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On polynomially integrable Birkhoff billiards on surfaces of constant curvature

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Abstract. The polynomial version of the Birkhoff Conjecture on integrable billiards on complete simply connected surfaces of constant curvature (plane, sphere, hyperbolic plane) was first stated, studied and solved in a particular case by Sergei Bolotin in 1990–1992. Here we present a complete solution of the polynomial version of the Birkhoff Conjecture. Namely, we show that *every polynomially integrable real bounded planar billiard with C^2-smooth connected boundary is an ellipse.* We extend this result to billiards with piecewise smooth and not necessarily convex boundary on an arbitrary two-dimensional simply connected complete surface of constant curvature: plane, sphere, Lobachevsky–Poincaré (hyperbolic) plane; each of them being modeled as a plane or a (pseudo-) sphere in $\mathbb{R}^3$ equipped with an appropriate quadratic form. Namely, we show that *a billiard is polynomially integrable if and only if its boundary is a union of confocal conical arcs and appropriate geodesic segments.* We also present a complexification of these results. These are joint results of Mikhail Bialy, Andrey Mironov and the author. The proof is split into two parts. The first part is given in two papers by Bialy and Mironov (in Euclidean and non-Euclidean cases respectively). Their geometric construction reduced the Polynomial Birkhoff Conjecture to a purely algebro-geometric problem to show that an irreducible algebraic curve in $\mathbb{CP}^2$ with certain properties is a conic. They have shown that its singular and inflection points lie in the complex light conic of the above-mentioned quadratic form. In the present paper we solve the above algebro-geometric problem completely.

Keywords. Billiard, geodesic billiard flow, polynomial integral, algebraic Birkhoff conjecture, surface of constant curvature, singularities of algebraic curves

Contents

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1. Introduction

1.1. Main results

The famous Birkhoff Conjecture deals with strictly convex bounded planar billiards with smooth boundary. Recall that a *caustic* of a planar billiard $\Omega \subset \mathbb{R}^2$ is a curve C such that each tangent line to C reflects from the boundary of the billiard to a line tangent to C. A billiard Ω is called *Birkhoff caustic-integrable* if a neighborhood of its boundary in Ω is foliated by closed caustics, and the boundary $\partial\Omega$ is a leaf of this foliation. It is well-known that each elliptic billiard is integrable [40, Section 4]. The **Birkhoff Conjecture** states the converse: *the only Birkhoff caustic-integrable convex bounded planar billiard with smooth boundary is an ellipse.*[1]

Let now Σ be a two-dimensional surface with a Riemannian metric, and $\Omega \subset \Sigma$ a connected domain[2] with piecewise smooth boundary. The *billiard flow* B_t acts on the tangent bundle $T\Sigma|_\Omega$ as follows. A point $(Q, P) \in T\Sigma|_\Omega$, $Q \in \Omega$, $P \in T_Q\Sigma$, moves along a trajectory of the geodesic flow of the surface Σ until Q hits the boundary $\partial\Omega$. When hitting the boundary, a point Q remains unchanged, and the velocity vector P is reflected from the boundary to a vector P^* according to the usual reflection law: the angle of incidence equals the angle of reflection; and $|P| = |P^*|$. Afterwards the new

[1] This conjecture, classically attributed to G. Birkhoff, was published in print only in the paper [37] by H. Poritsky, who worked with Birkhoff as a post-doctoral fellow in the late 1920s.

[2] Everywhere in the paper a billiard is a **connected domain** $\Omega \subset \Sigma$.

point (Q, P^*) again moves along a trajectory of the geodesic flow etc. The billiard flow thus defined, which can be viewed as a geodesic flow with impacts on $T\Sigma|_{\Omega}$, has an obvious first integral: the absolute value $|P|$ of the velocity. A strictly convex billiard Ω with smooth boundary is called *integrable in the Liouville sense* if its flow has an additional first integral functionally independent with $|P|$ on the intersection with $T\Sigma|_{\overline{\Omega}}$ of a neighborhood of the unit tangent bundle to the boundary.

The notions of a caustic and of Birkhoff caustic-integrability extend to the case of a strictly convex domain Ω on an arbitrary surface Σ equipped with a Riemannian metric, with lines replaced by geodesics. *Liouville integrability and Birkhoff caustic-integrability are equivalent:* it is a well-known folklore fact.

There is an analogue of the Birkhoff Conjecture for billiards on a simply connected complete Riemannian surface of non-zero constant curvature, sphere or hyperbolic (Lobachevsky–Poincaré) plane. This is also an open problem.

The particular case of the Birkhoff Conjecture, when the additional first integral is supposed to be polynomial in the velocity components, motivates the next definition and conjecture.

Definition 1.1. Let Σ be a two-dimensional surface with Riemannian metric, and let $\Omega \subset \Sigma$ be a domain with piecewise smooth boundary. We say that the billiard in Ω is *polynomially integrable* if its flow has a first integral on $T\Sigma|_{\Omega}$ that is polynomial in the velocity P and whose restriction to the hypersurface $\{|P| = 1\}$ is non-constant.

Definition 1.2. Let Σ be as above, and let $\Omega \subset \Sigma$ be a domain with piecewise smooth boundary. We say that Ω is *analytically integrable* if there exists a first integral analytic in P on a neighborhood in $T\Sigma|_{\Omega}$ of the zero section of the tangent bundle $T\Sigma|_{\Omega}$ that is not a function of just the modulus $|P|$. In addition, it is required that there exists an $r > 0$ such that the integral is defined for all (Q, P) with $Q \in \Omega$ and $|P| \leq r$ and its Taylor series in P converges uniformly in those (Q, P).

Note that all the integrals in question, which are defined over an open domain Ω, should be invariant under the geodesic flow in Ω and under the reflections from its boundary.

Remark 1.3. The following facts are well-known:

- Analytic integrability implies polynomial integrability, since each homogeneous part in P of an analytic integral is a first integral itself [32, p. 107] (the converse is obvious).
- When Σ is a simply connected *complete surface of constant curvature* and the boundary $\partial\Omega$ is smooth and connected, polynomial integrability is equivalent to the existence of a polynomial integral as above in a neighborhood of the unit tangent bundle to $\partial\Omega$ in $T\Sigma|_{\Omega}$, by S. V. Bolotin's results [15, 16, 17]; see Theorem 1.22 below. In this case each first integral that is just polynomial in P is *globally analytic* on $T\Sigma$; see [17, proof of Proposition 2] and Theorem 1.22.

The **Polynomial Birkhoff Conjecture** states that *if a convex planar billiard with smooth boundary is polynomially integrable, then its boundary is a conic.* The Polynomial Birkhoff Conjecture together with its generalization to billiards with piecewise smooth (maybe non-convex) boundaries on simply connected complete surfaces of constant curvature was first stated and studied by S. V. Bolotin [16, 17] and later studied by M. Bialy and A. E. Mironov [10, 11, 12]. In the present paper we give a complete solution of the Polynomial Birkhoff Conjecture in full generality (Theorems 1.6 and 1.21).

Remark 1.4. The Polynomial Birkhoff Conjecture and its generalization are important and interesting in themselves, *independently of a potential solution of the classical Birkhoff Conjecture.* They lie at the crossroads of different domains of mathematics, first of all, dynamical systems, algebraic geometry and singularity theory. *They are not implied by the classical Birkhoff Conjecture.* For the general case of piecewise smooth boundaries this is obvious. Even in the case of a smooth convex boundary, where polynomiality is a very strong restriction, the condition of just non-constance of a polynomial integral on the unit velocity hypersurface is topologically weaker than the independence condition in the Liouville integrability, which requires independence of the additional integral and the energy on a whole neighborhood in $T\mathbb{R}^2|_{\overline{\Omega}}$ of the unit tangent bundle to the boundary.

Without loss of generality we consider simply connected complete surfaces Σ of constant curvature 0 or ± 1: one can make non-zero constant curvature equal to ± 1 by multiplying the metric by a constant factor; this changes neither geodesics nor polynomial integrability. Thus, Σ is either the Euclidean plane, or the unit sphere, or the Lobachevsky–Poincaré hyperbolic plane. It is modeled as one of the following three surfaces in the space $\mathbb{R}^3$ with coordinates $x = (x_1, x_2, x_3)$ equipped with the quadratic form

$$\langle Ax, x\rangle, \quad A \in \{\mathrm{diag}(1, 1, 0), \mathrm{diag}(1, 1, \pm 1)\}, \quad \langle x, x\rangle = x_1^2 + x_2^2 + x_3^2.$$

- Euclidean plane: $\Sigma = \{x_3 = 1\}$, $A = \mathrm{diag}(1, 1, 0)$.
- The unit sphere: $\Sigma = \{x_1^2 + x_2^2 + x_3^2 = 1\}$, $A = \mathrm{Id}$.
- The hyperbolic plane: $\Sigma = \{x_1^2 + x_2^2 - x_3^2 = -1\} \cap \{x_3 > 0\}$, $A = \mathrm{diag}(1, 1, -1)$.

The metric of constant curvature on the surface Σ in question is induced by the quadratic form $\langle Ax, x\rangle$. The *geodesics* on Σ are its intersections with two-dimensional vector subspaces in $\mathbb{R}^3$. The *conics* on Σ are its intersections with quadrics $\{\langle Cx, x\rangle = 0\} \subset \mathbb{R}^3$, where C is a real symmetric 3×3-matrix.

Example 1.5. The billiard in a disk in $\mathbb{R}^2_{(x_1,x_2)}$ centered at 0 has first integral $x_1 P_2 - P_1 x_2$ linear in P. The billiard in any conic in any of the above surfaces Σ has an integral quadratic in P [17, Proposition 1].

Theorem 1.6. *Suppose a billiard Ω in Σ with C^2-smooth connected boundary is polynomially integrable. Assume that the boundary of Ω is not contained in a geodesic. Then $\partial\Omega$ is a conic (or a connected component of a conic).*

Corollary 1.7. *Every bounded polynomially integrable planar billiard with C^2-smooth connected boundary is an ellipse.*

Below we extend the above theorem to billiards with countably piecewise smooth boundaries:

Definition 1.8. A domain $\Omega \subset \Sigma$ has *countably piecewise (C^r-) smooth boundary* if $\partial\Omega$ consists of the following two parts:

- the *regular part:* an open and dense subset $\partial\Omega_{\text{reg}} \subset \partial\Omega$, where each point $X \in \partial\Omega_{\text{reg}}$ has a neighborhood $U = U(X) \subset \Sigma$ such that $U \cap \partial\Omega$ is a (C^r-) smooth one-dimensional submanifold in U;
- the *singular part:* the closed subset $\partial\Omega_{\text{sing}} = \partial\Omega \setminus \partial\Omega_{\text{reg}} \subset \partial\Omega$.

Remark 1.9. In the above definition the regular part of the boundary is always a dense and at most countable disjoint union of (C^2-) smooth arcs (taken without endpoints). The particular case of domains with piecewise smooth boundaries corresponds to the case when the above union is finite, the arcs are smooth up to their endpoints and the singular part of the boundary is a finite set (which consists of endpoints and may be empty). For a general billiard with countably piecewise smooth boundary the billiard flow is well-defined on a residual set for all time values. When the singular part of the boundary has zero one-dimensional Hausdorff measure, the billiard flow is well-defined as a flow of measurable transformations.

Remark 1.10. The notions of polynomially (analytically) integrable billiards obviously extend to billiards with countably piecewise smooth boundaries, and these two notions are equivalent, as in the piecewise smooth case.

Definition 1.11. A billiard in $\mathbb{R}^2$ with countably piecewise smooth boundary is called *countably confocal* if the regular part of its boundary consists of arcs of confocal conics and may be some straight-line segments such that

- at least one conical arc is present;
- when the common foci of the conics are distinct and finite (i.e., the conics are ellipses and (or) hyperbolas), the ambient line of each straight-line segment of the boundary is either the line through the foci, or the middle orthogonal line to the segment connecting the foci (see Fig. 1(a));
- when the conics are concentric circles, the above ambient lines may be any lines through their common center (see Fig. 1(b));
- when the conics are confocal parabolas, the ambient line of each straight-line segment of the boundary is either the common axis of the parabolas, or the line through the focus that is orthogonal to the axis (Fig. 1(c, d).

Let us extend the above definition to the non-Euclidean case. To do this, let us recall the following definition.

Definition 1.12 ([46, p. 84]). Let $\Sigma \subset \mathbb{R}^3$ be one of the standard surfaces of constant curvature defined by a quadratic form $\langle Ax, x\rangle$. Let B be a real symmetric 3×3-matrix that is not proportional to A. In the Euclidean case, when $A = \operatorname{diag}(1, 1, 0)$, we require in addition that the x_3-axis does not lie in $\operatorname{Ker} B$. The *pencil of confocal conics* in Σ defined by B consists of the conics

$$\Gamma_\lambda = \Sigma \cap \{\langle B_\lambda x, x\rangle = 0\}, \qquad B_\lambda = (B - \lambda A)^{-1}. \tag{1.1}$$

For those λ for which $\det(B - \lambda A) = 0$ and the kernel $K_\lambda = \mathrm{Ker}(B - \lambda A)$ is one-dimensional, we set Γ_λ to be the geodesic[3]

$$\Gamma_\lambda = \Sigma \cap K_\lambda^\perp, \tag{1.2}$$

provided that the intersection is non-empty. When $\dim K_\lambda = 2$, for every two-dimensional vector subspace $H \subset \mathbb{R}^3$ orthogonal to K_λ the intersection $\Sigma \cap H$ will also be denoted $\Gamma_\lambda = \Gamma_\lambda(H)$.

Remark 1.13. In the setting of Definition 1.12 the confocal conic pencil is well-defined: $\det(B - \lambda A) \not\equiv 0$ as a function of λ. In the non-Euclidean cases this is obvious, since the matrix A is non-degenerate. In the Euclidean case one has $A = \mathrm{diag}(1, 1, 0)$ and the x_3-axis does not lie in $\mathrm{Ker}\, B$, that is, some of the matrix elements B_{13}, B_{23}, B_{33} is non-zero. One has

$$\begin{aligned}\det(B - \lambda A) &= -\lambda^3 \det(A - \lambda^{-1} B) \\ &= \lambda^2 B_{33} + \lambda(B_{13}^2 + B_{23}^2 - B_{33}(B_{11} + B_{22})) + \det B \not\equiv 0;\end{aligned} \tag{1.3}$$

in the above right-hand side the identical vanishing of the coefficients at λ^2 and at λ would imply that $B_{33} = B_{13} = B_{23} = 0$, which is forbidden by our assumptions. Hence, $\det(B - \lambda A) \not\equiv 0$. Conversely, if in the Euclidean case the x_3-axis were contained in the kernel of the matrix B, then obviously $\det(B - \lambda A) \equiv 0$, and the confocal pencil would not be well-defined.

Remark 1.14. The matrix B is uniquely defined modulo the transformation $B \mapsto \mu B - \lambda A$, $\mu \neq 0$ (i.e., modulo $\mathbb{R}A$ and up to a constant factor) by the corresponding confocal conic pencil. In the Euclidean case, when $\Sigma = \{x_3 = 1\}$, $A = (1, 1, 0)$, the above notion of confocal conics coincides with the classical one. In the Euclidean case the kernel K_λ is two-dimensional for some λ if and only if the confocal conics in question are concentric circles; then the corresponding geodesics $\Gamma_\lambda(H)$ are the lines through their common center.

Definition 1.15. Consider a confocal conic pencil (1.1) defined by a matrix B. The corresponding *admissible geodesics* are the following:

(1) Each geodesic Γ_λ in (1.2) and (or) $\Gamma_\lambda(H)$ (if any) is admissible.

(2) Consider the special case, when $B = Aa \otimes b + b \otimes Aa$ (modulo $\mathbb{R}A$, see Remark 1.14) where $a, b \in \mathbb{R}^3 \setminus \{0\}$, $\langle a, b\rangle = 0$.

(2a) When Σ is non-Euclidean, those of the geodesics

$$\{r \in \Sigma \mid \langle r, a\rangle = 0\}, \qquad \{r \in \Sigma \mid \langle r, Ab\rangle = 0\} \tag{1.4}$$

that are well-defined (i.e., non-empty) are also admissible.

(2b) When $\Sigma = \{x_3 = 1\}$ is Euclidean and b is not parallel to Σ, only Γ_λ and the first geodesic in (1.4) are admissible.

[3] Everywhere below, unless otherwise specified, the orthogonal complement sign $\perp$ and the vector product are understood with respect to the standard Euclidean scalar product on $\mathbb{R}^3$.

Remark 1.16. Note that in (2) the subcase when $\Sigma = \{x_3 = 1\}$ and b is parallel to Σ is impossible, since the x_3-axis would then lie in the kernel Ker B, which is forbidden by our assumptions. This implies that in subcase (2b) the first geodesic in (1.4) is non-empty: the vector a is not vertical, since its orthogonal b is not horizontal. In subcases (2a), (2b) the corresponding admissible geodesics from (1.4) do not coincide with the geodesics Γ_λ or $\Gamma_\lambda(H)$. Indeed, suppose the contrary, say, the first geodesic $a^\perp \cap \Sigma$ in (1.4) is non-empty and coincides with some Γ_λ or $\Gamma_\lambda(H)$. Then $a \in \operatorname{Ker}(B - \lambda A)$, that is,

$$\langle Aa, a\rangle b + \langle b, a\rangle Aa = \langle Aa, a\rangle b = \lambda Aa.$$

Thus, either $\langle Aa, a\rangle = 0$ and $\lambda Aa = 0$, or the vector b, which is orthogonal to a, is proportional to Aa, thus $\langle Aa, a\rangle = 0$ again. But then $a^\perp \cap \Sigma = \emptyset$ [17, p. 122], a contradiction. The case when the second geodesic in (1.4) coincides with Γ_λ is treated analogously. The above non-coincidence statement can also be deduced from the next proposition.

Remark 1.17. In subcase (2a) set $\widetilde{a} = Ab$ and $\widetilde{b} = Aa$. Then $B = A\widetilde{a} \otimes \widetilde{b} + \widetilde{b} \otimes A\widetilde{a}$, and $\langle \widetilde{a}, \widetilde{b}\rangle = 0$, since $A^2 = \mathrm{Id}$. The geodesics in (1.4) are written in terms of the new vectors $\widetilde{a}$ and $\widetilde{b}$ in the opposite order. Thus, each geodesic of type (1.4) can be represented by the first equation in (1.4) for an appropriate presentation $B = Aa \otimes b + b \otimes Aa$.

Definition 1.18. A billiard $\Omega \subset \Sigma$ with a countably piecewise smooth boundary is *countably confocal* if its boundary consists of arcs of confocal conics (at least one conical arc is present) and maybe some segments of geodesics admissible with respect to the confocal conic pencil given by the conical arcs in $\partial\Omega$ (see Definition 1.15).

Confocal billiards with piecewise smooth boundaries were introduced by S. V. Bolotin [17], who proved their polynomial integrability with integrals of first, second or fourth degree. See the following proposition, whose proof presented in loc. cit. remains valid in the countably piecewise smooth case.

Proposition 1.19 ([17, Proposition 1 in Section 2 and Theorem in Section 4]). *Each countably confocal billiard is polynomially integrable: it has a non-trivial first integral that is either linear, or quadratic, or a degree* 4 *polynomial in the velocity components that is non-constant on the unit velocity hypersurface. If all the geodesic pieces of its boundary lie in* Γ_λ *or* $\Gamma_\lambda(H)$*, then the integral can be chosen of degree at most* 2*. The case of a degree* 4 *integral that cannot be reduced to an integral of degree at most* 2 *is exactly the case when the conics forming the billiard boundary are contained in a confocal pencil of types* (2a) *or* (2b) *from Definition* 1.15 *and the billiard boundary contains at least one segment of some of the admissible geodesics from* (1.4) *mentioned in* (2a) *and* (2b) *respectively.*

Example 1.20. For Euclidean billiards the two countable confocality notions given by Definitions 1.11 and 1.18 are equivalent. A Euclidean billiard whose boundary contains an arc of parabola and a segment of the line through the focus that is orthogonal to the axis of the parabola, as in Fig. 1(d), is exactly a billiard of type (2b) (see the end of [17]);

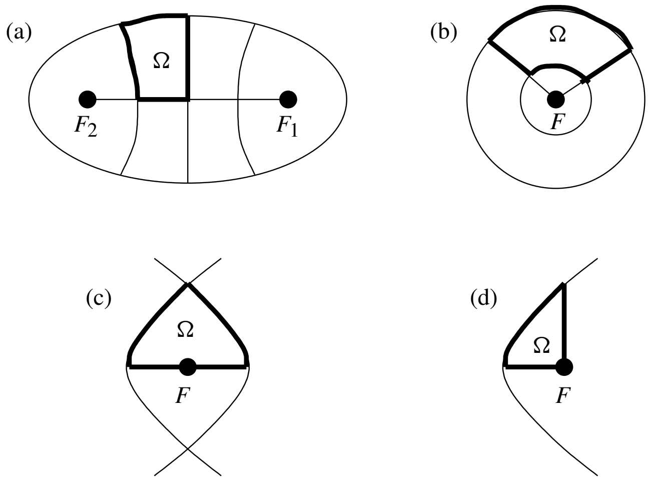

Fig. 1. Examples of confocal planar billiards; F_1, F_2, F are the foci; the conics in (c) and (d) are parabolas. All of these billiards have quadratic integrals, except for the billiard in Fig. 1(d), which has a degree 4 integral.

the above line is the first geodesic in (1.4). This example of a billiard having a degree 4 integral was first discovered in [38]. Analogous billiards on surfaces of non-zero constant curvature were constructed in [2].

The main result of the paper is the following theorem.

Theorem 1.21.[4] *Suppose a billiard in Σ with countably piecewise C^2-smooth boundary is polynomially integrable (or equivalently analytically integrable, see Definition 1.2), and suppose the regular part of its boundary contains at least one non-geodesic arc. Then the billiard is countably confocal.*

Theorem 1.21 is a joint result of M. Bialy, A. E. Mironov and the author. Its proof sketched below consists of the following two parts:

- The papers [10, 11] of Bialy and Mironov, whose geometric construction reduced the proof of Theorem 1.21 to a purely algebro-geometric problem that was partially investigated by them.
- The complete solution of the above-mentioned algebro-geometric problem obtained in the present paper (Theorem 1.25).

1.2. Sketch of proof of Theorem 1.21 and plan of the paper

In what follows, a point $r \in \Sigma$ will be identified with its radius-vector in $\mathbb{R}^3$.

[4] Theorem 1.21 with a brief proof was announced in the author's note [26].

Theorem 1.22 (S. V. Bolotin, see [16], [17, p. 118; Proposition 2 and its proof on p. 119], [33, Chapter 5, Section 3, Proposition 5]). *For every polynomially integrable billiard $\Omega \subset \Sigma$ with countably piecewise C^2-smooth boundary, there is a polynomial integral which is non-constant on the unit velocity hypersurface $\{|P| = 1\}$ and is a homogeneous polynomial $\Psi(M)$ of even degree in the components of the moment vector*

$$M = [r, P] = (x_2 P_3 - x_3 P_2, -x_1 P_3 + x_3 P_1, x_1 P_2 - x_2 P_1),$$
$$r = (x_1, x_2, x_3) \in \Sigma, \qquad P = (P_1, P_2, P_3) \text{ is the velocity vector.} \tag{1.5}$$

(*This statement is local and holds for reflection from an arbitrary smooth curve in Σ.*) *Each C^2-smooth arc of the boundary $\partial\Omega$ with non-zero geodesic curvature lies on an algebraic curve.*

Theorem 1.23 (see [17, Section 4]). *Suppose a billiard on Σ with a countably piecewise C^2-smooth boundary is polynomially integrable, and the boundary contains a non-geodesic* ***conical*** *arc. Then the billiard is countably confocal.*

Remark 1.24. S. V. Bolotin's theorems implying Theorems 1.22 and 1.23 were stated and proved in loc. cit. for piecewise smooth boundaries, but their proofs remain valid in the countably piecewise smooth case. To make the paper self-contained and to extend the main results to the complex domain, we give a proof of Theorem 1.23 in Subsection 2.2. It follows the arguments from [17, Section 4], but here it is done in dual terms using the results of Bialy and Mironov [10, 11].

The boundary $\partial\Omega$ is countably piecewise C^2-smooth. Therefore, it contains an open and dense subset contained in $\partial\Omega_{\mathrm{reg}}$ that is a disjoint union of geodesic segments and C^2-smooth arcs with non-zero geodesic curvature.

Let $\alpha \subset \partial\Omega$ be a C^2-smooth arc with non-zero geodesic curvature; its existence follows from the assumptions. By Bolotin's Theorem 1.23, for the proof of Theorem 1.21 it suffices to show that α contains a conical subarc. To do this, we use Bialy–Mironov's construction of the dual billiard and their results presented in Subsection 2.1. Let us describe them briefly.

In what follows, $\pi : \mathbb{R}^3 \setminus \{0\} \to \mathbb{RP}^2$ denotes the tautological projection. Its complexification and restriction to Σ will also be denoted by π.

Recall that the standard Euclidean scalar product $\langle x, x\rangle$ on $\mathbb{R}^3$ defines the *orthogonal polarity:* the correspondence sending each two-dimensional vector subspace in $\mathbb{R}^3$ to its Euclidean-orthogonal one-dimensional subspace. This together with the projection π induces a projective duality $\mathbb{RP}^{2*}_{(x_1:x_2:x_3)} \to \mathbb{RP}^2_{(M_1:M_2:M_3)}$ sending lines to points. Namely, each projective line, which is the projection of a two-dimensional vector subspace H (punctured at the origin), is dual to the point $\pi(H^\perp \setminus \{0\})$. (It is well-known that in the affine chart $(x_1 : x_2 : 1)$ the projective duality defined by the orthogonal polarity is the composition of the polar duality with respect to the unit circle and the central symmetry with respect to the origin.)

For simplicity, the curve dual to the projection $\pi(\alpha) \subset \mathbb{RP}^2$ with respect to the above projective duality will be denoted by α^* and called the *curve Σ-dual to α*. By definition, the dual curve α^* is the family of points in $\mathbb{RP}^2$ that are dual to the projective tangent lines to the curve $\pi(\alpha) \subset \mathbb{RP}^2$. The curve α^* is C^1-smooth, since the curve $\pi(\alpha)$ is C^2-smooth and has no inflection points: the geodesic curvature of α is non-zero.

Bialy and Mironov proved the following results in [10, 11]:

- Let $\Psi(M)$ be the homogeneous first integral of even degree $2n$ from Bolotin's Theorem 1.22. For every point $B \in \alpha^*$ the restriction to the projective tangent line $T_B\alpha^*$ of the rational function
$$G(M) = \frac{\Psi(M)}{\langle AM, M\rangle^n} \tag{1.6}$$
is invariant under a special projective involution $T_B\alpha^* \to T_B\alpha^*$ fixing B, the so-called angular symmetry centered at B. More precisely, invariance of G is equivalent to the statement that for every $r \in \alpha$ the function $\Psi(M) = \Psi([r, v])$ of $v \in T_r\Sigma$ is invariant under reflection of the vector v from the line $T_r\alpha$.
- Consider the so-called *absolute*, the complex conic
$$\mathbb{I} = \{\langle AM, M\rangle = 0\} \subset \mathbb{CP}^2_{(M_1:M_2:M_3)}. \tag{1.7}$$
The above angular symmetry coincides with the restriction to $T_B\alpha^*$ of the unique projective involution $\mathbb{CP}^2 \to \mathbb{CP}^2$ fixing B that fixes each line through B and permutes its intersection points with the conic $\mathbb{I}$, the so-called $\mathbb{I}$-*angular symmetry* centered at B.
- Consider the complex projective Zariski closure of the curve α^*, which is an algebraic curve, by Theorem 1.22. Each of its non-linear irreducible components, γ, *generates a rationally integrable* $\mathbb{I}$-*angular billiard* (see Definition 2.10): for every point $B \in \gamma \setminus \mathbb{I}$ the restriction of a rational function G to the projective tangent line $T_B\gamma$ is invariant under the $\mathbb{I}$-angular symmetry centered at B; the function G is non-constant on $\mathbb{CP}^2$ and has poles in $\mathbb{I}$.
- For every curve γ generating a rationally integrable $\mathbb{I}$-angular billiard all its singular and inflection points (if any) lie in $\mathbb{I}$.

The main algebro-geometric result of the present paper, which implies the main results, is the following theorem.

Theorem 1.25. *Let $\mathbb{I} \subset \mathbb{CP}^2$ be a conic* (*either regular, or a union of two distinct lines*). *Every irreducible algebraic curve $\gamma \subset \mathbb{CP}^2$ different from a line and from $\mathbb{I}$ and generating a rationally integrable $\mathbb{I}$-angular billiard is a conic.*

For the proof of Theorem 1.25 we study *local branches* of the curve γ at points $C \in \gamma \cap \mathbb{I}$, the irreducible components of the germ (γ, C). Each local branch is holomorphically bijectively parametrized in so-called adapted affine coordinates by a small complex parameter t as follows:
$$t \mapsto (t^q, ct^p(1 + o(1))) \text{ for } t \to 0; \quad q, p \in \mathbb{N},\ 1 \leq q < p,\ c \neq 0.$$

In Section 4 we prove Theorem 4.1 giving a list of statements on p and q satisfied by local branches of appropriate type (see Cases (1) and (2) below). In Section 5 we prove the following general algebro-geometric theorem. It states that the Bialy–Mironov inclusions $\mathrm{Sing}(\gamma), \mathrm{Infl}(\gamma) \subset \mathbb{I}$ and the statements of Theorem 4.1 on local branches together imply that γ is a conic.

Theorem 1.26. *Let $\mathbb{I} \subset \mathbb{CP}^2$ be a conic (either regular, or a union of two distinct lines). Suppose $\gamma \subset \mathbb{CP}^2$ is an irreducible complex algebraic curve different from a line and from $\mathbb{I}$, with all of its singularities and inflection points (if any) lying in $\mathbb{I}$. Assume that for every $C \in \gamma \cap \mathbb{I}$ the local branches b of γ at C satisfy the following statements:*

Case (1): *C is a regular point of the conic $\mathbb{I}$. If b is tangent to $\mathbb{I}$, then it is quadratic: $p = 2q$. If b is transversal to $\mathbb{I}$, then it is regular and quadratic: $q = 1$, $p = 2$.*

Case (2): *$\mathbb{I}$ is a union of two distinct lines intersecting at C. If b is transversal to both lines, then b is subquadratic: $p \leq 2q$.*

Then γ is a conic.

The proof of Theorem 1.26, which will be given in Section 5, is based on the ideas and arguments due to E. Shustin on plane curve invariants from the proof of its analogue for the outer billiards case [27, Subsections 4.1, 4.2].

The most technical part of the paper is the proof of statement (ii-b) of Theorem 4.1, which asserts that each local branch of the curve γ that is transversal to $\mathbb{I}$ and is based at a regular point of the conic $\mathbb{I}$ is regular and quadratic. Its proof is based on a remarkable formula of Bialy and Mironov for the Hessian of the function defining γ ([10, Theorem 6.1] and [11, formulas (16) and (32)]). This formula is recalled as formula (3.4) below. We use asymptotic formulas for both sides of the Bialy–Mironov formula along the transversal local branches, which are stated and proved in Subsection 3.4. In their proofs we use asymptotic formulas for the defining functions and their Hessians stated and proved in Subsections 3.2 and 3.3 respectively.

In Section 6 we prove the main results: Theorems 1.25, 1.21 and 1.6, and the complexification of Theorem 1.21 stated in the next subsection.

1.3. Complexification

Here we state a complexification of Theorem 1.21, which deals with the space $\mathbb{C}^3_{(x_1,x_2,x_3)}$ equipped with a quadratic form $\langle Ax, x\rangle$, $x = (x_1, x_2, x_3)$, $A \in \{\mathrm{diag}(1, 1, 0), \mathrm{diag}(1, 1, \pm 1)\}$, and a complex surface $\Sigma \subset \mathbb{C}^3$.

- Euclidean case: $\Sigma = \{x_3 = 1\}$, $A = \mathrm{diag}(1, 1, 0)$.
- Non-Euclidean case: $\Sigma = \Sigma_\pm = \{\langle Ax, x\rangle = \pm 1\}$, $A = \mathrm{diag}(1, 1, \pm 1)$.

We equip the surface Σ in question with the complex bilinear quadratic form induced by the form $\langle Ax, x\rangle$ on its tangent planes.

Note that the surfaces $\Sigma_\pm$ are regular, connected and obtained from each other by the transformation $(x_1, x_2, x_3) \mapsto (ix_1, ix_2, x_3)$, but this transformation changes the sign of the quadratic form $\langle Ax, x\rangle$ on $T\Sigma_\pm$.

Recall that a one-dimensional subspace Λ in a complex linear space equipped with a $\mathbb{C}$-bilinear scalar product is *isotropic* if each vector in Λ has zero scalar square. A holomorphic curve Λ in a complex manifold Σ equipped with a $\mathbb{C}$-bilinear quadratic form on $T\Sigma$ is *isotropic* if for every $x \in \Lambda$ the tangent subspace $T_x\Lambda \subset T_x\Sigma$ is isotropic.

A *complex geodesic* is

- a non-isotropic line in $\Sigma = \mathbb{C}^2$ in the Euclidean case;
- the intersection of the surface Σ with a two-dimensional subspace in $\mathbb{C}^3$ that is not tangent to the light cone $\widehat{\mathbb{I}} = \{\langle Ax, x\rangle = 0\}$ in the non-Euclidean case.

The reason to leave out the planes tangent to $\widehat{\mathbb{I}}$ is the following.

Proposition 1.27. *Consider the non-Euclidean case: $A = \operatorname{diag}(1, 1, \pm 1)$. For every two-dimensional vector subspace $H \subset \mathbb{C}^3$ tangent to the light cone $\widehat{\mathbb{I}}$ the intersection $H \cap \Sigma$ is a union of two parallel isotropic straight lines. Each isotropic holomorphic curve in Σ is a line contained in a two-dimensional vector subspace in $\mathbb{C}^3$ tangent to $\widehat{\mathbb{I}}$. For every $r \in \Sigma$ the one-dimensional isotropic vector subspaces in the plane $T_r\Sigma$ are exactly its intersections with two-dimensional vector subspaces in $\mathbb{C}^3$ containing r and tangent to $\widehat{\mathbb{I}}$; there are exactly two of them.*

Proof. For every $r \in \Sigma$ the quadratic form on $T_r\Sigma$ induced by $\langle Ax, x\rangle$ is non-degenerate, since $T_r\Sigma$ is $\langle Ax, x\rangle$-orthogonal to the radius-vector of the point r and transversal to it: $\langle Ar, r\rangle = \pm 1 \neq 0$. For every two-dimensional subspace H tangent to $\widehat{\mathbb{I}}$ the restriction of the form $\langle Ax, x\rangle$ to H is non-zero and has a non-zero kernel K, the tangency line of the plane H with $\widehat{\mathbb{I}}$. Hence, in appropriate affine coordinates (z_1, z_2) on H centered at 0 one has $\langle Ax, x\rangle|_H = z_1^2$, $K = \{z_1 = 0\}$, $H \cap \Sigma = \{z_1^2 = \pm 1\}$. Therefore, $H \cap \Sigma$ is a union of two lines parallel to K, which are thus isotropic. The first statement of the proposition is proved.

Let us now prove the third and the second statements. For every point $r \in \Sigma$ the tangent plane $T_r\Sigma$ equipped with the quadratic form induced by $\langle Ax, x\rangle$ contains two distinct one-dimensional isotropic vector subspaces, by non-degeneracy. Each ot them is the line of intersection of the plane $T_r\Sigma$ with a two-dimensional subspace H through r that is tangent to $\widehat{\mathbb{I}}$. This follows from the first statement of the proposition and the fact that there are two distinct two-dimensional subspaces through r that are tangent to $\widehat{\mathbb{I}}$. This implies the third statement of the proposition. This also implies that each isotropic curve in Σ is locally a phase curve of a (double-valued) holomorphic line field defined by the above intersections. The only phase curves of that field are the isotropic lines in $H \cap \Sigma$, H being tangent to $\widehat{\mathbb{I}}$, by the first statement of the proposition and the uniqueness theorem in ordinary differential equations. This proves the proposition. □

Definition 1.28. Consider the surface Σ in the non-Euclidean case. Let $\gamma \subset \Sigma$ be a complex geodesic. Let $\mathcal{G}_\gamma$ denote the stabilizer of the geodesic γ in the group of automorphisms $\mathbb{C}^3 \to \mathbb{C}^3$ preserving the form $\langle Ax, x\rangle$. Its identity component, which will be denoted by $\mathcal{G}_\gamma^0$, will be called the *group of translations along the geodesic* γ. A translation of the complex Euclidean plane along a complex line L is the translation by a vector parallel to L.

Remark 1.29. In the above definition in the non-Euclidean case let $H \subset \mathbb{C}^3$ denote the corresponding two-dimensional vector subspace; so $\gamma = H \cap \Sigma$. The geodesic γ

is thus a regular conic in the plane H that is biholomorphically parametrized by $\mathbb{C}^*$. Its projective closure $\hat{\gamma}$ in $\mathbb{CP}^2 \supset H$ intersects the infinity line $\mathbb{CP}^2 \setminus H$ in two distinct points. The restrictions to γ of translations along the geodesic γ are exactly those conformal automorphisms $\hat{\gamma} \to \hat{\gamma}$ that fix those intersection points. One has $\hat{\gamma} \simeq \overline{\mathbb{C}}$, $\gamma \simeq \mathbb{C}^*$. This leads to a natural isomorphism $\mathcal{G}_\gamma^0 \simeq \mathbb{C}^*$.

Definition 1.30. A *complex billiard* on Σ is a collection (finite or infinite, countable or uncountable) of holomorphic curves $\Gamma_t \subset \Sigma$ distinct from isotropic lines (see [25, Definition 1.3] for finite collections in the Euclidean case). A complex billiard is said to be *polynomially integrable* if there exists a function $\Phi(r, P)$ on $T\Sigma$ (called a *polynomial integral*) that is polynomial in $P \in T_r\Sigma$ with the following properties:

- $\Phi|_{\{\langle AP,P\rangle=1\}} \not\equiv \text{const}$;
- the restriction of Φ to the tangent bundle of every complex geodesic is invariant under translations along the geodesic;
- for every point $r \in \Gamma_t$ such that the line $T_r\Gamma_t$ is non-isotropic for the quadratic form on $T_r\Sigma$ induced by $\langle Ax, x\rangle$, the restriction $\Phi|_{T_r\Sigma}$ is invariant under the *symmetry* with respect to the complex line $T_r\Gamma_t$ (see [25, Definition 2.1]), the unique non-trivial $\mathbb{C}$-linear involution $T_r\Sigma \to T_r\Sigma$ that preserves the form $\langle Ax, x\rangle$ on $T_r\Sigma$ and fixes the points of the line $T_r\Gamma_t$.

Example 1.31. Consider a polynomially integrable billiard with countably piecewise smooth boundary in a real surface of constant curvature. Then the smooth part of the boundary is contained in a union of arcs of conics and segments of admissible geodesics (Theorem 1.21). Their complexifications form a complex billiard having a polynomial integral that is the complexification of the real polynomial integral of the real billiard; it can be chosen of degree no greater than four (see Proposition 1.19).

Remark 1.32. The confocality notion from Definition 1.12 for real conics extends to the case of complex conics in Σ without changes in both the non-Euclidean and Euclidean cases with B being a complex symmetric matrix and $\lambda \in \mathbb{C}$. In the Euclidean case this complex confocality notion is equivalent to the one given in [25, Definition 2.24], which follows from the definition and Remark 1.14.

Remark 1.33. A pencil of confocal conics given by a matrix B is well-defined if and only if (1.3) holds: $\det(B - \lambda A) \not\equiv 0$ as a function of λ. In the real case, (1.3) is equivalent to the condition that the x_3-axis is not contained in the kernel of the matrix B, i.e., $(B_{13}, B_{23}, B_{33}) \neq (0, 0, 0)$ (see Remark 1.13). In the complex case, (1.3) is equivalent to the following stronger condition: for every choice of $\pm$ the equalities

$$B_{33} = 0, \qquad B_{23} = \pm i B_{13}, \qquad B_{13}^2(B_{11} - B_{22} \pm 2i B_{12}) = 0$$

do not hold simultaneously.

Definition 1.34. A complex billiard Γ_t is said to be *confocal* if the set of its curves different from complex geodesics is non-empty, all of them are confocal complex conics, and the complex geodesics from the family Γ_t are admissible with respect to the corresponding confocal conic pencil in the sense of Definition 1.15, where now everything is complex: B, λ, a, b, etc.

Remark 1.35. A priori, some complex curves Γ_λ in (1.2), $\Gamma_\lambda(H)$ and some subsets in (1.4) may be isotropic lines; then they are not complex geodesics, and we do not call them admissible. For example, in the non-Euclidean case let $\lambda \in \mathbb{C}$ be such that the kernel $K_\lambda = \operatorname{Ker}(B - \lambda A)$ is one-dimensional. The corresponding intersection $\Gamma_\lambda = K_\lambda^\perp \cap \Sigma$ is isotropic if and only if $K_\lambda \subset \widehat{\mathbb{I}}$. This follows from Proposition 1.27 and since the Euclidean orthogonal $K_\lambda^\perp$ is tangent to the light cone $\widehat{\mathbb{I}}$ if and only if $K_\lambda \subset \widehat{\mathbb{I}}$ (see the last statement of Corollary 2.15).

Theorem 1.36. *Every polynomially integrable complex billiard Γ_t on Σ containing at least one non-geodesic curve is confocal and has an integral $\Phi(r, P) = \Psi(M)$ (where $M = [r, P]$ is the complexified Euclidean vector product) that is a homogeneous polynomial in M of degree at most* 4. *The integral can be chosen quadratic in M, except for the cases* (2a), (2b) *in Definition* 1.15, *when Γ_t contains a corresponding admissible complex geodesic of type* (1.4)*; in those cases there is an integral of degree* 4.

Theorem 1.36 will be proved in Subsection 6.4.

1.4. Historical remarks

Existence of caustics in any strictly convex planar billiard with sufficiently smooth boundary was proved by V. F. Lazutkin [34]. Non-existence of caustics in higher-dimensional billiards with boundaries different from quadrics was proved by M. Berger [6].

The Birkhoff Conjecture was studied by many mathematicians. In 1950 H. Poritsky [37] proved it under the additional assumption that the billiard in each closed caustic near the boundary has the same closed caustics as the initial billiard. Later in 1988 another proof of the same result was obtained by E. Amiran [4]. Recall that the reflection from the boundary of a convex planar billiard Ω acts on the space of oriented lines intersecting Ω, and their space is called the *phase cylinder*: each line is reflected from the boundary $\partial\Omega$ at its last point of intersection with $\partial\Omega$ (with respect to its orientation), and its reflected image is directed inside the domain Ω at this point. In 1993 M. Bialy [7] proved that if the phase cylinder of the billiard is foliated by non-contractible continuous closed curves which are invariant under the billiard map, then the boundary $\partial\Omega$ is a circle. (Another proof of the same result was later obtained in [47].) In particular, Bialy's result implies the Birkhoff Conjecture under the assumption that the foliation by caustics extends to the whole billiard domain punctured at one point; then the boundary is a circle. In 2012 he proved a similar result for billiards on constant curvature surfaces [8] and also for magnetic billiards [9]. In 1995 A. Delshams and R. Ramírez-Ros [19] suggested an approach to proving splitting of separatrices for generic perturbations of an ellipse. In 2013 D. V. Treschev [42] made a numerical experiment indicating that there should exist analytic *locally integrable* billiards with the billiard reflection map having a two-periodic point where the germ of its second iterate is analytically conjugate to a disk rotation. Recently Treschev studied the billiards from [42] in more detail in [43] and their multi-dimensional versions in [44]. A similar effect for a ball rolling on a vertical cylinder under gravitation was discovered in [3]: the authors showed that the ratio between its vertical and horizontal oscillation periods is a universal irrational constant, the number $\sqrt{7/2}$. Recently V. Kaloshin and A. Sorrentino [31] have proved a *local version* of the Birkhoff Conjecture: *an integrable deformation of an ellipse is an ellipse*. (The case of ellipses with small eccentricities was treated in the previous paper by A. Avila, J. De Simoi and V. Kaloshin [5].) A dynamical entropic version of the Birkhoff Conjecture was stated and partially studied by J.-P. Marco [35].

In 1988 A. P. Veselov [45, Proposition 4] proved that every billiard bounded by confocal quadrics in any dimension has a complete system of first integrals in involution that are quadratic in P. In 1990 he studied a billiard in a non-Euclidean ellipsoid: in the sphere and in the Lobachevsky (i.e., hyperbolic) space of any dimension n. He proved its complete integrability and provided an explicit complete list of first integrals [46, corollary on p. 95]. In the same paper he proved that all the sides of a billiard trajectory are tangent to the same $n-1$ quadrics confocal to the boundary of the ellipsoid and the billiard dynamics corresponds to a shift of the Jacobi variety corresponding to an appropriate hyperelliptic curve [46, Theorems 3, 2 on p. 99]. The Polynomial Birkhoff Conjecture together with its generalization to surfaces of constant curvature was stated and studied by S. V. Bolotin [16], who proved in 1990 that under the assumptions of the conjecture, the billiard boundary lies on an algebraic curve. In the same paper and in [17, Section 4] he proved the conjecture under the assumption that at least one irreducible component of the corresponding complex projective planar algebraic curve is non-linear and non-singular (in the non-Euclidean case it is also required that in addition, at least one intersection point of that component with the absolute be transversal). In [17] Bolotin proved integrability of countably confocal billiards with piecewise smooth boundaries with integrals of degrees 2 or 4 and a similar statement in higher-dimensional spaces of constant curvature. M. Bialy and A. E. Mironov proved the planar Polynomial Birkhoff Conjecture in the case of integrals of degree 4 [12]. A version of the planar Polynomial Birkhoff Conjecture for *families* of billiards sharing the same polynomial integral (with boundaries depending continuously on one parameter) was solved in [1]: in [1, bottom of p. 30] it was shown that it is sufficient to require that the union of the boundaries does not lie on an algebraic curve in $\mathbb{R}^2$. Dynamics in countably confocal billiards with piecewise smooth boundaries in two and higher dimensions was studied in [20, 21, 22, 23, 24]. Dynamics in so-called pseudo-integrable billiards (more precisely, confocal billiards with non-convex angles) was studied in [21, 22, 23, 24]. For further results on the Polynomial Birkhoff Conjecture and its version for magnetic billiards see the above-mentioned papers [10, 11, 12] by M. Bialy and A. E. Mironov, and also [13, 14] and references therein.

The analogue of the Birkhoff Conjecture for outer billiards was stated by S. L. Tabachnikov [41] in 2008. Its polynomial version was stated by Tabachnikov and proved by himself under genericity assumptions in the same paper, and recently solved completely in the joint work of the author of the present paper with E. I. Shustin [27].

2. Preliminaries: from polynomially integrable to $\mathbb{I}$-angular billiards

2.1. Reflection and $\mathbb{I}$-angular symmetry

Here we present the results of M. Bialy and A. E. Mironov mentioned in Subsection 1.2 and give self-contained proofs of some of them.

Proposition 2.1 (S. V. Bolotin, see [17, formula (15), p. 23], [33, formula (3.12), p. 140]). *For every $r \in \Sigma$ the linear operator $\mathcal{M}_r : T_r\Sigma \to V_r = r^{\perp}$, $v \mapsto [r, v]$, is an isomorphism preserving the quadratic form $\langle Ax, x\rangle$. Here the orthogonal complement and the vector product are taken with respect to the standard Euclidean scalar product* (*see footnote* 3).

Definition 2.2. Let the space $\mathbb{R}^n$ be equipped with a quadratic form $\langle Ax, x\rangle$, A being a symmetric $n \times n$-matrix, and let $\ell \subset \mathbb{R}^n$ be a one-dimensional vector subspace such that $\ell \not\subset \{\langle Ax, x\rangle = 0\}$. The *pseudo-symmetry* of $\mathbb{R}^n$ with respect to ℓ is the linear involution $\mathbb{R}^n \to \mathbb{R}^n$ preserving the quadratic form, fixing the points of ℓ and acting as central symmetry in its orthogonal complement with respect to the form. The definition of complex pseudo-symmetry of the space $\mathbb{C}^n$ equipped with a $\mathbb{C}$-bilinear quadratic form is analogous.

Corollary 2.3. *For every $r \in \Sigma$ and one-dimensional subspace $\ell \subset T_r\Sigma$ the mapping $\mathcal{M}_r : T_r\Sigma \to V_r$, $v \mapsto M$, conjugates the pseudo-symmetry $T_r\Sigma \to T_r\Sigma$ with respect to ℓ and the pseudo-symmetry $V_r \to V_r$ with respect to the one-dimensional subspace orthogonal to both r and ℓ.*

Definition 2.4. Let $\mathbb{I} \subset \mathbb{CP}^2$ be a conic (either a smooth conic, or a union of two distinct lines). Let $B \in \mathbb{CP}^2 \setminus \mathbb{I}$. For every complex line L through B consider its complex projective involution fixing B and permuting its intersection points with $\mathbb{I}$. (If L is tangent to $\mathbb{I}$, the involution in question is the unique non-trivial projective involution $L \to L$ fixing B and the tangency point.) The transformation thus constructed for each L is a projective involution $\mathbb{CP}^2 \to \mathbb{CP}^2$ fixing B, which will be called the $\mathbb{I}$-*angular symmetry* with center B. See Fig. 2 in the Euclidean case.

Proposition 2.5. *Consider the space $\mathbb{C}^3_{(M_1,M_2,M_3)}$ equipped with a quadratic form $\langle AM, M\rangle$,* $\dim(\operatorname{Ker} A) \leq 1$. *The absolute $\mathbb{I} = \{\langle AM, M\rangle = 0\} \subset \mathbb{CP}^2_{(M_1:M_2:M_3)}$ (see* (1.7)) *is either a regular conic, or a union of two distinct lines. The projectivization of the pseudo-symmetry $\mathbb{C}^3 \to \mathbb{C}^3$ with respect to a one-dimensional subspace ℓ is the $\mathbb{I}$-angular symmetry with center $\pi(\ell)$.*

The proposition follows from the definitions.

Theorem 2.6 (see [11, Theorem 1.3, p. 151] in the non-Euclidean case). *Let $\Omega \subset \Sigma$ be a polynomially integrable billiard with countably piecewise smooth boundary and a homogeneous polynomial integral $\Psi(M)$ of even degree. Let r be a point in a smooth arc in $\partial\Omega$. Set $V_r = r^{\perp} \subset \mathbb{R}^3$. Let $L \subset V_r$ be the one-dimensional subspace Euclidean-orthogonal to both r and the tangent line $T_r\partial\Omega$. The restriction $\Psi|_{V_r}$ is invariant under the pseudo-symmetry of the plane V_r equipped with the form $\langle Ax, x\rangle$ with respect to the line L.*

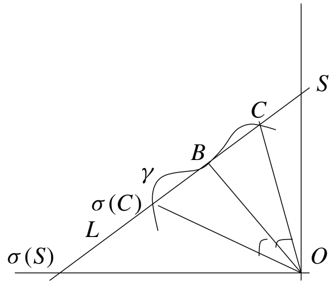

Fig. 2. The $\mathbb{I}$-angular symmetry $\sigma : \mathbb{CP}^2 \to \mathbb{CP}^2$ with center B in the Euclidean case, when $\mathbb{I} = \{x_1^2 + x_2^2 = 0\}$: the action in the affine chart $\mathbb{C}^2_{(x_1,x_2)}$; $O = (0, 0)$. The lines OC and $O\sigma(C)$ are symmetric with respect to the line OB. The projective lines OS and $O\sigma(S)$ are isotropic for the complex Euclidean metric $dx_1^2 + dx_2^2$ on $\mathbb{C}^2$, that is, $\mathbb{I} = OS \cup O\sigma(S)$.

Proof. The polynomial integral $\Psi([r, v])$ is invariant under the action on v of the pseudo-symmetry $T_r\Sigma \to T_r\Sigma$ with respect to the line $\ell = T_r\partial\Omega$ (invariance under reflection). This together with Corollary 2.3 implies the statement of the theorem. □

Convention 2.7. Recall that for every C^2-smooth curve $\alpha \subset \Sigma$ with non-zero geodesic curvature its Σ-*dual* is the curve $\alpha^* \subset \mathbb{RP}^2$ orthogonal-polar-dual to the projection $\pi(\alpha) \subset \mathbb{RP}^2$ (see Subsection 1.2). For every $r \in \Sigma$ each one-dimensional vector subspace $\ell \subset T_r\Sigma$ is the intersection of the tangent plane $T_r\Sigma$ with a two-dimensional subspace $H \subset \mathbb{R}^3$ containing r. The intersection $\widehat{\ell} = H \cap \Sigma$ is the geodesic tangent to ℓ. The point $\pi(H^\perp \setminus \{0\}) \in \mathbb{RP}^2$ will be called the *point* Σ-*dual* to the subspace ℓ and to the geodesic $\widehat{\ell}$. It will be denoted by $\widehat{\ell}^*$.

Theorem 2.8. *Let $\Omega \subset \Sigma$ be a polynomially integrable billiard with countably piecewise C^2-smooth boundary. Let $\Psi(M)$ be its homogeneous polynomial integral of even degree $2n$. The function $G = \Psi(M)/\langle AM, M\rangle^n$ from (1.6) treated as a rational function on $\mathbb{CP}^2_{(M_1:M_2:M_3)}$ satisfies the following statements.*

(1) *For every C^2-smooth arc $\alpha \subset \partial\Omega$ with non-zero geodesic curvature, let $\alpha^* \subset \mathbb{RP}^2$ be its Σ-dual curve. For every point $C \in \alpha^*$ the restriction of the function G to the projective line $T_C\alpha^*$ is invariant under the $\mathbb{I}$-angular symmetry with center C. One has $G|_{\alpha^*} \equiv$* const.
(2) *For every geodesic $\widehat{\ell} \subset \Sigma$ that contains a segment of the boundary $\partial\Omega$ the function G is invariant under the $\mathbb{I}$-angular symmetry of the whole projective plane $\mathbb{CP}^2$ with center $\widehat{\ell}^*$, the point Σ-dual to $\widehat{\ell}$.*

Remark 2.9. A version of statement (1) of Theorem 2.8 in the Euclidean case was proved in [10, Theorem 3] for convex domains with smooth boundary. But its proof remains valid in the general Euclidean case.

Proof of Theorem 2.8. Each point $C \in \alpha^*$ is dual to the projective line tangent to the curve $\pi(\alpha)$ at some point $\pi(r)$, $r \in \alpha$, by definition. Consider the projective line $T_C\alpha^*$

and set $V = \pi^{-1}(T_C\alpha^*) \cup \{0\} \subset \mathbb{R}^3$. It is the two-dimensional subspace orthogonal to the line Or, by definition. Set $L = \pi^{-1}(C) \cup \{0\} \subset V$; it is the one-dimensional subspace orthogonal to both lines $T_r\alpha$ and Or, by definition. The restrictions to V of both functions $\Psi(M)$ and $\langle AM, M\rangle$ are invariant under the pseudo-symmetry of the plane V with respect to the line L, by Theorem 2.6 and isometry. Hence, the restriction to V of the ratio $G(M) = \Psi(M)/\langle AM, M\rangle^n$ is also invariant. Therefore, the restriction to $\pi(V \setminus \{0\}) = T_C\alpha^*$ of G treated as a rational function on $\mathbb{CP}^2$ is invariant under the projectivized pseudo-symmetry, which coincides with the $\mathbb{I}$-angular symmetry centered at C, by Proposition 2.5. The equality $G|_{\alpha^*} \equiv \text{const}$ holds since the derivative of G at C along a vector tangent to $T_C\alpha^*$ vanishes. Indeed, the function $G|_{T_C\alpha^*}$, which is invariant under a projective involution fixing C, has zero derivative at C, similarly to vanishing of the derivative of an even function at 0. Statement (1) is proved.

The proof of (2) is analogous. In more detail, let $\Lambda \subset \Sigma$ be a geodesic whose segment $I \subset \Lambda$ is contained in $\partial\Omega$. For every point $Q \in I$ the projective line Q^* dual to $\pi(Q)$ passes through the point Λ^* Σ-dual to Λ. The restriction $G|_{Q^*}$ is invariant under the $\mathbb{I}$-angular symmetry with center Λ^*, as in the above argument. Therefore, this holds for the restriction of G to every complex line through Λ^*, and hence on all of $\mathbb{CP}^2$, by uniqueness of analytic extension. Statement (2) is proved. □

Definition 2.10. Let $\mathbb{I} \subset \mathbb{CP}^2$ be a conic (either a regular conic, or a pair of distinct lines). Let $\gamma \subset \mathbb{CP}^2$ be an irreducible algebraic curve different from a line and from $\mathbb{I}$. We say that γ *generates a rationally integrable* $\mathbb{I}$*-angular billiard* if there exists a non-constant rational function G on $\mathbb{CP}^2$ with poles contained in $\mathbb{I}$ (called the *integral of the* $\mathbb{I}$*-angular billiard*) such that for every $C \in \gamma \setminus \mathbb{I}$ the restriction of G to the projective tangent line $T_C\gamma$ is invariant under the $\mathbb{I}$-angular symmetry with center C.

Corollary 2.11. *Let $\mathbb{I} \subset \mathbb{CP}^2_{(M_1:M_2:M_3)}$ be the absolute (see (1.7)). Let $\Omega \subset \Sigma$ be a polynomially integrable billiard with a non-trivial homogeneous integral $\Psi(M)$ of even degree $2n$. Let $\alpha \subset \partial\Omega$ be a C^2-smooth arc with non-zero geodesic curvature, and let $\alpha^* \subset \mathbb{RP}^2 \subset \mathbb{CP}^2$ be its Σ-dual curve. The complex projective Zariski closure of the curve α^* is an algebraic curve. Each of its non-linear irreducible components generates a rationally integrable $\mathbb{I}$-angular billiard with integral $G(M) = \Psi(M)/\langle AM, M\rangle^n$ (see (1.6)).*

Proof. The function G is non-constant on $\mathbb{CP}^2$, since $\Psi|_{\{\langle AM,M\rangle=1\}} \not\equiv \text{const}$: this follows from non-constancy of the function $\Psi([r, v])$ on the hypersurface $\{\langle Av, v\rangle = 1\}$ (non-triviality of the integral) and Proposition 2.1. The first statement of the corollary, which follows from Bolotin's Theorem 1.22, also follows from the constancy of G on α^* (see Theorem 2.8(1)).

The second statement follows from the invariance of G in Theorem 2.8(1) by a straightforward analytic extension argument. □

Proposition 2.12. *Suppose an irreducible algebraic curve $\gamma \subset \mathbb{CP}^2$ generates a rationally integrable $\mathbb{I}$-angular billiard with integral G. Then $G|_\gamma \equiv$ const.*

The proof repeats literally the above proof of the analogous statement from Theorem 2.8(1).

2.2. *Duality and $\mathbb{I}$-angular billiards. Proof of Theorem 1.23*

For the proof of Theorem 1.23 we use the well-known classical properties of orthogonal polarity given by the following proposition and its corollary. We give the proof of the proposition for completeness.

Proposition 2.13. *Let B be a non-degenerate complex symmetric 3×3-matrix. Consider the complex space $\mathbb{C}^3$ with coordinates $x = (x_1, x_2, x_3)$ equipped with the complex-bilinear Euclidean quadratic form $dx_1^2 + dx_2^2 + dx_3^2$. The complex orthogonal-polar-dual to the conic in $\mathbb{CP}^2_{(x_1:x_2:x_3)}$ given by the equation $\langle Bx, x\rangle = 0$ is the conic given by the equation $\langle B^{-1}x, x\rangle = 0$.*

Proof. Consider the cone $K = \{x \in \mathbb{C}^3 \setminus \{0\} \mid \langle Bx, x\rangle = 0\}$ and its tautological projection $\Gamma = \pi(K) \subset \mathbb{CP}^2$, which is the conic under consideration. Let $x \in K$. The projective tangent line $L = T_{\pi(x)}\Gamma$ is defined by the tangent plane T_xK considered as a vector subspace in $\mathbb{C}^3$. It follows from the definitions that T_xK consists of those vectors v for which $\langle Bx, v\rangle = 0$. Thus, $(T_xK)^{\perp} = \mathbb{C}(Bx)$, and the dual L^* is $\pi(Bx)$. Therefore, the dual Γ^* is the projection $\pi(B(K))$, which is obviously defined by the equation $\langle B(B^{-1}y), B^{-1}y\rangle = \langle B^{-1}y, y\rangle = 0$. This proves the proposition. □

Definition 2.14 ([46, p. 84]). Let A, B be two real non-proportional symmetric 3×3-matrices. They define a *pseudo-Euclidean pencil* of conics in $\mathbb{RP}^2$ by

$$\{\langle (B - \lambda A)M, M\rangle = 0\} \subset \mathbb{RP}^2_{(M_1:M_2:M_3)}, \qquad \lambda \in \mathbb{R}.$$

The same pencil of complex conics in $\mathbb{CP}^2$ depending on $\lambda \in \mathbb{C}$ will also be called pseudo-Euclidean.

Corollary 2.15. *The Σ-duality transforms every confocal pencil of conics to the corresponding pseudo-Euclidean pencil. Namely, for every real symmetric 3×3-matrix B satisfying the conditions of Definition 1.12 for any two conics in Σ lying in the confocal pencil (1.1) defined by B their Σ-dual curves lie in conics belonging to the pseudo-Euclidean pencil defined by the same matrix B. In the non-Euclidean case, when the absolute $\mathbb{I}$ is a regular conic, $\mathbb{I}$ is self-dual with respect to complex orthogonal polarity.*

The first statement of the corollary is obvious. The self-duality follows from Proposition 2.13 and involutiveness $A^2 = \mathrm{Id}$ in the non-Euclidean case.

Proof of Theorem 1.23. Let $\Omega \subset \Sigma$ be a polynomially integrable billiard. Let $\Psi(M_1, M_2, M_3)$ be a non-trivial homogeneous polynomial integral of the billiard Ω of even degree $2n$. Consider the affine chart $M_3 \neq 0$ on $\mathbb{CP}^2_{(M_1:M_2:M_3)}$ with coordinates (x, y) where $x = M_1/M_3$, $y = M_2/M_3$. Set

$$\mathcal{Q}(x, y) = \langle AM, M\rangle, \text{ where } M = (x, y, 1);$$

$$\mathcal{Q}(x, y) = x^2 + y^2 \text{ in the Euclidean case; otherwise } \mathcal{Q}(x, y) = x^2 + y^2 \pm 1.$$

In this affine chart the function G on $\mathbb{CP}^2$ from (1.6) takes the form

$$G(x,y) = F(x,y)/(\mathcal{Q}(x,y))^n, \quad F(x,y) = \Psi(x,y,1), \quad \deg F \le 2n.$$

In what follows, for every conic $\alpha \subset \Sigma$ the corresponding complex conic containing its Σ-dual α^* will be denoted by $\widetilde{\alpha}^*$.

Suppose the boundary $\partial\Omega$ contains an arc of a conic α. Let $\mathcal{C}$ be the confocal conic pencil containing α, and let $\mathcal{C}^*$ denote the corresponding (Σ-dual) pseudo-Euclidean pencil of conics containing $\widetilde{\alpha}^*$:

$$\kappa_\lambda = \{\langle B_\lambda X, X\rangle = 0\} \subset \mathbb{R}^3_{(X_1,X_2,X_3)}, \quad B_\lambda = (B-\lambda A)^{-1}, \quad \mathcal{C}_\lambda = \kappa_\lambda \cap \Sigma,$$
$$\kappa^*_\lambda = \{\langle (B-\lambda A)M, M\rangle = 0\} \subset \mathbb{C}^3_{(M_1,M_2,M_3)}, \quad \mathcal{C}^*_\lambda = \pi(\kappa^*_\lambda \setminus \{0\}) \subset \mathbb{CP}^2,$$
$$\kappa^*_\infty = \widehat{\mathbb{I}} = \{\langle AM, M\rangle = 0\} \subset \mathbb{C}^3, \quad \mathcal{C}^*_\infty = \pi(\kappa^*_\infty \setminus \{0\}) = \mathbb{I}.$$

Claim 1. *Each C^2-smooth arc of the boundary $\partial\Omega$ with non-zero geodesic curvature lies in a conic confocal to α.*

Proof. The conic $\widetilde{\alpha}^*$ generates a rationally integrable $\mathbb{I}$-angular billiard with integral G, by Corollary 2.11. On the other hand, it is known that the billiard on a conic α admits a non-trivial quadratic homogeneous first integral $\widetilde{\Phi} = \widetilde{\Phi}(M)$ (see [17, Proposition 1]). Set

$$\widetilde{F}(x,y) = \widetilde{\Phi}(x,y,1), \quad \widetilde{G}(x,y) = \widetilde{F}(x,y)/\mathcal{Q}(x,y).$$

Claim 2. *The level curves of the function $\widetilde{G}$ are conics from the pencil $\mathcal{C}^*$, and the function G is constant on each of them.*

Proof. For every conic β confocal to α the quadratic integral $\widetilde{\Phi}$ is also an integral for the billiard on the conic β. This is well-known [17], and follows from the explicit formula [17, formula (12)] for the quadratic integral. Therefore, both the complexified dual conics $\widetilde{\alpha}^*$ and $\widetilde{\beta}^*$ generate rationally integrable $\mathbb{I}$-angular billiards with a common quadratic rational integral $\widetilde{G}$ having a first order pole at $\mathbb{I}$, by Corollary 2.11. Hence, $\widetilde{G}$ is constant on $\widetilde{\alpha}^*$ and $\widetilde{\beta}^*$, by Proposition 2.12. Thus, the integral $\widetilde{G}$ is constant on every conic from the complex pseudo-Euclidean pencil $\mathcal{C}^*$, since the above conics $\widetilde{\beta}^*$ with β being confocal to α form a real one-dimensional subfamily in $\mathcal{C}^*$. Let us normalize the integral $\widetilde{G}$ by an additive constant (or equivalently, the integral $\widetilde{\Phi}$ by addition of $c\langle AM, M\rangle$, $c = \text{const}$) so that $\widetilde{G}|_{\widetilde{\alpha}^*} \equiv 0$. After this normalization one has $\widetilde{F}|_{\widetilde{\alpha}^*} \equiv 0$, that is, $\widetilde{F}$ is the quadratic polynomial defining the conic $\widetilde{\alpha}^*$. On the other hand, $\widetilde{\alpha}^*$ generates a rationally integrable $\mathbb{I}$-angular billiard with integral G (Corollary 2.11). Hence, $G|_{\widetilde{\alpha}^*} \equiv c_1 = \text{const}$, by Proposition 2.12. Therefore,

$$G(x,y) = c_1 + G_1(x,y)\widetilde{G}(x,y),$$
$$G_1(x,y) = f_1(x,y)/(\mathcal{Q}(x,y))^{n-1}, \quad \deg f_1 \le 2n-2.$$

Hence, the fraction G_1 is also a rational integral of the $\mathbb{I}$-angular billiard generated by $\widetilde{\alpha}^*$, as are G and $\widetilde{G}$. Thus, $G_1|_{\widetilde{\alpha}^*} \equiv c_2 = \text{const}$, by Proposition 2.12. Similarly we get

$$G_1(x,y) = c_2 + G_2(x,y)\widetilde{G}(x,y),$$
$$G_2(x,y) = f_2(x,y)/(\mathcal{Q}(x,y))^{n-2}, \quad \deg f_2 \le 2n-4,$$

and G_2 is an integral of the $\mathbb{I}$-angular billiard generated by $\widetilde{\alpha}^*$, as are G_1 and $\widetilde{G}$. Continuing this procedure we find that G is a polynomial in $\widetilde{G}$. Hence, $G \equiv$ const on the level curves of the function $\widetilde{G}$, that is, on the conics from the pencil $\mathcal{C}^*$. Claim 2 is proved. □

Let ϕ be a C^2-smooth arc in $\partial\Omega$ with non-zero geodesic curvature, and let $\phi^* \subset \mathbb{RP}^2 \subset \mathbb{CP}^2$ denote its Σ-dual curve. The curve ϕ^* lies on a level curve of the function G, by Theorem 2.8(1). Hence, it lies in a finite union of conics from the pencil $\mathcal{C}^*$, since each level curve of G is a finite union of conics in $\mathcal{C}^*$ (follows from Claim 2). Therefore, ϕ lies in just one conic confocal to α, by smoothness, since any two intersecting confocal conics are orthogonal. This proves Claim 1. □

Now it remains to show that if $\partial\Omega$ contains geodesic segments, then their ambient geodesics are admissible with respect to the pencil $\mathcal{C}$ (see Definition 1.15). As shown below, this is a consequence of the following proposition.

Proposition 2.16. *Let B be a real symmetric 3×3-matrix as in Definition 1.12. Let $\mathcal{C}$ denote the corresponding pencil (1.1) of confocal conics in Σ. The corresponding admissible geodesics in Σ from Definition 1.15 are exactly those geodesics $\widehat{l}$ for which the symmetry of the surface Σ with respect to $\widehat{l}$ leaves the pencil $\mathcal{C}$ invariant: the symmetry permutes confocal conics. Or equivalently, the geodesics $\widehat{l}$ for which the $\mathbb{I}$-angular symmetry with center $\widehat{l}^*$ Σ-dual to $\widehat{l}$ leaves the Σ-dual pseudo-Euclidean pencil $\mathcal{C}^*$ invariant.*

Remark 2.17. We will be using only the second statement of Proposition 2.16 characterizing admissible geodesics $\widehat{l}$ in terms of $\mathbb{I}$-angular symmetry with center $\widehat{l}^*$ of the pencil $\mathcal{C}^*$. Their characterization in terms of symmetry of the pencil $\mathcal{C}$ will be proved just for completeness of presentation.

Proof of Proposition 2.16. Let us first prove that for every given geodesic $\widehat{l} \subset \Sigma$ the two statements of the proposition are indeed equivalent. As shown below, this is a consequence of the following proposition.

Proposition 2.18. *Consider the action of the symmetry with respect to a given geodesic $\widehat{l} \subset \Sigma$ on the space of all the geodesics in Σ. The Σ-duality conjugates this action to the $\mathbb{I}$-angular symmetry with center $\widehat{l}^*$.*

Proof. It suffices to prove the above conjugacy on the space of those geodesics that intersect $\widehat{l}$, by analyticity and since they form an open subset in the connected manifold of geodesics. Each geodesic through a point $r \in \widehat{l}$ is uniquely determined by its tangent line, a one-dimensional subspace $\Lambda \subset T_r\Sigma$. Thus, it suffices to show that the Σ-duality conjugates the symmetry action on the projectivized tangent plane $\mathbb{P}(T_r\Sigma)$ with the $\mathbb{I}$-angular symmetry centered at $\widehat{l}^*$. Indeed, the Σ-duality sends each one-dimensional subspace $\Lambda \subset T_r\Sigma$ to the point $\widehat{\Lambda}^* \in \mathbb{RP}^2$ represented by the one-dimensional vector subspace $\Lambda^r \subset \mathbb{R}^3$ orthogonal to both r and Λ (see Convention 2.7). The linear isomorphism $\mathcal{M}_r : T_r\Sigma \to V_r = r^\perp$, $v \mapsto [r, v]$, sends each subspace Λ to Λ^r and conjugates the pseudo-symmetries with respect to the lines $T_r\widehat{l} \subset T_r\Sigma$ and $(T_r\widehat{l})^r \subset V_r$, by definition and Corollary 2.3. Therefore, its projectivization realizes the Σ-duality $\mathbb{P}(T_r\Sigma) \to \mathbb{P}(V_r)$

and conjugates the action of the symmetry with respect to the line $T_r\widehat{l}$ on the source with the projectivized pseudo-symmetry of the image, the $\mathbb{I}$-angular symmetry with center $\widehat{l}^* = \pi((T_r\widehat{l})^r)$ (Proposition 2.5). Proposition 2.18 is proved. □

Note that for every curve $\gamma \subset \Sigma$ the Σ-duality sends the family of geodesics tangent to γ to the Σ-dual curve γ^* (see Convention 2.7). This together with the above proposition implies equivalence of the two statements of Proposition 2.16. Thus, it suffices to prove its second statement: those geodesics $\widehat{l}$ for which the pseudo-Euclidean pencil $\mathcal{C}^*$ is invariant under the $\mathbb{I}$-angular symmetry with center $\widehat{l}^*$ are exactly the admissible geodesics from Definition 1.15.

Fix a geodesic $\widehat{l}$. Let $H \subset \mathbb{R}^3$ denote the two-dimensional vector subspace containing $\widehat{l}$. Fix a vector $a \in H^{\perp} \subset \mathbb{R}^3$, $a \neq 0$. It represents the Σ-dual $\widehat{l}^* = \pi(a)$. The vector a lies in a unique cone κ^*_λ with $\lambda \neq \infty$, since $\langle Aa, a\rangle \neq 0$: otherwise, if $\langle Aa, a\rangle = 0$, then the intersection $\widehat{l} = H \cap \Sigma$ would be empty. Indeed, in the Euclidean case the equality $\langle Aa, a\rangle = 0$ for a real vector a holds exactly when a lies on the x_3-axis; then H is parallel to the plane Σ, $H \cap \Sigma = \emptyset$. In the non-Euclidean case the equality $\langle Aa, a\rangle = 0$ implies that $A = \operatorname{diag}(1, 1, -1)$ and the projective line $a^* = \pi(H \setminus \{0\})$ is tangent to the real absolute $\{\langle Ax, x\rangle = 0\} \subset \mathbb{RP}^2_{(x_1:x_2:x_3)}$, by self-duality (Corollary 2.15). Then H is tangent to the cone $\{\langle Ax, x\rangle = 0\} \subset \mathbb{R}^3$, and hence it is disjoint from the inner component containing Σ of the complement of that cone. Thus, $H \cap \Sigma = \emptyset$, a contradiction.

Without loss of generality we will assume that $a \in \kappa^*_0$, after replacing B by $B - \lambda A$ for appropriate λ, by the condition $\langle Aa, a\rangle \neq 0$. Let $S : \mathbb{C}^3 \to \mathbb{C}^3$ denote the pseudo-symmetry with respect to the line $\mathbb{C}a$.

Claim 3. *The pseudo-Euclidean pencil $\mathcal{C}^*$ is invariant under the $\mathbb{I}$-angular symmetry with center $\widehat{l}^*$ if and only if $S(\kappa^*_0) = \kappa^*_0$.*

Proof. The above $\mathbb{I}$-angular symmetry is the projectivization of the pseudo-symmetry S. Therefore, invariance of the pencil $\mathcal{C}^*$ under the $\mathbb{I}$-angular symmetry is equivalent to the S-invariance of the family of cones κ^*_λ, that is, to the existence of an involution $h : \lambda \mapsto h(\lambda)$ such that $S(\kappa^*_\lambda) = \kappa^*_{h(\lambda)}$. In the latter case one has $S(\kappa^*_0) = \kappa^*_0$, since $S(a) = a$ for $a \in \kappa^*_0 \setminus \kappa^*_\lambda$ for every $\lambda \neq 0$. Conversely, let $S(\kappa^*_0) = \kappa^*_0$. This means that the involution S sends the quadratic form $\langle Bx, x\rangle$ to itself up to sign. Hence, $S(\kappa^*_\lambda) = \kappa^*_{\pm\lambda}$ for every λ, since S preserves the quadratic form $\langle Ax, x\rangle$. This together with the previous equivalence statement proves the claim. □

Claim 4. *One has $S(\kappa^*_0) = \kappa^*_0$ if and only if κ^*_0 is a union of a pair of 2-planes through the origin in $\mathbb{C}^3$ that has one of the following types:*

(α) *both planes contain the line $\mathbb{C}a$ (they may coincide);*

(β) *one plane in κ^*_0 contains the line $\mathbb{C}a$, and the other coincides with the two-dimensional subspace $H_A \subset \mathbb{C}^3$ that is orthogonal to the vector a with respect to the scalar product $\langle Ax, x\rangle$.*

Proof. Every hyperplane $W \subset \mathbb{C}^3$ parallel to the plane H_A is S-invariant, and S acts there as the central symmetry with respect to the point C_W of the intersection $W \cap (\mathbb{C}a)$. The S-invariance of the cone κ^*_0 is equivalent to the invariance of each intersection $I_W = W \cap \kappa^*_0$

under the latter symmetry for every W as above. The intersection I_W is either all of W, or a line through C_W, or a conic in W containing the center of its symmetry C_W, since $\mathbb{C}a \subset \kappa_0^*$. In the latter case I_W is a union of two lines through C_W, since a planar conic central-symmetric with respect to some of its points C is a union of two lines through C (the lines may coincide). Note that all the intersections I_W with $W \neq H_A$ are naturally isomorphic via homotheties centered at the origin, since κ_0^* is a cone. Therefore, the following two cases are possible.

(α) I_W is a union of two (maybe coinciding) lines through C_W for every W; then κ_0^* is a union of two planes containing the line $\mathbb{C}a$.

(β) I_W is a line for all $W \neq H_A$, and $I_W = W$ for $W = H_A$; then κ_0^* is the union of the plane H_A and another plane containing $\mathbb{C}a$.

This proves the claim. □

Now let us return to the proof of Proposition 2.16. Suppose the pencil $\mathcal{C}^*$ is invariant under the $\mathbb{I}$-angular symmetry centered at $\widehat{l}^*$; or equivalently, the cone $\kappa_0^* = \{\langle Bx, x\rangle = 0\}$ is a union of two planes, as in Claim 4.

Case (α): The above planes both contain a, thus $a \in \operatorname{Ker} B$; $\dim(\operatorname{Ker} B) = 1$ if the planes are distinct; $\dim(\operatorname{Ker} B) = 2$ if they coincide. Hence, the hyperplane H orthogonal to a with respect to the standard Euclidean scalar product is orthogonal to $\operatorname{Ker} B$. Therefore, the geodesic $\widehat{l} = H \cap \Sigma$ is admissible of type (1) in Definition 1.15. Conversely, each admissible geodesic of type (1) can be represented as above after replacing B by $B - \lambda A$.

Case (β): Then the cone κ_0^* is the union of the plane H_A and a plane Π containing the line $\mathbb{C}a$. The plane Π is the complexification of a real plane, which will be here also denoted by Π, since κ_0^* is defined by a quadratic equation over the real numbers and H_A is the complexification of a real plane. Let $b \in \mathbb{R}^3\setminus\{0\}$ denote a vector Euclidean-orthogonal to Π. Thus, $\langle a, b\rangle = 0$. Note that the vector Aa is non-zero, since $\langle Aa, a\rangle \neq 0$, as was shown above, and it is Euclidean-orthogonal to H_A, by definition. Therefore, $\langle BM, M\rangle = c\langle Aa, M\rangle\langle b, M\rangle$, $c \in \mathbb{R} \setminus \{0\}$. Let us normalize the vectors a and b by constant factors so that $c = 2$. Then the quadratic form $\langle BM, M\rangle$ can be represented in the tensor form as $Aa \otimes b + b \otimes Aa$. The plane H defining the geodesic $\widehat{l}$ is orthogonal to the vector a, by definition. Hence, $\widehat{l}$ is an admissible geodesic of type (2) in Definition 1.15, the first geodesic in (1.4). Conversely, each geodesic of type (2) can be represented as above (see Remark 1.17). Proposition 2.16 is proved. □

Let now $\widehat{l} \subset \Sigma$ be a geodesic some of whose segments is contained in the boundary of the polynomially integrable billiard in question. The $\mathbb{I}$-angular symmetry with center $\widehat{l}^*$ leaves invariant the rational integral G, by Theorem 2.8. Hence, it permutes the level curves of the quadratic rational function $\widetilde{G}$, and the pencil $\mathcal{C}^*$ is invariant, by Claim 2. Thus, the geodesic $\widehat{l}$ is admissible, by Proposition 2.16. Theorem 1.23 is proved. □

3. Bialy–Mironov Hessian formula and asymptotics of Hessians

The material of the present section will be used in Section 4 in the proof of Theorem 4.1(ii-b). It includes:

- the Bialy–Mironov Hessian formula (3.4) recalled in Subsection 3.1;
- the asymptotics of its left- and right-hand sides along those local branches of the curve γ that are transversal to $\mathbb{I}$ (Subsection 3.4).

In the proof of the above asymptotics we use general asymptotic formulas

- for the defining function of an irreducible germ a of analytic curve along another irreducible germ b (Subsection 3.2);
- for the Hessian $H(f)$ of a defining function of a given germ b along b (Subsection 3.3).

3.1. Bialy–Mironov formula

Let $\gamma \subset \mathbb{CP}^2$ be an irreducible algebraic curve generating a rationally integrable $\mathbb{I}$-angular billiard with integral G. The function G has poles contained in $\mathbb{I}$ and is constant on γ, by Proposition 2.12. In what follows we normalize it so that $G|_\gamma \equiv 0$, and set

$$\Gamma = \{G = 0\} \supset \gamma.$$

Fix an affine chart $\mathbb{C}^2 \subset \mathbb{CP}^2$ with coordinates (x, y) such that the infinity line is not contained in $\mathbb{I}$. In this chart the function G takes the form

$$\begin{aligned}
&G(x, y) = F_1(x, y)/(Q(x, y))^n, \quad \text{where}\\
&F_1 \text{ is a polynomial of degree at most } 2n,\\
&Q \text{ is a fixed quadratic polynomial defining } \mathbb{I}\text{: } \mathbb{I} = \{Q = 0\}.
\end{aligned}$$

Let $f(x, y)$ be the polynomial defining the curve γ, which is irreducible, as is $\gamma = \{f = 0\}$, the differential df being non-zero on a Zariski open subset in γ. Recall that the polynomial F_1 vanishes on γ. Therefore,

$$F_1 = f^k g_1, \quad k \in \mathbb{N}, \quad g_1 \text{ is a polynomial coprime to } f. \tag{3.1}$$

Set

$$g = g_1^{1/k}, \quad F = F_1^{1/k} = fg, \quad m = n/k. \tag{3.2}$$

We consider the Hessian quadratic form of the function $f(x, y)$ evaluated on an appropriately normalized tangent vector to $\gamma = \{f = 0\}$ at a point (x, y), namely, the skew gradient $(f_y, -f_x)$ with respect to the standard complex symplectic form $dx \wedge dy$:

$$H(f) = f_{xx} f_y^2 - 2 f_{xy} f_x f_y + f_{yy} f_x^2. \tag{3.3}$$

Theorem 3.1 (see [10, Theorem 6.1], [11, formulas (16) and (32)]). *The following formula holds for all $(x, y) \in \gamma$:*

$$(g(x, y))^3 H(f)(x, y) = H(gf) = c(Q(x, y))^{3m-3}, \quad c \equiv \text{const} \neq 0. \tag{3.4}$$

Remark 3.2. In 2008 S.Tabachnikov obtained a version of formula (3.4) with $k = 1$ and constant right-hand side for polynomially integrable outer billiards satisfying some non-degeneracy assumptions [41, p. 102]. Theorem 6.1 in [10] deals with a polynomially integrable planar billiard $\Omega \subset \mathbb{R}^2$, a curve $\Gamma_1 \subset \mathbb{R}^2$ that is polar-dual to a C^2-smooth arc in $\partial\Omega$ with non-zero geodesic curvature, and the absolute $\mathbb{I} = \{x^2 + y^2 = 0\}$. It states that formula (3.4) holds along the curve Γ_1. Then it holds automatically on every irreducible component γ of its complex Zariski closure. Its proof given in [10] remains valid for every irreducible algebraic curve γ generating a rationally integrable $\mathbb{I}$-angular billiard. The same remark concerns formulas (16) and (32) from [11], which deal with the non-Euclidean case and the corresponding absolute $\mathbb{I} = \{x^2 + y^2 \pm 1 = 0\}$. These results from [10, 11] together cover Theorem 3.1 in the general case, since every conic different from a double line is projectively equivalent to some of the above absolutes.

Without loss of generality we will assume that G is an irreducible fraction, that is, its numerator $F_1(x, y)$ does not vanish identically on $\mathbb{I}$ when $\mathbb{I}$ is regular, and if $\mathbb{I}$ is a union of two lines Λ_1 and Λ_2, one has $F_1 \not\equiv 0$ on each Λ_j. In the former case we can do this by irreducibility of the conic $\mathbb{I}$: if F_1 vanishes on $\mathbb{I}$ with a certain multiplicity s, then we can divide both the numerator and denominator in G by $(\mathcal{Q}(x, y))^s$ and achieve the desired property. In the latter case we can do this because both lines Λ_1 and Λ_2 forming $\mathbb{I}$ enter the divisor of the function G (the zero-pole divisor) with the same multiplicity. Indeed, for every $u \in \gamma \setminus \mathbb{I}$ the tangent line $T_u\gamma$ intersects both lines Λ_1 and Λ_2, and their intersection points with the line $T_u\gamma$ are permuted by its $\mathbb{I}$-angular symmetry with center u, by definition. Both intersection points enter the divisor of the function $G|_{T_u\gamma}$ with the same multiplicity, by its invariance under the $\mathbb{I}$-angular symmetry. This implies the above statement on coincidence of multiplicities of the lines Λ_1 and Λ_2.

The above discussion implies that G has a pole along each irreducible component of the conic $\mathbb{I}$. Therefore, no component in $\mathbb{I}$ is contained in Γ. We choose the above affine chart $\mathbb{C}^2_{(x,y)}$ so that the finite intersection $\Gamma \cap \mathbb{I}$ lies in $\mathbb{C}^2$, in particular, $G \not\equiv 0$ on the infinity line, hence $\deg F_1 = 2n$. Let Δ denote the zero divisor of the function G. Finally, under our assumptions without loss of generality one has $F_1 \not\equiv 0$ on every irreducible component of the conic $\mathbb{I}$,

$$\begin{aligned} &\Gamma = \{F_1 = 0\}, \qquad \deg F_1 = 2n, \\ &\Delta \text{ is the zero divisor of the polynomial } F_1, \end{aligned} \tag{3.5}$$

and the intersection $\Gamma \cap \mathbb{I}$, and hence $\gamma \cap \mathbb{I}$, lies in the affine chart $\mathbb{C}^2_{(x,y)}$.

3.2. *Asymptotics of the defining function*

Definition 3.3. Let b be a non-linear irreducible germ of analytic curve at a point C in $\mathbb{CP}^2$. An *adapted system of coordinates* to b is a system of affine coordinates (z, w) centered at C such that the z-axis is tangent to b. In adapted coordinates the germ b can be locally holomorphically and bijectively parametrized by a small complex parameter t:

$$t \mapsto (t^q, ct^p(1+o(1))) \quad \text{as } t \to 0;\ q, p \in \mathbb{N},\ 1 \le q < p,\ c \neq 0, \tag{3.6}$$
$$q = q_b, \quad p = p_b, \quad c = c_b,$$
$$q = 1 \text{ if and only if } b \text{ is a regular germ.}$$

The *projective Puiseux exponent* [25, p. 250, Definition 2.9] of the germ b is the ratio

$$r = r_b = p_b/q_b.$$

The germ b is called *quadratic* if $r_b = 2$, and *subquadratic* if $r_b \le 2$ [27, Definition 3.5]. When b is a germ of line, it is parametrized by $t \mapsto (t, 0)$; then we set $q_b = 1$, $p_b = \infty$, and put the Puiseux exponent r_b to be equal to infinity, as in loc. cit.

Proposition 3.4. *Let a, b be irreducible germs of holomorphic curves at a point $C \in \mathbb{C}^2$, and suppose b is non-linear. Let f_a, f_b be the irreducible germs of holomorphic functions defining them: $g = \{f_g = 0\}$ for $g = a, b$. Set*

$$\rho_a = \begin{cases} 1 & \text{if } a \text{ is transversal to } b, \\ r_a & \text{if } a \text{ is tangent to } b. \end{cases} \tag{3.7}$$

Let (z, w) be affine coordinates centered at C that are adapted to b. Then

$$f_a(u) = O((z(u))^{q_a \min\{\rho_a, r_b\}}) \quad \text{as } u \in b \text{ tends to } C. \tag{3.8}$$

The proof of Proposition 3.4 is based on the following property of Newton diagrams of irreducible germs of analytic curves.

Proposition 3.5. *Let $b \subset \mathbb{CP}^2$ be a non-linear irreducible germ of analytic curve at a point C, and let (z, w) be local affine coordinates adapted to it. Let $t \mapsto (t^q, ct^p(1+o(1)))$ be its local parametrization with $1 \le q < p$, $c \neq 0$ (see (3.6)). Let f be an irreducible germ of analytic function at C defining b: $b = \{f = 0\}$. The Newton diagram of the function f consists of one edge, the segment connecting the points $(p, 0)$ and $(0, q)$. More precisely, the Taylor series of the function $f(z, w)$ contains only monomials $z^\alpha w^\beta$ such that*

$$q\alpha + p\beta \ge qp. \tag{3.9}$$

Proof. Without loss of generality we will assume that f is a Weierstrass polynomial,

$$f(z, w) = \phi_z(w) = w^d + h_1(z)w^{d-1} + \cdots + h_d(z), \quad h_j(0) = 0, \tag{3.10}$$

since each germ of holomorphic function at 0 that vanishes at 0 and does not vanish identically on the w-axis is the product of a unique polynomial as above (called a Weierstrass polynomial) and a non-zero holomorphic function, by the Weierstrass Preparation Theorem [29, Chapter 0, Section 1]. For every z small enough the polynomial $\phi_z(w) = f(z, w)$ has q roots $\zeta_l(z)$, $l = 1, \dots, q$: $\zeta_l(z) = ct_l^p(1+o(1))$, $t_l^q = z$, as $z \to 0$; thus, $\zeta_l(z) \simeq cz^{p/q}$. This implies that the Weierstrass polynomial (3.10) is the product of q factors $w - \zeta_l(z)$ with $\zeta_l(z) \simeq cz^{p/q}$ as $z \to 0$. Hence, in formula (3.10) one has $d = q$,

$$h_q(z) = (-1)^q \prod_{l=1}^{q} \zeta_l(z) = (-1)^{q+p(q+1)} c^q z^p (1 + o(1)).$$

The last equality follows from $\prod_{l=1}^{q} t_l = (-1)^{q+1}z$: the product of the q-th roots of unity equals $(-1)^{q+1}$. One has

$$h_s(z) = O(z^{(p/q)s}) \quad \text{for } 1 \le s < q \text{ as } z \to 0, \tag{3.11}$$

since $h_s(z) = (-1)^s\sigma_s$, where σ_s is the s-th elementary symmetric polynomial in the roots $\zeta_l(z) \simeq cz^{p/q}$. Formula (3.11) implies that the Taylor series of the Weierstrass polynomial (3.10) contains only the monomials w^q, z^p and those monomials $z^\alpha w^\beta$ for which $\beta < q$ (set $s = q - \beta$) and $\alpha \ge \frac{p}{q}s = \frac{p}{q}(q-\beta)$, i.e., $q\alpha + p\beta \ge pq$. □

Proof of Proposition 3.4. Case 1: the curve a is transversal to b. Then $\rho_a = 1 < r = r_b = p_b/q_b$, and we have to show that $f_a|_b = O(z^{q_a})$. To do this, let us take the coordinates (z_a, w_a) adapted to a so that the w_a-axis coincides with the z-axis $T_C b$, $w_a = z$ on $T_C b$ and $z_a = w$; one can do this by transversality. One has

$$w_a \simeq z, \qquad z_a = w \simeq c_b z^r \qquad \text{along the curve } b. \tag{3.12}$$

When the germ a is linear, one has $q_a = 1$ and $f_a = O(w_a) = O(z) = O(z^{q_a})$ on b. Let now a be non-linear. Each Taylor monomial $z_a^\alpha w_a^\beta$ of the function f_a has asymptotics $O(z^{\alpha r+\beta})$ along the curve b, by (3.12). Now it suffices to show that $\alpha r + \beta \ge q_a$. Recall that $\alpha q_a + \beta p_a \ge p_a q_a$, by (3.9). Dividing this inequality by p_a yields $\nu = \alpha r_a^{-1} + \beta \ge q_a$. Hence, $\alpha r + \beta \ge \nu \ge q_a$, since $r_a, r > 1$. This proves the proposition.

Case 2: the curve a is tangent to b, thus $\rho_a = r_a$. Then the coordinates (z, w) are adapted for both curves b and a. Each Taylor monomial $z^\alpha w^\beta$ of the function $f_a(z, w)$ is asymptotic to cz^ν, $\nu = \alpha + \beta r$, $c = \text{const}$, along the curve b, since $w \simeq c_b z^r$. It suffices to show that $\alpha + \beta r \ge s = q_a \min\{r_a, r\}$. When the germ a is linear, the last inequality is obvious, since $\beta \ge 1$ ($f_a = O(w)$) and $q_a = 1$. Let now a be non-linear.

Subcase 2a: $r_a \le r$. Thus, $s = q_a r_a = p_a$. One has $\alpha + \beta r \ge \alpha + \beta r_a \ge p_a = s$, by inequality (3.9) divided by q.

Subcase 2b: $r_a > r$. Thus, $\min\{\rho_a, r\} = r$, $s = q_a r$,

$$\frac{r_a}{r}(\alpha + \beta r) = \alpha\frac{r_a}{r} + \beta r_a \ge \alpha + \beta r_a \ge p_a = q_a r_a,$$

by (3.9). Multiplying the last inequality by r/r_a yields $\alpha + \beta r \ge q_a r = s$. Proposition 3.4 is proved. □

3.3. *Asymptotics of the Hessian of a local defining function*

Proposition 3.6. *Let $b \subset \mathbb{CP}^2$ be a non-linear irreducible germ of analytic curve at a point C. Let f be the irreducible germ of its defining function, $b = \{f = 0\}$, and let $H(f)$ be its Hessian defined in (3.3) in some affine chart $\mathbb{C}^2_{(x,y)}$ containing C. Let (z, w) be an affine chart on $\mathbb{CP}^2$ centered at C that is adapted to b: the projective line $T_C b$ is the z-axis. Then*

$$H(f)(u) = O((z(u))^{3q_b r - 2(r+1)}), \qquad r = r_b, \text{ as } u \in b \text{ tends to } C. \tag{3.13}$$

Proof. Everywhere below we denote by $\nabla_{\text{skew}} f = \left(\frac{\partial f}{\partial w}, -\frac{\partial f}{\partial z}\right)$ the skew gradient with respect to the standard symplectic form $dz \wedge dw$ in the coordinates (z, w). It is obtained from the previous skew gradient taken with respect to the symplectic form $dx \wedge dy$ by multiplication by the ratio of the above symplectic forms, the Jacobian of the coordinate change $(z, w) \mapsto (x, y)$. For every $u \in b$ let $L_u \subset \mathbb{C}^2$ denote the affine line tangent to b at u, and let v denote the extension of the vector $\nabla_{\text{skew}} f(u) \in T_u b = T_u L_u$ to a constant vector field on L_u. It suffices to prove formula (3.13) with its left-hand side replaced by the derivative $\frac{d^2 f}{dv^2}(u)$: for $u \in b$ the ratio of the absolute values of the last second derivative and the expression $H(f)(u)$ equals the squared modulus of the above Jacobian, which is a non-zero holomorphic function on a neighborhood of the base point C.

We evaluate the Hessian quadratic form of each Taylor monomial of the function f on $\nabla_{\text{skew}} f(u)$. We show that the expression thus obtained has asymptotics given by the right-hand side in (3.13). This will prove the proposition.

Let $z^\alpha w^\beta$ be the Taylor monomials of f. The skew gradient $(\nabla_{\text{skew}} f)|_b$ is a linear combination of the vector monomials

$$h_{\alpha,\beta} = \widetilde{h}_{\alpha,\beta} \frac{\partial}{\partial w}, \quad \widetilde{h}_{\alpha,\beta} = z^{\alpha-1} w^\beta \simeq c z^{\alpha+\beta r-1},$$
$$v_{\alpha,\beta} = \widetilde{v}_{\alpha,\beta} \frac{\partial}{\partial z}, \quad \widetilde{v}_{\alpha,\beta} = z^\alpha w^{\beta-1} \simeq c' z^{\alpha+\beta r-r}, \quad c, c' \neq 0;$$

both the above asymptotics are written along the curve b. The restrictions to the curve b of the second derivatives of a monomial $z^\alpha w^\beta$ are asymptotic to

$$\frac{\partial^2(z^\alpha w^\beta)}{\partial w^2} = \beta(\beta-1) z^\alpha w^{\beta-2} = O(z^{\alpha+\beta r-2r}),$$
$$\frac{\partial^2(z^\alpha w^\beta)}{\partial z^2} = \alpha(\alpha-1) z^{\alpha-2} w^\beta = O(z^{\alpha+\beta r-2}),$$
$$\frac{\partial^2(z^\alpha w^\beta)}{\partial z \partial w} = \alpha\beta z^{\alpha-1} w^{\beta-1} = O(z^{\alpha+\beta r-r-1}).$$

Therefore, applying the Hessian of each monomial $z^\alpha w^\beta$ to a linear combination of the vectors $h_{\alpha',\beta'}$ and $v_{\alpha',\beta'}$ yields a linear combination of expressions of the following three types:

$$\frac{\partial^2(z^\alpha w^\beta)}{\partial w^2} \widetilde{h}_{\alpha',\beta'} \widetilde{h}_{\alpha'',\beta''} = O(z^\nu), \quad \nu = (\alpha' + \beta' r - 1) + (\alpha'' + \beta'' r - 1)$$
$$+ \alpha + \beta r - 2r = (\alpha' + \beta' r) + (\alpha'' + \beta'' r) + (\alpha + \beta r) - 2(r+1); \tag{3.14}$$

$$\frac{\partial^2(z^\alpha w^\beta)}{\partial z^2} \widetilde{v}_{\alpha',\beta'} \widetilde{v}_{\alpha'',\beta''} = O(z^{\nu_2}), \quad \nu_2 = (\alpha'+\beta' r)+(\alpha''+\beta'' r)-2r+\alpha+\beta r-2 = \nu;$$
$$\frac{\partial^2(z^\alpha w^\beta)}{\partial z \partial w} \widetilde{h}_{\alpha',\beta'} \widetilde{v}_{\alpha'',\beta''} = O(z^{\nu_3}), \quad \nu_3 = (\alpha'+\beta' r)+(\alpha''+\beta'' r)+\alpha+\beta r-2r-2 = \nu.$$

Let us now estimate ν from below. Recall that for every Taylor monomial $z^\alpha w^\beta$ of the function f one has

$$\alpha + \beta r = \frac{1}{q_b}(\alpha q_b + \beta p_b) \geq p_b = q_b r$$

by (3.9), and hence the same inequality holds for (α', β') and (α'', β''). This together with formula (3.14) for the number ν implies that $\nu \geq 3q_b r - 2(r+1)$. This together with the above discussion proves formula (3.13). □

3.4. *Asymptotics of the Bialy–Mironov formula*

Everywhere below in this subsection, $C \in \gamma \cap \mathbb{I}$ is a regular point of the conic $\mathbb{I}$, and b is a local branch of the curve γ at C that is transversal to $\mathbb{I}$; (z, w) are affine coordinates centered at C and adapted to b. Recall that Δ is the zero divisor of the function G, it coincides with the zero divisor of the polynomial F_1, and $\deg F_1 = \deg(\Delta) = 2n$ (see (3.5)).

Proposition 3.7. *The right-hand side in (3.4) has the following asymptotics as $u = (x, y) \in b$ tends to C:*

$$(\mathcal{Q}(u))^{3m-3} \simeq c(z(u))^{3m-3}, \qquad c \neq 0,\ m = \frac{n}{k} = \frac{1}{2k}\deg(\Delta). \tag{3.15}$$

Proof. The degree equality in (3.15) follows from the definitions. The restriction to $T_C b$ of the differential $d\mathcal{Q}(C)$ does not vanish, since C is a regular point of the conic $\mathbb{I} = \{\mathcal{Q} = 0\}$ and b is transversal to $\mathbb{I}$. Recall that the tangent line $T_C b$ is the z-axis. Therefore, $\mathcal{Q}(u)|_b \simeq cz(u)$, $c \neq 0$, as $u \to C$. This implies the asymptotic formula in (3.15). □

Let $\sum_{j=1}^{l} s_j b_j$ denote the germ at C of the divisor Δ. Here $s_j \in \mathbb{N}$, and b_j are distinct irreducible germs of analytic curves in Δ at C numbered so that $b_1 = b$; thus, $s_1 = k$. For $j = 1, \dots, l$ let f_j denote the germ at C of a defining function of the curve b_j. Set

$$k_j = \frac{s_j}{k}, \qquad \widetilde{g} = \prod_{j=2}^{l} f_j^{k_j}; \qquad k_1 = \frac{s_1}{k} = 1; \qquad k_j = 1 \text{ whenever } b_j \subset \gamma,$$

by definition. Let F be as in (3.2).

Proposition 3.8. *Set $r = r_b$. As $u \in b$ tends to C, one has*

$$H(F)(u) \simeq c_1 \widetilde{g}^3 H(f_1)(u) = O((z(u))^{\eta}), \qquad c_1 \neq 0,$$
$$\eta = \eta(b) = 3\sum_{j=1}^{l} k_j q_{b_j} \min\{\rho_{b_j}, r\} - 2(r+1). \tag{3.16}$$

Here ρ_{b_j} are as in (3.7); $\rho_{b_1} = \rho_b = r$.

Proof. We use [11, formula (17)] valid for any two functions f_1 and β:

$$H(f_1(x, y)\beta(x, y))|_{\{f_1=0\}} = (\beta(x, y))^3 H(f_1(x, y)). \tag{3.17}$$

One has

$$F(x, y) = h(x, y) f_1(x, y) \widetilde{g}(x, y), \tag{3.18}$$

where h is a germ of holomorphic function at C, $h(C) \neq 0$. Formula (3.18) follows from the definitions (3.2). This together with (3.17) implies that

$$H(F)(u) \simeq c_1(\widetilde{g}^3 H(f_1))(u) = c_1\Big(H(f_1)\prod_{j=2}^{l} f_j^{3k_j}\Big)(u), \qquad c_1 = (h(C))^3 \neq 0.$$

Substituting formula (3.8) with $a = b_j$ and (3.13) to the above right-hand side leads to (3.16), taking into account that $k_1 = 1$ and $\rho_{b_1} = \rho_b = r$. □

Corollary 3.9. *For every local branch b as at the beginning of this subsection, the corresponding exponent $\eta = \eta(b)$ satisfies the inequality*

$$\eta = 3\sum_{j=1}^{l} k_j q_{b_j} \min\{\rho_{b_j}, r\} - 2(r+1) \leq 3m - 3 = 3\frac{\deg(\Delta)}{2k} - 3. \tag{3.19}$$

Proof. If the contrary inequality were true, then the left-hand side in (3.4) would be asymptotically dominated by the right-hand side along the branch b; this follows from formulas (3.15) and (3.16). This contradiction to (3.4) proves the corollary. □

4. Local branches and relative $\mathbb{I}$-angular symmetry

In this section we prove the following theorem.

Theorem 4.1. *Let $\mathbb{I} \subset \mathbb{CP}^2$ be a conic (either regular, or a pair of distinct lines). Let $\gamma \subset \mathbb{CP}^2$ be an irreducible algebraic curve different from a line and from $\mathbb{I}$ that generates a rationally integrable $\mathbb{I}$-angular billiard. Then every intersection point $C \in \gamma \cap \mathbb{I}$ satisfies the following statements:*

- (i) *If $\mathbb{I}$ is a union of two distinct lines through C, let b be a local branch of the curve γ at C that is transversal to both lines forming $\mathbb{I}$. Then b is quadratic.*
- (ii) *If C is a regular point of $\mathbb{I}$, then*
  - (ii-a) *each local branch of the curve γ at C that is tangent to $\mathbb{I}$ is quadratic;*
  - (ii-b) *each branch of γ at C that is transversal to $\mathbb{I}$ is regular and quadratic.*

In our assumptions for every $u \in \gamma$ the restriction to $T_u\gamma$ of the rational function G is invariant under the $\mathbb{I}$-angular symmetry with center u, and $\gamma \subset \Gamma = \{G = 0\}$. This implies that the following *relative projective symmetry property* holds: *for every $u \in \gamma$ the intersection of the projective tangent line $T_u\gamma$ with a bigger algebraic curve $\Gamma \supset \gamma$ (or a divisor) is invariant under a projective involution $T_u\gamma \to T_u\gamma$ fixing u,* the $\mathbb{I}$-angular symmetry in our case.

In Subsection 4.4 we state and prove Theorem 4.17, which unifies and generalizes statements (i) and (ii-a) of Theorem 4.1, and deduce those statements. Theorem 4.17 is stated for a non-linear germ of analytic curve b at $C \in \mathbb{CP}^2$ (which need not be algebraic) that has the local relative projective symmetry property with respect to a bigger finite collection Γ of irreducible germs of analytic curves at points in $T_C b$ (called a local multigerm) and projective involutions $T_u b \to T_u b$ fixing u with appropriate asymptotics as $u \to C$. The formal definitions of a local multigerm and the local symmetry property will be given in Subsections 4.1 and 4.3 respectively.

For the proof of Theorem 4.1 we first describe those points of intersection $T_u b \cap \Gamma$ whose z-coordinates (resp. w-coordinates) in the chart (z, w) adapted to b have asymptotics linear, sublinear and superlinear in $z(u)$ (resp. $w(u)$) as $u \in b$ tends to C. Their description, which mostly follows from the results of [25, 27], is presented in Subsection 4.1. Then in Subsection 4.3 we show that for every local branch b as in Theorem 4.1 the $\mathbb{I}$-angular symmetries of the tangent lines $T_u b$ written in appropriate affine coordinate form families of degenerating conformal involutions of two possible asymptotic types A or B. Those families of involutions are introduced in Subsection 4.2, where we prove general Propositions 4.13 and 4.14 on their asymptotics. In Subsection 4.4 we show that the collection (divisor) of asymptotic factors of points of the intersection $T_u b \cap \Gamma$ with linear asymptotics in $z(u)$ (or $w(u)$) is symmetric with respect to an appropriate conformal involution $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$, and then deduce Theorem 4.17.

The proof of statement (ii-b) takes the rest of the section, Subsections 4.5–4.8. First in Subsection 4.5 we prove that the branch b in question is subquadratic. In Subsection 4.6 we prove that every local branch of the curve Γ that is tangent to b (if any) has Puiseux exponent no greater than r_b. In Subsection 4.7 we deal with the zero divisor $\widetilde{\Delta} = \frac{1}{k}\Delta$ of the function $F_1^{1/k}$, whose germ at C contains b with multiplicity 1. We prove that its local intersection index with the tangent line to b at its base point C is no less than its half-degree plus 1, and this inequality is strict unless the germ b is regular and quadratic. The above-mentioned Puiseux exponent and intersection index inequalities will be proved in a general situation, for a germ b having the local projective symmetry property, with the projective symmetries forming a family of involutions of type A in the adapted coordinate z.

In Subsection 4.8 we prove statement (ii-b). Namely, we show that the above-mentioned Puiseux exponent and intersection index inequalities together would bring a contradiction to the upper bound (3.19) of the exponent η in the asymptotics of the Bialy–Mironov formula, unless the germ b is regular and quadratic. This will finish the proof of Theorem 4.1.

4.1. Local multigerms and asymptotics of intersections with a tangent line

Let a, b be irreducible germs of planar complex analytic curves at the origin in $\mathbb{C}^2$. Let p_g, q_g, c_g, $g = a, b$, be respectively the corresponding exponents and constants from the parametrizations (3.6) in adapted coordinates. Let t be the corresponding local parameter of the germ b. We identify points of the curve b with the corresponding local parameter values t. We use the following statements on the asymptotics of the points of the intersection $T_t b \cap a$.

Proposition 4.2 ([27, Proposition 3.8]). *Let a, b be transversal irreducible germs of holomorphic curves at the origin in $\mathbb{C}^2$, and suppose b is non-linear. Let (z, w) be affine coordinates centered at 0 and adapted to b: the germ b is tangent to the z-axis. Then for every t small enough the intersection $T_t b \cap a$ consists of q_a points $\xi_1, \dots, \xi_{q_a}$ whose coordinates have the following asymptotics as $t \to 0$:*

$$\begin{aligned} z(\xi_j) &= O(t^{p_b}) = O(w(t)) = o(z(t)) = o(t^{q_b}), \\ w(\xi_j) &= (1 - r_b) w(t)(1 + o(1)) = (1 - r_b) c_b t^{p_b}(1 + o(1)). \end{aligned} \tag{4.1}$$

(*Recall that $q_a = 1$ if a is a germ of line.*)

Proposition 4.3 ([25, p. 268, Proposition 2.50], [27, Proposition 3.10]). *Let a, b be irreducible tangent germs of holomorphic curves at the origin O in the plane $\mathbb{C}^2$, and let b be non-linear. Consider their parametrizations (3.6) in common adapted coordinates (z, w). Let c_a and c_b be the corresponding constants from (3.6). Then for every t small enough the intersection $T_t b \cap a$ consists of p_a points $\xi_1, \dots, \xi_{p_a}$ (or just one point ξ_1, if a is the germ of the line $T_O b$) whose coordinates have the following asymptotics as $t \to 0$.*

Case 1: $r_a > r_b$ *(including the case when a is linear, i.e., $r_a = \infty$). Then there are two types of intersection points ξ_j:*

$$\text{for } j \le q_a: \quad z(\xi_j) = \frac{r_b - 1}{r_b} z(t)(1 + o(1)) = \frac{r_b - 1}{r_b} t^{q_b}(1 + o(1)), \tag{4.2}$$
$$w(\xi_j) = O(t^{q_b r_a}) = o(t^{p_b}) = o(w(t));$$

$$\text{for } j > q_a: \quad z(t) = O((z(\xi_j))^{\frac{r_a - 1}{r_b - 1}}) = o(z(\xi_j)), \tag{4.3}$$
$$w(t) = O(z^{r_b}(t)) = O((z(\xi_j))^{\frac{r_b(r_a - 1)}{r_b - 1}}) = o(z^{r_a}(\xi_j)) = o(w(\xi_j)).$$

(Points satisfying (4.3) exist if and only if a is non-linear.)

Case 2: $r_a = r_b = r$. *Then*

$$z(\xi_j) = \zeta_j^{q_a} z(t)(1 + o(1)) = \zeta_j^{q_a} t^{q_b}(1 + o(1)), \tag{4.4}$$
$$w(\xi_j) = c_a \zeta_j^{p_a} t^{p_b}(1 + o(1)) = c \zeta_j^{p_a} w(t)(1 + o(1)),$$

where ζ_j are the roots of the polynomial

$$R_{p_a, q_a, c}(\zeta) = c\zeta^{p_a} - r\zeta^{q_a} + r - 1; \quad r = p_a/q_a, \quad c = c_a/c_b. \tag{4.5}$$

(When $b = a$, one has $c = 1$, and the above polynomial has double root 1 corresponding to the tangency point t.)

Case 3: $r_a < r_b$. *Then*

$$z(\xi_j) = O((z(t))^{r_b/r_a}) = o(z(t)), \tag{4.6}$$
$$w(\xi_j) = (1 - r_b) w(t)(1 + o(1)) = (1 - r_b) c_b t^{p_b}(1 + o(1)).$$

The formulas for $w(\xi_j)$ in (4.2)–(4.4), (4.6) are not contained in loc.cit. Those in (4.2)–(4.4) follow immediately from the corresponding formulas for $z(\xi_j)$. The formula for $w(\xi_j)$ in (4.6) follows from the formula for $z(\xi_j)$ and formula (4.1) applied to the curve a being the w-axis.

Definition 4.4 ([27, Definition 3.3]). Let $L \subset \mathbb{CP}^2$ be a line, and let $C \in L$. An (L, C)-*local multigerm* (resp. *divisor*) is a finite union (resp. a linear combination $\sum_j k_j b_j$ with $k_j \in \mathbb{R} \setminus \{0\}$) of distinct irreducible germs of analytic curves b_j (called *components*) at base points $C_j \in L$ such that each germ at $C_j \neq C$ is different from the line L. (A germ at C can be arbitrary, in particular, it may coincide with the germ (L, C).) The (L, C)-*localization* of an algebraic curve (resp. divisor) in $\mathbb{CP}^2$ is the corresponding (L, C)-local multigerm (resp. divisor) formed by all its local branches of the above type.

Everywhere below in the present subsection, b is a non-linear irreducible germ of analytic curve at a point $C \in \mathbb{CP}^2$, Γ is a $(T_C b, C)$-local multigerm (or divisor), and (z, w) is a local affine chart centered at C that is adapted to b: $T_C b$ is the z-axis. For every affine coordinate h, which will be either z or w, we consider its restriction to the projective lines $T_u b$.

Definition 4.5. Let h be an affine coordinate on a neighborhood of the point C in $\mathbb{CP}^2$ centered at C: $h(C) = 0$. *The points of* $\Gamma \cap T_u b$ *with linear* h*-asymptotics* are those intersection points whose h-coordinates have asymptotics $\tau_j h(u)(1 + o(1))$, $\tau_j \neq 0$, as $u \to C$; the corresponding constant factors τ_j are called the *asymptotic* h*-factors*. When Γ is a divisor, we take each factor τ_j with multiplicity which is the total multiplicity n_j of all the intersection points with the same asymptotic factor τ_j. The formal linear combination $M_h = \sum_j n_j[\tau_j]$, which is a divisor in $\mathbb{C}^*$, will be called the *asymptotic* h*-divisor.*

Definition 4.6. We say that a continuous family of points $Q = Q(u)$ of $T_u b \cap \Gamma$ has *sublinear* (resp. *superlinear*) h*-asymptotics* if $h(Q(u)) = o(h(u))$ (resp. $h(u) = o(h(Q(u)))$) as $u \to C$.

Remark 4.7. In general, the function $h(Q(u))$ can be multivalued. It can be always written as a Puiseux series in $z(u)$ (after multiplication by a power $z^s(u)$, $s \in \mathbb{Q}_{>0}$, if $h(Q(u)) \to \infty$ as $u \to C$). The above notions of family of points with sublinear, linear and superlinear h-asymptotics and the asymptotic factors are well-defined in this general case. For every given affine coordinate h on a neighborhood of the point C in $\mathbb{CP}^2$ with $h(C) = 0$ each (multivalued) continuous family of intersection points $Q(u)$ has one of the above three types.

In what follows, for a multigerm (or divisor) Γ we will denote by $\Gamma(C)$ its part consisting of the irreducible germs based at C. Recall that for every irreducible germ a in $\Gamma(C)$ we define the number ρ_a by (3.7): $\rho_a = 1$ if a is transversal to b; $\rho_a = r_a$ if a is tangent to b. Set

$$\Gamma_{\rho<r_b} = \text{ the collection (resp. divisor) of germs } a \text{ in } \Gamma(C) \text{ with } \rho_a < r_b, \tag{4.7}$$

$$\Gamma_{\rho>r_b} = \text{ the collection (resp. divisor) of germs } a \text{ in } \Gamma(C) \text{ with } \rho_a > r_b, \tag{4.8}$$

$$\Gamma_{\rho=r_b} = \text{ the collection (resp. divisor) of germs } a \text{ in } \Gamma(C) \text{ with } \rho_a = r_b, \tag{4.9}$$

$$\Gamma_{\text{out}} = \Gamma \setminus \Gamma(C), \text{ which consists of germs that are not based at } C. \tag{4.10}$$

Thus, $\Gamma_{\rho<r_b}$ consists of exactly those germs a in Γ that are based at C, and such that either

- a is transversal to b, or
- a is tangent to b and $r_a < r_b$.

All the germs in $\Gamma_{\rho>r_b}$ and $\Gamma_{\rho=r_b}$ are tangent to b.

Proposition 4.8. (1) *The points of $T_u b \cap \Gamma$ with sublinear z-asymptotics are exactly the points of intersection of the line $T_u b$ with $\Gamma_{\rho<r_b}$.*

(2) *If $\Gamma_{\rho>r_b} \neq \emptyset$, then $T_u b \cap \Gamma_{\rho>r_b}$ is split into two parts,*

$$T_u b \cap \Gamma_{\rho>r_b} = \mathcal{L}_u^< \sqcup \mathcal{L}_u^>, \qquad \mathcal{L}_u^< \neq \emptyset, \tag{4.11}$$

where

- *the points in $\mathcal{L}_u^<$ have linear z-asymptotics with z-factors equal to $\frac{r_b-1}{r_b}$;*
- *$\mathcal{L}_u^> \neq \emptyset$ if and only if $\Gamma_{\rho>r_b}$ contains a non-linear germ; the points in $\mathcal{L}_u^>$ have superlinear z-asymptotics.*

(3) *The set of points in $T_u b \cap \Gamma$ with superlinear z-asymptotics is $\mathcal{L}_u^> \sqcup (T_u b \cap \Gamma_{\text{out}})$.*

(4) *The set of points of $T_u b \cap \Gamma$ with linear z-asymptotics is $(T_u b \cap \Gamma_{\rho=r_b}) \sqcup L_u^<$.*

(5) *Let*

$$r = r_b = p/q$$

be the irreducible fraction presentation of the Puiseux exponent r_b. Let $a_1, \dots, a_N$ denote the germs forming $\Gamma_{\rho=r_b}$; they are tangent to b and $r_{a_i} = r$. Let $p_{a_i}, q_{a_i}, c_{a_i}$ be respectively the asymptotic exponents and coefficients in their parametrizations (3.6):

$$p_{a_i} = s_i p, \quad q_{a_i} = s_i q, \quad s_i \in \mathbb{N}, \quad s_i = \gcd(p_{a_i}, q_{a_i}), \quad c_{a_i} \in \mathbb{C}^*. \tag{4.12}$$

Let ζ_{ij} $(i = 1, \dots, N,\ j = 1, \dots, p)$ be the roots of the polynomials

$$R_{p,q,c(i)}(\zeta) = c(i)\zeta^p - r\zeta^q + r - 1, \qquad c(i) = c_{a_i}/c_b \in \mathbb{C}^*. \tag{4.13}$$

The asymptotic z-factors of points of $T_u b \cap \Gamma_{\rho=r_b}$ are ζ_{ij}^q.

(6) *One has*

$$\zeta_{ij}^q \neq \frac{r-1}{r} \quad \textit{for all } i \textit{ and } j. \tag{4.14}$$

Addendum to Proposition 4.8. *Under the assumptions of Proposition 4.8 in the case when Γ is a divisor, let $m_i \in \mathbb{N}$ denote the multiplicity of the germ a_i in $\Gamma_{\rho=\rho_b}$. The asymptotic z-divisor of Γ equals*

$$M_z = \sum_{i=1}^{N} \sum_{j=1}^{p} \ell_i [\zeta_{ij}^q] + \kappa_z \left[\frac{r-1}{r}\right], \qquad \ell_i = m_i s_i \in \mathbb{N},\ \kappa_z \in \mathbb{Z}_{\geq 0}, \tag{4.15}$$

$$\kappa_z = |\mathcal{L}_u^<| > 0 \quad \textit{if and only if} \quad \Gamma_{\rho>r_b} \neq \emptyset. \tag{4.16}$$

Proof. All the statements of Proposition 4.8, except for (4.14), follow from Propositions 4.2 and 4.3 (see details below). The statement (4.14) is a consequence of the following general proposition.

Proposition 4.9. *For every $p, q \in \mathbb{N}$ with $1 \leq q < p$, $c \in \mathbb{C}^*$, $r = p/q$, and every root ζ of the polynomial $R_{p,q,c}(z) = cz^p - rz^q + r - 1$ one has*

$$\zeta^q \neq \frac{r-1}{r}, \qquad c\zeta^p \neq 1 - r. \tag{4.17}$$

Proof. The proof of the first relation repeats the proof of an equivalent statement from [27, proof of Proposition 3.13]. Suppose the contrary: $\zeta^q = \frac{r-1}{r}$ for some root ζ. Then

$$R_{p,q,c}(\zeta) = c\zeta^p - r\zeta^q + r - 1 = c\zeta^p = c\left(\frac{r-1}{r}\right)^r \neq 0,$$

a contradiction. To prove the second relation, suppose the contrary: $c\zeta^p = 1-r$ for some root ζ. Then

$$R_{p,q,c}(\zeta) = c\zeta^p - r\zeta^q + r - 1 = -r\zeta^q \neq 0,$$

a contradiction again. □

Set $W_i = R_{p,q,c(i)}$ and $\widetilde{W}_i = R_{p_{a_i},q_{a_i},c(i)}$. Proposition 4.8(5) follows from Proposition 4.3, Case 2, and the relation $\widetilde{W}_i(h) = W_i(h^{s_i})$, which implies that to every root ζ of the polynomial W_i correspond s_i roots ζ^{1/s_i} of the polynomial $\widetilde{W}_i$ whose q_{a_i}-th powers are equal to ζ^q. Statements (4.15) and (4.16) follow from Proposition 4.8(4, 5), the above discussion and (4.14). □

Recall that $a_1, \dots, a_N$ denote the germs forming $\Gamma_{\rho=r_b}$.

Proposition 4.10. (1) *The set of points of $T_u b \cap \Gamma$ with sublinear w-asymptotics is exactly the set $\mathcal{L}_u^<$ from (4.11).*

(2) *The set of points of $T_u b \cap \Gamma$ with superlinear w-asymptotics is $\mathcal{L}_u^> \sqcup (T_u b \cap \Gamma_{\text{out}})$.*

(3) *The set of points of $T_u b \cap \Gamma$ with linear w-asymptotics is $T_u b \cap (\Gamma_{\rho<r_b} \sqcup \Gamma_{\rho=r_b})$. The asymptotic w-factors of the points in $T_u b \cap \Gamma_{\rho<r_b}$ are all equal to $1-r$, $r = r_b$. The asymptotic w-factors of the points in $T_u b \cap a_i$ are equal to $c(i)\zeta_{ij}^p$, $i = 1, \dots, N$, $j = 1, \dots, p$, where ζ_{ij} are the roots of the polynomials $R_{p,q,c(i)}$ (see (4.13)). One has*

$$c(i)\zeta_{ij}^p \neq 1 - r \quad \textit{for all } i \textit{ and } j. \tag{4.18}$$

(4) *When Γ is a divisor, let m_i, s_i be as in (4.15). The asymptotic w-divisor of the multigerm Γ equals*

$$M_w = \sum_{i=1}^{N}\sum_{j=1}^{p} \ell_i[c(i)\zeta_{ij}^p] + \kappa_w[(1-r)], \qquad \ell_i = m_i s_i \in \mathbb{N},\ \kappa_w \in \mathbb{Z}_{\geq 0}, \tag{4.19}$$

$$\kappa_w = |T_u b \cap \Gamma_{\rho<r_b}| > 0 \quad \textit{if and only if} \quad \Gamma_{\rho<r_b} \neq \emptyset. \tag{4.20}$$

All the statements of Proposition 4.10 follow from Propositions 4.2 and 4.3, except for (4.18) (which follows from (4.17)) and the part of (2) saying that the points in $T_u b \cap \Gamma_{\text{out}}$ have superlinear w-asymptotics which is given by the following proposition.

Proposition 4.11. *For every irreducible germ a of analytic curve at any point $B \in T_C b$, $B \neq C$, the points of $T_u b \cap a$ have superlinear w-asymptotics as $u \in b$ tends to C.*

Proof. For $u \in b$ close enough to C, let $Q_1 = Q_1(u)$ denote the point of intersection of the line $T_u b$ with the z-axis. Fix an arbitrary family of points $Q_2(u)$ of $T_u b \cap a$. Their limits $Q_1(C) = C$ and $Q_2(C) = B$ lie on the z-axis and are distinct, by assumption: $z(C) = 0 \neq z(B)$. Let us show that $w(u) = o(w(Q_2(u)))$ as $u \to C$.

Let $T = T(u)$ and $O = O(u)$ denote respectively the projections of the points u and Q_2 onto the z-axis: $z(T) = z(u)$, $z(O) = z(Q_2)$. Consider the triangles TQ_1u and OQ_1Q_2. They are similar in the following complex sense. Their edges Tu and OQ_2 lie on complex lines parallel to the w-axis. Their edges TQ_1, OQ_1 lie on the complex z-axis. Their edges uQ_1 and Q_2Q_1 lie on the same complex line Q_1Q_2. The parallelness of the complexified edges of the above triangles implies that

$$\frac{w(u) - w(T)}{w(Q_2) - w(O)} = \frac{z(T) - z(Q_1)}{z(O) - z(Q_1)}. \tag{4.21}$$

Substituting the equalities and asymptotics $w(T) = w(O) = 0$, $z(Q_1(u)) \to 0$, $z(T) = z(u) \to 0$, and $z(O(u)) - z(Q_1(u)) \to z(O(C)) = z(B) \neq 0$ into (4.21) yields $w(u)/w(Q_2) \to 0$. This proves Propositions 4.11 and 4.10. □

4.2. *Families of degenerating conformal involutions*

In Subsection 4.3 we will show that for every local branch b as in Theorem 4.1 the corresponding family of $\mathbb{I}$-angular symmetries $T_u b \to T_u b$ with center u written in appropriate coordinates becomes a degenerating family of conformal involutions $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of one of the following types.

Definition 4.12. Consider a family of non-trivial conformal involutions $\sigma_u : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of the Riemann sphere with coordinate z that are parametrized by a small complex parameter u with a given family of fixed points $\zeta(u)$:

$$\sigma_u(\zeta(u)) = \zeta(u), \qquad \zeta(u) \to 0 \quad \text{as } u \to 0.$$

The family σ_u is said to be

- *of type A* if there exist families of points $\alpha(u), \omega(u) \in \overline{\mathbb{C}}$ such that

$$\sigma_u(\alpha(u)) = \omega(u), \quad \alpha(u) = o(\zeta(u)), \quad \zeta(u) = o(\omega(u)) \quad \text{as } u \to 0;$$

- *of type B* if there exist families of points $\alpha(u), \omega(u) \in \overline{\mathbb{C}}$ such that

$$\sigma_u(\alpha(u)) = \omega(u), \quad \alpha(u), \omega(u) = o(\zeta(u)) \quad \text{as } u \to 0.$$

Proposition 4.13. *Each family of involutions $\sigma_u : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of type A with given fixed points $\zeta(u)$ satisfies the following statements:*

(a) *The involutions σ_u converge to the constant mapping $\overline{\mathbb{C}} \to 0$ uniformly on compact subsets of $\overline{\mathbb{C}} \setminus \{0\}$.*
(b) *Fix a $c \in \mathbb{C}^*$ and a family of points $z_u \in \mathbb{C}$ with asymptotics $z_u = c\zeta(u)(1 + o(1))$ as $u \to 0$. Then*

$$\sigma_u(z_u) = c^{-1}\zeta(u)(1 + o(1)) \quad \textit{as } u \to 0. \tag{4.22}$$

Proof. The scalings $\phi_u : z \mapsto \tilde{z} = z/\zeta(u)$ conjugate the involutions σ_u to the conformal involutions $\Sigma_u = \phi_u \circ \sigma_u \circ \phi_u^{-1} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ fixing 1 and permuting the points $\alpha(u)/\zeta(u)$ and $\omega(u)/\zeta(u)$, satisfying $\alpha(u)/\zeta(u) \to 0$ and $\omega(u)/\zeta(u) \to \infty$ as $u \to 0$. Hence, $\Sigma_u(z) \to 1/z$ in $\mathrm{Aut}(\overline{\mathbb{C}})$, and thus uniformly on $\overline{\mathbb{C}}$. For every $\delta > 0$ the mapping $\sigma_u = \phi_u^{-1} \circ \Sigma_u \circ \phi_u$ converges to the constant mapping $\overline{\mathbb{C}} \to 0$ uniformly on $\overline{\mathbb{C}} \setminus D_\delta$. Indeed, $\phi_u(z) = z/\zeta(u) \to \infty$ uniformly on $\overline{\mathbb{C}} \setminus D_\delta$, since $\zeta(u) \to 0$. Hence $f_u = \Sigma_u \circ \phi_u \to 0$ and $\sigma_u = \phi_u^{-1} \circ f_u = \zeta(u) f_u \to 0$. This proves (a).

For $z_u = c\zeta(u)(1 + o(1))$ with $c \neq 0$ one has

$$\sigma_u(z_u) = \zeta(u)\Sigma_u((\zeta(u))^{-1} z_u) = \zeta(u)\Sigma_u(c + o(1)) = \zeta(u)(c^{-1} + o(1)).$$

This proves (b). □

Proposition 4.14. *Each family of involutions $\sigma_u : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of type B with given fixed points $\zeta(u)$ satisfies the following statements:*

(a) *The coordinate change $\tilde{z} = \zeta(u)/z$ conjugates the involutions σ_u to conformal involutions $\Sigma_u : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ that converge in* $\mathrm{Aut}(\overline{\mathbb{C}})$ *to the central symmetry with respect to 1, $\tilde{z} \mapsto 2 - \tilde{z}$.*
(b) *For every $c \in \mathbb{C} \setminus \{0, 2\}$ and every family of points $z_u = c^{-1}\zeta(u)(1 + o(1))$ one has $\sigma_u(z_u) = d^{-1}\zeta(u)(1 + o(1))$, where $d = 2 - c$.*

Proof. The above change of coordinate $z \mapsto \tilde{z}$ sends the fixed point $\zeta(u)$ of the involution σ_u to 1, and $\tilde{z}(\alpha(u)), \tilde{z}(\omega(u)) \to \infty$ as $u \to 0$, since $\alpha(u), \omega(u) = o(\zeta(u))$. Therefore, the involution σ_u written in the coordinate $\tilde{z}$ fixes 1 and permutes two points converging to infinity. Its derivative at the fixed point 1 equals -1, since the involution is non-trivial. Therefore, it converges to the unique non-trivial involution fixing 1 and ∞, the central symmetry with respect to 1. This proves (a), which immediately implies (b). □

4.3. Relative projective symmetry properties and their types

Definition 4.15. Let b be a non-linear irreducible germ of analytic curve at a point $C \in \mathbb{CP}^2$. Let $\Delta = \sum_{j=1}^{l} k_j b_j$ be a $(T_C b, C)$-local divisor containing b, say the $b_1 = b$. We say that the germ b has the *relative projective symmetry property* with respect to the divisor Δ if for every $u \in b \setminus \{C\}$ there exists a projective involution $\sigma_u : T_u b \to T_u b$ with fixed point u such that $\Delta \cap T_u b$ treated as a divisor on $T_u b$ is σ_u-invariant. (We identify a point $u \in b$ with the corresponding value of the small complex parameter t of the curve b, $t(C) = 0$; thus, $t(u) \to 0$ as $u \to C$.) For any given affine coordinate h on a neighborhood of C in $\mathbb{CP}^2$ with $h(C) = 0$ we say that b has the relative projective symmetry property *of type A-h* (resp. *B-h*) if the family of involutions σ_u written in the coordinate h on the lines $T_u b$ is of type A (resp. B) in the sense of Definition 4.12, with $\zeta(u) = h(u)$ (the specified fixed point family).

Proposition 4.16. *Let $\mathbb{I} \subset \mathbb{CP}^2$ be a conic* (*either a regular conic, or a pair of distinct lines*)*. Suppose an irreducible algebraic curve $\gamma \subset \mathbb{CP}^2$ generates a rationally integrable $\mathbb{I}$-angular billiard with integral G, and let $C \in \gamma$. Let Δ denote the zero divisor of the* *function G. Every local branch b of the curve γ at C has the relative projective symmetry property with respect to the $(T_C b, C)$-localization (see Definition 4.4) of each of the divisors Δ and $\Delta + \mathbb{I}$; the corresponding projective involution from Definition 4.15 is the $\mathbb{I}$-angular symmetry centered at u. When $C \in \gamma \cap \mathbb{I}$, the following statements hold in the cases listed below; here (z, w) is a system of affine coordinates centered at C and adapted to b.*

Case 1: *C is a regular point of the conic $\mathbb{I}$, and b is transversal to $\mathbb{I}$. Then b has the relative projective symmetry property of type A-z.*

Case 2: *$\mathbb{I}$ is a pair of lines through the point C that are both transversal to b. Then b has the relative projective symmetry property of type B-z.*

Case 3: *C is a regular point of the conic $\mathbb{I}$, and b is tangent to $\mathbb{I}$.*

Subcase 3a: *$\mathbb{I}$ is a pair of lines. Then b has the relative projective symmetry property of type A-w.*

Subcase 3b: *$\mathbb{I}$ is a regular conic and $r_b < 2$. Then b has the relative projective symmetry property of type A-w.*

Subcase 3c: *$\mathbb{I}$ is a regular conic and $r_b > 2$. Then b has the relative projective symmetry property of type B-z.*

Proof. The first statement of the proposition follows immediately from the definitions. Let us prove other statements case by case.

Case 1: Then the line $T_C b$ intersects $\mathbb{I}$ in two points: the point C and a point $B \neq C$. Let $\mathbb{I}_C$ and $\mathbb{I}_B$ denote the germs of the conic $\mathbb{I}$ at C and B respectively. As $u \in b$ tends to C, the $\mathbb{I}$-angular symmetry of the line $T_u b$ with center u permutes its points C_u, B_u of intersection with $\mathbb{I}_C$ and $\mathbb{I}_B$. The coordinate $z(B_u)$ tends to a non-zero (may be infinite) limit, and $z(C_u) = o(z(u))$ as $u \to C$, by transversality of the germs $\mathbb{I}_C$ and b and Proposition 4.2. Therefore, the $\mathbb{I}$-angular symmetries in question written in the coordinate z form a family of conformal involutions of type A.

Case 2: As $u \to b$, the line $T_u b$ intersects $\mathbb{I}$ in two points permuted by the $\mathbb{I}$-angular symmetry. These intersection points tend to C, and their z-coordinates are $o(z(u))$, by transversality, as in the above case. Hence, the $\mathbb{I}$-angular symmetries of the lines $T_u b$ written in the coordinate z form a family of involutions of type B.

Case 3:

Subcase 3a: Then the conic $\mathbb{I}$ consists of two distinct lines intersecting at some point $B \neq C$: the line $\mathbb{I}_C = T_C b$ and a line $\mathbb{I}_B$. The $(T_C b, C)$-localization of the conic $\mathbb{I}$ consists of two germs: the germ of the line $\mathbb{I}_C$ at C and the germ of the line $\mathbb{I}_B$ at B. As $u \in b$ tends to C, the line $T_u b$ intersects $\mathbb{I}_C$ and $\mathbb{I}_B$ at points C_u and B_u respectively, which are permuted by the $\mathbb{I}$-angular symmetry with center u; $C_u \to C$ and $B_u \to B$ as $u \to C$. One has $w(C_u) = 0$, since $\mathbb{I}_C = T_C b$ is the z-axis, and $w(u) = o(w(B_u))$, by Proposition 4.11. Therefore, the $\mathbb{I}$-angular symmetries of the lines $T_u b$ written in the coordinate w form a family of involutions of type A.

Subcase 3b: Then the $(T_C b, C)$-localization of the conic $\mathbb{I}$ consists of just one regular germ at C, whose Puiseux exponent 2 is greater than r_b. As $u \in b$ tends to C, the line $T_u b$ intersects $\mathbb{I}$ in two points C_u and B_u tending to C so that $w(C_u) = o(w(u))$ and $w(u) = o(w(B_u))$, by Proposition 4.3, Case 1. The points C_u and B_u are permuted by the

$\mathbb{I}$-angular symmetry with center u. Therefore, the $\mathbb{I}$-angular symmetries of the lines $T_u b$ written in the coordinate w form a family of conformal involutions of type A.

Subcase 3c: Then $r_b > 2 = r_{\mathbb{I}}$. As $u \in b$ tends to C, both points of $T_u b \cap \mathbb{I}$ tend to C so that their z-coordinates are $o(z(u))$, by Proposition 4.3, Case 3. The latter points are permuted by the $\mathbb{I}$-angular symmetry centered at u. Therefore, these $\mathbb{I}$-angular symmetries of the lines $T_u b$ written in the coordinate z form a family of conformal involutions of type B. This proves Proposition 4.16. □

4.4. Symmetry of asymptotic divisors. Proof of (i) and (ii-a)

Here we prove the following theorem generalizing statements (i) and (ii-a) of Theorem 4.1.

Theorem 4.17. *Let b be a non-linear irreducible germ of analytic curve in $\mathbb{CP}^2$ at a point C, and let (z, w) be affine coordinates centered at C that are adapted to b. Suppose b has the local relative projective symmetry property of type either A-w or B-z. Then b is quadratic.*

We will deduce Theorem 4.17 from invariance of asymptotic divisors under appropriate conformal involutions:

Proposition 4.18. *Suppose an irreducible germ $b \subset \mathbb{CP}^2$ of analytic curve at a point C has the local relative projective symmetry property of type A-h for some affine coordinate h with $h(C) = 0$. Then its asymptotic h-divisor is invariant under the involution $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$, $z \mapsto z^{-1}$.*

Proposition 4.18 follows from Proposition 4.13(b).

Definition 4.19. For a divisor $M = \sum_j k_j[z_j]$ on $\overline{\mathbb{C}}$ its *inverse divisor* is

$$M^{-1} = \sum_j k_j[z_j^{-1}].$$

For every divisor M on $\overline{\mathbb{C}}$ and every subset $K \subset \overline{\mathbb{C}}$ we denote by $M \setminus K$ the divisor obtained from M by deleting those points that lie in K (taken with their total multiplicities).

Proposition 4.20. *Suppose an irreducible germ $b \subset \mathbb{CP}^2$ of analytic curve at a point C has the local relative projective symmetry property of type B-h for some affine coordinate h with $h(C) = 0$. Let M_h^{-1} denote the inverse to its asymptotic h-divisor M_h. The divisor $M_h^{-1} \setminus \{2\}$ is invariant under the central symmetry $\mathbb{C} \to \mathbb{C}$, $z \mapsto 2 - z$.*

Proposition 4.20 follows from Proposition 4.14(b).

Proof of Theorem 4.17. Case 1 of symmetry property of type A-w: The asymptotic w-divisor M_w being invariant under taking inverse (Proposition 4.18), the product of its points equals 1. On the other hand, that product equals the product of natural powers of expressions

$$U_i = \prod_{j=1}^{p}(c(i)\zeta_{ij}^{p}) = (c(i))^p\Big(\prod_{j=1}^{p}\zeta_{ij}\Big)^p \tag{4.23}$$

and a non-negative integer power of the number $1-r$ (see (4.19)). One has $\prod_{j=1}^{p}\zeta_{ij} = (c(i))^{-1}(r-1)$ up to sign, by Vieta's formula. Therefore, in (4.23) the number $c(i)$ cancels out and $U_i = \pm(1-r)^p$. Finally, the product of the points of the divisor M_w, which is 1, equals a natural power of $1-r$, up to sign. Hence, $r=2$ and the germ b is quadratic.

Case 2 of symmetry property of type B-z: The divisor $M_z^{-1}\setminus\{2\}$ being invariant under the symmetry with respect to 1 (Proposition 4.20), the sum of its points equals its degree. Let us write this equation explicitly and deduce that $r = r_b = 2$.

The divisor M_z^{-1} has the form

$$M_z^{-1} = \sum_i \ell_i \sum_{j=1}^{p}[\theta_{ij}^q] + \kappa_z\Big[\frac{r}{r-1}\Big], \qquad \theta_{ij} = \zeta_{ij}^{-1},\ \kappa_z \in \mathbb{Z}_{\geq 0},\ \ell_i \in \mathbb{N}$$

(see (4.15)). The numbers θ_{ij} are the roots of the polynomials

$$H_{p,q,c(i)}(\theta) = \theta^p R_{p,q,c(i)}(\theta^{-1}) = (r-1)\theta^p - r\theta^{p-q} + c(i).$$

The points of the divisor M_z^{-1} are distinct from zero. Those that are powers θ_{ij}^q are different from $\frac{r}{r-1}$, by Proposition 4.9. A priori, M_z^{-1} may contain some of the points 2 and $\frac{r-2}{r-1} = 2 - \frac{r}{r-1}$, which are symmetric to 0 and $\frac{r}{r-1}$, respectively. Define $M = M_z^{-1}\setminus\big\{2, \frac{r}{r-1}, \frac{r-2}{r-1}\big\}$, so

$$M = \text{the sum of those terms } \ell_i[\theta_{ij}^q] \text{ for which } \theta_{ij}^q \neq 2, \frac{r-2}{r-1}. \tag{4.24}$$

The divisor M is symmetric with respect to 1, as is $M_z^{-1}\setminus\{2\}$.

Lemma 4.21 ([27, Lemma 3.16]). *Let $r = p/q > 1$, $p, q \in \mathbb{N}$, $(p,q) = 1$. Consider a finite collection of polynomials $H_{p,q,c(i)}(\theta)$, $c(i) \neq 0$, and numbers $\ell_i \in \mathbb{N}$, $i = 1,\dots,N$. Let θ_{ij} denote the roots of the polynomials $H_{p,q,c(i)}$. Suppose the divisor M given by (4.24) is invariant under the symmetry of the line $\mathbb{C}$ with respect to 1. Then $r = 2$.*

Remark 4.22. In fact, Lemma 3.16 in [27] was stated in a slightly different but equivalent form. It dealt with a collection of polynomials $H_{p_i,q_i,c(i)}$, $q_i, p_i \in \mathbb{N}$, $p_i/q_i = r > 1$, $c(i) \neq 0$, and the divisor M of those q_i-th powers of their roots that are distinct from the numbers 2 and $\frac{r-2}{r-1}$. Set $s_i = \gcd(p_i, q_i)$. Those q_i-th powers of roots coincide with the q-th powers of roots of the corresponding polynomials $H_{p,q,c(i)}$, $p = p_i/s_i$, $q = q_i/s_i$, and the divisor M contains each of them s_i times. Hence, M is given by (4.24) with $\ell_i = s_i$, and this yields the equivalence of the above lemma to [27, Lemma 3.16].

Lemma 4.21 together with the symmetry of the divisor M given by (4.24) implies that $r = 2$. Theorem 4.17 is proved. □

Proof of statements (i) and (ii-a) of Theorem 4.1. Every branch b satisfying the conditions of (i) has the local relative projective symmetry property of type B-z, by Proposition 4.16, Case 2. Hence, it is quadratic, by Theorem 4.17. Statement (i) is proved.

To prove (ii-a), let b be a branch satisfying the conditions of (ii-a). Then its base point C is a regular point of the conic $\mathbb{I}$, and b is tangent to $\mathbb{I}$. We treat the following two cases separately.

Case 1: $\mathbb{I}$ is a union of two lines. Then b has the local relative projective symmetry property of type A-w, by Proposition 4.16, Subcase 3a. Hence, it is quadratic, by Theorem 4.17.

Case 2: $\mathbb{I}$ is a regular conic. Suppose the contrary: $r = r_b \neq 2$. We treat the following two subcases separately.

Subcase 2a: $r < 2$. Then b has the local relative projective symmetry property of type A-w, by Proposition 4.16, Subcase 3b. Hence, it is quadratic, by Theorem 4.17, a contradiction.

Subcase 2b: $r > 2$. Then b has the local relative projective symmetry property of type B-z, by Proposition 4.16, Subcase 3c. Hence, it is quadratic, by Theorem 4.17, a contradiction. Statements (i) and (ii-a) are proved. □

4.5. *Subquadraticity*

Here we prove the following theorem implying that every local branch b satisfying the conditions of statement (ii-b) of Theorem 4.1 is subquadratic. Recall that such a branch has the local relative projective symmetry property of type A-z (Proposition 4.16, Case 1).

In what follows, $b \subset \mathbb{CP}^2$ is a non-linear irreducible germ of analytic curve at a point C, and (z, w) are affine coordinates centered at C and adapted to b.

Theorem 4.23. *Every germ b having the local relative projective symmetry property of type A-z with respect to some $(T_C b, C)$-local divisor Γ is subquadratic.*

Proof. For a given divisor M on $\mathbb{C}$, we denote by $S(M)$ the sum of its points. The asymptotic z-divisor M_z is invariant under taking inverse (Proposition 4.18). Therefore, $S(M_z) = S(M^{-1}(z))$. Let us write down this equality explicitly. Let $a_1, \dots, a_N$ be the germs in Γ that are tangent to b and have the same Puiseux exponent $r = r_b$. Let ζ_{ij} be as in (4.15), and set $\theta_{ij} = \zeta_{ij}^{-1}$. One has

$$S(M_z) = \sum_{ij} \ell_i \zeta_{ij}^q + \kappa_z \frac{r-1}{r} = S(M_z^{-1}) = \sum_{ij} \ell_i \theta_{ij}^q + \kappa_z \frac{r}{r-1}, \tag{4.25}$$

by (4.15). Recall that for every fixed i the numbers θ_{ij} are the roots of the polynomial $(r-1)\theta^p - r\theta^{p-q} + c(i)$. Hence, the sum of their q-th powers equals $\frac{p}{r-1}$, by [27, (3.17)], and

$$S(M_z^{-1}) = \frac{\Pi}{r-1} + \kappa_z \frac{r}{r-1}, \qquad \Pi = p \sum_i \ell_i. \tag{4.26}$$

Suppose the contrary: $r > 2$, i.e., $p > 2q$. Then $\sum_j \zeta_{ij}^q = 0$ for every $i = 1, \dots, N$. Indeed, the last sum can be expressed as a polynomial in the symmetric polynomials in ζ_{ij} of degrees $1, \dots, q$. All of these symmetric polynomials vanish, as do the coefficients of the polynomial $R_{p,q,c(i)}(\zeta) = c(i)\zeta^p - r\zeta^q + r - 1$, at monomials of degrees $p-1, \dots, p-q > q$. Hence, $S(M_z) = \kappa_z \frac{r-1}{r}$. Substituting this equality and (4.26) to (4.25) yields

$$S(M_z) = \kappa_z \frac{r-1}{r} = S(M_z^{-1}) = \frac{\Pi}{r-1} + \kappa_z \frac{r}{r-1} > \kappa_z \frac{r}{r-1}.$$

The last inequality is strict, since $\Pi > 0$: the collection of germs a_i contains b, and hence is non-empty. But the right-hand side is no less than the left-hand side, since $\frac{r}{r-1} > 1 > \frac{r-1}{r}$. This contradiction proves that $r \leq 2$. □

Open problem. *Is it true that every germ b having the local relative projective symmetry property of type A-z is (a) quadratic, or (b) regular and quadratic?*

4.6. Puiseux exponents

Here we prove the following theorem implying that for every local branch b of the curve γ satisfying the conditions of statement (ii-b) one has $\Gamma_{\rho > r_b} = \emptyset$, that is, b has the maximal Puiseux exponent among all the local branches of the curve Γ that are tangent to b.

Theorem 4.24. *Let $b \subset \mathbb{CP}^2$ be a non-linear irreducible germ of analytic curve at a point C, and let (z, w) be affine coordinates centered at C and adapted to b. Suppose b has the local relative projective symmetry property of type A-z with respect to a $(T_C b, C)$-local divisor Δ. Then each irreducible germ at C tangent to b in the divisor Δ has Puiseux exponent no greater than r_b.*

The existence of a germ a in Δ tangent to b with $r_a > r = r_b$ is equivalent to the statement that the asymptotic z-divisor M_z contains the point $\theta = \frac{r-1}{r}$. Recall that its other points are the q-th powers of roots of a finite collection of polynomials $R_{p,q,c(i)}$ (see Addendum to Proposition 4.8).

We will deduce Theorem 4.24 from the following proposition.

Proposition 4.25. *Let $p, q \in \mathbb{N}$, $1 \leq q < p$, $r = p/q$, and*

$$W(z) = R_{p,q,c}(z) = cz^p - rz^q + r - 1, \quad \phi = \left(\frac{r-1}{r}\right)^{1/q}.$$

The polynomial $W(z)$ has a real root $z > \phi$ if and only if $0 < c \leq 1$. In this case it has a pair of roots $z_0 = z_0(c)$ and $z_1 = z_1(c)$ in $(\phi, +\infty)$ that are separated by 1 if $0 < c < 1$, and both equal to 1 if $c = 1$:

$$\phi < z_0(c) < 1 < z_1(c) \quad \textit{whenever } 0 < c < 1. \tag{4.27}$$

The functions $z_0(c)$ and $z_1(c)$ of $c \in (0, 1)$ are strictly increasing (resp. decreasing) homeomorphisms of $(0, 1)$ onto $(\phi, 1)$ (resp. $(1, +\infty)$).

Proof. For $c \notin \mathbb{R}_+$ one has $W|_{\{z>\phi\}} \neq 0$, since $-rz^q + r - 1 < 0$ for every $z > \phi$. Therefore, we assume that $c > 0$. As $W'(z) = cpz^{p-1} - rqz^{q-1} = pz^{q-1}(cz^{p-q} - 1)$, $c^{-1/(p-q)}$ is the unique local extremum point of W on the positive semiaxis, and it is obviously a local minimum point. For $c = 1$ one has $W(1) = 0$, and $z = 1$ is exactly the minimum point. Therefore, as c increases, the graph of W becomes disjoint from the positive coordinate semiaxis, and it has no positive root if $c > 1$. As c decreases from 1 to 0, the graph intersects the coordinate axis on both sides of 1 at two points $z_0(c)$ and $z_1(c)$ separated by the minimum point and by 1, $\phi < z_0(c) < 1 < z_1(c)$; $z_0(c)$ moves to the left, and $z_1(c)$ moves to the right. This follows from Proposition 4.9 (which implies that $z_0(c) \neq \phi$, hence $z_0(c)$ remains greater than ϕ) and the inequality $W'(z_0(c)) < 0 < W'(z_1(c))$ (which holds since $z_0(c)$ and $z_1(c)$ lie on different sides of the minimum point). The root $z_1(c)$ cannot disappear to infinity before c reaches 0, since $W(z) \to +\infty$ as $z \to +\infty$, for every fixed $c > 0$. The above discussion implies that $z_0(c)$ and $z_1(c)$ are strictly increasing (resp. decreasing) continuous mappings from $(0, 1)$ to $(\phi, 1)$ and $(1, +\infty)$ respectively. These mappings are "onto" homeomorphisms, since each $x \in (\phi, +\infty)$ is a root of $R_{p,q,c}$ with $c = \frac{rx^q - r + 1}{x^p} > 0$, and one has $c \leq 1$, as shown above. This implies the statements of Proposition 4.25. □

Proof of Theorem 4.24. Suppose the contrary. Then the asymptotic z-divisor M_z contains the point $\theta = \frac{r-1}{r}$, as noted after Theorem 4.24. There exists a strictly decreasing homeomorphism $J : [1, +\infty) \to (\theta, 1]$ such that $J(z_1^q(c)) = z_0^q(c)$ for every $c \in (0, 1]$, by Proposition 4.25. Set

$$\sigma(z) := z^{-1}, \qquad \beta := J \circ \sigma.$$

Then β is a strictly increasing mapping $[\theta, 1] \to (\theta, 1]$, and $\beta(\theta) = J(\theta^{-1}) \in (\theta, 1)$. Hence, the iterates $\beta^n(\theta) \in (\theta, 1)$ form an infinite increasing sequence of points. All of them lie in M_z, by σ-symmetry of the divisor M_z (Proposition 4.18), the inclusion $\theta \in M_z$ and the fact that the points in M_z different from θ are exactly the q-th powers of the roots of a finite collection of polynomials $W_i = R_{p,q,c(i)}$ (Addendum to Proposition 4.8). Indeed, if a point $\zeta \in [\theta, 1)$ lies in M_z, then $\sigma(\zeta) \in (1, +\infty) \cap M_z$, by symmetry. Hence, $\sigma(\zeta)$ is a q-th power of a root of some polynomial W_i. But we already know that $(\sigma(\zeta))^{1/q} > 1$ is a root of a real polynomial $W_0 = R_{p,q,c_0}$ with $0 < c_0 < 1$ (Proposition 4.25). This implies that the ratio of c_0 and $c(i)$ is a $\frac{p}{q}$-th power of a unity, and the polynomials W_i and W_0 have the same collection of q-th powers of roots. But then $\beta(\zeta) = J(\sigma(\zeta)) \in (\theta, 1)$ is a q-th power of root of the same polynomial W_0, or equivalently W_i, hence $\beta(\zeta) \in M_z$. Finally, the finite divisor M_z contains an infinite sequence of points $\beta^n(\theta)$. This contradiction proves Theorem 4.24. □

4.7. *Concentration of intersection index*

Under the conditions of Theorem 4.1(ii-b) let Δ be the zero divisor of a rational integral of the $\mathbb{I}$-angular billiard generated by γ; we normalize Δ by a positive rational factor so that b is included in Δ with multiplicity 1. Here we prove the following theorem implying that more than half of the intersection index $(\Delta, T_C b)$ is concentrated at the base point C.

Theorem 4.26. *Let $b \subset \mathbb{CP}^2$ be a non-linear irreducible germ of analytic curve at a point C. Let (z, w) be affine coordinates centered at C and adapted to b. Suppose b has the local relative projective symmetry property of type A-z with respect to an* ***effective*** *$(T_C b, C)$-local divisor $\Delta = \sum_{j=1}^{N} k_j b_j$, i.e., $k_j > 0$. Suppose Δ includes the germ b with coefficient 1. Set $D = \deg(\Delta)$: this is the intersection index $(\Delta, T_C b)$. Then the local intersection index of the projective tangent line $T_C b$ with Δ at C is no less than $D/2 + 1$. Equality may take place only when b is quadratic and regular, and Δ contains no other germs tangent to b at C with the same Puiseux exponent as b.*

Proof. Everywhere below for any effective divisor $\mathcal{D} = \sum_j n_j[\tau_j]$ on $\mathbb{C}$, $n_j > 0$, we denote by $|\mathcal{D}| = \sum_j n_j$ its degree. For every $u \in b$ close to C let $\mathcal{X} = \mathcal{X}(u)$ denote the part of the divisor $T_u b \cap \Delta$ on $T_u b$ consisting of those its points that tend to C as $u \to C$. Let $\Psi(u)$ denote the remaining part of $T_u b \cap \Delta$, consisting of those points that do not tend to C; they tend to other base points of the germs in Δ. The local intersection index $(T_C b, \Delta)_C$ at C equals the degree $|\mathcal{X}(u)|$ of the divisor $\mathcal{X}(u)$ whenever u is close enough to C.

Let $\mathcal{X}_1 = \mathcal{X}_1(u)$ and $\mathcal{X}_0 = \mathcal{X}_0(u)$ denote the parts of $\mathcal{X}(u)$ formed respectively by the points with and without the linear z-asymptotics.

Recall that the divisors $T_u b \cap \Delta$ are invariant under the projective involutions $\sigma_u : T_u b \to T_u b$ fixing u and forming a family of type A in the coordinate z.

Claim 1. *The involution σ_u sends the points of the divisor $\Psi(u)$ to some points in $\mathcal{X}_0(u)$, and $|\mathcal{X}_0(u)| \geq |\Psi(u)|$.*

Proof. The involutions σ_u written in the coordinate z converge to the constant mapping $\overline{\mathbb{C}} \to 0$ uniformly on compact subsets of $\overline{\mathbb{C}} \setminus \{0\}$ as $u \to C$, by Proposition 4.13(a). Therefore, the image of a point converging to a limit distinct from C as $u \to C$ is a point converging to C. This implies that each point of $\Psi(u)$ is sent to a point in $\mathcal{X}(u)$. That image in $\mathcal{X}(u)$ cannot lie in $\mathcal{X}_1(u)$, since the divisor $\mathcal{X}_1(u)$ of points with linear z-asymptotics is σ_u-invariant, by Proposition 4.13(b). Hence, σ_u sends $\Psi(u)$ to a part of $\mathcal{X}_0(u)$. This proves the claim. □

Thus,

$$\Delta \cap T_u b = \mathcal{X}_0(u) + \mathcal{X}_1(u) + \Psi(u), \qquad |\mathcal{X}_0(u)| \geq |\Psi(u)|,$$
$$|\mathcal{X}_0(u)| + \frac{1}{2}|\mathcal{X}_1(u)| \geq \frac{|\mathcal{X}_0(u)| + |\mathcal{X}_1(u)| + |\Psi(u)|}{2} = \frac{1}{2}|\Delta \cap T_u b| = \frac{D}{2}.$$

This implies that

$$(T_C b, \Delta)_C = |\mathcal{X}(u)| = |\mathcal{X}_0(u)| + |\mathcal{X}_1(u)| \geq \frac{D}{2} + \frac{1}{2}|\mathcal{X}_1(u)|. \tag{4.28}$$

One has $|\mathcal{X}_1(u)| \geq 2$. Indeed, the divisor $\mathcal{X}_1(u)$ of points with linear z-asymptotics includes $b \cap T_u b$ (which has degree at least two) with coefficient 1 and the intersections (with positive coefficients) of the line $T_u b$ with those germs in Δ that are tangent to b and have the same Puiseux exponent $r = r_b$. Equality may take place only if b is regular and quadratic and there are no other such germs. This together with (4.28) implies that $(T_C b, \Delta)_C \geq D/2 + 1$ and proves Theorem 4.26. □

4.8. Exponent in the asymptotics of the Bialy–Mironov formula. Proof of (ii-b)

Let b be a local branch of the curve γ at a point $C \in \gamma \cap \mathbb{I}$ that is a regular point of the conic $\mathbb{I}$, and suppose b is transversal to $\mathbb{I}$. Let $\sum_{j=1}^{l} k_j b_j$, $b_1 = b$, $k_1 = 1$, be the germ at C of the divisor $\frac{1}{k}\Delta$ (see (3.5)); here $k_j > 0$ for all j. Let ρ_{b_j} and η be the corresponding constants from formulas (3.7) and (3.16) respectively. Let us show that the upper bound (3.19) on η proved in Subsection 3.4 cannot hold unless b is regular and quadratic. Indeed, let (z, w) be affine coordinates adapted to b. The branch b has the local relative projective symmetry property of type A-z, by Proposition 4.16, Case 1. Therefore,

- $r = r_b \leq 2$, by Theorem 4.23;
- $\rho_{b_j} \leq r$ for all $j = 1, \dots, l$, by Theorem 4.24.

Substituting these into formula (3.16), one gets

$$\eta = 3\sum_{j=1}^{l} k_j q_{b_j} \min\{\rho_{b_j}, r\} - 2(r+1) \geq 3\sum_{j=1}^{l} k_j q_{b_j} \rho_{b_j} - 6. \tag{4.29}$$

The sum on the right-hand side in (4.29) equals the local intersection index of the divisor $\frac{1}{k}\Delta$ with $T_C b$ at the point C, by definition. The index is no less than $\frac{\deg(\Delta)}{2k} + 1$, by Theorem 4.26. Therefore,

$$\eta \geq 3\left(\frac{\deg(\Delta)}{2k} + 1\right) - 6 = 3\frac{\deg(\Delta)}{2k} - 3.$$

The inequality is strict unless the local branch b is regular and quadratic, as in Theorem 4.26. The strict inequality would obviously contradict inequality (3.19), and hence b is regular and quadratic. Statement (ii-b) is proved. The proof of Theorem 4.1 is complete.

5. Generalized genus and Plücker formulas. Proof of Theorem 1.26

The proof of Theorem 1.26 is based on generalized Plücker and genus formulas for planar algebraic curves and their corollaries (see, e.g., [27, Subsection 4.1]). It makes use of a modified version of Eugenii Shustin's arguments from [27, Subsection 4.2]. The main observation is that the assumptions of Theorem 4.1 on the Puiseux exponents of local branches of the curve and Plücker formulas imply that the singularity invariants of the curve γ must have a relatively high lower bound. On the other hand, the contribution of its potential singular and inflection points, which lie in the conic $\mathbb{I}$, appears not to be sufficient to fit that lower bound unless the curve is a conic.

5.1. Invariants of plane curve singularities

The material of the present subsection is contained in [27, Subsection 4.1]. It recalls classical results on invariants of singularities presented in [18, Chapter III], [36, §10]; see also a modern exposition in [28, Section I.3]. Let $\gamma \subset \mathbb{CP}^2$ be a non-linear irreducible algebraic curve.[5] Let d denote its degree. The intersection index of the curve γ with its Hessian H_γ equals $3d(d-2)$, by the Bézout Theorem. On the other hand, it is equal to the sum of the contributions $h(\gamma, C)$, which are called the *Hessians of the germs* (γ, C), over all the singular and inflection points C of the curve γ:

$$3d(d-2) = \sum_{C\in\gamma} h(\gamma, C). \tag{5.1}$$

An explicit formula for the Hessians $h(\gamma, C)$ was found in [39, (2) and Theorem 1]. To recall it, let us introduce the following notations. For every local branch b of the curve γ at C let $s(b)$ denote its multiplicity, the intersection index with a generic line through C. Let $s^*(b)$ denote the analogous multiplicity of the dual germ. Note that

$$s(b) = q, \qquad s^*(b) = p - q,$$

where p and q are the exponents in the parametrization $t \mapsto (t^q, c_b t^p(1 + o(1)))$ of the local branch b in adapted coordinates. Thus,

$$s(b) = s^*(b) \quad \text{if and only if} \quad b \text{ is quadratic,} \tag{5.2}$$

$$s(b) \geq s^*(b) \quad \text{if and only if} \quad b \text{ is subquadratic.} \tag{5.3}$$

Let $b_{C1}, \dots, b_{Cn(C)}$ denote the local branches of the curve γ at C; here $n(C)$ denotes the number of the branches. The above-mentioned formula for $h(\gamma, C)$ from [39] has the form

$$h(\gamma, C) = 3\kappa(\gamma, C) + \sum_{j=1}^{n(C)} (s^*(b_{Cj}) - s(b_{Cj})), \tag{5.4}$$

where $\kappa(\gamma, C)$ is the κ-invariant, the class of the singular point, defined as follows. Consider the germ of function f defining the germ (γ, C), i.e. $(\gamma, C) = \{f = 0\}$. Fix a line L through C that is transversal to all the local branches of γ at C. Fix a small ball $U = U(C)$ centered at C and consider a level curve $\gamma_\varepsilon = \{f = \varepsilon\} \cap U$ with small $\varepsilon \neq 0$, which is non-singular. Then $\kappa(C) = \kappa(\gamma, C)$ is the number of points of γ_ε where the tangent line is parallel to L. (One has $\kappa(C) = 0$ for C non-singular.) It is well-known that

$$\kappa(\gamma, C) = 2\delta(\gamma, C) + \sum_{j=1}^{n(C)} (s(b_{Cj}) - 1) \tag{5.5}$$

(see, for example, [28, Propositions I.3.35 and I.3.38]), where $\delta(\gamma, C) = \delta(C)$ is the δ-invariant, defined as follows. Consider the curve γ_ε, which is a Riemann surface whose boundary is a finite collection of closed curves; their number equals $n(C)$. Let us take the 2-sphere with $n(C)$ deleted disks. Let us paste it to γ_ε; this yields a compact surface.

[5] Everything stated in the present subsection holds for every algebraic curve in $\mathbb{CP}^2$ with no multiple components and no straight-line components [39, Theorem 1].

By definition, $\delta(C)$ is its genus. One has $\delta(C) \geq 0$, and $\delta(C) = 0$ whenever C is a non-singular point. Hironaka's genus formula [30] implies that

$$\sum_{C \in \mathrm{Sing}(\gamma)} \delta(\gamma, C) \leq (d-1)(d-2)/2. \tag{5.6}$$

Formulas (5.1), (5.4) and (5.5) together imply that

$$3d(d-2) = 6\sum_C \delta(\gamma, C) + 3\sum_C \sum_{j=1}^{n(C)} (s(b_{Cj}) - 1) + \sum_C \sum_{j=1}^{n(C)} (s^*(b_{Cj}) - s(b_{Cj})).$$

The first term on the right-hand side is no greater than $3(d-1)(d-2)$, by (5.6). This implies that

$$\begin{aligned} 3d(d-2) - 3(d-1)(d-2) &= 3(d-2) \\ &\leq 3\sum_C \sum_{j=1}^{n(C)} (s(b_{Cj}) - 1) + \sum_C \sum_{j=1}^{n(C)} (s^*(b_{Cj}) - s(b_{Cj})). \end{aligned} \tag{5.7}$$

5.2. *Proof of Theorem 1.26 for a union $\mathbb{I}$ of two lines*

Let $\mathbb{I}$ be a union of two distinct lines Λ_1 and Λ_2 through the point O. We know that all the singular and inflection points of the curve γ (if any) lie in $\mathbb{I} = \Lambda_1 \cup \Lambda_2$. Set

$$\begin{aligned} \mathcal{B}_{\tan} &= \{\text{the local branches of } \gamma \text{ at points } C \in \mathbb{I} \setminus \{O\} \text{ tangent to } \mathbb{I}\}, \\ \mathcal{B}_{O,\mathrm{tr}} &= \{\text{the branches of } \gamma \text{ at } O \text{ transversal to both } \Lambda_1,\ \Lambda_2\}, \\ \mathcal{B}_{O,\tan,j} &= \{\text{the branches of } \gamma \text{ at } O \text{ tangent to } \Lambda_j\}, \\ \mathcal{B}_{O,\tan} &= \bigsqcup_{j=1,2} \mathcal{B}_{O,\tan,j}, \qquad \mathcal{B}_O = \mathcal{B}_{O,\mathrm{tr}} \sqcup \mathcal{B}_{O,\tan}. \end{aligned}$$

All the local branches $b \notin \mathcal{B}_{O,\tan}$ of γ at points in $\gamma \cap \mathbb{I}$ are subquadratic, by the assumptions of Theorem 1.26. Therefore, their contributions $s^*(b) - s(b)$ to the right-hand side of (5.7), are non-positive, by (5.3). Every local branch $b \notin \mathcal{B}_{\tan} \cup \mathcal{B}_O$ is regular, by assumption, hence its contribution $s(b) - 1$ to (5.7) vanishes. This together with (5.7) implies that

$$\begin{aligned} d - 2 &\leq \sum_{b \in \mathcal{B}_{\tan} \cup \mathcal{B}_{O,\mathrm{tr}} \cup \mathcal{B}_{O,\tan}} (s(b) - 1) + \frac{1}{3} \sum_{b \in \mathcal{B}_{O,\tan}} (s^*(b) - s(b)) \\ &= \sum_{b \in \mathcal{B}_{\tan} \cup \mathcal{B}_{O,\mathrm{tr}} \cup \mathcal{B}_{O,\tan}} s(b) - |\mathcal{B}_{\tan}| - |\mathcal{B}_{O,\mathrm{tr}}| - |\mathcal{B}_{O,\tan}| + \frac{1}{3} \sum_{b \in \mathcal{B}_{O,\tan}} (s^*(b) - s(b)), \end{aligned} \tag{5.8}$$

where $|\mathcal{B}_s|$, $s \in \{\tan, (O, \mathrm{tr}), (O, \tan)\}$, are the cardinalities of the sets $\mathcal{B}_s$.

Let us estimate the right-hand side in (5.8) from above. To do this, we use the next equality, which follows from the Bézout Theorem.

For every $j = 1, 2$ we denote by $\mathcal{B}_{\mathrm{reg},j}$ the collection of local branches of the curve γ at points in $\Lambda_j \setminus \{O\}$ that are transversal to Λ_j. Recall that they are regular, by assumption. Set

$$\nu_j = |\mathcal{B}_{\mathrm{reg},j}|,$$
$$\mathcal{B}_{\tan,j} = \{b \in \mathcal{B}_{\tan} \mid b \text{ is tangent to } \Lambda_j\}, \qquad \mathcal{B}_{\tan} = \mathcal{B}_{\tan,1} \sqcup \mathcal{B}_{\tan,2}.$$

Claim 1. *For every $j = 1, 2$ one has*

$$\sum_{b\in\mathcal{B}_{\tan,j}} s(b) + \frac{1}{2}\sum_{b\in\mathcal{B}_{O,\tan,3-j}} s(b) + \frac{1}{2}\sum_{b\in\mathcal{B}_{O,\mathrm{tr}}} s(b) + \frac{\nu_j}{2} + \frac{1}{2}\sum_{b\in\mathcal{B}_{O,\tan,j}} (s^*(b) + s(b)) = \frac{d}{2}. \tag{5.9}$$

Proof. The intersection index of the curve γ with each line Λ_j equals d (Bézout Theorem). It is the sum of the intersection indices of the line Λ_j with the branches from the collections $\mathcal{B}_{\tan,j}$, $\mathcal{B}_{O,\mathrm{tr}}$, $\mathcal{B}_{O,\tan}$, $\mathcal{B}_{\mathrm{reg},j}$. Let us calculate those indices. The contribution of each branch from $\mathcal{B}_{\mathrm{reg},j}$ equals 1, by regularity and transversality. The intersection index of each branch $b \in \mathcal{B}_{O,\mathrm{tr}}$ with Λ_j equals $s(b)$. The intersection index with Λ_j of each branch $b \in \mathcal{B}_{\tan,j}$ equals $p_b = 2s(b)$, by quadraticity (assumption of Theorem 1.26). The intersection index with Λ_j of each branch $b \in \mathcal{B}_{O,\tan,j}$ equals $p_b = s(b) + s^*(b)$. The remaining branches $b \in \mathcal{B}_{O,\tan,3-j}$ are transversal to Λ_j, and their intersection indices with Λ_j are equal to $s(b)$. Summing the above intersection indices, writing that their sum should be equal to d and dividing the equality thus obtained by 2 yields (5.9). □

Summing equalities (5.9) for $j = 1, 2$ yields

$$\sum_{b\in\mathcal{B}_{\tan}\cup\mathcal{B}_{O,\mathrm{tr}}\cup\mathcal{B}_{O,\tan}} s(b) = d - \frac{1}{2}\sum_{b\in\mathcal{B}_{O,\tan}} s^*(b) - \frac{\nu_1+\nu_2}{2}. \tag{5.10}$$

Substituting (5.10) into (5.8) together with elementary inequalities yields

$$\begin{aligned} d - 2 \le d &- \frac{1}{2}\sum_{b\in\mathcal{B}_{O,\tan}} s^*(b) - \frac{\nu_1+\nu_2}{2} - |\mathcal{B}_{\tan}| - |\mathcal{B}_{O,\mathrm{tr}}| - |\mathcal{B}_{O,\tan}| \\ &+ \frac{1}{3}\sum_{b\in\mathcal{B}_{O,\tan}} (s^*(b) - s(b)) = d - |\mathcal{B}_{\tan}| - |\mathcal{B}_{O,\mathrm{tr}}| - |\mathcal{B}_{O,\tan}| \\ &- \frac{\nu_1+\nu_2}{2} - \sum_{b\in\mathcal{B}_{O,\tan}} \left(\frac{1}{6}s^*(b) + \frac{1}{3}s(b)\right), \end{aligned}$$

and so

$$|\mathcal{B}_{\tan}| + |\mathcal{B}_{O,\mathrm{tr}}| + |\mathcal{B}_{O,\tan}| + \frac{\nu_1+\nu_2}{2} + \sum_{b\in\mathcal{B}_{O,\tan}} \left(\frac{1}{6}s^*(b) + \frac{1}{3}s(b)\right) \le 2. \tag{5.11}$$

Claim 2. *The cardinality of the set of singular and inflection points of the curve γ is at most 2. Two cases are possible: either*

- *there are no inflection points, and each local branch of γ at every singular point is subquadratic; or*
- *there is just one special point (singular or inflection point), and γ has one local branch at it.*

Proof. Let Φ denote the collection of all local branches of γ at points in $\mathbb{I}$. Recall that $\mathbb{I}$ contains all the singular and inflection points of γ.

Case 1: $\mathcal{B}_{O,\tan} = \emptyset$. Then all local branches in Φ are subquadratic, and there are no inflection points; $|\mathcal{B}_{\tan}| + |\mathcal{B}_{O,\mathrm{tr}}| \leq 2$, by (5.11).

Subcase 1.1: $\mathcal{B}_{\tan} = \mathcal{B}_{O,\mathrm{tr}} = \emptyset$. Then all branches in Φ are regular and quadratic, and there are at most four of them: $\nu_1 + \nu_2 \leq 4$, by (5.11). Thus, the only possible candidates to be singular points of γ are intersections of branches. Since the total number of branches is at most 4, the number of singular points is at most 2.

Subcase 1.2: $|\mathcal{B}_{\tan}|+|\mathcal{B}_{O,\mathrm{tr}}| = 1$. The branches from $\Phi \setminus (\mathcal{B}_{\tan} \cup \mathcal{B}_{O,\mathrm{tr}})$ are transversal to the lines Λ_j, quadratic and regular, and there are at most two of them: $\nu_1 + \nu_2 \leq 2$, by (5.11). Thus, Φ consists of at most three branches, and at most one of them is singular. Thus, the only possible candidates to be singular points of γ are the base point of the unique branch from $\mathcal{B}_{\tan} \cup \mathcal{B}_{O,\mathrm{tr}}$ and a point of intersection of quadratic regular branches (if it is different from that base point). Finally, we have at most two singular points.

Subcase 1.3: $|\mathcal{B}_{\tan}| + |\mathcal{B}_{O,\mathrm{tr}}| = 2$. Then $\Phi = \mathcal{B}_{\tan} \cup \mathcal{B}_{O,\mathrm{tr}}$, by (5.11), the number of base points of the branches from the collection Φ is at most 2, and they are the only potential singular points.

Case 2: $|\mathcal{B}_{O,\tan}| \geq 1$. Then $|\mathcal{B}_{O,\tan}| = 1$, and $\Phi = \mathcal{B}_{O,\tan}$. This follows from (5.11) and positivity of the sum over $b \in \mathcal{B}_{O,\tan}$ in its left-hand side. Thus, the set Φ consists of just one branch, and we have at most one singular (or inflection) point. The claim is proved. □

Theorem 5.1 ([27, Theorem 1.6]). *Let $\gamma \subset \mathbb{CP}^2$ be an irreducible algebraic curve such that there exists a projective line L satisfying the following statements:*

- *all the singular and inflection points of γ (if any) lie in L;*
- *each local branch of γ at every point of $\gamma \cap L$ that is transversal to L is subquadratic.*

Then γ is a conic.

There exists a line L satisfying the conditions of Theorem 5.1 for the curve γ under consideration. Namely, in the first case of Claim 2 the line L is the line passing through (at most two) singular points of γ. In the second case we choose L to be the tangent line to the unique local branch at the unique special point. This together with Theorem 5.1 implies that γ is a conic. Theorem 1.26 is proved.

5.3. *Proof of Theorem 1.26: case when $\mathbb{I}$ is a regular conic*

Let $\mathbb{I} \subset \mathbb{CP}^2$ be a regular conic, and let $\gamma \subset \mathbb{CP}^2$ be an irreducible algebraic curve, $\gamma \neq \mathbb{I}$, $d = \deg \gamma$. Let $\mathcal{B}_{\mathrm{tr}}$, $\mathcal{B}_{\tan}$ denote respectively the set of those local branches of γ at base points in $\gamma \cap \mathbb{I}$ that are transversal (respectively, tangent) to $\mathbb{I}$. Let $|\mathcal{B}_{\mathrm{tr}}|$, $|\mathcal{B}_{\tan}|$ denote their cardinalities.

The proof of Theorem 1.26 in the case under consideration is based on the following inequality.

Proposition 5.2. *Let $\mathbb{I}$, γ, d be as above. Suppose each local branch in $\mathcal{B}_{\tan}$ is quadratic, and each branch in $\mathcal{B}_{\mathrm{tr}}$ is regular. Then*

$$\frac{1}{2}|\mathcal{B}_{\mathrm{tr}}| + \sum_{b\in\mathcal{B}_{\tan}} s(b) \leq d. \tag{5.12}$$

Proof. The intersection index of γ and $\mathbb{I}$ equals $2d$ (Bézout Theorem). On the other hand, it equals the sum of the intersection indices of $\mathbb{I}$ with the local branches from $\mathcal{B}_{\mathrm{tr}}$ and $\mathcal{B}_{\tan}$. Each branch in $\mathcal{B}_{\mathrm{tr}}$ has intersection index 1 with $\mathbb{I}$, since it is regular and transversal to $\mathbb{I}$, by the assumptions. Each branch $b \in \mathcal{B}_{\tan}$ has intersection index at least $2s(b)$ with $\mathbb{I}$. Indeed, b is quadratic, as is the branch of the conic $\mathbb{I}$ at the same base point. Therefore, applying coordinate change rectifying the germ of the conic $\mathbb{I}$ transforms b to a branch $\widetilde{b}$ with the same local degree $s(\widetilde{b}) = s(b)$ and Puiseux exponent $r \geq 2$. The intersection index of b and $\mathbb{I}$ equals the intersection index of $\widetilde{b}$ with its tangent line at the base point, that is, $rs(\widetilde{b}) = rs(b) \geq 2s(b)$. Finally, $2d \geq |\mathcal{B}_{\mathrm{tr}}| + 2\sum_{b\in\mathcal{B}_{\tan}} s(b)$. This proves (5.12). □

Now let us prove Theorem 1.26. Let γ be a curve as in Theorem 1.26. Recall that all the singular and inflection points of γ (if any) lie in the conic $\mathbb{I}$, and its local branches in $\mathcal{B}_{\tan}$ (resp. $\mathcal{B}_{\mathrm{tr}}$) are quadratic (resp. quadratic and regular). Let us calculate their contributions to the right-hand side of inequality (5.7) and substitute inequality (5.12). The second sum on the right-hand side in (5.7) vanishes, by quadraticity. The contribution of each $b \in \mathcal{B}_{\mathrm{tr}}$ to the first sum also vanishes, since $s(b) = 1$. The total contribution of the branches from $\mathcal{B}_{\tan}$ to the first sum equals $\sum_{b\in\mathcal{B}_{\tan}} s(b) - |\mathcal{B}_{\tan}|$. This together with (5.7) implies that

$$d - 2 \leq \sum_{b\in\mathcal{B}_{\tan}} s(b) - |\mathcal{B}_{\tan}|.$$

The right-hand side is no greater than $d - \frac{1}{2}|\mathcal{B}_{\mathrm{tr}}| - |\mathcal{B}_{\tan}|$, by (5.12). Therefore,

$$\tfrac{1}{2}|\mathcal{B}_{\mathrm{tr}}| + |\mathcal{B}_{\tan}| \leq 2. \tag{5.13}$$

Let us show that this together with Theorem 5.1 implies that γ is a conic.

Inequality (5.13) implies that the following three cases are possible.

Case 1: $|\mathcal{B}_{\mathrm{tr}}| \leq 4$, $\mathcal{B}_{\tan} = \emptyset$. Thus, all the local branches of γ at its intersection points with $\mathbb{I}$ lie in $\mathcal{B}_{\mathrm{tr}}$, and hence they are quadratic and regular. A point of $\gamma \cap \mathbb{I}$ can be singular only when it is a point of intersection of some two of (at most four) branches in $\mathcal{B}_{\mathrm{tr}}$. Hence, γ has at most two singular points (thus, all of them lie on a line), and all the local branches of γ at those points are quadratic. This together with Theorem 5.1 implies that γ is a conic.

Case 2: $|\mathcal{B}_{\tan}| = 1$, $|\mathcal{B}_{\mathrm{tr}}| \leq 2$. Let C denote the base point of the unique branch in $\mathcal{B}_{\tan}$. Each point of $\gamma \cap \mathbb{I}$ distinct from C lies in the union of (at most two) branches in $\mathcal{B}_{\mathrm{tr}}$. It is singular if and only if it is the intersection point of two such branches. Thus, γ has at most two singular points, the local branches at these points are quadratic, and hence γ is a conic, by Theorem 5.1, as in the above case.

Case 3: $|\mathcal{B}_{\tan}| = 2$, $\mathcal{B}_{\mathrm{tr}} = \emptyset$. Then γ has at most two singular points, and all the branches at those points, which lie in $\mathcal{B}_{\tan}$, are quadratic. Hence, γ is a conic, as in Case 1. Theorem 1.26 is proved.

6. Proof of the main theorems

6.1. *Rationally integrable $\mathbb{I}$-angular billiards. Proof of Theorem 1.25*

Let $\mathbb{I} \subset \mathbb{CP}^2$ be a conic (regular or a pair of distinct lines), and let $\gamma \subset \mathbb{CP}^2$ be an irreducible algebraic curve different from a line and from $\mathbb{I}$ and generating a rationally integrable $\mathbb{I}$-angular billiard.

Theorem 6.1 ([10, Theorem 1], [11, Theorem 1.2]). *All the singular and inflection points* (*if any*) *of the curve γ lie in $\mathbb{I}$.*

Remark 6.2. The above-cited theorems from [10, 11] are stated for a polynomially integrable billiard Ω: namely, for every C^2-smooth arc $\alpha \subset \partial\Omega$ with non-zero geodesic curvature the statement of Theorem 6.1 is proved there for each non-linear irreducible component γ of the Zariski closure of the Σ-dual curve α^*. But the proofs given in [10, 11] remain valid in the general context of Theorem 6.1.

Each local branch of the curve γ at a base point in $\gamma \cap \mathbb{I}$ that satisfies the conditions of some of the statements (i), (ii-a), or (ii-b) of Theorem 4.1 also satisfies the corresponding statement, by Theorem 4.1. Therefore, γ satisfies the conditions of Theorem 1.26, by Theorem 6.1. Hence, it is a conic, by Theorem 1.26. This proves Theorem 1.25.

6.2. *Confocal billiards. Proof of Theorem 1.21*

Let $\Omega \subset \Sigma$ be a polynomially integrable billiard with countably piecewise C^2-smooth boundary that contains a C^2-smooth arc α with non-zero geodesic curvature. Let $\Psi(M)$ be a non-trivial homogeneous polynomial integral of Ω of even degree $2n$: $M = [r, v]$, and $\Psi([r, v])$ is not a function of the squared norm $\|v\|^2 = \langle Av, v\rangle$ in the metric of the surface Σ. One has $\Psi(M) \not\equiv c\langle AM, M\rangle^n$, since $\langle AM, M\rangle = \langle Av, v\rangle$, by Proposition 2.1. Let G be the corresponding rational function (1.6). Then $G \not\equiv \mathrm{const}$. The complex Zariski closure of the Σ-dual curve α^* is an algebraic curve that contains at least one non-linear irreducible component. Each such component generates a rationally integrable $\mathbb{I}$-angular billiard with integral G, by Corollary 2.11. Hence, it is a conic, by Theorem 1.25. Therefore, α contains a non-geodesic conical arc. This together with Theorem 1.23 implies that the billiard Ω is countably confocal and proves Theorem 1.21.

6.3. *Case of smooth connected boundary. Proof of Theorem 1.6*

Let $\Omega \subset \Sigma$ be a polynomially integrable billiard, and suppose that $\partial\Omega$ is C^2-smooth, connected and does not lie on a geodesic. Then the billiard Ω is countably confocal, by Theorem 1.21. This means that $\partial\Omega$ contains an open dense subset R that is a disjoint union of open arcs of confocal conics and geodesic segments, including at least one non-geodesic conical arc. Let us fix such an arc and denote it by c, and let $\mathcal{C} \supset c$ denote the ambient conic. Let us show that $\partial\Omega$ coincides either with the whole conic $\mathcal{C}$, or with its connected component. We assume that c is a maximal arc of $\mathcal{C}$ that is contained in the C^2-smooth one-dimensional submanifold $\partial\Omega \subset \Sigma$. Suppose the contrary: c has an endpoint Q. The point Q cannot be an accumulation point of the union of the geodesic segments in $\partial\Omega$, by C^2-smoothness and since $\partial\Omega$ has non-zero geodesic curvature at Q, as does $\mathcal{C}$: it has quadratic tangency at Q to the geodesic tangent to $T_Q\partial\Omega$. Therefore, Q has a neighborhood U in Σ such that $I_U = \partial\Omega \cap U$ and $c_U = \mathcal{C} \cap U$ are connected, ∂U is transversal to $\partial\Omega$, and $R \cap U \subset I_U$ consists of arcs of conics confocal to $\mathcal{C}$. The ambient conics intersect U in leaves of an analytic foliation having c_U as a leaf, since each confocal conic pencil is locally given by a pair of orthogonal foliations and all the conics in question are C^1-close to $\mathcal{C}$. Thus, the C^2-smooth connected submanifold $I_U \subset U$ contains an open and dense subset $R \cap U$ where it is tangent to the above foliation. Therefore, I_U is a leaf of this foliation. The leaves $I_U = \partial\Omega \cap U$ and c_U coincide, since both contain an arc adjacent to Q of the conic $\mathcal{C}$, by construction. Finally, a neighborhood I_U of Q in $\partial\Omega$ is contained in $\mathcal{C}$. This contradicts the maximality of the conical arc $c \subset \partial\Omega$ and proves Theorem 1.6.

6.4. *Proof of complexification: Theorem 1.36*

The fact that each polynomially integrable complex billiard admits a homogeneous polynomial integral of the form $\Psi(M)$ is proved by a straightforward complexification of Bolotin's proof of the same statement in the real case [16, 17]. This implies that the curves Γ_t are algebraic, as in loc. cit., and the curves Σ-dual to the non-geodesic curves Γ_t generate rationally integrable $\mathbb{I}$-angular billiards with a common rational integral, as in the proofs of [10, Theorem 3], [11, Theorem 1.3] and Theorem 2.8. Next, confocality of the billiard is deduced from Theorem 1.25 in the same way as in Subsection 6.2, by a straightforward complexification of Theorem 1.23 and its proof. When the billiard contains no admissible complex geodesic of type (1.4), it has a non-trivial integral of degree 2 in P, as in [17, Proposition 1]. Otherwise, if it contains a complex geodesic of type (1.4), it has a non-trivial integral of degree 4 and no non-trivial integral of lower degree; the proof of this statement given in [17, p. 123] in the real case remains valid in the complex case without changes.

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