J. Eur. Math. Soc. 23, 503–556 (2021) DOI 10.4171/JEMS/1016

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Renormalization of Feynman amplitudes on manifolds by spectral zeta regularization and blow-ups

Received December 18, 2017

Abstract. Our goal in this paper is to present a generalization of the spectral zeta regularization for general Feynman amplitudes on Riemannian manifolds. Our method uses complex powers of elliptic operators but involves several complex parameters in the spirit of *analytic renormalization* by Speer, to build mathematical foundations for the renormalization of perturbative interacting quantum field theories. Our main result shows that spectrally regularized Feynman amplitudes admit analytic continuation as meromorphic germs with linear poles in the sense of the works of Guo–Paycha and the second author. We also give an explicit determination of the affine hyperplanes supporting the poles. Our proof relies on suitable resolution of singularities of products of heat kernels to make them smooth.

As an application of the analytic continuation result, we use a universal projection from meromorphic germs with linear poles on holomorphic germs to construct renormalization maps which subtract singularities of Feynman amplitudes of Euclidean fields. Our renormalization maps are shown to satisfy consistency conditions previously introduced in the work of Nikolov–Todorov–Stora in the case of flat space-times.

Keywords. Renormalization in quantum field theory, zeta regularization

Contents

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Mathematics Subject Classification (2020): 81T20, 81T15, 58J50, 81T18, 11M36

1. Introduction

Zeta regularization. Let M be a smooth, compact, connected manifold without boundary and P be a symmetric, positive, elliptic pseudodifferential operator on M. Later on, we will specialize to Schrödinger operators of the form $P = -\Delta_g + V$ where $-\Delta_g$ is a Laplace operator and V is a smooth nonnegative potential. But the present discussion applies to any symmetric, positive, elliptic pseudodifferential operator P. Such a P admits a discrete spectral resolution [36, Lemma 1.6.3 p. 51], which means there is an increasing sequence of eigenvalues

$$\sigma(P) = \{0 \leq \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_n \to +\infty\}$$

and corresponding L^2-basis of eigenfunctions $(e_\lambda)_{\lambda\in\sigma(P)}$ so that $Pe_\lambda = \lambda e_\lambda$. In his seminal work [72], Seeley constructed the complex powers $(P^{-s})_{s\in\mathbb{C}}$ of P as a holomorphic family of continuous linear operators acting on suitable scales of Sobolev spaces on the manifold M. In particular for $\mathrm{Re}(s) \geq 0$, P^{-s} is bounded in $L^2(M)$. Now let us consider the *spectral zeta function* $\zeta_P(s)$ which is defined as the trace $\mathrm{TR}(P^{-s})$ coinciding with the series

$$\zeta_P(s) = \mathrm{TR}(P^{-s}) = \sum_{\lambda\in\sigma(P)\setminus\{0\}} \lambda^{-s}. \tag{1}$$

By Weyl's law on the growth of eigenvalues of P [36, Lemma 1.12.6 p. 113], the operator P^{-s} is trace class and the series $\zeta_P(s) = \sum_{\lambda>0} \lambda^{-s}$ converges as a holomorphic function of s on the half-plane $\mathrm{Re}(s) > \dim(M)/\deg(P)$. Then Seeley showed that $\zeta_P(s)$

admits an **analytic continuation** on the complex plane as a **meromorphic function** [36, Thm. 1.12.2 p. 108] with simple poles. In case P is a **differential operator**, $\zeta_P(s)$ is holomorphic at $s = 0$. This result shows one of the first instances of the power of zeta regularization, where we can regularize the divergent series $\sum_{\lambda\in\sigma(P)} 1$ and obtain the value $\zeta_P(0)$ of the spectral zeta function ζ_P at $s = 0$. More importantly, the residues of $\zeta_P(s)$ at its poles can be expressed as multiples of integrals over M of **local invariants** of the operator P [6, pp. 299–303] and are intimately related to the heat invariants of P [36, Thm. 1.12.2 p. 108].

From zeta regularization to regularized traces. In the same spirit, zeta regularization techniques were also used in global analysis to construct regularized traces for certain algebras of pseudodifferential operators. The above result of Seeley on the analytic continuation of $\mathrm{TR}(P^{-s})$ has been generalized to *canonical traces* on pseudodifferential operators by Kontsevich–Vishik [49], whose work was partly clarified by Lesch [53] and used to study anomalies of regularized zeta determinants. Then general types of tracial anomalies were discussed in [55, 18, 61], sometimes in relation to quantum field theory, and finally a general notion of trace for *holomorphic families of pseudodifferential operators* appears in the work of Paycha–Scott [62]. An important object underlying all these constructions is the notion of noncommutative residue for any pseudodifferential operator A. This residue can be defined by zeta regularization using complex powers of elliptic operators as follows. Choose any symmetric, positive, elliptic differential operator P. Then the noncommutative residue of A is defined as the residue at $s = 0$ of the meromorphic continuation of the trace $\mathrm{TR}(AP^{-s})$, and is given by a local formula in terms of the symbol of A. In his seminal works, Wodzicki [78, 79] proved that up to some constant factor, this residue is the unique trace on the algebra of pseudodifferential operators. It plays a central role in global analysis and noncommutative geometry. We refer the reader to the monographs [60, 71] for further details on these topics.

Zeta regularization for partition functions. Already in the simple case of spectral zeta functions of the Laplace–Beltrami operator, these regularization methods turn out to be extremely useful to study Euclidean quantum fields on Riemannian manifolds. In the mathematical physics literature, zeta regularization was first applied to quantum field theory on curved spaces by Hawking [43] to give a definition of the partition function of Euclidean QFT. It can also be used to give a mathematical model of the Casimir effect [31]. For topological quantum field theories, following the seminal work of Ray–Singer [66] on analytic torsion, it was soon realized by Schwarz [70] that one can define and calculate the partition function of some abelian BF theories using zeta regularized determinants. Formally, for some flat bundle (E, ∇) over a smooth compact manifold M of dimension d, his formula for the partition function of the BF theory reads

$$\int_{(A,B)\in\Omega^k(M,E)\times\Omega^{n-k-1}(M,E)} \exp\left(-\int_M B\wedge d^\nabla A\right) = \prod_{k=0}^{d} \det(\Delta^{(k)})^{(-1)^k k/2}$$

where d^∇ is the twisted differential acting on $\Omega^\bullet(M, E)$ and the right hand side is the Ray–Singer analytic torsion of the flat bundle $(E, \nabla) \to M$, which is a topological invariant [56, (10) p. 9]. Then Witten generalized the above work of Schwarz by showing that the perturbative partition function of Chern–Simons theory involves the Ray–Singer analytic torsion and also the eta invariant of Atiyah–Patodi–Singer. Since the formula is quite complicated, we refer the reader to [56, (12) p. 9]. But the important point is that the formula involves zeta regularized determinants. The main idea underlying the above results is that partition functions are **formally** expressed as functional integrals on some space of fields; these partition functions are then identified with regularized determinants of elliptic operators. For instance, in the case of the Dirichlet action functional $S(\varphi) = \frac{1}{2}\int_M \varphi(-\Delta_g)\varphi\, dv(x)$ where $-\Delta_g$ is the Laplace–Beltrami operator and dv the Riemannian volume, the partition function Z reads

$$Z = \int d\varphi \exp\left(-\frac{1}{2}\int_M \varphi(-\Delta_g)\varphi\, dv\right) = \det(-\Delta_g)^{-1/2}$$

where $\det(-\Delta_g)$ may be defined as $\exp(-\zeta'(0))$ where ζ is the regularized zeta function of the elliptic operator $-\Delta_g$ appearing in the definition of the partition function.

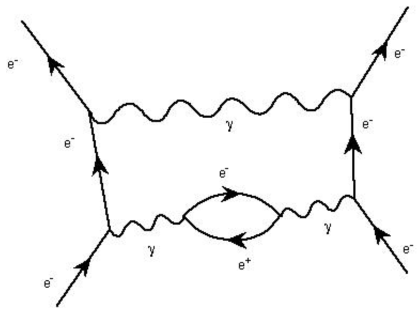
For applications in mathematical physics and in the present work, a particular role will be played by complex powers of generalized Laplacians (more generally elliptic, positive, self-adjoint operators of order 2) and their relation to the heat kernel asymptotics. These methods based on the local asymptotic expansion of the heat kernel are crucial in the local index theory [14] and are also used in [8, 4] to give a purely spectral definition of the Einstein–Hilbert action functional following [47, 1, 48].

Another interesting physical property of zeta regularization is its natural covariance, which is why it was used by Hawking in the first place. Indeed, for any diffeomorphism $\Phi : M \to M$, the spectrum $\sigma(-\Delta_{\Phi^*g})$ of $-\Delta_{\Phi^*g}$ on (M, Φ^*g) coincides with the spectrum of $-\Delta_g$ on (M, g) and $\sigma(-\Delta_g)$ is thus an **invariant of the Riemannian structure** (M, g).[1] Therefore zeta regularization is a coordinate independent regularization scheme which depends only on the spectral properties of the Laplacian, which in turn is entirely specified by the Riemannian structure (M, g).

Renormalization in quantum field theory. The present paper is written for analysts and does not require any background in physics or quantum field theory. We present our results in a purely mathematical form. However, we felt that for readers with some interest in QFT, it would be preferable to present some physical motivations, and the uninterested reader can skip the present paragraph. QFT is a general framework aimed at describing the fundamental forces and particles. In QFT, we are given some graphs called *Feynman graphs* which pictorially represent complicated interaction processes between various particles and we associate to every graph G some number c_G, called the *Feynman amplitude*, which is often given by some divergent integral when the graph G contains loops. The issue is that the above zeta regularization methods can only be used to renormalize one-loop graphs as discussed in [17, 1.4 p. 10]. For interacting QFT's, it is not enough to regularize only the partition function and one-loop graphs, one must renormalize amplitudes whose corresponding graphs contain an arbitrary number of loops. For

[1] The space of Riemannian structures is the set of pairs (M, g) quotiented by isometries.

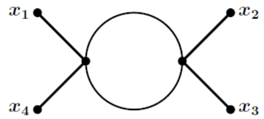
instance in quantum electrodynamics (QED), which is the QFT describing the interaction of light and matter, the computation of the probability amplitude of some scattering process for two incoming and two outgoing electrons is represented by the following Feynman diagram:



where the electrons are denoted by e^-, positrons by e^+ and photons by γ. The corresponding Feynman amplitude is given by some product of electron propagators, represented by the straight lines, and photon propagators represented by the wiggly lines. These propagators are distributions on $\mathbb{R}^4 \times \mathbb{R}^4$ valued in 4×4 matrices.

For the sake of simplicity, we limit ourselves to scalar theories in the present paper. In these theories, unlike in gauge theories, there is only one scalar valued propagator which is denoted by $\mathfrak{G}$ in what follows. The topology of the Feynman graphs that we encounter is dictated by the interactions of the theory. For instance in the massless ϕ^4 theory, the only Feynman graphs that we encounter have vertices of degree 4. Our goal is to use spectral zeta regularization to renormalize multiple loop amplitudes for Euclidean QFT on Riemannian manifolds with the aim to relate them to geometric invariants of Riemannian manifolds, which is the subject of future work of the authors. Our starting point is the work of Eugene Speer on analytic renormalization in QFT [73, 74, 75] who found an alternative formulation of the usual BPHZ renormalization algorithm, based on analytic regularization with several complex parameters. The analytic structure of the regularized amplitude in these variables encodes a rich algebraic structure so that a renormalized amplitude may be defined by the application of a universal projector, independent of the graph in question, to the regularized amplitude. Indeed, we will show that regularized amplitudes are **meromorphic germs with linear poles**, and in Subsection 6.3, we will describe a straightforward way of subtracting the divergent part of the regularized amplitudes while keeping only the holomorphic part. Then renormalization will be reformulated in Definition 6.10 as the evaluation at some poles of the holomorphic part of the regularized amplitude. This projection is a useful substitute to the BPHZ algorithm and the method pioneered by Connes–Kreimer based on Hopf algebras and Birkhoff factorizations. In our work, a common point with the BPHZ algorithm and Speer's work is that we rely on Hepp sectors and resolution of singularities arguments.

Let us show how the idea of *analytic renormalization* works in an example on flat space. On Euclidean space $\mathbb{R}^4$, the Green function of the Laplace operator reads $\mathfrak{G}(x, y) = CQ^{-1}(x-y)$ where C is some constant and Q is the quadratic form $Q(v) = \sum_{i=1}^4 v_i^2$. On configuration space $(\mathbb{R}^4)^6$, the Feynman rules assign to the graph



the amplitude

$$T(x_1, x_2, x_3, x_4) = \int_{(y_1,y_2)\in(\mathbb{R}^4)^2} \mathfrak{G}(x_1, y_1)\mathfrak{G}(x_2, y_1)\mathfrak{G}^2(y_1, y_2)\mathfrak{G}(y_2, x_3)\mathfrak{G}(y_2, x_4)\, d^4y_1\, d^4y_2,$$

which is given by some formal product of Green functions. To get rid of the infrared divergence due to the fact that we integrate over infinite volume $(\mathbb{R}^4)^2$, one may either introduce a sharp cut-off by replacing $\mathbb{R}^4$ by a finite box, or insert some smooth compactly supported cut-off function $g \in C_c^\infty(\mathbb{R}^4)$ for each variable $(y_i)_{i\in\{1,2\}}$ corresponding to the internal vertices of the Feynman graph, as follows:

$$T(x_1, x_2, x_3, x_4) = \int_{(\mathbb{R}^4)^2} \mathfrak{G}(x_1, y_1)\mathfrak{G}(x_2, y_1)\mathfrak{G}^2(y_1, y_2)\mathfrak{G}(y_2, x_3)\mathfrak{G}(y_2, x_4)g(y_1)g(y_2)\, d^4y_1\, d^4y_2.$$

In fact, it is natural to view the full amplitude $\mathfrak{G}(x_1, y_1)\mathfrak{G}(x_2, y_1)\mathfrak{G}^2(y_1, y_2)\mathfrak{G}(y_2, x_3) \times \mathfrak{G}(y_2, x_4)$ as a **distribution** in $\mathcal{D}'((\mathbb{R}^4)^6)$, so we may think that we insert some smooth compactly supported cut-off function $g(y_1)g(y_2)$ on $(\mathbb{R}^4)^2$ so that $T(x_1, x_2, x_3, x_4)$ is well-defined as the **push-forward** of the product $\mathfrak{G}(x_1, y_1)\mathfrak{G}(x_2, y_1)\mathfrak{G}^2(y_1, y_2)\mathfrak{G}(y_2, x_3) \times \mathfrak{G}(y_2, x_4)g(y_1)g(y_2)\, d^4y_1\, d^4y_2$ along the fibers of the projection $(\mathbb{R}^4)^6 \to (\mathbb{R}^4)^4$.

In terms of the quadratic function Q, the above amplitude reads

$$\int_{(y_1,y_2)\in(\mathbb{R}^4)^2} Q^{-1}(x_1, y_1)Q^{-1}(x_2, y_1)Q^{-1}(y_1, y_2)Q^{-1}(y_1, y_2) \times Q^{-1}(y_2, x_3)Q^{-1}(y_2, x_4)g(y_1)g(y_2)\, d^4y_1\, d^4y_2.$$

Now for each Q^{-1} factor in the amplitude, we shall introduce a complex power s as follows:

$$T(\mathbf{s}) = \int_{(y_1,y_2)\in(\mathbb{R}^4)^2} Q^{-s_1}(x_1, y_1)Q^{-s_2}(x_2, y_1)Q^{-s_3}(y_1, y_2)Q^{-s_4}(y_1, y_2) \times Q^{-s_5}(y_2, x_3)Q^{-s_6}(y_2, x_4)g(y_1)g(y_2)\, d^4y_1\, d^4y_2$$

where the new amplitude depends on $\mathbf{s} = (s_1, \dots, s_6) \in \mathbb{C}^6$. For $\mathrm{Re}(s_i)_{1\le i\le 6}$ large enough, one can easily see that the amplitude defining T is integrable. The main result of Speer is that $T(\mathbf{s})$ admits an analytic continuation in $\mathbf{s} \in \mathbb{C}^6$ as a meromorphic function with linear poles. Then he shows that $T(\mathbf{s})$ decomposes as the sum of a singular part and a holomorphic part at $\mathbf{s} = (1, \dots, 1) \in \mathbb{C}^6$ and renormalization consists in subtracting the singular part and evaluating at $(1, \dots, 1) \in \mathbb{C}^6$.

The main goal of the present paper is to combine the methods from zeta regularization with multiscale analysis of Feynman amplitudes, to present a generalization of analytic renormalization to general Riemannian manifolds. Then we will show that the renormalization defined satisfies the consistency axioms of Nikolov–Todorov–Stora in [59] inspired by the seminal works of Epstein–Glaser [30].

2. Main results

In the present section, we introduce the main objects of study and state the main results of our work. We first define Feynman amplitudes, next we explain how to implement zeta regularization with several complex parameters; then we state the first main analytic continuation theorem and finally we give a simplified version of our second main theorem concerning applications of the analytic continuation result to renormalization in QFT.

2.1. Feynman amplitudes

We work on a compact, connected Riemannian manifold (M, g) without boundary, the Laplace–Beltrami operator is denoted by Δ_g, and $C^\infty_{\geq 0}(M)$ denotes the smooth, nonnegative functions on M. For a potential $V \in C^\infty_{\geq 0}(M)$, it is well-known that the Schrödinger operator $P = -\Delta_g + V$ is a second order, symmetric, positive, elliptic differential operator which defines a unique unbounded, self-adjoint operator acting on $L^2(M)$ [76, pp. 34–35]. We now generalize the Feynman rules to this case. That is, to every graph we associate a formal product of Green kernels of the operator P. Since on a general manifold, there is no Fourier transform, our Feynman rules are just the Riemannian versions of the Euclidean Feynman rules in **position space** of [19, Definition 2.1] (see also [24]).

Definition 2.1 (Feynman rules). Let $\mathfrak{G}(x, y)$ denote the Green kernel of the operator P. Then for a graph G with the set of vertices $V(G)$ and the set of edges $E(G)$, if for any edge $e \in E(G)$, the vertices incident to e are $i(e)$ and $j(e)$ and G has no loops, then the *Feynman amplitude* associated to G is defined as

$$t_G = \prod_{e\in E(G)} \mathfrak{G}(x_{i(e)}, x_{j(e)}) \tag{2}$$

as a C^∞ function on $M^{V(G)} \setminus \{\text{all diagonals}\}$.

Remark 2.2. Since the Green kernel $\mathfrak{G}$ is symmetric in its variables, t_G is well-defined. The graphs are not allowed to have self-loops since the Green function $\mathfrak{G}$ is not well-defined on the diagonal, hence cannot be evaluated at coinciding points. The above Feynman rules correspond to a perturbative Euclidean QFT where the Lagrangian is already Wick renormalized, which explains why self-loops (also called tadpoles in the physics literature) are excluded. In the physics literature, the amplitude reads

$$t_G = \frac{1}{|\mathrm{Aut}(G)|} \prod_{e\in E(G)} \mathfrak{G}(x_{i(e)}, x_{j(e)})$$

where the combinatorial factor $|\mathrm{Aut}(G)| \in \mathbb{N}$ counts the number of automorphisms of G. We drop this combinatorial factor for simplicity since this does not affect our discussion.

2.2. *Multiple spectral zeta regularization*

The operator P^{-s} is defined as a spectral function of the operator P in a very simple way following [36, equation (1.12.13) p. 107]:

Definition 2.3 (Complex powers). For $\mathrm{Re}(s) \geq 0$, every $u \in L^2(M)$ can be decomposed in the orthonormal basis $(e_\lambda)_\lambda$ of $L^2(M)$ given by the eigenfunctions of P. Then

$$P^{-s}u = \sum_{\lambda \in \sigma(P)\setminus\{0\}} \lambda^{-s} \langle u, e_\lambda \rangle e_\lambda$$

where the sum on the right hand side converges absolutely in $L^2(M)$ since the eigenvalues λ_i tend to $+\infty$ as $i \to +\infty$, hence the sequence $(\lambda_i^{-s})_i$ remains bounded.

The Schwartz kernel of P^{-s} is then by definition

$$\mathfrak{G}^s(x, y) = \sum_{\lambda \in \sigma(P)\setminus\{0\}} \lambda^{-s} e_\lambda(x) e_\lambda(y) \tag{3}$$

where we abusively denote by $\mathfrak{G}^s(x, y)$ an actual *distribution* $\mathfrak{G}^s \in \mathcal{D}'(M \times M)$ and the series on the r.h.s. converges in $\mathcal{D}'(M \times M)$. We will later see that $\mathfrak{G}^s$ is actually a function on $M \times M$ for $\mathrm{Re}(s) > d/2$ where $d = \dim(M)$. We shall generalize this regularization to the case $M = \mathbb{R}^d$ with flat Euclidean metric g and $P = -\Delta_g + m^2$, $m \in \mathbb{R}_{\geq 0}$. Our definition of $\mathfrak{G}^s$ in the flat case is similar to the compact case since we define $\mathfrak{G}^s$ using complex powers of the Laplace operator:

Definition 2.4 (Complex powers for flat space). If $M = \mathbb{R}^d$, g is a constant quadratic form and $m \in \mathbb{R}_{\geq 0}$ is a mass, then we set

$$\begin{aligned}\mathfrak{G}^s(x, y) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{e^{i\langle (x-y), \xi \rangle}}{(g^{\mu\nu}\xi_\mu \xi_\nu + m^2)^s} d^d\xi \\ &= \frac{1}{\Gamma(s)} \int_0^\infty \frac{1}{(4\pi t)^{d/2}} e^{-\frac{\langle x-y, x-y \rangle_g}{4t}} e^{-tm^2} t^{s-1} \, dt.\end{aligned}$$

It is immediate from the above formulas that $\mathfrak{G}^s$ is the Schwartz kernel of $(-\Delta_g + m^2)^{-s}$ and that when $s = 1$, we recover the Green function of the operator $-\Delta_g + m^2$.

Definition 2.5 (Regularized Feynman rules). Under the above assumptions, we denote by P^{-s} the complex powers of P and by $\mathfrak{G}^s(x, y) \in \mathcal{D}'(M \times M)$ the corresponding Schwartz kernel. Then for a graph G with vertex set $V(G)$ and edge set $E(G)$, the *regularized Feynman amplitude* reads

$$t_G(s) = \prod_{e \in E(G)} \mathfrak{G}^{s_e}(x_{i(e)}, x_{j(e)}), \tag{4}$$

which is in $C^\infty(M^{V(G)} \setminus \{\text{all diagonals}\})$, where $s = (s_e)_{e \in E(G)} \in \mathbb{C}^{E(G)}$.

Remark 2.6. We will see later in Lemma 4.1 that $\mathfrak{G}^s$ is actually in $C^k(M \times M)$ for $\mathrm{Re}(s)$ large enough, hence the above Feynman rules also make sense for graphs G with self-loops when $\mathrm{Re}(s)$ is large enough, which was not true for $s = 1$ since $\mathfrak{G}$ would be a distribution singular on the diagonal.

Let us state our first main theorem:

Theorem 2.7. *Let (M, g) be a smooth, compact, connected Riemannian manifold without boundary of dimension d, $dv(x)$ the Riemannian volume and $P = -\Delta_g + V$, $V \in C^\infty_{\geq 0}(M)$ or $M = \mathbb{R}^d$ with a constant metric g and $P = -\Delta_g + m^2$, $m \in \mathbb{R}_{\geq 0}$. Then for every graph G, on the configuration space $(M^{V(G)}, g_{V(G)})$ endowed with the product metric $g_{V(G)}$ and product volume form $dv_{M^{V(G)}}$, for any test function $\varphi \in C^\infty(M^{V(G)})$,*

$$\mathbb{C}^{E(G)} \ni s \mapsto \int_{M^{V(G)}} t_G(s)\varphi \, dv_{M^{V(G)}} \tag{5}$$

*can be analytically continued near $(s_e = 1)_{e\in E(G)}$ as a **meromorphic germ with possible linear poles** on the hyperplanes of equation $\sum_{e\in G'} s_e - |E(G')| = 0$ where G' is a subgraph of G such that $2|E(G')| - b_1(G')d \leq 0$, $|E(G')|$ is the number of edges in G' and $b_1(G')$ the first Betti number of G'.*

To recover renormalized Feynman amplitudes, we follow the strategy of [25, 2.2]. We cannot evaluate $t_G(s)$ at $(s_e = 1)_{e\in E(G)}$ since it might belong to the polar set of t_G. The notion of polar set is defined in reference [39]. However, applying the machinery from [39] allows us to subtract the polar part of $t_G(s)$ at $(s_e = 1)_{e\in E(G)}$ while keeping the holomorphic part. This is based on an extension of the framework of [39] to distributions valued in meromorphic germs with linear poles, constructed in §6.2. Then to recover the renormalized Feynman amplitude, it suffices to evaluate the holomorphic part at $(s_e = 1)_{e\in E(G)}$. Following Speer [74, Section 3], analytic renormalization will be reformulated in Definition 6.10 as the evaluation at some poles of the holomorphic part of the regularized amplitude. This idea was recently abstracted in the works [20, 40] in a purely algebraic way where the composition of a projection on the holomorphic part and the evaluation at $(s_e = 1)_{e\in E(G)}$ is called an **evaluator** [40, 1.3 p. 6]. The renormalization $\mathcal{R}(t_G)$ of some amplitude t_G is the composition of the operations summarized in the following diagram:

$$t_G \xrightarrow{\text{regularization}} t_G(s) \xrightarrow{\text{projection on holomorphic part}} \pi(t_G(s))$$
$$\xrightarrow{\text{evaluation at } s=s_0} \mathbf{ev}|_{s_0}(\pi(t_G(s))) = \mathcal{R}(t_G),$$

where $s_0 = (s_e = 1)_{e\in E(G)}$.

In Section 6, we apply the above ideas to the renormalization of quantum field theories on Riemannian manifolds and show the existence of a collection $(\mathcal{R}_{M^I})_{I\subset\mathbb{N}}$ of renormalization maps that roughly assign to each graph G a renormalized amplitude in $\mathcal{D}'(M^{V(G)})$ such that the renormalization maps satisfy the consistency axioms 6.2 which come from the work of Nikolov–Todorov–Stora [59]. Let us state a simplified version of our second main Theorem 6.11:

Theorem 2.8. *Let (M, g) be a smooth, compact, connected Riemannian manifold without boundary of dimension d, $dv(x)$ the Riemannian volume and $P = -\Delta_g + V$, $V \in C^\infty_{\geq 0}(M)$ or $M = \mathbb{R}^d$ with a constant metric g and $P = -\Delta_g + m^2, m \in \mathbb{R}_{\geq 0}$. Then for every graph G, on the configuration space $M^{V(G)}$ endowed with the product volume form $dv_{M^{V(G)}}$:*

- *there exist distributions $\pi(t_G(s))$, $(1-\pi)(t_G(s))$ such that for any $\varphi \in C^\infty(M^{V(G)})$, we have a decomposition*

$$\int_{M^{V(G)}} t_G(s)\varphi\, dv_{M^{V(G)}} = \underbrace{\int_{M^{V(G)}} (1-\pi)(t_G(s))\varphi\, dv_{M^{V(G)}}}_{\text{meromorphic germ}} + \int_{M^{V(G)}} \pi(t_G(s))\varphi\, dv_{M^{V(G)}}$$

  where $s \mapsto \int_{M^{V(G)}} \pi(t_G(s))\varphi\, dv_{M^{V(G)}}$ is a holomorphic germ at $s_0 = (s_e = 1)_{e\in E(G)} \in \mathbb{C}^{|E(G)|}$.
- *If $\varphi \in C^\infty_c(M^{V(G)} \setminus \{\text{all diagonals}\})$ then*

$$\lim_{s\to s_0} \int_{M^{V(G)}} \pi(t_G(s))\varphi\, dv_{M^{V(G)}} = \int_{M^{V(G)}} t_G\varphi\, dv_{M^{V(G)}} \tag{6}$$

  which means $\lim_{s\to s_0} \pi(t_G(s))$ is a distributional extension of $t_G \in C^\infty(M^{V(G)} \setminus \{\text{all diagonals}\})$.

The reader is referred to Theorem 6.11 where we prove many important properties enjoyed by the renormalized amplitudes $\lim_{s\to s_0} \pi(t_G(s))$, the most important being a factorization equation appearing in Definition 6.2 which translates in mathematical terms the essential property of locality in Euclidean QFT.

Related works. In recent works of Hairer [42] and Pottel [64, 65], the authors give analytic treatments of the BPHZ algorithm. As in the present paper, they also start from Feynman amplitudes in **position space** but Hairer works on $\mathbb{R}^d$ with abstract kernels K with specific singularity along diagonals, whereas we work on Riemannian manifolds but we limit our discussion to Green kernels of Laplace type operators. He also uses Hepp sectors to perform some kind of multiscale analysis to analyze the divergences of the Feynman amplitudes. It would be interesting to compare the renormalization maps defined in the present paper with the valuations in Hairer's paper [42] and Definition 6.2 with the consistency axioms of [42].

Our treatment of renormalization bears a strong inspiration from the seminal work of Epstein–Glaser [30] who were among the first to understand the central role of causality (this is replaced in the current work by locality) in perturbative renormalization. Their work was generalized by Brunetti–Fredenhagen [16] to curved space-times while the crucial physical notions of covariance of the renormalization were addressed in the works of Hollands–Wald [45, 46]. A recent investigation of the Epstein–Glaser renormalization using resolution of singularities can be found in the thesis of Berghoff [12, 13] clarifying some previous attempts [11, 10]. Our results seem to be more general since we work in the manifold case and we resolve singularities by hand instead of using compactifications of configuration space of Fulton–McPherson and de Concini–Procesi.

There is a famous interpretation of BPHZ renormalization in terms of Hopf algebras pioneered by Connes–Kreimer [21, 22, 23]. This approach using dimensional regularization works essentially in momentum space and does not generalize in a straightforward way to curved spaces. Motivated by problems from number theory, Marcolli–Ceyhan [19] managed to reformulate the Hopf-algebraic approach on configuration space.

Other sources of inspiration for us are the recent works [29, 58], where renormalization is discussed from the point of view of distributional extensions in position space à la Epstein–Glaser, using also several complex parameters to perform some analytic continuation of the Feynman amplitude. In particular the paper [58] relates Epstein–Glaser renormalization to the analytic renormalization by Speer.

Renormalization in the Riemannian setting was recently discussed in the book by Costello [24], but it seems his proof of subtraction of counterterms contains some gaps that were fixed by Albert who also extended Costello's work to manifolds with boundary [2, 3].

Perspectives. A natural extension of our results would be to prove an analytic continuation result for Feynman amplitudes coming from Schwartz kernels of holomorphic families of pseudodifferential operators in the sense of Paycha–Scott [62] generalizing the Schwartz kernels of complex powers of Laplace operators. For the sake of simplicity, we limited ourselves to complex powers of Laplace operators because of their explicit relation to heat kernels and leave to another work the investigation of the more general case. Another interesting situation is when the manifold M is noncompact with specific asymptotic structure, as in scattering theory. Probably in this case, we would need to use resolvents to define complex powers.

It would also be very interesting to test our proof in the Lorentz case with the *Feynman propagator* instead of the Green function of the Laplacian. Then a natural question would be: what is the substitute in the Lorentz case for complex powers of Laplace operators? The first author defined complex regularization of Feynman propagators in a previous work [25] on **analytic Lorentzian space-times** under some very restrictive assumptions of **geodesic convexity**. This was based on the Hadamard parametrix for the Feynman propagator. From our point of view, it would be preferable to define a complex regularization scheme on **smooth** Lorentzian space-times which are **not necessarily geodesically convex**. The scheme should be manifestly covariant since it is spectral regularization on Riemannian manifolds. Probably, this could be based on the recent results of [9, 33, 34, 32, 26, 27, 77] on the analytic structure of Feynman propagators. It is a reasonable idea to replace the heat kernel asymptotic expansion by the Hadamard parametrix for the Feynman propagator as in [7].

Another interesting direction is to investigate if it is possible to renormalize the amplitudes in Euclidean theory and then perform a geometric Wick rotation as in Gérard–Wrochna [35] to build renormalized amplitudes of the corresponding Lorentzian QFT.

3. Preliminaries

The goal of the present section is to introduce the language of meromorphic germs with linear poles and give the main definitions, since meromorphic germs appear in the formulation of Theorem 2.7. We also introduce their distributional counterpart which we call meromorphic germs of distributions, which is the fundamental object needed for the proof of Theorem 2.7. Meromorphic germs of distributions are essentially distributions depending on some parameter $s \in \mathbb{C}^p$, $p \in \mathbb{N}$, which when paired with some test function φ, give meromorphic germs in s.

3.1. Meromorphic functions with linear poles

In this paper, all meromorphic functions of several variables $s = (s_1, \ldots, s_p) \in \mathbb{C}^p$ have singularities along unions of affine hyperplanes. In fact, we will work with **meromorphic germs with linear poles** in the terminology of [39]. We work in the space $\mathbb{R}^p$, and with the standard complex structure on $\mathbb{C}^p = \mathbb{R}^p \otimes \mathbb{C}$. Let $(\mathbb{R}^p)^*$ be the dual space. In what follows, holomorphic functions on a domain $\Omega \subset \mathbb{C}^p$ and holomorphic germs at $s_0 \in \mathbb{C}^p$ are denoted by $\mathcal{O}(\Omega)$ and $\mathcal{O}_{s_0}(\mathbb{C}^p)$ respectively.

Definition 3.1 (Meromorphic germs). Let $s_0 \in \mathbb{R}^p$. Then f is a *meromorphic germ with* (real) *linear poles* at s_0 if there are vectors $(L_i)_{1 \leq i \leq m}$ in $(\mathbb{R}^p)^*$ such that

$$\Big(\prod_{i=1}^{m} L_i(\cdot - s_0)\Big) f \in \mathcal{O}_{s_0}(\mathbb{C}^p). \tag{7}$$

The set of meromorphic germs with linear poles at $s_0 \in \mathbb{C}^p$ is denoted by $\mathcal{M}_{s_0}(\mathbb{C}^p)$.

Geometrically such a meromorphic germ f is singular along some arrangement of affine hyperplanes $\{s \in \mathbb{C}^p : L_i(s - s_0) = 0\}_{1 \leq i \leq m}$, intersecting at the point s_0.

3.2. Meromorphic germs of distributions

In this paper, we deal with families of distributions $t(s)$ on a smooth second countable manifold X without boundary, depending meromorphically on the parameter s and whose poles are linear. We will also call them distributions valued in meromorphic germs with linear poles and will denote the space of such families by $\mathcal{D}'(X, \mathcal{M})$. We devote this subsection to their proper definition. Our plan is to give the definition gradually, starting from holomorphic objects. For a smooth manifold X with given smooth density dv, we will use $\mathcal{D}'(X)$ to denote the space of distributions on X, defined in the present paper as the topological dual of $C_c^\infty(X) \otimes dv$, which is the space of smooth, compactly supported densities. But in many situations where the density is explicitly given by a geometric problem, we may equivalently think of the distributions as the dual of $C_c^\infty(X)$.

Holomorphic families of distributions. Before we discuss meromorphic germs of distributions, let us start smoothly by defining distributions depending holomorphically on some extra parameter.

Definition 3.2 (Holomorphic families). Let $\Omega \subset \mathbb{C}^p$ be a complex domain, and X be a smooth manifold. A *holomorphic family of distributions* on X parametrized by Ω is a family $(t(s))_{s\in\Omega}$ of distributions on X such that for every test function $\varphi \in C_c^\infty(X)$, $s \mapsto \langle t(s), \varphi\rangle$ is a holomorphic function on Ω. The set of such holomorphic families of distributions will be denoted by $\mathcal{D}'(X, \mathcal{O}(\Omega))$.

We next introduce a variant of the above definition involving distributions whose distributional order is bounded by some integer m.

Definition 3.3 (Holomorphic families with bounded order). Let m be a nonnegative integer and X a smooth manifold. Then a distribution t is *of order bounded above by m* on X if t defines a continuous linear function on $C_c^m(X)$. For a complex domain Ω, a *holomorphic family of distributions* $(t(s))_{s\in\Omega}$ *of order bounded above by m* on X is a family $(t(s))_{s\in\Omega}$ of distributions of order bounded above by m such that for every test function $\varphi \in C_c^m(X)$, $s \mapsto \langle t(s), \varphi\rangle$ is a holomorphic function on Ω. The set of such families is denoted by $\mathcal{D}'^{,m}(X, \mathcal{O}(\Omega))$.

Once we have defined holomorphic families of distributions where the complex parameter lives on some domain Ω containing some element s_0, it is natural to give a definition where we want to forget about Ω and localize around s_0. We thus work at the level of holomorphic germs near s_0.

Definition 3.4 (Holomorphic germs). A *holomorphic germ* at a point $s_0 \in \mathbb{C}^p$ of distributions on X is an equivalence class of holomorphic families of distributions on X with respect to the natural equivalence relation $(t(s))_{s\in\Omega_1} \sim (u(s))_{s\in\Omega_2}$ if there exists $\Omega_3 \subset \Omega_1 \cap \Omega_2$ such that $s_0 \in \Omega_3$ and $t(s) = u(s)$ for all $s \in \Omega_3$. The set of such germs is denoted by $\mathcal{D}'(X, \mathcal{O}_{s_0}(\mathbb{C}^p))$.

Example 3.5. The family of distributions $t(s) : C_c^\infty(\mathbb{R}) \ni \varphi \mapsto \int_{\mathbb{R}} e^{sx}\varphi(x)\, dx$ defines a holomorphic germ of distributions at $s = 0$ with real coefficients.

Meromorphic germs of distributions. Once we have a proper definition for holomorphic families of distributions, we can give a very natural definition of meromorphic families of distributions as follows:

Definition 3.6 (Meromorphic family of distributions with linear poles). For a complex domain $\Omega \subset \mathbb{C}^p$, a *meromorphic family of distributions* on Ω is a holomorphic family $(t(s))_{s\in\Omega\setminus\{s:\, L_1=\cdots=L_k(s)=0\}}$ of distributions, where $L_1, \dots, L_k$ are linear functions on $\mathbb{C}^p$, such that

$$t(s) = (L_1(s)\dots L_k(s))^{-1}h(s), \qquad \forall s \in \Omega \setminus \{s : L_1 = \cdots = L_k(s) = 0\}, \tag{8}$$

where $(h(s))_{s\in\Omega} \in \mathcal{D}'(X, \mathcal{O}(\Omega))$.

Now we localize the above definition to germs at $s_0 \in \mathbb{C}^p$:

Definition 3.7 (Meromorphic germs of distributions). A *meromorphic germ of distributions* at s_0 with linear poles is an equivalence class of meromorphic families of distributions on some neighborhood Ω of s_0 with linear poles under the equivalence relation $t(s)_{s\in\Omega_1\setminus Y_1} \sim t'(s)_{s\in\Omega_2\setminus Y_2}$, where $Y_i = \{s : L^i_1(s) = \cdots = L^i_{k_i}(s) = 0\}$, $i = 1, 2$, with linear functions L^i_j, $j = 1, \ldots, k_i$, $i = 1, 2$, if there exist a complex domain $\Omega_3 \subset \Omega_1 \cap \Omega_2$ and linear functions $L^3_1, \ldots, L^3_{k_3}$ such that $s_0 \in \Omega_3$, $s_0 \in Y_1 \cap Y_2$, $Y_3 = \{s : L^3_1(s) = \cdots = L^3_{k_3}(s) = 0\} \subset Y_1$, $Y_3 \subset Y_2$, and $t(s)_{s\in\Omega_3\setminus Y_3} = t'(s)_{s\in\Omega_3\setminus Y_3}$. The set of meromorphic germs of distributions with real coefficients will be denoted by $\mathcal{D}'(X, \mathcal{M}_{s_0}(\mathbb{C}^p))$.

It is simple to show

Proposition 3.8. *The set $\mathcal{D}'(X, \mathcal{O}_{s_0}(\mathbb{C}^p))$ is a vector subspace of $\mathcal{D}'(X, \mathcal{M}_{s_0}(\mathbb{C}^p))$.*

3.3. Power expansions of holomorphic germs

Let us state a convenient proposition about power series expansion of holomorphic families of distributions whose proof is given in the appendix.

Proposition 3.9. *Let X be a smooth manifold, $\Omega \subset \mathbb{C}^p$ and $(t(s))_{s\in\Omega} \in \mathcal{D}'(X, \mathcal{O}(\Omega))$ be a holomorphic family of distributions. Then near every $s_0 \in \Omega$, t_s admits a power series expansion*

$$t(s) = \sum_\alpha \frac{(s - s_0)^\alpha}{\alpha!} t_\alpha$$

where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ and t_α is a distribution in $\mathcal{D}'(X)$ such that for all test functions φ, $\sum_\alpha \frac{(s-s_0)^\alpha}{\alpha!} t_\alpha(\varphi)$ converges as a power series near s_0.

This classical result is just a multivariable version of [37, Theorem 1] which is stated for general locally convex spaces E; we include a proof in the appendix to make our text self-contained.

3.4. From Green functions to the heat kernel

The fundamental tool we use to investigate the singularities of Feynman amplitudes is the heat kernel. In this section, we recall its main properties and explain how one can express regularized Green functions and Feynman amplitudes in terms of the heat kernel.

3.4.1. Heat kernels. The complex powers of $P = -\Delta_g + V$ are related to the heat kernel e^{-tP} in the following way (see also [36, §1.12.14 p. 112]):

Proposition 3.10. *Let (M, g) be a smooth compact, connected Riemannian manifold without boundary and let $P = -\Delta_g + V$, $V \in C^\infty_{\geq 0}(M)$ or $M = \mathbb{R}^d$ with constant metric g and $P = -\Delta_g + m^2$, $m \in \mathbb{R}_{\geq 0}$. Set Π to be the spectral projector on* $\ker(P)$*, and $s \in \mathbb{C}$ with* $\mathrm{Re}(s) > 0$*. Then*

$$P^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty (e^{-tP} - \Pi) t^s \frac{dt}{t} \tag{9}$$

is a bounded operator $L^2(M,\mathbb{C}) \to L^2(M,\mathbb{C})$ *where* Γ *is the Euler Gamma function. In the sense of Schwartz kernels,*

$$\mathfrak{G}^s(x,y) = \frac{1}{\Gamma(s)} \int_0^\infty (K_t(x,y) - \Pi(x,y)) t^s \frac{dt}{t}$$

where $K_t(x,y) \in C^\infty((0,\infty) \times M \times M)$ *is the heat kernel.*

Note that when $0 \notin \sigma(P)$ and M is compact or when $M = \mathbb{R}^d$, we can set $\Pi = 0$.

Proof. The proposition is clear when $M = \mathbb{R}^d$, hence we just discuss the compact case. As a consequence of the compactness of M and the fact that P is an elliptic, positive, self-adjoint operator, P has discrete spectrum denoted by $\sigma(P)$, the eigenfunctions $(e_\lambda)_{\lambda\in\sigma(P)}$ of P form an orthonormal basis of $L^2(M,\mathbb{C})$, so for any $u \in L^2(M,\mathbb{C})$, $u = \sum \langle u, e_\lambda\rangle e_\lambda$. By definition, $P^{-s}u = \sum_{\lambda\in\sigma(P),\,\lambda\neq 0} \lambda^{-s}\langle u, e_\lambda\rangle e_\lambda$ where the sum converges absolutely in $L^2(M,\mathbb{C})$. And the spectral projector Π on $\ker(P)$ is simply $\Pi(u) = \sum_{\lambda=0}\langle u, e_\lambda\rangle e_\lambda$.

The heat operator e^{-tP} is a strongly continuous semigroup acting on $L^2(M,\mathbb{C})$. For every $u \in L^2(M,\mathbb{C})$,

$$(e^{-tP} - \Pi)u = \sum_{\lambda\in\sigma(P)\backslash 0} e^{-t\lambda}\langle u, e_\lambda\rangle e_\lambda$$

where the sum converges in $L^2(M,\mathbb{C})$.

Therefore for $\lambda > 0$, by a change of variable in the Γ function $\lambda^{-s} = \frac{1}{\Gamma(s)}\int_0^\infty e^{-t\lambda} t^s \frac{dt}{t}$, it follows that the identity $P^{-s} = \frac{1}{\Gamma(s)}\int_0^\infty (e^{-tP} - \Pi) t^s \frac{dt}{t}$ holds true in the operator sense where the integral converges in operator norm. Hence the same identity should hold true for the corresponding Schwartz kernels. □

3.5. Local asymptotic expansions of heat kernels

We will use the following property of the heat kernel asymptotics [14, Thm. 2.30] (see also [69, Thm. 7.15]):

Theorem 3.11 (Minakshisundaram–Pleijel). *Let* (M,g) *be a compact Riemannian manifold without boundary,* ε *the* ***injectivity radius*** *of* M *and* $P = -\Delta_g + V$, $V \in C^\infty_{\geq 0}(M)$. *Choose some cut-off function* $\psi : \mathbb{R}_+ \to [0,1]$ *such that* $\psi(s) = 1$ *if* $s \leq \varepsilon^2/4$ *and* $\psi(s) = 0$ *if* $s > 4\varepsilon^2/9$. *Let* $K_t(x,y) \in C^\infty((0,\infty) \times M \times M)$ *denote the heat kernel. Then there exist smooth* ***real valued*** *functions* $a_k \in C^\infty(M \times M)$, $k = 0,1,\ldots,$ *with* $a_0(x,y) = 1$, *such that for all* $(n,p) \in \mathbb{N}^2$, *and all differential operators* $Q(x,D_x)$ *of degree* m, *there exists a constant* $C > 0$ *such that for all* $t \in (0,1]$,

$$\sup_{(x,y)\in M^2} \left| Q(x,D_x)\partial_t^p \left(K_t(x,y) - \sum_{k=0}^n \psi(\mathbf{d}^2(x,y)) \frac{e^{-\frac{\mathbf{d}^2(x,y)}{4t}}}{(4\pi t)^{d/2}} a_k(x,y) t^k \right) \right| \leq C t^{n-d/2-m/2-p} \tag{10}$$

where $\mathbf{d}(\cdot,\cdot) : M \times M \to \mathbb{R}_{\geq 0}$ *is the Riemannian distance function.*

Note that our statement differs from the statement in [69] in that we use a cut-off function ψ since outside some neighborhood of the diagonal $\Delta \subset M \times M$, K_t vanishes to infinite order in t when $t \to 0$ (see [69, proof of Thm. 7.15 p. 102]). In case $M = \mathbb{R}^d$ with constant metric g and $P = -\Delta_g + m^2$, $m \in \mathbb{R}_{\geq 0}$, we have the well-known exact formula

$$K_t(x, y) = \frac{1}{(4\pi t)^{d/2}} \exp\left(-\frac{|x-y|_g^2}{4t} - tm^2\right) = \frac{\exp(-\frac{|x-y|_g^2}{4t})}{(4\pi t)^{d/2}} \sum_{k=0}^{\infty} \frac{(-1)^k t^k m^{2k}}{k!},$$

which already appeared in Definition 2.4.

4. Reduction of regularized Feynman amplitudes

Recall that our aim was to prove analytic continuation of the regularized amplitude

$$t_G = \prod_{e \in E(G)} \mathfrak{G}^{s_e}(x_{i(e)}, x_{j(e)})$$

in $\mathcal{D}'(M^{V(G)}, \mathcal{M}_{s_0}(\mathbb{C}^{E(G)}))$ where $s_0 = (s_e = 1)_{e \in E(G)}$ and $\mathfrak{G}^s$ is the Schwartz kernel of the complex power P^{-s}. The main goal of this section is to prove a technical Theorem 4.10 which allows us to reduce our main Theorem 2.7 to the proof of an analytic continuation theorem for simpler analytic objects. These are some kind of Feynman amplitudes introduced in Definition 4.8 corresponding to *labelled Feynman graphs* defined in Definition 4.7 which are graphs whose edges are decorated by some integer. Intuitively, the amplitude of the labelled graph is obtained from the regularized amplitude $t_G(s)$ where we replace the heat kernels appearing in the formula for the Green function $\mathfrak{G}^s = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}(e^{-tP} - \Pi)\, dt$ by the heat kernel asymptotic expansion. The integers decorating the edges exactly correspond to the heat coefficients in the heat kernel asymptotic expansion.

4.1. Holomorphicity of Green function

The next lemma discusses analytical properties of the regularized Green function of the Schrödinger operator P which is elliptic since its leading part coincides with the Laplace operator, it is therefore automatically self-adjoint by the symmetry assumption [76, p. 35].

Lemma 4.1. *Let (M, g) be a smooth compact, connected Riemannian manifold without boundary and let $P = -\Delta_g + V$, $V \in C^\infty_{\geq 0}(M)$ or $M = \mathbb{R}^d$ with constant metric g and $P = -\Delta_g + m^2$, $m \in \mathbb{R}_{\geq 0}$. Denote by K_t the corresponding heat kernel. Then*

1. *For all $k \in \mathbb{N}$, if* Re$(s) > d/2 + k$*, then $\mathfrak{G}^s$ is a C^k function on $M \times M$.*
2. *For all $k \in \mathbb{N}$ and any compact subset $K \subset M \times M \setminus$* Diagonal*, the kernel $\mathfrak{G}^s$ is holomorphic in s and valued in $C^k(K)$.*

3. *If we write*

$$\mathfrak{G}^s(x,y) = \mathfrak{G}^s_{\leq}(x,y) + \mathfrak{G}^s_{\geq}(x,y)$$

where

$$\mathfrak{G}^s_{\geq}(x,y) = \frac{1}{\Gamma(s)}\int_0^1 (K_t - \Pi)(x,y)t^{s-1}\,dt,$$
$$\mathfrak{G}^s_{\leq}(x,y) = \frac{1}{\Gamma(s)}\int_1^\infty (K_t - \Pi)(x,y)t^{s-1}\,dt$$

then $\mathfrak{G}^s_{\leq}(x,y)$ *is holomorphic in* s *and valued in* $C^\infty(M\times M)$*, which is denoted by* $\mathfrak{G}^s_{\leq} \in C^\infty(M\times M, \mathcal{O}(\mathbb{C}^p))$.

The proof of these classical properties, when M is compact, is recalled in the appendix. For $M = \mathbb{R}^d$, they follow from straightforward computations.

4.2. Reduction to local charts and localization near deepest diagonal

The purpose of the next two lemmas is to localize the proof of our main theorem about the analytic continuation of the distribution $t_G(s)$ to neighborhoods of the deepest diagonals in $M^{V(G)}$.

Lemma 4.2. *Let* X *be a manifold without boundary, and* $s_0 \in \mathbb{C}^p$*. Then* $t(s) \in \mathcal{D}'(X, \mathcal{M}_{s_0}(\mathbb{C}^p))$ *iff for every* $x \in N$*, there exists a neighborhood* U_x *of* x *such that* $t(s)|_{U_x} \in \mathcal{D}'(U_x, \mathcal{M}_{s_0}(\mathbb{C}^p))$.

If X *is noncompact, we require that there are linear functions* $(L_i)_{i=1}^k$ *corresponding to a fixed polar set* $Y = \{s : L_1(s) = \cdots = L_k(s) = 0\} \subset \mathbb{C}^p$ *with* $s_0 \in Y$ *such that* $t(s)|_{U_x} \in \mathcal{D}'(U_x, \mathcal{M}_{s_0}(\mathbb{C}^p))$ *is singular along the polar set* Y.

Proof. The direct implication is straightforward. Assume that for every $x \in X$, there exists a neighborhood U_x of x such that $t(s)|_{U_x} \in \mathcal{D}'(U_x, \mathcal{M}_{s_0}(\mathbb{C}^p))$. Then by local compactness, there is a locally finite subcover $(U_i)_i$ of N such that $t(s)|_{U_i} \in \mathcal{D}'(U_i, \mathcal{M}_{s_0}(\mathbb{C}^p))$. Let $(\chi_i)_i$ be a partition of unity where each χ_i is supported in U_i. Then for every test function $\varphi \in C^\infty_c(X)$, $\langle t(s), \varphi\rangle = \sum_i \langle t(s), \chi_i\varphi\rangle$ is a finite sum of meromorphic germs with linear poles at s_0. In the noncompact case, the polar set Y is fixed. Therefore the sum is a meromorphic germ with linear poles at s_0. This is a finite sum, hence $s \mapsto \langle t(s), \varphi\rangle$ is meromorphic with linear poles at s_0. □

The next lemma is inspired by the seminal work of Popineau and Stora [63] and it states that it is enough to solve our analytic continuation problem for the distributions $t_G \in \mathcal{D}'(M^{V(G)})$ near the deepest diagonals:

Lemma 4.3 (Popineau–Stora lemma). *If for any graph* G *and any* $x \in M$*, there is some neighborhood* U_x *of* x *such that* $t_G|_{U_x^{V(G)}} \in \mathcal{D}'(U_x^{V(G)}, \mathcal{M}_{s_0}(\mathbb{C}^p))$ *where* $s_0 = (s_e = 1)_{e\in E(G)}$*, then* $t_G \in \mathcal{D}'(M^{V(G)}, \mathcal{M}_{s_0}(\mathbb{C}^p))$*,* $s_0 = (s_e = 1)_{e\in E(G)}$*, for every graph* G.

Proof. We use induction on the number of vertices of G. For $|V(G)| = 2$, we have $t_G = \prod_{e=1}^{E} \mathfrak{G}^{s_e}(x, y)$. For a point $(x, y) \in M^2 = M^{V(G)}$, if $x \neq y$, consider neighborhoods V_x of x in M and V_y of y in M such that $V_x \cap V_y = \emptyset$. Then $\{V_x \times V_y\}_{(x,y)\in M^2,\, x\neq y} \cup \{U_x \times U_x\}_{x\in M}$ form an open cover of M^2; it has a locally finite subcover $\{W_i\}_i$ with $W_i = V_{x_i} \times V_{y_i}$ or $U_{x_i} \times U_{x_i}$. Let $\{\chi_i\}_i$ be a partition of unity subordinate to $\{W_i\}_i$. Then $t_G = \sum_i t_G\chi_i$, where each $t_G\chi_i$ is holomorphic at s_0 if the support of χ_i does not intersect the diagonal by Lemma 4.1 or has meromorphic continuation at s_0 by assumption. Now the claim follows from Lemma 4.2 applied to the manifold $X = M^{V(G)}$.

Now $|V(G)| = n > 2$ and assume the result holds for all graphs whose number of vertices is strictly less than n. Denote $d_n = \{x_1 = \cdots = x_n\} \subset M^n$, the *deepest diagonal* in the configuration space M^n. For $(x_1, \ldots, x_n) \in M^n \setminus d_n$, let $I = \{i : x_i = x_1\}$ and $I^c = \{1, \ldots, n\} \setminus I$. Then $I \neq \emptyset$, $I^c \neq \emptyset$, and for any $j \in I^c$ there are neighborhoods U_j of x_1 and V_j of x_j such that $U_j \cap V_j = \emptyset$. Let $\mathcal{V} = (\bigcap_{j\in I^c} U_j)^{|I|} \times \prod_{j\in I^c} V_j$. Then $\mathcal{V} \subset M^n \setminus d_n$, and $x_i \neq x_j$ for $(i, j) \in I \times I^c$ for all $(x_1, \ldots, x_n) \in \mathcal{V}$. Then we partition the set of edges $E(G)$ into three parts: $E(G) = E_I \cup E_{I^c} \cup E_{II^c}$, where E_I (resp. E_{I^c}) is the set of edges of G whose incident vertices are in I (resp. I^c), i.e. every edge $e \in E_I$ (resp. $e \in E_{I^c}$) is bounded by vertices in I (resp. I^c). The remaining subset of edges is denoted by E_{II^c} and is made up of all edges $e \in E(G)$ which are neither in E_I nor in E_{I^c}. Thus each edge in E_{II^c} connects a vertex in I with a vertex of I^c. So we write $(x_1, \ldots, x_n) = (x_i, x_j)_{i\in I, j\in I^c} \in M^I \times M^{I^c}$. Similarly the complex variables $(s_e)_{e\in E(G)}$ attached to the edges of G will be divided into three groups corresponding to E_I, E_{I^c} and E_{II^c}. Then we decompose the amplitude t_G as a product of three factors:

$$t_G((s_e)_{e\in E(G)}; (x_i, x_j)_{i\in I, j\in I^c}) = t_I((s_e)_{e\in E_I}; (x_i)_{i\in I}) t_{I^c}(((s_e)_{e\in E_{I^c}}; (x_j)_{j\in I^c}) \\ \times t_{II^c}((s_e)_{e\in E_{II^c}}; (x_i, x_j)_{i\in I, j\in I^c})$$

where $t_I = \prod_{e\in E_I} \mathfrak{G}^{s_e}$, $t_{I^c} = \prod_{e\in E_{I^c}} \mathfrak{G}^{s_e}$, $t_{II^c} = \prod_{e\in E_{II^c}} \mathfrak{G}^{s_e}$.

By the induction assumption, t_I and t_{I^c} are distributions in $\mathcal{D}'(M^I, \mathcal{M}_{s_{0I}}(\mathbb{C}^{E_I}))$, $s_{0I} = (s_e = 1)_{e\in E_I}$, and $\mathcal{D}'(M^{I^c}, \mathcal{M}_{s_{0I^c}}(\mathbb{C}^{E_{I^c}}))$, $s_{0I^c} = (s_e = 1)_{e\in E_{I^c}}$, respectively. Then by Lemma 7.1, the external product $t_I((s_e)_{e\in E_I}; (x_i)_{i\in I}) t_{I^c}(((s_e)_{e\in E_{I^c}}; (x_j)_{j\in I^c})$ of distributions depending on different variables is an element in $\mathcal{D}'(M^n, \mathcal{M}_{s_{0II^c}}(\mathbb{C}^{E_I\cup E_{I^c}}))$, $s_{0II^c} = (s_e = 1)_{e\in E_I\cup E_{I^c}}$. Now the factor t_{II^c} contains only products of propagators $\mathfrak{G}^{s_e}(x_i, x_j)$ where $x_i \neq x_j$, so in the open subset $\mathcal{V} \subset M^n$ we have $\mathfrak{G}^{s_e}(x_i, x_j) \in C^\infty(M \times M \setminus \text{Diagonal}, \mathcal{O}(\mathbb{C}))$ in the variables (x_i, x_j) by Lemma 4.1. Thus on $\mathcal{V}$, $t_{II^c} \in C^\infty(\mathcal{V}, \mathcal{O}(\mathbb{C}^{E_{II^c}}))$, which means t_{II^c} is holomorphic in the parameter $(s_e)_{e\in E_{II^c}} \in \mathbb{C}^{E_{II^c}}$. We conclude that near any element of M^n, there is some open neighborhood $U \subset M^n$ such that $t_G \in \mathcal{D}'(U, \mathcal{M}_{s_0}(\mathbb{C}^{E(G)}))$, $s_0 = (s_e = 1)_{e\in E(G)}$. Then by Lemma 4.2, $t_G \in \mathcal{D}'(M^n, \mathcal{M}_{s_0}(\mathbb{C}^{E(G)}))$, $s_0 = (s_e = 1)_{e\in E(G)}$. □

4.3. Reductions to integrals on cubes

In the representation of the Green function as integral of the heat kernel over $(0, \infty)$, we would like to get rid of the low energy part which is $\mathfrak{G}^s_{\leq} = \int_1^\infty dt\, (K_t - \Pi)t^{s-1}$, which is smooth and holomorphic in s so it does not contribute to the singularities of t_G. We thus reduce the study of t_G to the study of some function P_G which contains only integrals over cubes which are easier to handle and contain all the singularities of t_G.

Definition 4.4. For a graph G, and $E \subset E(G)$, the *subgraph induced* by E is the subgraph H of G such that $E(H) = E$ and $V(H) = \{v \in V(G) : v$ is a vertex incident to some $e \in E\}$.

Proposition 4.5. *If for every graph G, the product*

$$P_G(s) = \prod_{e\in E(G)} \frac{1}{\Gamma(s_e)} \int_0^1 (K_{\ell_e} - \Pi)(x_{i(e)}, x_{j(e)}) \ell_e^{s_e-1}\, d\ell_e \tag{11}$$

extends to an element of $\mathcal{D}'(M^{V(G)}, \mathcal{M}_{s_0}(\mathbb{C}^{E(G)}))$ *at* $s_0 = (s_e = 1)_{e\in E(G)}$, *then* $t_G(s)$ *extends to an element of* $\mathcal{D}'(M^{V(G)}, \mathcal{M}_{s_0}(\mathbb{C}^{E(G)}))$ *at* $s_0 = (s_e = 1)_{e\in E(G)}$.

Proof. For $\mathrm{Re}(s_e) > d/2$, since $\mathfrak{G}^s$, $\mathfrak{G}^s_{\geq}$, $\mathfrak{G}^s_{\leq}$ all belong to $C^0(M \times M)$ by Lemma 4.1, the following product makes sense:

$$\begin{aligned} t_G(s) &= \prod_{e\in E(G)} (\mathfrak{G}^{s_e}_{\leq} + \mathfrak{G}^{s_e}_{\geq}) \\ &= \prod_{e\in E(G)} \mathfrak{G}^{s_e}_{\geq} + \prod_{e\in E(G)} \mathfrak{G}^{s_e}_{\leq} + \sum_{E_1\cup E_2=E(G)} \Big(\prod_{e\in E_1} \mathfrak{G}^{s_e}_{\geq}\Big)\Big(\prod_{e\in E_2} \mathfrak{G}^{s_e}_{\leq}\Big) \end{aligned}$$

where the sum runs over all partitions $E(G) = E_1 \cup E_2$ with $E_1, E_2 \neq \emptyset$. Therefore

$$t_G(s) = P_G(s) + \underbrace{\prod_{e\in E(G)} \mathfrak{G}^{s_e}_{\leq}} + \sum_{E_1\cup E_2=E(G)} P_{G(E_1)}(s) \underbrace{\Big(\prod_{e\in E_2} \mathfrak{G}^{s_e}_{\leq}\Big)}$$

where $G(E_1)$ is the subgraph of G induced by the subset E_1. The terms underbraced are C^∞ functions depending holomorphically on the parameters $s \in \mathbb{C}^{E(G)}$ near $(s_e = 1)_{e\in E(G)}$ since each $\mathfrak{G}^s_{\leq}$ is in $C^\infty(M\times M, \mathcal{O}(\mathbb{C}))$. By assumption, for every induced subgraph $G(E_1) \subset G$, $P_{G(E_1)}$ extends to $\mathcal{D}'(M^{V(G(E_1))}, \mathcal{M}_{s_0}(\mathbb{C}^{E_1}))$, $s_0 = (s_e = 1)_{e\in E_1}$. Therefore by Lemma 7.3, each product $P_{G(E_1)}(s)(\prod_{e\in E_2} \mathfrak{G}^{s_e}_{\leq})$ has analytic continuation in $\mathcal{D}'(M^{V(G)}, \mathcal{M}_{s_0}(\mathbb{C}^{E(G)}))$, $s_0 = (s_e = 1)_{e\in E(G)}$. □

Therefore it is sufficient to study the product (11).

Lemma 4.6. *Let G be a graph with E edges and*

$$P_G(s) = \left(\prod_{e\in E(G)} \frac{1}{\Gamma(s_e)}\right) \int_{[0,1]^{E(G)}} \prod_{e\in E(G)} (K_{\ell_e} - \Pi)(x_{i(e)}, x_{j(e)}) \ell_e^{s_e-1}\, d\ell_e. \tag{12}$$

If $\mathrm{Re}(s_e) > d/2$ *for all e, then the integral defining* $P_G(s)$ *converges absolutely in* $[0, 1]^{E(G)}$ *uniformly in* $(x_1, \ldots, x_{|V(G)|}) \in M^{V(G)}$.

Proof. First if $M = \mathbb{R}^d$ or if M is compact and $0 \notin \ker(P)$ then $\Pi = 0$. Otherwise, if $0 \in \ker(P)$, then the Schwartz kernel of Π consists of constant functions (see Appendix 7.3). Therefore, it is sufficient that $\mathrm{Re}(s) > 0$ for the Riemann integral $\int_0^1 \Pi(x, y)\ell^{s-1}d\ell$ to be convergent. Now by Theorem 3.11, there exists a constant $C_0 > 0$ such that for $\ell \in (0, 1]$ and all $(x, y) \in M^2$,

$$\left| K_\ell(x, y) - \frac{1}{(4\pi\ell)^{d/2}}\psi(\mathbf{d}^2(x, y))e^{-\frac{\mathbf{d}^2(x,y)}{4\ell}} \sum_{0\leq k\leq d/2+1} a_k(x, y)\ell^k \right| \leq C_0.$$

So by the triangular inequality and by positivity of the heat kernel, we have the bound

$$0 \leq K_\ell(x, y) \leq \frac{1}{(4\pi\ell)^{d/2}}\psi(\mathbf{d}^2(x, y))e^{-\frac{\mathbf{d}^2(x,y)}{4\ell}} \sum_{0\leq k\leq d/2+1} |a_k(x, y)|\ell^k + C_0,$$

from which we can bound the integral

$$\begin{aligned}
&\int_0^1 K_\ell(x, y)|\ell^{s-1}|\, d\ell \\
&\leq \int_0^1 \left(\frac{1}{(4\pi)^{d/2}}\psi(\mathbf{d}^2(x, y))e^{-\frac{\mathbf{d}^2(x,y)}{4\ell}} \sum_{0\leq k\leq d/2+1} |a_k(x, y)\ell^{k+s-d/2-1}| + C_0|\ell^{s-1}| \right) d\ell \\
&\leq \int_0^1 \left(\frac{1}{(4\pi)^{d/2}} \sum_{0\leq k\leq d/2+1} |a_k(x, y)\ell^{k+s-d/2-1}| + C_0|\ell^{s-1}| \right) d\ell
\end{aligned}$$

since $\psi e^{-\frac{\mathbf{d}^2}{4\ell}} \leq 1$ and the right hand side is absolutely integrable when $\mathrm{Re}(s) > d/2$. Therefore

$$\int_{[0,1]^{E(G)}} \prod_{e\in E(G)} (K_{\ell_e} - \Pi)(x_{i(e)}, x_{j(e)})\ell_e^{s_e-1}\, d\ell_e = \prod_{e\in E(G)} \int_0^1 (K_{\ell_e} - \Pi)(x_{i(e)}, x_{j(e)})\ell_e^{s_e-1}\, d\ell_e$$

is a product of convergent Riemann integrals, the above integral inversions make sense by Fubini, which yields the claim of the lemma. □

Now we set

$$I_G(s) = \prod_{e\in E(G)} \frac{1}{\Gamma(s_e)} \int_0^1 K_{\ell_e}(x_{i(e)}, x_{j(e)})\ell^{s_e-1}\, d\ell_e, \tag{13}$$

which is well-defined as soon as $\mathrm{Re}(s_e) > d/2$ for all $e \in E(G)$ by the above arguments. Then

$$P_G(s) = \sum_{E\subset E(G)} I_{G(E)}(s) \prod_{e\in E(G)\setminus E} \frac{\Pi(x_{i(e)}, x_{j(e)})}{s_e} \tag{14}$$

where $G(E)$ is the subgraph induced by the subset of edges $E \subset E(G)$. By the fact that $\int_0^1 \Pi\ell^{s-1}d\ell = \Pi/s \in C^\infty(M \times M, \mathcal{O}_1(\mathbb{C}))$, which is holomorphic near $s = 1$, the products of spectral projectors do not contribute to the poles. So we can further reduce our study to the analytic continuation of $I_G(s)$.

4.4. Distributional order

In this step, we introduce a further reduction by replacing each K_ℓ in the integral formula of $I_G(s)$ by its heat asymptotic expansion and try to control the remainders. We have

$$\begin{aligned}\frac{1}{\Gamma(s)}\int_0^1 K_\ell \ell^{s-1} d\ell \\ = \frac{1}{\Gamma(s)}\int_0^1 \left(\frac{e^{-\frac{\mathbf{d}^2(x,y)}{4\ell}}}{(4\pi\ell)^{d/2}}\Big(\sum_{k=0}^{p} a_k(x,y)\psi(\mathbf{d}^2(x,y))\ell^k\Big) + h_p(\ell,x,y)\ell^{s-1}\right) d\ell\end{aligned}$$

where $h_p(\ell, x, y)$ is the remainder in the heat asymptotics which satisfies the estimate $\|h_p\|_m \leq C\ell^{p-d/2-m/2}$ by Theorem 3.11 and ψ is the cut-off function from Theorem 3.11.

We first introduce some refinement of Feynman graphs to keep track of the information on the heat coefficients for every edge. These are basically Feynman graphs whose edges are decorated by integers which correspond to heat coefficients.

Definition 4.7 (Labelled graph). For a set S, an *S-labelled graph* is a pair $(G, \vec{k})$ where $\vec{k}$ is a map $E(G) \to S$. If S is $\mathbb{N}$, we call it briefly a *labelled graph*, and for $e \in E(G)$, we use k_e to denote the element $\vec{k}(e) \in \mathbb{N}$. If $S = \mathbb{R}_{>0}$, then the map $E(G) \to \mathbb{R}_{>0}$, called the *length function*, is denoted by ℓ and we call the pair (G, ℓ) a *metric graph*. If ℓ is injective, then (G, ℓ) is called a *strict metric graph*.

We next define Feynman amplitudes attached to labelled graphs.

Definition 4.8. For every labelled graph $(G, \vec{k})$, we define the corresponding amplitude $I_{G,\vec{k}}(s)$ as follows:

$$I_{G,\vec{k}}(s) = \prod_{e\in E(G)} \frac{1}{\Gamma(s_e)} \int_{[0,1]^E} \prod_{e\in E(G)} \left(\frac{e^{-\frac{\mathbf{d}^2}{4\ell_e}}}{(4\pi)^{d/2}} a_{k_e}\psi(\mathbf{d}^2)\right)(x_{i(e)}, x_{j(e)})\ell_e^{k_e-d/2+s_e-1}\, d\ell_e, \tag{15}$$

which is well-defined and holomorphic in $(s_e)_{e\in E(G)} \in \mathbb{C}^{E(G)}$ on the domain $s_e > d/2$, $e \in E(G)$, by exactly the same proof as in Lemma 4.6.

Proposition 4.9. *If for every graph G, there is $m \in \mathbb{N}$ depending on G such that for all labels $\vec{k} \in \mathbb{N}^{E(G)}$ and all $x \in M$, there is an open neighborhood $U_x \subset M$ of x such that $I_{G,\vec{k}}(s)$ has analytic continuation to an element in $\mathcal{D}'^{,m}(U_x^{V(G)}, \mathcal{M}_{s_0}(\mathbb{C}^{E(G)}))$, $s_0 = (s_e = 1)_{e\in E(G)}$, then for all G, $t_G(s)$ extends to an element in $\mathcal{D}'(M^{V(G)}, \mathcal{M}_{s_0}(\mathbb{C}^{E(G)}))$, $s_0 = (s_e = 1)_{e\in E(G)}$.*

Proof. Let $n = |V(G)|$ and $U = U_x$. By Lemma 4.3 which allows us to localize our analytic continuation proof near the deepest diagonal of M^n, we only need to prove that $t_G(s)$ extends as a meromorphic germ of distributions at $(s_e = 1)_{e\in E(G)}$ on U^n. For a test function φ,

$$\begin{aligned}\langle I_G(s),\varphi\rangle &= \int_{U^n}\Big(\prod_{e\in E(G)}\frac{1}{\Gamma(s_e)}\int_0^1 K_{\ell_e}\ell_e^{s-1}\,d\ell_e\Big)\varphi\,dv(x_1)\dots dv(x_n)\\ &= \int_{U^n}\prod_{e\in E(G)}\Big(\sum_{k_e=0}^{p}\frac{1}{\Gamma(s_e)}\int_0^1\frac{e^{-\frac{\mathbf{d}^2}{4\ell_e}}}{(4\pi)^{d/2}}a_{k_e}\psi(\mathbf{d}^2)\ell_e^{k_e+s_e-d/2-1}\,d\ell_e\\ &\quad+\frac{1}{\Gamma(s_e)}\int_0^1 h_p\ell_e^{s_e-1}\,d\ell_e\Big)\varphi\,dv(x_1)\dots dv(x_n)\\ &= \int_{U^n}\Big(\sum_{E_1\cup E_2=E(G)}\prod_{e\in E_1}\Big(\sum_{k_e=0}^{p}\frac{1}{\Gamma(s_e)}\int_0^1\frac{e^{-\frac{\mathbf{d}^2}{4\ell_e}}}{(4\pi)^{d/2}}a_{k_e}\psi(\mathbf{d}^2)\ell_e^{k_e+s_e-d/2-1}\,d\ell_e\Big)\\ &\quad\times\prod_{e\in E_2}\Big(\frac{1}{\Gamma(s_e)}\int_0^1 h_p\ell_e^{s_e-1}\,d\ell_e\Big)\Big)\varphi\,dv(x_1)\dots dv(x_n)\end{aligned}$$

where the sum runs over partitions $E_1\cup E_2=E(G)$. Therefore we obtain

$$\begin{aligned}\langle I_G(s),\varphi\rangle = \int_{U^n}&\sum_{E_1\cup E_2=E(G)}\sum_{\vec{k}\in\{0,\dots,p\}^{E_1}} I_{G(E_1),\vec{k}}(s)\\ &\times\underbrace{\prod_{e\in E_2}\Big(\frac{1}{\Gamma(s_e)}\int_0^1 h_p(\ell_e,x_{i(e)},x_{j(e)})\ell_e^{s_e-1}\,d\ell_e\Big)}\varphi\,dv(x_1)\dots dv(x_n)\end{aligned}$$

where the summation is over all $\vec{k}\in\{0,1,\dots,p\}^{E(G)}$. By Theorem 3.11, we have

$$|D_x^m h_p(\ell_e,x,y)\ell_e^{s_e-1}|\le C\ell_e^{p-d/2-m/2+s_e-1},$$
$$|D_x^m h_p(\ell_e,x,y)\ell_e^{s_e-1}\log\ell_e|\le C\ell_e^{p-d/2-m/2+s_e-1+\varepsilon},$$

for some $\varepsilon>0$. So when $p>(d+m)/2-1$, for every $e\in E_2$, there is a small neighborhood of $s_e=1$ such that the integral $\int_0^1 h_p(\ell_e,x_{i(e)},x_{j(e)})\ell_e^{s_e-1}\,d\ell_e$ is absolutely convergent and depends holomorphically on s_e. Hence the term underbraced above belongs to $C^m(U^n,\mathcal{O}_{s_0}(\mathbb{C}^{E_2}))$, $s_0=(s_e=1)_{e\in E_2}$, where $G(E_2)$ is the graph induced by E_2. Now we conclude the proof by noticing that the product of $I_{G(E_1),\vec{k}}(s)\in\mathcal{D}'^{,m}(U^n,\mathcal{M}_{s_0}(\mathbb{C}^{E_1}))$, $s_0=(s_e=1)_{e\in E_1}$, and some element in $C^m(U^n,\mathcal{O}_{s_0}(\mathbb{C}^{E_2}))$, $s_0=(s_e=1)_{e\in E_2}$, yields an element of $\mathcal{D}'^{,m}(U^n,\mathcal{M}_{s_0}(\mathbb{C}^{E_1\cup E_2}))$, $s_0=(s_e=1)_{e\in E_1\cup E_2}$, by Lemma 7.3 proved in the appendix. □

The next theorem is the main result of the present section and summarizes all the reduction steps performed above:

Theorem 4.10 (Reduction theorem). *Assume that for every graph G, there is an integer $m(G)$ such that for any $x\in M$, there is a chart U_x of M around x such that for all $\vec{k}$, $I_{G,\vec{k}}(s)|_{U_x^n}$ admits an analytic continuation in $\mathcal{D}'^{,m(G)}(U_x^n,\mathcal{M}_{s_0}(\mathbb{C}^{E(G)}))$, $s_0=(s_e=1)_{e\in E(G)}$. Then for a given graph G and $m=\sup_{G'\subset G}m(G')$, for any $p>(d+m)/2-1$ we have a decomposition*

$$t_G(s)=\sum_{G'\subset G}\mathbf{m}_{G'}(s)h_{G\setminus G'}(s)\qquad(16)$$

where the sum runs over induced subgraphs G' of G, $\mathbf{m}_{G'}(s) = \sum_{\vec{k}\in\{0,\dots,p\}^{E(G')}} I_{G',\vec{k}}(s) \in \mathcal{D}'^{,m}(M^{V(G')}, \mathcal{M}_{s_0}(\mathbb{C}^{E(G')}))$ and $h_{G\setminus G'}(s) \in C^m(M^{V(G\setminus G')}, \mathcal{O}_{s_0}(\mathbb{C}^{E(G)\setminus E(G)'}))$. In particular, $t_G(s)$ extends to an element in $\mathcal{D}'(M^{V(G)}, \mathcal{M}_{s_0}(\mathbb{C}^{E(G)}))$, $s_0 = (s_e = 1)_{e\in E(G)}$.

The above theorem allows us to reduce the proof of Theorem 2.7 to the analytic continuation of the simpler objects $I_{G,\vec{k}}(s)$ if we can control the distributional order of $I_{G,\vec{k}}(s)$ independently of the label $\vec{k} \in \mathbb{N}^{E(G)}$.

Proof of Theorem 4.10. We use the following decomposition formula which summarizes the above three reduction steps, namely the reduction on cubes, the elimination of the spectral projector and the extraction of labelled graphs:

$$\begin{aligned} t_G(s)|_{U_x^n} &= \sum_{E_1\cup E_2\cup E_3\cup E_4=E(G)} \Big(\sum_{\vec{k}\in\{0,\dots,p\}^{E_1}} \underbrace{I_{G(E_1),\vec{k}}(s)} \Big) \prod_{e\in E_2} \left(\frac{1}{\Gamma(s_e)} \int_0^1 h_p(\ell_e, x, y) \ell_e^{s_e-1}\, d\ell_e \right) \\ &\quad \times \prod_{e\in E_3} \frac{\Pi(x_{i(e)}, y_{j(e)})}{s_e} \prod_{e\in E_4} \mathfrak{G}_{\leq}^{s_e} \end{aligned}$$

where the sum runs over partitions $E_1 \cup E_2 \cup E_3 \cup E_4 = E(G)$. Then consider the supremum $m = \sup_{E_1\subset E(G)} m(G(E_1))$ of the distributional orders $m(G(E_1))$ for $E_1 \subset E(G)$; m is finite by assumption and bounds the distributional order of all the terms $I_{G(E_1),\vec{k}}(s)$ underbraced. Moreover, we saw in the proof of Proposition 4.9 that if we choose $p > (d+m)/2 - 1$, then the product

$$\prod_{e\in E_2} \left(\frac{1}{\Gamma(s_e)} \int_0^1 h_p(\ell_e, x, y) \ell_e^{s_e-1}\, d\ell_e \right) \prod_{e\in E_3} \frac{\Pi(x_{i(e)}, y_{j(e)})}{s_e} \prod_{e\in E_4} \mathfrak{G}_{\leq}^{s_e}$$

is in $C^m(M^{V(G(E_2\cup E_3\cup E_4))}, \mathcal{O}_{s_0}(\mathbb{C}^{E_2\cup E_3\cup E_4}))$, $s_0 = (s_e = 1)_{e\in E_2\cup E_3\cup E_4}$. Therefore the whole product $t_G(s)$ is in $\mathcal{D}'(M^{V(G)}, \mathcal{M}_{s_0}(\mathbb{C}^{E(G)}))$, $s_0 = (s_e = 1)_{e\in E(G)}$. The above complicated formula can be written very concisely as

$$t_G(s) = \sum_{G'\subset G} \mathbf{m}_{G'}(s) h_{G\setminus G'}(s)$$

where the sum runs over induced subgraphs G' of G, $\mathbf{m}_{G'}(s) = \sum_{\vec{k}\in\{0,\dots,p\}^{E(G')}} I_{G',\vec{k}}(s)$ and $h_{G\setminus G'} = \prod_{e\in E_2} \big(\frac{1}{\Gamma(s_e)} \int_0^1 h_p(\ell_e, x, y) \ell_e^{s_e-1}\, d\ell_e\big) \prod_{e\in E_3} \frac{\Pi(x_{i(e)}, x_{j(e)})}{s_e} \prod_{e\in E_4} \mathfrak{G}_{\leq}^{s_e}$ where $E_2 \cup E_3 \cup E_4$ forms a partition of $E(G) \setminus E(G')$. □

5. Desingularization of parameter space

Now that we have reduced the proof of Theorem 2.7 to the proof of Theorem 4.10, we start by studying in local coordinates the amplitudes $I_{G,\vec{k}}(s) \in \mathcal{D}'(M^{V(G)})$ corresponding to the labelled graphs $(G, \vec{k})$.

Fixing charts. For any $x \in M$, take a coordinate chart U around x such that $U \cong \mathbb{R}^d$ and $\bar{U} \subset M$ is compact and $\mathbf{d}(y_1, y_2) < \varepsilon/2$ for any $y_1, y_2 \in U$. Since the volume form dv

on a Riemannian manifold reads $|\det(g)|d^dx$ in a local coordinate chart, we may absorb the smooth function $|\det(g)|$ in the test function φ and forget about the determinant of the metric. We number the vertices of G by $\{1,\dots,n\}$, and the edges by $\{1,\dots,E\}$. Let $i(e)$, $j(e)$ be the vertices of the edge e. Then for a test function φ with $\mathrm{supp}(\varphi)\subset U^n$,

$$\langle I_{G,\vec{k}}(s),\varphi\rangle = \frac{1}{(4\pi)^{dE/2}}\prod_{e=1}^{E}\frac{1}{\Gamma(s_e)}\int_{[0,1]^E} d\ell_1\dots d\ell_E \\ \left(\int_{U^n}\prod_{e=1}^{E}\exp\left(-\frac{\mathbf{d}^2}{4\ell_e}\right)\ell_e^{s_e+k_e-d/2-1}a_{k_e}\psi(\mathbf{d}^2)\tilde{\varphi}\, d^dx_1\dots d^dx_n\right)$$

where $\tilde{\varphi} = |\det(g)|\varphi$. This formula is well-defined when $\mathrm{Re}(s_e) > d/2$ for all $e\in\{1,\dots,E\}$ since the integration on $[0,1]^E$ is absolutely convergent, the integral on U^n converges absolutely by compactness of the support of φ, hence we can integrate by the Fubini theorem. Furthermore, arguing as in the proof of Proposition 4.9 shows that $\langle I_{G,\vec{k}}(s),\varphi\rangle$ is holomorphic in $s\in\mathbb{C}^E$ when $\mathrm{Re}(s_e) > d/2$.

By our choice of U, $\mathbf{d}^2$ is smooth on $U\times U$, so it is enough to prove that

$$\frac{1}{(4\pi)^{dE/2}}\int_{[0,1]^E} d\ell_1\dots d\ell_E \\ \left(\int_{\mathbb{R}^{dn}}\prod_{e=1}^{E}\exp\left(-\frac{\mathbf{d}^2}{4\ell_e}\right)\ell_e^{s_e+k_e-d/2-1}a_{k_e}\psi(\mathbf{d}^2)\tilde{\varphi}\, d^dx_1\dots d^dx_n\right)$$

extends to a meromorphic germ of distribution at $(s_e = 1)$. Note that this argument also applies to the case where $M=\mathbb{R}^d$ with constant metric g and $P=-\Delta_g+m^2$, $m\in\mathbb{R}_{\geq 0}$.

5.1. Smoothness problems and the need to resolve singularities

Assume we work on flat space $\mathbb{R}^d$. Then to study the analytic continuation of $I_{G,\vec{k}}$, we need to study integrals of the form

$$\int_{[0,1]^E} d\ell_1\dots d\ell_E \\ \left(\int_{\mathbb{R}^{dn}}\prod_{e=1}^{E}\exp\left(-\frac{|x_{i(e)}-x_{j(e)}|^2}{4\ell_e}\right)\ell_e^{s_e+k_e-d/2-1}a_{k_e}\psi(\mathbf{d}^2)\tilde{\varphi}\, d^dx_1\dots d^dx_n\right).$$

The analytic continuation would come from integration by parts on the cube $[0,1]^E$ in the variables $(\ell_1,\dots,\ell_E)$. However, we see immediately that $e^{-|x-y|^2/(4\ell)}$ is not a smooth function of $(\ell,x,y)\in[0,1]\times\mathbb{R}^d\times\mathbb{R}^d$. The problem occurs on the set $X=\{\ell=0, x-y=0\}\subset[0,1]\times\mathbb{R}^d\times\mathbb{R}^d$. A solution in global analysis is to consider the following smooth map:

$$\pi : [0,1]\times(\mathbb{R}^d)^2 \ni (t,x,h)\mapsto (t,x,x+\sqrt{t}\,h)\in[0,1]\times(\mathbb{R}^d)^2.$$

Note that after pull-back by π, we find that $e^{-|x-y|^2/(4\ell)}\circ\pi(x,t,h) = e^{-|h|^2/4}$, which is now a smooth function near the preimage $\pi^{-1}(X)=\{t=0\}$ in $[0,1]\times(\mathbb{R}^d)^2$. We say that we have resolved the singularities of $e^{-|x-y|^2/(4\ell)}$. For a discussion of why one needs to use blow-ups to study heat kernels and applications to index theory, the reader is referred to [54, p. 253]. Similarly, the product of exponentials $\prod_{e=1}^{E}\exp\big(-\frac{|x_{i(e)}-x_{j(e)}|^2}{4\ell_e}\big)$ appearing in Feynman amplitudes is not smooth on the whole domain of integration $(\ell_1,\ldots,\ell_E,x_1,\ldots,x_n)\in[0,1]^E\times\mathbb{R}^{dn}$ and integration by parts cannot be done. It follows that we must resolve the products $\prod_{e=1}^{E}\exp\big(-\frac{|x_{i(e)}-x_{j(e)}|^2}{4\ell_e}\big)$ to make them smooth, which is discussed in §5.4. Such a resolution of singularities was studied by Speer on flat space building on the work of Hepp. Also, when (M,g) is an analytic Riemannian manifold or when $M=\mathbb{R}^d$ with constant Euclidean metric, one can use Hironaka's resolution of singularities as in [5] or Bernstein–Sato polynomials to regularize Feynman amplitudes [25, 44]. However, on a Riemannian manifold (M,g), if for all $m\in M$, there is an open subset U containing m and a local coordinate system $(x_1,\ldots,x_d):M\supset U\rightarrow\mathbb{R}^d$ such that for every $(m_1,m_2)\in U^2$, $\mathbf{d}(m_1,m_2)=\sqrt{\sum_{i=1}^{d}(x_i(m_1)-x_i(m_2))^2}$ **then** (M,g) **is flat**. Otherwise for **generic** Riemannian manifolds (M,g), it is not possible to find good coordinates to make the distance function locally quadratic because of curvature. This makes our resolution of singularities more difficult to handle than the one appearing in the work of Speer, and the fact that we work in the C^∞ case and not in the analytic or algebraic category prevents us from using directly Hironaka's resolution of singularities or Bernstein–Sato polynomials. Following the tradition in QFT [41, 68], our strategy is essentially combinatorial and our blow-ups are encoded by spanning trees of Feynman graphs whose definition is recalled in the next subsection.

5.2. Spanning trees of metric graphs

Let us first collect some definitions and classical results on graphs which are close to [50, §2.1]. Recall that for all graphs we consider in the present paper, since we assume the graph has no self-loop, every edge e is adjacent to two different vertices.

Definition 5.1. Let G be a graph.

- A *path* from a vertex u to w in G is a sequence $(u=v_1,e_1,v_2,\ldots,v_n,e_n,v_{n+1}=w)$, where $v_i\in V(G)$, $e_j\in E(G)$ such that the vertices bounding e_i are v_i and v_{i+1}; u is the initial vertex of the path, w is its terminal vertex, and n is its length. A path is *simple* if all the edges are distinct. If $u=w$, it is called a *cycle*.
- The set of subgraphs is ordered as follows: for two subgraphs G_1,G_2 of G we write $G_1\subset G_2$ if $E(G_1)\subset E(G_2)$. A *forest* is a graph without any simple cycle and a *tree* is a connected forest.
- A *spanning forest* T of G is a subgraph of G which is a forest and is maximal for inclusion among subgraphs which are forests. If T is a tree, it is called a *spanning tree*. For any graph G, we will often use the following equivalent characterization of spanning forests, which is a classical result in graph theory [52, pp. 40–41]: A subgraph $T\subset G$ is a spanning forest if and only if T is a forest whose complement contains $b_1(G)$ edges:
$$b_1(G)=|E(G)|-|E(T)| \tag{17}$$
where $b_1(G)$ is the first Betti number of G.

- For a metric graph (G, ℓ), a *metric filtration* of G is an increasing family of subgraphs $G_1 \subset \cdots \subset G_E$ where G_i is *induced by the i shortest edges* where $E = |E(G)|$ is the number of edges in G. For a strict metric graph, the metric filtration is unique.
- For every forest $T \subset G$ and every subgraph G_i of G, we define $T|_{G_i}$ as the subgraph of G_i induced by the subset of edges $E(T) \cap E(G_i) \subset E(G_i)$. We will call $T|_{G_i}$ the *trace* of T in G_i.
- If T is a subgraph of G induced by $E(T)$, and $e \in E(G) \setminus E(T)$, then we define $T \cup e$ as the subgraph of G induced by $E(T) \cup e$. For every edge $e \in E(G)$, the graph $G \setminus e$ is the subgraph induced by $E(G) \setminus e$.

Definition 5.2. For any permutation $\sigma \in S_E$ of $\{1, \ldots, E\}$, the simplex

$$\Delta_\sigma = \{\ell_{\sigma(1)} < \cdots < \ell_{\sigma(E)}\} \tag{18}$$

is called a *sector* of $[0, 1]^E$.

Before we proceed, let us remark that for a graph G, an element $\ell \in [0, 1]^{E(G)}$, which is a map $\ell : E(G) \to [0, 1]$, naturally defines a metric graph (G, ℓ). To a strict metric graph G, the metric induces a natural strict ordering of edges by the length which defines an element $\ell \in [0, 1]^{E(G)}$ in a unique sector. The next theorem, due to Kruskal, aims to show how from a strict connected metric graph (G, ℓ), one can produce some algorithm which extracts a unique spanning tree T in G.

Theorem 5.3 (Kruskal). *For a connected strict metric graph (G, ℓ), let $G_1 \subset \cdots \subset G_E = G$ be the unique metric filtration of G. Then there exists a unique spanning tree T of G such that for all $i \in \{1, \ldots, E\}$, its trace $T|_{G_i}$ is a spanning forest of G_i.*

Proof. Let $\ell : E(G) \to (0, \infty)$ be the length function. We shall assume that the edges $E(G)$ are numbered as $\{e_1, \ldots, e_E\}$ in such a way that $i < j \Rightarrow \ell(e_i) < \ell(e_j)$. We construct the tree by the Kruskal algorithm [51] as described in [41, p. 107]. Notice that the requirement that T is a tree implies that T together with all traces $T|_{G_i}$ contain no simple cycles. Denote by N_i the first Betti number $b_1(G_i)$. So $T|_{G_i}$ is a forest and its complement in G_i contains at least N_i edges of G_i. Also notice that for any graph G and every $e \in E(G)$, we have the inequality $0 \le b_1(G) - b_1(G \setminus e) \le 1$. This implies that the sequence $N_1, N_2, \ldots$ is increasing.

Now we can construct the desired spanning tree T: start from G_1 which has only one edge $\{e_1\}$, hence contains no simple cycle, $N_1 = 0$. Let us denote by i_1 the first integer such that $b_1(G_{i_1}) = 1$, similarly define $\{i_2, \ldots, i_{N_E}\} \subset \{1, \ldots, E\}$ such that $b_1(G_{i_2}) = 2, \ldots, b_1(G_{i_{N_E}}) = N_E = b_1(G)$ and every i_j is the smallest integer such that $b_1(G_{i_j}) = j$ for any $j = 1, \ldots, N_E$. Set $i_0 = 1$. Then we have an increasing family of subgraphs $G_{i_0} \subset G_{i_1} \subset \cdots \subset G_{i_{N_E}} \subset G$. Let $G_i = G_{i-1} \cup e_i$ and set $T = G \setminus \bigcup_{j=1}^{N_E} e_{i_j}$. We prove that the subgraph T constructed above has the property that its trace $T|_{G_i}$ on every subgraph G_i is a spanning forest in G_i by induction for $j = 1, \ldots, E$. First, for $j = 1$, G_1 contains just one edge, hence $T|_{G_1} = G_1$ is a spanning tree in G_1. Assume that $T|_{G_k}$ is a spanning forest in G_k. Then there are two cases:

Case 1: $b_1(G_k) = b_1(G_{k+1})$, i.e. $N_k = N_{k+1}$ so $e_{k+1} \in T$, and $T|_{G_{k+1}} = T|_{G_k} \cup e_{k+1}$, and let us prove that $T|_{G_k} \cup e_{k+1}$ is a spanning forest in G_{k+1}. First $T|_{G_k} \cup e_{k+1}$ contains no simple cycle. Indeed, if it contained a simple cycle γ, then e_{k+1} would belong to γ and therefore $T|_{G_k}$ would be a spanning forest in G_{k+1}, so $b_1(G_{k+1}) = b_1(G_k) + 1$ by (17), which contradicts $b_1(G_k) = b_1(G_{k+1})$. Thus $T|_{G_k} \cup e_{k+1}$ is a forest. Since $T|_{G_k}$ is spanning in G_k it follows that $T|_{G_k} \cup e_{k+1}$ meets all vertices of G_{k+1} and is spanning in G_{k+1}.

Case 2: $b_1(G_k) + 1 = b_1(G_{k+1})$ and by definition $T|_{G_{k+1}} = T|_{G_k}$. Then $T|_{G_k}$ is obviously a forest in G_{k+1}, and its complement in G_{k+1} contains $b_1(G_k)+1 = b_1(G_{k+1})$ edges by construction, which implies it is spanning by (17).

Now we use induction to prove the uniqueness of T, in fact, we prove $T|_{G_k}$ is unique for any k. The base case is trivial, and in general there are two cases. Either $b_1(G_k) = b_1(G_{k+1})$, and then $T|_{G_{k+1}} = T|_{G_k} \cup e_{k+1}$, or $b_1(G_k) + 1 = b_1(G_{k+1})$, and then $T|_{G_{k+1}} = T|_{G_k}$, so our algorithm produces a unique spanning tree. □

Corollary 5.4. *Let (G, ℓ) be a strict metric graph and T be the unique spanning forest in T from Theorem 5.3. Then for every edge $e \in E(G) \setminus E(T)$, there is a unique simple cycle γ_e in $T \cup e$ such that $\ell(e) > \ell(e')$ for any edge $e' \in \gamma_e \setminus \{e\}$.*

Proof. Since T is a spanning tree and $e \in E(G) \setminus E(T)$, there is a unique simple cycle γ_e in $T \cup e$. By our construction, if $e = e_{i_j}$, then $T|_{G_{i_j-1}} = T|_{G_{i_j}}$, and $T|_{G_{i_j-1}} \cup e_{i_j}$ contains only one simple cycle γ_e, so $\ell(e) > \ell(e')$ for any edge $e' \in \gamma_e$, $e' \neq e$. □

5.3. *Approximation of the Riemannian distance in normal coordinates*

For a smooth Riemannian manifold (M, g), and any $x \in M$, $g(x)$ is an inner product in T_xM which induces an isomorphism $g(x) : T_xM \to T_x^*M$ and thus an inner product $g^{-1}(x)$ on T_x^*M by $g^{-1}(x)(w_1, w_2) = g(x)(g^{-1}(x)(w_1), g^{-1}(x)(w_2))$. This defines a smooth metric g^{-1} on T^*M.

For every $x \in M$, we will use normal coordinates (U, ϕ, x^μ) around x, and without loss of generality U will be assumed to be geodesically convex. The use of normal coordinates will be crucial since it allows us to approximate the squared distance $\mathbf{d}^2(x, y)$ by $|x - y|^2$ in local coordinates in Lemma 5.6. In some other coordinate chart, this approximation might not be as good. On (U, ϕ, x^μ), there are two metrics: the Riemannian metric g and the Euclidean metric h:

$$h\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right) = h_{\mu\nu} = \delta_{\mu\nu}.$$

For this Euclidean metric, we will use $|x - y|$ to denote the induced distance. We recall that at the origin, we have the identity $g_{\mu\nu}(0) = h_{\mu\nu} = \delta_{\mu\nu}$. The following lemma which dates back to Hadamard can be found in [28, Lemma 8.3 p. 90], [57, (A.3) p. 31], [67, (38) p. 171]:

Lemma 5.5 (Hadamard). *Denote by $\mathbf{d} : M \times M \to \mathbb{R}$ the Riemannian distance and $\phi = \mathbf{d}^2$. Then there exists a neighborhood U of the diagonal in $M \times M$ such that $\phi \in C^\infty(U)$ and ϕ is symmetric, that is, $\phi(x, y) = \phi(y, x)$, ϕ vanishes along the diagonal to order 2 and ϕ satisfies the Hamilton–Jacobi equation*

$$g^{-1}(d_x\phi(x, y), d_x\phi(x, y)) = 4\phi(x, y). \tag{19}$$

Next we state an important lemma which gives information on the jets of the function $\phi = \mathbf{d}^2$ along the diagonal in $M \times M$.

Lemma 5.6. *For $x_0 \in M$, if (U, ϕ) is a normal coordinate system around x_0 such that $\phi(x_0) = 0$ and the square of the Riemannian distance ϕ is smooth on $U \times U$, then on $U \times U$,*

$$\phi(x, y) - g_{\mu\nu}(x)(x^\mu - y^\mu)(x^\nu - y^\nu) \tag{20}$$

vanishes along the diagonal to order 3.

The proof can be found in Appendix 7.5.

5.4. *Resolving singularities using spanning trees*

Let (G, ℓ) be a connected strict metric graph with edge set $E(G)$ identified with the set of integers $\{1, \dots, E\}$ such that $0 \le \ell_1 < \cdots < \ell_E \le 1$. This means that the metric graph (G, ℓ) lies in a fixed sector $\Delta = \{0 < \ell_1 < \cdots < \ell_E < 1\} \subset [0, 1]^E$; let $\overline{\Delta}$ denote its closure $\{0 \le \ell_1 \le \cdots \le \ell_E \le 1\}$. It is associated with a unique spanning tree T by Theorem 5.3 and the vertices of both graphs G and T are numbered by $\{1, \dots, n\}$. For any $(i, j) \in \{1, \dots, n\}^2$, we denote by $\overrightarrow{ij}$ the unique simple path in T from i to j.

The product $\prod_{e \in E(G)} e^{-\frac{\mathbf{d}^2(x_{i(e)}, x_{j(e)})}{4\ell_e}} \psi(\mathbf{d}^2(x_{i(e)}, x_{j(e)}))$, where ψ is the cut-off function from Theorem 3.11, is not smooth near the algebraic set $X = \bigcup_{e \in E(G)} \{\ell_e(x_{i(e)} - x_{j(e)}) = 0\} \subset \mathbb{R}^{dn} \times \overline{\Delta}$. Next we give a recipe to resolve the singularities of such products by some explicit map π which is defined as follows:

Definition 5.7. In the above notation, define

$$\pi : \underbrace{(x, (h_e)_{e \in E(T)}, (t_k)_{k=1}^E)}_{\in \mathbb{R}^d \times (\mathbb{R}^d)^{E(T)} \times [0,1]^E} \mapsto \underbrace{\left(\left(x + \sum_{e \in \overrightarrow{1i}} \Big(\prod_{j \ge e} t_j \Big) h_e \right)_{i=1}^n, \left(\prod_{k \ge e} t_k^2 \right)_{e=1}^E \right)}_{\in \mathbb{R}^{dn} \times \overline{\Delta}} \tag{21}$$

where the sum runs over all edges e in the path $\overrightarrow{1i}$. The map π depends on the spanning tree T, hence on the strict ordering of $E(G)$ induced by the metric ℓ.

We shall denote elements of the target space $\mathbb{R}^{dn} \times \overline{\Delta}$ by $(x_i, \ell_e)_{1 \le i \le n, 1 \le e \le E}$. We check that the map $\pi : \mathbb{R}^d \times (\mathbb{R}^d)^{E(T)} \times [0, 1]^E \to \mathbb{R}^{dn} \times \overline{\Delta}$ is a diffeomorphism outside some subset of measure zero.

Proposition 5.8. *The map π is a smooth diffeomorphism from $\mathbb{R}^d \times \mathbb{R}^{d(n-1)} \times (0, 1)^E$ to $\mathbb{R}^{dn} \times \Delta$.*

Proof. It is one-to-one since we can explicitly invert π as $t_E = \ell_E^{1/2}$, $t_e = (\ell_{e+1}/\ell_e)^{1/2}$ for $e \le E - 1$ and the linear map

$$\mathbb{R}^d \times \mathbb{R}^{d(n-1)} \ni (x, (h_e)_{e \in E(T)}) \mapsto \left(x_i = x + \sum_{e \in \overrightarrow{1i}} \Big(\prod_{j \ge e} t_j \Big) h_e \right)_{i=1}^n \in \mathbb{R}^{dn}$$

is **invertible** when $(t_j)_j \in (0,1)^E$. Then the diffeomorphism property follows from an explicit calculation of the differential of π whose determinant does not vanish when $t \in (0,1)^E$. □

Finally, we may state the main theorem of this section:

Theorem 5.9 (Resolution of singularities). *Let g be a Riemannian metric on $\mathbb{R}^d$ and $\mathbf{d} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ be the Riemannian distance whose injectivity radius is ε. Let π be the map defined by (21). For any $\psi \in C_c^\infty(\mathbb{R})$ such that $\psi(t) = 1$ when $t \leq \varepsilon^2/4$ and $\psi(t) = 0$ when $t > 4\varepsilon^2/9$, for every edge $e \in E(G)$ with vertices $(i(e), j(e))$, the pull-back*

$$\pi^*\left(\psi(\mathbf{d}^2(x_{i(e)}, x_{j(e)}))\frac{\mathbf{d}^2(x_{i(e)}, x_{j(e)})}{\ell_e}\right) \tag{22}$$

defines a smooth function in $\mathbb{R}^d \times \mathbb{R}^{d(n-1)} \times [0,1]^E$.

Proof. For $e \in E(T)$, set $x_{j(e)} - x_{i(e)} = \pm h_e$. Recall that h_e has in fact d components $(h_e^\mu)_{\mu=1}^d$. Then by Lemma 5.6

$$\pi^*\left(\frac{\mathbf{d}^2(x_{i(e)}, x_{j(e)})}{\ell_e}\right) = \frac{g_{\mu\nu}(x_{i(e)})h_e^\mu h_e^\nu(\prod_{i\geq e} t_i)^2 + R(\pi^* x_{i(e)}, \pi^* x_{i(e)} \pm (\prod_{i\geq e} t_i)h_e)}{(\prod_{i\geq e} t_i)^2}$$
$$= g_{\mu\nu}(x_{i(e)})h_e^\mu h_e^\nu + r_e(t,x,h)$$

where

$$r_e(t,x,h) = \frac{R(\pi^* x_{i(e)}, \pi^* x_{i(e)} + (\prod_{i\geq e} t_i)h_e)}{(\prod_{i\geq e} t_i)^2} = \frac{O((\prod_{i\geq e} t_i)^3\|h_e\|^3)}{(\prod_{i\geq e} t_i)^2}$$
$$= O\left(\left(\prod_{i\geq e} t_i\right)\|h_e\|^3\right)$$

vanishes to order 3 in $(h_e)_{e\in T}$ and to order 1 in $(t_e)_{e=1}^E$ by Lemma 5.6 and r_e is smooth.

If $e \notin E(T)$, then

$$\pi^*\left(\frac{\mathbf{d}^2(x_{i(e)}, x_{j(e)})}{\ell_e}\right) = \pi^*\left(\frac{g_{\mu\nu}(x_{i(e)})(x_{i(e)}^\mu - x_{j(e)}^\mu)(x_{i(e)}^\nu - x_{j(e)}^\nu)}{\ell_e}\right)$$
$$+ \pi^*\left(\frac{R(x_{i(e)}, x_{j(e)})}{\ell_e}\right)$$

where

$$\pi^*\left(\frac{g_{\mu\nu}(x_{i(e)})(x_{i(e)}^\mu - x_{j(e)}^\mu)(x_{i(e)}^\nu - x_{j(e)}^\nu)}{\ell_e}\right)$$
$$= \frac{g_{\mu\nu}(x_{i(e)})(\sum_{e'\in\gamma_e\setminus e}\epsilon(e')(\prod_{j\geq e'} t_j)h_{e'}^\mu)(\sum_{e'\in\gamma_e\setminus e}\epsilon(e')(\prod_{j\geq e'} t_j)h_{e'}^\nu)}{(\prod_{i\geq e} t_i)^2}$$

where $\epsilon(e') = \pm 1$ and γ_e is the unique simple cycle in $T \cup e$ in Corollary 5.4. The important fact is that for every edge e' in the path $\gamma_e \setminus \{e\}$, we have $e' < e$. It follows that

$$\pi^*\Big(\frac{g_{\mu\nu}(x_{i(e)})(x^{\mu}_{i(e)}-x^{\mu}_{j(e)})(x^{\nu}_{i(e)}-x^{\nu}_{j(e)})}{\ell_e}\Big) \\ = g_{\mu\nu}(x_{i(e)})\Big(\sum_{e'\in\gamma_e\setminus e}\epsilon(e')\Big(\prod_{e'\leq j<e}t_j\Big)h^{\mu}_{e'}\Big)\Big(\sum_{e'\in\gamma_e\setminus e}\epsilon(e')\Big(\prod_{e'\leq j<e}t_j\Big)h^{\nu}_{e'}\Big),$$

which is smooth since the products $(\prod_{i\geq e}t_i)^2$ in the denominator cancel out with the same powers appearing in the numerator. The same argument applies to the remainder term $\pi^*(R(x_{i(e)},x_{j(e)})/\ell_e)$. □

5.5. *Change of variables*

For a test function φ supported in $\mathbb{R}^{dn}$, since the map π is a smooth diffeomorphism from $\mathbb{R}^d\times\mathbb{R}^{d(n-1)}\times(0,1)^E$ to $\mathbb{R}^{dn}\times\Delta$, we can take it as a change of variables for integration:

$$\langle I_{G,\vec{k}}(s),\varphi\rangle \\ = \frac{1}{(4\pi)^{dE/2}}\Big(\prod_{e=1}^{E}\frac{1}{\Gamma(s_e)}\Big)\int_{[0,1]^E}d\ell_1\dots d\ell_E\int_{\mathbb{R}^{dn}}\prod_{e=1}^{E}\exp\Big(-\frac{\mathbf{d}^2(x_{i(e)},x_{j(e)})}{4\ell_e}\Big) \\ \times\psi(\mathbf{d}^2(x_{i(e)},x_{j(e)}))\ell_e^{s_e+k_e-d/2-1}a_{k_e}(x_{i(e)},x_{j(e)})\tilde{\varphi}\,d^dx_1\dots d^dx_n \\ = \frac{1}{(4\pi)^{dE/2}}\Big(\prod_{e=1}^{E}\frac{1}{\Gamma(s_e)}\Big)\sum_{\sigma\in S_E}\int_{\Delta_\sigma}d\ell_1\dots d\ell_E\int_{\mathbb{R}^{dn}}\prod_{e=1}^{E}\exp\Big(-\frac{\mathbf{d}^2(x_{i(e)},x_{j(e)})}{4\ell_e}\Big) \\ \times\psi(\mathbf{d}^2(x_{i(e)},x_{j(e)}))\ell_e^{s_e+k_e-d/2-1}a_{k_e}(x_{i(e)},x_{j(e)})\tilde{\varphi}\,d^dx_1\dots d^dx_n$$

where σ runs over the group S_E of permutations of $\{1,\dots,E\}$. The open simplices Δ_σ do not cover $[0,1]^E$, but the complement of $\bigcup_\sigma\Delta_\sigma$ in $[0,1]^E$ has zero Lebesgue measure. Since for $\mathrm{Re}(s_e)_{e=1}^E$ large enough, the integral $\int_{[0,1]^E}\prod_{e\in E(G)}\frac{e^{-\frac{\mathbf{d}^2}{4\ell_e}}}{(4\pi)^{d/2}}a_{k_e}\psi(\mathbf{d}^2)\ell_e^{k_e-d/2+s_e-1}\,d\ell_e$ is absolutely convergent and depends holomorphically on $s\in\mathbb{C}^E$, we have the equality of integrals

$$\int_{[0,1]^E}\prod_{e\in E(G)}\frac{e^{-\frac{\mathbf{d}^2}{4\ell_e}}}{(4\pi)^{d/2}}a_{k_e}\psi(\mathbf{d}^2)\ell_e^{k_e-d/2+s_e-1}\,d\ell_e \\ = \sum_{\sigma\in S(E)}\int_{\Delta_\sigma}\prod_{e\in E(G)}\frac{e^{-\frac{\mathbf{d}^2}{4\ell_e}}}{(4\pi)^{d/2}}a_{k_e}\psi(\mathbf{d}^2)\ell_e^{k_e-d/2+s_e-1}\,d\ell_e,$$

where both sides depend holomorphically on $(s_e)_e$ for $\mathrm{Re}(s_e)_{e=1}^E$ large enough.

Now we can carry out the change of variables in a fixed sector $\Delta=\{0<\ell_1<\cdots<\ell_E<1\}$ (the other terms will be obtained by permutation), which yields an expression of the form

$$\int_\Delta \prod_{e=1}^E \frac{d\ell_e}{\ell_e}\Big(\int_{(\mathbb{R}^d)^n} \prod_{e=1}^E \exp\Big(-\frac{\mathbf{d}^2(x_{i(e)}, x_{j(e)})}{4\ell_e}\Big)\psi(\mathbf{d}^2(x_{i(e)}, x_{j(e)}))\ell_e^{s_e+k_e-d/2} \times a_{k_e}(x_{i(e)}, x_{j(e)})\tilde{\varphi}\, d^d x_1 \dots d^d x_n\Big)$$

$$= 2^E \int_{[0,1]^E} \prod_{e=1}^E \frac{dt_e}{t_e} \int_{(\mathbb{R}^d)^n} \pi^*\Big(\prod_{e=1}^E \exp\Big(-\frac{\mathbf{d}^2(x_{i(e)}, x_{j(e)})}{4\ell_e}\Big)\psi(\mathbf{d}^2(x_{i(e)}, x_{j(e)}))\Big) \times \pi^*\Big(\tilde{\varphi} \prod_{e\in E(G)} a_{k_e}\Big)\Big(\prod_{e\in E(G)} \ell_e^{(s_e+k_e)-d/2}\Big)\Big(\prod_{e\in E(T)} \ell_e^{d/2}\Big) d^d x \prod_{e\in E(T)} d^d h_e.$$

We can further simplify the product $\prod_{e\in E(G)} \ell_e(t)^{(s_e+k_e)-d/2} \prod_{e\in E(T)} \ell_e(t)^{d/2}$ as

$$\prod_{e\in E(G)} \Big(\prod_{i\geq e} t_i\Big)^{2(s_e+k_e)-d} \prod_{e\in E(T)} \Big(\prod_{i\geq e} t_i\Big)^{d} = \prod_{e\in E(T)} \Big(\prod_{i\geq e} t_i\Big)^{2s_e+2k_e} \prod_{e\notin E(T)} \Big(\prod_{i\geq e} t_i\Big)^{2s_e-d+2k_e}$$

$$= \prod_{e\in E(G)} \Big(\prod_{i\geq e} t_i\Big)^{2s_e+2k_e} \prod_{e\notin E(T)} \Big(\prod_{i\geq e} t_i\Big)^{-d}$$

$$= t_E^{2s_E+2k_E}(t_E t_{E-1})^{2s_{E-1}+2k_{E-1}} \dots (t_E \dots t_1)^{2s_1+2k_1}(t_E \dots t_{i_k})^{-d} \dots (t_E \dots t_{i_1})^{-d}$$

where $(i_1 < \dots < i_k) \subset \{1, \dots, E\}$ are the numbers decorating the edges in the complement of $E(T)$ and $k = b_1(G)$, hence

$$\prod_{e\in E(T)} \ell_e(t)^{2s_e+2k_e} \prod_{e\notin E(T)} \ell_e(t)^{2s_e-d+2k_e} = \prod_{e=1}^E t_e^{\sum_{i\leq e} 2s_i+2k_i-db_1(G_e)} \tag{23}$$

where G_e denotes the graph induced by the first e edges $\{1, \dots, e\}$. This in turns implies that we obtain the simplified form

$$\int_\Delta \prod_{e=1}^E \frac{d\ell_e}{\ell_e}\Big(\int_{(\mathbb{R}^d)^n} \prod_{e=1}^E \exp\Big(-\frac{\mathbf{d}^2(x_{i(e)}, x_{j(e)})}{4\ell_e}\Big)\psi(\mathbf{d}^2(x_{i(e)}, x_{j(e)}))\ell_e^{s_e+k_e-d/2} \times a_{k_e}(x_{i(e)}, x_{j(e)})\tilde{\varphi}\, d^d x_1 \dots d^d x_n\Big)$$

$$= \int_{[0,1]^E} \prod_{e=1}^E \frac{dt_e}{t_e}\, t_e^{(\sum_{i\leq e} 2s_i+2k_i)-db_1(G_e)} \int_{(\mathbb{R}^d)^n} A(t_e, x, h_e)\, d^d x \prod_{e\in E(T)} d^d h_e \tag{24}$$

where

$$A((t_e)_{e=1}^E, x, (h_e)_{e\in E(T)}) = 2^E \pi^*\Big(\Big(\prod_{e=1}^E \exp\Big(-\frac{\mathbf{d}^2(x_{i(e)}, x_{j(e)})}{4\ell_e}\Big)\psi(\mathbf{d}^2(x_{i(e)}, x_{j(e)}))a_{k_e}\Big)\tilde{\varphi}\Big).$$

The coordinates $((t_e)_{e=1}^E, x, (h_e)_{e\in E(T)})$ on $[0,1]^E \times \mathbb{R}^d \times (\mathbb{R}^d)^{E(T)}$ will be briefly denoted by (t, x, h) for simplicity. We now prove the smoothness in $t \in [0,1]^E$ of the partial integral $\int_{(\mathbb{R}^d)^n} A(t,x,h)\, d^dx \prod_{e\in E(T)} d^dh_e$, which is needed to ensure analytic continuation.

Lemma 5.10. *The map* $t \mapsto \int_{(\mathbb{R}^d)^n} A(t,x,h)\, d^dx \prod_{e\in E(T)} d^dh_e$ *belongs to* $C^\infty([0,1]^E)$.

Proof. The smoothness of A is a direct consequence of Theorem 5.9. We start from the definition of A:

$$A(t,x,h) = 2^E \pi^*\Big(\prod_{e\in E(T)} \exp\Big(-\frac{\mathbf{d}^2(x_{i(e)},x_{j(e)})}{4\ell_e}\Big)\psi(\mathbf{d}^2(x_{i(e)},x_{j(e)}))a_{k_e}\Big)$$
$$\times \pi^*\Big(\prod_{e\notin E(T)} \exp\Big(-\frac{\mathbf{d}^2(x_{i(e)},x_{j(e)})}{4\ell_e}\Big)\psi(\mathbf{d}^2(x_{i(e)},x_{j(e)}))a_{k_e}\Big)$$
$$\times \tilde{\varphi}\Big(x + \sum_{e\in\overrightarrow{11}}\Big(\prod_{j\geq e} t_j\Big)h_e, \dots, x + \sum_{e\in\overrightarrow{1n}}\Big(\prod_{j\geq e} t_j\Big)h_e\Big).$$

Then we use the key fact that since the open neighborhood ($\cong \mathbb{R}^d$) is chosen small enough and has compact closure, there exists a fixed constant $\delta > 0$ such that for all $x, y \in \mathbb{R}^d$, we have the following bound on the Riemannian distance:

$$\delta|x-y| \leq \mathbf{d}(x,y) = |x-y| + o(|x-y|) \leq \delta^{-1}|x-y|,$$

which follows from Lemma 5.6 since $\mathbf{d}^2(x,y) - \sum g_{\mu\nu}(x)(x^\mu - y^\mu)(x^\nu - y^\nu)$ vanishes along the diagonal to order 3, and locally $\sum g_{\mu\nu}(x)(x^\mu - y^\mu)(x^\nu - y^\nu)$ is bounded by some multiple of $|x-y|$ by compactness of the neighborhood. It follows that

$$\frac{\delta^2|x-y|^2}{4\ell_e} \leq \frac{\mathbf{d}^2(x,y)}{4\ell_e} \leq \frac{\delta^{-2}|x-y|^2}{4\ell_e},$$

which implies that for all edges $e \in E(T)$, we have the bound

$$\pi^*\exp\Big(-\frac{\mathbf{d}^2(x_{i(e)},x_{j(e)})}{4\ell_e}\Big) \leq \pi^*\exp\Big(-\frac{\delta^2|x-y|^2}{4\ell_e}\Big) = \exp\Big(-\frac{\delta^2|h_e|^2}{4}\Big).$$

This allows us to use the product $\pi^*(\prod_{e\in E(T)}\exp(-\frac{\mathbf{d}^2(x_{i(e)},x_{j(e)})}{4\ell_e}))$ to control the exponential decay of A:

$$\pi^*\Big(\prod_{e\in E(T)} \exp\Big(-\frac{\mathbf{d}^2(x_{i(e)},x_{j(e)})}{4\ell_e}\Big)\Big) \leq \prod_{e\in E(T)} \exp\Big(-\frac{\delta^2|h_e|^2}{4}\Big).$$

By smoothness of the heat coefficients a_k, by compactness of the support of $\varphi \in C_c^\infty(U^n)$ and thus of $\tilde{\varphi}$, and by the definition of π, for every multi-index α, there exists some constant $C_\alpha > 0$ such that

$$\Big|\partial_t^\alpha \pi^*\Big(\tilde{\varphi} \prod_{e\notin E(T)} \exp\Big(-\frac{\mathbf{d}^2(x_{i(e)},x_{j(e)})}{4\ell_e}\Big)\prod_{e=1}^{E} a_{k_e}\psi(\mathbf{d}^2(x_{i(e)},x_{j(e)}))\Big)\Big|$$
$$\leq C_\alpha\Big(1 + \sum_{e\in E(T)} |h_e|\Big)^{|\alpha|},$$

where the partial derivatives in t contribute the powers of h.

Therefore, since A is compactly supported in x, for all (t, x, h) we have the bound

$$|\partial_t^\alpha A(t_e, x, h_e)| \leq C_\alpha \prod_{e\in E(T)} \exp\left(-\frac{\delta^2|h_e|^2}{4}\right)\left(1+\sum_{e\in E(T)}|h_e|\right)^{|\alpha|}.$$

Then smoothness of $t \mapsto \int_{(\mathbb{R}^d)^n} A(t, x, h)\, d^d x \prod_{e\in E(T)} d^d h_e$ follows from smoothness of the integrand which is the function $A \in C^\infty([0, 1]^E \times (\mathbb{R}^d)^n)$, and any derivative $\partial_t^\alpha A$ has fast decrease in h when $|h| \to +\infty$ and compact support in $x \in \mathbb{R}^d$. Therefore all derivatives in $\partial_t^\alpha A$ are integrable, uniformly in $t \in [0, 1]^E$, and the conclusion follows from classical results on integrals depending smoothly on parameters. □

Lemma 5.11 (Jet lemma). *Fix the sector Δ corresponding to the system of inequalities $\{0 \leq \ell_1 \leq \cdots \leq \ell_E\}$. Let*

$$A((t_e)_{e=1}^E, x, (h_e)_{e\in T}) = \pi^*\left(\prod_{e=1}^E\left(\exp\left(-\frac{\mathbf{d}^2(x_{i(e)}, x_{j(e)})}{4\ell_e}\right)\psi(\mathbf{d}^2(x_{i(e)}, x_{j(e)}))a_{k_e}\right)\tilde{\varphi}\right).$$

Then the k-jet of

$$\chi(t_1, \ldots, t_E) = \int_{(\mathbb{R}^d)^n} A((t_e)_{e=1}^E, x, (h_e)_{e\in E(T)})\, d^d x \prod_{e\in E(T)} d^d h_e$$

depends continuously on the k-jet of $(a_{k_i}, i = 1, \ldots, E, \varphi, \mathbf{d}^2, g)$.

Proof. This follows from the formulas defining A and the change of variables π and repeated application of the chain rule to the $\pi^*(\ldots)$ term. □

Recall that by Theorem 4.10, the main Theorem 2.7 reduces to an analytic continuation result for the amplitudes $I_{G,\vec{k}}(s)$ corresponding to the labelled Feynman graphs $(G, \vec{k})$. The problem was that the integral formula for $I_{G,\vec{k}}(s)$ involved some product of heat kernels which required blow-ups performed in sectors. The following proposition shows how the integral expression $I_{G,\vec{k}}(s)$ simplifies after blow-up:

Proposition 5.12. *Let $I_{G,\vec{k}}(s)$ be as in (15), and for any $e \in \{1, \ldots, E\} \simeq E(G)$ and any permutation $\sigma \in S_E$, let $G_{\sigma(e)}$ be the subgraph of G induced by the collection of edges $\{\sigma(1), \ldots, \sigma(e)\} \subset E(G)$. Then for every $\varphi \in C_c^\infty(U^n)$, there exists a family $\chi_\sigma \in C^\infty([0, 1]^E)$ indexed by $\sigma \in S_E$ such that*

$$\int_{\mathbb{R}^{dn}} I_{G,\vec{k}}(s)\varphi\, d^{nd}x = \prod_{e=1}^E \frac{1}{\Gamma(s_e)} \sum_{\sigma\in S_E} \int_{[0,1]^E} \prod_{e=1}^E \frac{dt_e}{t_e}\, t_e^{\sum_{i\leq e}(2s_{\sigma(i)}+2k_{\sigma(i)})-db_1(G_{\sigma(e)})} \chi_\sigma(t) \qquad (25)$$

for $\operatorname{Re}(s_e)$, *$e \in \{1, \ldots, E\}$, large enough, and both sides are holomorphic in $s \in \mathbb{C}^E$.*

In the next subsection, we proceed to the meromorphic continuation of the r.h.s. of (25) as a meromorphic function with linear poles in s and we also bound the distributional order of $I_{G,\vec{k}}(s)$ independently of the label $\vec{k}$.

5.6. Integration by parts, bounding orders and pole decomposition

Now the proof of Theorem 2.7 on the analytic continuation of $t_G(s)$ is reduced to the meromorphic continuation in $s \in \mathbb{C}^E$ of the right hand side of (25). The meromorphic continuation comes from integration by parts, as shown in the next lemma and its corollary.

Lemma 5.13. *Let E be a positive integer. Then for any smooth function ψ on $[0, 1]$,*

$$I_s(\psi) = \int_{[0,1]^E} t_1^{s_1} \dots t_E^{s_E} \psi(t_1, \dots, t_E)\, d^E t$$

can be analytically extended to a meromorphic germ at $(s_e = p_e)_{e=1}^E \in \mathbb{Z}^E$; more precisely, if $I = \{i : p_i < 0\}$ then

$$\Big(\prod_{i\in I}(s_i - p_i)\Big) I_s(\psi) \tag{26}$$

extends to a holomorphic germ at $(s_e = p_e)_e$ and $I_s \in \mathcal{D}'^{,m}([0, 1]^E, \mathcal{M}_{s_0}(\mathbb{C}^E))$, $s_0 = (p_1, \dots, p_E)$, $m = \sum_{i\in I} |p_i|$.

The proof of this lemma, given in the appendix, follows from integration by parts. One consequence of this lemma is

Corollary 5.14. *Denote by $1_{[0,1]^E}$ the indicator function of the unit cube $[0, 1]^E \subset \mathbb{R}^E$. Let $(L_1, \dots, L_E)$ be linear functions of $s \in \mathbb{C}^E$ with coefficients $L_i \in (\mathbb{R}^E)^*$, $1 \le i \le E$, and let $(a_1, \dots, a_E) \in \mathbb{Z}^E$. Set $I = \{i : a_i < 0\}$. Then*

$$\Big(\prod_{i\in I} L_i(s)\Big) \int_{[0,1]^E} t_1^{L_1(s)+a_1} \dots t_E^{L_E(s)+a_E} \psi(t_1, \dots, t_E)\, d^E t$$

is a holomorphic germ at $s = 0 \in \mathbb{C}^E$, and

$$1_{[0,1]^E} t_1^{L_1(s)+a_1} \dots t_E^{L_E(s)+a_E}$$

extends to an element in $\mathcal{D}'^{,m}(\mathbb{R}^E, \mathcal{M}_0(\mathbb{C}^E))$ where $m = \sum_{i\in I} |a_i|$ and the polar set is contained in $\{\prod_{i\in I} L_i = 0\}$.

Applying Corollary 5.14 to the r.h.s. of (25) shows

Lemma 5.15. *Let S_σ be the set of all subgraphs $H \in \{G_{\sigma(1)} \subset \dots \subset G_{\sigma(E)} = G\}$ such that $b_1(H) \ge 1$. Then*

$$\prod_{H\in S_\sigma} \Big(\sum_{e\in E(H)} s_e - |E(H)| \Big) \Big(\int_{[0,1]^E} \prod_{e=1}^{E} \frac{dt_e}{t_e} t_e^{\sum_{i\le e}(2s_{\sigma(i)}+2k_{\sigma(i)})-db_1(G_{\sigma(e)})} \chi_\sigma(t) \Big)$$

*is a **holomorphic germ** at $(s_e = 1)_e$.*

Proof. By Corollary 5.14 and a change of variables $s'_e = s_e - 1$, $e \in \{1, \ldots, E\}$, we need to consider the following set of indices:

$$I = \left\{ e : e + \sum_{i \le e} k_{\sigma(i)} - \frac{d}{2} b_1(G_{\sigma(e)}) \le 0 \right\} \subset \{1, \ldots, E\}.$$

It is contained in $\{e : b_1(G_{\sigma(e)}) \ge 1\}$, which yields the conclusion. □

Let us comment on the above bound on the location of the pole. First the bound seems suboptimal since the set of indices $I = \{e : e + \sum_{i \le e} k_{\sigma(i)} - \frac{d}{2} b_1(G_{\sigma(e)}) \le 0\} \subset \{1, \ldots, E\}$ is only a subset of $\{e : b_1(G_{\sigma(e)}) \ge 1\} \subset \{1, \ldots, E\}$. However, it is important for us that we can give a bound on the location of the poles which does not depend on the multi-index $\vec{k}$ since poles from the original Feynman amplitude $t_G(s)$ do not depend on $\vec{k}$. The formula of Theorem 4.10 expresses t_G as a sum of $I_{G,\vec{k}}$ for some $\vec{k}$. Hence the poles of $t_G(s)$ come from contributions from the poles of $I_{G,\vec{k}}$. Therefore it is convenient to have a $\vec{k}$-independent bound for poles of $I_{G,\vec{k}}$. Finally, we bound the distributional order of $I_{G,\vec{k}}$ and also give a precise location of the affine planes supporting the poles of $I_{G,\vec{k}}$:

Proposition 5.16 (Poles of $I_{G,\vec{k}}$ and distributional order). *Let G be a graph whose set of edges is in bijection with $\{1, \ldots, E\}$. For any $e \in \{1, \ldots, E\} \simeq E(G)$ and any permutation $\sigma \in S_E$, let $G_{\sigma(e)}$ be the subgraph of G induced by $\{\sigma(1), \ldots, \sigma(e)\} \subset E(G)$. To every permutation $\sigma \in S_E$, we associate the filtration $\{G_{\sigma(1)} \subset \cdots \subset G_{\sigma(E)} = G\}$, and consider the set S_σ of all subgraphs $H \in \{G_{\sigma(1)} \subset \cdots \subset G_{\sigma(E)} = G\}$ such that $b_1(H) \ge 1$. For every $\vec{k} \in \mathbb{N}^{E(G)}$, the distribution $I_{G,\vec{k}}(s)$ defined in (15) can be analytically continued to $\mathcal{D}'^{,m}(U^n, \mathcal{M}_{s_0}(\mathbb{C}^E))$, $s_0 = (s_e = 1)_{e=1}^E$, where*

$$m = \sum_{H \subset G,\, 2|E(H)| - db_1(H) - 1 < 0} \big(db_1(H) - 2|E(H)| + 1 \big). \tag{27}$$

For every test function φ,

$$\int_{\mathbb{R}^{dn}} I_{G,\vec{k}}(s) \varphi \, d^{nd}x = \sum_{\sigma \in S_E} \prod_{H \in S_\sigma} \frac{1}{\sum_{i \in H} s_i - E(H)} f_\sigma(s)$$

where f_σ is a holomorphic germ at $(s_e = 1)_e$.

Remark 5.17. The bound on the distributional order depends only on the topology of the graph G and the dimension d and not on $\vec{k} \in \mathbb{N}^E$. The bound on the distributional order is not sharp since we should only sum over subgraphs $G' \in \{G_{\sigma(1)}, \ldots, G_{\sigma(E)}\}$ such that $2|E(G')| - db_1(G') - 1 < 0$ then take the supremum over all permutations σ.

Proof of Proposition 5.16. We proved in Proposition 5.12 that for every labelled graph $(G, \vec{k})$, and every test function $\varphi \in C_c^\infty(U^n)$, $n = |V(G)|$, there exists a family $\chi_\sigma(t)$ of smooth functions on the cube $[0, 1]^E$ indexed by permutations $\sigma \in S_E$ such that

$$\int_{\mathbb{R}^{dn}} I_{G,\vec{k}}(s) \varphi d^{nd}x = \prod_{e=1}^{E} \frac{1}{\Gamma(s_e)} \sum_{\sigma \in S_E} \int_{[0,1]^E} \prod_{e=1}^{E} \frac{dt_e}{t_e} t_e^{\sum_{i \le e} (2s_{\sigma(i)} + 2k_{\sigma(i)}) - db_1(G_{\sigma(e)})} \chi_\sigma(t).$$

By applying Lemma 5.15 to

$$\sum_{\sigma\in S_E}\int_{[0,1]^E}\prod_{e=1}^{E}\frac{dt_e}{t_e}\, t_e^{\sum_{i\leq e}(2s_{\sigma(i)}+2k_{\sigma(i)})-db_1(G_{\sigma(e)})}\chi_\sigma(t),$$

we obtain the meromorphic continuation of $s\mapsto\int_{\mathbb{R}^{dn}} I_{G,\vec{k}}(s)\varphi\, d^{nd}x$ with a bound on the location of poles. To show that $I_{G,\vec{k}}(s)$ is actually an element of $\mathcal{D}'^{,m}(U^n,\mathcal{M}_{s_0}(\mathbb{C}^E))$, $s_0=(s_e=1)_{e=1}^E$, we need to show that

$$C_c^\infty(U^n)\ni\varphi\mapsto\int_{\mathbb{R}^{dn}} I_{G,\vec{k}}(\cdot)\varphi\, d^{nd}x\in\mathcal{M}_{s_0}(\mathbb{C}^E)$$

depends linearly on the m-jet of φ for some m. By Corollary 5.14, the integral

$$\int_{[0,1]^E}\prod_{e=1}^{E}\frac{dt_e}{t_e}\, t_e^{\sum_{i\leq e}(2s_{\sigma(i)}+2k_{\sigma(i)})-db_1(G_{\sigma(e)})}\chi_\sigma(t)$$

depends linearly on the m-jet of χ_σ for

$$m=\sum_{G'\subset G,\,2|E(G')|-db_1(G')-1<0}\big(db_1(G')-2|E(G')|+1\big).$$

Then by Lemma 5.11, the m-jet of χ_σ depends continuously on the m-jet of φ, which yields the result. □

Now let us restate our first main theorem and conclude its proof:

Theorem 5.18. *Let (M,g) be a smooth, compact, connected Riemannian manifold without boundary of dimension d, $dv(x)$ the Riemannian volume and $P=-\Delta_g+V$, $V\in C^\infty_{\geq 0}(M)$, or $M=\mathbb{R}^d$ with a constant metric g and $P=-\Delta_g+\lambda^2$, $\lambda\in\mathbb{R}_{\geq 0}$. Then for every graph G,*

$$t_G(s)=\prod_{e\in E(G)}\mathfrak{G}^{s_e}(x_{i(e)},x_{j(e)}) \tag{28}$$

can be analytically continued to an element of $\mathcal{D}'(M^{V(G)},\mathcal{M}_{s_0}(\mathbb{C}^{E(G)}))$ where $s_0=(s_e=1)_{e\in E(G)}$, with linear poles supported on the union of affine hyperplanes

$$\bigcup_{G'}\Big\{\sum_{e\in G'}s_e-|E(G')|=0\Big\}$$

where the union runs over subgraphs G' of G such that $2|E(G')|-b_1(G')d\leq 0$.

Proof. From Theorem 4.10, one has a decomposition

$$t_G(s)|_{U_x^n}=\sum_{G'\subset G}\Big(\sum_{\vec{k}\in\{0,\dots,p\}^{E_1}}\underbrace{I_{G',\vec{k}}(s)}\Big)\times h_{G\setminus G'}(s)$$

where G' are subgraphs and $h_{G\setminus G'}(s) \in C^m(M^{V(G)}, \mathcal{O}_{s_0}(\mathbb{C}^{E(G)\setminus E(G')}))$, $s_0 = (s_e = 1)_{e\in E(G)\setminus E(G')}$ if $p > (d+m)/2-1$ for $m = \sum_{G'\subset G,\, 2|E(G')|-db_1(G')-1<0}(db_1(G') - 2|E(G')|+1)$. Therefore, the analytic continuation of $t_G(s)$ should follow from the analytic continuation of $I_{G',\vec{k}}$ for all subgraphs G' of G and the fact that the distributional order of $I_{G',\vec{k}}$ is bounded from above by some integer $m(G')$ independent of $\vec{k} \in \mathbb{N}^{E(G')}$. But Proposition 5.16 states precisely that the distributional order of $I_{G',\vec{k}}$ is bounded from above by some integer which depends only on the topology of G'. Now following the notation from Proposition 5.16, for every subgraph $G' \subset G$, we denote by $S_{E(G')}$ the permutations of the edges $E(G') = \{1, \ldots, E'\}$. To every $\sigma \in S_{E(G')}$ corresponds a canonical filtration $\{G'_{\sigma(1)} \subset \cdots \subset G'_{\sigma(E')}\}$ of G' and S_σ denotes the set of all subgraphs $H \in \{G'_{\sigma(1)} \subset \cdots \subset G'_{\sigma(E')}\}$ such that $b_1(H) \geq 1$. Finally doing all the bookkeeping, we find that

$$t_G(s) = \sum_{G'\subset G} \sum_{\sigma\in S_{E(G')}} \left(\prod_{H\in S_\sigma} \frac{1}{\sum_{i\in H} s_i - E(H)} \right) h_\sigma(s)$$

where $h_\sigma(s) \in \mathcal{D}'(M^{V(G)}, \mathcal{O}_{s_0}(\mathbb{C}^{E(G)}))$, $s_0 = (s_e = 1)_{e\in E(G)}$. □

6. Renormalization of Feynman amplitudes

In this second part of the paper, we shall apply the analytic continuation results derived to the renormalization of Feynman amplitudes on Riemannian manifolds.

6.1. *Renormalization maps*

For a smooth manifold (M, g) and every finite $I \subset \mathbb{N}$, we denote by M^I the configuration space of points labelled by I. For $J \subset I$ with $|J| \geq 2$, D_J is the subset $\{(x_i)_{i\in I} : x_j = x_k$ for $j, k \in J\}$ of M^I, called the J-*diagonal*. Let $\Delta_I = \bigcup_{J\subset I,\, |J|\geq 2} D_J$ be the maximal diagonal.

Definition 6.1 (Labelling vertices). Let $I \subset \mathbb{N}$ be finite. A *graph with vertices labelled by* I is a pair (G, ι), where G is a graph and ι is an injective map from $V(G)$ to I.

For a graph with vertices labelled by I, (G, ι), define

$$t_G = \prod_{e\in E(G)} \mathfrak{G}(x_{i(e)}, x_{j(e)}),$$

where $(i(e), j(e)) \in I^2$ and t_G is a smooth function on $M^I \setminus \Delta_I$. For a finite subset I of $\mathbb{N}$, let $\mathcal{F}(M^I)$ be the linear span of t_G of all (G, ι) with $\iota(V(G)) \subset I$ as smooth functions on $M^I \setminus \Delta_I$.

For a linear map $\mathcal{R} : E \to \mathcal{D}'(M)$ where E is a vector space and M is a smooth manifold, and any open subset $U \subset M$, let $i_U : U \hookrightarrow M$ denote the inclusion map. Then $\mathcal{R}|_U = i_U^*\mathcal{R} : E \to \mathcal{D}'(U)$ is the pull-back of $\mathcal{R}$ by i_U. Following recent work by Nikolov–Stora–Todorov [59], we can give a definition of renormalization as follows:

Definition 6.2. A *renormalization* is a sequence of (not necessarily continuous) linear maps $\mathcal{R}_I : \mathcal{F}(M^I) \to \mathcal{D}'(M^I)$ indexed by finite subsets I of $\mathbb{N}$, which satisfies the following system of *functional equations*

- For $I \subset J$ and $t \in \mathcal{F}(M^I)$,
$$\mathcal{R}_J(t) = \mathcal{R}_I(t). \tag{29}$$
This is the *compatibility* condition for the family of linear maps.
- For all $t \in \mathcal{F}(M^I)$ and $\varphi \in C^\infty_c(M^I \setminus \Delta_I)$,
$$\langle \mathcal{R}_I(t), \varphi\rangle = \langle t, \varphi\rangle. \tag{30}$$
This means that $\mathcal{R}_I(t)$ is a *distributional extension* of $t \in C^\infty(M^I \setminus \Delta_I)$.
- For a graph (G, ι) with vertices labelled by $J \subset \mathbb{N}$, and $I \subset J = \iota(V(G))$, set $I^c = J\setminus I$, let $E_I = \{e \in E(G) : i(e), j(e) \in I\}$, $E_{I^c} = \{e \in E(G) : i(e), j(e) \in I^c\}$, $E_{II^c} = E(G) \setminus (E_I \cup E_{I^c})$, and denote by G_I, G_{I^c}, G_{II^c} the corresponding induced subgraphs of G. For open subsets U, V of M with $\mathrm{dist}(U, V) > 0$, denote by $U^I \times V^{I^c}$ the subset $\{(x_j)_{j\in J} \in M^J : x_i \in U, \forall i \in I, x_i \in V, \forall i \in I^c\} \subset M^J$. Then
$$\mathcal{R}_J|_{U^I\times V^{I^c}}(t_G) = (\mathcal{R}_J|_{U^I}(t_{G_I}) \boxtimes \mathcal{R}_J|_{V^{I^c}}(t_{G_{I^c}}))t_{G_{II^c}}$$
as distributions in $\mathcal{D}'(U^I \times V^{I^c})$. This means that renormalization must preserve **locality**.
- Let $\Phi : M \to M$ be an orientation preserving diffeomorphism and denote by $\Phi_I : M^I \to M^I$ the induced diffeomorphism on the configuration space M^I. Assume that the renormalization maps depend on the Riemannian metric g and write $\mathcal{R}[g] = (\mathcal{R}[g]_I)_I$ to stress this dependence. Then the covariance equation for renormalization maps reads, for all graphs (G, ι) with vertices labelled by I,
$$\mathcal{R}[\Phi^* g]_I(\Phi_I^* t_G) = \Phi_I^*(\mathcal{R}[g]_I(t_G)). \tag{31}$$
This axiom of functorial nature ensures that the renormalization is *covariant*.

The following property follows from the locality condition: for a graph (G, ι) with vertices labelled by I, if G is the disjoint union of G_1 and G_2, $\iota(V(G_1)) \subset I_1$, $\iota(V(G_2)) \subset I_2$, $I_1 \cap I_2 = \emptyset$, then
$$\mathcal{R}_{I_1\cup I_2}(t_G) = \mathcal{R}_{I_1}(t_{G_1}) \boxtimes \mathcal{R}_{I_2}(t_{G_2})$$
as distributions in $\mathcal{D}'(M^{I_1\cup I_2})$.

6.2. *Decompositions of meromorphic germs of distributions*

Our goal in this subsection is to extend the decomposition of [39] (see also [15, Appendix]) of the space $\mathcal{M}_{s_0}$ of meromorphic germs with linear poles at $s_0 \in \mathbb{R}^p \subset \mathbb{C}^p$ to their distributional counterpart $\mathcal{D}'(\cdot, \mathcal{M}_{s_0})$ defined in §3.2. This decomposition plays an essential role in our definition of renormalization maps by projections. Recall we denote by $\mathcal{O}_{s_0}$ the space of holomorphic germs at s_0.

Let us fix a nondegenerate bilinear form

$$Q(\cdot,\cdot):\mathbb{R}^p\times\mathbb{R}^p\to\mathbb{R},$$

which induces a nondegenerate bilinear form

$$Q^*(\cdot,\cdot):(\mathbb{R}^p)^*\times(\mathbb{R}^p)^*\to\mathbb{R}.$$

We can now define the concept of polar germ [39, Definition 2.3]. A *polar germ* at s_0 is a meromorphic germ of the form $\frac{1}{L_1^{n_1}(s-s_0)\ldots L_k^{n_k}(s-s_0)}h(\ell(s-s_0))$, where $L_1,\ldots,L_k$ are linearly independent linear functions in $(\mathbb{R}^p)^*$, $(n_1,\ldots,n_k)\in\mathbb{N}^k_{>0}$, $\ell=(\ell_1,\ldots,\ell_n):\mathbb{R}^p\to\mathbb{R}^n$ defined by linear functions $\ell_1,\ldots,\ell_n$, and h is a holomorphic germ at $0\in\mathbb{C}^n$, such that $Q^*(L_i,\ell_j)=0$, $1\le i\le k$, $1\le j\le n$. Let $\mathcal{P}_{s_0}$ be the linear span of polar germs at s_0 in $\mathcal{M}_{s_0}$.

Notice that the polar set is defined by real linear functions. By similar proof to [39] using geometry of cones, we have the following:

Proposition 6.3. *There is a decomposition*

$$\mathcal{M}_{s_0}=\mathcal{O}_{s_0}\oplus\mathcal{P}_{s_0}.$$

Proof. We may assume that $s_0=0$. Our setting is slightly more general than that in [39] since we are dealing with complex coefficients whereas [39] deals with real coefficients. However, the proofs can be readily adapted in a straightforward way as we shall indicate below. Just as in [39, Lemma 2.9 and Thm. 2.10], we have $\mathcal{M}_0=\mathcal{O}_0+\mathcal{P}_0$, where the proof is the same for all coefficients since it relies on the decomposition of the meromorphic germ $\frac{1}{L_1^{n_1}\ldots L_k^{n_k}}$ as a sum of polar germs and Taylor expansion of the numerator.

To prove that the sum is direct, we need to show $\mathcal{O}_0\cap\mathcal{P}_0=\{0\}$, where we use two properties:

First, a *projectively properly positioned* family of simplicial real fractions is linearly independent over $\mathbb{C}$. This is an analog of [39, Proposition 3.5], following from it by taking real and imaginary parts.

A projectively properly positioned family of polar germs at zero in our sense is nonholomorphic. As in [39, Thm. 3.6], we can show that if a linear combination of projectively properly positioned family of polar germs is holomorphic then it is zero.

We may conclude as in [39, Thm. 4.15]. Take any decomposition of $f\in\mathcal{P}_0\cap\mathcal{O}_0$ as $f=\sum g_i$ where $(g_i)_i$ is a finite set of polar germs. By [39, Lemmas 4.10 and 4.11] there is a family of supporting cones of the polar germs such that the union of the cones does not contain any nonzero linear subspace and this family of supporting cones has a properly positioned subdivision. Then we transform the finite sum of polar germs $\sum h_i$ into a sum $\sum\tilde{h}_j$ of *projectively properly positioned* polar germs $(\tilde{h}_j)_j$ using the subdivision operator from [39, p. 18]. Finally, we get $\sum\tilde{h}_j=f\in\mathcal{O}_0$ where the r.h.s. is holomorphic, hence $\sum\tilde{h}_j=0$ by nonholomorphicity [39, Thm. 3.6]. Finally, this yields $\mathcal{O}_0\cap\mathcal{P}_0=\{0\}$. □

Now we extend the concept of polar germs to distributions valued in polar germs.

Definition 6.4. Let M be a smooth manifold and $p \in \mathbb{N}$. A *polar germ of distributions* at $s_0 \in \mathbb{R}^p \subset \mathbb{C}^p$ is an element of $\mathcal{D}'(M, \mathcal{M}_{s_0}(\mathbb{C}^p))$ of the form $\frac{1}{L_1^{n_1}(s-s_0)\ldots L_k^{n_k}(s-s_0)}$ $\times\, h(\ell(s-s_0))$, where $L_1, \ldots, L_k$ are linearly independent linear functions in $(\mathbb{R}^p)^*$, $(n_1, \ldots n_k) \in \mathbb{N}_{>0}^k$, $\ell = (\ell_1, \ldots, \ell_n) : \mathbb{R}^p \to \mathbb{R}^n$ defined by linear functions $\ell_1, \ldots, \ell_n$, and $h \in \mathcal{D}'(M, \mathcal{O}_0(\mathbb{C}^p))$ such that $Q^*(L_i, \ell_j) = 0$, $1 \le i \le k$, $1 \le j \le n$. We denote by $\mathcal{D}'(M, \mathcal{P}_{s_0}(\mathbb{C}^p))$ the linear span of polar germs of distributions at s_0 in $\mathcal{D}'(M, \mathcal{M}_{s_0}(\mathbb{C}^p))$.

Lemma 6.5. *If* $\frac{1}{L_1^{n_1}(s-s_0)\ldots L_k^{n_k}(s-s_0)} h(\ell(s-s_0))$ *and* $\frac{1}{M_1^{m_1}(s-s_0)\ldots M_p^{m_p}(s-s_0)} g(\ell'(s-s_0))$ *represent the same nonzero meromorphic germ of distributions, then* $k = p$, *and* $M_1, \ldots, M_m, L_1, \ldots, L_k$ *can be rearranged in such a way that* L_i *is a multiple of* M_i *and* $n_i = m_i$ *for* $1 \le i \le k$.

Proof. Since this meromorphic germ is not zero, we can take a test function φ such that $h(\ell(s-s_0))(\varphi)$ is not identically zero. Then $\frac{1}{L_1^{n_1}(s-s_0)\ldots L_k^{n_k}(s-s_0)} h(\ell(s-s_0))(\varphi)$ and $\frac{1}{M_1^{m_1}(s-s_0)\ldots M_p^{m_p}(s-s_0)} g(\ell'(s-s_0))(\varphi)$ represent the same polar germ, and by the same proof as in [39, Lemma 2.8], we have the conclusion. □

We now prove the promised decomposition theorem for $\mathcal{D}'(M, \mathcal{M}_{s_0})$ which generalizes the result in [39].

Theorem 6.6. *Let* M *be a smooth manifold and* $s_0 \in \mathbb{R}^p \subset \mathbb{C}^p$. *We have the direct sum decomposition*

$$\mathcal{D}'(M, \mathcal{M}_{s_0}(\mathbb{C}^p)) = \mathcal{D}'(M, \mathcal{O}_{s_0}(\mathbb{C}^p)) \oplus \mathcal{D}'(M, \mathcal{P}_{s_0}(\mathbb{C}^p)).$$

Proof. We can assume that $s_0 = 0$. For $t \in \mathcal{D}'(M, \mathcal{M}_0(\mathbb{C}^p))$, by definition, there exist $L_1, \ldots, L_k \in (\mathbb{R}^p)^*$ such that $L_1 \ldots L_k t \in \mathcal{D}'(M, \mathcal{O}_0(\mathbb{C}^p))$. By partial fractions decompositions as in the proof of [39, Lemma 2.9(a)], we may assume there is $(n_1, \ldots, n_k) \in \mathbb{N}_{>0}^k$ such that $L_1^{n_1} \ldots L_k^{n_k} t \in \mathcal{D}'(M, \mathcal{O}_0(\mathbb{C}^p))$ with $L_1, \ldots, L_k$ linearly independent and $(n_1, \ldots, n_k) \in \mathbb{N}_{>0}^k$.

Now let us extend $L_1, \ldots, L_k$ to a basis $(e_1, \ldots, e_p)$ of $(\mathbb{R}^p)^*$ with $e_i = L_i$, $1 \le i \le k$, and $Q(e_i, e_j) = 0$ for $1 \le i \le k$, $k+1 \le j \le p$. Then by Proposition 3.9, we have the power series expansion

$$z_1^{n_1} \ldots z_k^{n_k} t = \sum_{\alpha \in \mathbb{N}^p} \frac{z^\alpha}{\alpha!} t_\alpha,$$

where $z = \sum z_i e_i^* \in (\mathbb{C}^p)^*$. So when we apply $z_1^{n_1} \ldots z_k^{n_k} t$ to the test function φ, we obtain $z_1^{n_1} \ldots z_k^{n_k} t(\varphi) = \sum_{\alpha \in \mathbb{N}^p} \frac{z^\alpha}{\alpha!} t_\alpha(\varphi)$, which is absolutely convergent in a small neighborhood of $0 \in \mathbb{C}^p$.

Let $S = \{d = (d_1, \ldots, d_p) \in \mathbb{N}^p : d \ne (0, \ldots, 0), 0 \le d_i \le n_i\}$. For $d \in S$, let $I(d) = \{i : d_i \ne 0\} \subset \{1, \ldots, p\}$, and set

$$N_d = \big\{\alpha \in \mathbb{N}^p : \alpha_i = n_i - d_i \text{ if } i \in I(d),\ \alpha_i \ge n_i \text{ if } i \in \{1, \ldots, k\} \setminus I(d), \\ \alpha_i \in \mathbb{N} \text{ if } i \in \{k+1, \ldots, p\}\big\}.$$

Then $N_d \cap N_e = \emptyset$ if $d \neq e \in S$, and most importantly, we have the partition $\mathbb{N}^p = \bigcup_{d\in S} N_d$.

Now for $z_i \neq 0$, $1 \leq i \leq k$,

$$t(\varphi) = \sum_{\alpha\in\mathbb{N}^p} \frac{z^{\alpha-\alpha_0}}{\alpha!} t_\alpha(\varphi) = \sum_{d\in S}\sum_{\alpha\in N_d} \frac{z^{\alpha-\alpha_0}}{\alpha!} t_\alpha(\varphi),$$

where $\alpha_0 = (n_1, \dots, n_k, 0, \dots, 0)$. And we have

$$\sum_{\alpha\in N_d} \frac{z^{\alpha-\alpha_0}}{\alpha!} t_\alpha(\varphi) = \frac{1}{z^d_{I(d)}} \sum_{\alpha\in N_d} \frac{z^{\alpha-\alpha_0}_{[p]\setminus I(d)}}{\alpha!} t_\alpha(\varphi),$$

where $z^d_{I(d)} = \prod_{i\in I(d)} z_i^{d_i}$, $z^{\alpha-\alpha_0}_{[p]\setminus I(d)} = \prod_{i\in\{1,\dots,k\}\setminus I(d)} z_i^{\alpha_i - s_i} \prod_{i\in\{k+1,\dots,p\}} z_i^{\alpha_i}$.

Let

$$h_d = \sum_{\alpha\in N_d} \frac{z^{\alpha-\alpha_0}_{[p]\setminus I}}{\alpha!} t_\alpha.$$

By Proposition 3.9, for every compact $K \subset U$ there exists $C > 0$ and some continuous seminorm P for the Fréchet topology of $C^\infty_K(U)$ such that for each $\varphi \in C^\infty_K(U)$, $|t_\alpha(\varphi)| \leq \frac{\alpha!}{r^{|\alpha|}} CP(\varphi)$ for all $\alpha \in \mathbb{N}^p$. This implies that for every $0 < R < r$ and all $|z| < R$,

$$\left|\sum_{\alpha\in\mathbb{N}^p} \frac{z^\alpha}{\alpha!} t_\alpha(\varphi)\right| \leq \sum_{\alpha\in\mathbb{N}^p} \left|\frac{z^\alpha}{\alpha!} t_\alpha(\varphi)\right| \leq \sum_{\alpha\in\mathbb{N}^p} \frac{R^{|\alpha|}}{\alpha!} \frac{\alpha!}{r^{|\alpha|}} CP(\varphi) = \left(1 - \frac{R}{r}\right)^{-p} CP(\varphi).$$

Therefore, $|(\prod_{i\in[k]\setminus I(d)} z_i^{s_i}) h_d(\varphi)| \leq \sum_{\alpha\in\mathbb{N}^p} \left|\frac{z^\alpha}{\alpha!} t_\alpha(\varphi)\right| \leq (1 - R/r)^{-p} CP(\varphi)$, hence $(\prod_{i\in[k]\setminus I(d)} z_i^{s_i}) h_d \in \mathcal{D}'(M, \mathcal{O}_0(\mathbb{C}^p))$, so $h_d \in \mathcal{D}'(M, \mathcal{O}_0(\mathbb{C}^p))$.

Now by definition, $\frac{1}{z^d_{I(d)}} h_d$ is a polar germ of distributions if $d \neq (0, \dots, 0)$, $h_{(0,\dots,0)} \in \mathcal{D}'(M, \mathcal{O}_0(\mathbb{C}^p))$, and

$$t = \sum_{d\in S} \frac{1}{z^d_{I(d)}} h_d \in \mathcal{D}'(M, \mathcal{P}_0(\mathbb{C}^p)) + \mathcal{D}'(M, \mathcal{O}_0(\mathbb{C}^p)) \tag{32}$$

where the singular part is a finite sum of polar germs, as a corollary of the above argument. So we have $\mathcal{D}'(M, \mathcal{M}_{s_0}(\mathbb{C}^p)) = \mathcal{D}'(M, \mathcal{O}_{s_0}(\mathbb{C}^p)) + \mathcal{D}'(M, \mathcal{P}_{s_0}(\mathbb{C}^p))$. To show it is a direct sum, if $t \in \mathcal{D}'(M, \mathcal{O}_{s_0}(\mathbb{C}^p)) \cap \mathcal{D}'(M, \mathcal{P}_{s_0}(\mathbb{C}^p))$, then for any test function ϕ, $t(\phi) \in \mathcal{P}_{s_0} \cap \mathcal{O}_{s_0}$, so $t(\phi) = 0$ by Proposition 6.3, which implies $t = 0$. □

A consequence of the decomposition theorem is

Proposition 6.7. *Let M be a smooth manifold, $p \in \mathbb{N}$ and $s_0 \in \mathbb{R}^p \subset \mathbb{C}^p$. There exists a projection*

$$\pi_p : \mathcal{D}'(M, \mathcal{M}_{s_0}(\mathbb{C}^p)) \to \mathcal{D}'(M, \mathcal{O}_{s_0}(\mathbb{C}^p))$$

which sends a distribution valued in meromorphic germs at s_0 to a distribution valued in holomorphic germs at s_0 such that $\ker(\pi_p) = \mathcal{D}'(M, \mathcal{P}_{s_0}(\mathbb{C}^p))$.

Remark 6.8. Note that π_p is uniquely determined by the vector subspace of polar germs, which are in turn uniquely determined by the choice of the canonical quadratic form $Q : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}$ that we fixed at the beginning of the present section.

In Appendix 7.2, we show some useful lemmas on the functorial properties of the projection π_p for $p \in \mathbb{N}$. As a consequence of [39], we have a similar projection, still denoted a bit abusively by π_p, at the germ level: $\pi_p : \mathcal{M}_{s_0} \to \mathcal{O}_{s_0}$. It follows that the two projectors are related by the following equation:

Corollary 6.9. *Let X be a smooth manifold and $s_0 \in \mathbb{R}^p \subset \mathbb{C}^p$. For all $t(s) \in \mathcal{D}'(X, \mathcal{M}_{s_0}(\mathbb{C}^p))$ and all $\varphi \in C_c^\infty(X)$,*

$$(\pi_p t(s))(\varphi) = \pi_p(t(s)(\varphi)).$$

6.3. A renormalization map by projections

From now on, for any integer p, we fix the canonical quadratic form Q on $\mathbb{R}^p$, $Q(x) = \sum_{i=1}^p |x_i|^2$, and we study germs at $s_0 = (1, \dots, 1) \in \mathbb{R}^p$. We denote by $\mathbf{ev}|_{s_0}$ evaluation of holomorphic germs at s_0. The properties of the family of projections $(\pi_p)_{p \in \mathbb{N}}$ allow us to give a definition of renormalization maps:

Definition 6.10 (Renormalization maps by projections). For $I \subset \mathbb{N}$, we define the renormalization map $\mathcal{R}_I$ as follows: for a graph (G, ι) with vertices labelled by I,

$$\mathcal{R}_I(t_G) = \mathbf{ev}|_{s_0}(\pi_{|E(G)|}(t_G(s)))$$

where $\mathfrak{G}^s$ is the Schwartz kernel of $(-\Delta)^{-s}$.

Theorem 6.11 (Renormalization theorem). *Let (M, g) be a smooth, compact, connected Riemannian manifold without boundary of dimension d, $dv(x)$ the Riemannian volume and $P = -\Delta_g + V$, $V \in C^\infty_{\geq 0}(M)$ or $M = \mathbb{R}^d$ with a constant metric g and $P = -\Delta_g + m^2$, $m \in \mathbb{R}_{\geq 0}$. For every finite $I \subset \mathbb{N}$ and every graph (G, ι) with vertices labelled by I, define $\mathcal{R}_I(t_G) \in \mathcal{D}'(M^I)$ as in Definition 6.10 and extend it by linearity to the vector space $\mathcal{F}(M^I)$. Then the collection $(\mathcal{R}_I)_{I \subset \mathbb{N}, |I| < \infty}$ of renormalization maps satisfies the functional equations of Definition 6.2.*

Proof. The compatibility condition is encoded in the family of projections. For simplicity of notation, we drop the index s_0 from the space of germs so we write $\mathcal{M}, \mathcal{O}$ instead of $\mathcal{M}_{s_0}, \mathcal{O}_{s_0}$ and it will always be understood from the context that we consider holomorphic and meromorphic germs localized at $s_0 = (1, \dots, 1) \in \mathbb{C}^p$ for some $p \in \mathbb{N}$. Furthermore, we also write π instead of $\pi_{|E(G)|}$ where it will be understood that for every graph G, $\pi(t_G(s))$ means $\pi_{|E(G)|}(t_G(s))$.

We now prove that $\mathcal{R}_I(t_G)$ is a distributional extension of t_G. By Lemma 4.1, on $M^I \setminus \Delta_I$, for every $e \in E(G)$, every Green function $\mathfrak{G}^{s_e} \in C^\infty(M^I \setminus \Delta_I, \mathcal{O})$ is in fact smooth and depends holomorphically on s_e. We also have the convergence $\mathfrak{G}^{s_e} \to \mathfrak{G}$ in $C^\infty(M^2 \setminus \Delta_2)$ when $s_e \to 1$. Therefore, for any $\varphi \in C_c^\infty(M^I \setminus \Delta_I)$, by Corollary 6.9, we have $\pi(t_G(s))(\varphi) = \pi(t_G(s)(\varphi)) = t_G(s)(\varphi)$ since $t_G(s)(\varphi)$ is holomorphic at $s_0 = (s_e = 1)_{e\in E(G)} \in \mathbb{C}^{E(G)}$, and

$$\mathcal{R}_I(t_G)(\varphi) = \mathbf{ev}|_{s_0}\pi(t_G(s))(\varphi) = \mathbf{ev}|_{s_0} t_G(s)(\varphi) = t_G(\varphi).$$

Now let us prove the locality. For a graph (G, ι) with vertices labelled by $J \subset \mathbb{N}$, and $I \subset J = \iota(V(G))$, set $I^c = J \setminus I$, let $E_I = \{e \in E(G) : i(e), j(e) \in I\}$, $E_{I^c} = \{e \in E(G) : i(e), j(e) \in I^c\}$, $E_{II^c} = E(G) \setminus (E_I \cup E_{I^c})$, and denote by (G_I, G_{I^c}, G_{II^c}) the corresponding induced subgraphs of G. Start from $t_G(s) = t_{G_I}(s_I)t_{G_{I^c}}(s_{I^c})t_{G_{II^c}}(s_{II^c})$ where $s = (s_e)_{e\in E(G)}$, $s_I = (s_e)_{e\in E_I}$, $s_{I^c} = (s_e)_{e\in E_{I^c}}$, $s_{II^c} = (s_e)_{e\in E_{II^c}}$. For a pair (U, V) of disjoint open subsets such that $\mathrm{dist}(U, V) > 0$, consider the open subset $\{(x_j)_{j\in J} \in M^J : x_i \in U, \forall i \in I, x_i \in V, \forall i \in I^c\}$ of the configuration space M^J. Then $t_{G_{II^c}}(s_{II^c}) = \prod_{e\in E_{II^c}} \mathfrak{G}^{s_e}(x_{i(e)}, x_{j(e)}) \in C^\infty(U^I \times V^{I^c}, \mathcal{O}(\mathbb{C}^{E_{II^c}}))$. It follows by Lemma 7.3 in the Appendix that

$$\pi(t_G(s)) = \pi(t_{G_I}(s_I)t_{G_{I^c}}(s_{I^c})t_{G_{II^c}}(s_{II^c})) = \pi(t_{G_I}(s_I)t_{G_{I^c}}(s_{I^c}))t_{G_{II^c}}(s_{II^c}).$$

Now the distributions $t_{G_I}(s_I) \in \mathcal{D}'(U^I, \mathcal{M}(\mathbb{C}^{E_I}))$ and $t_{G_{I^c}}(s_{I^c}) \in \mathcal{D}'(V^{I^c}, \mathcal{M}(\mathbb{C}^{E_{I^c}}))$ depend on different variables, therefore by Lemma 7.2,

$$\pi(t_{G_I}(s_I) \boxtimes t_{G_{I^c}}(s_{I^c})) = \pi(t_{G_I}(s_I)) \boxtimes \pi(t_{G_{I^c}}(s_{I^c})).$$

Then as distributions on $U^I \times V^{I^c}$,

$$\begin{aligned}\mathcal{R}_J(t_G) &= \mathbf{ev}|_{(s_e=1)_{e\in E(G)}}(\pi(t_{G_I}(s_I)) \boxtimes \pi(t_{G_{I^c}}(s_{I^c})) \times t_{G_{II^c}}(s_{II^c}))\\ &= \mathbf{ev}|_{(s_e=1)_{e\in E_I}}\pi(t_{G_I}(s_I))\mathbf{ev}|_{(s_e=1)_{e\in E_{I^c}}}\pi(t_{G_{I^c}}(s_{I^c}))\mathbf{ev}|_{(s_e=1)_{e\in E_{II^c}}} t_{G_{II^c}}(s_{II^c})\\ &= \mathcal{R}_I(t_{G_I})\mathcal{R}_{I^c}(t_{G_{I^c}})t_{G_{II^c}}|_{U^I\times V^{I^c}}\end{aligned}$$

where $t_{G_{II^c}}$ is smooth on $U^I \times V^{I^c}$, which yields the desired equation. □

7. Appendix: technical details

7.1. *Proof of Proposition 3.9*

Proof. We can assume that $z = 0$. By definition and the multidimensional Cauchy formula [38, p. 3], for any polydisc $D_1 \times \cdots \times D_p$ around $z = 0$, any $\varphi \in C_c^\infty(M)$ and any λ in the polydisc,

$$\begin{aligned}t(\lambda)(\varphi) &= \frac{1}{(2\pi i)^p}\int_{\partial D_1}\cdots\int_{\partial D_p}\frac{t(z)(\varphi)dz_1\dots dz_p}{(z_1-\lambda_1)\dots(z_p-\lambda_p)}\\ &= \frac{1}{(2\pi i)^p}\int_{\partial D_1}\cdots\int_{\partial D_p}\sum_\alpha \lambda^\alpha\frac{t(z)(\varphi)dz_1\dots dz_p}{z_1^{\alpha_1+1}\dots z_p^{\alpha_p+1}}.\end{aligned}$$

For any multi-index α and any test function φ, we define the functional t_α by

$$t_\alpha(\varphi) = \frac{\alpha!}{(2\pi i)^p}\int_{\partial D_1}\cdots\int_{\partial D_p}\frac{t(z)(\varphi)\,dz_1\dots dz_p}{z_1^{\alpha_1+1}\dots z_p^{\alpha_p+1}}.$$

Then the series $\sum_{|\alpha|\geq 0}\frac{s^\alpha}{\alpha!}t_\alpha(\varphi)$ converges absolutely to $t(s)(\varphi)$.

First note that the functional t_α is linear; it remains to prove that it is continuous. For that, it suffices to show that for every compact $K \subset M$, the restriction of t_α to the Fréchet space $C^\infty_K(M)$ of test functions supported in K is continuous. For fixed s, $t(s)$ is linear continuous on $C^\infty_K(M)$, so there exists a constant $C(s)$ and a continuous seminorm P of $C^\infty_K(M)$ such that $|t(s)(\varphi)| \leq C(s)P(\varphi)$ for all $\varphi \in C^\infty_K(M)$. Conversely, for fixed φ, $D_1 \times \cdots \times D_p \ni s \mapsto t(s)(\varphi)$ is bounded by holomorphicity. By an application of the uniform boundedness principle since $C^\infty_K(M)$ is Fréchet, for every compact $K \subset M$ there exists $C > 0$ and a continuous seminorm P for the Fréchet topology of $C^\infty_K(M)$ such that

$$\forall \varphi \in C^\infty_K(M), \qquad \sup_{s \in \partial D_1 \times \cdots \times \partial D_p} |t(s)(\varphi)| \leq C P(\varphi).$$

Assuming that all discs D_i have radius r, it immediately follows that t_α satisfies a distributional version of Cauchy's bound:

$$\forall \varphi \in C^\infty_K(M), \qquad |t_\alpha(\varphi)| \leq \frac{\alpha!}{r^{|\alpha|}} C P(\varphi). \tag{33}$$

This also implies that for all $\varphi \in C^\infty_K(M)$, the power series $\sum_\alpha \frac{s^\alpha}{\alpha!} t_\alpha(\varphi)$ converges for $|\lambda| < r$, i.e. the convergence radius equals r. □

7.2. Products of meromorphic germs of distributions in different variables

In this subsection, we prove some useful lemmas on products of meromorphic germs of distributions in different variables.

Lemma 7.1. *Let (X_1, X_2) be smooth manifolds, $\mu_1 \in \mathbb{R}^{p_1} \subset \mathbb{C}^{p_1}$ and $\mu_2 \in \mathbb{R}^{p_2} \subset \mathbb{C}^{p_2}$. If $t_1(s_1) \in \mathcal{D}'(X_1, \mathcal{M}_{\mu_1}(\mathbb{C}^{p_1}))$ and $t(s_2) \in \mathcal{D}'(X_2, \mathcal{M}_{\mu_2}(\mathbb{C}^{p_2}))$ then the external tensor product $t_1(s_1) \boxtimes t_2(s_2)$ is a well-defined element in $\mathcal{D}'(X_1 \times X_2, \mathcal{M}_{(\mu_1,\mu_2)}(\mathbb{C}^{p_1+p_2}))$.*

Proof. Denote by dv_1, dv_2 some smooth densities on X_1, X_2 respectively. Since every compact subset $K \subset X_1 \times X_2$ can be covered by a finite number of products of compacts of the form $K_1 \times K_2$, by Lemma 4.2 it suffices to show that for all compacts $K_1 \subset X_1, K_2 \subset X_2$, the element $t(s_1; x)t(s_2; y)|_{K_1 \times K_2}$ is a well-defined meromorphic family of distributions in $\mathcal{D}'(K_1 \times K_2)$ at $(\mu_1, \mu_2) \in \mathbb{C}^{p_1} \times \mathbb{C}^{p_2}$ with linear poles. Hence we can assume that we work over some product $K_1 \times K_2 \subset X_1 \times X_2$ of compact subsets and that we work around $(\mu_1, \mu_2) = (0, 0)$. There exist mononomials $P(s_1) = L_1(s_1) \ldots L_k(s_1)$ and $Q(s_2) = M_1(s_2) \ldots M_l(s_2)$, where $(L_i)_{i=1}^k, (M_i)_{i=1}^l$ are linear functions, such that $P(s_1)t_1(s_1)$ and $Q(s_2)t_2(s_2)$ are holomorphic germs of distributions at $s_1 = \mu_1$ and $s_2 = \mu_2$ respectively. Therefore by Proposition 3.9, $P(s_1)t_1(s_1)$ and $Q(s_2)t_2(s_2)$ admit Laurent series expansions $P(s_1)t_1(s_1) = \sum_{\alpha_1} s_1^{\alpha_1} u_{\alpha_1}$ and $Q(s_2)t_2(s_2) = \sum_{\alpha_2} s_2^{\alpha_2} v_{\alpha_2}$ where there exist integers (m_1, m_2) corresponding to the distributional orders of $(t_1|_{K_1}, t_2|_{K_2})$ and positive real numbers r_1, r_2 such that for all $(\alpha_1, \alpha_2) \in \mathbb{N}^{p_1+p_2}$,

$$\|u_{\alpha_1}\|_{(C^{m_1})'} \leq C_1 r_1^{|\alpha_1|}, \qquad \|v_{\alpha_2}\|_{(C^{m_2})'} \leq C_2 r_2^{|\alpha_2|}. \tag{34}$$

We define the series $P(s_1)t_1(s_1) \boxtimes Q(s_2)t_2(s_2) = \sum_{\alpha_1,\alpha_2} s_1^{\alpha_1} s_2^{\alpha_2} u_{\alpha_1} \boxtimes v_{\alpha_2}$; we shall prove that it converges for $|s_1| + |s_2|$ small enough in the sense that for every test function $\varphi(x_1, x_2)$ supported in $K_1 \times K_2$, the series

$$\sum_{\alpha_1,\alpha_2} s_1^{\alpha_1} s_2^{\alpha_2} u_{\alpha_1} \boxtimes v_{\alpha_2}(\varphi) = \sum_{\alpha_1,\alpha_2} s_1^{\alpha_1} s_2^{\alpha_2} \int_{X_1\times X_2} u_{\alpha_1}(x_1) v_{\alpha_2}(x_2) \varphi(x_1, x_2)\, dv_1(x_1)\, dv_2(x_2)$$

converges absolutely. We first prove it for $\varphi = \varphi_1 \boxtimes \varphi_2 \in C^\infty_{K_1}(X_1) \boxtimes C^\infty_{K_2}(X_2) \subset C^\infty_{K_1\times K_2}(X_1 \times X_2)$. For $s \in \mathbb{C}^p$, we shall use the notation $\|s\| = \sup_{j\in\{1,\ldots,p\}} |s_j|$ and for $\alpha \in \mathbb{N}^p$, we set $|\alpha| = \sum_{j=1}^p \alpha_j$. Then the series converges thanks to the bound

$$\begin{aligned}
&\left| \sum_{\alpha_1,\alpha_2} s_1^{\alpha_1} s_2^{\alpha_2} \int_{X_1\times X_2} u_{\alpha_1}(x_1) v_{\alpha_2}(x_2) \varphi(x_1, x_2)\, dv_1(x_1)\, dv_2(x_2) \right| \\
&\quad \le \sum_{\alpha_1,\alpha_2} \|s_1^{\alpha_1} s_2^{\alpha_2}\| \left| \int_{X_1} u_{\alpha_1}(x_1)\varphi_1(x_1)\, dv_1(x_1) \int_{X_2} v_{\alpha_2}(x_2)\varphi_2(x_2)\, dv_2(x_2) \right| \\
&\quad \le \sum_{\alpha_1,\alpha_2} \|s_1\|^{|\alpha_1|} C_1 r_1^{|\alpha|} \|\varphi_1\|_{C^{m_1}(X_1)} \|s_2\|^{|\alpha_2|} C_2 r_2^{|\alpha|} \|\varphi_2\|_{C^{m_2}(X_2)} \\
&\quad \le \sum_{\alpha_1,\alpha_2} C_1 (\|s_1\| r_1)^{|\alpha_1|} C_2 (\|s_2\| r_2)^{|\alpha_2|} \|\varphi\|_{C^m(X_1\times X_2)}
\end{aligned}$$

for any $m \ge \sup(m_1, m_2)$ where the r.h.s. is absolutely convergent for s_1, s_2 small enough. Then we conclude by using the fact that the completed tensor product $C^\infty_{K_1}(X_1)\widehat{\boxtimes} C^\infty_{K_2}(X_2)$ coincides with $C^\infty_{K_1\times K_2}(X_1 \times X_2)$ where the topology for which we do the completion does not matter since the $C^\infty_{K_i}(X_i)$ are Fréchet nuclear spaces. Therefore the algebraic tensor product $C^\infty_{K_1}(X_1) \boxtimes C^\infty_{K_2}(X_2)$ is dense in $C^\infty_{K_1\times K_2}(X_1 \times X_2)$ and

$$\begin{aligned}
&\left| \sum_{\alpha_1,\alpha_2} s_1^{\alpha_1} s_2^{\alpha_2} \int_{X_1\times X_2} u_{\alpha_1}(x_1) v_{\alpha_2}(x_2) \varphi(x_1, x_2)\, dv_1(x_1)\, dv_2(x_2) \right| \\
&\qquad\qquad \le \sum_{\alpha_1,\alpha_2} C_1 (\|s_1\| r_1)^{|\alpha_1|} C_2 (\|s_2\| r_2)^{|\alpha_2|} \|\varphi\|_{C^m(X_1\times X_2)}
\end{aligned}$$

for all $\varphi \in C^\infty_{K_1\times K_2}(X_1 \times X_2)$. □

For every $p \in \mathbb{C}^p$, $s_0 \in \mathbb{R}^p \subset \mathbb{C}^p$, let $\pi_p : \mathcal{D}'(M, \mathcal{M}_{s_0}(\mathbb{C}^p)) \to \mathcal{D}'(M, \mathcal{O}_{s_0}(\mathbb{C}^p))$ be the projection from Proposition 6.7.

Lemma 7.2. *Under the assumptions of the previous lemma,*

$$\pi_{p_1+p_2}(t_1 \boxtimes t_2) = \pi_{p_1}(t_1) \boxtimes \pi_{p_2}(t_2). \tag{35}$$

Proof. We decompose t_1 and t_2 as $t_1 = \pi_{p_1}(t_1) + (1 - \pi_1)(t_1)$ and $t_2 = \pi_{p_2}(t_2) + (1 - \pi_2)(t_2)$ where $(\pi_{p_1}(t_1), \pi_{p_2}(t_2)) \in \mathcal{D}'(X_1, \mathcal{O}_{\mu_1}(\mathbb{C}^{p_1})) \times \mathcal{D}'(X_2, \mathcal{O}_{\mu_2}(\mathbb{C}^{p_2}))$ and $((1 - \pi_1)(t_1), (1 - \pi_2)(t_2)) \in \mathcal{D}'(X_1, \mathcal{P}_{\mu_1}(\mathbb{C}^{p_1})) \times \mathcal{D}'(X_2, \mathcal{P}_{\mu_2}(\mathbb{C}^{p_2}))$. Then

$$t_1 \boxtimes t_2 = \underbrace{\pi_{p_1}(t_1) \boxtimes \pi_{p_2}(t_2)}_{\in \mathcal{D}'(X_1\times X_2, \mathcal{O}_{(\mu_1,\mu_2)}(\mathbb{C}^{p_1+p_2}))} + \underbrace{(1-\pi_1)(t_1) \boxtimes \pi_2(t_2) + \pi_1(t_1) \boxtimes (1-\pi_2)(t_2) + (1-\pi_1)(t_1) \boxtimes (1-\pi_2)(t_2)}_{\in \mathcal{D}'(X_1\times X_2, \mathcal{P}_{(\mu_1,\mu_2)}(\mathbb{C}^{p_1+p_2}))}$$

where the last underbraced term is a finite sum of polar germs by (32). It follows that $\pi_{p_1+p_2}(t_1 \boxtimes t_2) = \pi_{p_1}(t_1) \boxtimes \pi_{p_2}(t_2)$ by the uniqueness of the decomposition which follows from Theorem 6.6. □

By a similar proof, we also have

Lemma 7.3. *Let X be a smooth manifold, $U \subset X$ an open subset and $m \in \mathbb{N}$. Let $(p_1, p_2) \in \mathbb{N}^2$ and $(\mu_1, \mu_2) \in \mathbb{R}^{p_1} \times \mathbb{R}^{p_2} \subset \mathbb{C}^{p_1+p_2}$. Let $t(s_1) \in \mathcal{D}'^{,m}(U, \mathcal{M}_{\mu_1}(\mathbb{C}^{p_1}))$ and $h(s_2) \in C^m(U, \mathcal{O}_{\mu_2}(\mathbb{C}^{p_2}))$. Then the product $t(s_1)h(s_2)$ is an element of $\mathcal{D}'^{,m}(U, \mathcal{M}_{(\mu_1,\mu_2)}(\mathbb{C}^{p_1+p_2}))$ which satisfies the equation*

$$\pi_{p_1+p_2}(t(s_1)h(s_2)) = \pi_{p_1}(t(s_1))h(s_2). \tag{36}$$

Proof. We can work locally since all local results can be glued together by a partition of unity thanks to Lemma 4.2. For every $\varphi \in C_c^\infty(U)$, we have $\langle t(s_1)h(s_2), \varphi\rangle = \langle t(s_1), \underbrace{h(s_2)\varphi}_{\in C_c^m(U)}\rangle$, hence the product $t(s_1)h(s_2)$ is well defined in $\mathcal{D}'^{,m}(U)$ as soon as both $t(s_1), h(s_2)$ exist. We now explain the meromorphicity of $(s_1, s_2) \mapsto \langle t(s_1)h(s_2), \varphi\rangle$ at $(\mu_1, \mu_2) \in \mathbb{C}^{p_1+p_2}$. Since $t(s) \in \mathcal{D}'^{,m}(U, \mathcal{M}_{\mu_1}(\mathbb{C}^{p_1}))$, there exists $u(s) \in \mathcal{D}'^{,m}(U, \mathcal{O}_{\mu_1}(\mathbb{C}^{p_1}))$ and linear functions $(L_1, \dots, L_k)$ such that $(L_1(s) \dots L_k(s))t(s) = u(s)$. Therefore the product $t(s_1)h(s_2)$ also reads $\frac{1}{L_1(s_1)\dots L_k(s_1)}u(s_1)h(s_2)$. Then using power expansions in $s_1 - \mu_1$ for $u(s_1)$ as in Theorem 3.9 and expanding $h(s_2)$ in powers of $s_2 - \mu_2$ where the coefficients are in $C^m(U)$, we easily show that $u(s_1)h(s_2) \in \mathcal{D}'^{,m}(U, \mathcal{O}_{(\mu_1,\mu_2)}(\mathbb{C}^{p_1+p_2}))$ for $(s_1, s_2) \in \mathbb{C}^{p_1+p_2}$ close enough to $(\mu_1, \mu_2) \in \mathbb{C}^{p_1+p_2}$, which proves $t(s_1)h(s_2) \in \mathcal{D}'^{,m}(U, \mathcal{M}_{(\mu_1,\mu_2)}(\mathbb{C}^{p_1+p_2}))$.

The equality $\pi_{p_1+p_2}(th) = h\pi_{p_1}(t)$ follows immediately from the fact that $\pi_{p_2}(h(s_2)) = h(s_2)$ since h is holomorphic and $h(s_2)(1-\pi_{p_1})(t(s_1))$ is valued in polar germs. □

7.3. Proof of Lemma 4.1

Since our Riemannian manifold (M, g) is connected, $\ker(P)$ contains only constant functions. Indeed, $Pu = 0$ implies that $u \in C^\infty$ by elliptic regularity and $0 = \langle u, -\Delta_g u\rangle + \langle u, Vu\rangle \Rightarrow \langle \nabla u, \nabla u\rangle = 0 \Rightarrow \nabla u = 0$, thus u is constant on connected components. Let us determine the spectral projector Π explicitly: it should satisfy, for all u,

$$0 = \langle 1, u - \Pi(u)\rangle = \int_M (u - \Pi(u)) = \int_M u\,dx - \Pi(u)\operatorname{Vol}(M), \quad \text{so} \quad \Pi(u) = \frac{\int_M u\,dx}{\operatorname{Vol}(M)}.$$

The Schwartz kernel of the spectral projector Π is therefore the constant function $\Pi(x, y) = \operatorname{Vol}(M)^{-1}$.

The first two claims about the Schwartz kernel $\mathfrak{G}^s(x, y)$ follow from [72, Theorem 4 p. 302] in the celebrated work of Seeley, by applying his theorem to $A = P - \Pi$, which is a well-defined elliptic pseudodifferential operator of order 2.

For the third claim, we start from the formula $\mathfrak{G}^s = \int_0^\infty (e^{-tP} - \Pi)(x, y)t^{s-1}dt$ and our proof follows the proof of [8, Proposition 1] in which we replace the heat semigroup $e^{t\Delta_g}$ by the semigroup $e^{-tP} - \Pi$ whose Schwartz kernel is $K_t - \Pi$ and is denoted by p_t. Start from $p_t(x, y) = \langle \delta_x, (e^{-tP} - \Pi)\delta_y\rangle_{L^2(M)} = \langle (e^{-\frac{t}{2}P} - \Pi)\delta_x, (e^{-\frac{t}{2}P} - \Pi)\delta_y\rangle_{L^2(M)}$. For any integers (k, l, m), $|\partial_t^m P_x^k P_y^l p_t(x, y)| = |P_x^{k+m} P_y^l p_t(x, y)|$ since $\partial_t^m(e^{-tP} - \Pi) = P^m(e^{-tP} - \Pi)$. Hence,

$$
\begin{aligned}
&|\partial_t^m P_x^k P_y^l p_t(x, y)| \\
&\quad \leq \|e^{-(t-\varepsilon)P} - \Pi\|_{B(L^2(M))} \|P_x^{k+m}(e^{-\frac{\varepsilon}{2}P} - \Pi)\delta_x\|_{L^2(M)} \|P_y^l(e^{-\frac{\varepsilon}{2}P} - \Pi)\delta_y\|_{L^2(M)}.
\end{aligned}
$$

Therefore taking the supremum over $(x, y) \in M \times M$ yields

$$
\begin{aligned}
&\|\partial_t^m P_x^k P_y^l p_t\|_{C^0(M\times M)} \\
&\quad \leq \|(e^{-(t-\varepsilon)P} - \Pi)\|_{B(L^2(M))} \|P_x^{k+m}(e^{-\frac{\varepsilon}{2}P} - \Pi)\delta_x\|_{L^2(M)} \|P_y^l(e^{-\frac{\varepsilon}{2}P} - \Pi)\delta_y\|_{L^2(M)}
\end{aligned}
$$

where both the norms $\|P_x^{k+m}(e^{-\frac{\varepsilon}{2}P} - \Pi)\delta_x\|_{L^2(M)}$ and $\|P_y^l(e^{-\frac{\varepsilon}{2}P} - \Pi)\delta_y\|_{L^2(M)}$ are finite since both $(e^{-\frac{\varepsilon}{2}P} - \Pi)\delta_x$ and $(e^{-\frac{\varepsilon}{2}P} - \Pi)\delta_y$ are smooth functions because the semigroup $(e^{tP} - \Pi)_{t\in\mathbb{R}_{\geq 0}}$ is smoothing. Furthermore, the term $\|e^{-(t-\varepsilon)P} - \Pi\|_{B(L^2(M))}$ has exponential decay as $t \to +\infty$ since $e^{-(t-\varepsilon)P} - \Pi$ is a smoothing operator which has a gap in the spectrum; indeed, by spectral theory $e^{-tP}u = \sum_{\lambda\in\sigma(P)} e^{-t\lambda}\Pi_\lambda(u)$ where Π_λ is the spectral projector on the eigenspace of eigenvalue λ and the r.h.s. converges absolutely in all Sobolev spaces $H^s(M)$, $s \geq 0$, when $t > 0$. More generally, we obtain decay estimates of the form

$$
\begin{aligned}
\|\partial_t^m p_t\|_{C^k(M\times M)} &\leq \sum_{l_1,l_2\leq k/2+1} \|\partial_t^m P_x^{k_1} P_y^{k_2} p_t\|_{C^0(M\times M)} \\
&\leq C_{k,m}\|(e^{-(t-\varepsilon)P} - \Pi)\|_{B(L^2(M))} \leq C_{k,m}e^{-(t-\varepsilon)\lambda_1}
\end{aligned}
$$

where $\lambda_1 > 0$ is the smallest nonzero eigenvalue of P which exists since $\sigma(P)$ is a discrete subset of $[0, \infty)$. It follows that the integral $\int_1^\infty t^{s-1}p_t\,dt$ converges absolutely for all $s \in \mathbb{C}$ and is valued in all Banach spaces $C^k(M \times M)$, $k \in \mathbb{N}$, since

$$
\left\| \int_1^\infty t^{s-1} p_t\,dt \right\|_{C^k(M\times M)} \leq \int_1^\infty t^{\mathrm{Re}(s)-1}\|p_t\|_{C^k(M\times M)}dt \leq C_k \int_1^\infty t^{\mathrm{Re}(s)-1}e^{-(t-\varepsilon)\lambda_1}\,dt.
$$

The integral $\int_1^\infty t^{s-1}p_t\,dt$ depends holomorphically on s since

$$
\left\| \int_1^\infty \left(\frac{d}{ds}\right)^l t^{s-1} p_t\,dt \right\|_{C^k(M\times M)} \leq C_k \int_1^\infty t^{\mathrm{Re}(s)-1}\log(t)^l e^{-(t-\varepsilon)\lambda_1}\,dt \qquad (37)
$$

where the r.h.s. is absolutely convergent and we can conclude by dominated convergence arguments. □

7.4. Proof of Lemma 5.13

First notice that when $\mathrm{Re}(s_i) > -1$, $i = 1, \dots, E$, the integral is absolutely convergent and holomorphic in s.

Now if $E = 1$, then by integration by parts, for $\mathrm{Re}(s) > -1$,

$$\int_{[0,1]} t^s \psi(t)\,dt = \sum_{i=0}^{k-1} (-1)^i \frac{1}{l_i(s)} \psi^{(i)}(1) + (-1)^k \frac{1}{l_{k-1}(s)} \int_{[0,1]} t^{s+k} \psi^{(k)}(t)\,dt,$$

where $l_i(s) = (s+1)\dots(s+i+1)$, and the l.h.s. is a meromorphic function when $\mathrm{Re}(s) > -k-1$ with possible poles at $s = -1, \dots, -k$, so it extends to a meromorphic function on $\mathrm{Re}(s) > -k-1$.

In general, for $\mathrm{Re}(s_i) > -1$, $i = 1, \dots, E$, and $k_1, \dots, k_E \in \mathbb{Z}_{>0}$,

$$\begin{aligned} I_s(\psi) &= \int_{[0,1]^E} t_1^{s_1} \dots t_E^{s_E} \psi(t_1, \dots, t_E)\, d^E t \\ &= \sum_{\{j_1,\dots,j_m\} \subset \{1,\dots,E\}} \sum_{\substack{j \neq j_1,\dots,j_m \\ i_j = 0,\dots,k_j - 1}} \frac{(-1)^{i_j}}{l_{i_j}(s_j)} \prod^{m} \frac{(-1)^{k_{j_i}}}{l_{k_{j_i}-1}(s_{j_i})} \\ &\times \int_{[0,1]^m} \prod_{j=j_1,\dots,j_m} t^{s_j + k_j} \Big(\prod_{j \neq j_1,\dots j_m} \partial_{t_j}^{i_j} \Big) \partial_{t_{j_1}}^{k_{j_1}} \dots \partial_{t_{j_m}}^{k_{j_m}} \psi|_{t_j = 1,\, j \neq j_1,\dots j_m} dt_{j_1} \dots dt_{j_s} \end{aligned} \tag{38}$$

the r.h.s. is a meromorphic function when $\mathrm{Re}(s_i) > -k_i - 1$. So $I_s(\psi)$ extends to a meromorphic germ at any point in $\mathbb{Z}^E$.

Now at a given point $(p_e)_e \in \mathbb{Z}^E$, $\frac{1}{s_e - a_e}$ is holomorphic except at $a_e = p_e$, therefore

$$\Big(\prod_{i \in I} (s_i - p_i) \Big) I_s(\psi)$$

is a holomorphic germ at $(p_e)_e$. The distribution order of $I_s(\psi)$ at (p_e) can be read off from (38) easily. □

7.5. Proof of Lemma 5.6

In the chart $(U \times U, (x^\mu, y^\nu))$, let us consider the Taylor expansion of $\phi(x, y)$, $\phi(x, y) = \sum_{k \geq 0} \phi_{[k]}(x, y)$, where $\phi_{[k]}(x, y) = \sum_{|\alpha|+|\beta|=k} \frac{x^\alpha y^\beta}{\alpha!\beta!} \partial_x^\alpha \partial_y^\beta \phi(0, 0)$. Obviously $\phi_{[0]}(x, y) = 0$. By symmetry and $\phi(x, x) = 0$, we know that

$$\phi_{[1]}(x, y) = 0.$$

By symmetry and $\phi(x, x) \equiv 0$, we know that $\phi_{[2]}(x, y) = \sum_\mu a_\mu (x^\mu - y^\mu)^2$. Now since $\phi(0, y) = \|y\|^2$, we find that

$$\phi_{[2]}(x, y) = \sum (x^\mu - y^\mu)^2. \tag{39}$$

In fact, let us take $x = y$ in (19),

$$g^{-1}(x)\left(\frac{\partial\phi}{\partial x^\mu}(x,x)dx^\mu, \frac{\partial\phi}{\partial x^\nu}(x,x)dx^\nu\right) = 0,$$

which means

$$\frac{\partial\phi}{\partial x^\mu}(x,x)dx^\mu \equiv 0, \quad \text{so} \quad \frac{\partial\phi}{\partial x^\mu}(x,x) \equiv 0.$$

By symmetry,

$$\frac{\partial\phi}{\partial y^\mu}(x,x) \equiv 0. \tag{40}$$

Now let us make a change of variables $V \times W \to U \times U$ given by

$$(v,h) \mapsto (v, v+h);$$

we can take V, W so small that $V \times W$ is a coordinate chart around (x_0, x_0). Let $\tilde\phi(v,h) = \phi(v, v+h)$. Take a partial Taylor expansion in h for $\tilde\phi$,

$$\tilde\phi(v,h) = \tilde\phi(v,0) + \frac{\partial\tilde\phi}{\partial h^\mu}(v,0)h^\mu + \frac{1}{2}\frac{\partial^2\tilde\phi}{\partial h^\mu \partial h^\nu}(v,0)h^\mu h^\nu + \varepsilon_3$$

where ε_3 vanishes to order 3 in h. We know

$$\tilde\phi(v,0) = \phi(v,v) = 0,$$

by (40), and

$$\frac{\partial\tilde\phi}{\partial h^\mu}(v,0) = \frac{\partial\phi}{\partial y^\mu}(v,v) = 0.$$

By the chain rule,

$$\frac{\partial^2\tilde\phi}{\partial h^\mu \partial h^\nu}(v,0) = \frac{\partial^2\phi}{\partial y^\mu \partial y^\nu}(v,v).$$

Equation (19) shows

$$\frac{\partial\phi}{\partial x^\mu}(x,y)g^{\mu\nu}(x)\frac{\partial\phi}{\partial x^\nu}(x,y) = 4\phi(x,y).$$

Taking $\frac{\partial^2}{\partial x^{\mu_1}\partial x^{\nu_1}}$ on both sides and letting $x = y = v$, we get

$$\begin{aligned}\frac{\partial^2\phi}{\partial x^\mu \partial x^{\mu_1}}(v,v)g^{\mu\nu}(v)\frac{\partial^2\phi}{\partial x^\nu \partial x^{\nu_1}}(v,v) + \frac{\partial^2\phi}{\partial x^\mu \partial x^{\nu_1}}(v,v)g^{\mu\nu}(v)\frac{\partial^2\phi}{\partial x^\nu \partial x^{\mu_1}}(v,v) \\ = 4\frac{\partial^2\phi}{\partial x^{\mu_1}\partial x^{\nu_1}}(v,v),\end{aligned}$$

that is,

$$\frac{\partial^2\phi}{\partial x^\mu \partial x^{\mu_1}}(v,v)g^{\mu\nu}(v)\frac{\partial^2\phi}{\partial x^\nu \partial x^{\nu_1}}(v,v) = 2\frac{\partial^2\phi}{\partial x^{\mu_1}\partial x^{\nu_1}}(v,v).$$

Notice that $\frac{\partial^2\phi}{\partial x^\mu \partial x^\nu}(v,v)$ is invertible since $\frac{\partial^2\phi}{\partial x^\mu \partial x^\nu}(0,0) = \delta_{\mu\nu}$ by (39) if U is chosen small enough. Then we get

$$\frac{\partial^2\phi}{\partial x^\mu \partial x^\nu}(v,v) = 2g_{\mu\nu}(v).$$

Since ϕ is symmetric, we know

$$\frac{\partial^2 \phi}{\partial y^\mu \partial y^\nu}(v, v) = \frac{\partial^2 \phi}{\partial x^\mu \partial x^\nu}(v, v) = 2 g_{\mu\nu}(v).$$

So

$$\tilde{\phi}(v, h) = g_{\mu\nu}(v) h^\mu h^\nu + \varepsilon_3,$$

which concludes the proof. □

Acknowledgments. We would like to thank M. Wrochna, D. Häfner, Y. Colin de Verdière and E. Herscovich for many interesting comments they made to one of us (NVD) when we first presented this work in Grenoble. Also great thanks to C. Guillarmou and S. Paycha for answering many of our questions on regularization using pseudodifferential powers. The first author wants to thank Dang Nguyen Bac for many discussions related to blow-ups and algebraic geometry. Finally, NVD acknowledges the ANR-16-CE40-0012-01 grant for financial support.

References

[1] Ackermann, T.: A note on the Wodzicki residue. J. Geom. Phys. **20**, 404–406 (1996) Zbl 0864.58057 MR 1419429

[2] Albert, B. I.: Heat kernel renormalization on manifolds with boundary. arXiv:1609.02220 (2016)

[3] Albert, B. I.: Effective field theory on manifolds with boundary. Ph.D. thesis, Univ. of Pennsylvania (2017) MR 3731923

[4] Ammann, B., Bär, C.: The Einstein–Hilbert action as a spectral action. In: Noncommutative Geometry and the Standard Model of Elementary Particle Physics, Springer, 75–108 (2002) Zbl 1255.81218 MR 1998531

[5] Atiyah, M. F.: Resolution of singularities and division of distributions. Comm. Pure Appl. Math. **23**, 145–150 (1970) Zbl 0188.19405 MR 0256156

[6] Atiyah, M. F., Bott, R., Patodi, V. K.: On the heat equation and the index theorem. Invent. Math. **19**, 279–330 (1973) Zbl 0257.58008 MR 0650828

[7] Bär, C., Ginoux, N., Pfäffle, F.: Wave Equations on Lorentzian Manifolds and Quantization. Eur. Math. Soc. (2007) Zbl 1118.58016 MR 2298021

[8] Bär, C., Moroianu, S.: Heat kernel asymptotics for roots of generalized Laplacians. Int. J. Math. **14**, 397–412 (2003) Zbl 1079.58020 MR 1984660

[9] Bär, C., Strohmaier, A.: An index theorem for Lorentzian manifolds with compact spacelike Cauchy boundary. Amer. J. Math. **141**, 1421–1455 (2019) Zbl 1429.83004 MR 4011805

[10] Bergbauer, C., Brunetti, R., Kreimer, D.: Renormalization and resolution of singularities. arXiv:0908.0633 (2009)

[11] Bergbauer, C., Kreimer, D.: The Hopf algebra of rooted trees in Epstein–Glaser renormalization. Ann. Henri Poincaré **6**, 343–367 (2005) Zbl 1065.81096 MR 2136195

[12] Berghoff, M.: Wonderful renormalization. PhD thesis, Humboldt-Univ. Berlin (2015)

[13] Berghoff, M.: Wonderful compactifications in quantum field theory. Comm. Number Theory Phys. **9**, 477–547 (2015) Zbl 1376.81052 MR 3399925

[14] Berline, N., Getzler, E., Vergne, M.: Heat Kernels and Dirac Operators. Springer, Berlin (2004) Zbl 1037.58015 MR 2273508

[15] Berline, N., Vergne, M.: Local asymptotic Euler–Maclaurin expansion for Riemann sums over a semi-rational polyhedron. In: Configuration Spaces, Springer, 67–105 (2016) Zbl 1387.52026 MR 3615728

[16] Brunetti, R., Fredenhagen, K.: Microlocal analysis and interacting quantum field theories: renormalization on physical backgrounds. Comm. Math. Phys. **208**, 623–661 (2000) Zbl 1040.81067 MR 1736329

[17] Bytsenko, A. A., Cognola, G., Elizalde, E., Moretti, V., Zerbini, S.: Analytic Aspects of Quantum Fields. World Sci. (2003) Zbl 1138.81035 MR 2022075

[18] Cardona, A., Ducourtioux, C., Paycha, S.: From tracial anomalies to anomalies in Quantum Field Theory. Comm. Math. Phys. **242**, 31–65 (2003) Zbl 1037.81089 MR 2018268

[19] Ceyhan, O., Marcolli, M.: Feynman integrals and motives of configuration spaces. Comm. Math. Phys. **313**, 35–70 (2012) Zbl 1256.81051 MR 2928218

[20] Clavier, P., Guo, L., Paycha, S., Zhang, B.: An algebraic formulation of the locality principle in renormalisation. Eur. J. Math. **5**, 356–394 (2019) Zbl 07073688 MR 3946584

[21] Connes, A., Kreimer, D.: Hopf algebras, renormalization and noncommutative geometry. Comm. Math. Phys. **199**, 203–42 (1998) Zbl 0932.16038 MR 1660199

[22] Connes, A., Kreimer, D.: Renormalization in quantum field theory and the Riemann–Hilbert problem I: The Hopf algebra structure of graphs and the main theorem. Comm. Math. Phys. **210**, 249–273 (2000) Zbl 1032.81026 MR 1748177

[23] Connes, A., Kreimer, D.: Renormalization in quantum field theory and the Riemann–Hilbert problem II: the β-function, diffeomorphisms and the renormalization group. Comm. Math. Phys. **216**, 215–241 (2001) Zbl 1042.81059 MR 1810779

[24] Costello, K.: Renormalization and Effective Field Theory. Math. Surveys Monogr. 170, Amer. Math. Soc. (2011) Zbl 1221.81004 MR 2778558

[25] Dang, N. V.: Complex powers of analytic functions and meromorphic renormalization in QFT. arXiv:1503.00995 (2015)

[26] Dereziński, J., Siemssen, D.: Feynman propagators on static spacetimes. Rev. Math. Phys. **30**, art. 1850006, 23 pp. (2018) Zbl 1394.35407 MR 3770965

[27] Dereziński, J., Siemssen, D.: An evolution equation approach to the Klein–Gordon operator on curved spacetime. Pure Appl. Anal. **1**, 215–261 (2019) Zbl 1423.58016 MR 3949374

[28] Duistermaat, J. J.: Fourier Integral Operators. Birkhäuser, Boston (1996) Zbl 0841.35137 MR 1362544

[29] Dütsch, M., Fredenhagen, K., Keller, K. J., Rejzner, K.: Dimensional regularization in position space and a forest formula for Epstein–Glaser renormalization. J. Math. Phys. **55**, art. 122303, 37 pp. (2014) Zbl 1309.81173 MR 3390545

[30] Epstein, H., Glaser, V.: The role of locality in perturbation theory. Ann. Inst. H. Poincaré Sect. A **19**, 211–295 (1973) Zbl 1216.81075 MR 0342091

[31] Fulling, S. A.: Vacuum energy as spectral geometry. Symmetry Integrability Geom. Methods Appl. **3**, art. 094, 23 pp. (2007) Zbl 1142.35052 MR 2366928

[32] Gérard, C., Oulghazi, O., Wrochna, M.: Hadamard states for the Klein–Gordon equation on Lorentzian manifolds of bounded geometry. Comm. Math. Phys. **352**, 519–583 (2017) Zbl 1364.35362 MR 3627405

[33] Gérard, C., Wrochna, M.: The massive Feynman propagator on asymptotically Minkowski spacetimes. Amer. J. Math. **141**, 1501–1546 (2019) Zbl 07155013 MR 4030522

[34] Gérard, C., Wrochna, M.: Feynman propagators and Hadamard states from scattering data for the Klein–Gordon equation on asymptotically Minkowski spacetimes. arXiv:1603.07465 (2016)

[35] Gérard, C., Wrochna, M.: Analytic Hadamard states, Calderón projectors and Wick rotation near analytic Cauchy surfaces. Comm. Math. Phys. **366**, 29–65 (2019) Zbl 1414.58002 MR 3919442

[36] Gilkey, P.: Invariance Theory, the Heat Equation and the Atiyah–Singer Index Theorem. CRC Press (1995) Zbl 0856.58001 MR 1396308

[37] Grothendieck, A.: Sur certains espaces de fonctions holomorphes. I. J. Reine Angew. Math. **192**, 35–64 (1953) Zbl 0051.08704 MR 0058865

[38] Gunning, R., Rossi, H.: Analytic Functions of Several Complex Variables. Prentice-Hall, Englewood Cliffs, NJ (1965) Zbl 0141.08601 MR 0180696

[39] Guo, L., Paycha, S., Zhang, B.: A conical approach to Laurent expansions for multivariate meromorphic germs with linear poles. Pacific J. Math. **307**, 159–196 (2020) Zbl 07229725 MR 4131805

[40] Guo, L., Paycha, S., Zhang, B.: Counting an infinite number of points: a testing ground for renormalization methods. In: Geometric, Algebraic, and Topological Methods for Quantum Field Theory (Villa de Leyva, 2013), World Sci., 309–352 (2017) Zbl 1369.81072 MR 3617360

[41] Gurau, R., Rivasseau, V., Sfondrini, A.: Renormalization: an advanced overview. arXiv:1401.5003 (2014)

[42] Hairer, M.: An analyst's take on the BPHZ theorem. In: Computation and Combinatorics in Dynamics, Stochastics and Control, Abel Sympos. 13, Springer, 429–476 (2018) Zbl 1408.60049 MR 3967393

[43] Hawking, S. W.: Zeta function regularization of path integrals in curved spacetime. Comm. Math. Phys. **55**, 133–148 (1977) Zbl 0407.58024 MR 0524257

[44] Herscovich, E.: Renormalization in quantum field theory (after R. Borcherds). Astérisque 412 (2019) Zbl 1425.81005 MR 3978423

[45] Hollands, S., Wald, R. M.: Local Wick polynomials and time ordered products of quantum fields in curved spacetime. Comm. Math. Phys. **223**, 289–326 (2001) Zbl 0989.81081 MR 1864435

[46] Hollands, S., Wald, R. M.: Existence of local covariant time ordered products of quantum fields in curved spacetime. Comm. Math. Phys. **231**, 309–45 (2002) Zbl 1015.81043 MR 1946335

[47] Kalau, W., Walze, M.: Gravity, non-commutative geometry and the Wodzicki residue. J. Geom. Phys. **16**, 327–344 (1995) Zbl 0826.58008 MR 1336738

[48] Kastler, D.: The Dirac operator and gravitation. Comm. Math. Phys. **166**, 633–643 (1995) Zbl 0823.58046 MR 1312438

[49] Kontsevich, M., Vishik, S.: Geometry of determinants of elliptic operators. In: Functional Analysis on the Eve of the 21st Century, Vol. 1, Progr. Math. 131, Birkhäuser, 173–197 (1995) Zbl 0920.58061 MR 1373003

[50] Krajewski, T., Rivasseau, V., Tanasă, A., Wang, Z.: Topological graph polynomials and quantum field theory. Part I: Heat kernel theories. J. Noncommut. Geom. **4**, 29–82 (2010) Zbl 1186.81095 MR 2575389

[51] Kruskal, J. B.: On the shortest spanning subtree of a graph and the traveling salesman problem. Proc. Amer. Math. Soc. **7**, 48–50 (1956) Zbl 0070.18404 MR 0078686

[52] Lefschetz, S.: Applications of Algebraic Topology. Springer (1975) Zbl 0328.55001 MR 0494126

[53] Lesch, M.: On the noncommutative residue for pseudodifferential operators with log-polyhomogeneous symbols. Ann. Global Anal. Geom. **17**, 151–187 (1999) Zbl 0920.58047 MR 1675408

[54] Melrose, R. B.: The Atiyah–Patodi–Singer Index Theorem. A K Peters, Wellesley, MA (1993) Zbl 0796.58050 MR 1348401

[55] Melrose, R. B., Nistor, V.: Homology of pseudodifferential operators I. Manifolds with boundary. arXiv:funct-an/9606005 (1996)

[56] Mnev, P.: Lecture notes on torsions. arXiv:1406.3705 (2014)

[57] Nicolaescu, L. I.: Random Morse functions and spectral geometry. arXiv:1209.0639 (2012)

[58] Nikolov, N., Nedanovski, D.: Analytic renormalization and residues of Feynman diagrams, C. R. Acad. Bulgare Sci. **70**, 1219–1226 (2017)

[59] Nikolov, N. M., Stora, R., Todorov, I.: Renormalization of massless Feynman amplitudes in configuration space. Rev. Math. Phys. **26**, no. 4, art. 1430002, 65 pp. (2014) Zbl 1303.81126 MR 3208883

[60] Paycha, S.: Regularised Integrals, Sums and Traces: An Analytic Point of View. Univ. Lecture Ser. 59, Amer. Math. Soc. (2012) Zbl 1272.11103 MR 2987296

[61] Paycha, S., Scott, S.: Chern–Weil forms associated with superconnections. In: Analysis, Geometry and Topology of Elliptic Operators, World Sci., 79–104 (2006) Zbl 1272.11103 MR 2246766

[62] Paycha, S., Scott, S.: A Laurent expansion for regularized integrals of holomorphic symbols. Geom. Funct. Anal. **17**, 491–536 (2007) Zbl 1125.58009 MR 2322493

[63] Popineau, G., Stora, R.: A pedagogical remark on the main theorem of perturbative renormalization theory. Nuclear Phys. B **912**, 70–78 (2016) Zbl 1349.81142 MR 3567574

[64] Pottel, S.: A BPHZ theorem in configuration space. arXiv:1706.06762 (2017)

[65] Pottel, S.: Configuration space BPHZ renormalization on analytic spacetimes. Nuclear Phys. B **927**, 274–293 (2018) Zbl 1380.81195 MR 3763330

[66] Ray, D. B., Singer, I. M.: R-torsion and the Laplacian on Riemannian manifolds. Adv. Math. **7**, 145–210 (1971) Zbl 0239.58014 MR 0295381

[67] Riesz, M.: L'intégrale de Riemann–Liouville et le problème de Cauchy. Acta Math. **81**, 1–222 (1949) Zbl 0033.27601 MR 0030102

[68] Rivasseau, V.: From Perturbative to Constructive Renormalization. Princeton Univ. Press, Princeton (1991) MR 1174294

[69] Roe, J.: Elliptic Operators, Topology, and Asymptotic Methods. Longman (1998) Zbl 0919.58060 MR 1670907

[70] Schwarz, A. S.: The partition function of degenerate quadratic functional and Ray–Singer invariants. Lett. Math. Phys. **2**, 247–252 (1978) Zbl 0383.70017 MR 0676337

[71] Scott, S.: Traces and Determinants of Pseudodifferential Operators. Oxford Univ. Press, Oxford (2010) Zbl 1216.35192 MR 2683288

[72] Seeley, R. T.: Complex powers of an elliptic operator. In: Singular Integrals (Chicago, IL, 1966), Proc. Sympos. Pure Math. 10, Amer. Math. Soc., 288–307 (1967) Zbl 0159.15504 MR 0237943

[73] Speer, E. R.: Analytic renormalization. J. Math. Phys. **9**, 1404–1410 (1968)

[74] Speer, E. R.: Dimensional and analytic renormalization. In: Renormalization Theory, Reidel, 25–93 (1976) MR 0522287

[75] Speer, E. R.: Lectures on analytic renormalization. Technical report 73-067, Univ. of Maryland (1972)

[76] Taylor, M.: Partial Differential Equations II: Qualitative Studies of Linear Equations. Appl. Math. Sci. 116, Springer (2011) Zbl 1206.35003 MR 2743652

[77] Vasy, A., Wrochna, M.: Quantum fields from global propagators on asymptotically Minkowski and extended de Sitter spacetimes. Ann. Henri Poincaré **19**, 1529–1586 (2018) Zbl 06871241 MR 3784921

[78] Wodzicki, M.: Noncommutative residue. Chapter I. Fundamentals. In: K-Theory, Arithmetic and Geometry, Springer, 320–399 (1987) Zbl 0649.58033 MR 0923140

[79] Wodzicki, M.: Local invariants of spectral asymmetry. Invent. Math. **75**, 143–177 (1984) Zbl 0538.58038 MR 0728144