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Antonin Chambolle · Vito Crismale

Compactness and lower semicontinuity in *GSBD*

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Abstract. In this paper we prove a compactness and semicontinuity result in *GSBD* for sequences with bounded Griffith energy. This generalises classical results in $(G)SBV$ by Ambrosio [1, 2, 4] and *SBD* by Bellettini–Coscia–Dal Maso [9]. As a result, the static problem in Francfort–Marigo's variational approach to crack growth [30] admits (weak) solutions.

Keywords. Generalised special functions of bounded deformation, brittle fracture, compactness

1. Introduction

The variational approach to fracture was introduced by Francfort and Marigo [30] in order to build crack evolutions in brittle materials, following Griffith's laws [36], without *a priori* knowledge of the crack path (or surface in higher dimension). It relies on successive minimisations of the *Griffith energy*:

$$(u, K) \mapsto \int_{\Omega\setminus K} \mathbf{C}e(u) : e(u)\,\mathrm{d}x + \gamma\mathcal{H}^{n-1}(K)$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set, the *reference configuration*, $u : \Omega \to \mathbb{R}^n$ is an (infinitesimal) *displacement*, $e(u)$ its symmetrised gradient (the *infinitesimal elastic strain*) and $\mathbf{C}$ the *Cauchy stress tensor* defining the *Hooke law* (in particular, $\mathbf{C}a : a$ defines a positive definite quadratic form of the $n \times n$ symmetric tensor a). The symmetrised gradient $e(u)$ is defined out of the *crack set* K, which is in the theory a compact $(n-1)$-dimensional set and is penalised by its surface (multiplied by a coefficient γ called the *toughness*).

The minimisation of the energy is under the constraint that K should contain a previously computed crack K_0, and that u should satisfy a Dirichlet condition $u = u_0$ on a subset $\partial_D\Omega \setminus K$ of $\partial\Omega$, where $\partial_D\Omega$ is a regular part of the boundary and u_0 a sufficiently

A. Chambolle, V. Crismale: CMAP, École Polytechnique, CNRS, 91128 Palaiseau Cedex, France; e-mail: antonin.chambolle@cmap.polytechnique.fr, vito.crismale@polytechnique.edu

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regular displacement. Hence an important question in the theory is whether the problem

$$\min_{\substack{u=u_0 \text{ on } \partial_D\Omega\setminus K\\ K_0\subset K \text{ compact}}} \int_{\Omega\setminus K} \mathbf{C}e(u):e(u)\,\mathrm{d}x + \gamma\mathcal{H}^{n-1}(K) \tag{1.1}$$

has a solution.

This problem however is not easy to analyse, since the energy controls very little of the function u: for instance if K almost cuts out from $\partial_D\Omega$ a connected component of Ω, the function u may have any (arbitrarily large) value in this component at small cost.

From a technical point of view, one cannot take truncations or compositions with bounded transformations to get an *a priori* L^∞ bound for minimisers. In fact, the integrability of $e(u)$ is in general lost by $e(\psi(u))$, unless $\psi(y) = y_0 + \lambda y$ for some $y_0 \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$ (see e.g. the introduction of [21]).

For this reason, most of the "sound" approaches to problem (1.1) consider additional assumptions. In particular, a global L^∞ bound on the displacements ensures one may work in the class *SBD* of *Special functions with Bounded Deformation* [5], provided one considers a *weak formulation* of the problem where K is replaced with the intrinsic jump set J_u of u (which need not to be closed anymore): in this space minimising sequences are shown to be compact [9], and the energy to be lower semicontinuous. Another possible assumption is, in $2d$, that the crack set K be connected [25, 12].

The natural space for studying (1.1) is not, in fact, $SBD(\Omega)$ (which assumes that the symmetrised gradient of u is a measure and hence u is in $L^{n/(n-1)}(\Omega;\mathbb{R}^n)$) but the space $GSBD(\Omega)$, introduced by Dal Maso in [21]. This space, defined by the slicing properties of the functions, is designed so as to contain "all" displacements u for which the energy is finite. Even if [21] proves compactness under very mild assumptions on the integrability of displacements, no compactness result was available in *GSBD* for minimizing sequences of (the weak formulation of) (1.1) until very recently.

The first existence result without further constraint has been proven indeed in [35], in *dimension two*. It relies on a delicate construction showing a *piecewise Korn inequality* [33] (for approximate Korn and Korn–Poincaré inequalities see also e.g. [19, 13, 32], for piecewise rigidity cf. [18]).

In this paper, we prove the following general compactness result for sequences bounded in energy, in the space $GSBD(\Omega)$, in any dimension.

Theorem 1.1. *Let $\phi\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a non-decreasing function with*

$$\lim_{t\to+\infty} \frac{\phi(t)}{t} = +\infty, \tag{1.2}$$

and let $(u_h)_h$ be a sequence in $GSBD(\Omega)$ such that

$$\int_\Omega \phi(|e(u_h)|)\,\mathrm{d}x + \mathcal{H}^{n-1}(J_{u_h}) < M \tag{1.3}$$

for some constant M independent of h. Then there exists a subsequence, still denoted by $(u_h)_h$, such that

$$A := \{x \in \Omega\colon |u_h(x)| \to +\infty\} \tag{1.4}$$

has finite perimeter, and $u \in GSBD(\Omega)$ *with* $u = 0$ *on* A *for which*

$$u_h \to u \qquad \mathcal{L}^n\text{-a.e. in } \Omega \setminus A, \tag{1.5a}$$

$$e(u_h) \rightharpoonup e(u) \quad \text{in } L^1(\Omega \setminus A; \mathbb{M}^{n\times n}_{\mathrm{sym}}), \tag{1.5b}$$

$$\mathcal{H}^{n-1}(J_u \cup \partial^* A) \le \liminf_{h\to\infty} \mathcal{H}^{n-1}(J_{u_h}). \tag{1.5c}$$

The proof of this theorem is in our opinion simpler than in [35], even if a fundamental tool is a quite technical Korn–Poincaré inequality for functions with small jump set, proved in [13] and employed also in [14, 15, 16]. We combine this inequality with arguments in the spirit of Rellich's type compactness theorems.

Theorem 1.1 gives the existence of minimisers for the Griffith energy with Dirichlet boundary conditions in the weak formulation (see Theorem 4.1), which by results in [20, 15] have the properties of strong solutions in the interior of Ω. In [17] we prove existence of solutions for the strong formulation (1.1) by extending the regularity theorems in [20, 15] up to the boundary, when $\partial_D\Omega$ is of class C^1 and u_0 is Lipschitz.

The major issue for establishing the compactness result of Theorem 1.1 comes from the lack of control on both the displacement and its full gradient, as is natural in the study of brittle fracture in small strain (linearised) elasticity [36].

A bound such as (1.3) for the full gradient in place of the symmetrised gradient is available for brittle fractures models in finite strain elasticity or in small strain elasticity in the simplified *antiplane case* (i.e. when the displacement u is vertical and depends only on the horizontal components). In these cases, the energy is closely related to the *Mumford–Shah functional* in image reconstruction [39] (which however includes a fidelity term, artificial from a mechanical standpoint). In this context, the original strategy of passing through a weak formulation in terms of u was first proposed by De Giorgi and realised by Ambrosio [1, 2, 3, 4], for the existence of weak solutions, and De Giorgi, Carriero and Leaci [27] (see also e.g. [11, 28]), for the regularity, giving the improvement to strong solutions (an alternative approach, where the discontinuity set is the main variable, has been successfully employed in [24, 38]).

Ambrosio's results are obtained in the space *GSBV* [26], and have been extended to *GSBD* by Dal Maso [21]. In both cases, control of the values is required to obtain compactness, guaranteeing that the set A in Theorem 1.1 is empty. Without such control, it is still relatively simple to obtain a *GSBV* version of Theorem 1.1. For instance, in the scalar case one can consider as in [1] the sequences of truncated functions $u_k^N := \max\{-N, \min\{u_k, N\}\}$ for any integer $N \ge 1$, which are compact in BV and converge up to subsequences. Then, by a diagonal argument, and then letting $N \to +\infty$, one builds a subsequence $(u_{k_h})_h$ which converges a.e. to some u, except on a possible set A where it goes to $+\infty$ or $-\infty$. The scalar version of (1.5b) is obtained exactly as in [1] (see in particular [1, Prop. 4.4]), by considering perturbations $w \in L^1(\Omega)$ with $w = 0$ a.e. in A. One possible way to derive inequality (1.5c) is then by slicing arguments, similar to (but simpler than) the arguments in Section 3 of the current paper. The extension to the vectorial case is not difficult in *GSBV*.

This strategy however fails in our case since, as already mentioned, the space *GSBD* is not stable by truncations. The way out to get compactness without any assumption on the displacements is to locally approximate *GSBD* functions with piecewise infinitesimal rigid motions, by means of the Korn–Poincaré inequality of [13], and use the fact that such motions belong to a finite-dimensional space. We then obtain compactness with respect to convergence in $\mathcal{L}^n$-measure, but still we cannot exclude the existence of a set A of points where the limit is not in $\mathbb{R}^n$. A slicing argument is then used to show that A has finite perimeter, whose measure is controlled by (1.5c). (Existence for (1.1) is then deduced by considering the limit of a minimising sequence and setting in A the limit function equal to 0, or to any ground state of the elastic energy.)

A more general (and difficult) approach, for $GSBV^p$, has been proposed by Friedrich [34]: there, the set A is a priori removed by a careful modification at the level of the minimising sequence, with control of the energy. Friedrich and Solombrino [35] also prove existence of quasistatic evolutions in dimension two, extending in that case the antiplane result by Francfort and Larsen [29] (see [8] for the existence of *strong* quasistatic evolutions in dimension two, and e.g. [22, 23] for quasistatic evolutions for brittle fractures with finite strain elasticity).

2. Notation and preliminaries

For every $x \in \mathbb{R}^n$ and $\varrho > 0$ let $B_\varrho(x)$ be the open ball with centre x and radius ϱ. For $x, y \in \mathbb{R}^n$, we write $x \cdot y$ for the scalar product and $|x|$ for the norm. We denote by $\mathcal{L}^n$ and $\mathcal{H}^k$ the n-dimensional Lebesgue measure and the k-dimensional Hausdorff measure. For any locally compact subset B of $\mathbb{R}^n$, the space of bounded $\mathbb{R}^m$-valued Radon measures on B is denoted by $\mathcal{M}_b(B;\mathbb{R}^m)$. For $m = 1$ we write $\mathcal{M}_b(B)$ for $\mathcal{M}_b(B;\mathbb{R})$ and $\mathcal{M}_b^+(B)$ for the subspace of positive measures of $\mathcal{M}_b(B)$. For every $\mu \in \mathcal{M}_b(B;\mathbb{R}^m)$, its total variation is denoted by $|\mu|(B)$. We write χ_E for the indicator function of any $E \subset \mathbb{R}^n$, which is 1 on E and 0 elsewhere. An *infinitesimal rigid motion* is any affine function with skew-symmetric gradient. Also set $\widetilde{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ and $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$.

Definition 2.1. Let $E \subset \mathbb{R}^n$, $v\colon E \to \mathbb{R}^m$ an $\mathcal{L}^n$-measurable function, and $x \in \mathbb{R}^n$ such that

$$\limsup_{\varrho \to 0^+} \frac{\mathcal{L}^n(E \cap B_\varrho(x))}{\varrho^n} > 0.$$

A vector $a \in \mathbb{R}^n$ is the *approximate limit* of v as y tends to x if for every $\varepsilon > 0$,

$$\lim_{\varrho \to 0^+} \frac{\mathcal{L}^n(E \cap B_\varrho(x) \cap \{|v - a| > \varepsilon\})}{\varrho^n} = 0,$$

and we then write

$$\operatorname*{ap\,lim}_{y \to x} v(y) = a. \tag{2.1}$$

Remark 2.2. Let E, v, x, and a be as in Definition 2.1 and let ψ be a homeomorphism between $\mathbb{R}^m$ and a bounded open subset of $\mathbb{R}^m$. Then (2.1) holds if and only if

$$\lim_{\varrho \to 0^+} \frac{1}{\varrho^n} \int_{E \cap B_\varrho(x)} |\psi(v(y)) - \psi(a)| \,\mathrm{d}y = 0.$$

Definition 2.3. Let $U \subset \mathbb{R}^n$ open, and $v\colon U \to \mathbb{R}^m$ be $\mathcal{L}^n$-measurable. The *approximate jump set* J_v is the set of points $x \in U$ for which there exist $a, b \in \mathbb{R}^m$, with $a \neq b$, and $\nu \in \mathbb{S}^{n-1}$ such that

$$\operatorname*{ap\,lim}_{(y-x)\cdot\nu>0,\ y\to x} v(y) = a \quad \text{and} \quad \operatorname*{ap\,lim}_{(y-x)\cdot\nu<0,\ y\to x} v(y) = b.$$

The triplet (a, b, ν) is uniquely determined up to a permutation of (a, b) and a change of sign of ν, and is denoted by $(v^+(x), v^-(x), \nu_v(x))$. The *jump* of v is the function defined by $[v](x) := v^+(x) - v^-(x)$ for every $x \in J_v$. Moreover, we define

$$J_v^1 := \{x \in J_v \colon |[v](x)| \geq 1\}. \tag{2.2}$$

Remark 2.4. By Remark 2.2, J_v and J_v^1 are Borel sets and $[v]$ is a Borel function. By Lebesgue's differentiation theorem, it follows that $\mathcal{L}^n(J_v) = 0$.

BV and BD functions. If $U \subset \mathbb{R}^n$ is open, a function $v \in L^1(U)$ is *of bounded variation* on U, and we write $v \in BV(U)$, if $\mathrm{D}_i v \in \mathcal{M}_b(U)$ for $i = 1, \dots, n$, where $\mathrm{D}v = (\mathrm{D}_1 v, \dots, \mathrm{D}_n v)$ is its distributional gradient. A vector-valued function $v\colon U \to \mathbb{R}^m$ is in $BV(U; \mathbb{R}^m)$ if $v_j \in BV(U)$ for every $j = 1, \dots, m$. The space $BV_{\mathrm{loc}}(U)$ is the space of $v \in L^1_{\mathrm{loc}}(U)$ such that $\mathrm{D}_i v \in \mathcal{M}_b(U)$ for $i = 1, \dots, n$.

An $\mathcal{L}^n$-measurable bounded set $E \subset \mathbb{R}^n$ is a set of *finite perimeter* if χ_E is a function of bounded variation. The *reduced boundary* of E, denoted by $\partial^* E$, is the set of points $x \in \operatorname{supp}|\mathrm{D}\chi_E|$ such that the limit $\nu_E(x) := \lim_{\varrho\to 0^+} \frac{\mathrm{D}\chi_E(B_\varrho(x))}{|\mathrm{D}\chi_E|(B_\varrho(x))}$ exists and satisfies $|\nu_E(x)| = 1$. The reduced boundary is countably $\mathcal{H}^{n-1}$-rectifiable (in the sense of [6, Definition 2.57], and the function ν_E is called the *generalised inner normal* to E.

A function $v \in L^1(U; \mathbb{R}^n)$ belongs to the space of *functions of bounded deformation* if its distributional symmetric gradient $\mathrm{E}v$ belongs to $\mathcal{M}_b(U; \mathbb{R}^n)$. It is well known (see [5, 40]) that for $v \in BD(U)$, J_v is countably $\mathcal{H}^{n-1}$-rectifiable, and that

$$\mathrm{E}v = \mathrm{E}^a v + \mathrm{E}^c v + \mathrm{E}^j v, \tag{2.3}$$

where $\mathrm{E}^a v$ is absolutely continuous with respect to $\mathcal{L}^n$, $\mathrm{E}^c v$ is singular with respect to $\mathcal{L}^n$ and such that $|\mathrm{E}^c v|(B) = 0$ if $\mathcal{H}^{n-1}(B) < \infty$, while $\mathrm{E}^j v$ is concentrated on J_v. The density of $\mathrm{E}^a v$ with respect to $\mathcal{L}^n$ is denoted by $e(v)$, and we find that (see [5, Theorem 4.3] and recall (2.1)), for $\mathcal{L}^n$-a.e. $x \in U$,

$$\operatorname*{ap\,lim}_{y\to x} \frac{(v(y) - v(x) - e(v)(x)(y - x)) \cdot (y - x)}{|y - x|^2} = 0. \tag{2.4}$$

The space $SBD(U)$ is the subspace of all functions $v \in BD(U)$ such that $\mathrm{E}^c v = 0$, while for $p \in (1, \infty)$

$$SBD^p(U) := \{v \in SBD(U) \colon e(v) \in L^p(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}}),\ \mathcal{H}^{n-1}(J_v) < \infty\}.$$

Analogous properties hold for BV, like the countable rectifiability of the jump set and the decomposition of $\mathrm{D}v$, and the spaces $SBV(U; \mathbb{R}^m)$ and $SBV^p(U; \mathbb{R}^m)$ are defined similarly, with ∇v, the density of $\mathrm{D}^a v$, in place of $e(v)$. For a complete treatment of BV, SBV functions and BD, SBD functions, we refer to [6] and to [5, 9, 7, 40], respectively.

GBD functions. We now recall the definition and the main properties of the space GBD of *generalised functions of bounded deformation*, introduced in [21], referring to that paper for a general treatment and more details. Since the definition of GBD is given by slicing (in contrast to the definition of *GBV*, cf. [26, 2]), we introduce before some notation.

For fixed $\xi \in \mathbb{S}^{n-1} := \{\xi \in \mathbb{R}^n : |\xi| = 1\}$, and for any $y \in \mathbb{R}^n$ and $B \subset \mathbb{R}^n$, let

$$\Pi^\xi := \{y \in \mathbb{R}^n : y \cdot \xi = 0\}, \qquad B^\xi_y := \{t \in \mathbb{R} : y + t\xi \in B\},$$

and for every function $v \colon B \to \mathbb{R}^n$ and $t \in B^\xi_y$ let

$$v^\xi_y(t) := v(y + t\xi), \qquad \widehat{v}^\xi_y(t) := v^\xi_y(t) \cdot \xi.$$

Definition 2.5 ([21]). Let $\Omega \subset \mathbb{R}^n$ be bounded and open, and $v \colon \Omega \to \mathbb{R}^n$ be $\mathcal{L}^n$-measurable. Then $v \in GBD(\Omega)$ if there exists $\lambda_v \in \mathcal{M}^+_b(\Omega)$ such that one of the following equivalent conditions holds true for every $\xi \in \mathbb{S}^{n-1}$:

(a) for every $\tau \in C^1(\mathbb{R})$ with $-1/2 \le \tau \le 1/2$ and $0 \le \tau' \le 1$, the partial derivative $\mathrm{D}_\xi(\tau(v \cdot \xi)) = \mathrm{D}(\tau(v \cdot \xi)) \cdot \xi$ belongs to $\mathcal{M}_b(\Omega)$, and for every Borel set $B \subset \Omega$,

$$|\mathrm{D}_\xi(\tau(v \cdot \xi))|(B) \le \lambda_v(B);$$

(b) $\widehat{v}^\xi_y \in BV_{\mathrm{loc}}(\Omega^\xi_y)$ for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$, and for every Borel set $B \subset \Omega$,

$$\int_{\Pi^\xi} \Big(|\mathrm{D}\widehat{v}^\xi_y|(B^\xi_y \setminus J^1_{\widehat{v}^\xi_y}) + \mathcal{H}^0(B^\xi_y \cap J^1_{\widehat{v}^\xi_y})\Big)\,\mathrm{d}\mathcal{H}^{n-1}(y) \le \lambda_v(B), \tag{2.5}$$

where $J^1_{\widehat{u}^\xi_y} := \{t \in J_{\widehat{u}^\xi_y} : |[\widehat{u}^\xi_y]|(t) \ge 1\}$.

The function v belongs to $GSBD(\Omega)$ if $v \in GBD(\Omega)$ and $\widehat{v}^\xi_y \in SBV_{\mathrm{loc}}(\Omega^\xi_y)$ for every $\xi \in \mathbb{S}^{n-1}$ and for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$.

$GBD(\Omega)$ and $GSBD(\Omega)$ are vector spaces, as stated in [21, Remark 4.6], and one has the inclusions $BD(\Omega) \subset GBD(\Omega)$, $SBD(\Omega) \subset GSBD(\Omega)$, which are in general strict (see [21, Remark 4.5 and Example 12.3]). For every $v \in GBD(\Omega)$ the *approximate jump set* J_v is still countably $\mathcal{H}^{n-1}$-rectifiable [21, Theorem 6.2] and can be reconstructed from the jump of the slices $\widehat{v}^\xi_y$ [21, Theorem 8.1]. Indeed, for every C^1 manifold $M \subset \Omega$ with unit normal ν, for $\mathcal{H}^{n-1}$-a.e. $x \in M$ there exist the *traces* $v^+_M(x), v^-_M(x) \in \mathbb{R}^n$ such that

$$\operatorname*{ap\,lim}_{\pm(y-x)\cdot\nu(x)>0,\, y\to x} v(y) = v^\pm_M(x) \tag{2.6}$$

and they can be reconstructed from the traces of the one-dimensional slices (see [21, Theorem 5.2]). Every $v \in GBD(\Omega)$ has an *approximate symmetric gradient* $e(v) \in L^1(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}})$, characterised by (2.4) and such that for every $\xi \in \mathbb{S}^{n-1}$ and $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$,

$$e(v)^\xi_y \xi \cdot \xi = \nabla \widehat{v}^\xi_y \qquad \mathcal{L}^1\text{-a.e. on } \Omega^\xi_y. \tag{2.7}$$

By these properties of slices it follows that if $v \in GSBD(\Omega)$ with $e(v) \in L^1(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}})$ and $\mathcal{H}^{n-1}(J_v) < +\infty$, then for every Borel set $B \subset \Omega$,

$$\mathcal{H}^{n-1}(J_v \cap B) = (2\omega_{n-1})^{-1} \int_{\mathbb{S}^{n-1}} \left(\int_{\Pi^\xi} \mathcal{H}^0(J_{v^\xi_y} \cap B^\xi_y)\, \mathrm{d}\mathcal{H}^{n-1}(y) \right) \mathrm{d}\mathcal{H}^{n-1}(\xi) \tag{2.8}$$

and the two conditions in the definition of *GSBD* for v hold for $\lambda_v \in \mathcal{M}^+_b(\Omega)$ such that

$$\lambda_v(B) \leq \int_B |e(v)|\, \mathrm{d}x + \mathcal{H}^{n-1}(J_v \cap B) \tag{2.9}$$

for every Borel set $B \subset \Omega$ (cf. also [31, Theorem 1] and [37, Remark 2]).

We now recall the following result, proven in [13, Proposition 2]. Notice that the proposition therein is stated in *SBD*, but the proof, which is based on the Fundamental Theorem of Calculus along lines, still holds for *GSBD*, with small adaptations.

Proposition 2.6 ([13]). *Let $Q_r = (-r, r)^n$, $v \in GSBD(Q)$, and $p \in [1, \infty)$. Then there exist a Borel set $\omega \subset Q_r$ and an affine function $a\colon \mathbb{R}^n \to \mathbb{R}^n$ with $e(a) = 0$ such that*

$$\mathcal{L}^n(\omega) \leq cr\mathcal{H}^{n-1}(J_v)$$

and

$$\int_{Q_r\setminus\omega} |v - a|^p\, \mathrm{d}x \leq cr^p \int_{Q_r} |e(v)|^p\, \mathrm{d}x. \tag{2.10}$$

The constant c depends only on p and n.

We conclude the section with a technical lemma.

Lemma 2.7. *Let $E \subset \mathbb{R}^n$ be Borel, $v_h\colon E \to \mathbb{R}^n$ for every h, and consider the n sequences $(v_h \cdot e_i)_h$, obtained by taking every component of v_h with respect to the canonical basis $\{e_1, \dots, e_n\}$ of $\mathbb{R}^n$. Assume that every $(v_h \cdot e_i)_h$ converges pointwise $\mathcal{L}^n$-a.e. to a $v_i\colon E \to \widetilde{\mathbb{R}}$, and that for $\mathcal{L}^n$-a.e. $x \in E$ there is $i \in \{1, \dots, n\}$ for which $v_i(x) \in \{-\infty, +\infty\}$. Then for $\mathcal{H}^{n-1}$-a.e. $\xi \in \mathbb{S}^{n-1}$,*

$$|v_h \cdot \xi| \to +\infty \quad \mathcal{L}^n\text{-a.e. in } E. \tag{2.11}$$

Proof. On the sets

$$E_i := \{|v_h \cdot e_i| \to +\infty\} \cap \bigcap_{j\neq i} \{\limsup_{h\to\infty} (|v_h \cdot e_j|/|v_h \cdot e_i|) < +\infty\},$$

(2.11) holds for every ξ in $\{\xi \in \mathbb{S}^{n-1}\colon\ \xi_i \neq 0\}$, which is of full $\mathcal{H}^{n-1}$-measure in $\mathbb{S}^{n-1}$.

Let us now consider the case when there are m components of v_h, with $1 < m \leq n$, that we may assume to be $v_h \cdot e_1, \dots, v_h \cdot e_m$, such that $\frac{v_h\cdot e_i}{v_h\cdot e_j} \to \xi_{i,j} \in \mathbb{R}^*$ for $1 \leq i < j \leq m$ and $|\frac{v_h\cdot e_i}{v_h\cdot e_j}| \to +\infty$ for $i \in \{1, \dots, m\}$ and $j \in \{m+1, \dots, n\}$ (if $m < n$). In this case (2.11) does not hold only on

$$\mathbb{S}^{n-1} \cap (1, \xi^{-1}_{1,2}, \dots, \xi^{-1}_{1,m}, 0 \dots, 0)^\perp,$$

which has dimension $n-2$. Notice now that for every m for which m components go faster to infinity than the others, there is an at most countable collection of $(\xi_{1,2},\dots,\xi_{1,m})\in(\mathbb{R}^*)^{m-1}$ for which $\frac{v_h\cdot e_1}{v_h\cdot e_j}\to\xi_{1,j}$ for $j\in\{2,\dots,m\}$ on a subset of E of positive $\mathcal{L}^n$-measure. Thus (2.11) holds for every ξ except for an at most countable union of $\mathcal{H}^{n-1}$-negligible sets of $\mathbb{S}^{n-1}$. □

3. The main compactness and lower semicontinuity result

In this section we prove Theorem 1.1, the main result of the paper.

Proof of Theorem 1.1. We divide the proof into three parts: compactness (with respect to convergence in measure, by means of approximation through piecewise infinitesimal rigid motions), lower semicontinuity, and closure (in *GSBD*).

Compactness. For every $k\in\mathbb{N}$ and $z\in(2k^{-1})\mathbb{Z}^n$ we consider the cubes of centre z

$$q_{k,z}:=z+(-k^{-1},k^{-1})^n.$$

Then $\Omega_k:=\Omega\setminus\bigcup_{q_{k,z}\not\subset\Omega}\overline{q_{k,z}}$ is essentially the union of the cubes which are contained in Ω.

We apply Proposition 2.6 with $p=1$ in any $q_{k,z}\subset\Omega$, so for $r=k^{-1}$. Then there exist sets $\omega^h_{k,z}\subset q_{k,z}$ with

$$\mathcal{L}^n(\omega^h_{k,z})\le ck^{-1}\mathcal{H}^{n-1}(J_{u_h}\cap q_{k,z})\tag{3.1}$$

and affine functions $a^h_{k,z}\colon\mathbb{R}^n\to\mathbb{R}^n$, with $e(a^h_{k,z})=0$, such that

$$\int_{q_{k,z}\setminus\omega^h_{k,z}}|u_h-a^h_{k,z}|\,\mathrm{d}x\le ck^{-1}\int_{q_{k,z}}|e(u_h)|\,\mathrm{d}x.\tag{3.2}$$

The functions $(a^h_{k,z})_{h\ge1}$ belong to the finite-dimensional space of affine functions. Any sequence of the i-th components $(a^h_{k,z}\cdot e_i)_h$, $i=1,\dots,n$, has one of the following properties:

- it is bounded, and then it converges uniformly (up to a subsequence) to an affine function;
- it is unbounded, and then either
  - it converges globally, up to a subsequence, to $+\infty$ or $-\infty$, or
  - there is a hyperplane $\{x\cdot\nu=t\}$ ($\nu\in\mathbb{R}^n$, $t\in\mathbb{R}$) and a subsequence such that $a^h_{k,z}(x)\cdot e_i\to+\infty$ if $x\cdot\nu>t$ and $a^h_{k,z}(x)\cdot e_i\to-\infty$ if $x\cdot\nu<t$.

  (To see this, consider the bounded sequence $\frac{a^h_{k,z}\cdot e_i}{\|a^h_{k,z}\cdot e_i\|}$, for any norm $\|\cdot\|$ on the space of affine functions, which has converging subsequences.)

Let τ denote the function tanh (or any smooth, 1-Lipschitz increasing function from -1 to 1 with $\tau(0) = 0$). Then, up to a subsequence, the function

$$a_k^h(x) := \sum_{q_{z,k} \subset \Omega} a_{k,z}^h(x) \chi_{q_{k,z}}(x)$$

is such that $(\tau(a_k^h \cdot e_i))_h$ converges to some function in $L^1(\Omega_k)$, for any $i = 1, \dots, n$. Indeed, we have

$$\tau(a_k^h \cdot e_i)(x) = \sum_{q_{z,k} \subset \Omega} \tau(a_{k,z}^h \cdot e_i)(x) \, \chi_{q_{k,z}}(x),$$

and in any cube $q_{k,z}$ the sequence $(\tau(a_{k,z}^h \cdot e_i))_h$ converges uniformly either to a function valued in $(-1, 1)$, if $(a_{k,z}^h \cdot e_i)_h$ is bounded, or to a function with values -1 and 1, attained where the limit of $(a_{k,z}^h \cdot e_i)_h$ is $+\infty$ or $-\infty$, respectively (notice that at this stage k is fixed).

Clearly the subsequence could be extracted from a previous subsequence built at stage $k-1$, hence by a diagonal argument, we may assume that for any k, $(\tau(a_k^h \cdot e_i))_h$ converges for all $i = 1, \dots, n$ in $L^1(\Omega_k)$.

For each $i = 1, \dots, n$, $k \geq 1$, and $l, m \geq 1$, we have

$$\begin{aligned}\int_\Omega |\tau(u_m \cdot e_i) - \tau(u_l \cdot e_i)| \,\mathrm{d}x &\leq 2|\Omega \setminus \Omega_k| + \int_{\Omega_k} |\tau(u_m \cdot e_i) - \tau(a_k^m \cdot e_i)| \,\mathrm{d}x \\ &\quad + \int_{\Omega_k} |\tau(a_k^m \cdot e_i) - \tau(a_k^l \cdot e_i)| \,\mathrm{d}x + \int_{\Omega_k} |\tau(u_l \cdot e_i) - \tau(a_k^l \cdot e_i)| \,\mathrm{d}x. \end{aligned} \tag{3.3}$$

By construction,

$$\lim_{l,m \to +\infty} \int_{\Omega_k} |\tau(a_k^m \cdot e_i) - \tau(a_k^l \cdot e_i)| \,\mathrm{d}x = 0.$$

On the other hand,

$$\begin{aligned}\int_{\Omega_k} |\tau(u_m \cdot e_i) - \tau(a_k^m \cdot e_i)| \,\mathrm{d}x &= \sum_{q_{k,z} \subset \Omega} \int_{q_{k,z}} |\tau(u_m \cdot e_i) - \tau(a_{k,z}^m \cdot e_i)| \,\mathrm{d}x \\ &\leq \sum_{q_{k,z} \subset \Omega} \left(2|\omega_{k,z}^m| + \int_{q_{k,z} \setminus \omega_{k,z}^m} |u_m - a_{k,z}^m| \,\mathrm{d}x \right) \\ &\leq \frac{2c}{k} \left(\mathcal{H}^{n-1}(J_{u_m}) + \int_{\Omega_k} |e(u_m)| \,\mathrm{d}x \right) \leq \frac{C}{k}. \end{aligned}$$

Since $|\Omega \setminus \Omega_k| \to 0$ as $k \to \infty$, we deduce from (3.3) that $(\tau(u_h \cdot e_i))_h$ is a Cauchy sequence (for each i) and therefore converges in $L^1(\Omega)$ to some limit which we denote $\tilde{\tau}_i$. Up to a further subsequence, we may assume that the convergence occurs almost everywhere and, by (1.2) and (1.3), that $(e(u_h))_h$ converges weakly in $L^1(\Omega; \mathbb{M}^{n \times n}_{\mathrm{sym}})$. This determines the (sub)sequence $(u_h)_h$ for which we are going to prove the result, fixed from now on. First notice that the set A defined in (1.4) (corresponding to the subsequence) is such that $(u_h)_h$ converges pointwise $\mathcal{L}^n$-a.e. in $\Omega \setminus A$ to a function with finite values (that is, in $\mathbb{R}^n$).

We define $\bar{u}\colon \Omega \to (\widetilde{\mathbb{R}})^n$ and $u\colon \Omega \to \mathbb{R}^n$ such that

$$\bar{u} := (\tilde{u}^1, \dots, \tilde{u}^n), \qquad \text{where } \tilde{u}^i = \tau^{-1}(\tilde{\tau}_i); \qquad u := \bar{u}\chi_{\Omega\setminus A}, \tag{3.4}$$

with the convention that $\tau^{-1}(\pm 1) = \pm\infty$.

The set A, which coincides with $\{x \in \Omega\colon \tilde{u}^i(x) \in \{-\infty, +\infty\}$ for some i in $\{1,\dots,n\}\}$, is measurable, since $\tilde{u}^i(x) \in \mathbb{R}$ if and only if $|\tau(\tilde{u}^i)| < 1$ and the functions $\tilde{\tau}_i\colon \Omega \to [-1,1]$ are measurable. Since $(u_h)_h$ converges pointwise $\mathcal{L}^n$-a.e. in $\Omega\setminus A$ to u, we find that for every $\xi \in \mathbb{S}^{n-1}$,

$$u_h\cdot\xi \to u\cdot\xi \qquad \mathcal{L}^n\text{-a.e. in } \Omega\setminus A. \tag{3.5}$$

Notice that we have not extracted further subsequences depending on ξ, and that the limit function u (equal to $\bar{u}$ since we are in $\Omega\setminus A$) does not depend on ξ. Eventually, by Lemma 2.7 we conclude that for $\mathcal{H}^{n-1}$-a.e. $\xi\in\mathbb{S}^{n-1}$,

$$|u_h\cdot\xi| \to +\infty \quad \mathcal{L}^n\text{-a.e. in } A. \tag{3.6}$$

Lower semicontinuity. Here we prove first (1.5c), which is specific to our approach due to the description of A, and then (1.5b), which follows the lines of [9, Theorem 1.1].

As in [9, Theorem 1.1] (see also [21, Theorem 11.3]), we introduce

$$\mathrm{I}^\xi_y(u_h) := \int_{\Omega^\xi_y} \phi(|(\dot{u}_h)^\xi_y|)\,\mathrm{d}t, \tag{3.7}$$

where $(\dot{u}_h)^\xi_y$ is the density of the absolutely continuous part of $\mathrm{D}(\widehat{u}_h)^\xi_y$, the distributional derivative of $(\widehat{u}_h)^\xi_y$ ($(\widehat{u}_h)^\xi_y \in SBV_{\mathrm{loc}}(\Omega^\xi_y)$ for every $\xi\in\mathbb{S}^{n-1}$ and for $\mathcal{H}^{n-1}$-a.e. $y\in\Pi^\xi$, since $u_h\in GSBD(\Omega)$). Thus for any $\xi\in\mathbb{S}^{n-1}$,

$$\int_{\Pi^\xi} \mathrm{I}^\xi_y(u_h)\,\mathrm{d}\mathcal{H}^{n-1}(y) = \int_\Omega \phi\big(|e(u_h)(x)\xi\cdot\xi|\big) \le \int_\Omega \phi(|e(u_h)|)\,\mathrm{d}x \le M, \tag{3.8}$$

by Fubini–Tonelli's theorem and (1.3), recalling that ϕ is non-decreasing. Moreover, since $u_h\in GSBD(\Omega)$, we have $\mathrm{D}_\xi(\tau(u_h\cdot\xi)) \in \mathcal{M}^+_b(\Omega)$ for every $\xi\in\mathbb{S}^{n-1}$ and

$$\int_{\Pi^\xi} |\mathrm{D}(\tau(u_h\cdot\xi)^\xi_y)|(\Omega^\xi_y)\,\mathrm{d}\mathcal{H}^{n-1}(y) = |\mathrm{D}_\xi(\tau(u_h\cdot\xi))|(\Omega) \le M, \tag{3.9}$$

by (2.9) and (1.3). We denote

$$\mathrm{II}^\xi_y(u_h) := |\mathrm{D}(\tau(u_h\cdot\xi)^\xi_y)|(\Omega^\xi_y). \tag{3.10}$$

Let $(u_k)_k = (u_{h_k})_k$ be a subsequence of $(u_h)_h$ such that

$$\lim_{k\to\infty} \mathcal{H}^{n-1}(J_{u_k}) = \liminf_{h\to\infty} \mathcal{H}^{n-1}(J_{u_h}) < +\infty, \tag{3.11}$$

so that, by (2.8), (3.8), and Fatou's lemma, for $\mathcal{H}^{n-1}$-a.e. $\xi \in \mathbb{S}^{n-1}$ we have

$$\liminf_{k\to\infty} \int_{\Pi^\xi} \big[\mathcal{H}^0(J_{(\widehat{u}_k)^\xi_y}) + \varepsilon\big(\mathrm{I}^\xi_y(u_k) + \mathrm{II}^\xi_y(u_k)\big)\big]\,\mathrm{d}\mathcal{H}^{n-1}(y) < +\infty, \tag{3.12}$$

for a fixed $\varepsilon \in (0,1)$. Fix $\xi \in \mathbb{S}^{n-1}$ such that (3.6) and (3.12) hold. Then there is a subsequence $(u_m)_m = (u_{k_m})_m$ of $(u_k)_k$, depending on ε and ξ, such that

$$\begin{aligned}\lim_{m\to\infty} \int_{\Pi^\xi} &\big[\mathcal{H}^0\big(J_{(\widehat{u}_m)^\xi_y}\big) + \varepsilon(\mathrm{I}^\xi_y(u_m) + \mathrm{II}^\xi_y(u_m))\big]\,\mathrm{d}\mathcal{H}^{n-1}(y)\\ &= \liminf_{k\to\infty} \int_{\Pi^\xi} \big[\mathcal{H}^0\big(J_{(\widehat{u}_k)^\xi_y}\big) + \varepsilon(\mathrm{I}^\xi_y(u_k) + \mathrm{II}^\xi_y(u_k))\big]\,\mathrm{d}\mathcal{H}^{n-1}(y).\end{aligned} \tag{3.13}$$

Therefore, by (3.13), (3.5), and (3.6), employing Fatou's lemma, for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ we have

$$\liminf_{m\to\infty} \big[\mathcal{H}^0\big(J_{(\widehat{u}_m)^\xi_y}\big) + \varepsilon(\mathrm{I}^\xi_y(u_m) + \mathrm{II}^\xi_y(u_m))\big] < +\infty, \tag{3.14}$$

$$(\widehat{u}_m)^\xi_y \to \widehat{u}^\xi_y \quad \mathcal{L}^1\text{-a.e. in } (\Omega\setminus A)^\xi_y, \qquad |(\widehat{u}_m)^\xi_y| \to \infty \quad \mathcal{L}^1\text{-a.e. in } A^\xi_y, \tag{3.15}$$

$$\tau(u_m\cdot\xi)^\xi_y \to \tilde{\tau}^\xi_y \quad \text{in } L^1(\Omega^\xi_y), \tag{3.16}$$

for a suitable $\tilde{\tau}^\xi_y \in L^1(\Omega^\xi_y)$. Now we employ (3.5), (3.6), and (3.15), (3.16) to get

$$\begin{cases} \tilde{\tau}^\xi_y = \tau(u\cdot\xi)^\xi_y & \mathcal{L}^1\text{-a.e. in } (\Omega\setminus A)^\xi_y,\\ |\tilde{\tau}^\xi_y| = 1 & \mathcal{L}^1\text{-a.e. in } A^\xi_y. \end{cases} \tag{3.17}$$

For fixed $y \in \Pi^\xi$ satisfying (3.14) and (3.15), and such that $(\widehat{u}_m)^\xi_y \in SBV_{\mathrm{loc}}(\Omega^\xi_y)$ for every m, we extract a subsequence $(u_j)_j = (u_{m_j})_j$ from $(u_m)_m$, depending also on y, for which

$$\lim_{j\to\infty} \big[\mathcal{H}^0\big(J_{(\widehat{u}_j)^\xi_y}\big) + \varepsilon(\mathrm{I}^\xi_y(u_j) + \mathrm{II}^\xi_y(u_j))\big] = \liminf_{m\to\infty} \big[\mathcal{H}^0\big(J_{(\widehat{u}_m)^\xi_y}\big) + \varepsilon(\mathrm{I}^\xi_y(u_m) + \mathrm{II}^\xi_y(u_m))\big]. \tag{3.18}$$

Then by (3.16) we have

$$\tau(u_j\cdot\xi)^\xi_y \overset{*}{\rightharpoonup} \tilde{\tau}^\xi_y \quad \text{in } SBV(\Omega^\xi_y). \tag{3.19}$$

In order to describe the set A, we consider its slices A^ξ_y and prove that for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$,

$$A^\xi_y \text{ is a finite union of intervals where } \tilde{\tau}^\xi_y \text{ has the value either 1 or } -1, \tag{3.20}$$

and

$$\partial A^\xi_y \subset J_{\tilde{\tau}^\xi_y}. \tag{3.21}$$

Recalling that $|\tilde{\tau}^\xi_y| < 1$ in $(\Omega\setminus A)^\xi_y$, by (3.17), the property above states that there is a jump each time one passes from values of $\tilde{\tau}^\xi_y$ with absolute value less than 1 to A^ξ_y, that is, the set where $|\tilde{\tau}^\xi_y| = 1$. In terms of the slices of u, one passes from finite to infinite values.

Let us show the claimed properties. Up to considering a subsequence of $(\widehat{u}_j)^\xi_y$, we may assume that for every j,

$$\mathcal{H}^0(J_{(\widehat{u}_j)^\xi_y}) = N_y \in \mathbb{N},$$

namely there is a fixed number N_y of jump points. These points tend to $M_y \le N_y$ points

$$t_1, \dots, t_{M_y}.$$

Then (recall that $\mathrm{II}^\xi_y(u_j)$ is equibounded in j by (3.18)) for every $l = 1, \dots, M_y - 1$,

$$\tau(u_j \cdot \xi)^\xi_y \rightharpoonup \tilde{\tau}^\xi_y \quad \text{in } W^{1,1}_{\mathrm{loc}}(t_l, t_{l+1}),$$

and the convergence is locally uniform (for the precise representatives). Moreover, since $\mathrm{I}^\xi_y(u_j)$ is equibounded again by (3.18), it follows that $x \mapsto (\widehat{u}_j)^\xi_y(x) - (\widehat{u}_j)^\xi_y(\overline{x})$ is locally uniformly bounded in (t_l, t_{l+1}), for any choice of $\overline{x} \in (t_l, t_{l+1})$ (by the Fundamental Theorem of Calculus). Hence for any l, either

- there is $\overline{x} \in (t_l, t_{l+1})$ such that

$$\lim_{j\to\infty} (\widehat{u}_j)^\xi_y(\overline{x}) = \widehat{u}^\xi_y(\overline{x}) \in \mathbb{R}$$

  (that is, $\overline{x} \notin A^\xi_y$), and then $(\widehat{u}_j)^\xi_y$ converges locally uniformly in (t_l, t_{l+1}) to $\widehat{u}^\xi_y$; or
- for $\mathcal{L}^1$-a.e. $x \in (t_l, t_{l+1})$,

$$\lim_{j\to\infty} |(\widehat{u}_j)^\xi_y(x)| = \infty,$$

  that is, $(t_l, t_{l+1}) \subset A^\xi_y$.

Therefore any (t_l, t_{l+1}) is contained either in $(\Omega \setminus A)^\xi_y$ or in A^ξ_y. Moreover, in the first case we have $\widehat{u}^\xi_y \in W^{1,1}(t_l, t_{l+1}) \subset L^\infty(t_l, t_{l+1})$; in particular, there is $\eta \in (0,1)$ such that

$$\tilde{\tau}^\xi_y(t_l, t_{l+1}) \subset [-1+\eta, 1-\eta]. \tag{3.22}$$

This implies (3.20) and (3.21).

By (3.18), (3.19), (3.21), and since the jump sets of $\tau(u_j \cdot \xi)^\xi_y$ and $(\widehat{u}_j)^\xi_y$ coincide, we deduce, by lower semicontinuity of *SBV* functions defined in one-dimensional domains (see [1, Proposition 4.2]), that

$$\begin{aligned}\mathcal{H}^0(J_{\widehat{u}^\xi_y} \cap (\Omega \setminus A)^\xi_y) + \mathcal{H}^0(\partial A^\xi_y) &\le \mathcal{H}^0(J_{\tilde{\tau}^\xi_y}) \\ &\le \liminf_{m\to\infty} \big[\mathcal{H}^0\big(J_{(\widehat{u}_m)^\xi_y}\big) + \varepsilon(\mathrm{I}^\xi_y(u_m) + \mathrm{II}^\xi_y(u_m))\big].\end{aligned} \tag{3.23}$$

We now integrate over $y \in \Pi^\xi$ and use Fatou's lemma with (3.13) to get

$$\begin{aligned}&\int_{\Pi^\xi} \big[\mathcal{H}^0\big(J_{\widehat{u}^\xi_y} \cap (\Omega \setminus A)^\xi_y\big) + \mathcal{H}^0(\partial A^\xi_y)\big] \,\mathrm{d}\mathcal{H}^{n-1}(y) \\ &\qquad \le \liminf_{k\to\infty} \int_{\Pi^\xi} \big[\mathcal{H}^0\big(J_{(\widehat{u}_k)^\xi_y}\big) + \varepsilon(\mathrm{I}^\xi_y(u_k) + \mathrm{II}^\xi_y(u_k))\big] \,\mathrm{d}\mathcal{H}^{n-1}(y)\end{aligned} \tag{3.24}$$

for $\mathcal{H}^{n-1}$-a.e. $\xi \in \mathbb{S}^{n-1}$. In particular we deduce that A has finite perimeter (cf. [6, Remark 3.104]).

We integrate (3.24) over $\xi \in \mathbb{S}^{n-1}$; by (2.8), (3.8), (3.9), and (3.11) we get

$$\mathcal{H}^{n-1}(J_u \cap (\Omega \setminus A)) + \mathcal{H}^{n-1}(\partial^* A) \le CM\varepsilon + \liminf_{h\to\infty} \mathcal{H}^{n-1}(J_{u_h}), \tag{3.25}$$

for a universal constant C. By the arbitrariness of ε and the definition of u we obtain (1.5c).

The property (1.5b) follows by an adaptation of the arguments in [9, Theorem 1.1] as in [21, Theorem 11.3] (which employ Ambrosio–Dal Maso's [1, Prop. 4.4]). We report the proof for the reader's convenience.

Fatou's lemma and (2.8) imply that for $\mathcal{H}^{n-1}$-a.e. $\xi \in \mathbb{S}^{n-1}$,

$$\liminf_{h\to\infty} \int_{\Pi^\xi} \mathcal{H}^0(J_{(\widehat{u}_h)^\xi_y} \cap \Omega^\xi_y)\, \mathrm{d}\mathcal{H}^{n-1}(y) < +\infty. \tag{3.26}$$

In particular there is a basis $\{\xi_1, \dots, \xi_n\}$ of $\mathbb{R}^n$ such that this holds for every ξ of the form $\xi = \xi_i + \xi_j$, $i, j = 1, \dots, n$. We fix a ξ of this type, and we find a subsequence $(u_k)_k = (u_{h_k})_k$ of $(u_h)_h$, depending on ξ, such that

$$\lim_{k\to\infty} \int_{\Pi^\xi} \mathcal{H}^0(J_{(\widehat{u}_k)^\xi_y} \cap \Omega^\xi_y)\, \mathrm{d}\mathcal{H}^{n-1}(y) = \liminf_{h\to\infty} \int_{\Pi^\xi} \mathcal{H}^0(J_{(\widehat{u}_h)^\xi_y} \cap \Omega^\xi_y)\, \mathrm{d}\mathcal{H}^{n-1}(y). \tag{3.27}$$

For a given $w \in L^1(\Omega)$ let (recall (3.7) for the definition of $(\dot{u}_k)^\xi_y$)

$$\mathrm{III}^\xi_y(u_k, w) := \int_{(\Omega\setminus A)^\xi_y} |(\dot{u}_k)^\xi_y - w^\xi_y|\, \mathrm{d}t.$$

By (2.7), (1.3) (the sequence $(u_h)_h$ has been fixed before (3.4)), and Fubini–Tonelli's theorem there is a subsequence $(u_l)_l = (u_{k_l})_l$ of $(u_k)_k$ such that

$$\lim_{l\to\infty} \int_{\Pi^\xi} \mathrm{III}^\xi_y(u_l, w)\, \mathrm{d}\mathcal{H}^{n-1}(y) = \liminf_{k\to\infty} \int_{\Omega\setminus A} |e(u_k)\xi\cdot\xi - w|\, \mathrm{d}x < +\infty. \tag{3.28}$$

Let us also fix $\varepsilon \in (0, 1)$. Again by Fubini–Tonelli's theorem, there is a subsequence $(u_m)_m = (u_{l_m})_m$ of $(u_l)_l$, depending on ξ, w, ε, such that (3.15) holds for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$ and

$$\begin{aligned}\lim_{m\to\infty} &\int_{\Pi^\xi} \mathrm{III}^\xi_y(u_m, w) + \varepsilon[\mathcal{H}^0(J_{(\widehat{u}_m)^\xi_y}) + \mathrm{I}^\xi_y(u_m) + \mathrm{II}^\xi_y(u_m)]\, \mathrm{d}\mathcal{H}^{n-1}(y)\\ &= \liminf_{l\to\infty} \int_{\Pi^\xi} \mathrm{III}^\xi_y(u_l, w) + \varepsilon[\mathcal{H}^0(J_{(\widehat{u}_l)^\xi_y}) + \mathrm{I}^\xi_y(u_l) + \mathrm{II}^\xi_y(u_l)]\, \mathrm{d}\mathcal{H}^{n-1}(y).\end{aligned} \tag{3.29}$$

By (3.8), (3.11), (3.27), (3.28), and Fatou's lemma, for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$,

$$\liminf_{m\to\infty} \big[\mathrm{III}^\xi_y(u_m, w) + \varepsilon[\mathcal{H}^0(J_{(\widehat{u}_m)^\xi_y}) + \mathrm{I}^\xi_y(u_m) + \mathrm{II}^\xi_y(u_m)]\big] < +\infty. \tag{3.30}$$

Let $y \in \Pi^\xi$ be such that (3.15) and (3.30) hold, and $(\widehat{u}_m)^\xi_y \in SBV_{\mathrm{loc}}(\Omega^\xi_y)$ for every m. We find a subsequence $(u_j)_j = (u_{m_j})_j$ of $(u_m)_m$, depending also on y, for which

$$\begin{aligned}\lim_{j\to\infty}\big[\mathrm{III}^\xi_y(u_j, w) + \varepsilon[\mathcal{H}^0(J_{(\widehat{u}_j)^\xi_y}) + \mathrm{I}^\xi_y(u_j) + \mathrm{II}^\xi_y(u_j)]\big]\\ = \liminf_{m\to\infty}\big[\mathrm{III}^\xi_y(u_m, w) + \varepsilon[\mathcal{H}^0(J_{(\widehat{u}_m)^\xi_y}) + \mathrm{I}^\xi_y(u_m) + \mathrm{II}^\xi_y(u_m)]\big]. \qquad (3.31)\end{aligned}$$

Recalling the form of A^ξ_y (and (3.22)) we deduce that $(\widehat{u}_j)^\xi_y$ converges to $\widehat{u}^\xi_y$ weakly* in $BV(I)$ for any I compactly contained in $(\Omega\setminus A)^\xi_y$, and then $(\dot{u}_j)^\xi_y \rightharpoonup \dot{u}^\xi_y$ in $L^1((\Omega\setminus A)^\xi_y)$, by (1.2). Together with (3.31) this gives

$$\begin{aligned}\mathrm{III}^\xi_y(u, w) &\le \liminf_{j\to\infty}\mathrm{III}^\xi_y(u_j, w)\\ &\le \liminf_{m\to\infty}\big[\mathrm{III}^\xi_y(u_m, w) + \varepsilon[\mathcal{H}^0(J_{(\widehat{u}_m)^\xi_y}) + \mathrm{I}^\xi_y(u_m) + \mathrm{II}^\xi_y(u_m)]\big].\end{aligned}$$

Integrating with respect to $y \in \Pi^\xi$, by Fatou's lemma and (3.28), (3.29) plus the bounds (3.8), (3.9), (3.12), we get

$$\begin{aligned}&\int_{\Omega\setminus A} |e(u)\xi\cdot\xi - w|\\ &\le \liminf_{k\to\infty}\int_{\Omega\setminus A}|e(u_k)\xi\cdot\xi - w|\,\mathrm{d}x + \varepsilon\Big(CM + \liminf_{h\to\infty}\int_{\Pi^\xi}\mathcal{H}^0(J_{(\widehat{u}_h)^\xi_y}\cap\Omega^\xi_y)\,\mathrm{d}\mathcal{H}^{n-1}(y)\Big).\end{aligned}$$

By (3.26) and the arbitrariness of ε, we deduce that for all $w \in L^1(\Omega)$,

$$\int_{\Omega\setminus A}|e(u)\xi\cdot\xi - w| \le \liminf_{k\to\infty}\int_{\Omega\setminus A}|e(u_k)\xi\cdot\xi - w|\,\mathrm{d}x.$$

Since the sequence $(e(u_h))_h$ weakly converges in $L^1(\Omega\setminus A;\mathbb{M}^{n\times n}_{\mathrm{sym}})$, [1, Proposition 4.4] gives

$$e(u_h)\xi\cdot\xi \rightharpoonup e(u)\xi\cdot\xi \quad \text{in } L^1(\Omega\setminus A),$$

and by the arbitrariness of $\xi = \xi_i + \xi_j$ we deduce (1.5b).

Closure. We now show that the limit function u, defined in (3.4), is in $GSBD(\Omega)$.

Employing (2.9) and recalling (1.3), we find that there exist $\lambda_{u_h} \in \mathcal{M}^+_b(\Omega)$ such that

$$\lambda_{u_h}(\Omega) \le M,$$

and for every $\xi \in \mathbb{S}^{n-1}$ and every Borel set $B \subset \Omega$,

$$|\mathrm{D}_\xi(\tau(u_h\cdot\xi))|(B) \le \lambda_{u_h}(B).$$

Let $\tilde{\lambda} \in \mathcal{M}^+_b(\Omega)$ be the weak* limit of a subsequence of $(\lambda_{u_h})_h$, so that $\tilde{\lambda}(\Omega) \le M$. Notice that

$$\mathrm{D}_\xi\tau(u\cdot\xi) \in \mathcal{M}_b(\Omega) \quad \text{for every } \xi \in \mathbb{S}^{n-1} \qquad (3.32)$$

and

$$|\mathrm{D}_\xi \tau(\tilde{u} \cdot \xi)|(B) \leq \tilde{\lambda}(B) =: \lambda_u(B) \tag{3.33}$$

for every Borel set $B \subset \Omega$, where $\tilde{\lambda}$ has been defined above. This follows by a slicing procedure and the use of Fatou's lemma for every ξ, to reconstruct at the end $|\mathrm{D}_\xi(\tau(u \cdot \xi))|(\Omega)$ from $\mathrm{II}^\xi_y(u) := |\mathrm{D}(\tau(u \cdot \xi)^\xi_y)|(\Omega^\xi_y)$ (see (3.10)), as in (3.9). The important point here is to get the semicontinuity

$$\mathrm{II}^\xi_y(u) \leq \liminf_{j\to\infty} \mathrm{II}^\xi_y(u_j) = \liminf_{j\to\infty} |\mathrm{D}(\tau(u_j \cdot \xi)^\xi_y)|(\Omega^\xi_y)$$

for the slices, which follows from (3.19). Indeed, $\mathrm{II}^\xi_y(u) \leq |\mathrm{D}(\tilde{\tau}^\xi_y))|(\Omega^\xi_y)$ because $\tau(u\cdot\xi)^\xi_y = \tilde{\tau}^\xi_y$ in $(\Omega \setminus A)^\xi_y$ by (3.17) and $\tau(u \cdot \xi) = 0$ in A^ξ_y, so we employ (3.21). Moreover, it is immediate that $\widehat{u}^\xi_y \in SBV_{\mathrm{loc}}(\Omega^\xi_y)$. Therefore $u \in GSBD(\Omega)$. □

4. Existence for minimisers of the Griffith energy

Employing Theorem 1.1, we deduce in this section the existence of weak solutions to the minimisation problem of the Griffith energy with Dirichlet boundary conditions.

Existence of weak solutions

Assume $\Omega \subset \mathbb{R}^n$ is an open, bounded domain for which

$$\partial\Omega = \partial_D\Omega \cup \partial_N\Omega \cup N,$$

with $\partial_D\Omega$ and $\partial_N\Omega$ relatively open, $\partial_D\Omega \cap \partial_N\Omega = \emptyset$, $\mathcal{H}^{n-1}(N) = 0$, $\partial_D\Omega \neq \emptyset$, and $\partial(\partial_D\Omega) = \partial(\partial_N\Omega)$. Let $u_0 \in W^{1,p}(\mathbb{R}^n;\mathbb{R}^n)$ and $W\colon \mathbb{M}^{n\times n}_{\mathrm{sym}} \to [0,\infty)$ be convex, with $W(0) = 0$ and

$$W(\xi) \geq \phi(|\xi|) \qquad \text{for } \xi \in \mathbb{M}^{n\times n}_{\mathrm{sym}}, \tag{4.1}$$

where ϕ satisfies (1.2).

Let $K_0 \subset \Omega \cup \partial_D\Omega$ be countably $\mathcal{H}^{n-1}$-rectifiable with $\mathcal{H}^{n-1}(K_0) < +\infty$, and consider the minimisation problem

$$\min_{v\in GSBD(\Omega)} \left\{ \int_\Omega W(e(v))\,\mathrm{d}x + \mathcal{H}^{n-1}\big(J_v \cup (\partial_D\Omega \cap \{\mathrm{tr}_\Omega\, v \neq \mathrm{tr}_\Omega\, u_0\}) \setminus K_0\big) \right\}. \tag{4.2}$$

Notice that, defining $\widetilde{\Omega} := \Omega \cup U$, where U is an open bounded set with $U \cap \partial\Omega = \partial_D\Omega$, we can recast the problem as

$$\min_{v\in GSBD(\widetilde{\Omega})} \left\{ \int_{\widetilde{\Omega}} W(e(v))\,\mathrm{d}x + \mathcal{H}^{n-1}(J_v \setminus K_0)\colon v = u_0 \text{ in } \widetilde{\Omega} \setminus (\Omega \cup \partial_D\Omega) \right\}. \tag{4.3}$$

Then we have the following existence result.

Theorem 4.1. *Problem (4.3) admits solutions.*

Proof. Let $u_h \in GSBD(\widetilde{\Omega})$ with $u_h = u_0$ in $\widetilde{\Omega} \setminus (\Omega \cup \partial_D\Omega)$ be the elements of a minimising sequence for (4.3). Observe that the infimum of problem (4.3) is finite, since the functional is non-negative and u_0 is an admissible competitor.

Assume for the moment that K_0 is compact. By (4.1) the functions u_h satisfy the hypotheses of Theorem 1.1 with $\Omega = \widetilde{\Omega} \setminus K_0$, so that there exist $A \subset \widetilde{\Omega} \setminus K_0$ with finite perimeter and a measurable function $u \colon \widetilde{\Omega} \setminus K_0 \to \mathbb{R}^n$ with $u = 0$ in A such that (up to a subsequence)

$$A = \{x \in \widetilde{\Omega} \setminus K_0 \colon |u_h(x)| \to \infty\}, \qquad u_h \to u \quad \mathcal{L}^n\text{-a.e. in } \Omega \setminus (K_0 \cup A) \tag{4.4}$$

(since $\mathcal{L}^n(K_0) = 0$ we could consider just $\widetilde{\Omega}$ above, but we keep $\widetilde{\Omega} \setminus K_0$ to indicate the set where we apply Theorem 1.1) and

$$\int_{\widetilde{\Omega}} W(e(u))\,\mathrm{d}x + \mathcal{H}^{n-1}(J_u \setminus K_0) \leq \liminf_{h\to\infty} \int_{\widetilde{\Omega}} W(e(u_h))\,\mathrm{d}x + \mathcal{H}^{n-1}(J_{u_h} \setminus K_0),$$

Moreover, by (4.4) and the admissibility condition for u_h it follows that $u = u_0$ in $\widetilde{\Omega} \setminus (\Omega \cup \partial_D \Omega)$, and in particular A does not intersect $\widetilde{\Omega} \setminus (\Omega \cup \partial_D \Omega)$. Since W is convex, we have lower semicontinuity for the bulk term, and u solves (4.3). This proves the theorem if K_0 is compact. Notice that this holds for any other function v which coincides with u in $\Omega \setminus A$ and is set equal to any fixed infinitesimal rigid motion in A, since the energy of v in A is null, and then by (1.5) the Griffith energy of v is less than the lim inf of the energies of u_h.

If K_0 is not compact, for any $\varepsilon > 0$ consider $\widehat{K_0} \subset K_0$, compact, such that $\mathcal{H}^{n-1}(K_0 \setminus \widehat{K_0}) < \varepsilon$. Then, arguing as above for the open set $\widetilde{\Omega} \setminus \widehat{K_0} \supset \widetilde{\Omega} \setminus K_0$, we still get

$$\int_{\widetilde{\Omega}} W(e(u))\,\mathrm{d}x \leq \liminf_{h\to\infty} \int_{\widetilde{\Omega}} W(e(u_h))\,\mathrm{d}x,$$

and

$$\begin{aligned}\mathcal{H}^{n-1}(J_u \setminus K_0) &\leq \mathcal{H}^{n-1}(J_u \setminus \widehat{K_0}) \leq \liminf_{h\to\infty} \mathcal{H}^{n-1}(J_{u_h} \setminus \widehat{K_0}) \\ &\leq \liminf_{h\to\infty} \mathcal{H}^{n-1}(J_{u_h} \setminus K_0) + \mathcal{H}^{n-1}(K_0 \setminus \widehat{K_0}) < \liminf_{h\to\infty} \mathcal{H}^{n-1}(J_{u_h} \setminus K_0) + \varepsilon,\end{aligned}$$

since $J_u \setminus K_0 \subset J_u \setminus \widehat{K_0}$ and $J_{u_h} \setminus \widehat{K_0} \subset (J_{u_h} \setminus K_0) \cup (K_0 \setminus \widehat{K_0})$ (cf. also [35, Theorem 2.5]). We conclude the proof since $\varepsilon > 0$ is arbitrary. □

Remark 4.2. Since, as observed in the proof, a family of minimisers is obtained by adding any fixed infinitesimal rigid motion in A to a given minimiser, we conclude that

$$\mathcal{H}^{n-1}(\partial^* A \cap \{\mathrm{tr}\, u = a\}) = 0$$

for every infinitesimal rigid motion a ($a(x) = \mathbf{a} \cdot x + b$, $\mathbf{a} + \mathbf{a}^T = 0$), where tr denotes the trace of u on $\partial^* A$ (which is countably $\mathcal{H}^{n-1}$-rectifiable) from $\Omega \setminus A$.

Existence of strong solutions

In recent works [20, 15], Chambolle, Conti, Focardi, and Iurlano have shown more regularity for the possible minimisers of (4.3) (or (4.2)) if $W(\xi) = \mathbf{C}e(\xi) : e(\xi)$ (in [15]), or $n = 2$ and

$$W(\xi) = f_\mu(\xi) := \frac{1}{p}((\mathbf{C}\xi : \xi + \mu)^{p/2} - \mu^{p/2}) \tag{4.5}$$

(in [20]), requiring that $\mathbf{C}\colon \mathbb{M}^{n\times n}_{\mathrm{sym}} \to \mathbb{M}^{n\times n}_{\mathrm{sym}}$ is a symmetric linear map with

$$\mathbf{C}(\xi - \xi^T) = 0 \quad \text{and} \quad \mathbf{C}\xi\cdot\xi \geq c_0|\xi + \xi^T|^2 \quad \text{for all } \xi \in \mathbb{M}^{n\times n}_{\mathrm{sym}}.$$

More precisely, the essential closedness of the jump set is established:

Theorem 4.3. *Let $K_0 \subset \Omega \cup \partial_D\Omega$ be closed with $\mathcal{H}^{n-1}(K_0) < +\infty$, and let $u \in GSBD^2(\Omega \setminus K_0)$ (or $u \in GSBD^p(\Omega\setminus K_0)$ in dimension 2) be a minimiser of*

$$\int_\Omega \mathbf{C}e(v) : e(v)\,\mathrm{d}x + \mathcal{H}^{n-1}\big(J_v \cup (\partial_D\Omega \cap \{\mathrm{tr}_\Omega\, v \neq \mathrm{tr}_\Omega\, u_0\}) \setminus K_0\big) \tag{4.6}$$

(or, in dimension 2, a minimiser of (4.3) with (4.5)). Then

$$\mathcal{H}^{n-1}\big((\Omega\setminus K_0)\cap(\overline{J}_u \setminus J_u)\big) = 0, \qquad u \in C^1(\Omega\setminus(K_0\cup\overline{J}_u)).$$

In [17], this is extended to $\Omega \cup \partial_D\Omega$, yielding the following result (see [8] for the *SBV* case):

Theorem 4.4. *Let $\partial_D\Omega$ be of class C^1, $u_0 \in W^{1,\infty}(\mathbb{R}^n;\mathbb{R}^n)$, and $u \in GSBD^2(\Omega\setminus K_0)$ be a minimiser of (4.6). Then, letting $J := J_u \cup (\partial_D\Omega \cap \{\mathrm{tr}_\Omega\, u \neq \mathrm{tr}_\Omega\, u_0\}$,*[1]

$$\mathcal{H}^{n-1}\big(((\Omega\cup\partial_D\Omega)\setminus K_0)\cap(\overline{J}\setminus J)\big) = 0$$

and

$$u \in C^1\big(\Omega\setminus(K_0\cup\overline{J})\big)\cap C\big((\Omega\cup\partial_D\Omega)\setminus(K_0\cup\overline{J})\big).$$

Another consequence of Theorem 1.1 is a compactness result for phase-field approximations of (1.1), which are used for the numerical simulations of evolutions in brittle fracture (such as in [10]); see [16] for details.

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