly challenging and has been at the origin of many rigorous works, computer simulations and videos, starting from the first example given by Arnold Shapiro (see for instance [36], [16], [38], [2], [17], [49], [1]).

Remark I.3. In a recent work [33] the author, in collaboration with Michelat, extends Bryant's classification to "true", possibly branched Willmore spheres.

Remark I.4. An interesting upper bound of the cost of the total curvature minmax sphere eversion is presented in [15]. It is proved in particular that

$$\inf_{\omega\in\Omega} \max_{\vec{\Phi}\in\omega} \int_{S^2} |K_{\vec{\Phi}}|\, d\mathrm{vol}_{g_{\vec{\Phi}}} \leq 8\pi$$

where $K_{\vec{\Phi}}$ is the Gauss curvature of $\vec{\Phi}$.

Open Problem 5. It would be interesting to study the cost of the frame energy $W + \mathcal{O}$ minmax sphere eversion. As we saw above, this energy is closely related to the minimizing Dirichlet energy among the bundle maps injections induced by $(\vec{\Phi}, d\vec{\Phi})$, used by Smale to compute the homotopy type of the space of immersions.

Open Problem 6. It would be interesting, besides the study of Willmore minmax sphere eversion, to also explore the Willmore minmax for $\mathcal{A}$ corresponding to other non-trivial classes in $\pi_k(\mathrm{Imm}(S^2,\mathbb{R}^3)) = \pi_k(\mathrm{SO}(3)\times\Omega^2(\mathrm{SO}(3)))$ for arbitrary k. How would their indices be related to k?

Open Problem 7. Explore the topology of $\mathrm{Imm}(S^2,\mathbb{R}^3)$ using W as some kind of "quasi[8] Morse function". For instance we could ask if all the Willmore immersions of S^2 into $\mathbb{R}^3$ (described by Robert Bryant) are related to a minmax procedure involving various classes of various groups $\pi_k(\mathrm{Imm}(S^2,\mathbb{R}^3))$. A first instructive step would consist in computing the indices of the Willmore sphere immersions in $\mathbb{R}^3$. Of course one would have also to complete the space of immersions by considering possibly branched immersions and, up to now, there is no general result about the extension of Bryant classification of Willmore immersed spheres in $\mathbb{R}^3$ to Willmore branched spheres, except some very special cases treated in [26].

Most of the proofs below are presented in the particular case $m = 3$ in order to make the presentation more accessible. The general case $m \geq 3$ is very similar but requires the use of the conservation laws in arbitrary codimensions introduced in [41], whose formulation involves the use of multi-vectors instead of vectors; this makes the argument a bit more tedious but does not bring any new fundamental difficulties.

II. The space of immersions into $\mathbb{R}^3$ with L^q bounded second fundamental form

For $k \in \mathbb{N}$ and $1 \leq q \leq +\infty$, we recall the definition of $W^{k,q}$ Sobolev functions on a closed smooth surface Σ (i.e. Σ is compact without boundary). To that end we take some reference smooth metric g_0 on Σ and set

$$W^{k,q}(\Sigma,\mathbb{R}) := \{f \text{ measurable}\,;\ (\nabla_{g_0})^k f \in L^q(\Sigma, g_0)\}$$

where $(\nabla_{g_0})^k$ denotes the k-th iteration of the Levi-Civita connection associated to g_0. Since the surface is closed, the space defined in this way is independent of g_0.

For $p \geq 1$, following [42] in which the case $p = 1$ was considered, we define the space $\mathcal{E}_{\Sigma,p}$ of *weak immersions of Σ into $\mathbb{R}^3$ with L^{2p} bounded second fundamental form* as follows:

$$\begin{aligned}\mathcal{E}_{\Sigma,p} := \{\vec{\Phi} \in W^{1,\infty}(\Sigma,\mathbb{R}^3)\,;\ \exists C > 1\ \ C^{-1} g_0 \leq \vec{\Phi}^* g_{\mathbb{R}^3} \leq C g_0 \text{ a.e.,}\\ \text{and the Gauss map } \vec{n}_{\vec{\Phi}} \text{ is in } W^{1,2p}(\Sigma, \mathrm{Gr}_2(\mathbb{R}^3))\}.\end{aligned}$$

For any $\vec{\Phi} \in \mathcal{E}_{\Sigma,p}$, starting from the equation $\Delta_{g_{\vec{\Phi}}}\vec{\Phi} = 2\vec{H}$, classical elliptic estimates permit one to bootstrap (in the case $p > 1$) and infer that $\vec{\Phi}$ is in fact in $W^{2,2p}(\Sigma,\mathbb{R}^3)$ (see [40]). It is then not difficult to observe that $\mathcal{E}_{\Sigma,p}$ is in fact, <u>for $p > 1$</u>, an open subset of the Banach space $W^{2,2p}(\Sigma,\mathbb{R}^3)$. The border line case $p = 1$ is more delicate; it was introduced first in [42] as being of primary interest for studying the variations of Willmore energy and is extensively presented also in [40].

[8] "Quasi" because we know that $\partial^2 W$ has at least the null directions given by the action of Möbius transformations.

III. Frame energies

For any weak immersion $\vec{\Phi}$ in $\mathcal{E}_{\Sigma,p}$ we denote by $A_{g_{\vec{\Phi}}}$ the connection 1-form—which lives on Σ since the tangent bundle is abelian—equal to the difference between the Levi-Civita connection $\nabla^{g_{\vec{\Phi}}}$ defined by $g_{\vec{\Phi}}$ and the Levi-Civita connection ∇^{g_0},

$$A_{g_{\vec{\Phi}}} := \nabla^{g_{\vec{\Phi}}} - \nabla^{g_0},$$

where g_0 is a constant scalar curvature metric of volume 1 on Σ for which there exists $\alpha : \Sigma \to \mathbb{R}$ such that

$$g_{\vec{\Phi}} = e^{2\alpha} g_0.$$

For $\Sigma \neq S^2$ the function α is defined without ambiguity, whereas in the sphere case we have to deal with the action of a "gauge group", the space $\mathcal{M}^+(S^2)$ of positive conformal transformations of S^2, and α is uniquely defined modulo this action. We shall now express $|A_{g_{\vec{\Phi}}}|^2_{g_{\vec{\Phi}}}$ locally using moving frames. Let (e_1, e_2) be an orthonormal local frame[9] for the metric $g_{\vec{\Phi}}$. We have[10]

$$|\nabla^{g_{\vec{\Phi}}} - \nabla^{g_0}|^2_{g_{\vec{\Phi}}} = \sum_{i,j=1}^{2} |((\nabla^{g_{\vec{\Phi}}} - \nabla^{g_0})e_i \cdot e_j)|^2_{g_{\vec{\Phi}}}.$$

Since e_i is a unit vector field for $g_{\vec{\Phi}}$ we have $(\nabla^{g_{\vec{\Phi}}} e_i, e_i) = 0$, and since $f_i := e^{\alpha} e_i$ is a unit vector field for g^0 we have

$$(\nabla^{g_0} e_i, e_i) = e^{-2\alpha}(\nabla^{g_0} f_i, f_i)_{g_{\vec{\Phi}}} - e^{-2\alpha} d e^{\lambda}(e_i, f_i)_{g_{\vec{\Phi}}} = -d\alpha.$$

Hence

$$\sum_{i=1}^{2} |((\nabla^{g_{\vec{\Phi}}} - \nabla^{g_0})e_i \cdot e_i)|^2_{g_{\vec{\Phi}}} = 2|d\alpha|^2_{g_{\vec{\Phi}}}.$$

In order to compute $|((\nabla^{g_{\vec{\Phi}}} - \nabla^{g_0})e_1 \cdot e_2)|^2_{g_{\vec{\Phi}}}$ we choose local conformal coordinates (x_1, x_2) for $g_{\vec{\Phi}}$. Then locally there exists μ such that $g_{\vec{\Phi}} = e^{2(\alpha+\mu)}[dx_1^2 + dx_2^2]$ and $g_0 = e^{2\mu}[dx_1^2 + dx_2^2]$. We choose $e_i := e^{-\alpha-\mu}\partial_{x_i}$ and thus $f_i = e^{-\mu}\partial_{x_i}$. Using the classical computation of the Levi-Civita connection of a metric in conformal charts (see [40]) we have

$$(\nabla^{g_{\vec{\Phi}}} e_1 \cdot e_2)_{g_{\vec{\Phi}}} = *d(\alpha + \mu) \quad \text{and} \quad (\nabla^{g_0} f_1 \cdot f_2)_{g_0} = *d\mu.$$

Since e_1 and e_2 are orthogonal to each other with respect to g_0, we have

$$(\nabla^{g_0} e_1 \cdot e_2)_{g_{\vec{\Phi}}} = e^{2\alpha}(\nabla^{g_0} e_1 \cdot e_2)_{g_0} = (\nabla^{g_0} f_1 \cdot f_2)_{g_0} = *d\mu.$$

9 We denote by $(\vec{e}_1, \vec{e}_2)$ the push-forward by $\vec{\Phi}$—its realization in $\mathbb{R}^3$—of the "abstract" orthonormal frame (e_1, e_2).

10 We shall denote by "·" the scalar product in the tangent space, by "," the scalar product in the cotangent space and by ";" the combination of the two scalar products.

Hence

$$((\nabla^{g_{\vec{\Phi}}} - \nabla^{g_0})e_1 \cdot e_2)_{g_{\vec{\Phi}}} = *d\alpha.$$

Combining the above we obtain

$$|\nabla^{g_{\vec{\Phi}}} - \nabla^{g_0}|^2_{g_{\vec{\Phi}}} = \sum_{i,j=1}^{2} |((\nabla^{g_{\vec{\Phi}}} - \nabla^{g_0})e_i, e_j)|^2_{g_{\vec{\Phi}}} = 4|d\alpha|^2_{g_{\vec{\Phi}}}. \tag{III.1}$$

For any C^1 function f we consider the following “frame energy”:

$$F_f(\vec{\Phi}) := \int_\Sigma [f(H_{g_{\vec{\Phi}}}) + 2^{-3}|A_{g_{\vec{\Phi}}}|^2_{g_{\vec{\Phi}}}]\, d\mathrm{vol}_{g_{\vec{\Phi}}}.$$

Observe that in the case of $\Sigma = T^2$ and $f(t) = t^2$, for some global Coulomb frame $\vec{e}$ (see [35]) we have

$$F_{t^2}(\vec{\Phi}) = \int_{T^2} [|H_{g_{\vec{\Phi}}}|^2 + 2^{-3}|A_{g_{\vec{\Phi}}}|^2_{g_{\vec{\Phi}}}]\, d\mathrm{vol}_{g_{\vec{\Phi}}} = \frac{1}{4}\int_{T^2} |d\vec{e}\,|^2\, d\mathrm{vol}_{g_{\vec{\Phi}}}, \tag{III.2}$$

which is nothing but the Dirichlet energy of the frame and justifies the name “frame energy”.

III.1. The first variation of frame energies

We shall now compute the first variation of F_f. We first concentrate on the second part

$$C(\vec{\Phi}) := \int_\Sigma |A_{g_{\vec{\Phi}}}|^2_{g_{\vec{\Phi}}}\, d\mathrm{vol}_{g_{\vec{\Phi}}},$$

which we call *connection energy*. We observe that locally for any unit vector field e for the metric $g_{\vec{\Phi}}$ one has

$$|d\alpha|^2_{g_{\vec{\Phi}}} = \big|d|e|_{g_0}\big|^2_{g_0}.$$

We consider a perturbation $\vec{\Phi}_t := \vec{\Phi} + t\vec{w}$. Recall the Liouville equation

$$-\Delta_{g_{\vec{\Phi}_t}}\alpha_t = K_{g_{\vec{\Phi}_t}} - e^{-2\alpha_t}K_{g_0}, \tag{III.3}$$

where $\Delta_{g_{\vec{\Phi}_t}}$ is the <u>negative</u> Laplace–Beltrami operator. Observe that K_{g_0} is independent of t. We then have

$$\frac{d}{dt}[\Delta_{g_{\vec{\Phi}_t}}\alpha_t] + 2K_{g_0}e^{-2\alpha}\frac{d\alpha}{dt} = -\frac{dK_{g_{\vec{\Phi}_t}}}{dt}. \tag{III.4}$$

Hence

$$\Delta_{g_{\vec{\Phi}_t}}\frac{d\alpha}{dt} + 2K_{g_0}e^{-2\alpha}\frac{d\alpha}{dt} = -\frac{d}{dt}[\Delta_{g_{\vec{\Phi}_t}}]\alpha - \frac{dK_{g_{\vec{\Phi}_t}}}{dt}. \tag{III.5}$$

In other words, we have

$$\Delta_{g_0}\frac{d\alpha}{dt}+2K_{g_0}\frac{d\alpha}{dt}=-e^{2\alpha}\frac{d}{dt}[\Delta_{g_{\vec{\Phi}_t}}]\alpha-e^{2\alpha}\frac{dK_{g_{\vec{\Phi}_t}}}{dt}. \tag{III.6}$$

In a local chart we have

$$K_{g_{\vec{\Phi}_t}}\,d\mathrm{vol}_{g_{\vec{\Phi}_t}}=\vec{n}\cdot\partial_{x_1}\vec{n}\times\partial_{x_2}\vec{n}\,dx_1\wedge dx_2. \tag{III.7}$$

Since

$$\frac{d\vec{n}}{dt}(0)=-\langle\vec{n}\cdot d\vec{w},d\vec{\Phi}\rangle_{g_{\vec{\Phi}}}, \tag{III.8}$$

we deduce

$$\begin{aligned}\frac{d(K_{g_{\vec{\Phi}_t}}\,d\mathrm{vol}_{g_{\vec{\Phi}_t}})}{dt}&=-\langle\vec{n}\cdot d\vec{w},\ d\vec{\Phi}\rangle_{g_{\vec{\Phi}}}\cdot\partial_{x_1}\vec{n}\times\partial_{x_2}\vec{n}\,dx_1\wedge dx_2\\ &\quad-[\vec{n}\cdot\partial_{x_1}\langle\vec{n}\cdot d\vec{w},\ d\vec{\Phi}\rangle_{g_{\vec{\Phi}}}\times\partial_{x_2}\vec{n}+\vec{n}\cdot\partial_{x_1}\vec{n}\times\partial_{x_2}\langle\vec{n}\cdot d\vec{w},\ d\vec{\Phi}\rangle_{g_{\vec{\Phi}}}]\,dx_1\wedge dx_2.\end{aligned}$$

We have

$$-\langle\vec{n}\cdot d\vec{w},\ d\vec{\Phi}\rangle_{g_{\vec{\Phi}}}\cdot\partial_{x_1}\vec{n}\times\partial_{x_2}\vec{n}=0. \tag{III.9}$$

We choose a chart in which $\vec{\Phi}$ is conformal and we denote

$$g_{\vec{\Phi}}=e^{2\lambda}[dx_1^2+dx_2^2]\quad\text{and}\quad g_0=e^{2\mu}[dx_1^2+dx_2^2],$$

thus $\alpha=\lambda-\mu$. We have on the one hand

$$\begin{aligned}-\vec{n}\cdot\partial_{x_1}\langle\vec{n}\cdot d\vec{w},\ d\vec{\Phi}\rangle_{g_{\vec{\Phi}}}\times\partial_{x_2}\vec{n}&=-\sum_{i=1}^{2}\partial_{x_1}(e^{-2\lambda}\vec{n}\cdot\partial_{x_i}\vec{w})\vec{n}\cdot\partial_{x_i}\vec{\Phi}\times\partial_{x_2}\vec{n}\\ &\quad-\sum_{i=1}^{2}e^{-2\lambda}\vec{n}\cdot\partial_{x_i}\vec{w}\ \vec{n}\cdot\partial^2_{x_ix_1}\vec{\Phi}\times\partial_{x_2}\vec{n}\\ &=\mathbb{I}_{22}\,\partial_{x_1}(e^{-2\lambda}\vec{n}\cdot\partial_{x_1}\vec{w})-\mathbb{I}_{21}\,\partial_{x_1}(e^{-2\lambda}\vec{n}\cdot\partial_{x_2}\vec{w})\\ &\quad-\langle\vec{n}\cdot d\vec{w},d\lambda\rangle_{g_{\vec{\Phi}}}\vec{n}\cdot\partial_{x_1}\vec{\Phi}\times\partial_{x_2}\vec{n}-\sum_{i=1}^{2}e^{-4\lambda}\vec{n}\cdot\partial_{x_i}\vec{w}\,\partial^2_{x_ix_1}\vec{\Phi}\cdot\partial_{x_2}\vec{\Phi}\,\mathbb{I}_{12},\end{aligned} \tag{III.10}$$

and on the other hand

$$\begin{aligned}-\vec{n}\cdot\partial_{x_1}\vec{n}\times\partial_{x_2}\langle\vec{n}\cdot d\vec{w},\ d\vec{\Phi}\rangle_{g_{\vec{\Phi}}}&=\sum_{i=1}^{2}\partial_{x_2}(e^{-2\lambda}\vec{n}\cdot\partial_{x_i}\vec{w})\vec{n}\cdot\partial_{x_i}\vec{\Phi}\times\partial_{x_1}\vec{n}\\ &\quad+\sum_{i=1}^{2}e^{-2\lambda}\vec{n}\cdot\partial_{x_i}\vec{w}\ \vec{n}\cdot\partial^2_{x_ix_2}\vec{\Phi}\times\partial_{x_1}\vec{n}\\ &=\mathbb{I}_{11}\,\partial_{x_2}(e^{-2\lambda}\vec{n}\cdot\partial_{x_2}\vec{w})-\mathbb{I}_{21}\,\partial_{x_2}(e^{-2\lambda}\vec{n}\cdot\partial_{x_1}\vec{w})\\ &\quad+\langle\vec{n}\cdot d\vec{w},d\lambda\rangle_{g_{\vec{\Phi}}}\vec{n}\cdot\partial_{x_2}\vec{\Phi}\times\partial_{x_1}\vec{n}-\sum_{i=1}^{2}e^{-4\lambda}\vec{n}\cdot\partial_{x_i}\vec{w}\ \ \partial^2_{x_ix_2}\vec{\Phi}\cdot\partial_{x_1}\vec{\Phi}\,\mathbb{I}_{12}.\end{aligned} \tag{III.11}$$

Summing (III.9)–(III.11) gives

$$\begin{aligned}\frac{d(K_{g_{\vec\Phi_t}}\, d\mathrm{vol}_{g_{\vec\Phi_t}})}{dt} &= [\mathbb{I}_{22}\partial_{x_1}(e^{-2\lambda}\vec n\cdot\partial_{x_1}\vec w)+\mathbb{I}_{11}\partial_{x_2}(e^{-2\lambda}\vec n\cdot\partial_{x_2}\vec w)]dx_1dx_2\\ &\quad-\mathbb{I}_{12}[\partial_{x_1}(e^{-2\lambda}\vec n\cdot\partial_{x_2}\vec w)+\partial_{x_2}(e^{-2\lambda}\vec n\cdot\partial_{x_1}\vec w)]dx_1dx_2\\ &\quad+2He^{2\lambda}\langle\vec n\cdot d\vec w, d\lambda\rangle_{g_{\vec\Phi}}dx_1dx_2.\end{aligned}\tag{III.12}$$

Recall Codazzi:

$$\begin{cases}\partial_{x_1}\mathbb{I}_{22}-\partial_{x_2}\mathbb{I}_{12}=H\partial_{x_1}e^{2\lambda},\\ \partial_{x_2}\mathbb{I}_{11}-\partial_{x_1}\mathbb{I}_{12}=H\partial_{x_2}e^{2\lambda}.\end{cases}\tag{III.13}$$

Hence we have proved the following lemma:

Lemma III.1. *Under the above notations we have*

$$\begin{aligned}\frac{d(K_{g_{\vec\Phi_t}}\, d\mathrm{vol}_{g_{\vec\Phi_t}})}{dt} &= [\partial_{x_1}(\mathbb{I}_{22}e^{-2\lambda}\vec n\cdot\partial_{x_1}\vec w)+\partial_{x_2}(\mathbb{I}_{11}e^{-2\lambda}\vec n\cdot\partial_{x_2}\vec w)]dx_1\wedge dx_2\\ &\quad-[\partial_{x_1}(\mathbb{I}_{12}e^{-2\lambda}\vec n\cdot\partial_{x_2}\vec w)+\partial_{x_2}(\mathbb{I}_{12}e^{-2\lambda}\vec n\cdot\partial_{x_1}\vec w)]dx_1\wedge dx_2.\end{aligned}\tag{III.14}$$

□

Recall

$$\frac{d}{dt}(d\mathrm{vol}_{g_{\vec\Phi}})(0)=\left[\sum_{i=1}^{2}\partial_{x_i}\vec\Phi\cdot\partial_{x_i}\vec w\right]dx_1\wedge dx_2=\langle d\vec\Phi, d\vec w\rangle_{g_{\vec\Phi}}\, d\mathrm{vol}_{g_{\vec\Phi}}.\tag{III.15}$$

Hence

$$\begin{aligned}e^{2\lambda}\frac{dK_{g_{\vec\Phi_t}}}{dt} &= \partial_{x_1}(\mathbb{I}_{22}e^{-2\lambda}\vec n\cdot\partial_{x_1}\vec w)+\partial_{x_2}(\mathbb{I}_{11}e^{-2\lambda}\vec n\cdot\partial_{x_2}\vec w)\\ &\quad-\partial_{x_1}(\mathbb{I}_{12}e^{-2\lambda}\vec n\cdot\partial_{x_2}\vec w)-\partial_{x_2}(\mathbb{I}_{12}e^{-2\lambda}\vec n\cdot\partial_{x_1}\vec w)-K_{g_{\vec\Phi}}\sum_{i=1}^{2}\partial_{x_i}\vec\Phi\cdot\partial_{x_i}\vec w.\end{aligned}\tag{III.16}$$

Recall

$$\Delta_g f := (\det(g_{kl}))^{-1/2}\sum_{i,j=1}^{2}\partial_{x_i}((\det(g_{kl}))^{1/2}g^{ij}\,\partial_{x_j}f)\tag{III.17}$$

and

$$\frac{dg_{ij}}{dt}(0)=\partial_{x_i}\vec w\cdot\partial_{x_j}\vec\Phi+\partial_{x_j}\vec w\cdot\partial_{x_i}\vec\Phi.\tag{III.18}$$

Since $\sum_i g_{ki}g^{ij}=\delta_{kj}$ and $g_{ki}=e^{2\lambda}\delta_{ki}$, we have

$$\frac{dg^{ij}}{dt}(0)=-e^{-4\lambda}[\partial_{x_i}\vec\Phi\cdot\partial_{x_j}\vec w+\partial_{x_j}\vec\Phi\cdot\partial_{x_i}\vec w].\tag{III.19}$$

Thus in particular

$$\frac{d}{dt}(\det(g_{ij}))^{1/2} = 2^{-1}(\det(g_{ij}))^{-1/2}e^{2\lambda}\left[\frac{dg_{11}}{dt}+\frac{dg_{22}}{dt}\right]$$
$$=\sum_{i=1}^{2}\partial_{x_i}\vec{\Phi}\cdot\partial_{x_i}\vec{w}, \tag{III.20}$$

and

$$\frac{d}{dt}(\det(g_{ij}))^{-1/2} = -e^{-4\lambda}\sum_{i=1}^{2}\partial_{x_i}\vec{\Phi}\cdot\partial_{x_i}\vec{w}. \tag{III.21}$$

Combining (III.17)–(III.21) we obtain

Lemma III.2. *Under the previous notations, for any function f independent of t on Σ we have*[11]

$$\frac{d(\Delta_{g_{\vec{\Phi}_t}})f}{dt} = \left\langle d\langle d\vec{\Phi}, d\vec{w}\rangle_{g_{\vec{\Phi}}}, df\right\rangle_{g_{\vec{\Phi}}} - *_{g_{\vec{\Phi}}}d*\left[[d\vec{\Phi}\dot{\otimes}d\vec{w}+d\vec{w}\dot{\otimes}d\vec{\Phi}]\llcorner_{g_{\vec{\Phi}}}df\right], \tag{III.22}$$

where we have explicitly in conformal coordinates

$$*_{g_{\vec{\Phi}}}d*\left[[d\vec{\Phi}\dot{\otimes}d\vec{w}+d\vec{w}\dot{\otimes}d\vec{\Phi}]\llcorner_{g_{\vec{\Phi}}}df\right]$$
$$= e^{-2\lambda}\sum_{i,j=1}^{2}\partial_{x_i}\left(e^{-2\lambda}(\partial_{x_i}\vec{\Phi}\cdot\partial_{x_j}\vec{w}+\partial_{x_j}\vec{\Phi}\cdot\partial_{x_i}\vec{w})\,\partial_{x_j}f\right). \qquad \square$$

We have

$$\frac{d}{dt}\left[\int_\Sigma |d\alpha_t|^2_{g_{\vec{\Phi}_t}}\, d\mathrm{vol}_{g_{\vec{\Phi}_t}}\right] = \sum_{i,j=1}^{2}\int_\Sigma \frac{dg^{ij}}{dt}\,\partial_{x_i}\alpha\,\partial_{x_j}\alpha\, d\mathrm{vol}_{g_{\vec{\Phi}}} + \int_\Sigma |d\alpha|^2_{g_{\vec{\Phi}}}\frac{d(d\mathrm{vol}_{g_{\vec{\Phi}_t}})}{dt}$$
$$-2\int_\Sigma \alpha\,\Delta_{g_{\vec{\Phi}}}\frac{d\alpha_t}{dt}\,d\mathrm{vol}_{g_{\vec{\Phi}}}. \tag{III.23}$$

We first have, using (III.19),

$$\sum_{i,j=1}^{2}\int_\Sigma \frac{dg^{ij}}{dt}\,\partial_{x_i}\alpha\,\partial_{x_j}\alpha\, d\mathrm{vol}_{g_{\vec{\Phi}}} = -2\int_\Sigma \langle d\vec{\Phi}, d\alpha\rangle_{g_{\vec{\Phi}}}\cdot\langle d\vec{w}, d\alpha\rangle_{g_{\vec{\Phi}}}\, d\mathrm{vol}_{g_{\vec{\Phi}}}. \tag{III.24}$$

Then, by (III.15),

$$\int_\Sigma |d\alpha|^2_{g_{\vec{\Phi}}}\frac{d(d\mathrm{vol}_{g_{\vec{\Phi}_t}})}{dt} = \int_\Sigma |d\alpha|^2_{g_{\vec{\Phi}}}\langle d\vec{\Phi}, d\vec{w}\rangle_{g_{\vec{\Phi}}}\, d\mathrm{vol}_{g_{\vec{\Phi}}}. \tag{III.25}$$

[11] The contraction operator $\llcorner_{g_{\vec{\Phi}}}$ between a quadratic form and a 1-form is defined as follows:

$$a\otimes b\llcorner_{g_{\vec{\Phi}}}c := \langle b, c\rangle_{g_{\vec{\Phi}}}\,a.$$

Using now (III.5) we have

$$
\begin{aligned}
-2\int_\Sigma \alpha\,\Delta_{g_{\vec\Phi}}\frac{d\alpha_t}{dt}\,d\mathrm{vol}_{g_{\vec\Phi}} &= -2\int_\Sigma \alpha\,\Delta_{g_0}\frac{d\alpha_t}{dt}\,d\mathrm{vol}_{g_0}\\
= 4K_{g_0}\int_\Sigma \alpha\frac{d\alpha}{dt}\,d\mathrm{vol}_{g_0} &+ 2\int_\Sigma \alpha\frac{d(\Delta_{g_{\vec\Phi_t}})\alpha}{dt}\,d\mathrm{vol}_{g_{\vec\Phi}} + 2\int_\Sigma \alpha\frac{dK_{g_{\vec\Phi_t}}}{dt}\,d\mathrm{vol}_{g_{\vec\Phi}}. \qquad \text{(III.26)}
\end{aligned}
$$

From Lemma III.2 we obtain

$$
\begin{aligned}
2\int_\Sigma \alpha\frac{d(\Delta_{g_{\vec\Phi_t}})\alpha}{dt}\,d\mathrm{vol}_{g_{\vec\Phi}} &= -2\int_\Sigma |d\alpha|^2_{g_{\vec\Phi}}\,\langle d\vec\Phi, d\vec w\rangle_{g_{\vec\Phi}}\,d\mathrm{vol}_{g_{\vec\Phi}}\\
-2\int_\Sigma \alpha\,\Delta_{g_{\vec\Phi}}\alpha\langle d\vec\Phi, d\vec w\rangle_{g_{\vec\Phi}}\,d\mathrm{vol}_{g_{\vec\Phi}} &+ 4\int_\Sigma \langle d\vec\Phi, d\alpha\rangle_{g_{\vec\Phi}}\cdot\langle d\vec w, d\alpha\rangle_{g_{\vec\Phi}}\,d\mathrm{vol}_{g_{\vec\Phi}}. \qquad \text{(III.27)}
\end{aligned}
$$

Now (III.16) yields

$$
\begin{aligned}
2\int_\Sigma \alpha\frac{dK_{g_{\vec\Phi_t}}}{dt}\,d\mathrm{vol}_{g_{\vec\Phi}} = &-2\sum_{i=1}^2\int_\Sigma \mathbb{I}_{ii}e^{-2\lambda}\vec n\cdot\partial_{x_{i+1}}\vec w\,\partial_{x_{i+1}}\alpha\,dx^2\\
&+2\int_\Sigma \mathbb{I}_{12}e^{-2\lambda}\vec n\cdot[\partial_{x_1}\vec w\,\partial_{x_2}\alpha+\partial_{x_2}\vec w\,\partial_{x_1}\alpha]\,dx^2\\
&-2\int_\Sigma \alpha K_{g_{\vec\Phi}}\langle d\vec\Phi, d\vec w\rangle_{g_{\vec\Phi}}\,d\mathrm{vol}_{g_{\vec\Phi}}. \qquad \text{(III.28)}
\end{aligned}
$$

Combining (III.26)–(III.28) gives, using $\Delta_{g_{\vec\Phi}}\alpha + K_{g_{\vec\Phi}} = e^{-2\alpha}\,K_{g_0}$,

$$
\begin{aligned}
-2\int_\Sigma \alpha\,\Delta_{g_{\vec\Phi}}\frac{d\alpha_t}{dt}\,d\mathrm{vol}_{g_{\vec\Phi}} = &-2\int_\Sigma |d\alpha|^2_{g_{\vec\Phi}}\,\langle d\vec\Phi, d\vec w\rangle_{g_{\vec\Phi}}\,d\mathrm{vol}_{g_{\vec\Phi}}\\
&+4\int_\Sigma \langle d\vec\Phi, d\alpha\rangle_{g_{\vec\Phi}}\cdot\langle d\vec w, d\alpha\rangle_{g_{\vec\Phi}}\,d\mathrm{vol}_{g_{\vec\Phi}} - 2\sum_{i=1}^2\int_\Sigma \mathbb{I}_{ii}e^{-2\lambda}\vec n\cdot\partial_{x_{i+1}}\vec w\partial_{x_{i+1}}\alpha\,dx^2\\
&+2\int_\Sigma \mathbb{I}_{12}e^{-2\lambda}\vec n\cdot[\partial_{x_1}\vec w\,\partial_{x_2}\alpha+\partial_{x_2}\vec w\,\partial_{x_1}\alpha]\,dx^2\\
&-2K_{g_0}\int_\Sigma \alpha e^{-2\alpha}\langle d\vec\Phi, d\vec w\rangle_{g_{\vec\Phi}}\,d\mathrm{vol}_{g_{\vec\Phi}} + 4K_{g_0}\int_\Sigma \alpha\frac{d\alpha}{dt}\,d\mathrm{vol}_{g_0}. \qquad \text{(III.29)}
\end{aligned}
$$

Observe that

$$
\begin{aligned}
-2K_{g_0}\int_\Sigma \alpha e^{-2\alpha}\langle d\vec\Phi, d\vec w\rangle_{g_{\vec\Phi}}\,d\mathrm{vol}_{g_{\vec\Phi}} &+ 4K_{g_0}\int_\Sigma \alpha\frac{d\alpha}{dt}\,d\mathrm{vol}_{g_0}\\
&= -2K_{g_0}\int_\Sigma \alpha\frac{d(d\mathrm{vol}_{g_0})}{dt}, \qquad \text{(III.30)}
\end{aligned}
$$

and since g_0 is normalized to have volume 1, we have

$$
\int_\Sigma \frac{d(d\mathrm{vol}_{g_0})}{dt} = 0, \qquad \text{(III.31)}
$$

which is consistent with the fact that the operation of adding a constant to α (i.e. dilations of $\vec{\Phi}$) generates a null direction for the Lagrangian C. Combining now (III.23)–(III.25) and (III.29) we obtain

$$\begin{aligned}\frac{d}{dt}\left[\int_\Sigma |d\alpha_t|^2_{g_{\vec{\Phi}_t}}\, d\mathrm{vol}_{g_{\vec{\Phi}_t}}\right] &= -\int_\Sigma |d\alpha|^2_{g_{\vec{\Phi}}} \langle d\vec{\Phi}, d\vec{w}\rangle_{g_{\vec{\Phi}}}\, d\mathrm{vol}_{g_{\vec{\Phi}}}\\ &\quad + 2\int_\Sigma \langle d\vec{\Phi}, d\alpha\rangle_{g_{\vec{\Phi}}}\cdot\langle d\vec{w}, d\alpha\rangle_{g_{\vec{\Phi}}}\, d\mathrm{vol}_{g_{\vec{\Phi}}} + 2\int_\Sigma (\vec{\mathbb{I}}\llcorner_{g_{\vec{\Phi}}}(*_{g_{\vec{\Phi}}} d\alpha))\,\dot{\wedge}\, d\vec{w}\\ &\quad - 2K_{g_0}\int_\Sigma \alpha e^{-2\alpha}\langle d\vec{\Phi}, d\vec{w}\rangle_{g_{\vec{\Phi}}}\, d\mathrm{vol}_{g_{\vec{\Phi}}} + 4K_{g_0}\int_\Sigma \alpha\frac{d\alpha}{dt}\, d\mathrm{vol}_{g_0}\end{aligned} \tag{III.32}$$

where we have explicitly, in positive conformal coordinates,

$$\begin{aligned}2\vec{\mathbb{I}}\llcorner_{g_{\vec{\Phi}}}(*_{g_{\vec{\Phi}}} d\alpha)\,\dot{\wedge}\, d\vec{w} &= -2\left[\sum_{i=1}^2 \mathbb{I}_{ii} e^{-2\lambda}\vec{n}\cdot\partial_{x_{i+1}}\vec{w}\,\partial_{x_{i+1}}\alpha\right] dx_1\wedge dx_2\\ &\quad + 2\mathbb{I}_{12} e^{-2\lambda}\vec{n}\cdot[\partial_{x_1}\vec{w}\,\partial_{x_2}\alpha + \partial_{x_2}\vec{w}\,\partial_{x_1}\alpha]\, dx_1\wedge dx_2\end{aligned} \tag{III.33}$$

where $\dot{\wedge}$ is the combination of the exterior product in the domain and the scalar product of vectors in the target. Moreover $d\alpha/dt$ solves the following PDE:

$$\begin{aligned}\Delta_{g_0}\frac{d\alpha}{dt} + 2K_{g_0}\frac{d\alpha}{dt} &= -\left\langle d(\langle d\vec{\Phi}, d\vec{w}\rangle_{g_{\vec{\Phi}}}), d\alpha\right\rangle_{g_0}\\ &\quad + *_{g_0} d *_{g_{\vec{\Phi}}}\left[[d\vec{\Phi}\,\dot{\otimes}\, d\vec{w} + d\vec{w}\,\dot{\otimes}\, d\vec{\Phi}]\llcorner_{g_{\vec{\Phi}}} d\alpha\right]\\ &\quad + *_{g_0} d[\mathbb{I}\llcorner_{g_{\vec{\Phi}}}(\vec{n}\cdot *_{g_{\vec{\Phi}}} d\vec{w})] + K_{g_{\vec{\Phi}}}\langle d\vec{\Phi}, d\vec{w}\rangle_{g_0}.\end{aligned} \tag{III.34}$$

Observe that

$$\begin{aligned}\frac{d}{dt}&\left[-\frac{1}{4}\int_\Sigma [2\alpha_t e^{-2\alpha_t} + e^{-2\alpha_t}]\, d\mathrm{vol}_{\vec{\Phi}_t}\right]\\ &= \int_\Sigma \alpha\frac{d\alpha}{dt}\, d\mathrm{vol}_{g_0} - \frac{1}{4}\int_\Sigma [2\alpha e^{-2\alpha} + e^{-2\alpha}]\langle d\vec{\Phi}, d\vec{w}\rangle_{g_{\vec{\Phi}}}\, d\mathrm{vol}_{g_{\vec{\Phi}}}.\end{aligned} \tag{III.35}$$

Hence

$$\begin{aligned}4K_{g_0}\int_\Sigma \alpha\frac{d\alpha}{dt}\, d\mathrm{vol}_{g_0} - 2K_{g_0}\int_\Sigma \alpha e^{-2\alpha}\langle d\vec{\Phi}, d\vec{w}\rangle_{g_{\vec{\Phi}}}\, d\mathrm{vol}_{g_{\vec{\Phi}}}&\\ = \frac{d}{dt}\left[-K_{g_0}\int_\Sigma [2\alpha_t e^{-2\alpha_t} + e^{-2\alpha_t}]\, d\mathrm{vol}_{\vec{\Phi}_t}\right]&\\ + K_{g_0}\int_\Sigma \alpha e^{-2\alpha}\langle d\vec{\Phi}, d\vec{w}\rangle_{g_{\vec{\Phi}}}\, d\mathrm{vol}_{g_{\vec{\Phi}}}.&\end{aligned} \tag{III.36}$$

Observe that

$$\int_\Sigma e^{-\alpha_t}\, d\mathrm{vol}_{\vec{\Phi}_t} = \int_\Sigma d\mathrm{vol}_{g_{0,t}} \equiv 1. \tag{III.37}$$

Combining (III.32), (III.36) and (III.37) we obtain

Lemma III.3. *Under the previous notations we have*

$$\frac{d}{dt}\left[\int_\Sigma [|d\alpha_t|^2_{g_{\vec{\Phi}_t}} + 2K_{g_0}\alpha_t e^{-2\alpha_t}]\, d\mathrm{vol}_{g_{\vec{\Phi}_t}}\right]$$
$$= \int_\Sigma (|d\alpha|^2_{g_{\vec{\Phi}}} *_{g_{\vec{\Phi}}} d\vec{\Phi}) \,\dot{\wedge}\, d\vec{w} - 2\int_\Sigma (\langle d\vec{\Phi}, d\alpha\rangle_{g_{\vec{\Phi}}} *_{g_{\vec{\Phi}}} d\alpha) \,\dot{\wedge}\, d\vec{w}$$
$$+ 2\int_\Sigma (\vec{\mathbb{I}}| \llcorner_{g_{\vec{\Phi}}} |(*_{g_{\vec{\Phi}}} d\alpha)) \,\dot{\wedge}\, d\vec{w} - K_{g_0}\int_\Sigma (\alpha e^{-2\alpha} *_{g_{\vec{\Phi}}} d\vec{\Phi}) \,\dot{\wedge}\, d\vec{w}.$$

This lemma justifies the introduction of the following modified *frame energy*:

$$\tilde{F}_f(\vec{\Phi}) := \int_\Sigma [f(H_{g_{\vec{\Phi}}}) + 2^{-3}|A_{g_{\vec{\Phi}}}|^2_{g_{\vec{\Phi}}} + K_{g_0}\alpha e^{-2\alpha}]\, d\mathrm{vol}_{g_{\vec{\Phi}}}$$
$$= \int_\Sigma [f(H_{g_{\vec{\Phi}}}) + 2^{-1}|d\alpha|^2_{g_{\vec{\Phi}}} + K_{g_0}\alpha e^{-2\alpha}]\, d\mathrm{vol}_{g_{\vec{\Phi}}}.$$

Lemma III.4. *Let $\vec{\Phi}$ be an immersion of the sphere S^2 and g_0 be a metric of constant curvature equal to 4π and volume 1 such that there exists a function α satisfying*

$$g_{\vec{\Phi}} = e^{2\alpha} g_0.$$

Then the Polyakov–Alvarez Lagrangian

$$L(\vec{\Phi}, g_0) := \int_{S^2} 2^{-1}|d\alpha|^2_{g_{\vec{\Phi}}}\, d\mathrm{vol}_{g_{\vec{\Phi}}} + 4\pi\int_{S^2} \alpha\, d\mathrm{vol}_{g_0}$$

is independent of the choice of g_0 and in this sense is gauge invariant for the gauge group given by the Möbius group of positive conformal transformations, $\mathcal{M}^+(S^2)$.

Proof. Let $\alpha(t)$ and $g_0(t)$ be smooth functions such that

$$g_{\vec{\Phi}} = e^{2\alpha(t)} g_0(t).$$

We have

$$\frac{d}{dt}L(\vec{\Phi}, g_0(t)) = -\int_{S^2} \Delta_{g_{\vec{\Phi}}}\alpha(t)\,\frac{d\alpha}{dt}\, d\mathrm{vol}_{g_{\vec{\Phi}}} + 4\pi\int_{S^2} \frac{d\alpha}{dt}[e^{-2\alpha} - 2\alpha e^{-2\alpha}]\, d\mathrm{vol}_{g_{\vec{\Phi}}}$$
$$= -\int_{S^2}\left[\Delta_{g_0(t)}\frac{d\alpha}{dt} + 8\pi\frac{d\alpha}{dt}\right]\alpha(t)\, d\mathrm{vol}_{g_0(t)} + 4\pi\int_{S^2}\frac{d\alpha}{dt}\, d\mathrm{vol}_{g_0(t)}.$$

Since $\int_{S^2} d\mathrm{vol}_{g_0(t)} \equiv 1$ we have on the one hand

$$0 = \frac{d}{dt}\int_{S^2} e^{-2\alpha(t)}\, d\mathrm{vol}_{g_{\vec{\Phi}}} = -2\int_{S^2}\frac{d\alpha}{dt}\, d\mathrm{vol}_{g_0(t)}.$$

The Liouville equation gives on the other hand

$$0 = \Delta_{g_{\vec{\Phi}}}\alpha(t) + K_{g_{\vec{\Phi}}} - 4\pi e^{-2\alpha(t)}.$$

Taking the derivative gives

$$0=\Delta_{g_{\vec{\Phi}}}\frac{d\alpha}{dt}+8\pi e^{-2\alpha(t)}\frac{d\alpha}{dt}=e^{-2\alpha(t)}\left[\Delta_{g_0(t)}\frac{d\alpha}{dt}+8\pi\frac{d\alpha}{dt}\right].$$

All the above says that $L(\vec{\Phi},g_0(t))$ is independent of t and Lemma III.4 is proved. □

Definition III.2. Let $\vec{\Phi}$ be a weak immersion in $\mathcal{E}_{S^2,1}$ and consider α such that there exists g_0 of constant curvature 4π and volume 1 such that

$$g_{\vec{\Phi}}=e^{2\alpha}g_0. \tag{III.38}$$

We define an *Aubin gauge* to be a choice of α and $\Psi\in\mathrm{Diff}(S^2)$ such that

$$\Psi^*g_0=\frac{g_{S^2}}{4\pi}\quad\text{and}\quad\forall j\in\{1,2,3\}\quad\int_{S^2}x_j e^{2\alpha\circ\Psi(x)}\,d\mathrm{vol}_{S^2}=0, \tag{III.39}$$

where g_{S^2} is the standard metric on S^2. □

We have the following theorem by E. Onofri.

Theorem III.1 ([37]). *For any weak immersion $\vec{\Phi}$ of S^2 and any α satisfying* (III.38)*,*

$$\int_{S^2}2^{-1}|d\alpha|^2_{g_{\vec{\Phi}}}\,d\mathrm{vol}_{g_{\vec{\Phi}}}+4\pi\int_{S^2}\alpha\,d\mathrm{vol}_{g_0}\geq 2\pi\log\left(\int_{S^2}e^{2\alpha}\,d\mathrm{vol}_{g_0}\right). \tag{III.40}$$

Moreover for any $\vec{\Phi}$ there exists an Aubin gauge (Ψ,α) satisfying (III.39).

We are going to use the following result proved by N. Ghoussoub and C. S. Lin.

Theorem III.2 ([19]). *For any weak immersion $\vec{\Phi}$ of S^2 and any α satisfying* (III.38) *and* (III.39)*,*

$$\int_{S^2}3^{-1}|d\alpha|^2_{g_{\vec{\Phi}}}\,d\mathrm{vol}_{g_{\vec{\Phi}}}+4\pi\int_{S^2}\alpha\,d\mathrm{vol}_{g_0}\geq 2\pi\log\int_{S^2}e^{2\alpha}\,d\mathrm{vol}_{g_0}. \tag{III.41}$$

□

It is suggested by A. Chang and P. Yang [13, Section 3] that the constant 3^{-1} could be replaced by 4^{-1} in (III.41).

III.2. The variation of the mean curvature

We have

$$\frac{d}{dt}H=\frac{1}{2}\sum_{i,j}\frac{dg^{ij}}{dt}\,\mathbb{I}_{ij}+\frac{1}{2}\sum_{i,j}g^{ij}\,\frac{d\mathbb{I}_{ij}}{dt}$$

and

$$\frac{dg_{ij}}{dt} = \frac{d}{dt}(\partial_{x_i}\vec{\Phi}_t \cdot \partial_{x_j}\vec{\Phi}_t) = \partial_{x_i}\vec{w} \cdot \partial_{x_j}\vec{\Phi} + \partial_{x_i}\vec{\Phi} \cdot \partial_{x_j}\vec{w}.$$

Since

$$e^{-2\lambda}\,\frac{dg_{ik}}{dt}\delta_{kj} + e^{2\lambda}\,\frac{dg^{kj}}{dt}\delta_{ik} = 0,$$

we have

$$\frac{dg^{ij}}{dt} = -e^{-4\lambda}[\partial_{x_i}\vec{w} \cdot \partial_{x_j}\vec{\Phi} + \partial_{x_i}\vec{\Phi} \cdot \partial_{x_j}\vec{w}].$$

So

$$\frac{1}{2}\sum_{i,j}\frac{dg^{ij}}{dt}\,\mathbb{I}_{ij} = e^{-4\lambda}\,\frac{1}{2}\sum_{i,j}[\partial_{x_i}\vec{w} \cdot \partial_{x_j}\vec{\Phi} + \partial_{x_i}\vec{\Phi} \cdot \partial_{x_j}\vec{w}]\,\partial_{x_i}\vec{n} \cdot \partial_{x_j}\vec{\Phi} = e^{-2\lambda}\nabla\vec{w} \cdot \nabla\vec{\Phi}.$$

We have moreover

$$\frac{d\mathbb{I}_{ij}}{dt} = -\frac{d}{dt}(\partial_{x_i}\vec{n}_t \cdot \partial_{x_j}\vec{\Phi}_t) = -\partial_{x_i}\frac{d\vec{n}_t}{dt} \cdot \partial_{x_j}\vec{\Phi} - \partial_{x_i}\vec{n} \cdot \partial_{x_j}\vec{w}.$$

Combining the previous assertions we get

$$\frac{dH}{dt} = -\frac{e^{-2\lambda}}{2}\nabla\vec{n} \cdot \nabla\vec{w} - \frac{e^{-2\lambda}}{2}\nabla\frac{d\vec{n}_t}{dt} \cdot \nabla\vec{\Phi} + e^{-2\lambda}\nabla\vec{n} \cdot \nabla\vec{\Phi}.$$

Since $\frac{d\vec{n}}{dt} \cdot \Delta\vec{\Phi} = 0$, we obtain

$$\frac{dH}{dt} = \frac{e^{-2\lambda}}{2}\nabla\vec{n} \cdot \nabla\vec{w} - \frac{e^{-2\lambda}}{2}\,\mathrm{div}\left(\frac{d\vec{n}_t}{dt} \cdot \nabla\vec{\Phi}\right).$$

As $\frac{d\vec{n}}{dt} = -e^{-2\lambda}\vec{n} \cdot \nabla\vec{w}\nabla\vec{\Phi}$, it follows that

$$\mathrm{div}\left(\frac{d\vec{n}_t}{dt} \cdot \nabla\vec{\Phi}\right) = -\,\mathrm{div}(\vec{n} \cdot \nabla\vec{w}).$$

Combining the above gives

$$\begin{aligned}\frac{dH}{dt} &= \frac{e^{-2\lambda}}{2}[\mathrm{div}(\vec{n} \cdot \nabla\vec{w}) + \nabla\vec{n} \cdot \nabla\vec{w}]\\ &= -2^{-1}d^{*_{g_{\vec{\Phi}}}}(\vec{n} \cdot d\vec{w}) + 2^{-1}\langle d\vec{w}; d\vec{n}\rangle_{g_{\vec{\Phi}}}.\end{aligned} \tag{III.42}$$

Hence, for any C^1 function f,

$$\begin{aligned}\frac{d}{dt}\left[\int_\Sigma f(H_{\vec{\Phi}_t})\,d\mathrm{vol}_{\vec{\Phi}_t}\right] &= -2^{-1}\int_\Sigma f'(H)d^{*_{g_{\vec{\Phi}}}}(\vec{n} \cdot d\vec{w})\,d\mathrm{vol}_{g_{\vec{\Phi}}}\\ &\quad + 2^{-1}\int_\Sigma f'(H)\langle d\vec{w}; d\vec{n}\rangle_{g_{\vec{\Phi}}}\,d\mathrm{vol}_{g_{\vec{\Phi}}} + \int_\Sigma f(H)\langle d\vec{w}; d\vec{\Phi}\rangle_{g_{\vec{\Phi}}}\,d\mathrm{vol}_{g_{\vec{\Phi}}}.\end{aligned} \tag{III.43}$$

We can therefore deduce the following result.

Lemma III.5. *Let f be a C^1 function. Under the above notations we have*

$$\frac{d}{dt}\left[\int_\Sigma f(H_{g_{\vec\Phi_t}})\, d\mathrm{vol}_{g_{\vec\Phi_t}}\right] = 2^{-1}\int_\Sigma (*_{g_{\vec\Phi}} d[f'(H)\vec n]) \dot\wedge d\vec w$$
$$- \int_\Sigma (f'(H) *_{g_{\vec\Phi}} d\vec n) \dot\wedge d\vec w - \int_\Sigma (f(H) *_{g_{\vec\Phi}} d\vec\Phi) \dot\wedge d\vec w. \qquad \square$$

Observe moreover that from (III.20) we can also deduce the following elementary lemma.

Lemma III.6. *Under the above notations we have*

$$\begin{aligned}\frac{d}{dt}\log\left[\int_\Sigma e^{2\alpha}\, d\mathrm{vol}_{g_0}\right] &= [A_{\vec\Phi}(\Sigma)]^{-1}\int_\Sigma \langle d\vec\Phi, d\vec w\rangle_{g_{\vec\Phi}}\, d\mathrm{vol}_{g_{\vec\Phi}}\\ &= -[A_{\vec\Phi}(\Sigma)]^{-1}\int_\Sigma [*_{g_{\vec\Phi}} d\vec\Phi] \dot\wedge d\vec w,\end{aligned} \tag{III.44}$$

where $A_{\vec\Phi}(\Sigma) := \int_\Sigma e^{2\alpha}\, d\mathrm{vol}_{g_0} = \int_\Sigma d\mathrm{vol}_{g_{\vec\Phi}}$ is the area of the immersion $\vec\Phi$.

III.3. The first variation of frame energies and conservation laws

Combining Lemmas III.3 and III.5 we obtain

Lemma III.7. *Let Σ be a closed oriented two-dimensional manifold. Let f be a C^1 function on $\mathbb{R}$, let $\vec\Phi$ be an immersion of Σ into $\mathbb{R}^3$, and let $g_{\vec\Phi}$ be the induced metric on Σ. Let g_0 be a constant Gauss curvature metric*[12] *of volume 1 on Σ such that there exists α with $g_{\vec\Phi} = e^{2\alpha} g_0$. The immersion is a critical point of*

$$\tilde F_f^\Lambda(\vec\Phi) := \int_\Sigma [f(H) + 2^{-1}|d\alpha|^2_{g_{\vec\Phi}} + K_{g_0}\alpha e^{-2\alpha}]\, d\mathrm{vol}_{g_{\vec\Phi}} - \Lambda\log\left(\int_\Sigma e^{2\alpha}\, d\mathrm{vol}_{g_0}\right) \tag{III.45}$$

if and only if the following conservation law (i.e. closedness of a 1-form) holds:

$$\begin{aligned}d\Big[&*_{g_{\vec\Phi}} d[f'(H)\vec n] - 2f'(H) *_{g_{\vec\Phi}} d\vec n\\ &+ [-2f(H) + |d\alpha|^2_{g_{\vec\Phi}} - K_{g_0}\alpha e^{-2\alpha} + 2\Lambda[A_{\vec\Phi}(\Sigma)]^{-1}] *_{g_{\vec\Phi}} d\vec\Phi\\ &- 2\langle d\vec\Phi, d\alpha\rangle_{g_{\vec\Phi}} *_{g_{\vec\Phi}} d\alpha + 2\vec{\mathbb{I}} \llcorner_{g_{\vec\Phi}} (*_{g_{\vec\Phi}} d\alpha)\Big] = 0.\end{aligned} \tag{III.46}$$

$\square$

[12] Observe that g_0 is unique if Σ has non-zero genus. When $\Sigma \simeq S^2$ it is unique modulo the action of the Möbius group $\mathcal{M}^+(S^2)$. Nevertheless, because of Lemma III.4, the Lagrangian, and hence the Euler–Lagrange equation, is insensitive to the gauge action.

Assume $\vec{\Phi}$ is a critical point of $\tilde{F}_f^\Lambda$ given by (III.45) and denote locally

$$
\begin{aligned}
d\vec{L} &:= *_{g_{\vec{\Phi}}} d[f'(H)\vec{n}] - 2f'(H) *_{g_{\vec{\Phi}}} d\vec{n} \\
&\quad + \left[-2f(H) + |d\alpha|^2_{g_{\vec{\Phi}}} - K_{g_0}\alpha e^{-2\alpha} + 2\Lambda [A_{\vec{\Phi}}(S^2)]^{-1}\right] *_{g_{\vec{\Phi}}} d\vec{\Phi} \\
&\quad - 2\langle d\vec{\Phi}, d\alpha\rangle_{g_{\vec{\Phi}}} *_{g_{\vec{\Phi}}} d\alpha + 2\vec{\mathbb{I}} \llcorner_{g_{\vec{\Phi}}} (*_{g_{\vec{\Phi}}} d\alpha).
\end{aligned}
\tag{III.47}
$$

In conformal coordinates this gives

$$
\begin{aligned}
\partial_{x_1}\vec{L} &= -\partial_{x_2} f'(H)\vec{n} + f'(H)\partial_{x_2}\vec{n} \\
&\quad - \left[-2f(H) + |d\alpha|^2_{g_{\vec{\Phi}}} - K_{g_0}\alpha e^{-2\alpha} + 2\Lambda [A_{\vec{\Phi}}(S^2)]^{-1}\right] \partial_{x_2}\vec{\Phi} \\
&\quad + 2\langle d\vec{\Phi}, d\alpha\rangle_{g_{\vec{\Phi}}}\, \partial_{x_2}\alpha + 2e^{-2\lambda} \sum_{i=1}^2 \vec{\mathbb{I}}_{1i} (*_{g_{\vec{\Phi}}} d\alpha)_i,
\end{aligned}
\tag{III.48}
$$

and

$$
\begin{aligned}
\partial_{x_2}\vec{L} &= \partial_{x_1} f'(H)\vec{n} - f'(H)\, \partial_{x_1}\vec{n} \\
&\quad + \left[-2f(H) + |d\alpha|^2_{g_{\vec{\Phi}}} - K_{g_0}\alpha e^{-2\alpha} + 2\Lambda [A_{\vec{\Phi}}(S^2)]^{-1}\right] \partial_{x_1}\vec{\Phi} \\
&\quad - 2\langle d\vec{\Phi}, d\alpha\rangle_{g_{\vec{\Phi}}}\, \partial_{x_1}\alpha + 2e^{-2\lambda} \sum_{i=1}^2 \vec{\mathbb{I}}_{2i} (*_{g_{\vec{\Phi}}} d\alpha)_i.
\end{aligned}
\tag{III.49}
$$

We have

$$
\begin{aligned}
d\vec{\Phi} \dot{\wedge} d\vec{L} &:= [\partial_{x_1}\vec{\Phi} \cdot \partial_{x_2}\vec{L} - \partial_{x_2}\vec{\Phi} \cdot \partial_{x_1}\vec{L}]\, dx_1 \wedge dx_2 \\
&= 2(f'(H)H - 2f(H) - K_{g_0}\alpha e^{-2\alpha} + 2\Lambda [A_{\vec{\Phi}}(S^2)]^{-1})\, d\mathrm{vol}_{g_{\vec{\Phi}}}.
\end{aligned}
\tag{III.50}
$$

We also have

$$
\begin{aligned}
d\vec{\Phi} \wedge d\vec{L} &:= [\partial_{x_1}\vec{\Phi} \times \partial_{x_2}\vec{L} - \partial_{x_2}\vec{\Phi} \times \partial_{x_1}\vec{L}]\, dx_1 \wedge dx_2 \\
&= \partial_{x_1}\vec{\Phi} \times [\partial_{x_1} f'(H)\vec{n} - f'(H)\partial_{x_1}\vec{n} - 2\langle d\vec{\Phi}, d\alpha\rangle_{g_{\vec{\Phi}}}\, \partial_{x_1}\alpha]\, dx_1 \wedge dx_2 \\
&\quad + 2e^{-2\lambda} \sum_{i=1}^2 \partial_{x_1}\vec{\Phi} \times \vec{\mathbb{I}}_{2i} (*_{g_{\vec{\Phi}}} d\alpha)_i\, dx_1 \wedge dx_2 \\
&\quad + \partial_{x_2}\vec{\Phi} \times [\partial_{x_2} f'(H)\vec{n} - f'(H)\partial_{x_2}\vec{n} - 2\langle d\vec{\Phi}, d\alpha\rangle_{g_{\vec{\Phi}}}\, \partial_{x_2}\alpha]\, dx_1 \wedge dx_2 \\
&\quad - 2e^{-2\lambda} \sum_{i=1}^2 \partial_{x_2}\vec{\Phi} \times \vec{\mathbb{I}}_{1i} (*_{g_{\vec{\Phi}}} d\alpha)_i\, dx_1 \wedge dx_2.
\end{aligned}
\tag{III.51}
$$

This gives

$$
\begin{aligned}
d\vec{\Phi} \wedge d\vec{L} &:= [\partial_{x_1}\vec{\Phi}\, \partial_{x_2} f'(H) - \partial_{x_2}\vec{\Phi}\, \partial_{x_1} f'(H)]\, dx_1 \wedge dx_2 \\
&\quad - 2e^{-2\lambda} \sum_{i,j=1}^2 \mathbb{I}_{ij} (*_{g_{\vec{\Phi}}} d\alpha)_j\, \partial_{x_i}\vec{\Phi}\, dx_1 \wedge dx_2.
\end{aligned}
\tag{III.52}
$$

We have

$$-2e^{-2\lambda}\sum_{i,j=1}^{2}\mathbb{I}_{ij}\,(*_{g_{\vec\Phi}}d\alpha)_j\,\partial_{x_i}\vec\Phi = 2\partial_{x_1}\alpha[e^{-2\lambda}\,\mathbb{I}_{12}\,\partial_{x_1}\vec\Phi + e^{-2\lambda}\,\mathbb{I}_{22}\,\partial_{x_2}\vec\Phi] \\ - 2\partial_{x_2}\alpha[e^{-2\lambda}\,\mathbb{I}_{11}\,\partial_{x_1}\vec\Phi + e^{-2\lambda}\,\mathbb{I}_{12}\,\partial_{x_2}\vec\Phi]. \quad \text{(III.53)}$$

We compute

$$\begin{aligned} &e^{2\lambda}\,\partial_{x_1}[e^{-2\lambda}\,\mathbb{I}_{12}\,\partial_{x_1}\vec\Phi + e^{-2\lambda}\,\mathbb{I}_{22}\,\partial_{x_2}\vec\Phi] - e^{2\lambda}\,\partial_{x_2}[e^{-2\lambda}\,\mathbb{I}_{11}\,\partial_{x_1}\vec\Phi + e^{-2\lambda}\,\mathbb{I}_{12}\,\partial_{x_2}\vec\Phi] \\ &\quad = \partial_{x_1}\vec\Phi[-2\partial_{x_1}\lambda\,\mathbb{I}_{12} + \partial_{x_1}\mathbb{I}_{12} + e^{-2\lambda}\,\mathbb{I}_{12}\,\partial^2_{x_1^2}\vec\Phi\cdot\partial_{x_1}\vec\Phi + e^{-2\lambda}\,\mathbb{I}_{22}\,\partial^2_{x_1x_2}\vec\Phi\cdot\partial_{x_1}\vec\Phi] \\ &\qquad + \partial_{x_2}\vec\Phi[-2\partial_{x_1}\lambda\,\mathbb{I}_{22} + \partial_{x_1}\mathbb{I}_{22} + e^{-2\lambda}\,\mathbb{I}_{22}\,\partial^2_{x_1x_2}\vec\Phi\cdot\partial_{x_2}\vec\Phi + e^{-2\lambda}\,\mathbb{I}_{12}\,\partial^2_{x_1^2}\vec\Phi\cdot\partial_{x_2}\vec\Phi] \\ &\qquad - \partial_{x_1}\vec\Phi[-2\partial_{x_2}\lambda\,\mathbb{I}_{11} + \partial_{x_2}\mathbb{I}_{11} + e^{-2\lambda}\,\mathbb{I}_{11}\,\partial^2_{x_1x_2}\vec\Phi\cdot\partial_{x_1}\vec\Phi + e^{-2\lambda}\,\mathbb{I}_{12}\,\partial^2_{x_2^2}\vec\Phi\cdot\partial_{x_1}\vec\Phi] \\ &\qquad - \partial_{x_2}\vec\Phi[-2\partial_{x_2}\lambda\,\mathbb{I}_{12} + \partial_{x_2}\mathbb{I}_{12} + e^{-2\lambda}\,\mathbb{I}_{12}\,\partial^2_{x_2^2}\vec\Phi\cdot\partial_{x_2}\vec\Phi + e^{-2\lambda}\,\mathbb{I}_{11}\,\partial^2_{x_1x_2}\vec\Phi\cdot\partial_{x_2}\vec\Phi]. \end{aligned} \quad \text{(III.54)}$$

Making use of the Codazzi identity (III.13) we finally obtain

$$\begin{aligned} &e^{2\lambda}\,\partial_{x_1}[e^{-2\lambda}\,\mathbb{I}_{12}\,\partial_{x_1}\vec\Phi + e^{-2\lambda}\,\mathbb{I}_{22}\,\partial_{x_2}\vec\Phi] - e^{2\lambda}\,\partial_{x_2}[e^{-2\lambda}\,\mathbb{I}_{11}\,\partial_{x_1}\vec\Phi + e^{-2\lambda}\,\mathbb{I}_{12}\,\partial_{x_2}\vec\Phi] \\ &\quad = \partial_{x_1}\vec\Phi\big[-H\,\partial_{x_2}e^{2\lambda} + \partial_{x_2}\lambda\,[\mathbb{I}_{22}+\mathbb{I}_{11}]\big] + \partial_{x_2}\vec\Phi\big[H\,\partial_{x_1}e^{2\lambda} - \partial_{x_1}\lambda\,[\mathbb{I}_{22}+\mathbb{I}_{11}]\big] \\ &\quad = 0. \end{aligned}$$

Hence there exists locally $\vec D$ such that (see also [35, Lemma III.2])

$$\begin{cases} \partial_{x_1}\vec D := [e^{-2\lambda}\,\mathbb{I}_{11}\,\partial_{x_1}\vec\Phi + e^{-2\lambda}\,\mathbb{I}_{12}\,\partial_{x_2}\vec\Phi], \\ \partial_{x_2}\vec D := [e^{-2\lambda}\,\mathbb{I}_{12}\,\partial_{x_1}\vec\Phi + e^{-2\lambda}\,\mathbb{I}_{22}\,\partial_{x_2}\vec\Phi]. \end{cases} \quad \text{(III.55)}$$

Combining all the above we obtain the following lemma.

Lemma III.8. *Let Σ be a closed two-dimensional manifold. Let f be a C^1 function on $\mathbb{R}$, let $\vec\Phi$ be an immersion of Σ into $\mathbb{R}^3$, let $g_{\vec\Phi}$ be the induced metric on Σ. Let g_0 be a constant Gauss curvature metric of volume 1 on Σ such that there exists α with $g_{\vec\Phi} = e^{2\alpha}g_0$. Assume the immersion $\vec\Phi$ is a critical point of*

$$\tilde F_f^{\Lambda}(\vec\Phi) := \int_\Sigma [f(H) + 2^{-1}|d\alpha|^2_{g_{\vec\Phi}} + K_{g_0}\alpha e^{-2\alpha}]\,d\mathrm{vol}_{g_{\vec\Phi}} - \Lambda\log\left(\int_\Sigma e^{2\alpha}\,d\mathrm{vol}_{g_0}\right),$$

and following Lemma III.7 introduce locally $\vec L$ by

$$\begin{aligned} d\vec L := {}& *_{g_{\vec\Phi}}d[f'(H)\vec n] - 2f'(H)*_{g_{\vec\Phi}}d\vec n \\ & + \big[-2f(H) + |d\alpha|^2_{g_{\vec\Phi}} - K_{g_0}\alpha e^{-2\alpha} + 2\Lambda[A_{\vec\Phi}(S^2)]^{-1}\big]*_{g_{\vec\Phi}}d\vec\Phi \\ & - 2\langle d\vec\Phi, d\alpha\rangle_{g_{\vec\Phi}}*_{g_{\vec\Phi}}d\alpha + 2\vec{\mathbb{I}}\llcorner_{g_{\vec\Phi}}(*_{g_{\vec\Phi}}d\alpha). \end{aligned} \quad \text{(III.56)}$$

Then the following almost conservation law holds:

$$d\vec{\Phi} \dot{\wedge} d\vec{L} = 2\big(f'(H)H - 2f(H) - K_{g_0}\alpha e^{-2\alpha} + 2\Lambda[A_{\vec{\Phi}}(S^2)]^{-1}\big)\, d\mathrm{vol}_{g_{\vec{\Phi}}}, \tag{III.57}$$

and the following exact conservation law holds:

$$d\vec{\Phi} \wedge d\vec{L} = d\vec{\Phi} \wedge df'(H) + 2\, d\alpha \wedge d\vec{D}, \tag{III.58}$$

where $\vec{D}$ satisfies

$$d\vec{D} = \mathbb{I} \llcorner_g \, d\vec{\Phi}. \tag{III.59}$$

□

Remark III.1. The three conservation laws or almost conservation laws (III.56)–(III.58) can be deduced from the Noether theorem (see [5]). More lrecisely, the existence of $\vec{L}$ satisfying the first conservation law (III.56) is due to the translation invariance of the Lagrangian $\tilde{F}_f^\Lambda$, (III.57) instead is related to the lack of invariance of the Lagrangian under dilation, whereas (III.58) is related to the rotation invariance of the Lagrangian. □

III.4. Various bounds involving frame energies

First of all we establish the following lemma.

Lemma III.9. *Under the previous notations we have, for any $\sigma > 0$,*

$$\left|\frac{\log(\int_{S^2} e^{2\alpha}\, d\mathrm{vol}_{g_0})}{\log \frac{1}{\sigma}}\right| \le 2 + \frac{\log(\sigma^2 \int_{S^2}(1+H^2)^2\, d\mathrm{vol}_{g_{\vec{\Phi}}})}{\log \frac{1}{\sigma}}. \tag{III.60}$$

Proof. Obviously

$$\log\left(\int_{S^2} e^{2\alpha}\, d\mathrm{vol}_{g_0}\right) \le \log\left(\sigma^2 \int_{S^2} (1+H^2)^2\, d\mathrm{vol}_{g_{\vec{\Phi}}}\right) + 2\log\frac{1}{\sigma}. \tag{III.61}$$

We also have

$$16\pi^2 \le \left(\int_{S^2} H_{\vec{\Phi}}^2\, d\mathrm{vol}_{g_{\vec{\Phi}}}\right)^2 \le \int_{S^2} e^{2\alpha}\, d\mathrm{vol}_{g_0} \int_{S^2} H_{\vec{\Phi}}^4\, d\mathrm{vol}_{g_{\vec{\Phi}}}. \tag{III.62}$$

Hence

$$2\log\left(\frac{1}{\sigma}\right) + \log\left(\sigma^2 \int_{S^2} (1+H^2)^2\, d\mathrm{vol}_{g_{\vec{\Phi}}}\right) \ge -\log\left(\int_{S^2} e^{2\alpha}\, d\mathrm{vol}_{g_0}\right) + \log 16\pi^2. \tag{III.63}$$

Combining (III.61) and (III.63) gives (III.60), and Lemma III.9 is proved. □

The following useful lemma is a direct consequence of Theorem III.2.

Lemma III.10. *Let $\vec{\Phi}$ be a weak immersion of S^2 and (Ψ, α) be an Aubin gauge satisfying (III.39). Then*

$$6^{-1}\int_{S^2}|d\alpha|^2_{g_0}\,d\mathrm{vol}_{g_0} \leq \int_{S^2}2^{-1}|d\alpha|^2_{g_{\vec{\Phi}}}\,d\mathrm{vol}_{g_{\vec{\Phi}}}+4\pi\int_{S^2}\alpha\,d\mathrm{vol}_{g_0}-2\pi\log\left(\int_{S^2}e^{2\alpha}\,d\mathrm{vol}_{g_0}\right). \tag{III.64}$$

□

For Σ being an arbitrary closed surface we denote

$$\begin{aligned} F^\sigma(\vec{\Phi}) &:= \left(\log\frac{1}{\sigma}\right)^{-1}\tilde{F}^{K_{g_0}}_{f_\sigma}(\vec{\Phi}) \\ &:= \left(\log\frac{1}{\sigma}\right)^{-1}\int_\Sigma[f_\sigma(H_{g_{\vec{\Phi}}})+2^{-1}|d\alpha|^2_{g_{\vec{\Phi}}}+K_{g_0}\alpha e^{-2\alpha}]\,d\mathrm{vol}_{g_{\vec{\Phi}}} \\ &\quad -2^{-1}K_{g_0}\left(\log\frac{1}{\sigma}\right)^{-1}\log\left(\int_\Sigma e^{2\alpha}\,d\mathrm{vol}_{g_0}\right) \end{aligned} \tag{III.65}$$

where $f_\sigma(t)=\log(\sigma^{-1})[t^2+\sigma^2(1+t^2)^2]$. First, in the particular case $\Sigma=S^2$ we have

Lemma III.11. *Let $\vec{\Phi}$ be a weak immersion of S^2 in $\mathcal{E}_{S^2,2}$ and g_0 be a constant Gauss curvature metric on S^2 of volume 1 such that $g_{\vec{\Phi}}=e^{2\alpha}g_0$. For $\sigma\in(0,1)$ we have*

$$W(\vec{\Phi})+\sigma^2\int_{S^2}(1+H^2)^2\,d\mathrm{vol}_{g_{\vec{\Phi}}}\leq F^\sigma(\vec{\Phi}) \tag{III.66}$$

and for any Aubin gauge we have

$$\left(\log\frac{1}{\sigma}\right)^{-1}\int_{S^2}|d\alpha|^2_{g_{\vec{\Phi}}}\,d\mathrm{vol}_{g_{\vec{\Phi}}}\leq 6[F^\sigma(\vec{\Phi})-W(\vec{\Phi})]. \tag{III.67}$$

Also, for any gauge,

$$\inf_{x\in S^2}\alpha\geq\frac{1}{2}\log A_{\vec{\Phi}}(S^2)-\frac{1}{2\pi}\int_{S^2}|d\alpha|^2_{g_0}\,d\mathrm{vol}_{g_0}\;-C[1+W(\vec{\Phi})] \tag{III.68}$$

and

$$\left(\log\frac{1}{\sigma}\right)^{-1}\|\alpha\|_{L^\infty(S^2)} \leq C\left(\log\frac{1}{\sigma}\right)^{-1}\left[\int_{S^2}|d\alpha|^2_{g_{\vec{\Phi}}}\,d\mathrm{vol}_{g_{\vec{\Phi}}}+W(\vec{\Phi})\right]+\left|\frac{\log A_{\vec{\Phi}}(S^2)}{2\log\frac{1}{\sigma}}\right| \tag{III.69}$$

where C is a positive universal constant. Moreover,

$$\frac{\log A_{\vec{\Phi}}(S^2)}{2\log\frac{1}{\sigma}}\leq 1+\left(\log\frac{1}{\sigma}\right)^{-1}\log[F^\sigma(\vec{\Phi})-W(\vec{\Phi})]. \tag{III.70}$$

□

Proof. First of all, the Onofri inequality (III.40) implies

$$F^\sigma(\vec{\Phi}) \geq W(\vec{\Phi}) + \sigma^2 \int_{S^2} (1 + H^2)^2 \, d\mathrm{vol}_{g_{\vec{\Phi}}}. \tag{III.71}$$

We also have

$$\begin{aligned} F^\sigma(\vec{\Phi}) - W(\vec{\Phi}) \geq & \left(\log \frac{1}{\sigma}\right)^{-1} \int_{S^2} [2^{-1} |d\alpha|^2_{g_{\vec{\Phi}}} + K_{g_0} \alpha e^{-2\alpha}] \, d\mathrm{vol}_{g_{\vec{\Phi}}} \\ & - 2\pi \left(\log \frac{1}{\sigma}\right)^{-1} \log\left(\int_{S^2} e^{2\alpha} \, d\mathrm{vol}_{g_0}\right). \end{aligned} \tag{III.72}$$

Then we obtain (III.67) directly from (III.64).

By the uniformization theorem on S^2, modulo the action of a conformal diffeomorphism, we can assume $g_0 = g_{S^2}/(4\pi)$ where g_{S^2} is the standard metric on S^2. The immersion $\vec{\Phi}$ is then conformal from the standard sphere into $\mathbb{R}^3$.

The Liouville equation reads in this gauge

$$-\Delta_{g_0} \alpha = e^{2\alpha} K_{\vec{\Phi}} - 4\pi.$$

A standard elliptic estimate on S^2 gives

$$\|d\alpha\|_{L^{2,\infty}_{g_0}(S^2)} \leq C(S^2) \|\Delta_{g_0} \alpha\|_{L^1_{g_0}(S^2)}.$$

Hence there exists a constant $C(S^2)$ such that

$$\|d\alpha\|_{L^{2,\infty}_{g_0}(S^2)} \leq C(S^2)[1 + W(\vec{\Phi})] \tag{III.73}$$

where the norm is taken with respect to the metric g_0. We cover S^2 by a finite, controlled family of geodesic convex balls for the metric g_0 and for each of these balls we choose a conformal chart $\Psi_i : D^2 \to \Sigma$ such that the $\Psi_i(D^2_{1/2})$ still cover S^2. We consider any of these balls and we continue to write $\vec{\Phi}$ for the composition of $\vec{\Phi}$ with Ψ_i. Denote $\vec{e}_j := e^{-\lambda} \partial_{x_j} \vec{\Phi}$ where $e^\lambda = \log |\partial_{x_j} \vec{\Phi}|$. Then $(\vec{e}_1, \vec{e}_2)$ realizes a moving frame and

$$|\nabla \vec{e}_j|^2 = |\vec{n} \cdot \nabla \vec{e}_j|^2 + |\nabla \lambda|^2.$$

We have

$$-\Delta\alpha = \Delta\mu + (\nabla \vec{e}_1; \nabla^\perp \vec{e}_2).$$

Using the Wente estimates together with the classical elliptic estimates we get the existence of $\overline{\alpha} \in \mathbb{R}$ such that

$$\begin{aligned} \|\alpha - \overline{\alpha}\|_{L^\infty(D^2_{3/4})} \leq & \, C[\|\nabla\alpha\|_{L^{2,\infty}(D^2)} + \|\nabla\mu\|_{L^\infty(D^2)} + \|\nabla\vec{n}\|^2_{L^2(D^2)}] \\ & + \frac{1}{2\pi} \int_{S^2} |d\alpha|^2_{g_0} \, d\mathrm{vol}_{g_0}. \end{aligned} \tag{III.74}$$

Since $\int_\Sigma d\mathrm{vol}_{g_0} = 1$, there exists $x \in \Sigma$ such that

$$\alpha(x) = \tfrac{1}{2} \log \mathrm{Area}(\vec{\Phi}(S^2)). \tag{III.75}$$

Combining (III.74) and (III.75) we get (III.68), and Lemma III.11 is proved.

Lemma III.12. *Let Σ be a closed oriented surface and $\vec{\Phi}$ be a weak immersion in $\mathcal{E}_{\Sigma,2}$. Let g_0 be a constant Gauss curvature metric of volume* 1 *on Σ such that there exists α for which $g_{\vec{\Phi}} = e^{2\alpha} g_0$. If* $\text{genus}(\Sigma) \geq 1$ *then*

$$K_{g_0} \int_\Sigma \alpha \, d\text{vol}_{g_0} \geq 2^{-1} K_{g_0} \log\left(\int_\Sigma d\text{vol}_{g_{\vec{\Phi}}} \right). \tag{III.76}$$

Proof. We have to bound

$$4\pi (1 - \text{genus}(\Sigma)) \int_\Sigma \alpha \, d\text{vol}_{g_0}$$

from below. Since we fixed $\int_\Sigma d\text{vol}_{g_0} = 1$, using the convexity of exp and the Jensen inequality we have

$$\exp\left(2 \int_\Sigma \alpha \, d\text{vol}_{g_0} \right) \leq \int_\Sigma e^{2\alpha} \, d\text{vol}_{g_0} = \int_\Sigma d\text{vol}_{g_{\vec{\Phi}}}. \tag{III.77}$$

Then for $K_{g_0} < 0$,

$$K_{g_0} \int_\Sigma \alpha \, d\text{vol}_{g_0} \geq 2\pi (1 - \text{genus}(\Sigma)) \log\left(\int_\Sigma d\text{vol}_{g_{\vec{\Phi}}} \right). \tag{III.78}$$

This concludes the proof. □

Combining (III.65), (III.69) and (III.76) we obtain

Lemma III.13. *Let Σ be a closed surface of non-zero genus, $\vec{\Phi}$ a weak immersion in $\mathcal{E}_{\Sigma,2}$, and g_0 the constant Gauss curvature metric on Σ of volume* 1 *such that $g_{\vec{\Phi}} = e^{2\alpha} g_0$. Then for $\sigma \in (0, 1)$,*

$$W(\vec{\Phi}) + \sigma^2 \int_\Sigma (1 + H^2)^2 \, d\text{vol}_{g_{\vec{\Phi}}} + \left(2 \log \frac{1}{\sigma} \right)^{-1} \int_\Sigma |d\alpha|^2_{g_{\vec{\Phi}}} \, d\text{vol}_{g_{\vec{\Phi}}} \leq F^\sigma(\vec{\Phi}) \tag{III.79}$$

and

$$\left(\log \frac{1}{\sigma} \right)^{-1} \|\alpha\|_{L^\infty(\Sigma)}$$
$$\leq C \left(\log \frac{1}{\sigma} \right)^{-1} \left[\int_\Sigma |d\alpha|^2_{g_{\vec{\Phi}}} \, d\text{vol}_{g_{\vec{\Phi}}} + W(\vec{\Phi}) \right] + \left| \frac{\log A_{\vec{\Phi}}(\Sigma)}{2 \log \frac{1}{\sigma}} \right| \tag{III.80}$$

where C is a positive universal constant. □

IV. Uniform regularity for critical points to frame energies approximating the Willmore energy

IV.1. Some Banach spaces relevant to the proof

We shall make use of some Banach spaces whose definitions we recall. For any domain $\Omega \in \mathbb{R}^2$ the *weak L^p space* $L^{p,\infty}(\Omega)$ is given by

$$L^{p,\infty}(\Omega) := \left\{ f \text{ measurable} \, ; \; |f|_{p,\infty} := \sup_{t>0} t |\{x \in \Omega \, ; \; |f(x)| > t\}|^{1/p} < +\infty \right\}.$$

For $p \in (1, +\infty)$ the quasi-norm $|\cdot|_{p,\infty}$ is equivalent to a norm and the normed vector space $L^{p,\infty}(\Omega)$ is complete (see for instance [20]). It is the dual space to $L^{p',1}(\Omega)$, the set of measurable functions f such that

$$|f|_{p',1} := \int_0^{+\infty} \left|\{x\,;\ |f(x)| > t\}\right|^{1/p'} dt < +\infty.$$

The quasi-norm $|\cdot|_{p',1}$ is equivalent to a norm denoted $\|\cdot\|_{p',1}$ and the space $L^{p',1}(\Omega)$ equipped with this norm is complete (see for instance [20]).

We shall also make use of the space

$$\begin{aligned}&L^{2,\infty} + \sigma^{-1/2} L^{4/3}(D^2)\\ &:= \left\{ f \text{ measurable}\,;\ \|f\|_{L^{2,\infty}+\sigma^{-1/2}L^{4/3}} := \inf_{f=f_1+f_2} (\|f_1\|_{2,\infty} + \sigma^{-1/2}\|f_2\|_{4/3}) < +\infty \right\},\end{aligned}$$

which is dual to

$$L^{2,1} \cap \sigma^{1/2} L^4(D^2) := \{f \text{ measurable}\,;\ \|f\|_{L^{2,1}\cap\sigma^{1/2}L^4} := \|f\|_{2,1} + \sigma^{1/2}\|f\|_4 < +\infty\}.$$

IV.2. Uniform ε-regularity

In this section we shall consider critical points of the following family of Lagrangians where $\sigma \in [0, \sigma_0)$:

$$\begin{aligned} F^\sigma(\vec\Phi) &:= \left(\log\frac{1}{\sigma}\right)^{-1} \tilde F^{K_{g_0}}_{f_\sigma}(\vec\Phi)\\ &:= \left(\log\frac{1}{\sigma}\right)^{-1} \int_\Sigma [f_\sigma(H_{g_{\vec\Phi}}) + 2^{-1}|d\alpha|^2_{g_{\vec\Phi}} + K_{g_0}\alpha e^{-2\alpha}]\, d\mathrm{vol}_{g_{\vec\Phi}}\\ &\quad - 2^{-1}K_{g_0}\left(\log\frac{1}{\sigma}\right)^{-1} \log\left(\int_\Sigma e^{2\alpha}\, d\mathrm{vol}_{g_0}\right) \end{aligned} \tag{IV.1}$$

where $f_\sigma(t) = \log(\sigma^{-1})[t^2 + \sigma^2(1+t^2)^2]$. Recall that for any domain U of S^2 we denote

$$A_{\vec\Phi}(U) := \mathrm{Area}(\vec\Phi(U)).$$

The goal of this subsection is to establish the following result:

Lemma IV.1 (Uniform ε-regularity). *For any $C_1 > 0$, there exist $\varepsilon > 0$ and $\sigma_0 > 0$ such that for any $\sigma \in (0, \sigma_0)$ the following holds. Let $\vec\Phi$ be a critical point of F^σ satisfying*

$$F^\sigma(\vec\Phi) \le C_1. \tag{IV.2}$$

We keep denoting by $\vec\Phi$ the expression of this immersion in a conformal chart from D^2 into $\mathbb{R}^3$. Write

$$g_{\vec\Phi} = e^{2\lambda}[dx_1^2 + dx_2^2] \quad \text{and} \quad g_0 = e^{2\mu}[dx_1^2 + dx_2^2]$$

and moreover $l_\sigma := \left(\log \frac{1}{\sigma}\right)^{-1}$. *Assume that* α, μ, $\vec{n}$ *and* H *satisfy*

$$\inf_{D^2} \alpha \geq \log \sigma - C_1, \quad \|\nabla \mu\|_{L^\infty(D^2)} \leq C_1, \tag{IV.3}$$

and

$$l_\sigma \|\alpha e^{2\mu}\|_{L^\infty(D^2)} + \int_{D^2} (|\nabla \vec{n}|^2 + \sigma^2 [1 + H^4] e^{2\lambda})\, dx^2 < \varepsilon. \tag{IV.4}$$

Then for any $j \in \mathbb{N}$,

$$\begin{aligned} &|\nabla^{j+1} \vec{n}|^2(0) + |e^\lambda \nabla^j (\vec{H}(1 + 2\sigma^2(1 + H^2))|^2(0) + \sigma^4 H^2 (1 + H^2)^2 e^{2\lambda}(0) \\ &\quad \leq \tilde{C}_j \int_{D^2} |\nabla \vec{n}|^2\, dx^2 + \tilde{C}_j \left[\sigma^2 \int_{D^2} H^4 e^{2\lambda}\, dx^2\right]^2 + \tilde{C}_j \left[l_\sigma \int_{D^2} |\nabla \alpha|^2\, dx^2\right]^2 \\ &\quad\quad + \tilde{C}_j\, [l_\sigma^2 |\overline{\alpha}|^2 + l_\sigma^2 \|e^{4\mu}\|_{L^\infty(D^2)}] \|e^{4\mu}\|_{L^\infty(D^2)} + \tilde{C}_j\, l_\sigma^2 \left[\frac{A_{\vec{\Phi}}(D^2)}{A_{\vec{\Phi}}(\Sigma)}\right]^2 \end{aligned} \tag{IV.5}$$

where $\overline{\alpha} = |D^2_{1/2}|^{-1} \int_{D^2_{1/2}} \alpha$ *and*

$$\begin{aligned} l_\sigma |\nabla^{j+1} \alpha|^2(0) &\leq \tilde{C}_j l_\sigma \int_{D^2} |\nabla \alpha|^2\, dx^2 + \tilde{C}_j \left[\int_{D^2} |\nabla \vec{n}|^2\, dx^2\right]^2 \\ &\quad + \tilde{C}_j \left[\sigma^2 \int_{D^2} H^4 e^{2\lambda}\, dx^2\right]^4 + \tilde{C}_j\, l_\sigma \|e^{4\mu}\|_{L^\infty(D^2)} + \tilde{C}_j\, l_\sigma^4 \left[\frac{A_{\vec{\Phi}}(D^2)}{A_{\vec{\Phi}}(\Sigma)}\right]^4 \end{aligned} \tag{IV.6}$$

where $\tilde{C}_j$ *only depends on* C *and* j. □

Proof. In the first part of the proof, following the original ideas of [41], we derive from (III.50) more conservation laws. We do it first formally, not worrying about regularity. In the second part of the proof we will revisit each step with estimates in relevant Banach spaces.

Step 1. Let

$$\begin{aligned} \nabla \vec{L} &:= l_\sigma \nabla^\perp (f'_\sigma(H) \vec{n}) - 2 l_\sigma f'_\sigma(H) \nabla^\perp \vec{n} - 2 e^{-2\lambda} l_\sigma \nabla \vec{\Phi} \cdot \nabla \alpha \nabla^\perp \alpha \\ &\quad + l_\sigma [-2 f_\sigma(H) + e^{-2\lambda} |\nabla \alpha|^2 - K_{g_0} \alpha e^{-2\alpha} + K_{g_0} A_{\vec{\Phi}}(\Sigma)^{-1}] \nabla^\perp \vec{\Phi} + 2 l_\sigma e^{-2\lambda} \vec{\mathbb{I}} \llcorner \nabla^\perp \alpha. \end{aligned} \tag{IV.7}$$

Equation (III.50) gives

$$\nabla \vec{\Phi} \cdot \nabla^\perp \vec{L} = 2 l_\sigma e^{2\lambda} (2 f_\sigma(H) - H f'_\sigma(H) + K_{g_0} \alpha e^{-2\alpha} - K_{g_0} A_{\vec{\Phi}}(\Sigma)^{-1}). \tag{IV.8}$$

Let Y be the solution of

$$\begin{cases} -\Delta Y = 2 l_\sigma e^{2\lambda} (2 f_\sigma(H) - H f'_\sigma(H) + K_{g_0} \alpha e^{-2\alpha} - K_{g_0} A_{\vec{\Phi}}(\Sigma)^{-1}) & \text{in } D^2, \\ Y = 0 & \text{on } \partial D^2. \end{cases} \tag{IV.9}$$

Observe that $2f_\sigma(H) - Hf'_\sigma(H) = 2l_\sigma^{-1}\sigma^2(1-H^4)$. So Y satisfies

$$\begin{cases} -\Delta Y = 4e^{2\lambda}\sigma^2(1-H^4) + 2l_\sigma K_{g_0}\alpha e^{2\mu} - 2K_{g_0}l_\sigma e^{2\lambda}A_{\vec{\Phi}}(\Sigma)^{-1} & \text{in } D^2, \\ Y = 0 & \text{on } \partial D^2. \end{cases} \tag{IV.10}$$

Using the Poincaré lemma we deduce the existence of a function S such that

$$\nabla S = \vec{L}\cdot\nabla\vec{\Phi} + \nabla^\perp Y. \tag{IV.11}$$

Equation (III.58) in conformal coordinates gives

$$\nabla\vec{\Phi}\times\nabla^\perp\vec{L} = -l_\sigma\nabla^\perp\vec{\Phi}\cdot\nabla f'_\sigma(H) + 2\nabla\alpha\cdot\nabla^\perp\vec{D}, \tag{IV.12}$$

where

$$\nabla\vec{D} = \left(l_\sigma e^{-2\lambda}\sum_{i=1}^2 \mathbb{I}_{1i}\,\partial_{x_i}\vec{\Phi},\, l_\sigma e^{-2\lambda}\sum_{i=1}^2 \mathbb{I}_{2i}\,\partial_{x_i}\vec{\Phi}\right). \tag{IV.13}$$

Using again the Poincaré lemma on D^2 we obtain the existence of $\vec{V}$ such that

$$\nabla\vec{V} := \vec{L}\times\nabla\vec{\Phi} + l_\sigma f'_\sigma(H)\nabla\vec{\Phi} - 2(\alpha-\overline{\alpha})\nabla\vec{D}. \tag{IV.14}$$

Using the explicit expression of $\nabla\vec{D}$ given by (III.59) we obtain

$$\vec{n}\cdot\nabla\vec{V} = \vec{n}\cdot(\vec{L}\times\nabla\vec{\Phi}) = \vec{L}\cdot\nabla^\perp\vec{\Phi} = \nabla^\perp S + \nabla Y. \tag{IV.15}$$

We also have

$$\vec{n}\times\nabla\vec{V} = -(\vec{L}\cdot\vec{n})\nabla\vec{\Phi} - l_\sigma f'_\sigma(H)\nabla^\perp\vec{\Phi} - 2(\alpha-\overline{\alpha})\vec{n}\times\nabla\vec{D}. \tag{IV.16}$$

Denoting by $\pi_T(\nabla^\perp\vec{V})$ the tangential projection of $\nabla^\perp\vec{V}$, we have

$$\pi_T(\nabla^\perp\vec{V}) = (\vec{L}\cdot\vec{n})\nabla\vec{\Phi} + l_\sigma f'_\sigma(H)\nabla^\perp\vec{\Phi} - 2(\alpha-\overline{\alpha})\nabla^\perp\vec{D}. \tag{IV.17}$$

Hence

$$\vec{n}\times\nabla\vec{V} = -\nabla^\perp\vec{V} - 2(\alpha-\overline{\alpha})(\nabla^\perp\vec{D} + \vec{n}\times\nabla\vec{D}) - \vec{n}(\nabla S - \nabla^\perp Y). \tag{IV.18}$$

Let $\vec{v}$ be the unique solution to

$$\begin{cases} \Delta\vec{v} = \nabla^\perp Y\cdot\nabla\vec{n} & \text{in } D^2, \\ \vec{v} = 0 & \text{on } \partial D^2. \end{cases} \tag{IV.19}$$

Using once more the Poincaré lemma we obtain $\vec{u}$ such that

$$\vec{n}\nabla^\perp Y = \nabla\vec{v} + \nabla^\perp\vec{u}. \tag{IV.20}$$

Finally, let $\vec{R} := \vec{V} - \vec{u}$. We have

$$\vec{n}\times\nabla\vec{V} = \vec{n}\times\nabla\vec{R} + \vec{n}\times\nabla\vec{u} = \vec{n}\times\nabla\vec{R} + \vec{n}\times\nabla^\perp\vec{v}. \tag{IV.21}$$

Hence (IV.18) becomes

$$\vec n\times\nabla\vec R+\vec n\times\nabla^{\perp}\vec v=-\nabla^{\perp}\vec R+\nabla\vec v-\vec n\nabla S-2(\alpha-\overline{\alpha})(\nabla^{\perp}\vec D+\vec n\times\nabla\vec D),\tag{IV.22}$$

which gives

$$\begin{cases}\Delta S=-\nabla\vec n\cdot\nabla^{\perp}\vec R+\mathrm{div}(\vec n\cdot\nabla\vec v-2(\alpha-\overline{\alpha})\vec n\cdot\nabla^{\perp}\vec D),\\ \Delta\vec R=\nabla\vec n\times\nabla^{\perp}\vec R+\nabla\vec n\cdot\nabla^{\perp}S+\mathrm{div}(-\vec n\times\nabla\vec v+2(\alpha-\overline{\alpha})\,(-\nabla\vec D+\vec n\times\nabla^{\perp}\vec D)).\end{cases}$$

Taking the vectorial product between (IV.14) and $\nabla^{\perp}\vec\Phi$ we obtain

$$\begin{aligned}\nabla\vec V\times\nabla^{\perp}\vec\Phi&=(\vec L\cdot\nabla^{\perp}\vec\Phi)\cdot\nabla\vec\Phi-2l_\sigma f'_\sigma(H)e^{2\lambda}\vec n-2(\alpha-\overline{\alpha})\nabla\vec D\times\nabla^{\perp}\vec\Phi\\&=\nabla^{\perp}S\cdot\nabla\vec\Phi+\nabla Y\cdot\nabla\vec\Phi-2l_\sigma f'_\sigma(H)e^{2\lambda}\vec n-2(\alpha-\overline{\alpha})\nabla\vec D\times\nabla^{\perp}\vec\Phi.\end{aligned}\tag{IV.23}$$

We also have

$$\begin{aligned}\nabla\vec V\times\nabla^{\perp}\vec\Phi&=\nabla\vec R\times\nabla^{\perp}\vec\Phi+\nabla\vec u\times\nabla^{\perp}\vec\Phi\\&=\nabla\vec R\times\nabla^{\perp}\vec\Phi+\nabla^{\perp}\vec v\times\nabla^{\perp}\vec\Phi+\nabla Y\cdot(\vec n\times\nabla^{\perp}\vec\Phi)\\&=\nabla\vec R\times\nabla^{\perp}\vec\Phi+\nabla\vec v\times\nabla\vec\Phi+\nabla Y\cdot\nabla\vec\Phi.\end{aligned}\tag{IV.24}$$

Combining (IV.23) and (IV.24) gives

$$2l_\sigma f'_\sigma(H)e^{2\lambda}\vec n=\nabla^{\perp}S\cdot\nabla\vec\Phi-2(\alpha-\overline{\alpha})\nabla\vec D\times\nabla^{\perp}\vec\Phi-\nabla\vec R\times\nabla^{\perp}\vec\Phi-\nabla\vec v\times\nabla\vec\Phi.\tag{IV.25}$$

We have explicitly $l_\sigma f'_\sigma(H)=2H(1+2\sigma^2(1+H^2))$; moreover, a straightforward computation gives

$$-\nabla\vec D\times\nabla^{\perp}\vec\Phi=2l_\sigma\vec H e^{2\lambda}=l_\sigma\Delta\vec\Phi.\tag{IV.26}$$

Inserting (IV.26) in (IV.25) gives

$$2(1+2\sigma^2(1+H^2)-l_\sigma(\alpha-\overline{\alpha}))\,\Delta\vec\Phi=\nabla^{\perp}S\cdot\nabla\vec\Phi-\nabla\vec R\times\nabla^{\perp}\vec\Phi-\nabla\vec v\times\nabla\vec\Phi.\tag{IV.27}$$

Step 2. We now prove that, for ε small enough, ∇S and $\nabla\vec R$ are in L^2.

Since $\int_{D^2}|\nabla\vec n|^2\,dx^2<\varepsilon$, for ε small enough, following Hélein's construction of energy controlled moving frames (see [21]), we get the existence of $\vec e_i$ such that

$$\vec e_1\times\vec e_2=\vec n\quad\text{and}\quad\int_{D^2}|\nabla\vec e_i|^2\,dx^2\le C\int_{D^2}|\nabla\vec n|^2\,dx^2.\tag{IV.28}$$

Using the assumptions (IV.2) we have

$$\begin{aligned}2C_1\ge W(\vec\Phi)+\sigma^2\int_{S^2}(1+H^2)^2\,d\mathrm{vol}_{g_{\vec\Phi}}\\+l_\sigma\Bigg[\int_{S^2}2^{-1}|d\alpha|^2_{g_{\vec\Phi}}\,d\mathrm{vol}_{g_{\vec\Phi}}+K_{g_0}\alpha e^{-2\alpha}-2^{-1}K_{g_0}\log A_{\vec\Phi}(\Sigma)\Bigg].\end{aligned}\tag{IV.29}$$

Using the Onofri inequality (III.40) we deduce that

$$W(\vec{\Phi}) + \sigma^2 \int_{S^2} (1 + H^2)^2 \, d\mathrm{vol}_{g_{\vec{\Phi}}} \leq 2C_1. \tag{IV.30}$$

Since $W(\vec{\Phi}) \leq 2C_1$ by (III.73), we obtain the existence of a constant C depending only on C_1 such that

$$\|\nabla\lambda\|_{L^{2,\infty}(D^2)} \leq C. \tag{IV.31}$$

Recall (see for instance [40]) that for $\vec{e}_i := e^{-\lambda}\, \partial_{x_i}\vec{\Phi}$ the Liouville equation giving the expression of the Gauss curvature in conformal coordinates is equivalent to

$$-\Delta\lambda = (\nabla\vec{e}_1; \nabla^{\perp}\vec{e}_2). \tag{IV.32}$$

Let ν be the solution of

$$\begin{cases} -\Delta\nu = (\nabla\vec{e}_1; \nabla^{\perp}\vec{e}_2) & \text{in } D^2, \\ \nu = 0 & \text{on } \partial D^2. \end{cases} \tag{IV.33}$$

Using the Wente inequality we get

$$\|\nabla\nu\|_{L^2(D^2)} + \|\nu\|_{L^\infty(D^2)} \leq C \sum_{i=1}^{2} \int_{D^2} |\nabla\vec{e}_i|^2 \, dx^2 \leq C \int_{D^2} |\nabla\vec{n}|^2 \, dx^2 \leq \varepsilon. \tag{IV.34}$$

Since $\lambda - \nu$ is harmonic and

$$\|\nabla(\lambda - \nu)\|_{L^{2,\infty}(D^2)} \leq C, \tag{IV.35}$$

we have

$$\|\nabla(\lambda - \nu)\|_{L^\infty(D^2_{1/2})} \leq C. \tag{IV.36}$$

Hence, by (IV.34), there exists $\overline{\lambda} = (\lambda - \nu)(0) \in \mathbb{R}$ such that

$$\|\lambda - \overline{\lambda}\|_{L^\infty(D^2_{1/2})} \leq C. \tag{IV.37}$$

Since $\|\nabla\mu\|_\infty \leq C_1$, we have the existence of $\overline{\mu} \in \mathbb{R}$ such that

$$\|\mu - \overline{\mu}\|_{L^\infty(D^2)} < C. \tag{IV.38}$$

Hence we deduce the existence of $\overline{\alpha} \in \mathbb{R}$ such that

$$\|\alpha - \overline{\alpha}\|_{L^\infty(D^2_{1/2})} \leq C. \tag{IV.39}$$

We rescale the domain so that $D^2_{1/2}$ becomes D^2. We now proceed to the introduction of $\vec{L}$, Y, $\vec{V}$, $\vec{v}$, $\vec{u}$, $\vec{R}$, S as in Step 1. First of all using classical elliptic estimates we deduce from (IV.10), using the hypothesis (IV.4) and for σ small enough

$$\|\nabla Y\|_{L^{2,\infty}(D^2)} \leq C \int_{D^2} \sigma^2[1 + H^4] e^{2\lambda} \, dx^2 + C l_\sigma |\overline{\alpha}| \int_{D^2} e^{2\mu} \, dx^2 + C l_\sigma \frac{A_{\vec{\Phi}}(D^2)}{A_{\vec{\Phi}}(\Sigma)} \leq C\varepsilon, \tag{IV.40}$$

and using (IV.19) we deduce by means of the Wente estimates

$$\|\nabla\vec{v}\|_{L^2(D^2)} \le C\|\nabla\vec{n}\|_{L^2(D^2)}\left[\int_{D^2}\sigma^2[1+H^4]e^{2\lambda}\,dx^2+l_\sigma|\overline{\alpha}|\int_{D^2}e^{2\mu}\,dx^2+l_\sigma\right]. \quad \text{(IV.41)}$$

Recall that for any $\vec{X}\in L^1(\mathbb{R}^2,\mathbb{R}^2)$ there exists a unique pair $a,b\in L^{2,\infty}$ such that

$$\vec{X}=\nabla a+\nabla^\perp b \quad \text{and} \quad \|a\|_{L^{2,\infty}(\mathbb{R}^2)}+\|b\|_{L^{2,\infty}(\mathbb{R}^2)}\le C\|\vec{X}\|_{L^1(\mathbb{R}^2)}.$$

This pair is explicitly given by

$$a:=-\frac{1}{2\pi}\,\frac{1}{r}\,\frac{\partial}{\partial r}\,\dot{\star}\,\vec{X} \quad \text{and} \quad b:=\frac{1}{2\pi}\,\frac{1}{r}\,\frac{\partial}{\partial r}\,\dot{\star}\,\vec{X}^\perp.$$

We apply this decomposition to each coordinate of the restriction to D^2 of

$$\begin{aligned}&-2l_\sigma e^{\overline{\lambda}}f'_\sigma(H)\nabla^\perp\vec{n}-2e^{-2\lambda+\overline{\lambda}}l_\sigma\nabla\vec{\Phi}\cdot\nabla\alpha\nabla^\perp\alpha+2l_\sigma e^{-2\lambda+\overline{\lambda}}\vec{\mathbb{I}}\llcorner\nabla^\perp\alpha\\&\quad+l_\sigma e^{\overline{\lambda}}[-2f_\sigma(H)+e^{-2\lambda}|\nabla\alpha|^2-K_{g_0}\alpha e^{-2\alpha}+K_{g_0}A_{\vec{\Phi}}(\Sigma)^{-1}]\nabla^\perp\vec{\Phi},\end{aligned} \quad \text{(IV.42)}$$

and we get the existence of $\vec{a}$ and $\vec{b}$ in $L^{2,\infty}(D^2,\mathbb{R}^3)$ such that

$$\nabla(e^{\overline{\lambda}}\vec{L})-\nabla^\perp(l_\sigma e^{\overline{\lambda}}f'_\sigma(H)\vec{n})=\nabla\vec{a}+\nabla^\perp\vec{b} \quad \text{in } D^2,$$

and

$$\begin{aligned}\|\vec{a}\|_{L^{2,\infty}(D^2)}+\|\vec{b}\|_{L^{2,\infty}(D^2)}&\le C\|\nabla\vec{n}\|_{L^2(D^2)}+Cl_\sigma\|\nabla\alpha\|^2_{L^2(D^2)}\\&\quad+Cl_\sigma|\overline{\alpha}|\int_{D^2_2}e^{2\mu}\,dx^2+C\left[\int_{D^2}\sigma^2|\nabla\vec{n}|^4e^{-2\lambda}\,dx^2\right]+Cl_\sigma\frac{A_{\vec{\Phi}}(D^2)}{A_{\vec{\Phi}}(\Sigma)}.\end{aligned} \quad \text{(IV.43)}$$

We know that both $e^{\overline{\lambda}}\vec{L}+\vec{a}$ and $l_\sigma e^{\overline{\lambda}}f'_\sigma(H)\vec{n}+\vec{b}$ are harmonic conjugate to each other. Let $\mathcal{H}$ be the operator which sends a harmonic function on D^2 of average 0 to its harmonic conjugate[13] of average 0. We have

$$e^{\overline{\lambda}}\vec{L}+\vec{a}=\mathcal{H}(l_\sigma e^{\overline{\lambda}}f'_\sigma(H)\vec{n}+\vec{b}).$$

Calderón–Zygmund theory shows that $\mathcal{H}$ maps continuously $L^{2,\infty}(D^2)$ into itself and $L^{4/3}(D^2)$ into itself: there exists C such that, for any harmonic function f in D^2,

$$\|\mathcal{H}(f)\|_{L^{2,\infty}(D^2)}\le C\|f\|_{L^{2,\infty}(D^2)} \quad \text{and} \quad \|\mathcal{H}(f)\|_{L^{4/3}(D^2)}\le C\|f\|_{L^{4/3}(D^2)}.$$

Let f be a harmonic function in $L^{2,\infty}(D^2)+\sigma^{-1/2}L^{4/3}(D^2)$. To any decomposition $f=f_1+f_2$ we assign $b(f_1)$ and $b(f_2)$, the respective Bergman projections of f_1 and f_2.

[13] $\mathcal{H}$ is also the operator which to the real part of a holomorphic function on D^2 assigns its imaginary part.

We have of course $b(f)=f$, and the L^p boundedness of the Bergman projection gives

$$\|b(f_1)\|_{2,\infty}\le C\|f_1\|_{2,\infty}\quad\text{and}\quad\|b(f_2)\|_{4/3}\le C\|f_2\|_{4/3}.$$

Combining all the above gives

$$\begin{aligned}\|\mathcal{H}(f)\|_{L^{2,\infty}(D^2)+\sigma^{-1/2}L^{4/3}(D^2)}&=\|\mathcal{H}(b(f))\|_{L^{2,\infty}(D^2)+\sigma^{-1/2}L^{4/3}(D^2)}\\&\le\|\mathcal{H}(b(f_1))\|_{L^{2,\infty}(D^2)}+\sigma^{-1/2}\|\mathcal{H}(b(f_2))\|_{L^{4/3}(D^2)}\\&\le C[\|b(f_1)\|_{L^{2,\infty}(D^2)}+\sigma^{-1/2}\|b(f_2)\|_{L^{4/3}(D^2)}]\\&\le C[\|f_1\|_{L^{2,\infty}(D^2)}+\sigma^{-1/2}\|f_2\|_{L^{4/3}(D^2)}].\end{aligned}$$

Since this holds for any decomposition $f=f_1+f_2$, we deduce that $\mathcal{H}$ maps continuously $L^{2,\infty}(D^2)+\sigma^{-1/2}L^{4/3}(D^2)$ into itself with a constant independent of σ. Hence

$$\begin{aligned}&\|e^{\overline{\lambda}}\vec{L}+\vec{a}\|_{L^{2,\infty}(D^2)+\sigma^{-1/2}L^{4/3}(D^2)}\\&\quad\le C[\|\vec{b}\|_{L^{2,\infty}(D^2)}+\|e^{\overline{\lambda}}\vec{H}\|_{L^{2,\infty}(D^2)}+\sigma^{-1/2}\|e^{\overline{\lambda}}\sigma^2\vec{H}(1+H^2)\|_{L^{4/3}(D^2)}].\end{aligned}$$

Combining this with (IV.43) gives

$$\begin{aligned}\|e^{\overline{\lambda}}\vec{L}\|_{L^{2,\infty}+\sigma^{-1/2}L^{4/3}(D^2)}&\le C\|\nabla\vec{n}\|_{L^2(D^2)}+Cl_\sigma\|\nabla\alpha\|^2_{L^2(D^2)}+C\int_{D^2}\sigma^2|\nabla\vec{n}|^4e^{-2\lambda}\,dx^2\\&\quad+Cl_\sigma|\overline{\alpha}|\int_{D^2_2}e^{2\mu}\,dx^2+C\left[\int_{D^2}\sigma^2|\nabla\vec{n}|^4e^{-2\lambda}\,dx^2\right]^{3/4}+Cl_\sigma\frac{A_{\vec{\Phi}}(D^2_2)}{A_{\vec{\Phi}}(\Sigma)}.\end{aligned}\tag{IV.44}$$

Combining Lemma VII.3 with (IV.3) gives

$$\sigma^2\int_{D^2}|\nabla\vec{n}|^4e^{-2\lambda}\,dx^2\le C\sigma^2\int_{D^2_2}H^4e^{2\lambda}\,dx^2+C\left[\int_{D^2_2}|\nabla\vec{n}|^2\,dx^2\right]^2<C\varepsilon.\tag{IV.45}$$

Hence, using the explicit expressions (IV.11), (IV.14) and (IV.20) we find that $\nabla\vec{R}$ and $\nabla\vec{S}$ are uniformly bounded in $L^{2,\infty}+\sigma^{-1/2}L^{4/3}$ and we have

$$\begin{aligned}\|\nabla\vec{R}\|_{L^{2,\infty}+\sigma^{-1/2}L^{4/3}(D^2)}+\|\nabla S\|_{L^{2,\infty}+\sigma^{-1/2}L^{4/3}(D^2)}&\le C\|\nabla\vec{n}\|_{L^2(D^2)}+Cl_\sigma\|\nabla\alpha\|^2_{L^2(D^2)}\\&\quad+Cl_\sigma|\overline{\alpha}|\int_{D^2_2}e^{2\mu}\,dx^2+\left[\int_{D^2}\sigma^2|\nabla\vec{n}|^4e^{-2\lambda}\,dx^2\right]^{3/4}+Cl_\sigma\frac{A_{\vec{\Phi}}(D^2_2)}{A_{\vec{\Phi}}(\Sigma)}.\end{aligned}\tag{IV.46}$$

Let φ be and $\vec{\Psi}$ the unique solutions in $W^{1,2}_0(D^2)$ of the linear system

$$\begin{cases}\Delta\varphi=-\nabla\vec{n}\cdot\nabla^\perp\vec{R}+\operatorname{div}\big(\vec{n}\cdot\nabla\vec{v}-2(\alpha-\overline{\alpha})\vec{n}\cdot\nabla^\perp\vec{D}\big),\\\Delta\vec{\Psi}=\nabla\vec{n}\times\nabla^\perp\vec{R}+\nabla\vec{n}\cdot\nabla^\perp S+\operatorname{div}\big(-\vec{n}\times\nabla\vec{v}+2(\alpha-\overline{\alpha})(-\nabla\vec{D}+\vec{n}\times\nabla^\perp\vec{D})\big).\end{cases}\tag{IV.47}$$

Using Lemma VII.1 and in particular (VII.2), together with the fact that both $\nabla\vec{v}$ and

$(\alpha-\overline{\alpha})\nabla\vec{D}$ are in L^2, we deduce using $l_\sigma\|\alpha-\overline{\alpha}\|_{L^\infty(D^2)}<\varepsilon$ that

$$\begin{aligned}&\|\nabla\varphi\|_{L^{2,\infty}(D^2)}+\|\nabla\vec{\Psi}\|_{L^{2,\infty}(D^2)}\\&\qquad\le\varepsilon^{1/2}\|\nabla\vec{n}\|_{L^2(D^2)}+C\varepsilon^{1/2}l_\sigma\|\nabla\alpha\|^2_{L^2(D^2)}+C\varepsilon^{1/2}l_\sigma|\overline{\alpha}|\int_{D^2_2}e^{2\mu}\,dx^2\\&\qquad\quad+\varepsilon^{1/2}\left[\int_{D^2}\sigma^2[e^{2\lambda}+|\nabla\vec{n}|^4e^{-2\lambda}]\,dx^2\right]^{3/4}+Cl_\sigma\varepsilon^{1/2}\frac{A_{\vec{\Phi}}(D^2_2)}{A_{\vec{\Phi}}(\Sigma)}.\end{aligned}$$

Recall that for any harmonic function v,

$$\left[\int_{D^2_{1/2}}|v|^2\,dx^2\right]^{1/2}\le C\left[\int_{D^2}|v|^{4/3}\,dx^2\right]^{3/4}.$$

Recall also the Hölder inequality in Lorentz spaces (see [20]):

$$\forall f\in L^{2,\infty}(D^2)\qquad\|f\|_{L^{4/3}(D^2)}\le C\|f\|_{L^{2,\infty}(D^2)}.$$

Observe that the triangle inequality gives

$$\begin{aligned}\|f\|_{L^{4/3}(D^2)}&\le\inf_{f=f_1+f_2}(\|f_1\|_{L^{4/3}(D^2)}+\|f_2\|_{L^{4/3}(D^2)})\\&\le\inf_{f=f_1+f_2}(\|f_1\|_{L^{4/3}(D^2)}+\sigma^{-1/2}\|f_2\|_{L^{4/3}(D^2)})\\&\le C\inf_{f=f_1+f_2}(\|f_1\|_{L^{2,\infty}(D^2)}+\sigma^{-1/2}\|f_2\|_{L^{4/3}(D^2)})=C\|f\|_{L^{2,\infty}+\sigma^{-1/2}L^{4/3}(D^2)}.\end{aligned}$$

Since $S-\varphi$ and $\vec{R}-\vec{\Psi}$ are harmonic, we have, using the previous two inequalities,

$$\begin{aligned}&\|\nabla(S-\varphi)\|_{L^{2,\infty}(D^2_{1/2})}+\|\nabla(\vec{R}-\vec{\Psi})\|_{L^{2,\infty}(D^2_{1/2})}\\&\quad\le C\|\nabla(S-\varphi)\|_{L^{4/3}(D^2)}+\|\nabla(\vec{R}-\vec{\Psi})\|_{L^{4/3}(D^2)}\\&\quad\le C\|\nabla S\|_{L^{4/3}(D^2)}+C\|\nabla\vec{R}\|_{L^{4/3}(D^2)}+C\|\nabla\varphi\|_{L^{2,\infty}(D^2)}+C\|\nabla\vec{\Psi}\|_{L^{2,\infty}(D^2)}\\&\quad\le C\|\nabla S\|_{L^{2,\infty}+\sigma^{-1/2}L^{4/3}(D^2)}+C\|\nabla\vec{R}\|_{L^{2,\infty}+\sigma^{-1/2}L^{4/3}(D^2)}\\&\qquad+C\|\nabla\varphi\|_{L^{2,\infty}(D^2)}+C\|\nabla\vec{\Psi}\|_{L^{2,\infty}(D^2)}.\end{aligned}\tag{IV.48}$$

Hence

$$\begin{aligned}&\|\nabla S\|_{L^{2,\infty}(D^2_{1/2})}+\|\nabla\vec{R}\|_{L^{2,\infty}(D^2_{1/2})}\\&\le\|\nabla(S-\varphi)\|_{L^{2,\infty}(D^2_{1/2})}+\|\nabla(\vec{R}-\vec{\Psi})\|_{L^{2,\infty}(D^2_{1/2})}+\|\nabla\varphi\|_{L^{2,\infty}(D^2_{1/2})}+2\|\nabla\vec{\Psi}\|_{L^{2,\infty}(D^2_{1/2})}\\&\le C\|\nabla S\|_{L^{2,\infty}+\sigma^{-1/2}L^{4/3}(D^2)}+C\|\nabla\vec{R}\|_{L^{2,\infty}+\sigma^{-1/2}L^{4/3}(D^2)}+C\|\nabla\varphi\|_{L^{2,\infty}(D^2)}\\&\quad+C\|\nabla\vec{\Psi}\|_{L^{2,\infty}(D^2)}\\&\le C\|\nabla\vec{n}\|_{L^2(D^2)}+Cl_\sigma\|\nabla\alpha\|^2_{L^2(D^2)}\\&\quad+Cl_\sigma|\overline{\alpha}|\int_{D^2_2}e^{2\mu}\,dx^2+\left[\int_{D^2}\sigma^2[e^{2\lambda}+|\nabla\vec{n}|^4e^{-2\lambda}]\,dx^2\right]^{3/4}+Cl_\sigma\frac{A_{\vec{\Phi}}(D^2_2)}{A_{\vec{\Phi}}(\Sigma)}.\end{aligned}\tag{IV.49}$$

Let φ_1 and $\vec{\Psi}_1$ be the unique solutions in $W^{1,2}_0(D^2_{1/2})$ of the linear system

$$\begin{cases}\Delta\varphi_1 = -\nabla\vec{n}\cdot\nabla^\perp\vec{R} + \operatorname{div}\big(\vec{n}\cdot\nabla\vec{v} - 2(\alpha-\overline{\alpha})\vec{n}\cdot\nabla^\perp\vec{D}\big),\\ \Delta\vec{\Psi}_1 = \nabla\vec{n}\times\nabla^\perp\vec{R} + \nabla\vec{n}\cdot\nabla^\perp S + \operatorname{div}\big(-\vec{n}\times\nabla\vec{v} + 2(\alpha-\overline{\alpha})(-\nabla\vec{D} + \vec{n}\times\nabla^\perp\vec{D})\big).\end{cases} \tag{IV.50}$$

The Wente estimates, combined with (IV.41) and the pointwise bound $|\nabla\vec{D}|^2 \leq Cl_\sigma^2\,|\nabla\vec{n}|^2$, give

$$\begin{aligned}&\|\nabla\varphi_1\|_{L^2(D^2_{1/2})} + \|\nabla\vec{\Psi}_1\|_{L^2(D^2_{1/2})} \leq C\|\nabla\vec{n}\|_{L^2(D^2_{1/2})}[\|\nabla S\|_{L^{2,\infty}(D^2_{1/2})} + \|\nabla\vec{R}\|_{L^{2,\infty}(D^2_{1/2})}]\\ &+ C\|\nabla\vec{n}\|_{L^2(D^2)}\\ &\quad\times\left[\int_{D^2}[|\nabla\vec{n}|^2 + \sigma^2[e^{2\lambda} + e^{-2\lambda}|\nabla\vec{n}|^4]\,dx^2 + Cl_\sigma|\overline{\alpha}|\int_{D^2}e^{2\mu}\,dx^2 + Cl_\sigma\frac{A_{\vec{\Phi}}(D^2_2)}{A_{\vec{\Phi}}(\Sigma)}\right]\\ &+ Cl_\sigma\|\alpha-\overline{\alpha}\|_{L^\infty(D^2)}\|\nabla\vec{n}\|^2_{L^2(D^2)}.\end{aligned} \tag{IV.51}$$

Since $S-\varphi_1$ and $\vec{R}-\vec{\Psi}_1$ are harmonic we finally obtain

$$\begin{aligned}\|\nabla S\|_{L^2(D^2_{1/4})} + \|\nabla\vec{R}\|_{L^2(D^2_{1/4})} &\leq \|\nabla S\|_{L^{2,\infty}(D^2_{1/2})} + \|\nabla\vec{R}\|_{L^{2,\infty}(D^2_{1/2})}\\ &\quad + 2\|\nabla\varphi_1\|_{L^2(D^2_{1/2})} + 2\|\nabla\vec{\Psi}_1\|_{L^2(D^2_{1/2})}\\ &\leq C\|\nabla\vec{n}\|_{L^2(D^2)} + Cl_\sigma\|\nabla\alpha\|^2_{L^2(D^2)} + Cl_\sigma|\overline{\alpha}|\int_{D^2_2}e^{2\mu}\,dx^2\\ &\quad + C\left[\int_{D^2}\sigma^2[e^{2\lambda} + |\nabla\vec{n}|^4e^{-2\lambda}]\,dx^2\right]^{3/4} + Cl_\sigma\frac{A_{\vec{\Phi}}(D^2_2)}{A_{\vec{\Phi}}(\Sigma)}.\end{aligned} \tag{IV.52}$$

Combining (IV.52) and (IV.45) gives, by changing 1 into 1/2,

$$\begin{aligned}\|\nabla S\|_{L^2(D^2_{1/8})} + \|\nabla\vec{R}\|_{L^2(D^2_{1/8})} &\leq C\|\nabla\vec{n}\|_{L^2(D^2)} + Cl_\sigma\|\nabla\alpha\|^2_{L^2(D^2)}\\ &\quad + Cl_\sigma|\overline{\alpha}|\int_{D^2_2}e^{2\mu}\,dx^2 + C\sigma^2\int_{D^2}[1+H^4]e^{2\lambda}\,dx^2 + Cl_\sigma\frac{A_{\vec{\Phi}}(D^2_2)}{A_{\vec{\Phi}}(\Sigma)}.\end{aligned} \tag{IV.53}$$

Hence ∇S and $\nabla\vec{R}$ are in $L^2(D^2_{1/8})$ and under the assumptions of the lemma, using also Lemma III.11, we see that $\|\nabla S\|_{L^2(D^2_{1/8})} + \|\nabla\vec{R}\|_{L^2(D^2_{1/8})}$ is bounded by a constant depending only on C_1. We rescale the domain in such a way that ∇S and $\nabla\vec{R}$ are in $L^2(D^2)$.

Step 3 (Uniform Morrey decrease of the Willmore energy). More precisely, we are going to prove the existence of $\gamma > 0$ independent of σ and of the solution such that

$$\sup_{x_0\in D^2_{1/2},\, r<1/4} r^{-\gamma}\int_{B^2_r(x_0)} H^2(1+\sigma^2(1+H^2))^2e^{2\lambda}\,dx^2 \leq C. \tag{IV.54}$$

Following Step 1 of the proof of the theorem the map $\vec{U} := (e^{-\overline{\lambda}}\vec{\Phi}, \vec{v}, S, \vec{R})$ satisfies the following system on $B_r(x_0)$:

$$\begin{cases} \Delta(e^{-\overline{\lambda}}\vec{\Phi}) = h_\sigma e^{-\overline{\lambda}}[\nabla^\perp S \cdot \nabla\vec{\Phi} - \nabla\vec{R} \times \nabla^\perp\vec{\Phi} - \nabla\vec{v} \times \nabla\vec{\Phi}], \\ \Delta\vec{v} = \nabla^\perp Y \cdot \nabla\vec{n}, \\ \Delta S = -\nabla\vec{n} \cdot \nabla^\perp\vec{R} + \operatorname{div}\big(\vec{n} \cdot \nabla\vec{v} - 2(\alpha - \overline{\alpha})\vec{n} \cdot \nabla^\perp\vec{D}\big), \\ \Delta\vec{R} = \nabla\vec{n} \times \nabla^\perp\vec{R} + \nabla\vec{n} \cdot \nabla^\perp S + \operatorname{div}\big(-\vec{n} \times \nabla\vec{v} + 2(\alpha - \overline{\alpha})(-\nabla\vec{D} + \vec{n} \times \nabla^\perp\vec{D})\big), \end{cases} \tag{IV.55}$$

where $0 \le h_\sigma := 2(1 + 2\sigma^2(1 + H^2) - l_\sigma(\alpha - \overline{\alpha}))^{-1} \le 1$ and where we use the fact that

$$\partial_{x_i}\vec{n} = -e^{-2(\lambda - \overline{\lambda})} \sum_{j=1}^{2} \vec{n} \cdot \partial^2_{x_i x_j}(e^{-\overline{\lambda}}\vec{\Phi}) e^{-\overline{\lambda}} \partial_{x_j}\vec{\Phi}, \tag{IV.56}$$

and

$$\nabla\vec{D} = l_\sigma e^{-2(\lambda - \overline{\lambda})} \Big(\sum_{i=1}^{2} \vec{n} \cdot \partial^2_{x_1 x_i}(e^{-\overline{\lambda}}\vec{\Phi}) e^{-\overline{\lambda}} \partial_{x_i}\vec{\Phi}, \vec{n} \cdot \partial^2_{x_2 x_i}(e^{-\overline{\lambda}}\vec{\Phi}) e^{-\overline{\lambda}} \partial_{x_i}\vec{\Phi} \Big). \tag{IV.57}$$

Let $\vec{w}$ in $W^{1,2}_0(B_r(x_0))$ be a solution of

$$\Delta\vec{w} = \nabla^\perp Y \cdot \nabla\vec{n}. \tag{IV.58}$$

Using the Wente estimates we obtain

$$\int_{B_r(x_0)} |\nabla\vec{w}|^2 \, dx^2 \le C \|\nabla Y\|^2_{L^{2,\infty}(D^2)} \int_{B_r(x_0)} |\nabla\vec{n}|^2 \, dx^2. \tag{IV.59}$$

Since $\vec{v} - \vec{w}$ is harmonic for any $t \in (0, 1)$, the monotonicity formula for harmonic functions gives

$$\int_{B_{tr}(x_0)} |\nabla(\vec{v} - \vec{w})|^2 \, dx^2 \le t^2 \int_{B_r(x_0)} |\nabla(\vec{v} - \vec{w})|^2 \, dx^2. \tag{IV.60}$$

We deduce from (IV.40), (IV.59) and (IV.60) that

$$\int_{B_{tr}(x_0)} |\nabla\vec{v}|^2 \, dx^2 \le t^2 \int_{B_r(x_0)} |\nabla\vec{v}|^2 \, dx^2 + C\sqrt{\varepsilon}\, \frac{\|\nabla Y\|^2_{L^{2,\infty}(D^2)}}{\sqrt{\varepsilon}} \int_{B_r(x_0)} |\nabla\vec{n}|^2 \, dx^2. \tag{IV.61}$$

Let T and $\vec{Q}$ in $W^{1,2}_0(B_{tr}(x_0))$ solve

$$\begin{cases} \Delta T = -\nabla\vec{n} \cdot \nabla^\perp\vec{R} + \operatorname{div}\big(\vec{n} \cdot \nabla\vec{v} - 2(\alpha - \overline{\alpha})\vec{n} \cdot \nabla^\perp\vec{D}\big), \\ \Delta\vec{Q} = \nabla\vec{n} \times \nabla^\perp\vec{R} + \nabla\vec{n} \cdot \nabla^\perp S + \operatorname{div}\big(-\vec{n} \times \nabla\vec{v} + 2(\alpha - \overline{\alpha})(-\nabla\vec{D} + \vec{n} \times \nabla^\perp\vec{D})\big). \end{cases} \tag{IV.62}$$

The Wente inequalities combined with classical elliptic estimates and the inequalities

$$\int_{D^2} |\nabla\vec{n}|^2 \, dx^2 + l^2_\sigma \|\alpha - \overline{\alpha}\|^2_{L^\infty(D^2)} < \varepsilon \quad \text{and} \quad |\nabla\vec{D}|^2 \le C l^2_\sigma |\nabla\vec{n}|^2$$

give

$$\int_{B_{tr}(x_0)} [|\nabla T|^2 + |\nabla \vec{Q}|^2]\, dx^2 \leq C\sqrt{\varepsilon} \int_{B_{tr}(x_0)} [|\nabla S|^2 + |\nabla \vec{R}|^2 + \delta^2 |\nabla \vec{n}|^2]\, dx^2$$
$$+ C \int_{B_{tr}(x_0)} |\nabla \vec{v}|^2\, dx^2, \tag{IV.63}$$

where

$$\delta^2 := \frac{l_\sigma^2 \|\alpha - \overline{\alpha}\|^2_{L^\infty(D^2)} + \|\nabla Y\|^2_{L^{2,\infty}(D^2)}}{\sqrt{\varepsilon}}.$$

Since $S - T$ and $\vec{R} - \vec{Q}$ are harmonic, the monotonicity formula gives

$$\int_{B_{t^2r}(x_0)} [|\nabla (S - T)|^2 + |\nabla (\vec{R} - \vec{Q})|^2]\, dx^2$$
$$\leq t^2 \int_{B_{tr}(x_0)} [|\nabla (S - T)|^2 + |\nabla (\vec{R} - \vec{Q})|^2]\, dx^2. \tag{IV.64}$$

Hence combining (IV.61), (IV.63) and (IV.64) we obtain

$$\int_{B_{t^2r}(x_0)} [|\nabla S|^2 + |\nabla \vec{R}|^2 + |\nabla \vec{v}|^2]\, dx^2 \leq t^2 \int_{B_r(x_0)} [|\nabla S|^2 + |\nabla \vec{R}|^2 + |\nabla \vec{v}|^2]\, dx^2$$
$$+ C\sqrt{\varepsilon} \int_{B_r(x_0)} [|\nabla S|^2 + |\nabla \vec{R}|^2 + \delta^2 |\nabla \vec{n}|^2]\, dx^2. \tag{IV.65}$$

We recall the structural equation (see [40])

$$\nabla \vec{n} = \nabla^\perp \vec{n} \times \vec{n} - 2H \nabla \vec{\Phi}. \tag{IV.66}$$

Taking the divergence gives

$$\Delta \vec{n} = \nabla^\perp \vec{n} \times \nabla \vec{n} - 2\, \mathrm{div}[2H \nabla \vec{\Phi}]. \tag{IV.67}$$

We introduce $\vec{\xi}$ to be the solution of

$$\begin{cases} \Delta \vec{\xi} = \nabla^\perp \vec{n} \times \nabla \vec{n} - 2\, \mathrm{div}[2H \nabla \vec{\Phi}] & \text{in } B_r(x_0), \\ \vec{\xi} = 0 & \text{on } \partial B_r(x_0). \end{cases} \tag{IV.68}$$

Classical elliptic estimates combined with the first equation of (IV.55) and the fact that $\|\nabla \vec{n}\|_2^2 < \varepsilon$ give

$$\int_{B_r(x_0)} |\nabla \vec{\xi}|^2\, dx^2 \leq C \int_{B_r(x_0)} [|\nabla S|^2 + |\nabla \vec{R}|^2 + |\nabla \vec{v}|^2]\, dx^2 + C\varepsilon \int_{B_r(x_0)} |\nabla \vec{n}|^2\, dx^2. \tag{IV.69}$$

Since $\vec{n} - \vec{\xi}$ is harmonic on $B_r(x_0)$, we have

$$\int_{B_{tr}(x_0)} |\nabla (\vec{n} - \vec{\xi})|^2\, dx^2 \leq t^2 \int_{B_r(x_0)} |\nabla (\vec{n} - \vec{\xi})|^2\, dx^2. \tag{IV.70}$$

Hence

$$\int_{B_{tr}(x_0)} |\nabla \vec{n}|^2\, dx^2 \leq [t^2 + C\varepsilon] \int_{B_r(x_0)} |\nabla \vec{n}|^2\, dx^2 + C \int_{B_r(x_0)} [|\nabla S|^2 + |\nabla \vec{R}|^2 + |\nabla \vec{v}|^2]\, dx^2. \tag{IV.71}$$

Inserting (IV.65) in (IV.71) we finally obtain, taking σ small enough that $\delta < 1$,

$$\int_{B_{t^2 r}(x_0)} [|\nabla S|^2 + |\nabla \vec{R}|^2 + |\nabla \vec{v}|^2 + \delta^2 |\nabla \vec{n}|^2]\, dx^2 \leq C[t^2 + C\sqrt{\varepsilon} + \delta^2] \int_{B_r(x_0)} [|\nabla S|^2 + |\nabla \vec{R}|^2 + |\nabla \vec{v}|^2 + \delta^2 |\nabla \vec{n}|^2]\, dx^2. \tag{IV.72}$$

We fix $t > 0$ and $\varepsilon > 0$ independent of r, x_0, $\sigma < \sigma_0$ and the solution such that $C[t^2 + C\sqrt{\varepsilon}] \leq 1/2$. By a classical iteration argument we deduce the existence of $\gamma > 0$ such that

$$\sup_{x_0 \in D^2_{1/8},\, r < 1/16} r^{-\gamma} \int_{B_r^2(x_0)} [|\nabla S|^2 + |\nabla \vec{R}|^2 + |\nabla \vec{v}|^2 + \delta^2 |\nabla^2 (e^{-\overline{\lambda}} \vec{\Phi})|^2]\, dx^2 \leq \int_{D^2_{1/4}} [|\nabla S|^2 + |\nabla \vec{R}|^2 + |\nabla \vec{v}|^2 + \delta^2 |\nabla \vec{n}|^2]\, dx^2. \tag{IV.73}$$

Combining (IV.41), (IV.53) and (IV.73) we obtain in particular

$$\begin{aligned} &\sup_{x_0 \in D^2_{1/8},\, r < 1/16} r^{-\gamma/2} \left[\int_{B_r^2(x_0)} [|\nabla S|^2 + |\nabla \vec{R}|^2 + |\nabla \vec{v}|^2]\, dx^2\right]^{1/2} \\ &\quad \leq C \|\nabla \vec{n}\|_{L^2(D^2)} + C l_\sigma \|\nabla \alpha\|^2_{L^2(D^2)} \\ &\qquad + C l_\sigma |\overline{\alpha}| \int_{D^2} e^{2\mu}\, dx^2 + C\sigma^2 \int_{D^2} [1 + H^4] e^{2\lambda}\, dx^2 + C l_\sigma \frac{A_{\vec{\Phi}}(D^2)}{A_{\vec{\Phi}}(\Sigma)}. \end{aligned} \tag{IV.74}$$

Combining (IV.74) and (IV.27) gives

$$\begin{aligned} &\sup_{x_0 \in D^2_{1/8},\, r < 1/16} r^{-\gamma} \int_{B_r^2(x_0)} H^2 [1 + \sigma^2 (1 + H^2)]^2\, dx^2] e^{2\lambda}\, dx^2 \\ &\quad \leq C \int_{D^2} |\nabla \vec{n}|^2\, dx^2 + C\left[l_\sigma \int_{D^2} |\nabla \alpha|^2\, dx^2\right]^2 + C l_\sigma^2 |\overline{\alpha}|^2 \left[\int_{D^2} e^{2\mu}\, dx^2\right]^2 \\ &\qquad + C\left[\sigma^2 \int_{D^2} [1 + H^4] e^{2\lambda}\, dx^2\right]^2 + C l_\sigma^2 \left[\frac{A_{\vec{\Phi}}(D^2)}{A_{\vec{\Phi}}(\Sigma)}\right]^2. \end{aligned} \tag{IV.75}$$

Step 4. Bootstraping (IV.75). Lemma VII.4 applied to (IV.67) implies that

$$\begin{aligned}
&\sup_{x_0\in D^2_{1/16},\, r<1/32} r^{-\gamma}\int_{B_r^2(x_0)}|\nabla\vec{n}|^2\,dx^2\\
&\quad\leq C\int_{D^2}|\nabla\vec{n}|^2\,dx^2+C\left[l_\sigma\int_{D^2}|\nabla\alpha|^2\,dx^2\right]^2+Cl_\sigma^2|\overline{\alpha}|^2\left[\int_{D^2}e^{2\mu}\,dx^2\right]^2\\
&\qquad+C\left[\sigma^2\int_{D^2}[1+H^4]e^{2\lambda}\,dx^2\right]^2+Cl_\sigma^2\left[\frac{A_{\vec{\Phi}}(D^2)}{A_{\vec{\Phi}}(\Sigma)}\right]^2.
\end{aligned}\tag{IV.76}$$

The Liouville equation reads

$$-\Delta\alpha=e^{2\lambda}K+\Delta\mu=e^{2\lambda}K-e^{2\mu}K_{g_0}.\tag{IV.77}$$

Thus

$$\int_{B_r^2(x_0)}|\Delta\alpha|\,dx^2\leq\int_{B_r^2(x_0)}|\nabla\vec{n}|^2\,dx^2+4\pi^2r^2\|e^{2\mu}\|_{L^\infty(D^2)}.\tag{IV.78}$$

Combining (IV.76), (IV.78) and Adams–Morrey embedding gives, for $p<\frac{2-\gamma}{1-\gamma}$,

$$\begin{aligned}
&\|\nabla\alpha\|_{L^p(D^2_{1/32})}\\
&\quad\leq C_p\sup_{x_0\in D^2_{1/16},\, r<1/32} r^{-\gamma}\int_{B_r^2(x_0)}|\nabla\vec{n}|^2\,dx^2+C\|e^{2\mu}\|_{L^\infty(D^2)}+\|\nabla\alpha\|_{L^2(D^2)}.
\end{aligned}\tag{IV.79}$$

This gives in particular

$$\begin{aligned}
\left[\int_{D^2_{1/32}}[l_\sigma|\nabla\alpha|^2]^{p/2}\,dx^2\right]^{2/p}&\leq Cl_\sigma\left[\sup_{x_0\in D^2_{1/16},\, r<1/32} r^{-\gamma}\int_{B_r^2(x_0)}|\nabla\vec{n}|^2\,dx^2\right]^2\\
&\quad+Cl_\sigma\|e^{2\mu}\|^2_{L^\infty(D^2)}+Cl_\sigma\int_{D^2}|\nabla\alpha|^2\,dx^2.
\end{aligned}\tag{IV.80}$$

Equation (IV.7) gives

$$\begin{aligned}
\Delta(2\vec{H}(1+2\sigma^2(1+H^2)))=\operatorname{div}\Big(&2l_\sigma f_\sigma'(H)\nabla\vec{n}+2e^{-2\lambda}l_\sigma\nabla\vec{\Phi}\cdot\nabla\alpha\nabla\alpha\\
&-l_\sigma[-2f_\sigma(H)+e^{-2\lambda}|\nabla\alpha|^2-K_{g_0}\alpha e^{-2\alpha}\\
&+K_{g_0}A_{\vec{\Phi}}(\Sigma)^{-1}]\nabla\vec{\Phi}-2l_\sigma e^{-2\lambda}(\vec{\mathbb{I}}\llcorner\nabla^\perp\alpha)^\perp\Big).
\end{aligned}\tag{IV.81}$$

This implies that $\vec{V}:=e^{\overline{\lambda}}2\vec{H}(1+2\sigma^2(1+H^2))$ satisfies an equation of the form

$$\Delta\vec{V}=\operatorname{div}(\vec{I}+\vec{J}),\tag{IV.82}$$

where

$$\vec{I}:=2l_\sigma e^{\overline{\lambda}}f_\sigma'(H)\nabla\vec{n}+2l_\sigma f_\sigma(H)\nabla\vec{\Phi}+l_\sigma K_{g_0}\alpha e^{\overline{\lambda}-2\alpha}\nabla\vec{\Phi}.$$

Thanks to (IV.75) we have on the one hand

$$\sup_{x_0\in D^2_{1/32},\, r<1/4} r^{-\gamma}\int_{B^2_r(x_0)}|\vec{I}|\,dx^2 \le C\int_{D^2}|\nabla\vec{n}|^2\,dx^2 + C\left[l_\sigma\int_{D^2}|\nabla\alpha|^2\,dx^2\right]^2$$
$$+ Cl_\sigma^2|\overline{\alpha}|^2\|e^{4\mu}\|_{L^\infty(D^2)} + C\left[\sigma^2\int_{D^2}[1+H^4]e^{2\lambda}\,dx^2\right]^2 + Cl_\sigma^2\left[\frac{A_{\vec{\Phi}}(D^2)}{A_{\vec{\Phi}}(\Sigma)}\right]^2 \qquad \text{(IV.83)}$$

where γ and C are independent of the solution and of σ but only depend on the constant C_1 in the statement of the lemma.

On the other hand, using the fact that $2-\gamma/(1-\gamma) > 2$ and using (IV.80) we deduce the existence of $q > 1$ such that

$$\left[\int_{D^2_{1/32}}|\vec{J}|^q\,dx^2\right]^{1/q} \le Cl_\sigma\left[\sup_{x_0\in D^2_{1/16},\, r<1/32} r^{-\gamma}\int_{B^2_r(x_0)}|\nabla\vec{n}|^2\,dx^2\right]^2$$
$$+ C[l_\sigma\|e^{2\mu}\|_{L^\infty(D^2)} + l_\sigma|\overline{\alpha}|]\|e^{2\mu}\|_{L^\infty(D^2)} + Cl_\sigma\int_{D^2}|\nabla\alpha|^2\,dx^2. \qquad \text{(IV.84)}$$

Using (III.67) and the classical Adams–Sobolev inequalities (see [40]) gives the existence of $p > 2$ such that

$$2\|e^{\overline{\lambda}}\vec{H}(1+2\sigma^2(1+H^2))\|^2_{L^p(D_{1/64})} = \|V\|^2_{L^p(D_{1/64})}$$
$$\le C\int_{D^2}|\nabla\vec{n}|^2\,dx^2 + C\left[l_\sigma\int_{D^2}|\nabla\alpha|^2\,dx^2\right]^2 + C\left[\sigma^2\int_{D^2}H^4e^{2\lambda}\,dx^2\right]^2$$
$$+ C[l_\sigma^2\,|\overline{\alpha}|^2 + l_\sigma^2\|e^{4\mu}\|_{L^\infty(D^2)}]\|e^{4\mu}\|_{L^\infty(D^2)} + Cl_\sigma^2\left[\frac{A_{\vec{\Phi}}(D^2)}{A_{\vec{\Phi}}(\Sigma)}\right]^2 \qquad \text{(IV.85)}$$

where we have used (III.69), (III.70) .

Bootstraping this information respectively in the three elliptic systems (IV.67), (IV.77) and (IV.81) (which are now becoming subcritical for $\vec{V} \in L^p$ with $p > 2$) one obtains (IV.5), (IV.6), and Lemma IV.1 is proved. □

V. The Palais–Smale condition for frame energies

V.1. Sequential weak compactness of weak immersions in $\mathcal{E}_{\Sigma,2}$ with uniformly bounded frame energies

In this section we work with the Lagrangian F^σ defined in the previous section but the parameter σ will be fixed all along the section. So, in order to simplify the presentation, we will simply work with the following Lagrangian:

$$F(\vec{\Phi}) := \int_\Sigma\left[H^2 + [(1+H^2)^2 + 2^{-1}|d\alpha|^2_{g_{\vec{\Phi}}} + K_{g_0}\alpha e^{-2\alpha}]\right]d\mathrm{vol}_{g_{\vec{\Phi}}}$$
$$- 2^{-1}K_{g_0}\log(\mathrm{Area}(\vec{\Phi}(\Sigma))), \qquad \text{(V.1)}$$

where as before g_0 is a constant Gauss curvature metric of volume 1 on Σ, and $g_{\vec{\Phi}} = e^{2\alpha} g_0$. The following lemma was proved in [35] when Σ is the torus. We extend it now to general surfaces following the same ideas.

Lemma V.1. *Let Σ be a closed surface and $\vec{\Phi}^k$ be a sequence of weak immersions in $\mathcal{E}_{\Sigma,2}$ satisfying*

$$\limsup_{k\to+\infty} F(\vec{\Phi}^k) < +\infty. \tag{V.2}$$

Then the conformal class of the associated sequence g_0^k of constant scalar curvature metrics of volume 1 such that $g_{\vec{\Phi}^k} = e^{2\alpha^k} g_0^k$ is precompact in the moduli space $\mathcal{M}(\Sigma)$. Moreover, there exists a sequence of diffeomorphisms Ψ^k of Σ such that $(\Psi^k)^* g_0^k$ converges strongly in any C^l topology to a limiting constant curvature metric h, $\vec{\Phi}^k \circ \Psi^k$ is conformal on $(\Sigma, (\Psi^k)^* g_0^k)$ and is sequentially weakly precompact in $W^{2,4}$ and for any weakly converging subsequence the limit $\vec{\xi}^\infty$ is still a weak immersion in $\mathcal{E}_{\Sigma,2}$ and

$$\log |d(\vec{\Phi}^k \circ \Psi^k)|^2_{(\Psi^k)^* g_0^k} \rightharpoonup \log |d\vec{\xi}^\infty|^2_h \quad \textit{weakly in } W^{1,4}_h. \tag{V.3}$$

Proof. We work with an Aubin gauge in the case $K_{g_0} > 0$. Using Lemma III.12 for $K_{g_0} < 0$ and Lemma III.11 for $K_{g_0} > 0$, we have in all cases

$$\limsup_{k\to+\infty} \int_\Sigma (1 + H^2_{\vec{\Phi}^k})^2 \, d\mathrm{vol}_{g_{\vec{\Phi}^k}} < +\infty. \tag{V.4}$$

Hence using again (III.76) for genus$(\Sigma) > 1$ or Lemma III.11 for $K_{g_0} > 0$ gives in all cases

$$\limsup_{k\to+\infty} \int_\Sigma |d\alpha^k|^2_{g_0^k} \, d\mathrm{vol}_{g_0^k} < +\infty. \tag{V.5}$$

Moreover since Σ is a closed surface, using the Willmore–Li–Yau universal lower bound of the Willmore energy (see [40]) we also have

$$16\pi^2 \le \left[\int_\Sigma e^{2\alpha^k} \, d\mathrm{vol}_{g_0^k}\right]\left[\int_\Sigma H^4_{\vec{\Phi}^k} \, d\mathrm{vol}_{g_{\vec{\Phi}^k}}\right]. \tag{V.6}$$

Combining (V.4) and (V.6) we have proved in all cases

$$\limsup_{k\to+\infty} |\log(\mathrm{Area}(\vec{\Phi}^k(\Sigma)))| < +\infty. \tag{V.7}$$

These preliminary estimates having been established we now prove the precompactness of the conformal class in the non-zero genus case. The case when Σ is a torus has already been considered in [35]. So we can restrict to the case where genus$(\Sigma) > 1$. Assuming the conformal class associated to $g_{\vec{\Phi}^k}$ and hence g_0^k would degenerate we have a rather precise description of this degeneration (see [23]). It requires the formation of at least a *collar* which is a subdomain of Σ diffeomorphic to an annulus that we identify to a cylinder of the form

$$\mathcal{C} := \left\{(x_1, x_2)\,;\ \frac{2\pi}{l^k}\varphi^k < x_2 < \frac{2\pi}{l^k}(\pi - \varphi^k),\ 0 \le x_1 \le 2\pi\right\}$$

where the vertical lines $x_1 = 0$ and $x_1 = 2\pi$ are identified, l^k is the length of a closed geodesic for the hyperbolic metric g_0^k,

$$l^k \to 0,$$

and $\varphi^k := \arctan(\sinh(l^k/2))$. The closed geodesic of length l^k is given by $x_2 = \pi^2/l^k$. On this cylinder the hyperbolic metric g_0^k has the following explicit expression:

$$g_0^k = \left(\frac{l^k}{2\pi\,\sin\left(\frac{l^k x_2}{2\pi}\right)}\right)^2 [dx_1^2 + dx_2^2].$$

Denote in these coordinates

$$g_{\vec{\Phi}^k} := e^{2\lambda^k}[dx_1^2 + dx_2^2] \quad \text{and} \quad \vec{e}_i^{\,k} := e^{-\lambda^k}\,\partial_{x_i}\vec{\Phi}^k.$$

The unit vector field $\vec{e}_1^{\,k}$ is tangent to a foliation by circles of the $\vec{\Phi}^k$-image in $\mathbb{R}^3$ of the collar region. We apply the Fenchel theorem to each of these circles. More precisely, for each $t \in \left(\frac{2\pi}{l^k}\varphi^k, \frac{2\pi}{l^k}(\pi - \varphi^k)\right)$ we have

$$2\pi \le \int_{\{x_2=t\}\cap\mathcal{C}} \left|\frac{\partial \vec{e}_1^{\,k}}{\partial x_1}\right| dx_1. \tag{V.8}$$

Integrating this inequality for x_2 between $\frac{2\pi}{l^k}\varphi^k$ and $\frac{2\pi}{l^k}(\pi - \varphi^k)$ and using Cauchy–Schwarz gives

$$\frac{(2\pi)^2}{l^k}(\pi - 2\varphi^k) \le \frac{2\pi}{\sqrt{l^k}}\sqrt{\pi - 2\varphi^k}\left[\int_{\mathcal{C}} |\nabla \vec{e}_1^{\,k}|^2\,dx^2\right]^{1/2}. \tag{V.9}$$

We have $|\nabla \vec{e}_1^{\,k}|^2 \le |\nabla \vec{n}_{\vec{\Phi}^k}|^2 + |\nabla\lambda^k|^2$, and moreover $\lambda^k = \alpha^k + \mu^k$ where

$$\mu^k = 2\log\left(\frac{l^k}{2\pi}\right) - 2\log\left(\sin\left(\frac{l^k x_2}{2\pi}\right)\right).$$

Thus

$$|\nabla\mu^k|^2 = \frac{(l^k)^2}{\pi^2}\,\frac{\cos^2\left(\frac{l^k x_2}{2\pi}\right)}{\sin^2\left(\frac{l^k x_2}{2\pi}\right)}.$$

Hence

$$\int_{\mathcal{C}} |\nabla\mu^k|^2\,dx^2 \le \frac{(l^k)^2}{\pi^2}\int_{2\pi\varphi^k/l^k}^{2\pi(\pi-\varphi^k)/l^k} \frac{dx_2}{\sin^2\left(\frac{l^k x_2}{2\pi}\right)} \le C(l^k)^2\int_{2\pi\varphi^k/l^k}^{\infty}\frac{dx_2}{(l^k)^2 x_2^2} \le C, \tag{V.10}$$

and since from (V.5) we have

$$\limsup_{k\to+\infty}\int_{\mathcal{C}} |\nabla\alpha^k|^2\,dx^2 = \limsup_{k\to+\infty}\int_{\mathcal{C}} |d\alpha^k|^2_{g_{\vec{\Phi}^k}}\,d\mathrm{vol}_{g_{\vec{\Phi}^k}} < +\infty,$$

we deduce that

$$\limsup_{k\to+\infty}\int_{\mathcal{C}}|\nabla\vec{e}_1^{\,k}|^2\,dx^2<+\infty. \tag{V.11}$$

Combining (V.9) and (V.11) shows that l^k is bounded from below by a positive number, which contradicts the formation of a collar and the degeneracy of the conformal class of $[g_0^k]$. Modulo composition with isometries, g_0^k strongly converges in every Banach space $C^l(\Sigma)$. In order to simplify the presentation we assume that g_0^k is fixed. We cover the Riemannian surface (Σ, g_0) by finitely many conformal charts $\phi_j : D^2 \to \phi_j(D^2)$ for $j \in J$ such that $\Sigma \subset \bigcup_j \phi_j(D^2_{1/2})$. Denote again by $\phi_j^* g_{\vec{\Phi}^k} = e^{2\lambda_j^k}\,[dx_1^2+dx_2^2]$ the expression of $g_{\vec{\Phi}^k}$ in the chart ϕ_j. We have

$$\limsup_{k\to+\infty}\int_{D^2}(|\nabla\vec{n}_{\vec{\Phi}^k\circ\phi_j}|^2+|\nabla\lambda_j^k|^2)\,dx^2<+\infty. \tag{V.12}$$

For $i=1,2$ we denote $\vec{e}^{\,k}_{j,i}:=e^{-\lambda_j^k}\,\partial_{x_i}(\vec{\Phi}^k\circ\phi_j)$ and the Liouville equation gives

$$-\Delta\lambda_j^k=(\nabla\vec{e}^{\,k}_{j,1};\nabla^\perp\vec{e}^{\,k}_{j,2})\quad \text{in } D^2.$$

Inequality (V.12) implies

$$\limsup_{k\to+\infty}\int_{D^2}|\nabla\vec{e}^{\,k}_{j,i}|^2\,dx^2<+\infty. \tag{V.13}$$

Combining (V.12) and (V.13) with the Wente estimates we obtain the existence of $\overline{\lambda_j^k}\in\mathbb{R}$ such that

$$\limsup_{k\to+\infty}\|\lambda_j^k-\overline{\lambda_j^k}\|_{L^\infty(D^2_{3/4})}<+\infty. \tag{V.14}$$

Due to the connectedness of Σ we deduce that

$$\sup_{j\neq l}\limsup_{k\to+\infty}\|\overline{\lambda_l^k}-\overline{\lambda_j^k}\|_{L^\infty(D^2_{3/4})}<+\infty. \tag{V.15}$$

Moreover

$$\limsup_{k\to+\infty}\int_{D^2}e^{2\lambda_j^k}\,dx^2<+\infty. \tag{V.16}$$

We have

$$\begin{aligned}4\pi&\le\sum_{j\in J}\int_{D^2_{1/2}}H^2_{\vec{\Phi}^k\circ\phi_j}\,e^{2\lambda_j^k}\,dx^2\\&\le\sum_{j\in J}\left(\int_{D^2_{1/2}}e^{2\lambda_j^k}\,dx^2\right)^{1/2}\left(\int_{D^2_{1/2}}H^4_{\vec{\Phi}^k\circ\phi_j}\,e^{2\lambda_j^k}\,dx^2\right)^{1/2}.\end{aligned} \tag{V.17}$$

Since

$$\limsup_{k\to+\infty}\int_{D^2_{1/2}}H^4_{\vec{\Phi}^k\circ\phi_j}\,e^{2\lambda_j^k}\,dx^2<+\infty, \tag{V.18}$$

we deduce from (V.15)–(V.17) that

$$\max_{j\in J}\limsup_{k\to+\infty}\|\lambda_j^k\|_{L^\infty(D^2_{3/4})}<+\infty. \tag{V.19}$$

Using

$$\Delta(\vec{\Phi}^k\circ\phi_j)=e^{2\lambda_j^k}H_{\vec{\Phi}^k\circ\phi_j}$$

together with (V.18) we deduce that $\vec{\Phi}^k\circ\phi_j$ is sequentially weakly precompact in $W^{2,4}(D^2_{1/2})$. Bootstrapping with (V.19) gives (V.3), and Lemma V.1 is proved. □

V.2. *The Finsler structure on the space of $W^{2,4}$-immersions*

While aiming to apply *Palais deformation theory* we are going to equip $\mathcal{E}_{\Sigma,2}$, the space of $W^{2,4}$-immersions, with a *Finsler structure* $\|\cdot\|_{\vec{\Phi}}$ given in [44] for which the metric space given by the *Palais distance* is complete.

V.3. *The Palais–Smale condition*

The aim of the present subsection is to establish the following lemma which was proved for the exact Willmore functional in [6].

Lemma V.2. *Let Σ be a closed surface and $\vec{\Phi}^k$ be a sequence of weak immersions in $\mathcal{E}_{\Sigma,2}$ satisfying*

$$\limsup_{k\to+\infty}F(\vec{\Phi}^k)<+\infty \tag{V.20}$$

where F is given by (IV.1), and

$$\lim_{k\to+\infty}\sup_{\|\vec{w}\|_{\vec{\Phi}^k}\le 1}\partial F(\vec{\Phi}^k)\cdot\vec{w}=0 \tag{V.21}$$

where for any $\vec{\Phi}\in\mathcal{E}_{\Sigma,2}$ we denote

$$\|\vec{w}\|^4_{\vec{\Phi}}:=\int_\Sigma[|\nabla^{g_{\vec{\Phi}}}d\vec{w}|^4_{g_{\vec{\Phi}}}+|d\vec{w}|^4_{g_{\vec{\Phi}}}+|w|^4]\,d\mathrm{vol}_{g_{\vec{\Phi}}}.$$

Then, modulo extraction of a subsequence, there exists a sequence of parametrizations Ψ^k such that $\vec{\Phi}^k\circ\Psi^k$ strongly converges in $W^{2,4}$ to a critical point of F in $\mathcal{E}_{\Sigma,2}$.

Proof. We take an Aubin gauge for the S^2 case. We denote $f(t):=t^2+(1+t^2)^2$. We consider the charts ϕ^j given by the previous lemma and we omit the index j. We also skip the index k unless absolutely necessary. In each of these charts, (V.21) says that

$$\begin{aligned}\vec{G}:=\operatorname{div}\Big[&\nabla(f'(H)\vec{n})-2f'(H)\nabla\vec{n}-2e^{-2\lambda}\nabla\vec{\Phi}\cdot\nabla\alpha\nabla\alpha\\&+[-2f(H)+e^{-2\lambda}|\nabla\alpha|^2-K_{g_0}\alpha e^{-2\alpha}+K_{g_0}A_{\vec{\Phi}}(\Sigma)^{-1}]\nabla\vec{\Phi}-2e^{-2\lambda}(\vec{\mathbb{I}}\llcorner\nabla^\perp\alpha)^\perp\Big]\\&\to 0\quad\text{strongly in }W^{-2,4/3}(D^2)=(W^{2,4}_0(D^2))^*.\end{aligned} \tag{V.22}$$

Let

$$\vec{\phi}_k := -\frac{1}{2\pi}\,\frac{1}{r}\,\frac{\partial}{\partial r} \star \big[-2f'(H)\nabla\vec{n} - 2e^{-2\lambda}\nabla\vec{\Phi}\cdot\nabla\alpha\nabla\alpha$$
$$+ [-2f(H) + e^{-2\lambda}|\nabla\alpha|^2 - K_{g_0}\alpha e^{-2\alpha} + K_{g_0}A_{\vec{\Phi}}(\Sigma)^{-1}]\nabla\vec{\Phi} - 2e^{-2\lambda}(\vec{\mathbb{I}}\llcorner\nabla^{\perp}\alpha)^{\perp}\big]\mathbf{1}_{D^2}.$$

This convolution is justified by observing that $\frac{1}{r}\frac{\partial}{\partial r} \in L^{2,\infty}(\mathbb{R}^2)$ and the other term in the convolution is uniformly bounded in $L^1(\mathbb{R}^2)$. Hence

$$\limsup_{k\to+\infty} \|\vec{\phi}_k\|_{2,\infty} < +\infty.$$

Obviously

$$\begin{cases} \vec{G}_k = \Delta(f'(H_k)\vec{n}_k + \vec{\phi}_k) \to 0 \quad \text{strongly in } W^{-2,4/3}(D^2) = (W_0^{2,4}(D^2))^*, \\ \limsup\limits_{k\to+\infty} \| f'(H_k)\vec{n}_k + \vec{\phi}_k\|_{L^{4/3}(D^2)} < +\infty. \end{cases}$$

Let $\vec{h}$ be a weak limit (of a subsequence) of $f'(H_k)\vec{n}_k + \vec{\phi}_k$ in $L^{4/3}(D^2)$. It is obviously harmonic, and using Rellich–Kondrashov we have

$$f'(H_k)\vec{n}_k + \vec{\phi}_k \to \vec{h} \quad \text{strongly in } L^{4/3}_{\mathrm{loc}}(D^2).$$

Let $\vec{M}_k := f'(H_k)\vec{n}_k + \vec{\phi}_k - \vec{h}$. Then

$$\vec{G}_k = \Delta\vec{M}_k \quad \text{and} \quad \vec{M}_k \to 0 \quad \text{strongly in } L^{4/3}_{\mathrm{loc}}(D^2). \tag{V.23}$$

Applying the Poincaré lemma we obtain the existence of $\vec{L}$ such that

$$\begin{aligned} \nabla\vec{L} := {} & \nabla^{\perp}(f'(H)\vec{n}) - 2f'(H)\nabla^{\perp}\vec{n} - 2e^{-2\lambda}\,\sigma^2\nabla\vec{\Phi}\cdot\nabla\alpha\nabla^{\perp}\alpha \\ & + [-2f(H) + e^{-2\lambda}\,|\nabla\alpha|^2 - K_{g_0}\alpha e^{-2\alpha} + K_{g_0}A_{\vec{\Phi}}(\Sigma)^{-1}]\nabla^{\perp}\vec{\Phi} \\ & + 2e^{-2\lambda}\vec{\mathbb{I}}\llcorner\nabla^{\perp}\alpha - \nabla^{\perp}\vec{M}. \end{aligned} \tag{V.24}$$

Equation (III.50) gives

$$\begin{aligned} & \nabla\vec{\Phi}\cdot\nabla^{\perp}\vec{L} - \mathrm{div}[\nabla\vec{\Phi}\cdot\vec{M}] \\ & \quad = 2e^{2\lambda}(2f(H) - H\,f'(H) + K_{g_0}\alpha e^{-2\alpha} - K_{g_0}A_{\vec{\Phi}}(\Sigma)^{-1}) - \vec{M}\cdot\Delta\vec{\Phi}. \end{aligned}$$

Let Y be the solution of

$$\begin{cases} \Delta Y = 2e^{2\lambda}\big(2f(H) - Hf'(H) + K_{g_0}\alpha e^{-2\alpha} - K_{g_0}A_{\vec{\Phi}}(\Sigma)^{-1}\big) - \vec{M}\cdot\Delta\vec{\Phi} & \text{in } D^2, \\ Y = 0 & \text{on } \partial D^2. \end{cases}$$

Observe that $2f(H) - H\,f'(H) = 2(1 - H^4)$. So Y satisfies

$$\begin{cases} -\Delta Y = 4e^{2\lambda}(1 - H^4) + 2K_{g_0}\alpha e^{2\mu} - 2K_{g_0}A_{\vec{\Phi}}(\Sigma)^{-1} - \vec{M}\cdot\Delta\vec{\Phi} & \text{in } D^2, \\ Y = 0 & \text{on } \partial D^2. \end{cases}$$

Since $\Delta\vec{\Phi}$ is uniformly bounded in L^4, we have $\vec{M}\cdot\Delta\vec{\Phi}\to 0$ strongly in L^1. Hence

$$\limsup_{k\to+\infty}\|\nabla Y_k\|_{L^{2,\infty}(D^2)}<+\infty. \tag{V.25}$$

Using the Poincaré lemma we deduce the existence of a function S such that

$$\nabla S=\vec{L}\cdot\nabla\vec{\Phi}-\vec{M}\cdot\nabla^\perp\vec{\Phi}+\nabla^\perp Y. \tag{V.26}$$

Equation (III.58) in conformal coordinates gives

$$\nabla\vec{\Phi}\times[\nabla^\perp\vec{L}-\nabla\vec{M}]=-\nabla^\perp\vec{\Phi}\cdot\nabla f'(H)+2\nabla\alpha\cdot\nabla^\perp\vec{D} \tag{V.27}$$

where

$$\nabla\vec{D}=\left(e^{-2\lambda}\sum_{i=1}^2\mathbb{I}_{1i}\,\partial_{x_i}\vec{\Phi},\,e^{-2\lambda}\sum_{i=1}^2\mathbb{I}_{2i}\,\partial_{x_i}\vec{\Phi}\right). \tag{V.28}$$

Let $\vec{W}$ be the solution of

$$\begin{cases}\Delta\vec{W}=\vec{M}\times\Delta\vec{\Phi} & \text{in } D^2,\\ \vec{W}=0 & \text{on } \partial D^2.\end{cases} \tag{V.29}$$

Since $\Delta\vec{\Phi}$ is uniformly bounded in L^4, we have $\vec{M}\times\Delta\vec{\Phi}\to 0$ strongly in L^1. Hence

$$\lim_{k\to+\infty}\|\nabla\vec{W}_k\|_{L^{2,\infty}(D^2)}=0. \tag{V.30}$$

Using again the Poincaré lemma on D^2 we obtain the existence of $\vec{V}$ such that

$$\nabla\vec{V}:=\vec{L}\times\nabla\vec{\Phi}-\vec{M}\times\nabla^\perp\vec{\Phi}+f'(H)\nabla\vec{\Phi}-2(\alpha-\overline{\alpha})\nabla\vec{D}+\nabla^\perp\vec{W}. \tag{V.31}$$

Using the explicit expression of $\nabla\vec{D}$ given by (III.59) we obtain

$$\begin{aligned}\vec{n}\cdot(\nabla\vec{V}-\nabla^\perp\vec{W})&=\vec{n}\cdot(\vec{L}\times\nabla\vec{\Phi}-\vec{M}\times\nabla^\perp\vec{\Phi})\\&=\vec{L}\cdot\nabla^\perp\vec{\Phi}+\vec{M}\cdot\nabla\vec{\Phi}=\nabla^\perp S+\nabla Y.\end{aligned} \tag{V.32}$$

We also have

$$\begin{aligned}\vec{n}\times(\nabla\vec{V}-\nabla^\perp\vec{W})&=-(\vec{L}\cdot\vec{n})\nabla\vec{\Phi}+(\vec{n}\cdot\vec{M})\nabla^\perp\vec{\Phi}\\&\quad-f'(H)\nabla^\perp\vec{\Phi}-2(\alpha-\overline{\alpha})\vec{n}\times\nabla\vec{D}.\end{aligned} \tag{V.33}$$

Denoting by $\pi_T(\nabla^\perp\vec{V}+\nabla\vec{W})$ the tangential projection of $\nabla^\perp\vec{V}+\nabla\vec{W}$, we have

$$\begin{aligned}\pi_T(\nabla^\perp\vec{V}+\nabla\vec{W})&=(\vec{L}\cdot\vec{n})\nabla\vec{\Phi}-(\vec{n}\cdot\vec{M})\nabla^\perp\vec{\Phi}\\&\quad+f'(H)\nabla^\perp\vec{\Phi}-2(\alpha-\overline{\alpha})\nabla^\perp\vec{D}.\end{aligned} \tag{V.34}$$

Hence

$$\vec{n}\times(\nabla\vec{V}-\nabla^\perp\vec{W})=-\nabla^\perp\vec{V}-\nabla\vec{W}-2(\alpha-\overline{\alpha})(\nabla^\perp\vec{D}+\vec{n}\times\nabla\vec{D})-\vec{n}(\nabla S-\nabla^\perp Y). \tag{V.35}$$

Let $\vec{v}$ be the unique solution to

$$\begin{cases} \Delta \vec{v} = \nabla^{\perp} Y \cdot \nabla \vec{n} & \text{in } D^2, \\ \vec{v} = 0 & \text{on } \partial D^2. \end{cases} \tag{V.36}$$

Using one more time the Poincaré lemma we obtain the existence of $\vec{u}$ such that

$$\vec{n} \nabla^{\perp} Y = \nabla \vec{v} + \nabla^{\perp} \vec{u}. \tag{V.37}$$

Finally, let $\vec{R} := \vec{V} - \vec{u}$. We have

$$\vec{n} \times \nabla \vec{V} = \vec{n} \times \nabla \vec{R} + \vec{n} \times \nabla \vec{u} = \vec{n} \times \nabla \vec{R} + \vec{n} \times \nabla^{\perp} \vec{v}. \tag{V.38}$$

Hence (V.35) becomes

$$\begin{aligned} \vec{n} \times (\nabla \vec{R} - \nabla^{\perp} \vec{W}) + \vec{n} \times \nabla^{\perp} \vec{v} = & -\nabla^{\perp} \vec{R} - \nabla \vec{W} + \nabla \vec{v} \\ & - \vec{n} \nabla S - 2(\alpha - \overline{\alpha})(\nabla^{\perp} \vec{D} + \vec{n} \times \nabla \vec{D}), \end{aligned} \tag{V.39}$$

which gives

$$\begin{cases} \Delta S = -\nabla \vec{n} \cdot \nabla^{\perp} \vec{R} + \operatorname{div}(\vec{n} \cdot \nabla \vec{z} - 2(\alpha - \overline{\alpha}) \vec{n} \cdot \nabla^{\perp} \vec{D}), \\ \Delta \vec{R} = \nabla \vec{n} \times \nabla^{\perp} \vec{R} + \nabla \vec{n} \cdot \nabla^{\perp} S + \operatorname{div}\big(-\vec{n} \times \nabla \vec{z} + 2(\alpha - \overline{\alpha})(-\nabla \vec{D} + \vec{n} \times \nabla^{\perp} \vec{D})\big), \end{cases}$$

where $\vec{z} := \vec{v} - \vec{W}$. Let $\vec{U} := \vec{R} + 2(\alpha - \overline{\alpha}) \vec{D}$. With this notation the above becomes

$$\begin{cases} \Delta S = -\nabla \vec{n} \cdot \nabla^{\perp} \vec{U} + \operatorname{div}(\vec{n} \cdot \nabla \vec{z} + 2 \vec{n} \cdot \vec{D} \nabla^{\perp} \alpha), \\ \Delta \vec{U} = \nabla \vec{n} \times \nabla^{\perp} \vec{U} + \nabla \vec{n} \cdot \nabla^{\perp} S - \operatorname{div}(\vec{n} \times \nabla \vec{z} - 2 \nabla \alpha \vec{D} + \vec{n} \times \vec{D} \nabla^{\perp} \alpha). \end{cases}$$

From Lemma V.1 we know that $\nabla \vec{n}$ is weakly sequentially precompact in $L^4(D^2)$. The factor α satisfies the Liouville type equation

$$e^{-2\mu} \Delta \alpha = e^{2\alpha} K_{g_{\vec{\Phi}}} - K_{g_0}.$$

Since α is uniformly bounded in L^{∞}, as also is μ on D^2, we deduce that $\Delta \alpha$ is uniformly bounded in L^2, and hence $\nabla \alpha$ is <u>strongly</u> precompact in $L^p_{\text{loc}}(D^2)$ for any $p < +\infty$. We know that $\nabla \vec{W}$ converges to zero strongly in $L^{2,\infty}$ and that $\nabla \vec{v}$ is strongly precompact in $L^q(D^2)$ for any $q < 4$. Since also $\nabla \vec{D}$ is weakly precompact in L^4 and since we have chosen $\vec{D}$ to be of mean zero on D^2, it is precompact in any $L^p_{\text{loc}}(D^2)$ for any $p < +\infty$. We can then apply Lemma VII.2 to deduce that ∇S and $\nabla \vec{U}$ are <u>strongly precompact</u> in $L^{4/3}_{\text{loc}}(D^2)$.

Taking now the vectorial product between (V.31) and $\nabla^{\perp} \vec{\Phi}$ we obtain

$$\begin{aligned} & (\nabla \vec{V} - \nabla^{\perp} \vec{W}) \times \nabla^{\perp} \vec{\Phi} \\ & = (\vec{L} \cdot \nabla^{\perp} \vec{\Phi} - \vec{M} \cdot \nabla \vec{\Phi}) \cdot \nabla \vec{\Phi} - 2 f'(H) e^{2\lambda} \vec{n} - 2(\alpha - \overline{\alpha}) \nabla \vec{D} \times \nabla^{\perp} \vec{\Phi} + 2 \vec{M} e^{2\lambda} \\ & = \nabla^{\perp} S \cdot \nabla \vec{\Phi} + \nabla Y \cdot \nabla \vec{\Phi} - 2(\vec{M} \cdot \nabla \vec{\Phi}) \cdot \nabla \vec{\Phi} - 2 f'(H) e^{2\lambda} \vec{n} \\ & \quad - 2(\alpha - \overline{\alpha}) \nabla \vec{D} \times \nabla^{\perp} \vec{\Phi} + 2 \vec{M} e^{2\lambda}. \end{aligned} \tag{V.40}$$

We also have

$$\begin{aligned}\nabla\vec{V}\times\nabla^\perp\vec{\Phi}&=\nabla\vec{R}\times\nabla^\perp\vec{\Phi}+\nabla\vec{u}\times\nabla^\perp\vec{\Phi}\\&=\nabla\vec{R}\times\nabla^\perp\vec{\Phi}+\nabla^\perp\vec{v}\times\nabla^\perp\vec{\Phi}+\nabla Y\cdot(\vec{n}\times\nabla^\perp\vec{\Phi})\\&=\nabla\vec{R}\times\nabla^\perp\vec{\Phi}+\nabla\vec{v}\times\nabla\vec{\Phi}+\nabla Y\cdot\nabla\vec{\Phi}.\end{aligned}\tag{V.41}$$

Combining (V.40) and (V.41) gives

$$\begin{aligned}2f'(H)e^{2\lambda}\vec{n}=\nabla^\perp S\cdot\nabla\vec{\Phi}-2(\alpha-\overline{\alpha})\nabla\vec{D}\times\nabla^\perp\vec{\Phi}-\nabla\vec{R}\times\nabla^\perp\vec{\Phi}-\nabla\vec{v}\times\nabla\vec{\Phi}\\+2\vec{M}e^{2\lambda}-2(\vec{M}\cdot\nabla\vec{\Phi})\cdot\nabla\vec{\Phi}+\nabla\vec{W}\times\nabla\vec{\Phi}.\end{aligned}\tag{V.42}$$

This implies

$$\begin{aligned}2f'(H)e^{2\lambda}\vec{n}=\nabla^\perp S\cdot\nabla\vec{\Phi}-\nabla\vec{U}\times\nabla^\perp\vec{\Phi}-\nabla\vec{v}\times\nabla\vec{\Phi}+2\vec{D}\times\nabla^\perp\vec{\Phi}\cdot\nabla\alpha\\+2\vec{M}e^{2\lambda}-2(\vec{M}\cdot\nabla\vec{\Phi})\cdot\nabla\vec{\Phi}+\nabla\vec{W}\times\nabla\vec{\Phi}.\end{aligned}\tag{V.43}$$

From the results established above we see that $f'(H)$ is strongly precompact in $L^{4/3}_{\text{loc}}(D^2)$. Explicitly, $f'(H)=2H(3+H^2)$. Denote

$$\vec{J}^\infty:=\lim_{k\to+\infty}\vec{H}^k(3+|H^k|^2)\quad\text{strongly in }L^{4/3}_{\text{loc}}(D^2).\tag{V.44}$$

Since $\nabla\alpha^k$ is strongly precompact in any $L^p_{\text{loc}}(D^2)$ for $p<+\infty$, this is also the case for $\nabla\lambda^k$. Moreover $\Delta\vec{\Phi}^k$ is uniformly bounded in L^4, so for a subsequence we have

$$\vec{H}^k\rightharpoonup\vec{H}^\infty:=2^{-1}e^{-2\lambda^\infty}\Delta\vec{\Phi}^\infty\quad\text{weakly in }L^4(D^2_{3/4}),$$

and $\vec{\Phi}^\infty$ is a conformal immersion of the disc $D^2_{1/2}$ in $\mathcal{E}_{D^2_{1/2},2}$. Observe that

$$\vec{H}^k\cdot\vec{H}^k(3+|H^k|^2)\rightharpoonup\vec{H}^\infty\cdot\vec{J}^\infty\in L^1(D^2_{1/2})\quad\text{weakly in }\mathcal{D}'(D^2_{1/2}).$$

Hence the sequence of non-negative measures $|H^k|^2(3+|H^k|^2)$ does not concentrate with respect to the Lebesgue measure:

$$\forall\varepsilon>0\;\exists\delta>0\;\forall E\subset D^2_{1/2}\text{ measurable,}$$
$$|E|\le\delta\implies\limsup_{k\to+\infty}\int_E|H^k|^4\,dx^2\le\varepsilon.\tag{V.45}$$

From the strong convergence (V.44) we deduce

$$\vec{H}^k\to\vec{I}^\infty\quad\text{a.e. in }D^2.$$

In view of the Egorov theorem, for any $\delta>0$ there exists $E_\delta\subset D^2$ such that $|E_\delta|<\delta$ and

$$\vec{H}^k\to\vec{I}^\infty\quad\text{uniformly in }D^2\setminus E_\delta.$$

Hence for any test function $\varphi \in C_0^\infty(D^2_{1/2})$ we have

$$\limsup_{k\rightarrow 0}\left|\int_{D^2}\varphi\vec{H}^k\,dx^2-\int_{D^2\setminus E_\delta}\varphi\vec{I}^\infty\,dx^2\right|\le\|\varphi\|_\infty\ \limsup_{k\rightarrow+\infty}\int_{E_\delta}|H^k|\,dx^2. \tag{V.46}$$

Combining (V.45) and (V.46) we deduce that

$$\vec{I}^\infty=\vec{H}^\infty, \tag{V.47}$$

and, using (V.44), that $\vec{H}^k$ converges strongly to $\vec{H}^\infty$ in L^4. Hence $\vec{\Phi}^k$ is strongly precompact in $W^{2,4}(D^2_{1/2})$. Inserting this information in (V.22) shows that the limiting immersion satisfies the Euler–Lagrange equation of F, which concludes the proof of Lemma V.2.

□

V.4. Minmax procedures for frame energies

V.4.1. The free case. Let $\mathcal{P}(\mathcal{E}_{S^2})$ the space of subsets of the space $\mathcal{E}_{S^2}$ of weak immersions.

Definition V.3. A non-empty subset $\mathcal{A}$ of $\mathcal{P}(\mathcal{E}_{\Sigma,2})$ is called *admissible* if for any homeomorphism $\varphi\in\mathrm{Hom}(\mathcal{E}_{S^2,2})$ isotopic to the identity we have

$$\forall A\in\mathcal{A}\qquad\varphi(A)\in\mathcal{A}.$$

Moreover, there exists a topological space X such that for any $A\in\mathcal{A}$ there exists $\vec{\Phi}^A$ in $C^0(X,\mathcal{E}_{\Sigma,2})$ such that

$$A=\vec{\Phi}^A(X).$$ □

Let now $\mathcal{A}$ be admissible. Because of Willmore's universal lower bounds of W on the space of closed surfaces, we obviously have

$$\beta(0):=\inf_{A\in\mathcal{A}}\max_{\vec{\Phi}\in A}W(\vec{\Phi})\ge 4\pi>0. \tag{V.48}$$

Since for any fixed $\vec{\Phi}$ the map $\sigma\mapsto F^\sigma(\vec{\Phi})$ is increasing, we can use a beautiful argument initially introduced by Michael Struwe [48] and follow word for word [32, proof of Theorem 6.4] using the Palais–Smale property of F^σ established in Lemma V.2 to deduce the following lemma which was the main goal of the present subsection.

Lemma V.3. *Let $\mathcal{A}$ be an admissible family. There exists a sequence $\sigma^k\rightarrow 0$ and a sequence of critical points $\vec{\Phi}^{\sigma^k}$ of F^{σ^k} such that*

$$\beta(\sigma^k)=F^{\sigma^k}(\vec{\Phi}^{\sigma^k})\quad\textit{and}\quad\partial_\sigma F^{\sigma^k}(\vec{\Phi}^{\sigma^k})=o\left(\frac{1}{\sigma^k\log(\sigma^k)^{-1}}\right), \tag{V.49}$$

so in particular

$$\lim_{k\rightarrow+\infty}W(\vec{\Phi}^{\sigma^k})=\beta(0).$$ □

V.4.2. The area constrained case. We are now going to adapt the previous case to the situation when we fix the area to be 1. More precisely, we define

$$\mathcal{E}^1_{\Sigma,2} := \left\{ \vec{\Phi} \in \mathcal{E}_{\Sigma,2}\, ;\ \int_\Sigma d\mathrm{vol}_{g_{\vec{\Phi}}} = 1 \right\}.$$

It is not difficult to check that this defines a Finsler manifold structure based on the $W^{2,4}$ topology. The notion of admissible set is defined as above but with general homeomorphisms of $\mathcal{E}_{\Sigma,2}$ replaced by homeomorphisms of $\mathcal{E}^1_{\Sigma,2}$. The construction of the pseudo-gradient restricted to $\mathcal{E}^1_{\Sigma,2}$ applies and we can again follow [32, proof of Theorem 6.4] word for word and use the Palais–Smale property of F^σ established in Lemma V.2 in order to deduce the statement of Lemma V.3 under the area constraint $\mathrm{Area}(\vec{\Phi}^{\sigma^k}) = 1$. We shall now establish the following lemma which is a consequence of the scaling invariance of the Willmore energy in $\mathbb{R}^m$.

Lemma V.4. *Let $\vec{\Phi}^{\sigma^k}$ be a critical point of F^{σ^k} under the constraint $\mathrm{Area}(\vec{\Phi}^{\sigma^k}) = 1$ and satisfying*

$$\partial_\sigma F^{\sigma^k}(\vec{\Phi}^{\sigma^k}) = o\left(\frac{1}{\sigma^k \log(\sigma^k)^{-1}} \right). \tag{V.50}$$

Then it satisfies the equation (for any choice of gauge α^k in the case $\Sigma = S^2$)

$$\begin{aligned} d\Big[&*_{g_{\vec{\Phi}^k}} d[l_{\sigma^k} f'_{\sigma^k}(H^k)\vec{n}_{\vec{\Phi}^k}] - 2l_{\sigma^k} f'_{\sigma^k}(H^k) *_{g_{\vec{\Phi}^k}} d\vec{n}_{\vec{\Phi}^k} \\ &+ l_{\sigma^k}[-2f_{\sigma^k}(H^k) + |d\alpha^k|^2_{g_{\vec{\Phi}^k}} - K_{g_0}\alpha^k e^{-2\alpha^k} + K_{g_0}[A_{\vec{\Phi}^k}(\Sigma)]^{-1}] *_{g_{\vec{\Phi}^k}} d\vec{\Phi}^k \\ &- 2l_{\sigma^k} \langle d\vec{\Phi}^k, d\alpha^k\rangle_{g_{\vec{\Phi}^k}} *_{g_{\vec{\Phi}^k}} d\alpha^k + 2l_{\sigma^k} \vec{\mathbb{I}}^k \llcorner_{g_{\vec{\Phi}^k}} (*_{g_{\vec{\Phi}^k}} d\alpha^k) \Big] = C^k d[*_{g_{\vec{\Phi}^k}} d\vec{\Phi}^k] \end{aligned} \tag{V.51}$$

where

$$C^k = 2(\sigma^k)^2 \int_\Sigma (1 - |H^k|^4)\, d\mathrm{vol}_{g_{\vec{\Phi}}} + l_{\sigma^k} \int_\Sigma K_{g_0}\alpha\, d\mathrm{vol}_{g_0} - l_{\sigma^k} K_{g_0}. \tag{V.52}$$

Hence (for the choice of an Aubin gauge in the case $\Sigma = S^2$) we have

$$\lim_{k\to+\infty} |C^k| = 0. \tag{V.53}$$

□

Remark V.1. Observe that a priori C^k depends on the choice of the gauge α^k.

Proof of Lemma V.4. We omit the index k. The fact that equation (V.51) is satisfied comes from (III.46) and classical Lagrange multiplier theory, bearing in mind that the first derivative of the fixed area constraint is proportional to $d[*_{g_{\vec{\Phi}^k}} d\vec{\Phi}^k]$ which cannot be zero since obviously there is no compact minimal immersion in $\mathbb{R}^m$ (hence the constraint is non-degenerate). We take the scalar product between (V.51) and $\vec{\Phi}$ and we integrate the resulting 2-form over the closed surface Σ. This gives

$$C = l_\sigma \int_\Sigma [2f_\sigma(H) - Hf'_\sigma(H)]\, d\mathrm{vol}_{g_{\vec{\Phi}}} + K_{g_0} \int_\Sigma \alpha e^{-2\alpha}\, d\mathrm{vol}_{g_{\vec{\Phi}}} + 4\pi l_\sigma(\mathrm{genus}(\Sigma) - 1). \tag{V.54}$$

Since $2f_\sigma(H) - Hf'_\sigma(H) = 2\sigma^2 l_\sigma^{-1}(1 - H^4)$, we obtain (V.52). For the choice of an Aubin gauge we have due to (V.50), combined with Theorem III.10 and (III.69),

$$l_\sigma \|\alpha\|_{L^\infty(S^2)} = o(1). \tag{V.55}$$

This implies (V.53) for $\Sigma = S^2$. For $\Sigma \neq S^2$, Lemma III.12 combined with the assumption (V.50) also implies

$$l_\sigma \|\alpha\|_{L^\infty(\Sigma)} = o(1), \tag{V.56}$$

and Lemma V.4 is proved in any case. □

For the area constrained critical point we then have the following ε-regularity lemma whose proof follows step by step the proof of Lemma IV.1 since (V.53) holds.

Lemma V.5 (Uniform ε-regularity under area constraint). *For any $C_0 > 0$, there exist $\varepsilon, \sigma_0 > 0$ such that for any $\sigma \in (0, \sigma_0)$ and any critical point $\vec{\Phi}$ of F^σ under the constraint* $\mathrm{Area}(\vec{\Phi}) = 1$ *satisfying*

$$F^\sigma(\vec{\Phi}) \leq C_0 \quad \textit{and} \quad \partial_\sigma F^\sigma(\vec{\Phi}) \leq \frac{\varepsilon}{\sigma \log(\sigma)^{-1}}, \tag{V.57}$$

if we assume moreover

$$\int_{D^2} |\nabla \vec{n}|^2\, dx^2 < \varepsilon, \tag{V.58}$$

then for any $j \in \mathbb{N}$ the estimates (IV.5) and (IV.6) hold for $\Sigma \neq S^2$, and for any Coulomb gauge in case $\Sigma = S^2$. □

VI. The passage to the limit as $\sigma \to 0$

We shall give two results regarding the passage to the limit in the equation. A subsection will be devoted to each of the two results.

VI.1. The limiting immersions

We denote by $\mathcal{M}^+(S^2)$ the non-compact Möbius group of positive conformal diffeomorphisms of the 2-sphere S^2.

Lemma VI.1. *Let $\sigma^k \to 0$ and let $\vec{\Phi}^k \in \mathcal{E}_{S^2}$ be a sequence of weak immersions which are critical points of F^{σ_k} under area constraint equal to 1 and such that*

$$\limsup_{k \to +\infty} F^{\sigma^k}(\vec{\Phi}^k) < +\infty \quad \textit{and} \quad \partial_\sigma F^{\sigma^k}(\vec{\Phi}^k) = o\left(\frac{1}{\sigma^k \log(\sigma^k)^{-1}}\right) \tag{VI.1}$$

Then modulo translation there exists a subsequence, still denoted $\vec{\Phi}^k$, there exists a family of bilipschitz homeomorphism Ψ^k, there exists a finite family of sequences $(f_i^k)_{i=1}^N$ of elements in $\mathcal{M}^+(S^2)$, there exists a finite family of integers $(N_i)_{i=1}^N$ and for each $i \in \{1, \dots, N\}$ there exist finitely many points of S^2, $b_{i,1}, \dots, b_{i,N_i}$, such that

$$\vec{\Phi}^k \circ \Psi^k \to \vec{f}^\infty \quad \textit{strongly in } C^0(S^2, \mathbb{R}^m) \tag{VI.2}$$

where $\vec{f}^\infty \in W^{1,\infty}(S^2, \mathbb{R}^m)$, and moreover

$$\vec{\Phi}^k \circ f_i^k \rightharpoonup \vec{\xi}_i^\infty \quad \textit{strongly in } C^l_{loc}(S^2 \setminus \{b_{i,1}, \dots, b_{i,N_i}\}) \tag{VI.3}$$

for any $l \in \mathbb{N}$ where $\vec{\xi}_i^\infty$ is a Willmore conformal possibly branched immersion of S^2. In addition

$$\vec{f}^\infty(S^2) = \bigcup_{i=1}^N \vec{\xi}_i^\infty(S^2), \tag{VI.4}$$

moreover

$$A(\vec{\Phi}_k) = \int_{S^2} 1\, d\mathrm{vol}_{g_{\vec{\Phi}_k}} \to A(\vec{f}^\infty) = \sum_{i=1}^N A(\vec{\xi}_i^\infty), \tag{VI.5}$$

and finally

$$(\vec{f}^\infty)_*[S^2] = \sum_{i=1}^N (\vec{\xi}_i^\infty)_*[S^2] \tag{VI.6}$$

where for any Lipschitz mapping $\vec{a}$ from S^2 into $\mathbb{R}^m$, $(\vec{a})_*[S^2]$ denotes the current given by the push-forward by $\vec{a}$ of the current of integration over S^2: for any smooth 2-form ω on $\mathbb{R}^m$,

$$\langle (\vec{a})_*[S^2], \omega \rangle := \int_{S^2} (\vec{a})^* \omega. \qquad \Box$$

Remark VI.1. Lemma VI.1 "detects" the Willmore spheres "visible" at the limit but ignores the possible "asymptotic Willmore spheres" which shrink and disappear in the limit. The detection of these asymptotic Willmore spheres is the purpose of the next subsection. Finally, the detection of the possible loss of energy in the so called "neck regions" and the energy quantization effect is going to be investigated in Section VII.

Proof of Lemma VI.1. We work modulo extraction of subsequences. Consider the various diffeomorphisms f_i^k of S^2 given by [34, Theorem I.2]. We choose the gauges given by f_i^k, that is, the pairs (f_i^k, α_i^k) satisfying

$$g_{\vec{\Phi}^k \circ f_i^k} = e^{2\alpha_i^k} g_{S^2}.$$

From the analysis in [34] we have the existence of N_i points of S^2, $b_{i,1}, \dots, b_{i,N_i}$, such that

$$\limsup_{k \to +\infty} \left(\|\alpha_i^k\|_{L^\infty_{loc}(S^2 \setminus \{b_{i,1}, \dots, b_{i,N_i}\})} + \|\nabla \alpha_i^k\|_{L^2_{loc}(S^2 \setminus \{b_{i,1}, \dots, b_{i,N_i}\})} \right) < +\infty. \tag{VI.7}$$

Hence in particular

$$\lim_{k \to +\infty} \left(l_{\sigma^k} \|\alpha_i^k\|_{L^\infty_{loc}(S^2 \setminus \{b_{i,1}, \dots, b_{i,N_i}\})} + \sqrt{l_{\sigma^k}}\, \|\nabla \alpha_i^k\|_{L^2_{loc}(S^2 \setminus \{b_{i,1}, \dots, b_{i,N_i}\})} \right) = 0. \tag{VI.8}$$

The assumptions (VI.1) imply moreover

$$(\sigma^k)^2 \int_{S^2} (1 + H_{\vec{\Phi}^k}^2)^2 \, d\mathrm{vol}_{\vec{\Phi}^k} = o(l_{\sigma^k}). \tag{VI.9}$$

Again from the analysis in [34] we find that the density of energy

$$|d\vec{n}_{\vec{\Phi}^k \circ f_i^k}|^2_{g_{\vec{\Phi}^k \circ f_i^k}} \, d\mathrm{vol}_{g_{\vec{\Phi}^k \circ f_i^k}}$$

remains uniformly absolutely continuous with respect to Lebesgue measure in $S^2 \setminus \bigcup_{j=1}^{N_i} B_\delta(b_{i,j})$. Hence all the assumptions which permit applying the uniform ε-regularity Lemma IV.1 are fulfilled and we deduce the strong convergence (VI.3) towards a Willmore sphere that can be possibly branched at the $b_{i,j}$.

We now claim that these Willmore spheres are true Willmore spheres in the sense that the Willmore residues are zero:

$$\int_{\partial B_\delta(b_{i,j})} (\partial_\nu \vec{H}_{\vec{\xi}_i^\infty} - 2 H_{\vec{\xi}_i^\infty} \, \partial_\nu \vec{n}_{\vec{\xi}_i^\infty} - 2 H_{\vec{\xi}_i^\infty}^2 \partial_\nu \vec{\xi}_i^\infty) \, dl = 0.$$

Indeed, because of the strong convergence (in any C^l norm) away from the $b_{i,j}$ we have, for any $\delta > 0$,

$$\int_{\partial B_\delta(b_{i,j})} (\partial_\nu \vec{H}_{\vec{\xi}_i^k} - 2 H_{\vec{\xi}_i^k} \, \partial_\nu \vec{n}_{\vec{\xi}_i^k} - 2 H_{\vec{\xi}_i^k}^2 \partial_\nu \vec{\xi}_i^k) \, dl$$
$$\to \int_{\partial B_\delta(b_{i,j})} (\partial_\nu \vec{H}_{\vec{\xi}_i^\infty} - 2 H_{\vec{\xi}_i^\infty} \, \partial_\nu \vec{n}_{\vec{\xi}_i^\infty} - 2 H_{\vec{\xi}_i^\infty}^2 \partial_\nu \vec{\xi}_i^\infty) \, dl = R$$

where $\vec{\xi}_i^k = \vec{\Phi}^k \circ f_i^k$. Since $\vec{\xi}_i^\infty$ is obviously Willmore, we see that R is independent of $\delta < \inf_{j \neq l} |b_{i,j} - b_{i,l}|$. For any (i,j) we choose $\chi_{i,j} \in C_0^\infty(0, \delta_i)$ such that $0 < \delta_i < \inf_{j \neq l} |b_{i,j} - b_{i,l}|$ and $\int_{\mathbb{R}_+} \chi_{i,j}(s) \, ds = 1$. Hence

$$R = \int_{S^2} \chi_{i,j}(|x - b_{i,j}|) \, d|x - b_{i,j}| \wedge [* d\vec{H}_{\vec{\xi}_i^\infty} - 2 H_{\vec{\xi}_i^\infty} * d\vec{n}_{\vec{\xi}_i^\infty} - 2 H_{\vec{\xi}_i^\infty}^2 * d\vec{\xi}_i^\infty].$$

Because of the strong C^l convergence we have

$$R = \lim_{k \to +\infty} \int_{S^2} \chi_{i,j}(|x - b_{i,j}|) \, d|x - b_{i,j}| \wedge [* d\vec{H}_{\vec{\xi}_i^k} - 2 H_{\vec{\xi}_i^k} * d\vec{n}_{\vec{\xi}_i^k} - 2 H_{\vec{\xi}_i^k}^2 * d\vec{\xi}_i^k].$$

Since we are on a sphere, and because of the Euler–Lagrange equation (IV.7), we have the existence of $\vec{L}_k$ on $B_{\delta_i}(b_{i,j})$ such that

$$\begin{aligned} d\vec{L}^k = {} & * d\vec{H}_{\vec{\xi}_i^k} - 2 H_{\vec{\xi}_i^k} * d\vec{n}_{\vec{\xi}_i^k} - 2 H_{\vec{\xi}_i^k}^2 * d\vec{\xi}_i^k + 4\sigma^2 * d(\vec{H}_{\vec{\xi}_i^k}(1 + H_{\vec{\xi}_i^k}^2)) \\ & - 8\sigma^2 \vec{H}_{\vec{\xi}_i^k}(1 + H_{\vec{\xi}_i^k}^2) * d\vec{n}_{\vec{\xi}_i^k} - 2\sigma^2 (1 + H_{\vec{\xi}_i^k}^2)^2 * d\vec{\xi}_i^k - 2 l_\sigma \langle d\vec{\Phi} \cdot d\alpha^k \rangle_{g_{\vec{\xi}_i^k}} * d\alpha^k \\ & + l_\sigma [|d\alpha^k|^2_{g_{\vec{\xi}_i^k}} - K_{g_0} \alpha^k e^{-2\alpha^k} + K_{g_0} A_{\vec{\xi}_i^k}(\Sigma)^{-1}] * d\vec{\xi}_i^k + 2 l_\sigma \vec{\mathbb{I}}_{\vec{\xi}_i^k} \llcorner * d\alpha^k. \end{aligned}$$

Since clearly

$$\int_{S^2} \chi_{i,j}(|x-b_{i,j}|)\, d|x-b_{i,j}| \wedge d\vec{L}^k = 0,$$

we have

$$\begin{aligned}
R = &\\
&-4\sigma^2 \lim_{k\to+\infty} \int_{S^2} \left[\chi_{i,j}(|x-b_{i,j}|)\, d|x-b_{i,j}| \wedge * d(\vec{H}_{\vec{\xi}_i^k}(1+H^2_{\vec{\xi}_i^k})) - 2\vec{H}_{\vec{\xi}_i^k}(1+H^2_{\vec{\xi}_i^k}) * d\vec{n}_{\vec{\xi}_i^k}\right]\\
&+2\sigma^2 \lim_{k\to+\infty} \int_{S^2} \chi_{i,j}(|x-b_{i,j}|)\, d|x-b_{i,j}| \wedge (1+H^2_{\vec{\xi}_i^k})^2 * d\vec{\xi}_i^k\\
&-2l_\sigma \lim_{k\to+\infty} \int_{S^2} \chi_{i,j}(|x-b_{i,j}|)\, d|x-b_{i,j}| \wedge [\langle d\vec{\Phi}\cdot d\alpha^k\rangle_{g_{\vec{\xi}_i^k}} * d\alpha^k - \vec{\mathbb{I}}_{\vec{\xi}_i^k} \llcorner * d\alpha^k]\\
&+l_\sigma \lim_{k\to+\infty} \int_{S^2} \chi_{i,j}(|x-b_{i,j}|)[|d\alpha^k|^2_{g_{\vec{\xi}_i^k}} - K_{g_0}\alpha^k e^{-2\alpha^k} + K_{g_0} A_{\vec{\xi}_i^k}(\Sigma)^{-1}]\, d|x-b_{i,j}| \wedge * d\vec{\xi}_i^k.
\end{aligned}$$

Using the strong precompactness of $\vec{\xi}_i^k$ in any C^l topology in the domain where $\chi_{i,j}(|x-b_{i,j}|) \neq 0$, we conclude that $R = 0$, and this finishes the proof of Lemma VI.1. □

VI.2. Bubble detection

Lemma VI.2 (Bubble detection lemma). *Let Σ be a closed surface, let $C_0 > 0$ and let $\varepsilon(C_0) > 0$ be given by Lemma V.5 . Let $\sigma^k \to 0$ and $\vec{\Phi}^k$ be a sequence of critical points of F^{σ^k} under the constraint* $\text{Area}(\vec{\Phi}^k) = 1$ *and satisfying*

$$F^{\sigma^k}(\vec{\Phi}^k) \leq C_0 \quad \textit{and} \quad \partial_\sigma F^{\sigma^k}(\vec{\Phi}^k) = o\left(\frac{1}{\sigma^k \log(\sigma^k)^{-1}}\right). \tag{VI.10}$$

When $\Sigma \neq S^2$ assume that the sequence of constant Gauss curvature metrics g_0^k such that $g_{\vec{\Phi}^k} = e^{2\alpha^k} g_0^k$ is precompact in any $C^l(\Sigma)$ topology. Let $B_{\rho^k}(p^k)$ be a geodesic ball for the metric g_0^k such that

$$\int_{B_{\rho^k}(p^k)} |d\vec{n}|^2_{g_{\vec{\Phi}^k}}\, d\text{vol}_{g_{\vec{\Phi}^k}} < \varepsilon. \tag{VI.11}$$

Compose $\vec{\Phi}^k$ with a sequence of conformal charts (x_1^k, x_2^k) from D^2 into $(B_{\rho^k}(x^k), g_0^k)$ such that there exists $\overline{\mu}_k \in \mathbb{R}$ satisfying $g_0^k = e^{2\mu^k}[dx_1^2 + dx_2^2]$ and $\mu^k - \overline{\mu}^k$ is weakly precompact for the $(L^\infty)^*$ topology. Let $\overline{\alpha}^k$ be the average of α^k on $D^2_{1/2}$. Then

$$e^{-\overline{\mu}^k - \overline{\alpha}^k}[\vec{\Phi}^k(x) - \vec{\Phi}^k(0)]$$

strongly converges in $D^2_{1/2}$ in any C^l norm towards a Willmore disc. □

Remark VI.2. We are mostly interested in the balls $B_{\rho^k}(p^k)$ for which

$$\limsup_{k\to 0}\int_{B_{\rho^k}(p^k)}|d\vec{n}|^2_{g_{\vec{\Phi}^k}}\,d\mathrm{vol}_{g_{\vec{\Phi}^k}}>0 \tag{VI.12}$$

where there is indeed a non-flat Willmore bubble forming which swallows part of the energy. The lemma however also applies when (VI.12) does not hold. □

Proof of Lemma VI.2. We keep denoting by $\vec{\Phi}^k$ the composition of $\vec{\Phi}^k$ with the given chart and we work on D^2. Because of Lemma V.4, $\vec{\Phi}$ satisfies the equation (where we omit the subscript k)

$$\begin{aligned}
&e^{\overline{\lambda}}\,\mathrm{div}[\nabla\vec{H}_{\vec{\Phi}}-2H_{\vec{\Phi}}\nabla\vec{n}_{\vec{\Phi}}-2H^2_{\vec{\Phi}}\nabla\vec{\Phi}]\\
&\quad=\mathrm{div}\big[-2e^{\overline{\lambda}}\sigma^2\nabla[\vec{H}_{\vec{\Phi}}(1+H^2_{\vec{\Phi}})]+4\sigma^2e^{\overline{\lambda}}H_{\vec{\Phi}}(1+H^2_{\vec{\Phi}})\nabla\vec{n}_{\vec{\Phi}}\big]\\
&\qquad+\mathrm{div}\big[\sigma^2(1+H^2_{\vec{\Phi}})^2e^{\overline{\lambda}}\nabla\vec{\Phi}+l_\sigma e^{\overline{\lambda}-2\lambda}[(\mathbb{I}_{\vec{\Phi}}\llcorner\nabla^\perp\alpha)^\perp+\nabla\vec{\Phi}\cdot\nabla\alpha\nabla\alpha]\big]\\
&\qquad-\mathrm{div}\big[2^{-1}l_\sigma[e^{-2\lambda}|\nabla\alpha|^2-K_{g_0}\alpha e^{-2\alpha}+K_{g_0}]e^{\overline{\lambda}}\big]+2^{-1}Ce^{\overline{\lambda}}\vec{H}_{\vec{\Phi}}
\end{aligned} \tag{VI.13}$$

where $\overline{\lambda}$ is the average of λ on $D^2_{1/2}$, equal to $\overline{\mu}+\overline{\alpha}$. The uniform ε-regularity under the area constraint (Lemma V.5), combined with our assumption (VI.10) and Lemma V.4, implies that the right hand side of (VI.13) converges to 0 in any C^l norm. Denoting $\vec{\xi}:=e^{-\overline{\lambda}}(\vec{\Phi}(x)-\vec{\Phi}(0))$ we have

$$\vec{n}_{\vec{\xi}}=\vec{n}_{\vec{\Phi}},\quad \vec{H}_{\vec{\xi}}=e^{\overline{\lambda}}\vec{H}_{\vec{\Phi}}\quad\text{and}\quad\nabla\vec{\xi}=e^{-\overline{\lambda}}\nabla\vec{\Phi}.$$

Hence

$$\mathrm{div}[\nabla\vec{H}_{\vec{\xi}^k}-2H_{\vec{\xi}^k}\nabla\vec{n}_{\vec{\xi}^k}-2H^2_{\vec{\xi}^k}\nabla\vec{\xi}^k]\to 0\quad\text{in } C^l(D^2_{1/2})\ \forall l\in\mathbb{N},$$

and

$$\limsup_{k\to+\infty}\int_{D^2_{1/2}}|\nabla\vec{n}_{\vec{\xi}^k}|^2\,dx^2<+\infty\quad\text{and}\quad\limsup_{k\to+\infty}\log|\nabla\vec{\xi}^k|^2<+\infty.$$

Adapting the arguments in [41] (see also [43, Theorem 7.11]) to this perturbed case gives the strong convergence of $\vec{\xi}^k$ to a limiting Willmore immersion in $C^l(D^2_{1/2})$ for all $l\in\mathbb{N}$, and Lemma VI.2 is proved. □

VI.3. Energy quantization

With Lemma VI.2 at hand, in order to prove our main result of Theorem I.1, following the scheme of [8] it remains to establish the vanishing of the energy in the so called *neck region*. We now restrict to the sphere case exclusively. More precisely, we are going to prove the following lemma.

Lemma VI.1. *Let $\sigma^k \to 0$ and let $\vec{\Phi}^k \in \mathcal{E}_{S^2}$ be a sequence of weak immersions which are critical points of F^{σ_k} under the area constraint and such that*

$$\limsup_{k\to+\infty} F^{\sigma^k}(\vec{\Phi}^k) < +\infty \quad and \quad \partial_\sigma F^{\sigma^k}(\vec{\Phi}^k) = o\left(\frac{1}{\sigma^k \log(\sigma^k)^{-1}}\right). \tag{VI.1}$$

Then there exist finitely Willmore immersions $\vec{\xi}_1, \ldots, \vec{\xi}_n$ of S^2 minus finitely many points such that

$$\lim_{k\to+\infty} W(\vec{\Phi}^k) = \sum_{i=1}^n W(\vec{\xi}_i) - 4\pi N \tag{VI.2}$$

for some $N \in \mathbb{N}$. □

Proof. We shall work with an Aubin gauge (Ψ^k, α^k) satisfying

$$g_{\vec{\Phi}^k \circ \Psi^k} = e^{2\alpha^k} \frac{g_{S^2}}{4\pi} \quad \text{and} \quad \forall i = 1, 2, 3 \quad \int_{S^2} x_i e^{2\alpha^k} \, d\mathrm{vol}_{S^2} = 0$$

where g_{S^2} and $d\mathrm{vol}_{S^2}$ are respectively the standard metric and the standard associated volume form on S^2. In order to simplify the notation we omit Ψ_k which we assume to be the identity. The assumption (VI.1) reads

$$o\left(\frac{1}{\sigma^k \log(\sigma^k)^{-1}}\right) = 2\sigma^k \int_{S^2} (1 + H^2_{\vec{\Phi}^k})^2 e^{2\alpha_k} \, d\mathrm{vol}_{g_{S^2}}$$
$$+ \frac{1}{\sigma^k (\log \sigma^k)^2} \left[\int_{S^2} 2^{-1} |d\alpha^k|_{g_{S^2}} \, d\mathrm{vol}_{g_{S^2}} + \int_{S^2} \alpha^k \, d\mathrm{vol}_{g_{S^2}} - 2\pi \log \mathrm{Area}(\vec{\Phi}_k)\right].$$

The non-negativity of both terms on the r.h.s. gives respectively

$$(\sigma^k)^2 \int_{S^2} (1 + H^2_{\vec{\Phi}^k})^2 e^{2\alpha_k} \, d\mathrm{vol}_{g_{S^2}} = o\left(\frac{1}{\log(\sigma^k)^{-1}}\right)$$

and

$$\frac{1}{\log(\sigma^k)^{-1}} \left[\int_{S^2} 2^{-1} |d\alpha^k|_{g_{S^2}} \, d\mathrm{vol}_{g_{S^2}} + \int_{S^2} \alpha^k \, d\mathrm{vol}_{g_{S^2}} - 2\pi \log \mathrm{Area}(\vec{\Phi}_k)\right] = o(1).$$

We keep the notation $l_{\sigma^k} := 1/\log((\sigma^k)^{-1})$ from the previous sections. Using Lemmas III.10 and III.11, we get for this Aubin gauge

$$l_{\sigma^k} \int_{S^2} |d\alpha^k|^2_{g_{S^2}} \, d\mathrm{vol}_{g_{S^2}} = o(1) \quad \text{and} \quad l_{\sigma^k} \|\alpha^k\|_{L^\infty(S^2)} = o(1).$$

Hence in order to apply the uniform ε-regularity Lemma V.5 on any geodesic ball $B_r(x_0)$ for the S^2 metric, it suffices to assume that σ is small enough and

$$\int_{B_r(x_0)} |d\vec{n}|^2_{g_{S^2}} \, d\mathrm{vol}_{g_{S^2}} < \varepsilon.$$

Then in particular

$$
\begin{aligned}
& r^4\|e^{\lambda}\nabla(\vec{H}_{\vec{\Phi}^k}(1+2\sigma^2(1+H^2_{\vec{\Phi}^k})))\|^2_{L^\infty(B_{r/2}(x_0))}+r^2\sigma^4\|e^{\lambda}\, H_{\vec{\Phi}^k}(1+H^2_{\vec{\Phi}^k})\|^2_{L^\infty(B_{r/2}(x_0))}\\
&\quad + r^2\|\nabla\vec{n}_{\vec{\Phi}^k}\|^2_{L^\infty(B_{r/2}(x_0))}\le C\int_{B_r^2(x_0)}|\nabla\vec{n}|^2\,dx^2\\
&\quad + C\left[\int_{B_r(x_0)}[l_{\sigma^k}e^{2\alpha^k}+l_{\sigma^k}\|\alpha^k\|_{L^\infty(S^2)}]\,d\mathrm{vol}_{g_{S^2}}\right]^2+C\left[\int_{B_r^2(x_0)}(\sigma^k)^2H^4e^{2\lambda}\,dx^2\right]^2\\
&\quad + C\left[\int_{B_r^2(x_0)}l_{\sigma^k}\,|\nabla\alpha^k|^2\,dx^2\right]^2+Cl_{\sigma^k}\|e^{4\mu_k}\|_{L^\infty(B_r(x_0))},
\end{aligned}
\tag{VI.3}
$$

and

$$
\begin{aligned}
r^2l_{\sigma^k}\|\nabla\alpha^k\|^2_{L^\infty(B_{r/2}(x_0))} &\le Cl_{\sigma^k}\int_{B_r^2(x_0)}|\nabla\alpha^k|^2\,dx^2+C\left[\int_{B_r^2(x_0)}|\nabla\vec{n}_{\vec{\Phi}^k}|^2\,dx^2\right]^2\\
&\quad + C\left[\int_{B_r^2(x_0)}(\sigma^k)^2H^4_{\vec{\Phi}^k}e^{2\lambda}\,dx^2\right]^4+C\left[\int_{B_r(x_0)}(\sqrt{l_{\sigma^k}}+l_{\sigma^k}e^{2\alpha^k})\,d\mathrm{vol}_{g_{S^2}}\right]^2\\
&\quad + Cl_{\sigma^k}\|e^{4\mu_k}\|_{L^\infty(B_r(x_0))}.
\end{aligned}
\tag{VI.4}
$$

Recall that $\vec{\Phi}^k$ being a critical point of F^{σ^k} is equivalent to the existence of $\vec{L}_{\vec{\Phi}^k}$ such that

$$
\begin{aligned}
&\nabla\vec{L}_{\vec{\Phi}^k}:=2\nabla^\perp\big(\vec{H}_{\vec{\Phi}^k}(1+2\sigma^2(1+H^2_{\vec{\Phi}^k}))\big)-4H_{\vec{\Phi}^k}(1+2\sigma^2(1+H^2_{\vec{\Phi}^k}))\nabla^\perp\vec{n}_{\vec{\Phi}^k}\\
&\quad -2e^{-2\lambda^k}\,l_{\sigma^k}\nabla\vec{\Phi}^k\cdot\nabla\alpha^k\nabla^\perp\alpha^k+2l_\sigma e^{-2\lambda^k}\mathbb{I}_{\vec{\Phi}^k}\llcorner\nabla^\perp\alpha^k-2(H^2_{\vec{\Phi}^k}+\sigma^2(1+H^2_{\vec{\Phi}^k})^2)\nabla^\perp\vec{\Phi}^k\\
&\quad +l_{\sigma^k}[e^{-2\lambda^k}\,|\nabla\alpha^k|^2-K_{g_0}\alpha e^{-2\alpha^k}+4\pi A_{\vec{\Phi}^k}(S^2)^{-1}]\nabla^\perp\vec{\Phi}^k.
\end{aligned}
$$

Since $2(\sigma^k)^2(1+H^2_{\vec{\Phi}^k})|H_{\vec{\Phi}^k}|\,|\nabla\vec{n}_{\vec{\Phi}^k}|e^{\lambda}\le|\nabla\vec{n}_{\vec{\Phi}^k}|^2+(\sigma^k)^4H^2_{\vec{\Phi}^k}(1+H^2_{\vec{\Phi}^k})^2e^{2\lambda^k}$, the previous estimates imply

$$
\begin{aligned}
r^2\|e^{\lambda^k}\nabla\vec{L}_{\vec{\Phi}^k}\|_{L^\infty(B_{r/2}(x_0))} &\le C\left[\int_{B_r^2(x_0)}|\nabla\vec{n}_{\vec{\Phi}^k}|^2\,dx^2\right]^{1/2}+Cl_{\sigma^k}\int_{B_r^2(x_0)}|\nabla\alpha^k|^2\,dx^2\\
+\int_{B_r^2(x_0)}(\sigma^k)^2H^4_{\vec{\Phi}^k}e^{2\lambda^k}\,dx^2 &+l_{\sigma^k}\int_{B_r(x_0)}d\mathrm{vol}_{g_{\vec{\Phi}^k}}+l_{\sigma^k}\|\alpha^k\|_\infty\int_{B_r(x_0)}d\mathrm{vol}_{g_{S^2}}.
\end{aligned}
\tag{VI.5}
$$

We now follow step by step the arguments from [8, Section VI] and check how each estimate is slightly modified by the viscous terms. We consider a neck region that is an annulus (for the S^2 metric) of the form $B^2_{R^k}(0)\setminus B^2_{r^k}(0)$ where

$$
\lim_{k\to+\infty}R^k=0\quad\text{and}\quad\lim_{k\to+\infty}\frac{r^k}{R^k}=0,
$$

and such that

$$
\lim_{k\to+\infty}\sup_{r^k<s<R^k/4}\int_{B^2_{4s}(0)\setminus B^2_s(0)}|\nabla\vec{n}_{\vec{\Phi}^k}|^2\,dx^2=0.
\tag{VI.6}
$$

For $s < R^k/4$ denote

$$
\begin{aligned}
s\delta^k(s) := &\left[\int_{B^2_{2s}(0)\setminus B^2_{s/2}(0)} |\nabla\vec{n}|^2\, dx^2\right]^{1/2} + C\left[l_{\sigma^k}\int_{B^2_{2s}(0)\setminus B^2_{s/2}(0)} |\nabla\alpha^k|^2\, dx^2\right]^{1/2}\\
&+\left[\int_{B^2_{2s}(0)\setminus B^2_{s/2}(0)} (\sigma^k)^2(1+H^2)^2 e^{2\lambda}\, dx^2\right]^{1/2} + \left[l_{\sigma^k}\int_{B^2_{2s}(0)\setminus B^2_{s/2}(0)} d\mathrm{vol}_{g_{\vec{\Phi}}}\right]^{1/2}\\
&+\left[l_{\sigma^k}\|\alpha\|_\infty\int_{B^2_{2s}(0)\setminus B^2_{s/2}(0)} d\mathrm{vol}_{g_{S^2}}\right]^{1/2}. \qquad \text{(VI.7)}
\end{aligned}
$$

We have

$$
\lim_{k\to+\infty}\ \sup_{s\in(r^k,R^k)} s\delta^k(s) = 0.
$$

We shall omit the superscript k unless it is necessary. From (VI.5), in the neck region we have

$$
|x|^2|\nabla\vec{L}|(x) \le |x|\delta(|x|)e^{-\lambda(x)}, \qquad \text{(VI.8)}
$$

and

$$
\limsup_{k\to+\infty}\int_r^{R/4}\delta^2(s)s\, ds < +\infty. \qquad \text{(VI.9)}
$$

Following [8] we introduce

$$
\vec{L}_t := \frac{1}{|\partial B_t(0)|}\int_{\partial B_t(0)}\vec{L}\, dl_{\partial B_t(0)} \quad\text{and}\quad \lambda(t) := \frac{1}{|\partial B_t(0)|}\int_{\partial B_t(0)}\lambda\, dl_{\partial B_t(0)}.
$$

Using (VI.5) we have

$$
\begin{aligned}
\frac{d\vec{L}_t}{dt} = &\frac{1}{2\pi}\int_0^{2\pi}\frac{\partial\vec{L}}{\partial t}(t,\theta)\, d\theta = -\frac{2}{\pi}\int_0^{2\pi} H(1+2\sigma^2(1+H^2))\frac{1}{t}\frac{\partial\vec{n}}{\partial\theta}\, d\theta\\
&-\frac{1}{\pi}\int_0^{2\pi} e^{-2\lambda} l_\sigma\nabla\vec{\Phi}\cdot\nabla\alpha\frac{1}{t}\frac{\partial\alpha}{\partial\theta}\, d\theta + \frac{1}{\pi}\int_0^{2\pi} e^{-2\lambda} l_\sigma(\vec{\mathbb{I}}\llcorner\nabla^\perp\alpha)\cdot\frac{\partial}{\partial r}\, d\theta\\
&-\frac{1}{\pi}\int_0^{2\pi}(H^2+\sigma^2(1+H^2)^2)\frac{1}{t}\frac{\partial\vec{\Phi}}{\partial\theta}\, d\theta\\
&+\frac{1}{2\pi}\int_0^{2\pi} l_\sigma[e^{-2\lambda}|\nabla\alpha|^2 - K_{g_0}\alpha e^{-2\alpha} + 4\pi A_{\vec{\Phi}}(S^2)^{-1}]\frac{1}{t}\frac{\partial\vec{\Phi}}{\partial\theta}\, d\theta. \qquad \text{(VI.10)}
\end{aligned}
$$

Using again $\sigma^2(1+H^2)^2 \le 2\sigma^4(1+H^2)^2H^2 + 3(1+H^2)$, this gives

$$
\begin{aligned}
e^{\lambda(t)}\left|\frac{d\vec{L}}{dt}\right|(t) \le &C\sigma^4\|e^\lambda H(1+H^2)\|^2_{L^\infty(\partial B_t(0))} + C\|\nabla\vec{n}\|^2_{L^\infty(\partial B_t(0))}\\
&+ Cl_\sigma\|\nabla\alpha\|^2_{L^\infty(\partial B_t(0))} + l_\sigma e^{2\lambda(t)} + l_\sigma\|\alpha\|_\infty. \qquad \text{(VI.11)}
\end{aligned}
$$

Combining the previous inequality with (VI.3), (VI.4) and (VI.7) we finally obtain

$$e^{\lambda(t)}\left|\frac{d\vec{L}}{dt}\right|(t) \le C\delta^2(t). \tag{VI.12}$$

Following [8, proof of Lemma VI.1] we can choose a normalization in such a way that

$$e^{\lambda(x)}|\vec{L}|(x) \le C|x|^{-1} \quad \text{on } B_{R/4}\setminus B_{2r}, \tag{VI.13}$$

where C is independent of k. We adopt the notations of the proof of Lemma IV.1. Let Y satisfy

$$\begin{cases} -\Delta Y = 4e^{2\lambda}\sigma^2(1-H^4) + 2l_\sigma K_{g_0}\alpha e^{2\mu} - 8\pi l_\sigma e^{2\lambda} A_{\vec{\Phi}}(S^2)^{-1} & \text{in } B_R(0),\\ Y = 0 & \text{on } \partial B_R(0). \end{cases} \tag{VI.14}$$

Inequality (IV.40) gives

$$\begin{aligned}\|\nabla Y\|_{L^{2,\infty}(B_R(0))} &\le C\int_{B_R(0)} \sigma^2[1+H^4]e^{2\lambda}\,dx^2\\ &\quad + Cl_\sigma\|\alpha\|_\infty\int_{B_R(0)} e^{2\mu}\,dx^2 + Cl_\sigma\frac{A_{\vec{\Phi}}(B_R(0))}{A_{\vec{\Phi}}(S^2)} = o(1).\end{aligned} \tag{VI.15}$$

On $B_{2t}\setminus B_{t/2}$ we have

$$\|\Delta Y\|_{L^\infty(B_{2t}\setminus B_{t/2})} \le C\delta^2(t). \tag{VI.16}$$

Hence using standard interpolation theory[14] we deduce

$$t^2\|\nabla Y\|^2_{L^\infty(\partial B_t(0))} \le C\|\nabla Y\|_{L^{2,\infty}(B_R(0))}\, t^2\|\Delta Y\|_{L^\infty(B_{2t}\setminus B_{t/2})} + \|\nabla Y\|^2_{L^{2,\infty}(B_R(0))}. \tag{VI.17}$$

Combining (VI.15)–(VI.17) gives

$$\|\nabla Y\|_{L^\infty(\partial B_t(0))} \le o(1)[\delta(t)+t^{-1}] = o(1)t^{-1}. \tag{VI.18}$$

Using the Poincaré lemma we deduce the existence of a function S such that

$$\nabla S = \vec{L}\cdot\nabla\vec{\Phi} + \nabla^\perp Y \quad \text{on } B_R(0). \tag{VI.19}$$

Combining (VI.13) and (VI.18) gives

$$\|\nabla S\|_{L^\infty(\partial B_t(0))} \le Ct^{-1} \quad \text{for } t\in[2r, R/2]. \tag{VI.20}$$

Let $\vec{v}$ satisfy

$$\begin{cases} \Delta\vec{v} = \nabla^\perp Y\cdot\nabla\vec{n} & \text{in } B_R(0),\\ \vec{v} = 0 & \text{on } \partial B_R(0). \end{cases} \tag{VI.21}$$

[14] See for instance [10, proof of Lemma A.1] with $\|u\|_\infty$ replaced by $\|\nabla u\|_{2,\infty}$ which has the same scaling in dimension 2.

Using the Wente estimates (see [40]) this gives

$$\int_{B_R(0)} |\nabla \vec{v}|^2\, dx^2 \leq C\|\nabla Y\|^2_{2,\infty} \int_{B_R(0)} |\nabla \vec{n}|^2\, dx^2 = o(1), \tag{VI.22}$$

and

$$t^2\|\nabla \vec{v}\|^2_{L^\infty(\partial B_t(0))} \leq C\|\nabla \vec{v}\|_{L^2} t^2 \|\Delta \vec{v}\|_{L^\infty(B_{2t}\setminus B_{t/2})} + \int_{B_{2t}\setminus B_{t/2}} |\nabla \vec{v}|^2\, dx^2. \tag{VI.23}$$

Let

$$\eta_1(t) := t^{-1}\left[\int_{B_{2t}\setminus B_{t/2}} |\nabla \vec{v}|^2\, dx^2\right]^{1/2} = o(1)t^{-1}.$$

We have

$$\int_r^R \eta_1^2(t) t\, dt = o(1). \tag{VI.24}$$

Hence using (VI.23) we have

$$\|\nabla \vec{v}\|_{L^\infty(\partial B_t(0))} \leq o(1)\delta(t) + \eta_1(t) = o(1)t^{-1}. \tag{VI.25}$$

Let now $\vec{u}$ be such that

$$\vec{n}\nabla^\perp Y = \nabla \vec{v} + \nabla^\perp \vec{u}. \tag{VI.26}$$

Then

$$\|\nabla \vec{u}\|_{L^\infty(\partial B_t(0))} \leq o(1)t^{-1}. \tag{VI.27}$$

Let

$$\nabla \vec{D} = \left(l_\sigma e^{-2\lambda} \sum_{i=1}^2 \mathbb{I}_{1i}\, \partial_{x_i} \vec{\Phi},\, l_\sigma e^{-2\lambda} \sum_{i=1}^2 \mathbb{I}_{2i}\, \partial_{x_i} \vec{\Phi}\right). \tag{VI.28}$$

Using again the Poincaré lemma on $B_R(0)$ we obtain the existence of $\vec{V}$ such that

$$\nabla \vec{V} := \vec{L} \times \nabla \vec{\Phi} + 2H(1 + 2\sigma^2(1 + H^2))\nabla \vec{\Phi} - 2\alpha \nabla \vec{D}. \tag{VI.29}$$

Using again $2|H|\sigma^2(1 + H^2)) \leq 2\sqrt{H^2 e^{2\lambda}\sigma^4(1 + H^2)^2}$ together with (VI.3) and using also (VI.13) we obtain

$$\|\nabla \vec{V}\|_{L^\infty(\partial B_t(0))} \leq C[t^{-1} + \delta(t)] \leq 2Ct^{-1}. \tag{VI.30}$$

We denote $\vec{R} := \vec{V} - \vec{u}$. Then (IV.22) implies

$$\vec{n} \times \nabla \vec{R} + \vec{n} \times \nabla^\perp \vec{v} = -\nabla^\perp \vec{R} + \nabla \vec{v} - \vec{n}\nabla S - 2\alpha(\nabla^\perp \vec{D} + \vec{n} \times \nabla \vec{D}). \tag{VI.31}$$

Combining (VI.27) and (VI.30) we have

$$\|\nabla \vec{R}\|_{L^\infty(\partial B_t(0))} \leq Ct^{-1}. \tag{VI.32}$$

Let $\vec{E}$ be the solution of

$$\begin{cases} \Delta\vec{E} = 2\nabla\alpha\cdot\nabla^\perp\vec{D} & \text{on } B_R(0), \\ \vec{E} = 0 & \text{on } \partial B_R(0). \end{cases} \tag{VI.33}$$

Using the Wente estimates (see [40]) we have

$$\|\nabla\vec{E}\|_{L^2(B_R(0))} \le \|\nabla\vec{E}\|_{L^{2,1}(B_R(0))} \le C l_\sigma \|\nabla\alpha\|_{L^2(B_R(0))}\|\nabla\vec{n}\|_{L^2(B_R(0))} = o(1). \tag{VI.34}$$

Denote

$$\eta_2(t) := t^{-1}\left[\int_{B_{2t}\setminus B_{t/2}} |\nabla\vec{E}|^2\,dx^2\right]^{1/2} = o(1)t^{-1}.$$

From (VI.34) we obtain

$$\int_{2r}^{R/2} \eta_2^2(t)t\,dt = o(1). \tag{VI.35}$$

Interpolation inequalities give again

$$\begin{aligned} t^2\|\nabla\vec{E}\|^2_{L^\infty(\partial B_t(0))} &\le C\|\nabla\vec{E}\|_{L^2(B_R(0))}t^2\|\Delta\vec{E}\|_{L^\infty(B_{2t}\setminus B_{t/2})} + C\int_{B_{2t}\setminus B_{t/2}} |\nabla\vec{E}|^2\,dx^2 \\ &\le Ct^2 o(1) l_\sigma\|\nabla\alpha\|_{L^\infty(B_{2t}\setminus B_{t/2})}\|\nabla\vec{n}\|_{L^\infty(B_{2t}\setminus B_{t/2})} + t^2\eta_2^2(t) \\ &\le Ct^2[o(1)\delta^2(t) + \eta_2^2(t)] = o(1). \end{aligned} \tag{VI.36}$$

Hence we deduce

$$\|\nabla\vec{E}\|_{L^\infty(\partial B_t(0))} \le C[o(1)\delta(t) + \eta_2(t)] = o(1)t^{-1}. \tag{VI.37}$$

Let $\vec{F}$ be such that

$$2\alpha\nabla^\perp\vec{D} = \nabla^\perp\vec{F} + \nabla\vec{E}.$$

Observe that on the one hand

$$\|\nabla\vec{F}\|_{L^2(B_R(0))} \le 2l_\sigma\|\alpha\|_\infty\|\nabla\vec{n}\|_{L^2(B_R(0))} + \|\nabla\vec{E}\|_{L^2(B_R(0))} = o(1), \tag{VI.38}$$

and on the other hand, for $t \in [4r, R/4]$,

$$\begin{aligned} \|\nabla\vec{F}\|_{L^\infty(\partial B_t(0))} &\le 2l_\sigma\|\alpha\|_\infty\|\nabla\vec{n}\|_{L^\infty(\partial B_t(0))} + \|\nabla\vec{E}\|_{L^\infty(\partial B_t(0))} \\ &= C[o(1)\delta(t) + \eta_2(t)] = o(1)t^{-1}. \end{aligned} \tag{VI.39}$$

Let $\vec{X} := \vec{R} + \vec{F}$. Using (VI.32) we have

$$\|\nabla\vec{X}\|_{L^\infty(\partial B_t(0))} \le Ct^{-1}, \tag{VI.40}$$

and $\vec{X}$ satisfies

$$\vec{n}\times\nabla\vec{X} + \vec{n}\times\nabla^\perp\vec{v} = -\nabla^\perp\vec{X} + \nabla\vec{v} - \vec{n}\nabla S - \nabla\vec{E} + \vec{n}\times\nabla^\perp\vec{E}. \tag{VI.41}$$

Let $\vec{w}$ be the solution of

$$\begin{cases} \Delta\vec{w} = \nabla\vec{n}\cdot\nabla^\perp(\vec{v}-\vec{E}) & \text{on } B_R(0), \\ \vec{w} = 0 & \text{on } \partial B_R(0). \end{cases} \tag{VI.42}$$

Using the Wente estimates we obtain

$$\|\nabla\vec{w}\|_{L^2(B_R(0))} \le C\|\nabla(\vec{v}-\vec{E})\|_{L^2(B_R(0))}\|\nabla\vec{n}\|_{L^2(B_R(0))} = o(1). \tag{VI.43}$$

Denote

$$\eta_3(t) := t^{-1}\left[\int_{B_{2t}\setminus B_{t/2}} |\nabla\vec{w}|^2\,dx^2\right]^{1/2} = o(1)t^{-1}.$$

From (VI.43) we obtain

$$\int_{2r}^{R/2} \eta_3^2(t)t\,dt = o(1). \tag{VI.44}$$

Interpolation inequalities again give, using (VI.25)

$$\|\nabla\vec{w}\|_{L^\infty(\partial B_t(0))} \le C[o(1)\delta(t)+\eta_3(t)] = o(1)t^{-1}. \tag{VI.45}$$

Let $\vec{Z}$ be such that

$$\vec{n}\times\nabla^\perp(\vec{v}-\vec{E}) = \nabla^\perp\vec{Z}+\nabla\vec{w}.$$

Clearly from the above we also have

$$\begin{aligned} &\|\nabla\vec{Z}\|_{L^2(B_R(0))} = o(1), \\ &\|\nabla\vec{Z}\|_{L^\infty(\partial B_t(0))} \le C\Big[o(1)\delta(t)+\sum_{i=1}^3 \eta_i(t)\Big] = o(1)t^{-1}. \end{aligned} \tag{VI.46}$$

Let $\vec{T} := \vec{X}+\vec{Z}$. We have

$$\nabla\vec{T} = -\nabla^\perp\vec{v}+\nabla^\perp\vec{E}+\vec{n}\nabla^\perp S+\vec{n}\times\nabla^\perp\vec{X}, \tag{VI.47}$$

which implies in particular

$$\nabla^\perp S+\vec{n}\cdot\nabla^\perp(\vec{E}-\vec{v}) = \vec{n}\cdot\nabla\vec{T}. \tag{VI.48}$$

From (VI.40) and (VI.46) we have

$$\|\nabla\vec{T}\|_{L^\infty(\partial B_t(0))} \le Ct^{-1}, \quad \text{so} \quad \|\nabla\vec{T}\|_{L^{2,\infty}(B_R\setminus B_r)} \le C, \tag{VI.49}$$

where the constant C is independent of k (as for all constants C above). Let B be the solution of

$$\begin{cases} \Delta B = \nabla\vec{n}\cdot\nabla^\perp(\vec{E}-\vec{v}) & \text{in } B_R(0), \\ B = 0 & \text{on } \partial B_R(0). \end{cases} \tag{VI.50}$$

Similarly to the above we obtain

$$\|\nabla B\|_{L^{2,1}(B_R(0))} = o(1) \quad \text{and} \quad \|\nabla B\|_{L^\infty(\partial B_t(0))} \le o(1)t^{-1}. \tag{VI.51}$$

Denote

$$\eta_4(t) := t^{-1}\left[\int_{B_{2t}\setminus B_{t/2}} |\nabla B|^2\, dx^2\right]^{1/2} = o(1)t^{-1}.$$

From (VI.51) we obtain

$$\int_{2r}^{R/2} \eta_4^2(t)t\, dt = o(1). \tag{VI.52}$$

Interpolation inequalities again give

$$\|\nabla B\|_{L^\infty(\partial B_t(0))} \leq C[o(1)\,\delta(t) + \eta_4(t)] \leq o(1)t^{-1}. \tag{VI.53}$$

Let D be such that

$$\vec{n}\cdot\nabla^\perp(\vec{E}-\vec{v}) = \nabla B + \nabla^\perp D. \tag{VI.54}$$

Similarly to the above we obtain

$$\begin{aligned} &\|\nabla D\|_{L^2(B_R(0))} = o(1),\\ &\|\nabla D\|_{L^\infty(\partial B_t(0))} \leq C[o(1)\,\delta(t) + \eta_1(t) + \eta_2(t) + \eta_4(t)] = o(1)t^{-1}. \end{aligned} \tag{VI.55}$$

Denoting $U := S + D$ we have, from (VI.20) and (VI.55),

$$\|\nabla U\|_{L^\infty(\partial B_t(0))} \leq Ct^{-1}, \quad \text{so} \quad \|\nabla U\|_{L^{2,\infty}(B_R\setminus B_r)} \leq C. \tag{VI.56}$$

The pair $(U, \vec{T})$ satisfies the following system

$$\begin{cases} \nabla U = \nabla^\perp C - \vec{n}\cdot\nabla^\perp\vec{T},\\ \nabla\vec{T} = -\nabla^\perp\vec{v} + \nabla^\perp\vec{E} + \vec{n}\nabla^\perp S + \vec{n}\times\nabla^\perp\vec{X}. \end{cases} \tag{VI.57}$$

Let $U_t := |\partial B_t|^{-1}\int_{\partial B_t} U$ and $\vec{T}_t := |\partial B_t|^{-1}\int_{\partial B_t}\vec{T}$. The system (VI.57) implies

$$\begin{cases} \dfrac{dU_t}{dt} = -\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}[\vec{n}-\vec{n}_t]\cdot\frac{1}{t}\frac{\partial\vec{T}}{\partial\theta}\,d\theta,\\ \dfrac{d\vec{T}_t}{dt} = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}[\vec{n}-\vec{n}_t]\frac{1}{t}\frac{\partial S}{\partial\theta}\,d\theta + \frac{1}{2\pi}\int_0^{2\pi}[\vec{n}-\vec{n}_t]\times\frac{1}{t}\frac{\partial\vec{X}}{\partial\theta}\,d\theta, \end{cases} \tag{VI.58}$$

where $\vec{n}_t := |\partial B_t|^{-1}\int_{\partial B_t}\vec{n}$. We have

$$\|\vec{n}-\vec{n}_t\|_{L^\infty(\partial B_t)} \leq Ct\delta(t).$$

Hence using the estimates above we obtain

$$\left|\frac{dU_t}{dt}\right| + \left|\frac{d\vec{T}_t}{dt}\right| \leq C\delta(t), \tag{VI.59}$$

which implies

$$\int_r^R\left(\left|\frac{dU_t}{dt}\right|^2 + \left|\frac{d\vec{T}_t}{dt}\right|^2\right)t\,dt \leq C, \tag{VI.60}$$

where C is again independent of k. The system (VI.57) implies

$$\begin{cases} \Delta U = -\nabla \vec{n} \cdot \nabla^\perp \vec{T}, \\ \Delta \vec{T} = \nabla \vec{n} \cdot \nabla^\perp S + \nabla \vec{n} \times \nabla^\perp \vec{X}. \end{cases} \tag{VI.61}$$

We can then make use of [28, Lemma 10] to deduce that ∇U and $\nabla \vec{T}$ are uniformly bounded in L^2,

$$\int_{B_R \setminus B_r} (|\nabla U|^2 + |\nabla \vec{T}|^2)\, dx^2 \leq C, \tag{VI.62}$$

which in turn implies, using (VI.55) and (VI.46),

$$\int_{B_R \setminus B_r} (|\nabla S|^2 + |\nabla \vec{X}|^2)\, dx^2 \leq C, \tag{VI.63}$$

where C is independent of k. We bootstrap this information in (VI.58) as follows:

$$\begin{aligned} \int_r^R \left(\left| \frac{dU_t}{dt} \right| + \left| \frac{d\vec{T}_t}{dt} \right| \right) dt &\leq \int_r^R \delta(t)\, dt \int_{\partial B_t} (|\nabla \vec{T}| + |\nabla \vec{X}| + |\nabla S|)\, dl \\ &\leq \left[\int_r^R \delta^2(t) t\, dt \right]^{1/2} \left[\int_r^R t^{-1} \left[\int_{\partial B_t} (|\nabla \vec{T}| + |\nabla \vec{X}| + |\nabla S|)\, dl \right]^2 dt \right]^{1/2} \\ &\leq \left[\int_r^R \delta^2(t) t\, dt \right]^{1/2} \left[\int_{B_R \setminus B_r} (|\nabla S|^2 + |\nabla \vec{T}|^2 + |\nabla \vec{X}|^2)\, dx^2 \right]^{1/2} \leq C. \end{aligned} \tag{VI.64}$$

We can choose S and $\vec{R}$ (which were fixed modulo addition of an arbitrary constant) in such a way that

$$0 = U_r = |\partial B_r|^{-1} \int_{\partial B_r} U\, dl \quad \text{and} \quad 0 = \vec{T}_r = |\partial B_r|^{-1} \int_{\partial B_r} \vec{T}\, dl. \tag{VI.65}$$

Combining this choice with (VI.64) we obtain

$$|U_t|_{L^\infty([r,R])} + |\vec{T}_t|_{L^\infty([r,R])} \leq C. \tag{VI.66}$$

We can then make use of [28, Lemma 8] to deduce

$$\|\nabla U\|_{L^{2,1}(B_R \setminus B_r)} + \|\nabla \vec{T}\|_{L^{2,1}(B_R \setminus B_r)} \leq C. \tag{VI.67}$$

From (IV.27) we have

$$\begin{aligned} 2(1 + 2\sigma^2(1 + H^2) - l_\sigma \alpha) e^{2\lambda} \vec{H} &= \nabla^\perp S \cdot \nabla \vec{\Phi} - \nabla \vec{R} \times \nabla^\perp \vec{\Phi} - \nabla \vec{v} \times \nabla \vec{\Phi} \\ &= \nabla^\perp U \cdot \nabla \vec{\Phi} - \nabla \vec{T} \times \nabla^\perp \vec{\Phi} - \nabla \vec{v} \times \nabla \vec{\Phi} - \nabla^\perp D \cdot \nabla \vec{\Phi} + \nabla(\vec{F} + \vec{Z}) \times \nabla^\perp \vec{\Phi}. \end{aligned}$$

Hence (VI.67) implies

$$\|h_\sigma^{-1} e^\lambda \vec{H} + [\nabla \vec{v} + \nabla^\perp (\vec{F} + \vec{Z})] \times \nabla \vec{\Phi} e^{-\lambda} + \nabla^\perp D \cdot \nabla \vec{\Phi} e^{-\lambda}\|_{L^{2,1}(B_R \setminus B_r)} \leq C, \tag{VI.68}$$

where $h_\sigma^{-1} := 2(1+2\sigma^2(1+H^2) - l_\sigma\alpha)$. For any $\varepsilon > 0$ we choose r^k and R^k such that

$$\|s\delta^k(s)\|_{L^\infty([r^k,R^k])} \le \varepsilon,$$

which implies in particular using (VI.3) that

$$\|(h_\sigma^k)^{-1} e^{\lambda^k} \vec{H}^k\|_{L^{2,\infty}(B_{R^k}\setminus B_{r^k})} \le C\varepsilon. \tag{VI.69}$$

Using (VI.22), (VI.38), (VI.46) and (VI.55), for k large enough we have

$$\|[\nabla\vec{v}^k + \nabla^\perp(\vec{F}^k + \vec{Z}^k)] \times \nabla\vec{\Phi}^k e^{-\lambda^k} + \nabla^\perp D^k \cdot \nabla\vec{\Phi}^k e^{-\lambda^k}\|_{L^2(B_{R^k}\setminus B_{r^k})} \le \varepsilon. \tag{VI.70}$$

Combining (VI.69) and (VI.70) we obtain in particular

$$\|(h_\sigma^k)^{-1} e^{\lambda^k} \vec{H}^k + [\nabla\vec{v}^k + \nabla^\perp(\vec{F}^k + \vec{Z}^k)] \times \nabla\vec{\Phi}^k e^{-\lambda^k} + \nabla^\perp D^k \cdot \nabla\vec{\Phi}^k e^{-\lambda^k}\|_{L^{2,\infty}(B_{R^k}\setminus B_{r^k})} \le C\varepsilon. \tag{VI.71}$$

Combining (VI.68) and (VI.71) we obtain

$$\|(h_\sigma^k)^{-1} e^{\lambda^k} \vec{H}^k + [\nabla\vec{v}^k + \nabla^\perp(\vec{F}^k + \vec{Z}^k)] \times \nabla\vec{\Phi}^k e^{-\lambda^k} + \nabla^\perp D^k \cdot \nabla\vec{\Phi}^k e^{-\lambda^k}\|_{L^2(B_{R^k}\setminus B_{r^k})} \le C\sqrt{\varepsilon}. \tag{VI.72}$$

Combining (VI.70) and (VI.72) we then get

$$\|(h_\sigma^k)^{-1} e^{\lambda^k} \vec{H}^k\|_{L^2(B_{R^k}\setminus B_{r^k})} \le C[\sqrt{\varepsilon} + \varepsilon], \tag{VI.73}$$

which implies that the Willmore energy is as small as we want in any neck region for k large enough. We deduce Lemma VI.1 from this fact and the final arguments of [8].

VII. Appendix

Lemma VII.1. *There exists $C > 0$ such that for any $\sigma \in (0,1)$ and any $\nabla a \in L^4(D^2)$ and $\nabla b \in L^{4/3}(D^2)$, denoting by φ the $W^{1,1}$ solution to the equation*

$$\begin{cases} -\Delta\varphi = \partial_{x_1} a\, \partial_{x_2} b - \partial_{x_2} a\, \partial_{x_1} b & \text{in } D^2, \\ \varphi = 0 & \text{on } \partial D^2, \end{cases} \tag{VII.1}$$

the following inequalities hold:

$$\|\nabla\varphi\|_{L^{2,\infty}} \le C\|\nabla a\|_{L^{2,\infty}\cap\sigma^{1/2}L^4(D^2)} \|\nabla b\|_{L^{2,\infty}+\sigma^{-1/2}L^{4/3}(D^2)}, \tag{VII.2}$$

$$\|\nabla\varphi\|_{L^{2,\infty}+\sigma^{-1/2}L^{4/3}(D^2)} \le C\|\nabla a\|_{L^{2,\infty}} \|\nabla b\|_{L^{2,\infty}+\sigma^{-1/2}L^{4/3}(D^2)}, \tag{VII.3}$$

where

$$\|f\|_{L^{2,\infty}\cap\sigma^{1/2}L^4(D^2)} := \|f\|_{L^{2,\infty}(D^2)} + \sigma^{1/2}\|f\|_{L^4(D^2)},$$

$$\|f\|_{L^{2,\infty}+\sigma^{-1/2}L^{4/3}(D^2)} := \inf\{\|f_1\|_{L^{2,\infty}(D^2)} + \sigma^{-1/2}\|f_2\|_{L^{4/3}(D^2)}\ ;\ f = f_1 + f_2\}.$$

Proof. Let X_1 and X_2 be such that

$$\nabla b = X_1 + X_2, \quad \|X_1\|_{L^{2,\infty}(D^2)} + \sigma^{-1/2}\|X_2\|_{L^{4/3}(D^2)} \le 2\|\nabla b\|_{L^{2,\infty}+\sigma^{-1/2}L^{4/3}(D^2)}. \tag{VII.4}$$

Let c_i for $i = 1, 2$ satisfy

$$\begin{cases} -\Delta c_i = \operatorname{div}(X_i^\perp) & \text{in } D^2, \\ c_i = 0 & \text{on } \partial D^2. \end{cases} \tag{VII.5}$$

There exists a constant C independent of σ such that

$$\|\nabla c_1\|_{L^{2,\infty}(D^2)} \le C\|X_1\|_{L^{2,\infty}(D^2)} \quad \text{and} \quad \|\nabla c_2\|_{L^{4/3}(D^2)} \le C\|X_2\|_{L^{4/3}(D^2)}.$$

Applying the Poincaré lemma we obtain the existence of b_i such that $\int_{D^2} b_i = 0$ and

$$X_i^\perp + \nabla c_i = \nabla^\perp b_i \iff X^i = \nabla b_i + \nabla^\perp c_i, \tag{VII.6}$$

and we have

$$\|\nabla b_1\|_{L^{2,\infty}(D^2)} \le (C+1)\|X_1\|_{L^{2,\infty}(D^2)} \quad \text{and} \quad \|\nabla b_2\|_{L^{4/3}(D^2)} \le (C+1)\|X_2\|_{L^{4/3}(D^2)}.$$

Observe that

$$\nabla b = \nabla b_1 + \nabla b_2 + \nabla^\perp c_1 + \nabla^\perp c_2, \qquad \Delta(c_1 + c_2) = 0. \tag{VII.7}$$

Since $c_1 + c_2 = 0$ on ∂D^2, we have $c_1 + c_2 \equiv 0$ on D^2 and

$$\nabla b = \nabla b_1 + \nabla b_2. \tag{VII.8}$$

Let

$$\begin{cases} -\Delta \varphi_i = \partial_{x_1} a\, \partial_{x_2} b_i - \partial_{x_2} a\, \partial_{x_1} b_i & \text{in } D^2, \\ \varphi_i = 0 & \text{on } \partial D^2. \end{cases} \tag{VII.9}$$

Using the Wente estimates (see for instance [40]) we obtain respectively

$$\|\nabla \varphi_1\|_{L^{2,\infty}(D^2)} \le C\|\nabla a\|_{L^{2,\infty}(D^2)}\|\nabla b_1\|_{L^{2,\infty}(D^2)}, \tag{VII.10}$$

$$\|\nabla \varphi_2\|_{L^2(D^2)} \le C\|\nabla a\|_{L^4(D^2)}\|\nabla b_2\|_{L^{4/3}(D^2)}. \tag{VII.11}$$

Combining (VII.10) and (VII.11) we obtain (VII.2). We now write

$$\begin{cases} -\Delta \varphi_2 = \operatorname{div}(b_2 \nabla^\perp a) & \text{in } D^2, \\ \varphi_2 = 0 & \text{on } \partial D^2. \end{cases} \tag{VII.12}$$

The Sobolev–Lorentz embedding theorem gives, since $\int_{D^2} b_2 = 0$,

$$\|b_2\|_{L^{4,4/3}(D^2)} \le C\|\nabla b_2\|_{L^{4/3}(D^2)}. \tag{VII.13}$$

Using fundamental properties of Lorentz spaces (see [20]) we have

$$\|b_2 \nabla^\perp a\|_{L^{4/3}(D^2)} \le C\|\nabla a\|_{L^{2,\infty}(D^2)}\|b_2\|_{L^{4,4/3}(D^2)} \le C\|\nabla a\|_{L^{2,\infty}(D^2)}\|\nabla b_2\|_{L^{4/3}(D^2)}.$$

Hence using the classical elliptic estimates we get

$$\|\nabla\varphi_2\|_{L^{4/3}(D^2)} \le C\|\nabla a\|_{L^{2,\infty}(D^2)}\|\nabla b_2\|_{L^{4/3}(D^2)}. \tag{VII.14}$$

Combining (VII.10) and (VII.14) we obtain, using (VII.4),

$$\begin{aligned}\|\nabla\varphi_1\|_{L^{2,\infty}(D^2)} &+ \sigma^{-1/2}\|\nabla\varphi_2\|_{L^{4/3}(D^2)}\\ &\le C\|\nabla a\|_{L^{2,\infty}(D^2)}[\|\nabla b_1\|_{L^{2,\infty}(D^2)} + \sigma^{-1/2}\|\nabla b_2\|_{L^{4/3}(D^2)}]\\ &\le C\|\nabla a\|_{L^{2,\infty}(D^2)}[\|X_1\|_{L^{2,\infty}(D^2)} + \sigma^{-1/2}\|X_2\|_{L^{4/3}(D^2)}]\\ &\le C\|\nabla a\|_{L^{2,\infty}(D^2)}\ \|\nabla b\|_{L^{2,\infty}+\sigma^{-1/2}L^{4/3}(D^2)}.\end{aligned}$$

This implies (VII.3), and Lemma VII.1 is proved. □

Lemma VII.2. *Let $m \in \mathbb{N}^*$ and $1 \le p < +\infty$. There exists $\varepsilon(m, p) > 0$ such that for any sequence of maps $\mathcal{A}_k \in W^{1,2p}(D^2, M_m(\mathbb{R}))$ satisfying*

$$\int_{D^2} |\nabla\mathcal{A}_k|^{2p}\, dx^2 \le \varepsilon(m, p) \tag{VII.15}$$

and weakly converging to $\mathcal{A}_\infty$ in $W^{1,2p}$, for any sequence of maps $\vec{\varphi}_k$ weakly converging in $W^{1,\frac{2p}{2p-1}}(D^2, \mathbb{R}^m)$ and any sequence of maps $\vec{F}_k$ strongly converging to $\vec{F}_\infty$ in $L^{\frac{2p}{2p-1}}(D^2, \mathbb{R}^2\otimes\mathbb{R}^m)$ and satisfying

$$-\Delta\vec{\varphi}_k = \nabla\mathcal{A}_k\cdot\nabla^\perp\vec{\varphi}_k + \operatorname{div}\vec{F}_k \quad \textit{in } \mathcal{D}'(D^2), \tag{VII.16}$$

$\vec{\varphi}_k$ strongly converges in $W^{1,\frac{2p}{2p-1}}_{\mathrm{loc}}(D^2, \mathbb{R}^m)$.[15] □

Proof. We consider the case $p = 1$, which is the most delicate. We first prove, assuming $\vec{F}_k \to \vec{F}_\infty$ in $W^{1,2}(D^2, \mathbb{R}^2\otimes\mathbb{R}^m)$, that there exists[16] $q > 1$ such that for any $\Omega \subset\subset D^2$,

$$\limsup_{k\to+\infty} \|\vec{\phi}_k\|_{W^{2,q}(\Omega)} < +\infty. \tag{VII.17}$$

This implies, using Rellich–Kondrashov, that $\vec{\phi}_k \to \vec{\phi}_\infty$ strongly in $W^{1,2}_{\mathrm{loc}}(D^2)$.

Proof of (VII.17). Let $\rho < 1$. We prove that there exists $\gamma > 0$ such that

$$\limsup_{k\to+\infty}\ \sup_{x_0\in B_\rho(0)}\ \sup_{r<1-\rho} r^{-\gamma}\int_{B_r(x_0)} |\nabla\vec{\varphi}_k|^2\, dx^2. \tag{VII.18}$$

Let $x_0 \in B_\rho(0)$ and $r < 1-\rho$. On $B_\rho(0)$ we decompose $\vec{\varphi}_k = \vec{\psi}_k + \vec{v}_k$ where

$$\begin{cases} -\Delta\vec{\psi}_k = \nabla\mathcal{A}_k\cdot\nabla^\perp\vec{\varphi}_k + \operatorname{div}\vec{F}_k & \text{in } B_r(x_0),\\ \vec{\psi}_k = 0 & \text{on } \partial B_r(x_0).\end{cases}$$

[15] If one assumes further $p > 1$, the smallness condition (VII.15) is not needed for the same result to hold.

[16] In fact, under the assumptions this is true for any $q < 2$.

Using the Wente estimate we obtain

$$\int_{B_r(x_0)} |\nabla\vec{\psi}_k|^2\, dx^2 \le C\sqrt{\varepsilon(2,m)} \int_{B_r(x_0)} |\nabla\vec{\varphi}_k|^2\, dx^2 + Cr^2\|\vec{F}_k\|^2_{W^{1,2}(D^2)}. \tag{VII.19}$$

Since $\vec{v}_k$ is harmonic, the monotonicity formula gives, for any $t < 1$,

$$\int_{B_{tr}(x_0)} |\nabla\vec{v}_k|^2\, dx^2 \le t^2 \int_{B_r(x_0)} |\nabla\vec{v}_k|^2\, dx^2 \le t^2 \int_{B_r(x_0)} |\nabla\vec{\varphi}_k|^2\, dx^2, \tag{VII.20}$$

where we have used the fact that the harmonic extension minimizes the Dirichlet energy. Combining (VII.19) and (VII.20) then gives

$$\int_{B_{2^{-1}r}(x_0)} |\nabla\vec{\varphi}_k|^2\, dx^2 \le [2C\sqrt{\varepsilon(2,m)} + 2^{-1}] \int_{B_r(x_0)} |\nabla\vec{\varphi}_k|^2\, dx^2 + Cr^2\|\vec{F}_k\|^2_{W^{1,2}(D^2)}.$$

By choosing $2C\sqrt{\varepsilon(2,m)} < 1/4$ we obtain

$$\int_{B_{2^{-1}r}(x_0)} |\nabla\vec{\varphi}_k|^2\, dx^2 \le \frac{3}{4} \int_{B_r(x_0)} |\nabla\vec{\varphi}_k|^2\, dx^2 + Cr^2\|\vec{F}_k\|^2_{W^{1,2}(D^2)}. \tag{VII.21}$$

Iteration of (VII.21) gives (VII.18). Inserting (VII.18) in the right hand side of (VII.16) gives

$$\limsup_{k\to+\infty} \sup_{x_0\in B_\rho(0)} \sup_{r<1-\rho} r^{-\gamma/2} \int_{B_r(x_0)} |\Delta\vec{\varphi}_k|\, dx^2$$

where we have used $\|\operatorname{div}\vec{F}_k\|_{L^1(B_r(x_0))} \le r\|\vec{F}_k\|_{W^{1,2}(D^2)}$. Using the Adams estimates we deduce the existence of $s > 2$ such that

$$\limsup_{k\to+\infty} \|\vec{\varphi}_k\|_{W^{1,s}(B_{1-2\rho}(0))} < +\infty.$$

Inserting this bound in the right hand side of (VII.16) gives (VII.17).

We now consider the general case: $\vec{F}_k \to \vec{F}_\infty$ strongly in $L^2(D^2)$. The weak convergence in $W^{1,2}$ of $\vec{\phi}_k$ to $\vec{\phi}_\infty$ and of $\mathcal{A}_k$ towards $\mathcal{A}_\infty$ implies that the limits, due to the jacobian structure of the r.h.s., satisfy the equation

$$-\Delta\vec{\varphi}_\infty = \nabla\mathcal{A}_\infty\cdot\nabla^\perp\vec{\varphi}_\infty + \operatorname{div}\vec{F}_\infty \quad \text{in } \mathcal{D}'(D^2).$$

Let $\vec{F}^s_\infty := \vec{F}_\infty \star \chi^s$ where $\chi^s(x) := s^{-2}\chi(x/s)$. Let $\vec{\varphi}^s_\infty$ be the unique solution of

$$\begin{cases} -\Delta\vec{\varphi}^s_\infty = \nabla\mathcal{A}_\infty\cdot\nabla^\perp\vec{\varphi}^s_\infty + \operatorname{div}\vec{F}^s_\infty & \text{in } D^2,\\ \vec{\varphi}^s_\infty = \vec{\varphi}_\infty & \text{on } \partial D^2.\end{cases}$$

We claim that $\vec{\varphi}^s_\infty$ strongly converges to $\vec{\varphi}_\infty$ in $W^{1,2}(D^2)$. Indeed,

$$\begin{cases} -\Delta(\vec{\varphi}^s_\infty - \vec{\varphi}_\infty) = \nabla\mathcal{A}_\infty\cdot\nabla^\perp(\vec{\varphi}^s_\infty - \vec{\varphi}_\infty) + \operatorname{div}(\vec{F}^s_\infty - \vec{F}_\infty) & \text{in } D^2,\\ \vec{\varphi}^s_\infty - \vec{\varphi}_\infty = 0 & \text{on } \partial D^2.\end{cases}$$

Multiplying by $\vec{\varphi}^s_\infty - \vec{\varphi}_\infty$ and integrating by parts gives

$$\int_{D^2} |\nabla(\vec{\varphi}^s_\infty - \vec{\varphi}_\infty)|^2\, dx^2 \le \int_{D^2} (\vec{\varphi}^s_\infty - \vec{\varphi}_\infty)\cdot \nabla\mathcal{A}_\infty \cdot \nabla^\perp(\vec{\varphi}^s_\infty - \vec{\varphi}_\infty)\, dx^2 - \int_{D^2} [\vec{F}^s_\infty - \vec{F}_\infty \cdot \nabla(\vec{\varphi}^s_\infty - \vec{\varphi}_\infty)]\, dx^2. \tag{VII.22}$$

Recall the Wente inequality

$$\forall a, b \in W^{1,2}(D^2)\ \forall c \in W^{1,2}_0(D^2) \qquad \left|\int_{D^2} c\nabla a \cdot \nabla^\perp b\right| \le C\|\nabla a\|_2 \|\nabla b\|_2 \|\nabla c\|_2.$$

Applying it to the first term on the r.h.s. of (VII.22) gives

$$\|\vec{\varphi}^s_\infty - \vec{\varphi}_\infty\|_{W^{1,2}(D^2)} \le C\varepsilon(m,2)\|\vec{\varphi}^s_\infty - \vec{\varphi}_\infty\|_{W^{1,2}(D^2)} + \|\vec{F}^s_\infty - \vec{F}_\infty\|_2.$$

Choosing $\varepsilon(m,1)$ so small that $C\varepsilon(m,1) < 1/2$ we obtain

$$\|\vec{\varphi}^s_\infty - \vec{\varphi}_\infty\|_{W^{1,2}(D^2)} \le 2\|\vec{F}^s_\infty - \vec{F}_\infty\|_2, \tag{VII.23}$$

which implies

$$\vec{\varphi}^s_\infty \to \vec{\varphi}_\infty \qquad \text{strongly in } W^{1,2}(D^2).$$

Let $\vec{\varphi}^s_k$ be the unique solution of

$$\begin{cases} -\Delta\vec{\varphi}^s_k = \nabla\mathcal{A}_k \cdot \nabla^\perp\vec{\varphi}^s_k + \operatorname{div}(\vec{F}_k \star \chi^s) & \text{in } D^2,\\ \vec{\varphi}^s_k = \vec{\varphi}_k & \text{on } \partial D^2. \end{cases}$$

For any fixed $s > 0$ and any $\Omega \subset\subset D^2$, using the first part of the proof we have the existence of $q > 1$ such that

$$\limsup_{k\to+\infty} \|\vec{\varphi}^s_k\|_{W^{2,q}(\Omega)} < +\infty.$$

Hence

$$\lim_{k\to+\infty} \|\vec{\varphi}^s_k - \vec{\varphi}^s_\infty\|_{W^{1,2}} = 0. \tag{VII.24}$$

Similarly to the proof of (VII.23) we have, for $\varepsilon(m,1)$ chosen as above,

$$\|\vec{\varphi}^s_k - \vec{\varphi}_k\|_{W^{1,2}(D^2)} \le 2\|\vec{F}^s_k - \vec{F}_k\|_{L^2(D^2)}. \tag{VII.25}$$

Using the triangle inequality and Young inequality we have

$$\begin{aligned} \|\vec{F}^s_k - \vec{F}_k\|_{L^2(D^2)} &\le \|(\vec{F}_k - \vec{F}_\infty)\star\chi^s\|_{L^2(D^2)} + \|\vec{F}^s_\infty - \vec{F}_\infty\|_{L^2(D^2)} + \|\vec{F}_k - \vec{F}_\infty\|_{L^2(D^2)}\\ &\le 2\|\vec{F}_k - \vec{F}_\infty\|_{L^2(D^2)} + \|\vec{F}^s_\infty - \vec{F}_\infty\|_{L^2(D^2)}. \end{aligned} \tag{VII.26}$$

From the triangle inequality, for any $s > 0$ and $\Omega \subset\subset D^2$, combining (VII.23), (VII.25) and (VII.26) we have

$$\begin{aligned}\|\nabla(\vec{\varphi}_k-\vec{\varphi}_\infty)\|_{L^2(\Omega)} &\le \|\nabla(\vec{\varphi}_k-\vec{\varphi}_k^s)\|_{L^2(\Omega)}+\|\nabla(\vec{\varphi}_k^s-\vec{\varphi}_\infty^s)\|_{L^2(\Omega)}+\|\nabla(\vec{\varphi}_\infty-\vec{\varphi}_\infty^s)\|_{L^2(\Omega)}\\ &\le 2\|\vec{F}_k^s-\vec{F}_k\|_{L^2(D^2)}+\|\nabla(\vec{\varphi}_k^s-\vec{\varphi}_\infty^s)\|_{L^2(\Omega)}+2\|\vec{F}_\infty^s-\vec{F}_\infty\|_{L^2(D^2)}\\ &\le 4\|\vec{F}_k-\vec{F}_\infty\|_{L^2(D^2)}+\|\nabla(\vec{\varphi}_k^s-\vec{\varphi}_\infty^s)\|_{L^2(\Omega)}+4\|\vec{F}_\infty^s-\vec{F}_\infty\|_{L^2(D^2)}.\end{aligned}$$

Let $\delta > 0$. There exists $s > 0$ such that $\|\vec{F}_\infty^s - \vec{F}_\infty\|_{L^2(D^2)} \le \delta/8$. Once s is fixed, by (VII.24), there exists k_δ such that

$$\forall k > k_\delta \quad \|\vec{F}_k - \vec{F}_\infty\|_{L^2(D^2)} < \delta/16 \quad \text{and} \quad \|\nabla(\vec{\varphi}_k^s - \vec{\varphi}_\infty^s)\|_{L^2(\Omega)} < \delta/16.$$

Hence $\|\nabla(\vec{\varphi}_k - \vec{\varphi}_\infty)\|_{L^2(\Omega)} < \delta$ for $k > k_\delta$. This implies the lemma for $p = 1$. □

Lemma VII.3. *For any $C_0 > 0$ there exists $\varepsilon > 0$ such that for any conformal weak immersion in $\mathcal{E}_{\Sigma,2}(D^2)$ satisfying*

$$\|\nabla\lambda\|_{L^{2,\infty}(D^2)} \le C_0 \quad \textit{and} \quad \int_{D^2} |\nabla\vec{n}|^2\, dx^2 < \varepsilon,$$

we have

$$\sigma^2 \int_{D^2_{1/2}} |\nabla\vec{n}|^4 e^{-2\lambda}\, dx^2 \le C\sigma^2 \int_{D^2} H^4 e^{2\lambda}\, dx^2 + C\sigma^2 e^{-2\overline{\lambda}}\left[\int_{D^2} |\nabla\vec{n}|^2\, dx^2\right]^2 \tag{VII.27}$$

where $e^\lambda = |\partial_{x_1}\vec{\Phi}| = |\partial_{x_2}\vec{\Phi}|$ and $\overline{\lambda} = |D^2_{1/2}|^{-1}\int_{D^2_{1/2}} \lambda(x)\, dx^2$. □

Proof. Arguing as at the beginning of the proof of Lemma IV.1 we get

$$\|\alpha - \overline{\alpha}\|_{L^\infty(D^2_{5/6})} \le C. \tag{VII.28}$$

We also have

$$\Delta\vec{\Phi} = 2e^{2\lambda}\vec{H} = 2e^{2\overline{\lambda}}e^{2(\lambda-\overline{\lambda})}\vec{H} \quad \text{in } D^2. \tag{VII.29}$$

This gives $\nabla^2\vec{\Phi} \in L^4(D^2_{3/4})$ and hence $\nabla\vec{n} \in L^4(D^2_{3/4})$. Let $\vec{a}$ satisfy

$$\begin{cases}\Delta\vec{a} = \operatorname{div}(\vec{n}\times\nabla\vec{n}) & \text{in } D^2,\\ \vec{a} = 0 & \text{on } \partial D^2,\end{cases} \tag{VII.30}$$

and let $\vec{b}$ be such that $\vec{n}\times\nabla\vec{n} = \nabla\vec{a} + \nabla^\perp\vec{b}$. Using classical elliptic estimates we have

$$\int_{D^2}(|\nabla\vec{a}|^2 + |\nabla\vec{b}|^2)\, dx^2 \le C\int_{D^2}|\nabla\vec{n}|^2\, dx^2 \le C\varepsilon. \tag{VII.31}$$

Let $\rho \in [1/2, 3/4]$ be such that

$$\int_{\partial D^2_\rho}(|\nabla\vec{a}|^2 + |\nabla\vec{b}|^2)\, dl \le 4\int_{D^2}|\nabla\vec{n}|^2\, dx^2 \le 4\varepsilon. \tag{VII.32}$$

Observe that $W^{1,2}(\partial D^2_\rho) \hookrightarrow W^{1-1/4,4}(\partial D^2_\rho)$. Hence

$$\|\vec{a}\|_{W^{1-1/4,4}(\partial D^2_\rho)} + \|\vec{b}\|_{W^{1-1/4,4}(\partial D^2_\rho)} \le C\left[\int_{D^2} |\nabla\vec{n}|^2\, dx^2\right]^{1/2}. \tag{VII.33}$$

Recall now from [40] the general formula

$$\vec{n} \times \nabla\vec{n} = 2H\nabla^\perp\vec{\Phi} + \nabla^\perp\vec{n}. \tag{VII.34}$$

Hence in particular on D^2_ρ,

$$\Delta\vec{a} = 2\,\mathrm{div}(H\nabla^\perp\vec{\Phi}). \tag{VII.35}$$

Classical elliptic estimates then give, using (VII.33),

$$\|\nabla\vec{a}\|_{L^4(D^2_\rho)} \le C e^{\overline{\lambda}}\left[\int_{D^2_\rho} H^4\, dx^2\right]^{1/4} + C\left[\int_{D^2} |\nabla\vec{n}|^2\, dx^2\right]^{1/2}. \tag{VII.36}$$

We also have, on D^2_ρ,

$$\Delta\vec{b} = \nabla^\perp\vec{n} \times \nabla\vec{n}. \tag{VII.37}$$

Classical elliptic estimates imply (using $H^{1/2}(D^2_\rho) \hookrightarrow L^4(D^2_\rho)$)

$$\|\nabla\vec{b}\|_{L^4(D^2_\rho)} \le C\|\Delta\vec{b}\|_{L^{4/3}(D^2_\rho)} + C\|\nabla\vec{b}\|_{L^2(\partial D^2_\rho)}.$$

Hence using (VII.32) and (VII.37) we obtain

$$\|\nabla\vec{b}\|_{L^4(D^2_\rho)} \le C\left[\int_{D^2_\rho} |\nabla\vec{n}|^{4/3}|\nabla\vec{n}|^{4/3}\, dx^2\right]^{3/4} + C\left[\int_{D^2} |\nabla\vec{n}|^2\, dx^2\right]^{1/2}.$$

Hence

$$\|\nabla\vec{b}\|_{L^4(D^2_\rho)} \le C\left[\int_{D^2_\rho} |\nabla\vec{n}|^2\, dx^2\right]^{1/2} \|\nabla\vec{n}\|_{L^4(D^2_\rho)} + C\|\nabla\vec{n}\|_{L^2(D^2)}.$$

Combining (VII.36) and (VII) with $\|\nabla\vec{n}\|_{L^2(D^2)} < \varepsilon$ gives

$$\|\nabla\vec{n}\|_{L^4(D^2_\rho)} \le C e^{\overline{\lambda}}\left[\int_{D^2_\rho} H^4\, dx^2\right]^{1/4} + C\varepsilon\|\nabla\vec{n}\|_{L^4(D^2_\rho)} + C\|\nabla\vec{n}\|_{L^2(D^2)}.$$

Hence for ε small enough we finally obtain

$$\sigma^2\int_{D^2_\rho} |\nabla\vec{n}|^4 e^{-2\lambda}\, dx^2 \le C\sigma^2\int_{D^2_\rho} H^4 e^{2\lambda}\, dx^2 + \sigma^2 e^{-2\overline{\lambda}}\left[\int_{D^2} |\nabla\vec{n}|^2\, dx^2\right]^2. \qquad \square$$

Lemma VII.4. *For any* $\gamma \in (0,1)$ *there exists* $\varepsilon > 0$ *such that for any* $\vec{\phi}$ *in* $W^{1,2}(D^2)$ *satisfying*

$$\Delta\vec{\phi} = \nabla^{\perp}\vec{n} \times \nabla\vec{\phi} + \operatorname{div} \vec{F} \tag{VII.38}$$

where

$$\int_{D^2} |\nabla\vec{n}|^2\, dx^2 \leq \varepsilon \quad \textit{and} \quad \sup_{B_r(x)\subset D^2} r^{-\gamma} \int_{B_r(x)} |\vec{F}|^2\, dx^2 < +\infty, \tag{VII.39}$$

we have

$$\sup_{B_r(x)\subset D^2_{1/2}} r^{-\gamma} \int_{B_r(x)} |\nabla\vec{\phi}|^2\, dx^2 \leq C_\gamma \left[\sup_{B_r(x)\subset D^2} r^{-\gamma} \int_{B_r(x)} |\vec{F}|^2\, dx^2 + \int_{D^2} |\nabla\vec{\phi}|^2\, dx^2\right] \tag{VII.40}$$

where C_γ *depends only on* $\gamma \in (0,1)$. □

Proof. For any $x_0 \in D^2_{1/2}$ and $r < 1/4$ we decompose $\vec{\phi} = \vec{\psi} + \vec{v}$ in $B_r(x_0)$ where $\vec{\psi}$ is the solution of

$$\begin{cases} \Delta\vec{\psi} = \nabla^{\perp}\vec{n} \times \nabla\vec{\phi} + \operatorname{div} \vec{F} & \text{in } B_r(x_0), \\ \vec{\psi} = 0 & \text{on } \partial B_r(x_0). \end{cases}$$

Using the Wente inequality we have

$$\int_{B_r(x_0)} |\nabla\vec{\psi}|^2\, dx^2 \leq C\varepsilon \int_{B_r(x_0)} |\nabla\vec{\phi}|^2\, dx^2 + C \int_{B_r(x_0)} |\vec{F}|^2\, dx^2.$$

Since $\vec{v}$ is harmonic we have, for any $t \in (0,1)$,

$$\int_{B_{tr}(x_0)} |\nabla\vec{v}|^2\, dx^2 \leq t^2 \int_{B_r(x_0)} |\nabla\vec{v}|^2\, dx^2 \leq t^2 \int_{B_r(x_0)} |\nabla\vec{\phi}|^2\, dx^2.$$

Hence in particular for $t = 1/2$,

$$\int_{B_{2^{-1}r}(x_0)} |\nabla\vec{\phi}|^2\, dx^2 \leq (2^{-1} + C\varepsilon) \int_{B_r(x_0)} |\nabla\vec{\phi}|^2\, dx^2 + C \int_{B_r(x_0)} |\vec{F}|^2\, dx^2. \tag{VII.41}$$

We choose ε such that $2^{-1} + C\varepsilon = 2^{-\gamma}$, and (VII.40) is obtained by iterating (VII.41). □

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References

[1] Aitchison, I. R.: The 'Holiverse': holistic eversion of the 2-sphere in $\mathbb{R}^3$. arXiv:1008.0916 (2010)

[2] Apéry, F.: An algebraic halfway model for the eversion of the sphere (with an appendix by B. Morin). Tohoku Math. J. **44,** 103–150 (1992) Zbl 0788.57020 MR 1145725

[3] Banchoff, T., Max, N.: Every sphere eversion has a quadruple point. In: Contributions to Analysis and Geometry (Baltimore, MD, 1980), Johns Hopkins Univ. Press, Baltimore, MD, 191–209 (1981) Zbl 0544.57012 MR 0648465

[4] Bauer, M., Kuwert, E.: Existence of minimizing Willmore surfaces of prescribed genus. Int. Math. Res. Notices **2003**, 553–576 Zbl 1029.53073 MR 1941840

[5] Bernard, Y.: Noether's theorem and the Willmore functional. Adv. Calc. Var. **9,** 217–234 (2016) Zbl 1343.58008 MR 3518329

[6] Bernard, Y., Rivière, T.: Local Palais–Smale sequences for the Willmore functional. Comm. Anal. Geom. **19,** 563–599 (2011) Zbl 1245.53051 MR 2843242

[7] Bernard, Y., Rivière, T.: Singularity removability at branch points for Willmore surfaces. Pacific J. Math. **265,** 257–311 (2013) Zbl 1304.30032 MR 3096502

[8] Bernard, Y., Rivière, T.: Energy quantization for Willmore surfaces and applications. Ann. of Math. (2) **180**, 87–136 (2014) Zbl 1325.53014 MR 3194812

[9] Bernard, Y., Rivière, T.: Uniform regularity results for critical and subcritical surface energies. Calc. Var. Partial Differential Equations **58** (2019), no. 1, art. 10, 39 pp. Zbl 06991669 MR 3881881

[10] Bethuel, F., Brezis, H., Hélein, F.: Asymptotics for the minimization of a Ginzburg–Landau functional. Calc. Var. Partial Differential Equations **1**, 123–148 (1993) Zbl 0834.35014 MR 1261720

[11] Bryant, R. L.: A duality theorem for Willmore surfaces. J. Differential Geom. **20**, 23–53 (1984) Zbl 0555.53002 MR 0772125

[12] Cervera, V., Mascaró, F., Michor, P. W.: The action of the diffeomorphism group on the space of immersions. Differential Geom. Appl. **1**, 391–401 (1991) Zbl 0783.58012 MR 1244452

[13] Chang, S.-Y. A., Yang, P. C.: Prescribing Gaussian curvature on S^2. Acta Math. **159**, 215–259 (1987) Zbl 0636.53053 MR 0908146

[14] Dolbeault, J., Esteban, M. J., Jankowiak, G.: The Moser–Trudinger–Onofri inequality. Chin. Ann. Math. Ser. B **36**, 777–802 (2015) Zbl 1347.46024 MR 3377875

[15] Ekholm, T.: Regular homotopy and total curvature. II. Sphere immersions into 3-space. Algebr. Geom. Topol. **6**, 493–512 (2006) Zbl 1113.53041 MR 2220686

[16] Francis, G. K., Morin, B.: Arnold Shapiro's eversion of the sphere. Math. Intelligencer **2**, 200–203 (1979) Zbl 0456.57008

[17] Francis, G., Sullivan, J. M., Hartman, C.: Computing sphere eversions. In: Mathematical Visualization (Berlin, 1997), Springer, Berlin, 237–255 (1998) Zbl 0931.68128 MR 1677675

[18] Francis, G., Sullivan, J. M., Kusner, R. B., Brakke, K. A., Hartman, C., Chappell, G.: The minimax sphere eversion. In: Visualization and Mathematics (Berlin-Dahlem, 1995), Springer, Berlin, 3–20 (1997) MR 1607221

[19] Ghoussoub, N., Lin C.-S.: On the best constant in the Moser–Onofri–Aubin inequality. Comm. Math. Phys. **298**, 869–878 (2010) Zbl 1211.58009 MR 2670931

[20] Grafakos, L.: Classical Fourier Analysis. 3rd ed., Grad. Texts in Math. 249, Springer, New York (2014) Zbl 1304.42001 MR 2445437

[21] Hélein, F.: Harmonic Maps, Conservation Laws, and Moving Frames. Cambridge Tracts in Math. 150, Cambridge Univ. Press (2002) Zbl 1010.58010 MR 1913803

[22] Hughes, J.: Another proof that every eversion of the sphere has a quadruple point. Amer. J. Math. **107**, 501–505 (1985) Zbl 0567.57023 MR 0784293

[23] Hummel, C.: Gromov's Compactness Theorem for Pseudo-Holomorphic Curves. Progr. Math. 151, Birkhäuser, Basel (1997) Zbl 0870.53002 MR 1451624

[24] Kuwert, E., Lamm, T., Li, Y.: Two-dimensional curvature functionals with super-quadratic growth. J. Eur. Math. Soc. **17**, 3081–3111 (2015) Zbl 1332.53011 MR 3429160

[25] Kuwert, E., Schätzle, R.: Removability of point singularities of Willmore surfaces. Ann. of Math. (2) **160**, 315–357 (2004) Zbl 1078.53007 MR 2119722

[26] Lamm, T., Nguyen, H.: The branched Willmore spheres. J. Reine Angew. Math. **701**, 169–194 (2015) Zbl 1314.53013 MR 3331730

[27] Langer, J.: A compactness theorem for surfaces with L^p-bounded second fundamental form. Math. Ann. **270**, 223–234 (1985) Zbl 0564.58010 MR 0771980

[28] Laurain, P., Rivière, T.: Angular energy quantization for linear elliptic systems with antisymmetric potentials and applications. Anal. PDE **7**, 1–41 (2014) Zbl 1295.35204 MR 3219498

[29] Laurain, P., Rivière, T.: Optimal estimate for the gradient of Green's function on degenerating surfaces and applications. Comm. Anal. Geom. **26**, 887–913 (2018) Zbl 1400.30049 MR 3853930

[30] Laurain, P., Rivière, T.: Willmore energy quantization at the boundary of the moduli space Duke Math. J. **167**, 2073–2124 (2018) Zbl 1401.53009 MR 3843372

[31] Marques, F. C., Neves, A.: min-max theory and the Willmore conjecture. Ann. of Math. (2) **179**, 683–782 (2014) Zbl 1297.49079 MR 3152944

[32] Michelat, A., Rivière, T.: A viscosity method for the min-max construction of closed geodesics. ESAIM Control Optim. Calc. Var. **22**, 1282–1324 (2016) Zbl 1353.49006 MR 3570503

[33] Michelat, A., Rivière, T.: The classification of branched Willmore spheres in the 3-sphere and the 4-sphere. arXiv:1706.01405 (2017)

[34] Mondino, A., Rivière, T.: Immersed spheres of finite total curvature into manifolds. Adv. Calc. Var. **7**, 493–538 (2014) Zbl 1306.53049 MR 3276119

[35] Mondino, A., Rivière, T.: A frame energy for immersed tori and applications to regular homotopy classes. J. Differential Geom. **104**, 143–186 (2016) Zbl 1356.53008 MR 3544288

[36] Morin, B., Petit, J.-P.: Problématique du retournement de la sphère. C. R. Acad. Sci. Paris Sér. A-B **287**, A767–A770 (1978) Zbl 0395.53025 MR 0516780

[37] Onofri, E.: On the positivity of the effective action in a theory of random surfaces. Comm. Math. Phys. **86**, 321–326 (1982) Zbl 0506.47031 MR 0677001

[38] Phillips, A.: Turning a sphere inside out. Sci. Amer. **214**, 112–120 (1966)

[39] Polyakov, A. M.: Quantum geometry of bosonic strings. Phys. Lett. B **103**, 207–210 (1981) MR 0623209

[40] Rivière, T.: Conformally invariant variational problems. In preparation

[41] Rivière, T.: Analysis aspects of Willmore surfaces. Invent. Math. **174**, 1–45 (2008) Zbl 1155.53031 MR 2430975

[42] Rivière, T.: Variational principles for immersed surfaces with L^2-bounded second fundamental form. J. Reine Angew. Math. **695**, 41–98 (2014) Zbl 1304.49095 MR 3276154

[43] Rivière, T.: Weak immersions of surfaces with L^2-bounded second fundamental form. In: Geometric Analysis, IAS/Park City Math. Ser. 22, Amer. Math. Soc., Providence, RI, 303–384 (2016) Zbl 1359.53002 MR 3524220

[44] Rivière, T.: A viscosity method in the min-max theory of minimal surfaces. Publ. Math. Inst. Hautes Études Sci. **126**, 177–246 (2017) Zbl 1387.53084 MR 3735867

[45] Simon, L.: Existence of surfaces minimizing the Willmore functional. Comm. Anal. Geom. **1**, 281–325 (1993) Zbl 0848.58012 MR 1243525

[46] Smale, S.: A classification of immersions of the two-sphere. Trans. Amer. Math. Soc. **90**, 281–290 (1958) Zbl 0089.18102 MR 0104227

[47] Smale, S.: Diffeomorphisms of the 2-sphere. Proc. Amer. Math. Soc. **10**, 621–626 (1959) Zbl 0118.39103 MR 0112149

[48] Struwe, M.: Positive solutions of critical semilinear elliptic equations on non-contractible planar domains. J. Eur. Math. Soc. **2**, 329–388 (2000) Zbl 0982.35041 MR 1796963

[49] Sullivan, J. M.: Sphere eversions: from Smale through "The Optiverse". In: Mathematics and Art: Mathematical Visualization in Art and Education (Maubeuge, 2000), C. P. Bruter (ed.), Springer, Berlin, 201–212 (2002)

[50] Tromba, A. J.: Teichmüller Theory in Riemannian Geometry. Lectures in Math. ETH Zürich, Birkhäuser, Basel (1992) Zbl 0785.53001 MR 1164870

[51] Willmore, T. J.: Note on embedded surfaces. An. Şti. Univ. "Al. I. Cuza" Iaşi Sect. I a Mat. (N.S.) **11B**, 493–496 (1965) Zbl 0171.20001 MR 0202066