J. Eur. Math. Soc. 23, 557–583 (2021) DOI 10.4171/JEMS/1017

Dario Bambusi · Benoît Grébert · Alberto Maspero · Didier Robert

Growth of Sobolev norms for abstract linear Schrödinger equations

Received January 18, 2018

Abstract. We prove an abstract theorem giving a $\langle t\rangle^{\epsilon}$ bound (for all $\epsilon > 0$) on the growth of the Sobolev norms in linear Schrödinger equations of the form $\mathrm{i}\dot{\psi} = H_0\psi + V(t)\psi$ as $t \to \infty$. The abstract theorem is applied to several cases, including the cases where (i) H_0 is the Laplace operator on a Zoll manifold and $V(t)$ a pseudodifferential operator of order smaller than 2; (ii) H_0 is the (resonant or nonresonant) harmonic oscillator in $\mathbb{R}^d$ and $V(t)$ a pseudodifferential operator of order smaller than that of H_0 depending in a quasiperiodic way on time. The proof is obtained by first conjugating the system to some normal form in which the perturbation is a smoothing operator and then applying the results of [MR17].

Keywords. Linear Schrödinger operators, time-dependent Hamiltonians, growth in time of Sobolev norms

1. Introduction

In this paper we study growth of Sobolev norms for solutions of the abstract linear Schrödinger equation

$$\mathrm{i}\partial_t\psi = H_0\psi + V(t)\psi, \tag{1.1}$$

in a scale of Hilbert spaces $\mathcal{H}^r$; here $V(t)$ is a time-dependent operator and H_0 a time independent linear operator. We will prove some abstract results ensuring that for any $r \geq 0$ and any $\epsilon > 0$, the $\mathcal{H}^r$ norm of the solution grows in time at most as $\langle t\rangle^{\epsilon}$ as $t \to \infty$, where $\langle t\rangle := \sqrt{1+t^2}$. The main novelty of our results is that they allow (1) weakening the standard gap assumptions on the spectrum of H_0, in particular dealing with some cases where the gaps are dense in $\mathbb{R}$, and (2) dealing with perturbations which are of any order strictly smaller than that of H_0 (see below for a precise definition).

D. Bambusi: Dipartimento di Matematica Federigo Enriques, Università degli Studi di Milano, Via Saldini 50, 20133 Milano, Italy; e-mail: dario.bambusi@unimi.it

B. Grébert, D. Robert: Laboratoire de Mathématiques Jean Leray, Université de Nantes, 2 rue de la Houssinière BP 92208, 44322 Nantes Cedex 3, France; e-mail: benoit.grebert@univ-nantes.fr, didier.robert@univ-nantes.fr

A. Maspero: International School for Advanced Studies (SISSA), Via Bonomea 265, 34136 Trieste, Italy; e-mail: alberto.maspero@sissa.it

Mathematics Subject Classification (2010): Primary 35Q41; Secondary 47G30

The main applications are to the cases where

(i) H_0 is either the Laplace operator on a Zoll manifold (e.g. the sphere) or an anharmonic oscillator in $\mathbb{R}$, while V is an operator depending arbitrarily on time and having order strictly smaller than that of H_0;
(ii) H_0 is the (possibly nonresonant) multidimensional harmonic oscillator and $V(t)$ is an operator which *depends on time in a quasiperiodic way* and has order strictly smaller than that of H_0.

Further applications will be presented in the main body of the paper.

We emphasize in particular the results (ii) which, as far as we know, are the first controlling growth of Sobolev norms in higher dimensional systems without any gap condition.

The proof is based on a combination of the ideas of [Bam18, Bam17, BGMR18] (which in turn are developments of the ideas of [BBM14], see also [PT01, IPT05]) and the results of [MR17]; more precisely, for any positive N, we construct a (finite) sequence of unitary time-dependent transformations conjugating $H_0 + V(t)$ to a Hamiltonian of the form

$$H_0 + Z^{(N)}(t) + V^{(N)}(t), \tag{1.2}$$

where $[H_0, Z^{(N)}] = 0$ and $V^{(N)}$ is a smoothing operator of order N, namely an operator belonging to $\mathcal{L}(\mathcal{H}^s, \mathcal{H}^{s+N})$ for any s (bounded linear operators from $\mathcal{H}^s$ to $\mathcal{H}^{s+N}$). Then we apply [MR17, Theorem 1.5] to (1.2) getting the $\langle t\rangle^{\epsilon}$ bound on the growth of Sobolev norms.

We think that a further point of interest is that the conjugation to a system of the form (1.2) is here developed in an abstract context, and not in the framework of classes of pseudodifferential operators adapted to the situation under study; this is the main reason why we get an abstract theory directly applicable to many different contexts.

The main point is that we introduce an abstract graded algebra of operators whose properties mimic the properties of pseudodifferential operators. The use of this framework is made possible by the technique we develop to solve the homological equations met in the construction of the conjugation of H to (1.2). Indeed, we recall that in previous papers the smoothing theorem, namely the result conjugating the original system to (1.2), was obtained by quantizing the procedure of classical normal form. Here instead, we work directly at the quantum level, in particular solving at this level the two homological equations that we find (see (3.17) and (3.24) below).

It is worth to add a few words on the way we solve the homological equations. When dealing with systems related to the applications (i), we assume that $H_0 = f(K_0)$ where f is a superlinear function and K_0 is an operator such that

$$\operatorname{spec}(K_0) \subset \mathbb{N} + \lambda, \quad \lambda > 0. \tag{1.3}$$

In this case we solve the homological equation essentially by averaging over the flow $\mathrm{e}^{-\mathrm{i}tK_0}$ of K_0. In turn this is made possible by the use of a commutator expansion lemma proved in [DG97]. When dealing with the d-dimensional harmonic oscillators instead, we take

$$H_0 = \sum_{j=0}^{d} \nu_j K_j$$

with K_j commuting linear operators, each fulfilling the property (1.3) (think of $K_j = -\partial_{x_j}^2 + x_j^2$) and $\nu_j > 0$; then we consider operators of the form

$$e^{i\tau \cdot K} A \, e^{-i\tau \cdot K}$$

(where of course $\tau \cdot K := \tau_1 K_1 + \cdots + \tau_d K_d$), remark that they are quasiperiodic in the "angles" τ, and use a Fourier expansion in τ in order to solve the homological equation.

The study of growth of Sobolev norms and the related results on the nature of the spectrum of the Floquet operator has a long history: we recall the results of [How89, How92, Joy94] showing that the Floquet spectrum of systems with growing gaps and bounded perturbations is pure point, a result which implies boundedness of the expectation value of the energy. The first $\langle t \rangle^\epsilon$ estimate on the expectation value of the energy for a system of the form (1.1) was obtained by Nenciu [Nen97] for the case of increasing gaps and bounded perturbations (see also [BJ98, Joy96] for similar results), and by Duclos, Lev and Šťovíček [DLS08] in the case of shrinking gaps. In the case of increasing gaps, such results were improved recently by two of us in [MR17] where we obtained the $\langle t \rangle^\epsilon$ growth of Sobolev norms also in the case of unbounded perturbations depending arbitrarily on time, for example in the case where $H_0 = -\partial_x^2 + x^{2k}$; the result of [MR17] allows one to deal with perturbations growing at infinity as $|x|^m$ with $m < k - 1$. In the present paper we get the result for any $m < 2k$. The result of [MR17] also applies to perturbations of the free Schrödinger equation on Zoll manifolds with perturbations of order strictly smaller than 1. Here we deal with perturbations of order strictly smaller than 2. A study of perturbations of maximal order has been done independently by Montalto [Mon18] who got control of the growth of Sobolev norms for the Schrödinger equation on $\mathbb{T}$ with $H = a(t,x)|-\partial_{xx}|^M + V(t)$ with $M > 1/2$, a a smooth positive function and V a pseudodifferential operator of order smaller than M.

Finally, we recall that in [MR17] logarithmic estimates for the growth of Sobolev norms were also obtained in the case of perturbations depending analytically on time. Here we do not attack the problem of getting logarithmic estimates, but we think that our technique would also yield such estimates.

A remarkable further result was obtained by Bourgain [Bou99a, Bou99b] who obtained a logarithmic bound on the growth of Sobolev norms for the Schrödinger equation on $\mathbb{T}^d$ ($d = 1, 2$) in the case of an analytic perturbation depending quasiperiodically on time. That result is based on the use of a lemma on the clustering of resonant sites (in a suitable space time lattice) which does not seem to extend to different geometries. The result of Bourgain was extended by Wang [Wan08] to Schrödinger equations on $\mathbb{T}$ perturbed by a potential analytic in time (but otherwise depending arbitrarily on time) and greatly simplified by Delort [Del10] who used it in an abstract framework which allows dealing with the case of $\mathbb{T}^d$ (any $d \geq 1$) and also with the case of Zoll manifolds, obtaining a $\langle t \rangle^\epsilon$ growth bound (see also [FZ12] for analytic potentials on $\mathbb{T}^d$). We also mention the reducibility result of [EK09] dealing with small quasiperiodic perturbations of the free Schrödinger equation on $\mathbb{T}^d$; for such a system, the authors prove that growth of Sobolev norms cannot happen, provided the frequency of the quasiperiodic solution is chosen in a nonresonant set. At present our method does not allow dealing with the Schrödinger equation on $\mathbb{T}^d$ for $d \geq 2$.

Concerning harmonic oscillators in $\mathbb{R}^d$ with $d > 1$, a couple of reducibility results are known, namely in [GP19] the authors study small bounded perturbations of the *completely resonant* harmonic oscillator, and in [BGMR18] we studied small *polynomial* perturbations of the resonant or nonresonant Harmonic oscillator.

As far as we know, no results are known on growth of Sobolev norms for perturbations of the harmonic oscillator

$$H_0 := -\Delta + \sum_{j=1}^{d} \nu_j^2 x_j^2 \tag{1.4}$$

with nonresonant frequencies ν_j. This is due to the fact that the differences between two of its eigenvalues $\{\lambda_a\}_{a\in\mathbb{N}^d}$,

$$\lambda_a - \lambda_b = \nu \cdot (a - b),$$

are dense on the real axis and this prevents the use of *any* previous technique. As anticipated above, here we obtain $\langle t\rangle^\epsilon$ growth for the case of a perturbation of order strictly smaller than the order of the harmonic oscillator.

2. Main results

2.1. An abstract graded algebra

We start with a Hilbert space $\mathcal{H}$ and a reference operator K_0, which we assume to be self-adjoint and positive, namely such that

$$\langle \psi, K_0\psi\rangle \geq c_K \|\psi\|^2, \qquad \forall \psi \in D(K_0^{1/2}), \qquad c_K > 0,$$

and define as usual a scale of Hilbert spaces by $\mathcal{H}^r = D(K_0^r)$ (the domain of the operator K_0^r) if $r \geq 0$, and $\mathcal{H}^r = (\mathcal{H}^{-r})'$ (the dual space) if $r < 0$. Finally, we denote $\mathcal{H}^{-\infty} = \bigcup_{r\in\mathbb{R}} \mathcal{H}^r$ and $\mathcal{H}^{+\infty} = \bigcap_{r\in\mathbb{R}} \mathcal{H}^r$. We endow $\mathcal{H}^r$ with the natural norm $\|\psi\|_r := \|K_0^r\psi\|_0$, where $\|\cdot\|_0$ is the norm of $\mathcal{H}^0 \equiv \mathcal{H}$. Notice that for any $m \in \mathbb{R}$, $\mathcal{H}^{+\infty}$ is a dense linear subspace of $\mathcal{H}^m$ (this is a consequence of the spectral decomposition of K_0).

We now introduce a graded algebra $\mathcal{A}$ of operators which mimic some fundamental properties of various classes of pseudodifferential operators. For $m \in \mathbb{R}$ let $\mathcal{A}_m$ be a linear subspace of $\bigcap_{s\in\mathbb{R}} \mathcal{L}(\mathcal{H}^s, \mathcal{H}^{s-m})$ and define $\mathcal{A} := \bigcup_{m\in\mathbb{R}} \mathcal{A}_m$. We notice that $\bigcap_{s\in\mathbb{R}} \mathcal{L}(\mathcal{H}^s, \mathcal{H}^{s-m})$ is a Fréchet space when equipped with the seminorms $\|A\|_{m,s} := \|A\|_{\mathcal{L}(\mathcal{H}^s,\mathcal{H}^{s-m})}$.

One of our aims is to control the smoothing properties of the operators in the scale $\{\mathcal{H}^r\}_{r\in\mathbb{R}}$. If $A \in \mathcal{A}_m$ then A is more and more smoothing as $m \to -\infty$ and the opposite as $m \to +\infty$. We will say that A is of *order* m if $A \in \mathcal{A}_m$.

Definition 2.1. We say that $S \in \mathcal{L}(\mathcal{H}^{+\infty}, \mathcal{H}^{-\infty})$ is N-*smoothing* if for each $\kappa \in \mathbb{R}$, it can be extended to an operator in $\mathcal{L}(\mathcal{H}^{\kappa}, \mathcal{H}^{\kappa+N})$. When this is true for every $N \geq 0$, we say that S is *smoothing*.

The first set of assumptions concerns the properties of $\mathcal{A}_m$:

Assumption I. (i) For each $m \in \mathbb{R}$, $K_0^m \in \mathcal{A}_m$; in particular K_0 is an operator of order 1.

(ii) For each $m \in \mathbb{R}$, $\mathcal{A}_m$ is a Fréchet space for a family $\{\wp_j^m\}_{j \geq 1}$ of filtering seminorms such that the embedding $\mathcal{A}_m \hookrightarrow \bigcap_{s \in \mathbb{R}} \mathcal{L}(\mathcal{H}^s, \mathcal{H}^{s-m})$ is continuous. If $m' \leq m$ then $\mathcal{A}_{m'} \subseteq \mathcal{A}_m$ with a continuous embedding.

(iii) $\mathcal{A}$ is a graded algebra, i.e. for all $m, n \in \mathbb{R}$, if $A \in \mathcal{A}_m$ and $B \in \mathcal{A}_n$ then $AB \in \mathcal{A}_{m+n}$ and the map $(A, B) \mapsto AB$ is continuous from $\mathcal{A}_m \times \mathcal{A}_n$ into $\mathcal{A}_{m+n}$.

(iv) $\mathcal{A}$ is a graded Lie-algebra:[1] if $A \in \mathcal{A}_m$ and $B \in \mathcal{A}_n$ then the commutator $[A, B]$ is in $\mathcal{A}_{m+n-1}$ and the map $(A, B) \mapsto [A, B]$ is continuous from $\mathcal{A}_m \times \mathcal{A}_n$ into $\mathcal{A}_{m+n-1}$.

(v) $\mathcal{A}$ is closed under perturbation by smoothing operators in the following sense: Let A be a linear map: $\mathcal{H}^{+\infty} \to \mathcal{H}^{-\infty}$. If there exists $m \in \mathbb{R}$ such that for every $N > 0$ we have a decomposition $A = A^{(N)} + S^{(N)}$, where $A^{(N)} \in \mathcal{A}_m$ and $S^{(N)}$ is N-smoothing, then $A \in \mathcal{A}_m$.

(vi) If $A \in \mathcal{A}_m$ then also the adjoint operator A^* is in $\mathcal{A}_m$. The duality here is defined by the scalar product $\langle \cdot, \cdot \rangle$ of $\mathcal{H} = \mathcal{H}^0$. The adjoint A^* is defined by $\langle u, Av \rangle = \langle A^* u, v \rangle$ for $u, v \in \mathcal{H}^{\infty}$ and extended by continuity.

It is well known that classes of pseudodifferential operators satisfy these properties provided one chooses for K_0 a suitable operator of the right order (see e.g. [Hör85]).

In [Gui85] V. Guillemin has introduced abstract pseudodifferential algebras, called generalized Weyl algebras. Guillemin [Gui85] needs different properties than ours, but obviously there is an overlap with our presentation.

Remark 2.2. For all $A \in \mathcal{A}_m$ and $B \in \mathcal{A}_n$,

$$\forall m, s\ \exists N : \|A\|_{m,s} \leq C_1\, \wp_N^m(A), \tag{2.1}$$

$$\forall m, n, j\ \exists N : \wp_j^{m+n}(AB) \leq C_2\, \wp_N^m(A)\, \wp_N^n(B), \tag{2.2}$$

$$\forall m, n, j\ \exists N : \wp_j^{m+n-1}([A, B]) \leq C_3\, \wp_N^m(A)\, \wp_N^n(B), \tag{2.3}$$

for some positive constants $C_1(s, m)$, $C_2(m, n, j)$, $C_3(m, n, j)$.

For $\Omega \subset \mathbb{R}^d$ and $\mathcal{F}$ a Fréchet space, we will denote by $C_b^m(\Omega, \mathcal{F})$ the space of C^m maps $f : \Omega \ni x \mapsto f(x) \in \mathcal{F}$ such that for every seminorm $\|\cdot\|_j$ of $\mathcal{F}$ one has

$$\sup_{x \in \Omega} \|\partial_x^{\alpha} f(x)\|_j < +\infty, \qquad \forall \alpha \in \mathbb{N}^d,\ |\alpha| \leq m. \tag{2.4}$$

If (2.4) is true for all m, we write $f \in C_b^{\infty}(\Omega, \mathcal{F})$.

The next property needed is the following Egorov property, also well known for pseudodifferential operators.

[1] This property will impose the choice of the seminorms $\{\wp_j^m\}_{j \geq 1}$. We will see in the examples that the natural choice $(\|\cdot\|_{m,s})_{s \geq 0}$ has to be refined.

Assumption II. For any $A \in \mathcal{A}_m$,

$$\mathbb{R} \ni \tau \mapsto A(\tau) := \mathrm{e}^{\mathrm{i}\tau K_0} A\, \mathrm{e}^{-\mathrm{i}\tau K_0} \in C_b^0(\mathbb{R}, \mathcal{A}_m).$$

Remark 2.3. From Assumption II one sees that for any $B \in \mathcal{A}_n$ and $\ell \in \mathbb{N}$, $\mathrm{ad}^{\ell}_{A(s)}(B) \in C_b^0(]-T, T[, \mathcal{A}_{n+(m-1)\ell})$ for all $T > 0$. Here $\mathrm{ad}_A(B) := \mathrm{i}[A, B]$.

Observe that Assumption II is a quantum property for the time evolution of observables. Practically it follows from the time evolution of classical observables (Hamilton equation) if some classes of symbols are preserved under the classical flows. Indeed, one might replace Assumption II by a weaker one (see Appendix B).

2.2. *Perturbations of systems of order larger than 1*

Now we state our spectral assumption on K_0:

Assumption A. The spectrum of K_0 is discrete and

$$\mathrm{spec}(K_0) \subseteq \mathbb{N} + \lambda \tag{2.5}$$

for some $\lambda > 0$.

Our second spectral assumption is essentially that the unperturbed operator H_0 is a function of K_0. To state it precisely we need the following definition.

Definition 2.4. A function $f \in C^\infty(\mathbb{R})$ will be said to be a *classical symbol of order* ρ (at $+\infty$) if there exist real numbers $\{c_j\}_{j\ge 0}$ such that $c_0 \ge 0$ and for all $k, N \ge 1$, there exists $C_{k,N}$ such that

$$\left| \frac{d^k}{dx^k}\Big(f(x) - \sum_{0\le j\le N-1} c_j x^{\rho-j} \Big) \right| \le C_{k,N} |x^{\rho-N-k}|, \qquad \forall x \ge 1.$$

We will denote by S^ρ the space of classical symbols of order ρ.

We shall say that f is an *elliptic classical symbol of order* ρ if moreover f is real and $c_0 > 0$. We shall then write $f \in S^\rho_+$.

We shall say that f is a *classical symbol of order* $-\infty$ if $f \in S^m$ for all $m < 0$. We shall then write $f \in S^{-\infty}$.

Some standard properties of classical symbols are recalled in Appendix A. We assume the following:

Assumption B. There exists an elliptic classical symbol f of order $\mu > 1$ such that

$$H_0 = f(K_0). \tag{2.6}$$

We will prove (see Lemma A.2) that (2.6) implies $H_0 \in \mathcal{A}_\mu$, i.e. H_0 is an operator of order $\mu > 1$.

We come back to the Schrödinger equation defined by the time-dependent Hamiltonian $H(t) := H_0 + V(t)$ (see (1.1)). When the solution $\psi(t)$ exists globally in time, we define the Schrödinger propagator $\mathcal{U}(t, s)$, generated by (1.1), such that

$$\psi(t) = \mathcal{U}(t, s)\psi, \quad \mathcal{U}(s, s) = \mathbf{1}. \tag{2.7}$$

We are ready to state our main result on systems with increasing gaps:

Theorem 2.5. *Assume that $\mathcal{A}$ is a graded algebra as defined in Section 2.1 and that K_0, H_0 satisfy Assumptions A and B. Furthermore assume that the perturbation $V(t)$ with domain $\mathcal{H}^\infty$ is symmetric for every $t \in \mathbb{R}$ and satisfies*

$$V \in C_b^\infty(\mathbb{R}, \mathcal{A}_\rho) \quad \textit{for some } \rho < \mu. \tag{2.8}$$

Then $H(t) = H_0 + V(t)$ generates a propagator $\mathcal{U}(t, s)$ such that $\mathcal{U}(t, s) \in \mathcal{L}(\mathcal{H}^r)$ for all $r \in \mathbb{R}$. Moreover for any $r > 0$ and any $\epsilon > 0$ there exists $C_{r,\epsilon} > 0$ such that

$$\|\mathcal{U}(t, s)\psi\|_r \le C_{r,\epsilon} \langle t - s\rangle^\epsilon \|\psi\|_r, \qquad \forall t, s \in \mathbb{R}. \tag{2.9}$$

This result extends a result by Nenciu [Nen97] for bounded perturbations ($\rho = 0$). Furthermore in [MR17] two of us had already extended Nenciu's result to unbounded perturbations with the constraint $\rho < \min(\mu - 1, 1)$. The main point is that we add here a stronger spectral assumption: essentially the spectrum of H_0 is $f(\mathbb{N}+\lambda)$ for some smooth function f (see Assumptions A and B).

As a final remark, we note that Theorem 2.5 also gives a proof of the existence and of some properties of the propagator $\mathcal{U}(t, s)$, which in the framework of Theorem 2.5 are not obvious.

2.3. *Applications (i)*

Zoll manifolds. Recall that a *Zoll manifold* is a compact Riemannian manifold (M, g) such that all the geodesic curves have the same period $T := 2\pi$. For example the d-dimensional sphere $\mathbb{S}^d$ is a Zoll manifold. We denote by $\triangle_g$ the positive Laplace–Beltrami operator on M and by $H^r(M) = \mathrm{Dom}\,(1 + \triangle_g)^{r/2}$, $r \ge 0$, the usual scale of Sobolev spaces. Finally, we denote by $S^m_{\mathrm{cl}}(M)$ the space of classical real valued symbols of order $m \in \mathbb{R}$ on the cotangent bundle $T^*(M)$ of M (see Hörmander [Hör85] for more details).

Definition 2.6. We write $A \in \mathcal{A}_m$ if it is a pseudodifferential operator (in the sense of Hörmander [Hör85]) with symbol of class $S^m_{\mathrm{cl}}(M)$.

In this case the operator K_0 is a perturbation of order -1 of $\sqrt{\triangle_g}$ (see Sect. 4.1), and the norms $\|\psi\|_r$ coincide with the standard Sobolev norms.

Corollary 2.7 (Zoll manifolds). *Let $V(t)$ be a symmetric pseudodifferential operator of order $\rho < 2$ on M with symbol $v \in C_b^\infty(\mathbb{R}, S^\rho_{\mathrm{cl}}(M))$. Then the propagator $\mathcal{U}(t, s)$ generated by $H(t) = \triangle_g + V(t)$ exists and satisfies (2.9).*

Anharmonic oscillators on $\mathbb{R}$. The second application concerns one-dimensional quantum anharmonic oscillators

$$\mathrm{i}\partial_t \psi = H_{k,l}\psi + V(t)\psi, \qquad x \in \mathbb{R}, \tag{2.10}$$

where $H_{k,l}$ is the one-degree-of-freedom Hamiltonian

$$H_{k,l} := D_x^{2l} + ax^{2k}, \quad k, l \in \mathbb{N},\ k + l \geq 3,\ a > 0. \tag{2.11}$$

Here $D_x := \mathrm{i}^{-1}\partial_x$. It is well known that $H_{k,\ell}$ is essentially self-adjoint in $L^2(\mathbb{R})$ [HR82b].

Define the Sobolev spaces $\mathcal{H}^r := \mathrm{Dom}(H_{k,l}^{\frac{k+l}{2kl}r})$ for $r \geq 0$. We now define suitable operator classes for the perturbation. Denote

$$\mathtt{k}_0(x, \xi) := (1 + x^{2k} + \xi^{2l})^{\frac{k+l}{2kl}}.$$

Definition 2.8. A function f will be called a *symbol of order* $\rho \in \mathbb{R}$ if $f \in C^\infty(\mathbb{R}_x \times \mathbb{R}_\xi)$ and for all $\alpha, \beta \in \mathbb{N}$, there exists $C_{\alpha,\beta} > 0$ such that

$$|\partial_x^\alpha \partial_\xi^\beta f(x, \xi)| \leq C_{\alpha,\beta}\, \mathtt{k}_0(x, \xi)^{\rho - \frac{k\beta + l\alpha}{k+l}}. \tag{2.12}$$

We will then write $f \in S^\rho_{\mathrm{an}}$.

As usual, to a symbol $f \in S^\rho_{\mathrm{an}}$ we associate the operator $f(x, D_x)$ which is obtained by standard Weyl quantization (see formula (4.2) below).

Definition 2.9. We write $F \in \mathcal{A}_\rho$ if F is a pseudodifferential operator with symbol of class S^ρ_{an}, i.e., there exist $f \in S^\rho_{\mathrm{an}}$ and S smoothing (in the sense of Definition 2.1) such that $F = f(x, D_x) + S$.

In this case the seminorms are defined by

$$\wp_j^\rho(F) := \sum_{|\alpha|+|\beta|\leq j} C_{\alpha\beta},$$

with $C_{\alpha\beta}$ the smallest constant such that (2.12) holds. If a symbol f depends on additional parameters (e.g. it is time-dependent), we require that the constants $C_{\alpha,\beta}$ should be uniform with respect to the parameters.

Remark 2.10. With this definition of symbols, one has $x \in S_{\mathrm{an}}^{\frac{l}{k+l}}$, $\xi \in S_{\mathrm{an}}^{\frac{k}{k+l}}$, $x^{2k} + \xi^{2l} \in S_{\mathrm{an}}^{\frac{2kl}{k+l}}$, $\mathtt{k}_0(x, \xi) \in S^1_{\mathrm{an}}$.

We get the following:

Corollary 2.11 (1-D anharmonic oscillators). *Consider equation* (2.10) *with the assumption* (2.11)*. Assume also that* $V \in C_b^\infty(\mathbb{R}, \mathcal{A}_\rho)$ *with* $\rho < \frac{2kl}{k+l}$*. Then the propagator* $\mathcal{U}(t, s)$ *generated by* $H(t) = H_{k,l} + V(t)$ *is well defined and satisfies* (2.9).

An example of an admissible perturbation is $V(t,x,\xi) = \sum_{l\alpha+k\beta<2kl} a_{\alpha,\beta}(t)x^\alpha\xi^\beta$ with $a_{\alpha,\beta} \in C^\infty_b(\mathbb{R},\mathbb{R})$. In particular if we choose $H_0 = -\frac{d^2}{dx^2} + x^4$, we can consider unbounded perturbations of the form $x^3 g(t)$ and of course also $xg(t)$ with $g \in C^\infty_b(\mathbb{R},\mathbb{R})$.

Remark 2.12. Our class of perturbations contains quite general pseudodifferential operators, but it is easy to see that multiplication operators (i.e. operators independent of ∂_x) must be polynomials in x with coefficients which are possibly time-dependent.

In the similar problem of reducibility more general classes of perturbations have been treated in [Bam17]. We have not tried to push the result in that direction. This is probably non-trivial in an abstract framework like the one we are using here.

Remark 2.13. We think that our method should also allow dealing with some perturbations of the same order as the main term. For example one should be able to treat the case where V is a quasihomogeneous polynomial of maximal order fulfilling some sign condition (more or less as in [Bam18, Theorem 2.12]).

2.4. *Perturbations of systems of order 1*

In order to deal with perturbations of operators of order 1 we have to restrict to the case where the dependence of the perturbation on time is quasiperiodic.

Let $\mathcal{A} := \bigcup_{m\in\mathbb{R}} \mathcal{A}_m$ be a graded Lie algebra satisfying Assumption I with a reference operator K_0. Let $K_1,\dots,K_d$ be self-adjoint positive operators such that $K_j \in \mathcal{A}_1$ for $1 \le j \le d$. Assume the following modified Assumption II:

Assumption II′. (i) $[K_j, K_\ell] = 0$ for any $0 \le j,\ell \le d$.
(ii) Denote $K = (K_1,\dots,K_d)$ and for $\tau \in \mathbb{R}^d$, $\tau\cdot K := \sum_{1\le j\le d} \tau_j K_j$. Then for any $A \in \mathcal{A}_m$, $\tau \mapsto A(\tau) := \mathrm{e}^{\mathrm{i}\tau\cdot K} A\,\mathrm{e}^{-\mathrm{i}\tau\cdot K} \in C^\infty_b(\mathbb{R}^d, \mathcal{A}_m)$.

Remark 2.14. For any $B \in \mathcal{A}_n$ and any $\ell \in \mathbb{N}$, one has $\mathrm{ad}^\ell_{A(s)}(B) \in C^\infty_b(\mathbb{R}^d, \mathcal{A}_{n+\ell(m-1)})$.

We also adapt our spectral conditions:

Assumption A′. $K = (K_1,\dots,K_d)$ has joint spectrum $\mathrm{spec}(K)$ contained in $\mathbb{N}^d + \lambda$ for some $\lambda \in \mathbb{R}^d$, $\lambda \ge 0$.

Assumption B′. There exist $\{\nu_j\}_{j=1}^d$, $\nu_j > 0$, such that

$$H_0 = \sum_{1\le j\le d} \nu_j K_j\ , \tag{2.13}$$

$$K_0 = H_0. \tag{2.14}$$

To fix ideas one can think of the case of harmonic oscillators, in which $K_j = -\partial_j^2 + x_j^2$, $1 \le j \le d$.

Remark 2.15. Since the operators K_j are positive, the norm $\|\cdot\|_r$ defined using the operator K_0 is equivalent to one defined using $K_0' := \sum_{j=1}^d K_j$.

We consider both the case where

$$\nu := (\nu_1, \dots, \nu_d)$$

is resonant and the case where it is nonresonant. To state the arithmetical assumptions on ν, we first recall the following well known lemma whose scheme of proof will be recalled in Appendix C.

Lemma 2.16. *There exist $\tilde{d} \le d$, a vector $\tilde{\nu} \in \mathbb{R}^{\tilde{d}}$ with components independent over the rationals, and vectors $\mathbf{v}_j \in \mathbb{Z}^d$, $j = 1, \dots, \tilde{d}$, such that*

$$\nu = \sum_{j=1}^{\tilde{d}} \tilde{\nu}_j \mathbf{v}_j. \tag{2.15}$$

Remark 2.17. For example,

(i) if ν is nonresonant (that is, $\nu \cdot k = 0$ for $k \in \mathbb{Z}^d$ implies $k = 0$), then $\tilde{\nu} = \nu$ and $\mathbf{v}_j = \mathbf{e}_j$, the standard basis of $\mathbb{R}^d$;
(ii) if ν is completely resonant (that is, for each j one has $\nu_j = \overline{\nu} k_j$ with $k_j \in \mathbb{Z}$), then $\tilde{d} = 1$; e.g. if $\nu = (1, \dots, 1)$, then $\tilde{\nu}_1 = 1$, $\mathbf{v}_1 = (1, \dots, 1)$.

Theorem 2.18. *Assume that $V(t) = W(\omega t)$ with $W \in C_b^\infty(\mathbb{T}^n, \mathcal{A}_\rho)$ a quasi-periodic operator of order $\rho < 1$. Assume furthermore that $(\tilde{\nu}, \omega) \in \mathbb{R}^{\tilde{d}+n}$ is a Diophantine vector, i.e., there exist $\gamma > 0$ and $\kappa \in \mathbb{R}$ such that*

$$|\omega \cdot k + \tilde{\nu} \cdot \ell| \ge \frac{\gamma}{(|\ell| + |k|)^\kappa}, \qquad 0 \ne (k, \ell) \in \mathbb{Z}^{n+\tilde{d}}. \tag{2.16}$$

Then the propagator $\mathcal{U}(t, s)$ generated by $H(t) = \nu \cdot K + W(\omega t)$ exists and satisfies (2.9).

Remark 2.19. The vector $\tilde{\nu}$ is defined up to linear combinations with integer coefficients; clearly condition (2.16) does not depend on the choice of $\tilde{\nu}$.

Remark 2.20. We recall that the Diophantine vectors form a subset of $\mathbb{R}^{n+\tilde{d}}$ of full measure if $\kappa > n + \tilde{d} - 1$.

2.5. Applications (ii)

Relativistic Schrödinger equation on Zoll manifolds. We consider the reduced Dirac equation on a Zoll manifold M with mass $\mu > 0$,

$$\mathrm{i}\partial_t \psi = \sqrt{\Delta_g + \mu}\, \psi + V(\omega t, x, D_x)\psi, \qquad t \in \mathbb{R},\ x \in M.$$

As in the case of the Schrödinger equation on Zoll manifolds, $\mathcal{A}_\rho$ is the class of pseudo-differential operators with symbols in $S^\rho_{\mathrm{cl}}(M)$ (see Definition 2.6).

In this case V is assumed to be quasiperiodic in time.

Corollary 2.21 (Relativistic Schrödinger equation on Zoll manifolds). *Assume that* $V(t) = W(\omega t)$ *with* $W \in C^\infty(\mathbb{T}^n, \mathcal{A}_\rho)$ *with* $\rho < 1$. *Assume furthermore that the nonresonance condition*

$$|\omega \cdot k + m| \geq \frac{\gamma}{1+|k|^\kappa}, \qquad \forall\, 0 \neq k \in \mathbb{Z}^n,\ \forall m \in \mathbb{Z}, \tag{2.17}$$

holds for some $\gamma > 0$ *and* κ. *Then the propagator* $\mathcal{U}(t,s)$ *generated by* $H(t) = \sqrt{\Delta_g + \mu} + W(\omega t)$ *exists and satisfies* (2.9).

Harmonic oscillator in $\mathbb{R}^d$. Consider the quantum harmonic oscillator

$$\mathrm{i}\partial_t \psi = H_\nu \psi + V(t)\psi, \quad x \in \mathbb{R}^d, \tag{2.18}$$

$$H_\nu := -\Delta + \sum_{j=1}^d \nu_j^2 x_j^2, \qquad V(t) = W(\omega t, x, D_x). \tag{2.19}$$

Here W is the Weyl quantization of a symbol belonging to the following class:

Definition 2.22. A function f will be called a *symbol of order* $\rho \in \mathbb{R}$ if $f \in C^\infty(\mathbb{R}^d_x \times \mathbb{R}^d_\xi)$ and for all $\alpha, \beta \in \mathbb{N}^d$, there exists $C_{\alpha,\beta} > 0$ such that

$$|\partial_x^\alpha \, \partial_\xi^\beta f(x,\xi)| \leq C_{\alpha,\beta}\, (1 + |x|^2 + |\xi|^2)^{\rho - (|\beta| + |\alpha|)/2}. \tag{2.20}$$

We will write $f \in S^\rho_{\mathrm{ho}}$.

The class (2.20) is the extension to higher dimensions of the class used in anharmonic oscillators (see Definition 2.8) and with $k = l = 1$.

Remark 2.23. With our numerology, the symbol of the harmonic oscillator is of order 1, $|\xi|^2 + \sum_j \nu_j^2 x_j^2 \in S^1_{\mathrm{ho}}$, and not of order 2 as typically in the literature.

The classes $\mathcal{A}_m$ are defined as in Definition 2.9, with symbols in the class S^m_{ho}.

Corollary 2.24. *Assume that* ν *is such that* $\tilde{\nu}$ *fulfills* (2.16), *and that* $W \in C^\infty(\mathbb{T}^n, \mathcal{A}_\rho)$ *with* $\rho < 1$. *Then the propagator* $\mathcal{U}(t,s)$ *of* $H(t) = H_\nu + W(\omega t)$ *exists and fulfills* (2.9).

Note that after a trivial rescaling of the spatial variables, $H_\nu = \sum_{j=1}^d \nu_j(-\partial_j^2 + x_j^2)$, thus the corollary is a trivial application of Theorem 2.18.

Remark 2.25. In the completely resonant case

$$H_{(1,\ldots,1)} = -\Delta + |x|^2,$$

one has $\tilde{\nu} = 1$ and the set of ω's for which (2.16) is fulfilled has full measure provided $\kappa > n$.

Remark 2.26. In the resonant case, examples of polynomial growth of Sobolev norms have already been exhibited. In particular see [Del14] and [BGMR18] for periodic in time perturbations; of course in such examples the frequency ω does not fulfill (2.16). Finally, we also recall [BJLPN], where some random in time perturbations are considered.

3. Proofs of the abstract theorems

3.1. Scheme of the proof

As explained in the introduction, the main step of the proof consists in proving a theorem conjugating the original Hamiltonian to a Hamiltonian of the form (1.2); this will be done in Theorem 3.8. Subsequently we will apply [MR17, Theorem 1.5], which essentially states that if $H(t)$ is such that for some $N > -1$,

$$[H(t), K_0]K_0^N \in C_b^0(\mathbb{R}, \mathcal{L}(\mathcal{H}^r)), \tag{3.1}$$

then there exists $C_{r,N} > 0$ such that

$$\|\mathcal{U}(t, s)\,\psi\|_r \leq C_{r,N}\,\langle t - s\rangle^{\frac{r}{1+N}}\,\|\psi\|_r, \qquad \forall t, s \in \mathbb{R}. \tag{3.2}$$

We come to the algorithm of conjugation of the original Hamiltonian to (1.2). Before discussing it, we need to know the way a Hamiltonian is changed by a time-dependent unitary transformation. This is the content of the following lemma.

Lemma 3.1. *Let $H(t)$ be a time-dependent self-adjoint operator, and $X(t)$ be a family of self-adjoint operators. Assume that $\psi(t) = \mathrm{e}^{-\mathrm{i}X(t)}\varphi(t)$. Then*

$$\mathrm{i}\dot{\psi} = H(t)\psi \iff \mathrm{i}\dot{\varphi} = \tilde{H}(t)\varphi \tag{3.3}$$

where

$$\tilde{H}(t) := \mathrm{e}^{\mathrm{i}X(t)}H(t)\,\mathrm{e}^{-\mathrm{i}X(t)} - \int_0^1 \mathrm{e}^{\mathrm{i}sX(t)}\dot{X}(t)\,\mathrm{e}^{-\mathrm{i}sX(t)}\,\mathrm{d}s. \tag{3.4}$$

This is seen by an explicit computation; for example see [Bam18, Lemma 3.2].

A further important property giving the expansion of an operator of the form $\mathrm{e}^{\mathrm{i}X(t)}A\,\mathrm{e}^{-\mathrm{i}X(t)}$ into operators of decreasing order is stated in the following lemma.

Lemma 3.2. *Let $X \in \mathcal{A}_\rho$ with $\rho < 1$ be a symmetric operator. Let $A \in \mathcal{A}_m$ with $m \in \mathbb{R}$. Then X is self-adjoint and for any $M \geq 1$ we have*

$$\mathrm{e}^{\mathrm{i}\tau X}A\,\mathrm{e}^{-\mathrm{i}\tau X} = \sum_{\ell=0}^{M}\frac{\tau^\ell}{\ell!}\,\mathrm{ad}_X^\ell(A) + R_M(\tau, X, A), \qquad \forall \tau \in \mathbb{R}, \tag{3.5}$$

where $R_M(\tau, X, A) \in \mathcal{A}_{m-(M+1)(1-\rho)}$. In particular $\mathrm{ad}_X^\ell(A) \in \mathcal{A}_{m-\ell(1-\rho)}$ and $\mathrm{e}^{\mathrm{i}\tau X}A\,\mathrm{e}^{-\mathrm{i}\tau X} \in \mathcal{A}_m$ for all $\tau \in \mathbb{R}$.

The proof will be given in Sect. 3.2.

We now describe the algorithm which will lead to the smoothing Theorem 3.8; the proof is slightly different according to the set of assumptions one chooses. We start by discussing it under the assumptions of Theorem 2.5, namely Assumptions A and B. Subsequently we will discuss the changes needed to deal with Theorem 2.18.

We look for a change of variables of the form $\varphi = \mathrm{e}^{\mathrm{i}X_1(t)}\psi$ where $X_1(t) \in \mathcal{A}_{\rho-\mu+1}$ is a self-adjoint operator which, due to the assumption $\rho < \mu$, has order smaller than 1. We will check that also $\dot{X}_1(t) \in \mathcal{A}_{\rho-\mu+1}$. Then φ fulfills the Schrödinger equation $\mathrm{i}\dot{\varphi} = H^+(t)\varphi$ with

$$\begin{aligned} H^+(t) &:= \mathrm{e}^{\mathrm{i}X_1(t)}\, H(t)\, \mathrm{e}^{-\mathrm{i}X_1(t)} - \int_0^1 \mathrm{e}^{\mathrm{i}sX_1(t)}\, \dot{X}_1(t)\, \mathrm{e}^{-\mathrm{i}sX_1(t)}\, \mathrm{d}s \\ &= H_0 + \mathrm{i}[X_1(t), H_0] + V(t) + \mathrm{i}[X_1(t), V(t)] - \tfrac{1}{2}[X_1(t), [X_1(t), H_0]] + \cdots \\ &\quad - \int_0^1 \mathrm{e}^{\mathrm{i}sX_1(t)}\, \dot{X}_1(t)\, \mathrm{e}^{-\mathrm{i}sX_1(t)}\, \mathrm{d}s. \end{aligned}$$

In view of the properties of the graded algebra we have $[X_1, V] \in \mathcal{A}_{2\rho-\mu}$, $[X_1, [X_1, H_0]] \in \mathcal{A}_{2\rho-\mu}$ (Assumption I(iv)) and $\mathrm{e}^{\mathrm{i}sX_1(t)}\dot{X}_1(t)\,\mathrm{e}^{-\mathrm{i}sX_1(t)} \in \mathcal{A}_{\rho-\mu+1}$ (Lemma 3.2), so

$$H^+(t) = H_0 + \mathrm{i}[X_1(t), H_0] + V(t) + V_1^+(t) \tag{3.6}$$

with $V_1^+(t) \in C_b^\infty(\mathbb{R}, \mathcal{A}_{\min(\rho-\mu+1, 2\rho-\mu)})$.

Now we look for $X_1(t)$ such that

$$\mathrm{i}[H_0, X_1(t)] = V(t) - \langle V(t)\rangle, \tag{3.7}$$

where $\langle V(t)\rangle$ is the average over τ of $\mathrm{e}^{\mathrm{i}\tau K_0}V(t)\,\mathrm{e}^{-\mathrm{i}\tau K_0}$ (see (3.18)), which in particular commutes with K_0. We will verify in Lemma 3.5 that there exists X_1 such that

$$\mathrm{i}[H_0, X_1(t)] - V(t) + \langle V(t)\rangle \in \mathcal{A}_{\rho-1}.$$

Using such an X_1 to generate a unitary transformation, we get

$$H^+(t) := H_0 + \langle V(t)\rangle + V^+(t), \tag{3.8}$$

where $V^+(t) \in C_b^\infty(\mathbb{R}, \mathcal{A}_{\rho-\delta})$ with

$$\delta := \min(1, \mu - 1, \mu - \rho) > 0. \tag{3.9}$$

Therefore $V^+(t)$ is a perturbation of order lower than that of $V(t)$. Furthermore $\langle V(t)\rangle$ commutes with K_0.

Iterating this procedure we will establish an "almost" reducibility result that will be stated and proved in Subsect. 3.4.

Then, using [MR17, Theorem 1.5], we immediately get Theorem 2.5.

In the case where $H_0 \in \mathcal{A}_1$ the procedure has to be slightly modified since in this case X_1 and therefore $\dot{X}_1$ has the same order as V and thus it cannot be considered as a remainder when analyzing H^+. In this case one rewrites

$$\begin{aligned} H^+(t) = H_0 &+ \mathrm{i}[X_1(t), H_0] + V(t) \\ &+ \mathrm{i}[X_1(t), V(t)] - \tfrac{1}{2}[X_1(t), [X_1(t), H_0]] + \cdots \\ &- \dot{X}_1 - \int_0^1 (\mathrm{i}\, s\, [X_1(t), \dot{X}_1(t)] + \ldots)\, \mathrm{d}s, \end{aligned}$$

so that (3.6) is replaced by

$$H^+(t) = H_0 + \mathrm{i}[X(t), H_0] + V(t) - \dot{X}_1(t) + V^+(t) \tag{3.10}$$

with $V^+ \in \mathcal{A}_{\rho-\delta_*}$ where

$$\delta_* := 1 - \rho > 0, \tag{3.11}$$

so again it is more regular than $V(t)$. Thus one is led to consider the new homological equation

$$\mathrm{i}[H_0, X_1(t)] + \dot{X}_1(t) = V(t) - \langle V(t)\rangle, \tag{3.12}$$

where $\langle V(t)\rangle$ has to commute with K_0. In order to be able to solve such an equation we restrict to the case of $V(t)$ quasiperiodic in t and, as explained in the introduction, we develop a procedure based on a suitable Fourier expansion to construct X_1 and $\langle V(t)\rangle$. The details are given in Lemma 3.7 which will ensure that such a homological equation has a smooth solution and thus the procedure is well defined also in the case of order 1.

3.2. A couple of lemmas on flows

Lemma 3.3. (i) *Let $X \in \mathcal{A}_1$ be symmetric with respect to the scalar product of $\mathcal{H}^0$. Then X has a unique self-adjoint extension and $\mathrm{e}^{-\mathrm{i}\tau X} \in \mathcal{L}(\mathcal{H}^r)$ for all $r \geq 0$ and $\tau \in \mathbb{R}$. Furthermore $\mathrm{e}^{-\mathrm{i}\tau X}$ is an isometry in $\mathcal{H}^0$.*

(ii) *Assume that $X(t)$ is a family of symmetric operators in $\mathcal{A}_1$ such that*

$$\sup_{t\in\mathbb{R}} \wp^1_j(X(t)) < \infty, \quad \forall j \geq 1. \tag{3.13}$$

Then there exist $c_r, C_r > 0$ such that

$$c_r\|\psi\|_r \leq \|\mathrm{e}^{-\mathrm{i}\tau X(t)}\psi\|_r \leq C_r\|\psi\|_r, \quad \forall t \in \mathbb{R}, \forall \tau \in [0, 1]. \tag{3.14}$$

Proof. (i) From the properties of the algebra $\mathcal{A}$ we find that XK_0^{-1} and $[X, K_0]K_0^{-1}$ are of order 0. Thus by definition these operators belong to $\mathcal{L}(\mathcal{H}^r)$ for all $r \in \mathbb{R}$. Then the result follows from [MR17, Theorem 1.2].

(ii) By item (i), for any $t \in \mathbb{R}$ and $\tau \in [0, 1]$ the operator $\mathrm{e}^{-\mathrm{i}\tau X(t)}$ is an isometry in $\mathcal{H}^0$, therefore

$$\|\mathrm{e}^{-\mathrm{i}\tau X(t)}\psi\|_r = \|\mathrm{e}^{\mathrm{i}\tau X(t)}\, K_0^r\, \mathrm{e}^{-\mathrm{i}\tau X(t)}\psi\|_0.$$

Then we have

$$\begin{aligned}\mathrm{e}^{\mathrm{i}\tau X(t)} K_0^r\, \mathrm{e}^{-\mathrm{i}\tau X(t)}\psi &= K_0^r\psi + \mathrm{i}\int_0^\tau \mathrm{e}^{\mathrm{i}\tau_1 X(t)}[X(t), K_0^r]\,\mathrm{e}^{-\mathrm{i}\tau_1 X(t)}\psi\, \mathrm{d}\tau_1\\ &= K_0^r\psi + \mathrm{i}\int_0^\tau \mathrm{e}^{\mathrm{i}\tau_1 X(t)}[X(t), K_0^r]K_0^{-r}K_0^r\,\mathrm{e}^{-\mathrm{i}\tau_1 X(t)}\psi\, \mathrm{d}\tau_1. \end{aligned} \tag{3.15}$$

By the properties of the algebra $\mathcal{A}$ and (3.13), using (2.1)–(2.3) one sees that

$$\sup_{t\in\mathbb{R}} \|[X(t), K_0^r]K_0^{-r}\|_{\mathcal{L}(\mathcal{H}^0)} < C_r < +\infty,$$

therefore taking the norm $\|\cdot\|_0$ of (3.15) yields

$$\|\mathrm{e}^{-\mathrm{i}\tau X(t)}\psi\|_r \leq \|\psi\|_r + \int_0^\tau C_r \|\mathrm{e}^{-\mathrm{i}\tau_1 X(t)}\psi\|_r \,\mathrm{d}\tau_1 .$$

Then by Gronwall we conclude that

$$\|\mathrm{e}^{-\mathrm{i}\tau X(t)}\psi\|_r \leq \mathrm{e}^{C_r}\|\psi\|_r, \qquad \forall t \in \mathbb{R},\ \forall \tau \in [-1, 1].$$

This proves the majorization in (3.14). The minorization follows simply by the identity $\psi = \mathrm{e}^{\mathrm{i}\tau X(t)}\,\mathrm{e}^{-\mathrm{i}\tau X(t)}\psi$ and the majorization. □

Proof of Lemma 3.2. Self-adjointness was proven in the previous lemma. Let us apply to the l.h.s. of (3.5) the Taylor formula at $\tau = 0$. Then we get, with $U_X(\tau) := \mathrm{e}^{-\mathrm{i}\tau X}$ and $\mathrm{ad}_X(A) := \mathrm{i}[X, A]$,

$$\begin{aligned} &U_X(-\tau) A U_X(\tau) \\ &\quad = \sum_{j=0}^{M} \frac{\tau^j}{j!}\,\mathrm{ad}_X^j(A) + \frac{\tau^{M+1}}{M!}\int_0^1 (1-s)^{M+1} U_X(-s\tau)\,\mathrm{ad}_X^{M+1}(A) U_X(s\tau)\,\mathrm{d}s. \end{aligned} \tag{3.16}$$

Using Assumption I(iv), we deduce that $\mathrm{ad}_X^j(A) \in \mathcal{A}_{m-j(1-\rho)}$. We define the remainder $R_M(\tau, X, A)$ to be the integral term in (3.16), which, also by Lemma 3.3, belongs to $\mathcal{L}(\mathcal{H}^s, \mathcal{H}^{s-m+(M+1)(1-\rho)})$ for all $s \in \mathbb{R}$. Therefore $R_M(\tau, X, A)$ is N-smoothing provided $M + 1 \geq \frac{N+m}{1-\rho}$. As M can be taken arbitrarily large, $\mathrm{e}^{\mathrm{i}\tau X} A\, \mathrm{e}^{-\mathrm{i}\tau X}$ fulfills Assumption I(v), thus it belongs to $\mathcal{A}_m$. □

3.3. *Solution of the homological equations*

The first homological equation. As we have seen in Section 3.1, to prove Theorem 2.5 we need to study a homological equation of the form

$$\mathrm{i}[H_0, X] = A - \langle A \rangle\,, \tag{3.17}$$

where $A \in \mathcal{A}_m$ and $\langle A \rangle$ is the average of A along the periodic flow of K_0:

$$\langle A \rangle := \frac{1}{2\pi}\int_0^{2\pi} A(\tau)\,\mathrm{d}\tau, \qquad A(\tau) = \mathrm{e}^{\mathrm{i}\tau K_0} A\, \mathrm{e}^{-\mathrm{i}\tau K_0}. \tag{3.18}$$

Notice that the assumption on the spectrum of K_0 (see Assumption A) entails that $\mathrm{e}^{2\mathrm{i}\pi K_0} = \mathrm{e}^{2\mathrm{i}\pi\lambda}$, thus for any $A \in \mathcal{A}$ one has $\mathrm{e}^{2\mathrm{i}\pi K_0} A\, \mathrm{e}^{-2\mathrm{i}\pi K_0} = A$, so $\tau \mapsto A(\tau)$ is 2π-periodic.

Lemma 3.4. *Let $A \in \mathcal{A}_m$ for some $m \in \mathbb{R}$. Then $\langle A \rangle \in \mathcal{A}_m$ and*

$$[K_0, \langle A \rangle] = 0. \tag{3.19}$$

Proof. $\langle A \rangle \in \mathcal{A}_m$ is a consequence of Assumption II. Identity (3.19) follows by a direct computation. □

Lemma 3.5. (i) *Let $A \in \mathcal{A}_m$ for some $m \in \mathbb{R}$. Then*

$$Y = \frac{1}{2\pi} \int_0^{2\pi} \tau\, (A - \langle A \rangle)(\tau)\, \mathrm{d}\tau \tag{3.20}$$

solves the homological equation

$$\mathrm{i}[K_0, Y] = A - \langle A \rangle\,. \tag{3.21}$$

Further $Y \in \mathcal{A}_m$ and if A is symmetric, so is Y.

(ii) *Let $A \in \mathcal{A}_m$ be symmetric. Choose $R > 0$ such that $f'(x) \geq 1$ if $x \geq R$ and $\eta \in C^\infty(\mathbb{R})$ such that $\eta(x) = 1$ if $x \in [0, R]$ and $\eta(x) = 0$ if $x \geq R + 1$. Define*

$$X_0 := (1 - \eta(K_0))(f'(K_0))^{-1} Y \tag{3.22}$$

with Y as in (3.20). Then $X_0 \in \mathcal{A}_{m-\mu+1}$ and $X := \frac{1}{2}(X_0 + X_0^*) \in \mathcal{A}_{m-\mu+1}$ is symmetric and solves (3.17) modulo an error term in $\mathcal{A}_{m-1}$. More precisely,

$$\mathrm{i}[H_0, X] = A - \langle A \rangle + \mathcal{A}_{m-1}. \tag{3.23}$$

We note for later reference that if $A \in \mathcal{A}_m$ then $X \in \mathcal{A}_{m-(\mu-1)}$, so we have a gain of $\mu - 1 > 0$ in the smoothing order.

Proof of Lemma 3.5. Assertion (i) is proved by integration by parts using the fact that $A(\tau)$ is 2π-periodic.

To prove (ii), first remark that by Assumption B and Lemma A.1, $f' \in S_+^{\mu-1}$, thus it is different from zero provided $x \geq R$ is large enough. It follows that $x \mapsto \frac{1-\eta(x)}{f'(x)} \in S^{-\mu+1}$. Therefore, by Lemma A.2, $(1 - \eta(K_0))(f'(K_0))^{-1} \in \mathcal{A}_{-\mu+1}$. Finally, since $Y \in \mathcal{A}_m$, it follows that $X_0, X \in \mathcal{A}_{m-\mu+1}$.

We show now that X_0 solves (3.23). This is a consequence of the commutator expansion lemma. Indeed, fix $N \geq 2$. Then by Lemma A.3,

$$\begin{aligned}[H_0, X_0] = [f(K_0), X_0] = f'(K_0)[K_0, X_0] \\ + \sum_{2 \leq j \leq N} \frac{1}{j!} f^{(j)}(K_0)\, \mathrm{ad}_{K_0}^j (X_0) + R_{N+1}(f, X_0)\end{aligned}$$

with $R_{N+1}(f, X_0) \in \mathcal{A}_{m-\mu+1+[\mu]-N} \subset \mathcal{A}_{m-1}$. By Lemma A.1 and Assumption I, for any integer $j \geq 2$ one has $f^{(j)}(K_0)\, \mathrm{ad}_{K_0}^j (X) \in \mathcal{A}_{m-\mu+1+\mu-j} \subset \mathcal{A}_{m-1}$. Then we get

$$\begin{aligned}\mathrm{i}[H_0, X_0] &= \mathrm{i} f'(K_0)[K_0, X_0] + A_{m-1} \\ &\overset{(3.22)}{=} (1 - \eta(K_0))\mathrm{i}[K_0, Y] + A_{m-1} \overset{(3.21)}{=} (1 - \eta(K_0))(A - \langle A \rangle) + A_{m-1},\end{aligned}$$

with $A_{m-1} \in \mathcal{A}_{m-1}$. Now put $R := -\eta(K_0)(A - \langle A \rangle)$. Since $x \mapsto \eta(x) \in S^{-\infty}$, R is a smoothing operator and thus $A_{m-1} + R \in \mathcal{A}_{m-1}$. Now since $X = X_0 + \mathcal{A}_{m-\mu}$ by construction, we easily see that X satisfies (3.23). ◻

The second homological equation. We want to solve equation (3.12). Using the quasiperiodicity assumption $V(t) = W(\omega t)$, we look for a quasiperiodic solution $X_1(t) = X(\omega t)$ of the equation

$$\omega \cdot \partial_\theta X(\omega t) + \mathrm{i}[H_0, X(\omega t)] = W(\omega t) - \langle W \rangle. \tag{3.24}$$

In order to define $\langle W \rangle$ precisely, consider again the vectors $\mathbf{v}_j$ and the frequencies $\tilde{\nu}_j$ of Lemma 2.16. First note that since $\nu = \sum_{j=1}^{\tilde{d}} \tilde{\nu}_j \mathbf{v}_j$, one has $\nu \cdot K = \sum_{j=1}^{\tilde{d}} (K \cdot \mathbf{v}_j)\tilde{\nu}_j$, so that defining

$$\tilde{K}_j := K \cdot \mathbf{v}_j, \quad \tilde{K} := (\tilde{K}_1, \dots, \tilde{K}_{\tilde{d}}), \tag{3.25}$$

one has

$$H_0 \equiv \nu \cdot K = \tilde{\nu} \cdot \tilde{K},$$

and furthermore, since $\mathbf{v}_j$ has integer entries, the joint spectrum of $\tilde{K} \equiv (\tilde{K}_1, \dots, \tilde{K}_{\tilde{d}})$ is contained in $\mathbb{Z}^{\tilde{d}} + \tilde{\lambda}$, therefore for each operator B the map $\mathbb{R}^{\tilde{d}} \ni \tau \mapsto B^\sharp(\tau) := \mathrm{e}^{\mathrm{i}\tau\cdot\tilde{K}} B\, \mathrm{e}^{-\mathrm{i}\tau\cdot\tilde{K}}$ is periodic in each τ_j. For $A \in C^\infty(\mathbb{T}^n, \mathcal{A}_m)$, denote $A^\sharp(\theta, \tau) := \mathrm{e}^{\mathrm{i}\tau\cdot\tilde{K}} A(\theta)\, \mathrm{e}^{-\mathrm{i}\tau\cdot\tilde{K}}$. By Assumption II$'$, $A^\sharp \in C^\infty(\mathbb{T}^{n+\tilde{d}}, \mathcal{A}_m)$. Define now

$$\langle A \rangle := \frac{1}{(2\pi)^{n+\tilde{d}}} \int_{\mathbb{T}^{n+\tilde{d}}} A^\sharp(\theta, \tau)\, \mathrm{d}\tau\, \mathrm{d}\theta. \tag{3.26}$$

Remark 3.6. Let $A \in C^\infty(\mathbb{T}^n, \mathcal{A}_m)$, $m \in \mathbb{R}$. Then by Assumption II$'$, $\langle A \rangle \in \mathcal{A}_m$ is independent of the angles and

$$[\tilde{K}_j, \langle A \rangle] = 0, \quad 1 \le j \le \tilde{d}, \quad [K_0, \langle A \rangle] = 0. \tag{3.27}$$

Lemma 3.7. *Let $A \in C_b^\infty(\mathbb{T}^n, \mathcal{A}_m)$ for some $m \in \mathbb{R}$. Provided (2.16) holds, the homological equation (3.24) has a solution $X \in C^\infty(\mathbb{T}^n, \mathcal{A}_m)$. Furthermore, if A is symmetric then X is symmetric as well.*

Proof. Since $A^\sharp$ is defined on $\mathbb{T}^{n+\tilde{d}}$, we can expand it in a Fourier series:

$$A^\sharp(\theta, \tau) = \sum_{(k,\ell)\in\mathbb{Z}^{n+\tilde{d}}} \hat{A}^\sharp_{k,\ell}\, \mathrm{e}^{\mathrm{i}(k\cdot\theta + \ell\cdot\tau)},$$

where

$$\hat{A}^\sharp_{k,\ell} := \frac{1}{(2\pi)^{n+\tilde{d}}} \int_{\mathbb{T}^{n+\tilde{d}}} A^\sharp(\theta, \tau)\, \mathrm{e}^{-\mathrm{i}(k\cdot\theta + \ell\cdot\tau)}\, \mathrm{d}\theta\, \mathrm{d}\tau.$$

Notice that

$$A(\theta) \equiv A^\sharp(\theta, 0) = \sum_{(k,\ell)\in\mathbb{Z}^{n+\tilde{d}}} \hat{A}^\sharp_{k,\ell}\, \mathrm{e}^{\mathrm{i}k\cdot\theta}, \tag{3.28}$$

$$\langle A \rangle = \hat{A}^\sharp_{0,0}. \tag{3.29}$$

Then, instead of solving directly the homological equation (3.24), we solve

$$\omega \cdot \partial_\theta X^\sharp(\theta, \tau) + \mathrm{i}[H_0, X^\sharp(\theta, \tau)] = (W - \langle W \rangle)^\sharp(\theta, \tau), \quad \forall \theta \in \mathbb{T}^n,\ \forall \tau \in \mathbb{T}^{\tilde{d}}. \tag{3.30}$$

Clearly if we find a smooth solution $X^\sharp(\theta,\tau)$ of this equation, then $X(\theta) := X^\sharp(\theta,0)$ solves the original homological equation (3.24). Now we remark that using

$$X^\sharp(\theta,\tau+\tau') = \mathrm{e}^{i\tau\tilde{K}} X^\sharp(\theta,\tau')\,\mathrm{e}^{-i\tau\tilde{K}}$$

we have

$$\begin{aligned}
\mathrm{i}[H_0, X^\sharp(\theta,\tau)] &= \sum_{j=1}^{\tilde{d}} \tilde{\nu}_j\, \mathrm{i}[\tilde{K}_j, X^\sharp(\theta,\tau)] = \sum_{j=1}^{\tilde{d}} \tilde{\nu}_j \frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0} \mathrm{e}^{\mathrm{i}\epsilon\tilde{K}_j}\, X^\sharp(\theta,\tau)\,\mathrm{e}^{-\mathrm{i}\epsilon\tilde{K}_j} \\
&= \sum_{j=1}^{\tilde{d}} \tilde{\nu}_j \frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0} X^\sharp(\theta,\tau+\epsilon\mathbf{e}_j) \\
&= \sum_{(k,\ell)\in\mathbb{Z}^{n+\tilde{d}}} \hat{X}^\sharp_{k,\ell} \frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0} \sum_{j=1}^{\tilde{d}} \tilde{\nu}_j\, \mathrm{e}^{\mathrm{i}(k\cdot\theta+\ell\cdot(\tau+\epsilon\mathbf{e}_j))} \\
&= \sum_{(k,\ell)\in\mathbb{Z}^{n+\tilde{d}}} \mathrm{i}\tilde{\nu}\cdot\ell\hat{X}^\sharp_{k,\ell}\,\mathrm{e}^{\mathrm{i}(k\cdot\theta+\ell\cdot\tau)}.
\end{aligned}$$

Therefore, expanding in a Fourier series, equation (3.30) is equivalent to

$$\mathrm{i}(\omega\cdot k+\tilde{\nu}\cdot\ell)\hat{X}^\sharp_{k,\ell} = \widehat{W}^\sharp_{k,\ell}, \qquad (k,\ell)\neq 0.$$

Hence we define

$$\hat{X}^\sharp_{k,\ell} = -\mathrm{i}\frac{\widehat{W}^\sharp_{k,\ell}}{\omega\cdot k+\tilde{\nu}\cdot\ell} \qquad \text{if } (k,\ell)\neq 0. \tag{3.31}$$

Since $W^\sharp$ is in $C^\infty(\mathbb{T}^{n+\tilde{d}},\mathcal{A}_m)$ we find that for any $j, N\geq 1$ there exists $C_{N,j}$ such that

$$\wp^m_j(\widehat{W}^\sharp_{k,\ell}) \leq C_{N,j}(|k|+|\ell|)^{-N}.$$

So we see easily that if X is defined by $X(\theta) = X^\sharp(\theta,0)$ and $X^\sharp$ has Fourier coefficients (3.31) with $X^\sharp_{0,0} = 0$, then $X\in C^\infty_b(\mathbb{T}^n,\mathcal{A}_m)$. □

3.4. The iterative lemma

We state and prove the iterative lemma which is the main step for the proof of our main results.

Theorem 3.8. *Assume that the assumptions of Theorem 2.5 or of Theorem 2.18 are satisfied. There exist $\delta>0$ and a sequence $\{X_j(t)\}_{j\geq1}$ of self-adjoint (time-dependent) operators in $\mathcal{H}$ with $X_j\in C^\infty_b(\mathbb{R},\mathcal{A}_{\rho-(\mu-1)-(j-1)\delta})$ such that for all j, the inequalities (3.14) are satisfied; for any $N\geq 1$ the change of variables*

$$\psi = \mathrm{e}^{-\mathrm{i}X_1(t)}\ldots\mathrm{e}^{-\mathrm{i}X_N(t)}\varphi \tag{3.32}$$

transforms $H_0 + V(t)$ into the Hamiltonian

$$H^{(N)}(t) := H_0 + Z^{(N)}(t) + V^{(N)}(t) \tag{3.33}$$

where $Z^{(N)} \in C_b^\infty(\mathbb{R}, \mathcal{A}_\rho)$ commutes with K_0, i.e. $[Z^{(N)}, K_0] = 0$, while $V^{(N)} \in C_b^\infty(\mathbb{R}, \mathcal{A}_{\rho-N\delta})$. Furthermore, under the assumptions of Theorem 2.18, $Z^{(N)}$ is independent of the angles and

$$[Z^{(N)}, \tilde{K}_j] = 0, \qquad \forall j = 1, \dots, \tilde{d}. \tag{3.34}$$

Proof. This is proved by recurrence. Consider first the assumptions of Theorem 2.5. Using Lemmas 3.1, 3.2, 3.3, 3.5, one gets the conclusion for $N = 1$ with $Z^{(1)}(t) := \langle V(t) \rangle \in C_b^\infty(\mathbb{R}, \mathcal{A}_\rho)$. By Lemma 3.4, $[Z^{(1)}(t), K_0] = 0$. In this case δ can be taken as in (3.9).

The iterative step $N \to N + 1$ is proved following the same lines, just adding the remark that $e^{\mathrm{i}X_{N+1}} Z^{(N)} \mathrm{e}^{-\mathrm{i}X_{N+1}} - Z^{(N)} \in \mathcal{A}_{\rho-(\mu-1)-N\delta+\rho-1} \subset \mathcal{A}_{\rho-(N+1)\delta}$.

Under the assumptions of Theorem 2.18, the result is proved along the same lines, with δ as in (3.11). The property (3.34) follows by Remark 3.6. □

3.5. *Proof of Theorem 2.5*

By Theorem 3.8, the operator $H(t)$ is conjugated to $H^{(N)}(t)$. So we apply [MR17, Theorem 1.5] to the Schrödinger equation for $H^{(N)}(t)$. More precisely, we have

$$[H^{(N)}(t), K_0] = [V^{(N)}(t), K_0] \in C_b^0(\mathbb{R}, \mathcal{A}_{\rho-N\delta}),$$

and thus, by choosing N large enough, (3.2) ensures the result for the propagator $\mathcal{U}_N(t, s)$ of $H^{(N)}(t)$.

Now since $H(t)$ is conjugated to $H^{(N)}(t)$, $H(t)$ generates a propagator $\mathcal{U}(t, s)$ in the Hilbert space scale $\mathcal{H}^r$ unitarily equivalent to the propagator $\mathcal{U}_N(t, s)$. Therefore, using also (3.14), we conclude that $\mathcal{U}(t, s)$ fulfills (2.9), thus yielding the result. □

4. Applications

In this section we prove Corollaries 2.7, 2.11 and 2.21.

4.1. *Zoll manifolds*

To begin, we show how to put ourselves in the abstract setup. So first we define the operator K_0. This will be achieved by exploiting the spectral properties of the operator $\triangle_g$. By Theorem 1 of Colin de Verdière [CdV79], there exists a pseudodifferential operator Q of order -1, commuting with $\triangle_g$, such that $\mathrm{spec}(\sqrt{\triangle_g} + Q) \subseteq \mathbb{N} + \lambda$ with some $\lambda \geq 0$. We can assume $\lambda > 0$. If not, denoting by Π_- the projector on the nonpositive eigenvalues, we replace Q by $Q + C\Pi_-$ with $C > 0$ large enough; we remark that Π_- commutes with $\triangle_g$ and is a smoothing operator. So we define

$$K_0 := \sqrt{\triangle_g} + Q, \qquad H_0 := K_0^2. \tag{4.1}$$

Now since $H_0 = \triangle_g + 2Q\sqrt{\triangle_g} + Q^2$, we have

$$H_0 = \triangle_g + Q_0$$

where Q_0 is a pseudodifferential operator of order 0 and therefore

$$H(t) = \triangle_g + V(t) \equiv H_0 + \tilde{V}(t), \qquad \tilde{V}(t) := V(t) - Q_0.$$

and we are in the setup of the abstract Schrödinger equation (1.1) with the new perturbation $\tilde{V}(t)$.

Note that $\mathcal{H}^r := \mathrm{Dom}((K_0)^r)$, $r \geq 0$, coincides with the classical Sobolev space $H^r(M)$ and one has the equivalence of norms

$$c_r\|\psi\|_{H^r(M)} \leq \|\psi\|_r \leq C_r\|\psi\|_{H^r(M)}, \qquad \forall r \in \mathbb{R}.$$

We define $\mathcal{A}_m$ to be the class of pseudodifferential operators whose (real valued) symbols belong to $S^m_{\mathrm{cl}}(M)$. Clearly $K_0 \in \mathcal{A}_1$ (recall that Π_- is a smoothing operator). It is classical that Assumptions I and II are fulfilled (see e.g. [Hör85] and Appendix B).

Remark 4.1. We have implicitly used here the fact that on a compact manifold any smoothing operator has a symbol in the class $S^{-\infty}_{\mathrm{cl}}(M)$. This is true because on a compact manifold any operator is properly supported [Hör85]. In particular Assumption I(v) is satisfied for $\mathcal{A}_m = \mathrm{Op}(S^m_{\mathrm{cl}}(M))$. Let us remark that this property fails for classical pseudodifferential operators on $M = \mathbb{R}^d$. Hence the topology on $\mathcal{A}^m$ is the topology defined on $S^m_{\mathrm{cl}}(M)$.

Moreover the uniform boundedness in Assumption II is checked using the periodicity of the classical flow.

Proof of Corollary 2.7. Assumption A holds true by construction of K_0; Assumption B holds with $f(x) = x^2$ and therefore $\mu := 2$. Since $V(t)$ is a pseudodifferential operator of order $\rho < 2$ whose symbol belongs to $C^\infty_b(\mathbb{R}, S^\rho_{\mathrm{cl}}(M))$, one verifies easily, using pseudodifferential calculus (in particular estimates (2.1)–(2.3)), that $\tilde{V}(t) = V(t) - Q_0 \in C^\infty_b(\mathbb{R}, \mathcal{A}_\rho)$. Hence the corollary follows from Theorem 2.5. $\square$

4.2. Anharmonic oscillators

We recall that for a symbol a (in the sense of Definition 2.8) we denote by $a(x, D_x)$ its Weyl quantization

$$(a(x, D_x)\psi)(x) := \frac{1}{2\pi}\iint_{y,\xi\in\mathbb{R}} \mathrm{e}^{\mathrm{i}(x-y)\xi}\, a\left(\frac{x+y}{2}, \xi\right)\psi(y)\,\mathrm{d}y\,\mathrm{d}\xi. \tag{4.2}$$

We endow S^ρ_{an} (defined in Definition 2.8) with the family of seminorms

$$\wp^\rho_j(a) := \sum_{|\alpha|+|\beta|\leq j}\ \sup_{(x,\xi)\in\mathbb{R}^2} \frac{|\partial^\alpha_x \partial^\beta_\xi a(x,\xi)|}{[\mathrm{k}_0(x,\xi)]^{\rho - \frac{k\beta + l\alpha}{k+l}}}, \qquad j \in \mathbb{N}. \tag{4.3}$$

Remark 4.2. As we have seen, on a compact manifold (see Remark 4.1) we can use seminorms on symbols in S^m_{an} on the corresponding operator classes which are defined modulo smoothing operators. The reason here is that smoothing operators A are in $\mathcal{L}(\mathcal{S}'(\mathbb{R}), \mathcal{S}(\mathbb{R}))$, where $\mathcal{S}'(\mathbb{R})$ is the Schwartz space of tempered distributions on $\mathbb{R}$. It is well known that equivalently the Schwartz kernel K_A of A is in $\mathcal{S}(\mathbb{R}\times\mathbb{R})$ so its Weyl symbol σ^w_A is also in $\mathcal{S}(\mathbb{R}\times\mathbb{R})$. These facts result from two formulas: $K_A(x, y) = \langle A\delta_y, \delta_x\rangle$ and

$$\sigma^w_A(x,\xi) = \int_{\mathbb{R}} \mathrm{e}^{-iu\xi} K_A(x+u/2, x-u/2)\, du.$$

Then we can easily check that Assumption I(v) is satisfied for $\mathcal{A}_m = \mathrm{Op}^w(S^m_{\mathrm{an}})$.

The operator K_0 is defined using the spectral properties of the Hamiltonian $H_{k,l}$ defined in (2.11) that were studied in detail in [HR82b]; in that paper an accurate Bohr–Sommerfeld rule for the the eigenvalues of $H_{k,l}$ was obtained and the existence of a pseudodifferential operator Q of order -1 such that $\mathrm{spec}(H_{k,l}^{\frac{k+l}{2kl}} + Q) \subseteq \mathbb{N} + \lambda$ ($\lambda \geq 0$) was proved.[2] Therefore we define

$$K_0 := H_{k,l}^{\frac{k+l}{2kl}} + Q, \qquad H_0 := K_0^{\frac{2kl}{k+l}}.$$

We define $\mathcal{A}_m$ to be the class of pseudodifferential operators with symbols in S^m_{an}. Notice that by construction $\mathcal{A}_m \subset \mathcal{L}(\mathcal{H}^s, \mathcal{H}^{s-m})$ for all $s \in \mathbb{R}$. It is classical that $\mathcal{A}$ fulfills Assumptions I and II (see [HR82b, HR82a]).

On the other hand, Assumptions A and B are fulfilled with $\mu := \frac{2kl}{k+l} > 1$ (as $k + l \geq 3$). Furthermore,

$$H_{k,\ell} = (K_0 - Q)^{\frac{2kl}{k+l}} = K_0^{\frac{2kl}{k+l}} + Q_0$$

where Q_0 is a pseudodifferential operator of order $\frac{2kl}{k+l} - 2$. Therefore

$$H(t) = H_{k,l} + V(t) \equiv H_0 + \tilde{V}(t), \qquad \tilde{V}(t) := V(t) + Q_0,$$

and once again we are in the setup of the abstract Schrödinger equation (1.1) with the new perturbation $\tilde{V}(t)$.

Proof of Corollary 2.11. Since $V(t)$ is a pseudodifferential operator of order $\rho < \frac{2kl}{k+l}$ whose symbol and its time-derivatives have uniformly (in time) bounded seminorms, one verifies that $\tilde{V}(t) = V(t) + Q_0 \in C^\infty_b(\mathbb{R}, \mathcal{A}_\rho)$. Hence the corollary follows from Theorem 2.5. □

[2] Actually [HR82b] proves that Q has a symbol which is quasi-homogeneous of degree $-k-l$. Here a symbol $f(x,\xi)$ is *quasi-homogeneous of degree m* if

$$f(\lambda^l x, \lambda^k \xi) = \lambda^m f(x,\xi), \qquad \forall\lambda > 0,\ \forall(x,\xi) \in \mathbb{R}^2 \setminus \{0\}.$$

It is classical [HR82b, HR82a] that if f is quasi-homogeneous of degree m, then it is a symbol in the class $S^{m/(k+l)}_{\mathrm{an}}$. Note that for our numerology $H_{k,l}^{\frac{k+l}{2kl}}$ is of order 1 by definition.

4.3. Relativistic Schrödinger equation on Zoll manifolds

The proof of Corollary 2.21 is along the lines of Subsection 4.1. Let us remark that the operator $\sqrt{\Delta_g + \mu} - \sqrt{\Delta_g}$ is of order -1. Hence, defining K_0 as in (4.1), one has again $\sqrt{\Delta_g + \mu} = K_0 + Q_0$ with Q_0 of order -1. Therefore

$$H(t) = \sqrt{\Delta_g + \mu} + V(\omega t, x, D_x) = K_0 + \tilde{V}(\omega t)$$

with the new perturbation $\tilde{V}(\omega t) \in C^\infty(\mathbb{T}^n, \mathcal{A}_\rho)$.

This time we verify Assumptions II′, A′ and B′ with $d = 1$ and $K_1 = K_0 = H_0$. Concerning the nonresonance condition, just note that in this case ν has only one component given by 1.

Thus Theorem 2.18 immediately yields Corollary 2.21.

Appendix A. Technical lemmas on classical symbols

We begin with the following lemma whose proof is completely standard (and we skip it).

Lemma A.1. (i) *If $f \in S^a$ and $g \in S^b$, then $fg \in S^{a+b}$.*
(ii) *If $f \in S^a$, then $f^{(j)} \in S^{a-j}$.*
(iii) *If $x \mapsto \eta(x)$ is a smooth cut-off function on $\mathbb{R}$, then $\eta \in S^{-\infty}$.*
(iv) *The function $f(x) = x^a$, $a > 0$, is a classical elliptic symbol in S^a_+.*

Lemma A.2. *If $g \in S^\mu$ for some $\mu \in \mathbb{R}$, then $g(K_0) \in \mathcal{A}_\mu$.*

Proof. By definition $g(x) = \sum_{0\le j\le N-1} c_j x^{\mu-j} + R(x)$ with $|R(x)| \le C_N |x^{\mu-N}|$ for $|x| \ge 1$. Then $g(K_0) = \sum_{0\le j\le N-1} c_j K_0^{\mu-j} + R(K_0)$, where $R(K_0)$ is defined by functional calculus as $R(K_0) := \int_0^\infty R(\lambda)\, \mathrm{d}E_{K_0}(\lambda)$, $\mathrm{d}E_{K_0}(\lambda)$ being the spectral resolution of K_0. By Assumption I, $\sum_{0\le j\le N-1} c_j K_0^{\mu-j} \in \mathcal{A}_\mu$ while the operator $R(K_0)$ is N-smoothing (in the sense of Definition 2.1). Since N can be taken arbitrarily large, $g(K_0)$ fulfills Assumption I(v), therefore it belongs to $\mathcal{A}_\mu$. The other properties are easily verified using such decomposition. □

Finally, we recall a commutator expansion lemma following from [DG97, Lemma C.3.1]:

Lemma A.3. *Let $f \in S^\rho_+$ and $W \in \mathcal{A}_m$. Then for all $N \ge [\rho]$ we have*

$$[f(K_0), W] = \sum_{1\le j\le N} \frac{1}{j!} f^{(j)}(K_0)\, \mathrm{ad}^j_{K_0} W + R_{N+1}(f, K_0, W),$$

where $R_{N+1}(f, K_0, W) \in \mathcal{A}_{[\rho]+m-N}$. Moreover if W depends on time t with uniform estimates in $\mathcal{A}_m$ then it is also true for $R_{N+1}(f, K_0, W)$.

Proof. Apply [DG97, Lemma C.3.1] to the bounded operator $B = K_0^{-m} W$. □

Appendix B. An abstract proof of the Egorov theorem

In order to check Assumption II, we introduce the following weaker condition.

Assumption II-CL. For every $m \in \mathbb{R}$ and every $A \in \mathcal{A}_m$ there exist $\Phi^{(t)}(A) \in C^1(\mathbb{R}_t, \mathcal{A}_m)$ and $R(A,t) \in C^0(\mathbb{R}_t, \mathcal{A}_{m-1})$ such that $\Phi^{(0)}(A) = A$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi^{(t)}(A) = \mathrm{i}^{-1}[\Phi^{(t)}(A), K_0] + R(A,t). \tag{B.1}$$

In applications in a pseudodifferential operator setting, we have $A = \mathrm{Op}(a)$, a is the symbol of A and one can choose $\Phi^{(t)}(A) = \mathrm{Op}(a \circ \phi^t)$ where ϕ^t is the classical flow of the symbol of K_0. Then one has to verify that $a \circ \phi^t$ belongs to the same symbol class as a using the periodicity of ϕ^t (see for example [Tay91]).

Theorem B.1 (Abstract Egorov theorem). *If Assumptions I and II-CL are satisfied then for any $A \in \mathcal{A}_m$,*

$$\tau \mapsto A(\tau) := \mathrm{e}^{\mathrm{i}\tau K_0} A\, \mathrm{e}^{-\mathrm{i}\tau K_0} \in C^0(\mathbb{R}, \mathcal{A}_m).$$

In particular, if $\tau \mapsto A(\tau)$ is periodic on $\mathbb{R}$ then $A(\cdot) \in C^0_b(\mathbb{R}, \mathcal{A}_m)$ and Assumption II holds true.

Remark B.2. Notice that if the spectrum of K_0 is discrete with eigenvalues $\{\lambda_j\}_{j\geq 0}$ such that $\lambda_j - \lambda_k \in \mathbb{Z}$ for all $j,k \in \mathbb{N}$ then $\tau \mapsto A(\tau)$ is periodic on $\mathbb{R}$.

Proof of Theorem B.1. We follow [Rob87, pp. 202–207]. Let $U(t) = \mathrm{e}^{-\mathrm{i}tK_0}$. We compute

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(U(\tau-t)\Phi^{(\tau)}(A)U(t-\tau)) = U(\tau-t)\underbrace{\left(\mathrm{i}[\Phi^{(\tau)}(A), K_0] + \frac{\mathrm{d}}{\mathrm{d}\tau}\Phi^{(\tau)}(A)\right)}_{R(A,\tau)}U(t-\tau).$$

So using (B.1) and integrating in τ between 0 and t we get

$$U(-t)AU(t) = \Phi^{(t)}(A) - \int_0^t U(\tau-t)R(A,\tau)U(t-\tau)\,\mathrm{d}\tau. \tag{B.2}$$

Now we iterate this formula. In the following step we apply the formula for every τ to $A_{\text{new}} = R(A,\tau)$. In particular, Assumption II-CL implies that $\frac{\mathrm{d}}{\mathrm{d}t}\Phi^{(t)}(A_{\text{new}}) = \mathrm{i}^{-1}[\Phi^{(t)}(A_{\text{new}}), K_0] + R(A_{\text{new}}, t)$, so we get

$$U(-t)AU(t) = A_0(t) + A_1(t) + \int_0^t \int_0^{t-\tau} U(\tau-\tau_1-t)R(R(A,\tau),\tau_1)U(t-\tau-\tau_1)\,\mathrm{d}\tau\,\mathrm{d}\tau_1.$$

where $A_0(t) = \Phi^{(t)}(A)$, $A_1(t) = \int_0^t \Phi^{(t-\tau)}(R(A,\tau))\,\mathrm{d}\tau \in \mathcal{A}_{m-1}$ and $R(R(A,\tau),\tau-\tau_1) \in \mathcal{A}_{m-2}$. At step N we easily get by induction

$$U(-t)AU(t) = A_0(t) + A_1(t) + \cdots + A_N(t)$$
$$+ \int_0^t \int_0^{t-\tau_0} \cdots \int_0^{t-\tau_0-\cdots-\tau_N} \mathrm{d}\tau_0\,\mathrm{d}\tau_1 \cdots \mathrm{d}\tau_N$$
$$U(\tau_0+\tau_1+\cdots+\tau_N-t)R^{(N)}(A,\tau_0,\tau_1,\ldots,\tau_N)U(t-\tau_0-\tau_1-\cdots-\tau_N),$$

where $A_j \in C^0(\mathbb{R}, \mathcal{A}_{m-j})$ and $R^{(N)}(A,\tau_1,\ldots,\tau_N) \in C^0(\mathbb{R}^{N+1}, \mathcal{A}_{m-N-1})$. Now we remark that the remainder term is as smoothing as we want by taking N large enough, so the algebra being stable by smoothing perturbations we get $A(\cdot) \in C_b^0(\mathbb{R}, \mathcal{A}_m)$. $\square$

Appendix C. Proof of Lemma 2.16

We reproduce here the proof given in the lecture notes by Giorgilli [Gio] (in particular the technical results are contained in Appendix A). A general presentation containing also the results that we use here can be found in [Sie89].

We start by stating a simple lemma without proof.

Lemma C.1. *Let $\mathbf{e}_1, \ldots, \mathbf{e}_d$ and $\mathbf{e}'_1, \ldots, \mathbf{e}'_d$ be two bases of $\mathbb{Z}^d$. Then the matrix $M = (M_{ij})$ such that $\mathbf{e}'_i = \sum_j M_{ij}\mathbf{e}_j$ is unimodular with integer entries.*

Then one has the following corollary.

Corollary C.2. *A collection of vectors $\mathbf{e}_j \in \mathbb{Z}^d$, $j = 1, \ldots, d$, is a basis of $\mathbb{Z}^d$ if and only if the determinant of the matrix having $\mathbf{e}_j$ as rows is* 1.

The corollary immediately follows from Lemma C.1 and the remark that such a property holds for the canonical basis of $\mathbb{Z}^d$.

Define now the resonance modulus $\mathcal{M}_\nu$ of ν by

$$\mathcal{M}_\nu := \{k \in \mathbb{Z}^d : \nu \cdot k = 0\}.$$

This is a discrete subgroup of $\mathbb{R}^d$ which satisfies

$$\operatorname{span}(\mathcal{M}_\nu) \cap \mathbb{Z}^d = \mathcal{M}_\nu. \tag{C.1}$$

Let $0 \le r \le d-1$ be the dimension of $\mathcal{M}_\nu$. It is well known that any discrete subgroup of $\mathbb{R}^d$ admits a basis. Let $\mathbf{e}_1, \ldots, \mathbf{e}_r$, be a basis of $\mathcal{M}_\nu$; note that the vectors $\mathbf{e}_j$ have integer components. Then the following result holds.[3]

Lemma C.3. *There exist $\tilde{d} := d - r$ vectors $\mathbf{u}_1, \ldots, \mathbf{u}_{\tilde{d}}$ with integer entries such that $\mathbf{e}_1, \ldots, \mathbf{e}_r, \mathbf{u}_1, \ldots, \mathbf{u}_{\tilde{d}}$ form a basis of $\mathbb{Z}^d$.*

[3] This can be found in [Sie89, Theorem 31], or in [Gio, Lemma A.6].

Then one immediately obtains the following

Corollary C.4. *Let M be the matrix with rows given by the vectors $\mathbf{e}_j$ and the vectors $\mathbf{u}_j$, and define $\check{\nu} := M\nu$. Then $\check{\nu}_i = 0$ for all $i = 1, \dots, r$, while $\tilde{\nu}_i := \check{\nu}_{r+i}$, $i = 1, \dots, \tilde{d}$, are independent over the rationals.*

Proof of Lemma 2.16. Consider the matrix M^{-1}; since M is unimodular with integer entries, the same is true for M^{-1}, and one has $\nu = M^{-1}\check{\nu}$; however, since the first r components of $\check{\nu}$ vanish, such an expression reduces to a linear combination of vectors with integer entries, the coefficients of the combination being $\tilde{\nu}_1, \dots, \tilde{\nu}_{\tilde{d}}$. □

Acknowledgments. During the preparation of this work, we were supported by ANR-15-CE40-0001-02 "BEKAM" of the Agence Nationale de la Recherche. A. Maspero is also partially supported by PRIN 2015 "Variational methods, with applications to problems in mathematical physics and geometry".

References

[Arn89] Arnold, V.: Mathematical Methods of Classical Mechanics. Grad. Texts in Math. 60, Springer, New York (1989) Zbl 0692.70003 MR 1345386

[Bam17] Bambusi, D.: Reducibility of 1-d Schrödinger equation with time quasiperiodic unbounded perturbations, II. Comm. Math. Phys. **353**, 353–378 (2017) Zbl 1367.35149 MR 3638317

[Bam18] Bambusi, D.: Reducibility of 1-d Schrödinger equation with time quasiperiodic unbounded perturbations, I. Trans. Amer. Math. Soc. **370**, 1823–1865 (2018) Zbl 1386.35058 MR 3739193

[BBM14] Baldi, P., Berti, M., Montalto, R.: KAM for quasi-linear and fully nonlinear forced perturbations of Airy equation. Math. Ann. **359**, 471–536 (2014) Zbl 1350.37076 MR 3201904

[BGMR18] Bambusi, D., Grebert, B., Maspero, A., Robert, D.: Reducibility of the quantum harmonic oscillator in d-dimensions with polynomial time dependent perturbation. Anal. PDE **11**, 775–799 (2018) Zbl 1386.35059 MR 3738262

[BJ98] Barbaroux, J. M., Joye, A.: Expectation values of observables in time-dependent quantum mechanics. J. Statist. Phys. **90**, 1225–1249 (1998) Zbl 0921.47059 MR 1628328

[Bou99a] Bourgain, J.: Growth of Sobolev norms in linear Schrödinger equations with quasi-periodic potential. Comm. Math. Phys. **204**, 207–247 (1999) Zbl 0938.35150 MR 1705671

[Bou99b] Bourgain, J.: Growth of Sobolev norms in linear Schrödinger equations with smooth time dependent potentials. J. Anal. Math. **77**, 315–348 (1999) Zbl 0938.35026 MR 1753490

[BJLPN] Bunimovich, L., Jauslin, H. R., Lebowitz, J. L, Pellegrinotti, A., Nielaba, P.: Diffusive energy growth in classical and quantum driven oscillators. J. Statist. Phys. **62**, 793–817 (1991) Zbl 0741.70017 MR 1105283

[CdV79] Colin de Verdière, Y.: Sur le spectre des opérateurs elliptiques à bicaractéristiques toutes périodiques. Comment. Math. Helv. **54**, 508–522 (1979) Zbl 0459.58014 MR 0543346

[Del10] Delort, J. M.: Growth of Sobolev norms of solutions of linear Schrödinger equations on some compact manifolds. Int. Math. Res. Notices **2010**, 2305–2328 Zbl 1229.35040 MR 2652223

[Del14] Delort, J. M.: Growth of Sobolev norms for solutions of time dependent Schrödinger operators with harmonic oscillator potential. Comm. Partial Differential Equations **39**, 1–33 (2014) Zbl 1291.35254 MR 3169778

[DG97] Dereziński, J., Gérard, C. Scattering Theory of Classical and Quantum N-particle Systems. Springer, Berlin (1997) Zbl 0899.47007 MR 1459161

[DLS08] Duclos, P., Lev, O., Šťovíček, P.: On the energy growth of some periodically driven quantum systems with shrinking gaps in the spectrum. J. Statist. Phys. **130**, 169–193 (2008) Zbl 1134.81023 MR 2375960

[EK09] Eliasson, H. L., Kuksin, S. B.: On reducibility of Schrödinger equations with quasiperiodic in time potentials. Comm. Math. Phys. **286**, 125–135 (2009) Zbl 1176.35141 MR 2470926

[FZ12] Fang, D., Zhang, Q.: On growth of Sobolev norms in linear Schrödinger equations with time dependent Gevrey potential. J. Dynam. Differential Equations **24**, 151–180 (2012) Zbl 1247.35118 MR 2915755

[Gio] Giorgilli, A.: Lecture notes on Hamiltonian systems and perturbations. http://www.mat.unimi.it/users/antonio/hamsys/hamsys.html

[GP19] Grébert, B., Paturel, E.: On reducibility of quantum harmonic oscillator on $\mathbb{R}^d$ with quasiperiodic in time potential. Ann. Fac. Sci. Toulouse Math. (6) **28**, 977–1014 (2019) Zbl 1439.81039 MR 4093983

[Gui85] Guillemin, V.: A new proof of Weyl's formula on the asymptotic distribution of eigenvalues. Adv. Math. **55**, 131–160 (1985) Zbl 0559.58025 MR 0772612

[HR82a] Helffer, B., Robert, D.: Asymptotique des niveaux d'énergie pour des hamiltoniens à un degré de liberté. Duke Math. J. **49**, 853–868 (1982) Zbl 0519.35063 MR 0683006

[HR82b] Helffer, B., Robert, D.: Propriétés asymptotiques du spectre d'opérateurs pseudodifférentiels sur $\mathbf{R}^n$. Comm. Partial Differential Equations **7**, 795–882 (1982) Zbl 0501.35081 MR 0662451

[Hör85] Hörmander, L.: The Analysis of Linear Partial Differential Operators. III. Grundlehren Math. Wiss. 274, Springer, Berlin (1985) Zbl 0601.35001 MR 0781536

[How89] Howland, J.: Floquet operators with singular spectrum. I, II. Ann. Inst. H. Poincaré Phys. Théor. **50**, 309–323, 325–334 (1989) Zbl 0689.34022 MR 1017967

[How92] Howland, J.: Stability of quantum oscillators. J. Phys. A **25**, 5177–5181 (1992) Zbl 0764.58039 MR 1184303

[IPT05] Iooss, G., Plotnikov, P. I., Toland, J. F.: Standing waves on an infinitely deep perfect fluid under gravity. Arch. Ration. Mech. Anal. **177**, 367–478 (2005) Zbl 1176.76017 MR 2187619

[Joy94] Joye, A.: Absence of absolutely continuous spectrum of Floquet operators. J. Statist. Phys. **75**, 929–952 (1994) Zbl 0835.47010 MR 1285294

[Joy96] Joye, A.: Upper bounds for the energy expectation in time-dependent quantum mechanics. J. Statist. Phys. **85**, 575–606 (1996) Zbl 0952.47500 MR 1418806

[Mon18] Montalto, R.: On the growth of Sobolev norms for a class of linear Schrödinger equations on the torus with superlinear dispersion. Asymptotic Anal. **108**, 85–114 (2018) Zbl 1398.35190 MR 3802067

[MR17] Maspero, A., Robert, D.: On time dependent Schrödinger equations: Global well-posedness and growth of Sobolev norms. J. Funct. Anal. **273**, 721–781 (2017) Zbl 1366.35153 MR 3648866

[Nen97] Nenciu, G.: Adiabatic theory: stability of systems with increasing gaps. Ann. Inst. H. Poincaré Phys. Théor. **67**, 411–424 (1997) Zbl 0884.34088 MR 1632240

[PT01] Plotnikov, P. I., Toland, J. F.: Nash–Moser theory for standing water waves. Arch. Ration. Mech. Anal. **159**, 1–83 (2001) Zbl 1033.76005 MR 1854060

[Rob87] Robert, D.: Autour de l'approximation semi-classique. Progr. Math. 68, Birkhäuser (1987) Zbl 0621.35001 MR 0897108

[Sie89] Siegel, C.: Lectures on the Geometry of Numbers. Springer, Berlin (1989) Zbl 0691.10021 MR 1020761

[Tay91] Taylor, M.: Pseudodifferential Operators and Nonlinear PDE. Progr. Math. 100, Birkhäuser (1991) Zbl 0746.35062 MR 1121019

[Wan08] Wang, W. M.: Logarithmic bounds on Sobolev norms for time dependent linear Schrödinger equations. Comm. Partial Differential Equations **33**, 2164–2179 (2008) Zbl 1154.35450 MR 2475334