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# The least prime number represented by a binary quadratic form

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**Abstract.** Let $D < 0$ be a fundamental discriminant and $h(D)$ be the class number of $\mathbb{Q}(\sqrt{D})$. Let $R(X, D)$ be the number of classes of the binary quadratic forms of discriminant $D$ which represent a prime number in the interval $[X, 2X]$. Moreover, assume that $\pi_D(X)$ is the number of primes which split in $\mathbb{Q}(\sqrt{D})$ with norm in the interval $[X, 2X]$. We prove that

$$\left(\frac{\pi_D(X)}{\pi(X)}\right)^2 \ll \frac{R(X, D)}{h(D)}\left(1 + \frac{h(D)}{\pi(X)}\right),$$

where $\pi(X)$ is the number of primes in the interval $[X, 2X]$ and the implicit constant in $\ll$ is independent of $D$ and $X$.

**Keywords.** Quadratic forms, ideal class group, sieve theory, Siegel mass formula, theta transfer, half-integral weight Maass forms, golden quantum gates

## Contents



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## 1. Introduction

### *1.1. Motivation*

In this paper, we consider the problem of giving the optimal upper bound on the least prime number represented by a binary quadratic form in terms of its discriminant. Giving a sharp upper bound on the least prime number represented by a binary quadratic form is crucial in the analysis of the complexity of some algorithms in quantum compiling, in particular, Ross and Selinger's algorithm for the optimal navigation of $z$-axis rotations in $SU(2)$ by quantum gates [RS16], and its $p$-adic analogue for finding the shortest path between two diagonal vertices of LPS Ramanujan graphs [Sar17]. In [Sar17], we proved that these heuristic algorithms run in polynomial time under a Cramér type conjecture on the distribution of the inverse image of integers representable as a sum of two squares (or primes $p \equiv 1 \bmod 4$) by a binary quadratic form; see [Sar17, Conjecture 1.4]. Given a discriminant $D < 0$, we show that this Cramér type conjecture holds for binary quadratic forms of discriminant $D$ with a positive probability that only depends on the density of the primes that split in $\mathbb{Q}(\sqrt{D})$.

More precisely, let $\pi(X)$ be the number of primes with norm in the interval $[X, 2X]$. Let $D < 0$ be a fundamental discriminant, which means $D$ is square-free and $D \equiv 1 \bmod 4$. Let $\pi_D(X)$ be the number of primes that split in $\mathbb{Q}(\sqrt{D})$ with norm in the interval $[X, 2X]$. In [Sar18], we proved that for a given fundamental discriminant $D$, by assuming the generalized Riemann hypothesis for the zeta function of the Hilbert class field of the imaginary quadratic field $\mathbb{Q}(\sqrt{D})$, 100% of the binary quadratic forms of discriminant $D$ represent a prime number less than $h(D)\log(|D|)^{2+\epsilon}$ as $|D| \to \infty$, where $h(D)$ is the class number of $\mathbb{Q}(\sqrt{D})$. In this paper, we remove the GRH assumption and show that unconditionally with probability at least $\alpha\left(\frac{\pi_D(X)}{\pi(X)}\right)^2$ a binary quadratic form of discriminant $D < 0$ represents a prime number smaller than any fixed scalar multiple of $h(D)\log(|D|)$, where $\alpha$ is an absolute constant independent of $D$. As a result, we prove that if $\left(\frac{\pi_D(X)}{\pi(X)}\right)^2 \gg 1$ for some $X \sim h(D)\log(|D|)$ then a positive proportion of the binary quadratic forms of discriminant $D < 0$ represent a prime number smaller than any fixed scalar multiple of $h(D)\log(|D|)$. Next, we state a form of our main theorem. Let $R(X, D)$ be the number of classes of binary quadratic forms of discriminant $D$ which represent a prime number in the interval $[X, 2X]$.

**Theorem 1.1.** *We have*

$$\left(\frac{\pi_D(X)}{\pi(X)}\right)^2 \ll \frac{R(X, D)}{h(D)}\left(1 + \frac{h(D)}{\pi(X)}\right),$$

*where the implicit constant is independent of $D$ and $X$.*

**Remark 1.2.** Note that by Dirichlet's theorem, we have $\pi_D(X)/\pi(X) \sim 1/2$ as $X \to \infty$. By assuming the Riemann hypothesis or even a zero-free region of width $O(\log\log(|D|)/\log(|D|))$ for the Dirichlet $L$-function $L(s, \chi_D)$, we have $\pi_D(X)/\pi(X) \sim 1/2$ for any $X \gg |D|^{\epsilon}$ where $\epsilon > 0$. Since $h(D) \gg |D|^{1/2-\epsilon}$, under GRH we have $\pi_D(X)/\pi(X) \sim 1/2$ for any $X \sim h(D)\log(|D|)$ and it follows that the above proposed algorithms give a probabilistic polynomial time algorithm for navigating $SU(2)$ and $PSL_2(\mathbb{Z}/q\mathbb{Z})$.

Next, we show that our result is optimal up to a scalar. Namely, if a positive proportion of the binary quadratic forms of discriminant $D$ represent a prime number less than $X$, then $h(D)\log|D| \ll X$. We give a proof of this claim in what follows. Let $H(D)$ be the equivalence classes of the binary quadratic forms of discriminant $D$ by the action of $PSL_2(\mathbb{Z})$. Let $r(n, [Q])$ be the number of representations of $n$ by $[Q] \in H(D)$, and define

$$r(n, D) := \sum_{[Q]\in H(D)} r(n, [Q]).$$

By the classical formula due to Dirichlet we have

$$r(n, D) = w_D \sum_{d|n} \chi_D(d), \tag{1.1}$$

where

$$w_D = \begin{cases} 6 & \text{if } D = -3, \\ 4 & \text{if } D = -4, \\ 2 & \text{if } D < -4. \end{cases}$$

This means that the multiplicity of representing a prime number $p$ by all the classes of the binary quadratic forms of a fixed negative discriminant $D < -4$ is bounded by 4:

$$r(p, D) \le 4. \tag{1.2}$$

Assume that a positive proportion of binary quadratic forms represent a prime number smaller than $X$. Let $N(X, D)$ denote the number of pairs $(p, Q)$ such that $X < p < 2X$ is a prime number represented by $Q \in H(D)$. We proceed by giving a double counting formula for $N(X, D)$. By our assumption a positive proportion of binary quadratic forms of discriminant $D$ represent a prime number in the interval $[X, 2X]$, so

$$h(D) \ll N(X, D). \tag{1.3}$$

On the other hand,

$$N(X, D) = \sum_{p<X} r(p, D).$$

By inequality (1.2),

$$N(X, D) \le 4\pi_D(X).$$

By the above inequality and inequality (1.3), we obtain

$$h(D) \ll \pi_D(X).$$

By Siegel's lower bound $|D|^{1/2-\varepsilon} \ll h(D)$, it follows that

$$h(D)\log(|D|) \ll X.$$

This completes the proof of our claim.

### *1.2. The generalized Minkowski bound for the prime ideals*

It follows from our result that a positive proportion of the ideal classes of $\mathbb{Q}(\sqrt{D})$ contain a prime ideal with norm less than the optimal bound $h(D)\log(|D|)$. More precisely, let $H_D$ be the ideal class group of $\mathbb{Q}(\sqrt{D})$ and $N_{\mathbb{Q}(\sqrt{D})}(x+y\sqrt{D}) = x^2 - Dy^2$ be the norm of the imaginary quadratic field $\mathbb{Q}(\sqrt{D})$. Given an integral ideal $I \subset \mathcal{O}_{\mathbb{Q}(\sqrt{D})}$ with an integral basis $I \cong \langle \alpha, \beta \rangle_{\mathbb{Z}}$, which identifies $I$ with $\mathbb{Z}^2$, we define

$$q_I(x, y) := \frac{N_{\mathbb{Q}(\sqrt{D})}(x\alpha + y\beta)}{N_{\mathbb{Q}(\sqrt{D})}(I)} \in \mathbb{Z},$$

where $x, y \in \mathbb{Z}$. It follows that $[q_I] \in H(D)$ and $[q_I]$ only depends on the ideal class $[I] \in H_D$ and not on the choice of the basis $\langle \alpha, \beta \rangle_{\mathbb{Z}}$. This gives an isomorphism between $H_D$ and $H(D)$. Note that if $q_I$ represents the prime number $p$ then $q_I(x, y) = p$ for some $x, y \in \mathbb{Z}$. Then the principal ideal $(x\alpha + y\beta) = IJ$ factors into the product of $I$ and $J$ where $N_{\mathbb{Q}(\sqrt{D})}(J) = p$ and $J$ belongs to the inverse of the ideal class $[I] \in H_D$. Let $h_D(X)$ be the number of ideal classes of $H_D$ that contain a prime ideal with norm in the interval $[X, 2X]$. Hence, we have the following corollary of Theorem 1.1.

**Corollary 1.3.** *We have*

$$\left(\frac{\pi_D(X)}{\pi(X)}\right)^2 \ll \frac{h_D(X)}{h(D)}\left(1 + \frac{h(D)}{\pi(X)}\right),$$

*where the implicit constant is independent of $D$ and $X$.*

More generally, let $K$ be a number field of bounded degree $n$ over $\mathbb{Q}$ with discriminant $D_K$ and class number $h_K$. Then we have the following conjecture which generalizes Minkowski's bound for prime ideals.

**Conjecture 1.4.** *A positive proportion (depending only on $n$) of ideal classes in the ideal class group of $K$ contain a prime ideal with norm less than any fixed scalar multiple of* $h_K \log(|D_K|)$.

Next, we show that these bounds are compatible with the random model for prime numbers known as Cramér's model. We cite the following formulation of the Cramér model from [Sou07].

**Cramér Model 1.5.** *The primes behave like independent random variables $X(n)$ $(n \geq 3)$ with $X(n) = 1$ (the number $n$ is 'prime') with probability $1/\log n$, and $X(n) = 0$ (the number $n$ is 'composite') with probability $1 - 1/\log n$.*

Note that each class of the integral binary quadratic forms is associated to a Heegner point in $SL_2(\mathbb{Z})\backslash\mathbb{H}$. By the equidistribution of Heegner points in $SL_2(\mathbb{Z})\backslash\mathbb{H}$, it follows that almost all classes of the integral quadratic forms have a representative $Q(x, y) := Ax^2 + Bxy + Cy^2$ such that the coefficients of $Q(x, y)$ are bounded by any function growing faster than $\sqrt{|D|}$:

$$\max(|A|, |B|, |C|) < \sqrt{|D|}\,\psi(D)$$

for any function $\psi(D)$ defined on integers such that $\psi(D) \to \infty$ as $|D| \to \infty$. We show this claim in what follows. We consider the set of representatives of the Heegner points inside the Gauss fundamental domain of $SL_2(\mathbb{Z})\backslash\mathbb{H}$ and denote them by $z_\alpha$ for $\alpha \in H_D$. They are associated to roots of representatives of binary quadratic forms in the ideal class group. By the equidistribution of Heegner points in $SL_2(\mathbb{Z})\backslash\mathbb{H}$ and the fact that the volume of the Gauss fundamental domain decays with rate $y^{-1}$ near the cusp, it follows that for almost all $\alpha \in H_D$, if $z_\alpha = a + ib$ is the Heegner point inside the Gauss fundamental domain associated to $\alpha$ then

$$|a| \le 1/2 \quad \text{and} \quad \sqrt{3}/2 \le b \le \psi(D), \tag{1.4}$$

where $\psi(D)$ is any function such that $\psi(D) \to \infty$ as $|D| \to \infty$. Let $Q_\alpha(x, y) := Ax^2 + Bxy + Cy^2$ be the quadratic form associated to $\alpha \in H_D$ that has $z_\alpha$ as its root. Then

$$z_\alpha = \frac{-B \pm i\sqrt{|D|}}{2A},$$

where $a = \frac{-B}{2A}$ and $b = \frac{\sqrt{|D|}}{2A}$. By inequality (1.4), we have

$$|B| \le |A| \quad \text{and} \quad \frac{\sqrt{|D|}}{2\psi(D)} \le A < \sqrt{|D|}.$$

By the above inequalities and $D = B^2 - 4AC$, it follows that

$$\max(|A|, |B|, |C|) < \sqrt{|D|}\psi(D). \tag{1.5}$$

This yields our claim. Next, we give a heuristic upper bound on the size of the smallest prime number represented by a binary quadratic forms of discriminant $D$ that satisfies (1.5). Since $D$ is square-free, there is no local restriction for representing prime numbers. So, by Cramér's model and consideration of the Hardy–Littlewood local measures, we expect that for a positive proportion of the classes of binary quadratic forms $Q$ there exists an integral point $(a, b) \in \mathbb{Z}^2$ such that $|(a, b)|^2 < L(1, \chi_D)\log(|D|)$ and $Q(a, b)$ is a prime number. We have

$$\begin{aligned} Q(a, b) &= Aa^2 + Bab + Cb^2 \\ &\le \max(|A|, |B|, |C|)|(a, b)|^2 \ll \sqrt{|D|}\,L(1, \chi_D)\psi(D)\log(|D|). \end{aligned} \tag{1.6}$$

We may take $\psi(D)$ to be any constant in the above estimate. Therefore, we expect that a positive proportion of quadratic forms of discriminant $D$ represent a prime number less than $h(D)\log(|D|)$. By a similar analysis, we expect that almost all binary quadratic forms of discriminant $D$ represent a prime number less than $h(D)\log(|D|)^{2+\epsilon}$. In other words, almost all ideal classes of $\mathbb{Q}(\sqrt{D})$ contain a prime ideal with norm less than $h(D)\log(|D|)^{2+\epsilon}$. In [Sar18], we proved this result by assuming the generalized Riemann hypothesis for the zeta function of the Hilbert class field of the imaginary quadratic field $\mathbb{Q}(\sqrt{D})$. We conjectured that this type of generalized Minkowski bound holds for every number field.

**Conjecture 1.6.** *Almost all ideal classes in the ideal class group of $K$ contain a prime ideal with norm less than $h_K\log(|D_K|)^A$ for some $A > 0$. Note that by the Brauer–Siegel Theorem and GRH, we have $h_K \ll \sqrt{|D_K|}\log(|D_K|)^{\epsilon}$.*

### *1.3. Repulsion of the prime ideals near the cusp*

As we noted above, based on Cramér's model we expect that the split prime numbers are randomly distributed among the ideal classes of $\mathbb{Q}(\sqrt{D})$, and hence with a positive probability that is independent of $D$, a quadratic form of discriminant $D$ represents a prime number less than a fixed scalar multiple of $h(D)\log(|D|)$. We may hope that every ideal class contain a prime ideal of size $h(D)|D|^{\epsilon}$. Note that Cramér's conjecture states that every short interval of size $\log(X)^{2+\epsilon}$ contains a prime number. By Linnik's conjecture, every congruence class modulo $q$ contains a prime number less than $q^{1+\epsilon}$. This shows that small prime numbers cover all short intervals and congruence classes. However, we note that the family of binary quadratic forms of discriminant $D < 0$ is different from the family of short intervals and its $p$-adic analogue. Small primes do not cover all the classes of binary quadratic forms. For example, the principal ideal class that is associated to the binary quadratic form $Q(x, y) = Dx^2 + y^2$ repels prime numbers, which means the least prime number represented by this form is bigger than $|D|$ compared to $\sqrt{|D|}\log(|D|)^{2+\varepsilon}$ that is the upper bound for almost all binary quadratic forms under GRH. This feature is different from the analogous conjectures for the size of the least prime number in a given congruence classes modulo an integer (Linnik's conjecture) and the distribution of prime numbers in short intervals (Cramér's conjecture). We call this new feature the *repulsion* of small primes by the cusp. In fact, the binary quadratic forms with the associated Heegner point near the cusp repel prime numbers. This can be seen in equation (1.6), where $\max(|A|, |B|, |C|)|$ could be as large as $|D|$ near the cusp but for a typical binary quadratic form it is bounded by $|D|^{1/2+\epsilon}$. This shows that the bound in Conjecture 1.6 does not hold for every ideal class.

### *1.4. Method of the proof*

Our method is based on our recent work on the distribution of prime numbers in short intervals. In [Sar], we proved that a positive proportion of the intervals of length equal to any fixed scalar multiple of $\log(X)$ in the dyadic interval $[X, 2X]$ contain a prime number. We also showed that a positive proportion of the congruence classes modulo $q$ contain a prime number smaller than any fixed scalar multiple of $\varphi(q)\log(q)$. These results are compatible with Cramér's model.

We briefly describe our method here. We proceed by introducing some new notations and follow the previous ones. Let $w(u)$ be a positive smooth weight function that is supported on $[1, 2]$ and $\int w(u)\,du = 1$. Let $w_X(u) := w(u/X)$ that is derived from $w(u)$ by scaling with $X$. Let $\pi(Q, w, X)$ denote the number of primes weighted by $w_X$ that are representable by the binary quadratic form $Q$. By the Cauchy–Schwarz inequality, we obtain

$$\Big(\sum_{Q\in H(D)} \pi(Q, w, X)\Big)^2 \leq R(X, D)\Big(\sum_{Q\in H(D)} \pi(Q, w, X)^2\Big). \tag{1.7}$$

By Dirichlet's formula in (1.1), $\sum_{Q\in H(D)} \pi(Q, w, X)$ is the weighted number of prime numbers inside the interval $[X, 2X]$ that split in the quadratic field $\mathbb{Q}(\sqrt{D})$. So, we have

$$\pi_D(X) \sim \sum_{Q\in H(D)} \pi(Q, w, X). \tag{1.8}$$

Next, we give a double counting formula for $\sum_{Q\in H(D)} \pi(Q, w, X)^2$. This sum counts pairs of primes $(p_1, p_2)$ weighted by $w_X(p_1)w_X(p_2)$ such that $p_1$ and $p_2$ are represented by the same binary quadratic form class $[Q] \in H(D)$. Assume that $Q$ is a representative of that class that represents two prime numbers $p_1$ and $p_2$. Without loss of generality, we assume that $Q(x, y) = p_1x^2 + \alpha xy + \beta y^2$ for some integers $\alpha$ and $\beta$ such that

$$D = \alpha^2 - 4p_1\beta, \tag{1.9}$$

since by the action of $SL_2(\mathbb{Z})$ on the space of the integral binary quadratic forms we can find a representative of $Q$ with the above form. Since $Q$ represents $p_2$, $p_2 = p_1u^2 + \alpha uv + \beta v^2$ for some integers $u$ and $v$. We multiply both side of the previous identity by $4p_1$ and obtain

$$4p_1p_2 = 4p_1^2u^2 + 4p_1\alpha uv + 4p_1\beta v^2.$$

We use identity (1.9), and substitute $\alpha^2 - D = 4p_1\beta$ in the above identity to obtain

$$4p_1p_2 = 4p_1^2u^2 + 4p_1\alpha uv + (\alpha^2 - D)v^2.$$

Hence,

$$4p_1p_2 = (2p_1u + \alpha v)^2 - Dv^2.$$

We change variables to $s := 2p_1u + \alpha v$, and obtain

$$4p_1p_2 = s^2 - Dv^2. \tag{1.10}$$

On the other hand, if $(p_1, p_2)$ is a solution to the equation (1.10) for prime numbers $X < p_1 < 2X$ and $X < p_2 < 2X$, then $p_1$ and $p_2$ are represented by the same binary quadratic form class in $H(D)$. Heuristically, this number is about $\pi_D(X)^2/h(D) + \pi_D(X)$, that is, the number of distinct pairs of split primes inside the interval $[X, 2X]$ divided by the number of classes of binary quadratic forms of discriminant $D$ plus the contribution of diagonal terms where $p_1 = p_2$. Therefore, we expect

$$\sum_{Q\in H(D)} \pi(Q, w, X)^2 \approx \frac{\pi_D(X)^2}{h(D)} + \pi_D(X).$$

In fact, by applying the Selberg upper bound sieve on the number of prime solutions $(p_1, p_2)$ to the equation (1.10), we show that

$$\sum_{Q \in H(D)} \pi(Q, w, X)^2 \ll \frac{\pi(X)^2}{h(D)} + \pi(X). \tag{1.11}$$

Therefore, by inequality (1.7), equation (1.8) and the above inequality, it follows that

$$\left(\frac{\pi_D(X)}{\pi(X)}\right)^2 \ll \frac{R(X, D)}{h(D)}\left(1 + \frac{h(D)}{\pi(X)}\right).$$

This gives a proof of Theorem 1.1. Next, we briefly explain how we prove inequality (1.11). We begin by counting the number of solutions $(p_1, p_2, s, v)$ to the equation (1.10) weighted by the smooth weight function $w_X$ where $v = 0$. We call them *diagonal solutions*. If $v = 0$, then $4p_1p_2 = s^2$. Hence, $p_1 = p_2 = p$ for some prime number $p$ and $s = \pm 2p$. Therefore, the number of diagonal solutions to the equation (1.10) is the number of prime numbers weighted by $w_X$, that is, $\pi(w_X) \approx \pi(X)$. Next, we give an upper bound on the number of non-diagonal terms $v \neq 0$ weighted by $w_X(p_1)w_X(p_2)$. Since $D < 0$ and $w_X(p_1)w_X(p_2) \neq 0$ only if $X < p_1, p_2 < 2X$ then

$$|s| \leq 4X \quad \text{and} \quad |v| \leq 4X/\sqrt{|D|}. \tag{1.12}$$

We fix $v = v_0$ for some $0 \leq v_0 \leq 4X/\sqrt{|D|}$, and let $m := Dv_0^2 < 0$. Let $\mathcal{P}(m, w_X)$ be the number of prime solutions $(p_1, p_2)$ to $s^2 - 4p_1p_2 = m$ weighted by $w_X(p_1)w_X(p_2)$. Let

$$V_m := \{(x, y, z) : q(x, y, z) = m\}, \tag{1.13}$$

where $q(x, y, z) := z^2 - 4xy$. Let $Y := |D|^\delta$, where $\delta > 0$ is a fixed small power; e.g. $\delta < 1/620$. Let $S(m, Y, w_X)$ denote the number of integral solutions $(x, y, z)$ to the equation in (1.13) weighted by $w_X(x)w_X(y)$, where $x$ and $y$ do not have any prime divisor smaller than $Y$. If $X < \sqrt{|m|}/2$, then $\mathcal{P}(m, w_X) = S(m, Y, w_X) = 0$. Otherwise, $X \geq \sqrt{|m|}/2 > Y$, and we have $\mathcal{P}(m, w_X) \leq S(m, Y, w_X)$. We apply the Selberg upper bound sieve to give a sharp upper bound up to a constant on $S(m, Y, w_X)$. We briefly discuss the Selberg upper bound sieve in what follows. Assume that $d_1, d_2$ and $d$ are square-free integers. Let $\#_{w_X} A_{d_1,d_2;m}$ denote the number of integral solutions $(x, y, z)$ to the equation (1.13) weighted by $w_X(x)w_X(y)$, where $d_1 \mid x$ and $d_2 \mid y$. Similarly, let $\#_{w_X} A_{d;m}$ be the same number where $d|xy$. We write $\#_{w_X} A_m$ for $\#_{w_X} A_{d;m}$ when $d = 1$. It follows from the inclusion-exclusion principle that (see [BF94, Lemma 8, p. 79])

$$\#_{w_X} A_{d;m} = \mu(d) \sum_{\mathrm{lcm}[d_1,d_2]=d} \mu(d_1)\mu(d_2)\#_{w_X} A_{d_1,d_2;m}. \tag{1.14}$$

Let $\chi_Y(\cdot)$ be the indicator function of integers with no prime divisor less than $Y$. Let $\{\lambda_d\}$ be any sequence of real numbers for $d \geq 1$, where $\lambda_1 = 1$. We have the following upper bound on $\chi_Y(n)$,

$$\chi_Y(n) \leq \Big(\sum_{d|\gcd(n, \prod_{p<Y} p)} \lambda_d\Big)^2.$$

Hence,

$$
\begin{aligned}
S(m, Y, w_X) &= \sum_{z^2-4xy=m} \chi_Y(xy) w_X(x) w_X(y) \\
&\le \sum_{z^2-4xy=m} \Big( \sum_{d \mid \gcd(xy, \prod_{p<Y} p)} \lambda_d \Big)^2 w_X(x) w_X(y) = \sum_d \mu^+(d) \#_{w_X} A_{d;m}, \qquad (1.15)
\end{aligned}
$$

where

$$\mu^+(d) := \sum_{\mathrm{lcm}[d_1,d_2]=d} \lambda_{d_1} \lambda_{d_2}.$$

In Theorem 3.2, we give an asymptotic formula for $\#_{w_X} A_{d_1,d_2;m}$ with a power saving error term if $d_1 d_2 \le |D|^{1/308}$. The proof of that theorem is the main technical part of our work. We apply the Siegel Mass formula to the ternary quadratic form $V_{m,d_1d_2} := \{(x, y, z) : z^2 - d_1 d_2 xy = 0\}$ in order to give the main term of $\#_{w_X} A_{d_1,d_2;m}$ as the product of Hardy–Littlewood local densities. To give a power saving upper bound on the error term we use the spectral theory of modular forms and Duke's subconvex upper bounds on the Fourier coefficients of weight 1/2 Maass forms and Eisenstein series and our upper bound on the $L^2$ norm of the theta lift of weight 1/2 Maass forms. We give the outline of the proof of Theorem 3.2 in the next section. By assuming these results the main term of the weighted number of integral points comes from the product of the local densities with a power saving error term Er,

$$\#_{w_X} A_{d_1,d_2;m} = \sigma_{\infty,w_X}(V_{m,d_1d_2}) \prod_p \sigma_p(V_{m,d_1d_2}) + \mathrm{Er}, \qquad (1.16)$$

where

$$
\begin{aligned}
\sigma_{\infty,w_X}(V_{m,d_1d_2}) &= \lim_{\epsilon\to 0} \frac{\int_{m<z^2-4d_1d_2xy<m+\varepsilon} w_{X/d_1}(x) w_{X/d_2}(y)\, dx\, dy\, dz}{\epsilon}, \\
\sigma_p(V_{m,d_1d_2}) &:= \lim_{k\to\infty} \frac{|V_{m,d_1d_2}(\mathbb{Z}/p^k\mathbb{Z})|}{p^{2k}}. \qquad (1.17)
\end{aligned}
$$

We explicitly compute these local densities in terms of the quadratic character $\chi_D$ and as a result we have

$$\#_{w_X} A_{d;m} = \#_{w_X} A_m \frac{\omega(d)}{d} + \mathrm{Er}, \qquad (1.18)$$

where $\omega(\cdot)$ is explicit and is called the *sieve density*. For a square-free integer $l$, define

$$g(l) := \frac{\omega(l)}{l} \prod_{p\mid l} \Big(1 - \frac{\omega(p)}{p}\Big)^{-1}, \qquad (1.19)$$

and let $G(Y) := \sum_{l=1}^{Y} g(l)$, where the sum is over square-free integers $l$. By the fundamental theorem for the Selberg sieve [FI10, Theorem 7.1], there exists a sequence $\lambda_d \in \mathbb{R}$ with $\lambda_1 = 1$ such that

$$S(m, Y, w_X) \le \sum_d \mu^+(d) \#_{w_X} A_{d;m} \le \frac{\#_{w_X} A_m}{G(Y)} + \mathrm{Er}.$$

In Lemma 2.7, we show that

$$L(1,\chi_D)^2\log(|D|)^2\frac{\varphi(v_0)}{v_0}\ll G(Y).$$

Finally, by summing over $|v_0|\ll X/\sqrt{|D|}$ and proving the analogue of Gallagher's result on the average size of the Hardy–Littlewood singular series [Gal76, equation (3)], we prove inequality (1.11) and hence Theorem 1.1.

### *1.5. Outline of the paper*

In Section 2, we give the proof of Theorem 1.1 by assuming Theorem 3.2. In Lemma 2.1, we compute $\sigma_{\infty,w_X}$, the Hardy–Littlewood measure at the archimedean place. In Lemma 2.3, we give an explicit formula for $\sigma_p$ in terms of the quadratic character $\chi_D$. In Lemma 2.4, we give an explicit formula for $\#_{w_X}A_{d;m}$ involving $L(1,\chi_D)$. In Lemma 2.6, we compute the sieve densities $\omega(d)$ defined in (1.18). In Lemma 2.7, we give a sharp upper bound on the main term of the Selberg sieve. Finally, we prove that the average size of these singular series is bounded (the analogue of Gallagher's theorem), and prove Theorem 1.1.

In Section 3, we prove Theorem 3.2 which implies equation (1.16). Let $q_k(x,y,z)=z^2-4kxy$, $V_{m,k}:=\{(x,y,z):q_k(x,y,z)=m\}$ and $\Gamma_k:=SO_{q_k}(\mathbb{Z})$ be the integral points of the orthogonal group of $q_k$. Then $\Gamma_k$ is a lattice and $\Gamma_k\backslash V_{m,k}$ has a natural hyperbolic structure with finitely many possible elliptic and cusp points. We construct an automorphic function $W$ defined on $\Gamma_k\backslash V_{m,k}$ from the smooth function $w_X$. We spectrally expand $W$ in the basis of eigenfunctions of the Laplace–Beltrami operator on $\Gamma_k\backslash V_{m,k}$. We denote the contribution of the constant function by the main term and the contribution of the non-trivial eigenfunctions (Maass cusp forms and Eisenstein series of $\Gamma_k\backslash V_{m,k}$) by Er.

In Section 3.1, we prove a generalized class number formula in Proposition 3.5. This theorem gives the main term of $\#_{w_X}A_{d_1,d_2;m}$ defined in equation (1.16). We briefly describe the proof of Proposition 3.5. The proof uses the Siegel Mass formula that gives a product formula for the sum of the representation number of an integer $n$ by a quadratic form averaged over the genus class of $q_k$. In Lemma 3.4, we show that the genus class of $q_k(x,y,z)=z^2-4kxy$ contains only one element for every $k\in\mathbb{Z}$, and Proposition 3.5 follows from the Siegel Mass formula.

Our goal for the rest of Section 3 is to give a power saving upper bound on Er. This power saving in the error term is crucial for the application of the Selberg sieve in Section 2. We write Er as the sum of the low and the high frequency eigenfunctions in the spectrum,

$$\mathrm{Er}=\mathrm{Er}_{\mathrm{high}}+\mathrm{Er}_{\mathrm{low}}+\mathrm{Er}_{\mathrm{cts,low}},$$

where the terms are defined in (3.3).

In Section 3.2, we give an upper bound on the contribution of $\mathrm{Er}_{\mathrm{high}}$. The upper bound follows from integration by parts. We show that $\mathrm{Er}_{\mathrm{high}}=O(1)$. In Section 3.3, we prove an explicit form of the Maass identity that relates the Weyl sums to the Fourier coefficients of the associated half-integral weight Maass form obtained by the theta transfer using the Siegel theta kernel. In Section 3.4, we give an upper bound on $\mathrm{Er}_{\mathrm{low}}$. We apply Duke's non-trivial upper bound on the Fourier coefficients of the weight $1/2$ Maass form and the upper bound on the $L^2$ norm of the theta transfer of a Maass form that we prove in Section 4 to bound $\mathrm{Er}_{\mathrm{low}}$.

In Section 3.5, we give an upper bound on $\mathrm{Er}_{\mathrm{cts,low}}$. We need to regularize the theta integral for bounding $\mathrm{Er}_{\mathrm{cts,low}}$. We use the center of the enveloping algebra (Casimir operator) for regularizing this theta integral. This method has been used in the work of Maass [Maa71], Deitmar and Krieg [DK91] and Kudla and Rallis [KR94, Section 5]. Finally, we use Duke's non-trivial upper bound on the Fourier coefficients of the weight $1/2$ Eisenstein series.

There is a technical issue in using Duke's result. The bound is exponentially growing in the eigenvalue aspect with the term $\cosh(\pi t/2)$ for the half-integral weight eigenfunctions $\psi_\lambda$ with norm 1 and eigenvalue $1/4 + t^2$. We show that this term cancels with the exponentially decaying factor $\cosh(-\pi t/2)$ that appears in $|\Theta * \psi_\lambda|^2$, the $L^2$ norm of the theta transfer of $\psi_\lambda$. This is the content of Section 4.

In Section 4, we give an upper bound on the $L^2$ norm of the theta transfer of a weight $1/2$ Maass form $f$ in the eigenvalue and the level aspect up to a polynomial in these parameters. In Lemma 4.1, we compute the Mellin transform of the theta lift of $f$ by a see-saw identity that is originally due to Niwa [Niw75] and used by Katok and Sarnak [KS93]. The see-saw identity in this case identifies the Mellin transform with the inner product of an Eisenstein series against the product of the weight 1/2 modular form $f$ and the complex conjugate of the Jacobi theta series $\bar{\theta}$. The last integral against the Eisenstein series is explicitly computable by unfolding the Eisenstein series. Hence, we obtain the Fourier coefficients of the theta transfer at the cusp at infinity. Finally, we bound the $L^2$ norm of a modular form by bounding the truncated sum of the squares of its Fourier coefficients; see [Iwa02, p. 110, equation 8.17]. Note that the $L^2$ norm of the theta transfer of a new form is given by the Rallis–Inner product formula. Since we also deal with old forms, we rather use a more direct approach.

## 2. Deducing Theorem 1.1 from Theorem 3.2 via the Selberg sieve

Recall (1.16),

$$\#_{w_X} A_{d_1,d_2;m} = \sigma_{\infty,w_X} \prod_p \sigma_p(V_{m,d_1 d_2}) + \mathrm{Er},$$

where $\sigma_{\infty,w_X}$ and $\sigma_p(V_{m,d_1 d_2})$ were defined in (1.17). In this section, we give the proof of Theorem 1.1 by assuming the above formula and a power saving upper bound on Er.

### 2.1. *Local densities*

We proceed by computing the local densities $\sigma_{\infty,w_X}$ and $\sigma_p$. Let

$$I(a) := \int_1^2 \int_1^2 \frac{1}{2\sqrt{4x_1x_2 + a}^{+}} w(x_1)w(x_2)\, dx_1\, dx_2,$$

where

$$\sqrt{y}^{+} := \begin{cases} \sqrt{y} & \text{if } y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

**Lemma 2.1.** *We have*

$$\sigma_{\infty, w_X} = \frac{X}{d_1 d_2} I\left(\frac{m}{X^2}\right).$$

*Proof.* We change the variables to $u := d_1 x$ and $v := d_2 y$. Then

$$\begin{aligned} \sigma_{\infty, w_X} &= \lim_{\epsilon \to 0} \frac{\int_{m < z^2 - 4d_1 d_2 xy < m + \varepsilon} w_{X/d_1}(x) w_{X/d_2}(y)\, dx\, dy\, dz}{\epsilon} \\ &= \frac{1}{d_1 d_2} \lim_{\epsilon \to 0} \frac{\int_{m < z^2 - 4uv < m + \varepsilon} w_X(u) w_X(v)\, du\, dv\, dz}{\epsilon}. \end{aligned}$$

Next, we scale the coordinates by $1/X$ and define $x_1 = u/X$, $x_2 = v/X$ and $x_3 = z/X$. Hence,

$$\begin{aligned} \sigma_{\infty, w_X} &= \frac{1}{d_1 d_2} \lim_{\epsilon \to 0} \frac{\int_{m < z^2 - 4uv < m + \varepsilon} w_X(u) w_X(v)\, du\, dv\, dz}{\epsilon} \\ &= \frac{X}{d_1 d_2} \lim_{\epsilon' \to 0} \frac{\int_{m/X^2 < x_3^2 - 4x_1 x_2 < m/X^2 + \varepsilon'} w(x_1) w(x_2)\, dx_1\, dx_2\, dx_3}{\epsilon'} \\ &= \frac{X}{d_1 d_2} \int_1^2 \int_1^2 \frac{1}{2\sqrt{4x_1 x_2 + m/X^2}^{+}} w(x_1) w(x_2)\, dx_1\, dx_2, \end{aligned}$$

where $\epsilon' := \epsilon/X^2$. Then $\sigma_{\infty, w_X} = \frac{X}{d_1 d_2} I\big(\frac{m}{X^2}\big)$. It follows that $I$ is smooth and is bounded by a constant that only depends on the smooth function $w$. □

Next, we compute explicitly the local density $\sigma_p$ at each odd prime $p$. We have

$$\sigma_p = \sum_{t=0}^{\infty} S(p^t),$$

where $S(1) := 1$ and

$$S(p^t) := \frac{1}{p^{3t}} \sum_a{}^{*} \sum_b e\left(\frac{a(q_{d_1 d_2}(b) - n)}{p^t}\right),$$

where $a$ runs over integers modulo $p^t$ with $\gcd(a, p) = 1$, and $b$ runs over vectors in $\mathbb{Z}^3$ modulo $p^t$. Since $p$ is an odd prime number, we can diagonalize our quadratic form $q_{d_1 d_2}(X)$ over the local ring $\mathbb{Z}_p$ by changing the variables to $x_1 = z$ , $x_2 = x - y$ and $x_3 = x + y$ and obtain

$$q_{d_1 d_2}(x_1, x_2, x_3) = x_1^2 + d_1 d_2 x_2^2 - d_1 d_2 x_3^2.$$

We apply the following lemma for the computation of local densities. For other versions for this lemma see [TS18, Lemma 3.1] and Blomer [Blo08, (1.8)].

**Lemma 2.2.** *Let*

$$Q(x_1, x_2, x_3) = x_1^2 + p^\alpha d x_2^2 - p^\alpha d x_3^2,$$

*where* $\alpha \in \mathbb{Z}$ *with* $\alpha \geq 0$ *and* $d \in \mathbb{Z}_p$ *with* $\gcd(d, p) = 1$. *Assume that* $n = p^\beta n'$ *where* $n' \in \mathbb{Z}_p$ *with* $\gcd(n', p) = 1$. *Let* $V_n$ *be the quadric*

$$V_n := \{(x_1, x_2, x_3) : Q(x_1, x_2, x_3) = n\},$$

*defined over* $\mathbb{Z}_p$. *Then*

$$\sigma_p(V_n) := \lim_{t\to\infty} \frac{V_n(\mathbb{Z}/p^t\mathbb{Z})}{p^{2t}} = 1 + \sum_{t=1}^{\infty} S(p^t),$$

*where*

$$S(p^t) := \frac{1}{p^{3t}} \sum_a{}^{*} \sum_b e\left(\frac{a(Q(b) - n)}{p^t}\right).$$

*Moreover if* $t$ *is odd, then*

$$S(p^t) = \begin{cases} \left(\frac{n'}{p}\right) \frac{p^{\min(\alpha+t,2t)} p^{t/2}}{p^{3t}} p^{t-1/2} & \text{if } \beta = t-1,\\ 0 & \text{otherwise.}\end{cases}$$

*where* $\left(\frac{n'}{p}\right)$ *denote the Legendre symbol of* $n'$ *modulo* $p$, *and if* $t$ *is even then*

$$S(p^t) = \begin{cases} 0 & \text{if } \beta < t-1,\\ -\frac{p^{\min(\alpha+t,2t)} p^{t/2}}{p^{3t}} p^{t-1} & \text{if } \beta = t-1,\\ \frac{p^{\min(\alpha+t,2t)} p^{t/2}}{p^{3t}} \phi(p^t) & \text{if } \beta \geq t.\end{cases}$$

*Proof.* We compute

$$\begin{aligned} S(p^t) &:= \frac{1}{p^{3t}} \sum_a{}^{*} \sum_{b \in (\mathbb{Z}/p^t\mathbb{Z})^3} e\left(\frac{a(Q(b)-n)}{p^t}\right)\\ &= \frac{1}{p^{3t}} \sum_a{}^{*} \sum_{b \in (\mathbb{Z}/p^t\mathbb{Z})^3} e\left(\frac{a(b_1^2 + p^\alpha d b_2^2 - p^\alpha d b_3^2 - n)}{p^t}\right)\\ &= \frac{1}{p^{3t}} \sum_a{}^{*} e\left(\frac{-an}{p^t}\right) \prod_{i=1}^{3} \sum_{b \bmod p^t} e\left(\frac{a a_i p^{\alpha_i} b^2}{p^t}\right), \end{aligned}$$

where $a_1 := 1$, $\alpha_1 := 0$, $a_2 := d$, $\alpha_2 := \alpha$, $a_3 := -d$ and $\alpha_3 = \alpha$. We note that the last summation is a Gauss sum. Let $G(h, m) := \sum_{x \bmod m} e(hx^2/m)$ be the Gauss sum, and let $\varepsilon_m = 1$ if $m \equiv 1 \bmod 4$ and $\varepsilon_m = i$ if $m \equiv 3 \bmod 4$. Then if $\gcd(h, m) = 1$, we have

$$G(h, m) := \begin{cases} \varepsilon_m \left(\frac{h}{m}\right) m^{1/2} & \text{if } m \text{ is odd},\\ (1 + \chi_{-4}(h)) m^{1/2} & \text{if } m = 4^\alpha,\\ (\chi_8(h) + i\chi_{-8}(h)) m^{1/2} & \text{if } m = 2\cdot 4^\alpha, \alpha \geq 1,\end{cases}$$

where $\left(\frac{h}{m}\right)$ is the Jacobi symbol. We define $G(h, p^{t-\alpha_i}) := 1$ when $t < \alpha_i$. We have

$$S(p^t) = \frac{1}{p^{3t}} \sum_a^* e\left(\frac{-an}{p^t}\right) \prod_{i=1}^{3} p^{\min(\alpha_i, t)} G(a a_i, p^{t-\alpha_i}).$$

We substitute the values of $G$ and obtain

$$S(p^t) = \frac{\prod_{i=1}^{3} p^{\min(\frac{\alpha_i+t}{2}, t)} \varepsilon_{p^{t-\alpha_i}}}{p^{3t}} \sum_a^* e\left(\frac{-an}{p^t}\right)\left(\frac{a}{p}\right)^t \left(\frac{-1}{p}\right)^{t-\alpha},$$

By our assumption we have $n = p^\beta n'$, where $\gcd(n', p) = 1$. If $t$ is an odd number, then the inner sum is a Gauss sum, and we obtain

$$\sum_{a \bmod p^t}^* e\left(\frac{-a p^\beta n'}{p^t}\right)\left(\frac{a}{p}\right) = \begin{cases} \varepsilon_p \left(\frac{-n'}{p}\right) p^{t-1/2} & \text{if } \beta = t-1, \\ 0 & \text{otherwise.} \end{cases}$$

Note $\varepsilon_p^2 \left(\frac{-1}{p}\right) = 1$ and $\varepsilon_{p^{t-\alpha}}^2 \left(\frac{-1}{p}\right)^{t-\alpha} = 1$. Hence if $t$ is odd, we deduce that

$$S(p^t) = \begin{cases} \left(\frac{n'}{p}\right) \frac{p^{\min(\alpha+t, 2t)} p^{t/2}}{p^{3t}} p^{t-1/2} & \text{if } \beta = t-1, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, if $t$ is even then the inner sum is a Ramanujan sum $c_{p^t}(n)$:

$$c_{p^t}(n) = \sum_a^* e\left(\frac{-an}{p^t}\right) = \begin{cases} 0 & \text{if } \beta < t-1, \\ -p^{t-1} & \text{if } \beta = t-1, \\ \phi(p^t) & \text{if } \beta \geq t. \end{cases}$$

Hence if $t$ is even, it follows that

$$S(p^t) = \begin{cases} 0 & \text{if } \beta < t-1, \\ -\frac{\prod_{i=1}^{3} p^{\min(\frac{\alpha_i+t}{2}, t)}}{p^{tk}} p^{t-1} & \text{if } \beta = t-1, \\ \phi(p^t) \frac{\prod_{i=1}^{3} p^{\min(\frac{\alpha_i+t}{2}, t)}}{p^{3t}} & \text{if } \beta \geq t. \end{cases}$$ □

In the following lemma, we apply Lemma 2.2 and give an explicit formula for the local densities $\sigma_p(V_{m,d_1d_2})$.

**Lemma 2.3.** *Let $\alpha(d_1d_2) := \mathrm{Ord}_p(d_1d_2)$, and $\beta(m) := \mathrm{Ord}_p(m)$. Then*

$$\sigma_p(V_{m,d_1d_2}) = \begin{cases} 1 + \frac{1}{p} + \frac{\chi_D(p)}{p^{k+1}} - \frac{1}{p^{k+1}} & \text{if } \alpha(d_1d_2) = 0 \text{ and } \beta(m) = 2k, \\ 2 + \frac{\chi_D(p)}{p^k} - \frac{1}{p^k} & \text{if } \alpha(d_1d_2) = 1 \text{ and } \beta(m) = 2k, \\ p + 1 + \frac{\chi_D(p)}{p^{k-1}} - \frac{1}{p^{k-1}} & \text{if } \alpha(d_1d_2) = 2 \text{ and } \beta(m) = 2k, \\ 1 + \frac{1}{p} - \frac{1}{p^{k+1}} - \frac{1}{p^{k+2}} & \text{if } \alpha(d_1d_2) = 0 \text{ and } \beta(m) = 2k+1, \\ 2 - \frac{1}{p^k} - \frac{1}{p^{k+1}} & \text{if } \alpha(d_1d_2) = 1 \text{ and } \beta(m) = 2k+1, \\ p + 1 - \frac{1}{p^{k-1}} - \frac{1}{p^k} & \text{if } \alpha(d_1d_2) = 2 \text{ and } \beta(m) = 2k+1. \end{cases}$$

*Proof.* By Lemma 2.2, we have

$$\sigma_p(V_{m,d_1d_2}) = \sigma_p(\alpha(d_1d_2), \beta(m)).$$

If $\alpha = 0$ and $\beta = 0$, it follows that

$$\sigma_p(0,0) = 1 + \frac{\chi_D(p)}{p}.$$

More generally, we have

$$\sigma(0,2k) = 1 + \frac{1}{p} + \frac{\chi_D(p) - 1}{p^{k+1}}.$$

Moreover, if $\alpha = 1$ or 2 and $\beta = 0$ then

$$\sigma_p(1,0) = \sigma_p(2,0) = 1 + \chi_D(p).$$

More generally,

$$\sigma_p(1,2k) = 2 + \frac{\chi_D(p)}{p^k} - \frac{1}{p^k}.$$

We also have, for $k \geq 1$,

$$\sigma_p(2,2k) = p + 1 + \frac{\chi_D(p)}{p^{k-1}} - \frac{1}{p^{k-1}}.$$

Next, we compute the local densities for $\beta = 2k+1$ and $\alpha = 0, 1, 2$. We have

$$\sigma(0,1) = 1 - 1/p^2, \qquad \sigma(1,1) = 1 - 1/p, \qquad \sigma(2,1) = 0.$$

In general, we have

$$\begin{aligned}
\sigma(0,2k+1) &= 1 + \frac{1}{p} - \frac{1}{p^{k+1}} - \frac{1}{p^{k+2}},\\
\sigma(1,2k+1) &= 2 - 1/p^k - 1/p^{k+1},\\
\sigma(2,2k+1) &= 1 + p - \frac{1}{p^{k-1}} - \frac{1}{p^k}.
\end{aligned}$$

□

In the following lemma, we give an asymptotic formula for $\#_{w_X} A_m = \#_{w_X} A_{d_1,d_2;m}$ where $d_1 = d_2 = 1$.

**Lemma 2.4.** *We have*

$$\#_{w_X} A_m = XW\left(\frac{m}{X^2}\right) L(1,\chi_D) \frac{6}{\pi^2} \prod_{\beta(p)\geq 2} \left(1 - \frac{1}{p^2}\right)^{-1} \left(1 - \frac{\chi_D(p)}{p}\right) \sigma_p + \mathrm{Er}, \quad (2.1)$$

*where $m = Dv_0^2$. As a result,*

$$\#_{w_X} A_m \ll XW\left(\frac{m}{X^2}\right) L(1,\chi_D) \left(\frac{v_0}{\varphi(v_0)}\right)^2.$$

*Proof.* By formula (1.16), we have

$$\#_{w_X} A_m = \sigma_{\infty, w_X} \prod_p \sigma_p + \mathrm{Er},$$

where $\sigma_p = \sigma_p(\alpha, \beta)$ for $\alpha(p) = 0$ and $\beta(p) = \mathrm{Ord}_p(Dv_0^2)$. By Lemmas 2.1 and 2.3,

$$\sigma_{\infty, w_X} = XW\left(\frac{m}{X^2}\right), \quad \sigma(0,0) = 1 + \frac{\chi_D(p)}{p}, \quad \sigma(0,1) = 1 - \frac{1}{p^2}.$$

By substituting the above values in the product formula, we obtain

$$\#_{w_X} A_m = XW\left(\frac{m}{X^2}\right) \prod_{\beta(p)=0} \left(1 + \frac{\chi_D(p)}{p}\right) \prod_{\beta(p)=1} \left(1 - \frac{1}{p^2}\right) \prod_{\beta(p)\geq 2} \sigma_p + \mathrm{Er}.$$

We simplify the above product formula by applying the following Euler product identities:

$$L(1, \chi_D) = \prod_p \left(1 - \frac{\chi_D(p)}{p}\right)^{-1}, \quad \prod_p \left(1 - \frac{1}{p^2}\right) = \frac{6}{\pi^2}.$$

Hence,

$$\begin{aligned}
\#_{w_X} A_m &= XW\left(\frac{m}{X^2}\right) L(1, \chi_D) \prod_p \left(1 - \frac{\chi_D(p)}{p}\right) \\
&\quad \times \prod_{\beta(p)=0} \left(1 + \frac{\chi_D(p)}{p}\right) \prod_{\beta(p)=1} \left(1 - \frac{1}{p^2}\right) \prod_{\beta(p)\geq 2} \sigma_p + \mathrm{Er} \\
&= XW\left(\frac{m}{X^2}\right) L(1, \chi_D) \frac{6}{\pi^2} \prod_{\beta(p)\geq 2} \left(1 - \frac{1}{p^2}\right)^{-1} \left(1 - \frac{\chi_D(p)}{p}\right) \sigma_p + \mathrm{Er}.
\end{aligned}$$

This completes the proof of the identity (2.1). By Lemma 2.3 if $\beta(p) \geq 2$, then

$$\sigma_p = 1 + 1/p + O(1/p^2).$$

Hence,

$$\#_{w_X} A_m \ll XW\left(\frac{m}{X^2}\right) L(1, \chi_D) \prod_{\beta(p)\geq 2} \left(1 + \frac{2}{p}\right) \ll XW\left(\frac{m}{X^2}\right) L(1, \chi_D) \left(\frac{v_0}{\varphi(v_0)}\right)^2.$$

This completes the proof of our lemma. ◻

Recall that from identity (1.14), we have

$$\#_{w_X} A_{d;m} = \mu(d) \sum_{\mathrm{lcm}[d_1, d_2] = d} \mu(d_1) \mu(d_2) \#_{w_X} A_{d_1, d_2; m}.$$

In the following lemma, we give an asymptotic formula for $\#_{w_X} A_{d;m}$.

**Lemma 2.5.** *We have*

$$\#_{w_X} A_{d;m} = \#_{w_X} A_m \frac{\omega(d)}{d} + \mathrm{Er},$$

*where*

$$\omega(d) = \prod_{p|d} \frac{2\sigma_p(1,\beta) - \sigma_p(2,\beta)/p}{\sigma_p(0,\beta)}. \tag{2.2}$$

*Proof.* Let $d_1$ and $d_2$ be two square-free integers. By product formula (1.16), we have

$$\#_{w_X} A_{d_1,d_2;m} = \sigma_{\infty,w_X} \prod_p \sigma_p(\alpha,\beta) + \mathrm{Er},$$

where $\alpha(p) = \mathrm{Ord}_p(d_1d_2)$ and $\beta(p) = \mathrm{Ord}_p(Dv_0^2)$. Hence,

$$\#_{w_X} A_{d_1,d_2;m} = \frac{\#_{w_X} A_m}{d_1d_2} \prod_{p|d_1d_2} \frac{\sigma_p(\alpha,\beta)}{\sigma_p(0,\beta)} + \mathrm{Er}.$$

We substitute the above product formula in (1.14) and obtain

$$\begin{aligned}
\#_{w_X} A_{d;m} &= \mu(d) \sum_{\mathrm{lcm}[d_1,d_2]=d} \mu(d_1)\mu(d_2)\#_{w_X} A_{d_1,d_2;m} + \mathrm{Er} \\
&= \mu(d)\#_{w_X} A_m \sum_{\mathrm{lcm}[d_1,d_2]=d} \frac{\mu(d_1)\mu(d_2)}{d_1d_2} \prod_{p|d_1d_2} \frac{\sigma_p(\alpha,\beta)}{\sigma_p(0,\beta)} + \mathrm{Er} \\
&= \frac{\#_{w_X} A_m}{d} \prod_{p|d} \frac{2\sigma(1,\beta) - \sigma_p(2,\beta)/p}{\sigma_p(0,\beta)} + \mathrm{Er}.
\end{aligned}$$

This completes the proof of our lemma. □

In the following lemma, we give an explicit formula for $\omega(p)$ that is defined in (2.2).

**Lemma 2.6.** *We have*

$$\omega(p) = \begin{cases} \frac{2+2\chi_D(p)-1/p-\chi_D(p)/p}{1+\chi_D(p)/p} & \text{if } \beta(p) = 0, \\ \frac{2}{1+1/p} & \text{if } \beta(p) = 1, \\ \frac{3-1/p+\chi_D(p)/p^k-1/p^k}{1+1/p+\chi_D(p)/p^{k+1}-1/p^{k+1}} & \text{if } \beta(p) = 2k \text{ for } k \geq 1, \\ \frac{3-1/p-3/p^k+2/p^{k+1}}{1+1/p-1/p^{k+1}-1/p^{k+2}} & \text{if } \beta(p) = 2k+1 \text{ for } k \geq 1. \end{cases}$$

*Proof.* By definition of $\omega(p)$ in (2.2), we have

$$\omega(p) = \frac{2\sigma_p(1,\beta) - \sigma_p(2,\beta)/p}{\sigma_p(0,\beta)}.$$

We substitute the explicit values of $\sigma_p(\alpha,\beta)$ from Lemma 2.3 and obtain the explicit values of $\omega(p)$. □

Finally, we give an upper bound on the main term of the sieve. For a square-free integer $l$, define

$$g(l) := \frac{\omega(l)}{l} \prod_{p|l} \left(1 - \frac{\omega(p)}{p}\right)^{-1}, \tag{2.3}$$

and let $G(Y) := \sum_{l=1}^{Y} g(l)$, where the sum is over square-free variables $l$. In the following lemma, we give an asymptotic formula for $G(Y)$.

**Lemma 2.7.** *Let $Y = |D|^{\delta}$ for some fixed $\delta > 0$ and $G(Y)$ be as above. Then*

$$L(1, \chi_D)^2 \log(|D|)^2 \frac{\varphi(v_0)}{v_0} \ll_{\delta} G(Y).$$

*Proof.* First, we estimate the value of $g(p)$ at primes $p$. By (2.3), we have

$$g(p) = \frac{\omega(p)}{p - \omega(p)} \geq 0.$$

By Lemma 2.6, we have

$$g(p) = \begin{cases} \frac{2(1+\chi_D(p))}{p} + O(1/p^2) & \text{if } \beta(p) = 0, \\ \frac{2}{p} + O(1/p^2) & \text{if } \beta(p) = 1, \\ \frac{3}{p} + O(1/p^2) & \text{if } \beta(p) = 2k \text{ for } k \geq 1, \\ \frac{3}{p} + O(1/p^2) & \text{if } \beta(p) = 2k+1 \text{ for } k \geq 1, \end{cases} \tag{2.4}$$

where the implicit constant involved in $O(1/p^2)$ is independent of all variables. Next, we apply Rankin's trick and relate the truncated sum $G(Y)$ to an Euler product. Note that

$$G(Y) \geq \sum_{\substack{n \\ p|n \Rightarrow p \leq Y^{1/10}}} \mu(n)^2 g(n) \left(\frac{1}{n^{10/\log(Y)}} - e^{-10}\right).$$

Then

$$G(Y) \geq \prod_{p \leq Y^{1/10}} \left(1 + \frac{g(p)}{p^{10/\log(Y)}}\right) - e^{-10} \prod_{p \leq Y^{1/10}} (1 + g(p)).$$

Since $\frac{\exp(x)}{1+x}$ is an increasing function in $x \geq 0$, we have

$$\begin{aligned} \prod_{p \leq Y^{1/10}} (1 + g(p)) \left(1 + \frac{g(p)}{p^{10/\log(Y)}}\right)^{-1} &\leq \exp\left(\sum_{p \leq Y^{1/10}} g(p) \left(1 - \frac{1}{p^{10/\log(Y)}}\right)\right) \\ &\leq \prod_{p \leq Y^{1/10}} \left(4 \sum_{p \leq Y^{1/10}} \frac{1}{p} \cdot \frac{10 \log(p)}{\log(Y)}\right) \sim e^4, \end{aligned}$$

where we used the prime number theorem and the fact that $g(p) \leq 4/p$. Hence,

$$G(Y) \geq \frac{1}{2} \prod_{p \leq Y^{1/10}} \left(1 + \frac{g(p)}{p^{10/\log(Y)}}\right).$$

Next, we complete the above Euler product by extending the product over primes $Y^{1/10} < p$ . Note that

$$\prod_{Y^{1/10}<p}\left(1+\frac{g(p)}{p^{10/\log(Y)}}\right)\leq\exp\left(\sum_{Y^{1/10}<p}\frac{g(p)}{p^{10/\log(Y)}}\right)\leq\exp\left(\sum_{Y^{1/10}<p}\frac{4}{p^{1+10/\log(Y)}}\right)$$
$$\leq 2\log(2),$$

where we used the fact that $\sum_{p<N}\frac{1}{p}=\log\log(n)+O(1)$ and $g(p)\leq\frac{4}{p}$. Therefore,

$$G(Y)\gg\prod_{p}\left(1+\frac{g(p)}{p^{10/\log(Y)}}\right). \tag{2.5}$$

Next, we complexify this Euler product and consider $G(s)$, the Dirichlet series associated to the multiplicative function $g$:

$$G(s):=\sum_{l}\mu(l)^2g(l)l^{-s}=\prod_{p}\left(1+\frac{g(p)}{p^s}\right).$$

We write

$$G(s)=\zeta(s+1)^2L(s+1,\chi_{-D})^2\eta(s)\tilde{G}(s), \tag{2.6}$$

where

$$\eta(s)=\prod_{\beta(p)\geq 2}\left(1+\frac{g(p)}{p^s}\right)\left(1-\frac{1}{p^{s+1}}\right)^2\left(1-\frac{\chi_{-D}(p)}{p^{s+1}}\right)^2,$$
$$\tilde{G}(s)=\prod_{\beta(p)\leq 1}\left(1+\frac{g(p)}{p^s}\right)\left(1-\frac{1}{p^{s+1}}\right)^2\left(1-\frac{\chi_{-D}(p)}{p^{s+1}}\right)^2.$$

We analyze the Dirichlet series $\eta(s)$ and $\tilde{G}(s)$. First, we give an upper bound on $|\eta(s)|$. Recall that $\beta(p)=\mathrm{Ord}_p(Dv_0^2)$ and $D$ is square-free. Let $p$ be a prime number such that $\beta(p)\geq 2$. Hence, $p\mid v_0^2$ and by equation (2.4), we have

$$\eta(s)=\prod_{p|v_0}\left(1+\frac{g(p)}{p^s}\right)\left(1-\frac{1}{p^{s+1}}\right)^2\left(1-\frac{\chi_{-D}(p)}{p^{s+1}}\right)^2$$
$$=\prod_{p|v_0}\left(1+\frac{1-2\chi_{-D}(p)}{p^{s+1}}+O\left(\frac{1}{p^{s+2}}\right)\right).$$

Hence, for $\sigma>0$ we have

$$\eta(\sigma+it)\gg\prod_{p|v_0}\left(1-\frac{1}{p}\right)=\frac{\varphi(v_0)}{v_0}.$$

In particular,

$$\eta(10/\log(Y))\gg\frac{\varphi(v_0)}{v_0}. \tag{2.7}$$

Next, we analyze $\tilde{G}(s)$. Assume that $p$ is a prime number such that $\beta(p) \leq 1$. By (2.4), it follows that

$$\left(1+\frac{g(p)}{p^s}\right)\left(1-\frac{1}{p^{s+1}}\right)^2(1-\frac{\chi_{-D}(p)}{p^{s+1}})^2 = 1+O\left(\frac{1}{p^{s+2}}\right).$$

Hence,

$$\tilde{G}(s) \ll 1 \quad \text{and} \quad \tilde{G}(s)^{-1} \ll 1$$

for $\mathrm{Re}(s) > -1+\epsilon$, where the implicit constants depend only on $\epsilon > 0$. In particular,

$$\tilde{G}\left(\frac{10}{\log(Y)}\right) \ll 1.$$

By (2.5)–(2.7) and the above inequality, it follows that

$$G(Y) \gg \zeta\left(1+\frac{10}{\log(Y)}\right)^2 L\left(1+\frac{10}{\log(Y)}, \chi_D\right)^2 \frac{\varphi(v_0)}{v_0}.$$

Since $Y = |D|^\delta$ we have $\zeta\big(1+\frac{10}{\log(Y)}\big)^2 \gg \big(\frac{\delta \log(|D|)}{10}\big)^2$ and it follows that

$$G(Y) \gg_\delta \log(|D|)^2 L\left(1+\frac{10}{\log(Y)}, \chi_D\right)^2 \frac{\varphi(v_0)}{v_0}. \tag{2.8}$$

Finally, we make the observation that any completed $L$-function is increasing in $\sigma \geq 1$. This is a consequence of the fact that all zeros are to the left of 1. More precisely, for $D$ a negative discriminant one looks at

$$\Lambda(s, \chi_D) := \frac{|D|^{s/2}}{\pi}\Gamma\left(\frac{s+1}{2}\right)L(s, \chi_D).$$

Then $\Lambda(\sigma, \chi_D)$ is increasing in $\sigma \geq 1$. The proof is an application of the Hadamard factorization formula, which shows that $\Lambda(\sigma, \chi_D) = \prod_\rho |1-\sigma/\rho|$, and since all the zeros have real part in $(0, 1)$, each term $|1-\sigma/\rho|$ is increasing in $\sigma \geq 1$. Therefore,

$$\Lambda(1, \chi_D) \leq \Lambda\left(1+\frac{10}{\log(Y)}, \chi_D\right).$$

In other words,

$$L(1, \chi_D) \ll |D|^{5/\log(Y)} L\left(1+\frac{10}{\log(Y)}, \chi_D\right).$$

Since $Y = |D|^\delta$ we get $|D|^{5/\log(Y)} = e^{5\delta}$. By the above inequality and (2.8), we have

$$L(1, \chi_D)^2 \log(|D|)^2 \frac{\varphi(v_0)}{v_0} \ll_\delta G(Y).$$

This completes the proof of our lemma. □

### 2.2. *Proof of Theorem 1.1*

Recall that $S(m, Y, w_X)$ is the weighted number of integral solutions $(x, y, z)$ to $z^2 - 4xy = m$, where $x$ and $y$ do not have a prime divisor smaller than $Y$ and $m = Dv_0^2$. By inequality (1.15), we have

$$S(m, Y, w_X) \le \sum_d \mu^+(d) \#_{w_X} A_{d;m}.$$

By the fundamental theorem for Selberg sieve [FI10, Theorem 7.1], we have

$$S(m, Y, w_X) \le \frac{\#_{w_X} A_m}{G(Y)} + O_\epsilon(X^{1-\epsilon}),$$

for some $\epsilon > 0$. By Lemmas 2.4 and 2.7, we have

$$\#_{w_X} A_m \ll X W\left(\frac{m}{X^2}\right) L(1, \chi_D) \left(\frac{v_0}{\varphi(v_0)}\right)^2,$$
$$L(1, \chi_D)^2 \log(|D|)^2 \frac{\varphi(v_0)}{v_0} \ll_\delta G(Y).$$

Therefore,

$$S(m, Y, w_X) \ll \frac{X W(m/X^2)}{\log(|D|)^2 L(1, \chi_D)} \left(\frac{v_0}{\varphi(v_0)}\right)^3,$$

where $m = Dv_0^2$. By inequality (1.12), we have $v_0 \le 4X/\sqrt{|D|}$. We sum the above inequality for $0 \le v_0 \le 4X/\sqrt{|D|}$, and obtain

$$\sum_{Q \in H(D)} \pi(Q, w, X)^2 \ll \pi(X) + \sum_{1 \le v_0 \le 4X/\sqrt{|D|}} \frac{X W(Dv_0^2/X^2)}{\log(|D|)^2 L(1, \chi_D)} \left(\frac{v_0}{\varphi(v_0)}\right)^3$$
$$\ll \pi(X) + \frac{X}{\log(|D|)^2 L(1, \chi_D)} \sum_{1 \le v_0 \le 4X/\sqrt{|D|}} W\left(\frac{Dv_0^2}{X^2}\right) \left(\frac{v_0}{\varphi(v_0)}\right)^3.$$

By Lemma 2.1, $W(\frac{Dv_0^2}{X^2}) = O(1)$. It is easy to check that

$$\sum_{1 \le v_0 \le 4X/\sqrt{|D|}} \left(\frac{v_0}{\varphi(v_0)}\right)^3 = O(X/\sqrt{|D|}).$$

Therefore, we obtain

$$\sum_{Q \in H(D)} \pi(Q, w, X)^2 \ll \pi(X) + \frac{X}{\log(|D|)^2 L(1, \chi_D)} \frac{X}{\sqrt{|D|}} \ll \pi(X) + \frac{\pi(X)^2}{h(D)}.$$

This proves inequality (1.11) and concludes the proof of Theorem 1.1. □

## 3. Quantitative equidistribution of integral points on hyperboloids

Recall that $q(\mathbf{v}) := z^2 - 4xy$, where $\mathbf{v} := (x, y, z)$, and that $V_m := \{\mathbf{v} \in \mathbb{R}^3 : q(\mathbf{v}) = m\}$, where $m := Dv_0^2$, and $D < 0$ is a fundamental discriminant and $v_0 \leq 4X/\sqrt{|D|}$. Assume that $d_1$ and $d_2$ are integers. Recall that $w(u)$ is a positive smooth weight function that is supported on $[1, 2]$ and $\int_u w(u)\,du = 1$. Let $X \gg \sqrt{|m|}$ and $w_X(u) := w(u/X)$. Recall that $\#_{w_X} A_{d_1,d_2;m}$ is the number of integral points lying on $V_m(\mathbb{R})$ which are weighted by $w_X(x)w_X(y)$ such that $x$ and $y$ are divisible by $d_1$ and $d_2$, respectively. In this section, we show that

$$\#_{w_X} A_{d_1,d_2;m} = \sigma_{\infty,w_X} \prod_p \sigma_p(V_{m,d_1 d_2}) + \mathrm{Er},$$

where $\sigma_{\infty,w_X}(V_{m,d_1d_2})$ and $\sigma_p(V_{m,d_1d_2})$ were defined in (1.17) and Er is the error term that we bound in this section. First, we reduce the problem to the case where $\gcd(d_1 d_2, v_0) = 1$. Recall that $k := d_1 d_2$, $q_k(x, y, z) := z^2 - 4kxy$, and $V_{m,k} := \{\mathbf{v} \in \mathbb{R}^3 : q_k(\mathbf{v}) = m\}$ and $V_{m,k}(\mathbb{Z})$ is the set of integral points of $V_{m,k}$. Note that $\#_{w_X} A_{d_1,d_2;m} = \#_{w_X} A_{d_1,k/d_1;m}$ is the number of integral points lying on $V_{m,k}$ which are weighted by $w_{X/d_1}(x)w_{Xd_1/k}(y)$. In this reinterpretation, we do not have any divisibility condition, and we have extended the definition of $\#_{w_X} A_{d_1,k/d_1;m}$ to any $d_1 \in \mathbb{R}^*$. By Section 2.2, Theorem 1.1 is reduced to proving a power saving asymptotic formula for $\#_{w_X} A_{d_1,k/d_1;m}$. In Lemma 3.1, we show that it is enough to prove such a formula when $\gcd(k, v_0) = 1$. We give an upper bound on Er by using Theorem 5.1, which we prove in our appendix. Theorem 5.1 extends Duke's Theorem [Duk88, Theorem] from square-free integers $m$ to any integer $m = Dv_0^2$, where $\gcd(k, v_0) = 1$.

**Lemma 3.1.** *Suppose that $p$ is an odd prime number where $p \mid v_0$. If $p^2 \mid k$, then*

$$\#_{w_X} A_{d_1,k/d_1;m} = \#_{w_{\frac{X}{p}}} A_{\frac{d_1}{p},\frac{k/p^2}{d_1/p};\frac{m}{p^2}}.$$

*If $p \mid k$ and $p^2 \nmid k$, then*

$$\#_{w_X} A_{d_1,k/d_1;m} = \#_{w_{\frac{X}{p}}} A_{d_1,\frac{k/p}{d_1};\frac{m}{p^2}} + \#_{w_{\frac{X}{p}}} A_{pd_1,\frac{k/p}{pd_1};\frac{m}{p^2}} - \#_{w_{\frac{X}{p}}} A_{d_1,\frac{k}{d_1};\frac{m}{p^2}}.$$

*Proof.* Suppose that $p^2 \mid k$. Then

$$\#_{w_X} A_{d_1,k/d_1;m} = \sum_{(x,y,z)\in V_{m,k}} w\left(\frac{d_1 x}{X}\right) w\left(\frac{ky}{Xd_1}\right).$$

Note that $z^2 - 4kxy = Dv_0^2$, where $p^2 \mid v_0^2$ and $p^2 \mid k$. Hence $p \mid z$ and by dividing both side of the equation by $p^2$, we have $(x, y, z/p) \in V_{m/p^2,k/p^2}$. This defines a one-to-one correspondence between $V_{m,k}$ and $V_{m/p^2,k/p^2}$. Then it is easy to check that

$$\#_{w_X} A_{d_1,k/d_1;m} = \#_{w_{\frac{X}{p}}} A_{\frac{d_1}{p},\frac{k}{pd_1};\frac{m}{p^2}}.$$

Next, suppose that $p \mid k$ and $p^2 \nmid k$. Similarly $z^2 - 4kxy = Dv_0^2$, where $p^2 \mid v_0^2$ and $p \mid k$. Hence, $p \mid z^2$, and we have

$$\left(\frac{z}{p}\right)^2 - 4\frac{k}{p}\frac{xy}{p} = \frac{m}{p^2}.$$

Hence, $p \mid xy$. By inclusion and exclusion, we deduce that

$$\#_{w_X} A_{d_1,k/d_1;m} = \#_{w_{\frac{X}{p}}} A_{d_1,\frac{k/p}{d_1};\frac{m}{p^2}} + \#_{w_{\frac{X}{p}}} A_{d_1/p,\frac{k}{d_1};\frac{m}{p^2}} - \#_{w_{\frac{X}{p}}} A_{d_1,\frac{k}{d_1};\frac{m}{p^2}}.$$

This completes the proof of our lemma. □

By repetitive use of Lemma 3.1, we may assume without loss of generality that $\gcd(k, v_0^2) = 1$. Let $\Gamma_k := SO_{q_k}(\mathbb{Z})$ and consider the surface $\Gamma_k \backslash V_{m,k}$. We equipped $\Gamma_k \backslash V_{m,k}$ with the hyperbolic metric. Let $d\mu$ be the Haar measure induced from the hyperbolic metric, and let $\langle f, g\rangle := \int_{\Gamma_k \backslash V_{m,k}} fg\, d\mu$ be the Petersson inner product. Let $\Delta$ be the Laplace–Beltrami operator. We assume that the reader is familiar with the spectral theory of $\Delta$; see [Iwa95, Sel43]. Let $\mathcal{S}_k := \{f_\lambda \in L^2(\Gamma_k \backslash V_{m,k}) : \Delta f_\lambda = \lambda f\}$ be an orthonormal basis of Maass cusp forms. Let $\mathcal{E}_k := \{\mathfrak{a} : \mathfrak{a}$ ranges over all inequivalent cusps of $\Gamma_k \backslash V_{m,k}\}$. For $\mathfrak{a} \in \mathcal{E}_k$, let $\sigma_{\mathfrak{a}}$ be a scaling matrix associated to $\mathfrak{a}$, which is an isometry between $V_{m,k}$ and the upper half-plane $\mathbb{H}$ such that $\sigma_{\mathfrak{a}}(\infty) = \mathfrak{a}$, and

$$\sigma_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}} = \left\{\begin{bmatrix}1 & n\\ 0 & 1\end{bmatrix} : n \in \mathbb{Z}\right\},$$

where $\Gamma_{\mathfrak{a}}$ is the stabilizer of $\mathfrak{a}$. For $\mathfrak{a} \in \mathcal{E}_k$, we define the height function $y_{\mathfrak{a}} : V_{m,k} \to \mathbb{R}^+$ as

$$y_{\mathfrak{a}}(\mathbf{v}) := \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}(\mathbf{v})).$$

For $\mathbf{v} \in \Gamma_k \backslash V_{m,k}$ and $s \in \mathbb{C}$, let $E_{\mathfrak{a}}(\mathbf{v}, s)$ be the Eisenstein series such that its constant Fourier coefficient at cusp $\mathfrak{b}$ is $\delta_{\mathfrak{a}\mathfrak{b}} y_{\mathfrak{b}}^s + \varphi_{\mathfrak{a}\mathfrak{b}}(s) y_{\mathfrak{b}}^{1-s}$, where $\delta_{\mathfrak{a}\mathfrak{b}} = 1$ if $\mathfrak{a} = \mathfrak{b}$ and $\delta_{\mathfrak{a}\mathfrak{b}} = 0$ otherwise. We define the $\Gamma_k$ periodic function $W$ on $\Gamma_k \backslash V_{m,k}$ by averaging the smooth weight function $w$ on $\Gamma_k$ orbits,

$$W(\Gamma_k \mathbf{v}) := \sum_{\gamma \in \Gamma_k} w(\gamma\mathbf{v}).$$

By Proposition 3.5, the action of $\Gamma_k$ on $V_{m,k}(\mathbb{Z})$ has finitely many orbits. Let $H_k(m) \subset \Gamma_k \backslash V_{m,k}(\mathbb{Z})$ be this finite set of orbits. We have

$$\#_{w_X} A_{d_1,d_2;m} = \sum_{\mathbf{h} \in V_{m,k}(\mathbb{Z})} w(\mathbf{h}) = \sum_{\mathbf{l} \in H_k(m)} \frac{1}{|\Gamma_{k,\mathbf{l}}|} W(\mathbf{l}),$$

where $\mathbf{l} = \Gamma_k \mathbf{h}$ is an orbit for some $\mathbf{h} \in V_{m,k}(\mathbb{Z})$, and $|\Gamma_{k,\mathbf{l}}| = |\Gamma_{k,\mathbf{h}}|$ is the size of the stabilizer of any representative $\mathbf{h} \in \mathbf{l}$, which is independent of the choice of $\mathbf{h} \in \mathbf{l}$. Define the $m$-th Weyl sum associated to a $\Gamma_k$ periodic function $f$ to be

$$R(m, f) := \sum_{\Gamma_k \mathbf{h} \in H_k(m)} \frac{1}{|\Gamma_{k,\mathbf{h}}|} f(\Gamma_k \mathbf{h}).$$

Hence, $\#_{w_X} A_{d_1,d_2;m} = R(m, W)$. By the spectral theory of Maass forms developed by Selberg [Sel43], we write $W$ in terms of Maass cusp forms, Eisenstein series and the constant function, and obtain

$$W(\mathbf{v}) = \frac{\int_{\Gamma_k \backslash V_{m,k}} W d\mu}{\mathrm{vol}(\Gamma_k \backslash V_{m,k})} + \sum_{f_\lambda} \langle W, f_\lambda \rangle f_\lambda(\mathbf{v}) + W_{\mathrm{cts}}(\mathbf{v}), \tag{3.1}$$

where the first term comes from the contribution of the constant function, and

$$W_{\mathrm{cts}}(\mathbf{v}) := \sum_{\mathfrak{a} \in \mathcal{E}_k} \int_{-\infty}^{\infty} \langle W, E_{\mathfrak{a}}(\cdot, 1/2 + it) \rangle E_{\mathfrak{a}}(\mathbf{v}, 1/2 + it)\, dt.$$

By (3.1), we have

$$R(m, W) = \frac{\int_{\Gamma_k \backslash V_{m,k}} W\, d\mu}{\mathrm{vol}(\Gamma_k \backslash V_{m,k})} \sum_{\Gamma_k \mathbf{h} \in H_k(m)} \frac{1}{|\Gamma_{k,\mathbf{h}}|} + \sum_{f_\lambda} \langle W, f_\lambda \rangle R(m, f_\lambda) + R(m, W_{\mathrm{cts}}).$$

Note that $\sum_{\Gamma_k \mathbf{h} \in H_k(m)} \frac{1}{|\Gamma_{k,\mathbf{h}}|}$ is the class number associated to the action of $\Gamma_k$ on $V_{m,k}(\mathbb{Z})$. By Proposition 3.5, the first term can be written as the product of local densities, and we obtain

$$\frac{\int_{\Gamma_k \backslash V_{m,k}} W\, d\mu}{\mathrm{vol}(\Gamma_k \backslash V_{m,k})} \sum_{\Gamma_k \mathbf{h} \in H_k(m)} \frac{1}{|\Gamma_{k,\mathbf{h}}|} = \frac{\int_{\Gamma_k \backslash V_{m,k}} W\, d\mu}{\mathrm{vol}(\Gamma_k \backslash V_{m,k})} \sigma_\infty \prod_p \sigma_p(V_{m,k}),$$

where $\sigma_p(V_{m,k}) := \lim_{l \to \infty} |V_{m,k}(\mathbb{Z}/p^l\mathbb{Z})|/p^{2l}$ and $\sigma_\infty := \int_{\Gamma \backslash V_{m,k}} d\sigma_\infty$. Therefore,

$$\#_{w_X} A_{d_1,d_2;m} = \sigma_{\infty, w_X} \prod_p \sigma_p(V_{m,k}) + \mathrm{Er}, \tag{3.2}$$

where $\mathrm{Er} := \sum_{f_\lambda} \langle f_\lambda, W \rangle R(m, f_\lambda) + R(m, W_{\mathrm{cts}})$. Our goal in this section is to give an upper bound on Er. Let $T := |m|^\delta$ for some $\delta > 0$. We write $\mathrm{Er} = \mathrm{Er}_{\mathrm{high}} + \mathrm{Er}_{\mathrm{low}} + \mathrm{Er}_{\mathrm{cts,low}}$, where

$$\begin{aligned}
\mathrm{Er}_{\mathrm{high}} &:= \sum_{\lambda \geq T} \langle W, f_\lambda \rangle R(m, f_\lambda) \\
&\quad + \sum_{\mathfrak{a} \in \mathcal{E}_k} \int_{|1/4+t^2| > T} \langle W, E_{\mathfrak{a}}(\cdot, 1/2 + it) \rangle R(m, E_{\mathfrak{a}}(\mathbf{v}, 1/2 + it))\, dt, \\
\mathrm{Er}_{\mathrm{low}} &:= \sum_{\lambda < T} \langle W, f_\lambda \rangle R(m, f_\lambda), \\
\mathrm{Er}_{\mathrm{cts,low}} &:= \sum_{\mathfrak{a} \in \mathcal{E}_k} \int_{|1/4+t^2| < T} \langle W, E_{\mathfrak{a}}(\cdot, 1/2 + it) \rangle R(m, E_{\mathfrak{a}}(\mathbf{v}, 1/2 + it))\, dt.
\end{aligned} \tag{3.3}$$

**Theorem 3.2.** *Let $D$ be a fundamental discriminant and $m = Dv_0^2$ where $v_0 < \log(|D|)^A$ for some fixed $A > 0$. Let $\#_{w_X} A_{d_1,d_2;m}$ be as above. Then, for every $\epsilon > 0$,*

$$\#_{w_X} A_{d_1,d_2;m} = \sigma_{\infty,w_X} \prod_p \sigma_p(V_{m,k}) + O(1 + |m|^{-1/28} k^{10} X^{1+\epsilon} |D|^{\epsilon}). \tag{3.4}$$

*As a result, for every $\delta > 0$ there exists an $\epsilon > 0$ such that if $k^{308+\delta} \leq D$ and $X \leq |D|^{1/2} \log(|D|)^B$ for some $B > 0$ then*

$$\#_{w_X} A_{d_1,d_2;m} = \sigma_{\infty,w_X} \times \prod_p \sigma_p(V_{m,k}) + O\left(1 + \frac{X}{d_1 d_2} |D|^{-\epsilon}\right), \tag{3.5}$$

*where $k = d_1 d_2$ and the implicit constant depends only on $\epsilon$ and $w$.*

*Proof.* By equation (3.2), we have

$$\#_{w_X} A_{d_1,d_2;m} = \sigma_{\infty,w_X} \prod_p \sigma_p(V_{m,k}) + \mathrm{Er},$$

where $\mathrm{Er} = \mathrm{Er}_{\mathrm{high}} + \mathrm{Er}_{\mathrm{low}} + \mathrm{Er}_{\mathrm{cts,low}}$. By Proposition 3.10, $\mathrm{Er}_{\mathrm{high}} = O(1)$, where the implicit constant in $O$ depends only on $\epsilon$ and $w$. By Proposition 3.15, we have

$$\mathrm{Er}_{\mathrm{low}} \ll |m|^{-1/28} k^{10} X^{1+\epsilon} T^7.$$

Let $T = |D|^{\epsilon/7}$. Then

$$\mathrm{Er}_{\mathrm{low}} = O(|m|^{-1/28} k^{10} X^{1+\epsilon} |D|^{\epsilon}).$$

By Proposition 3.20,

$$\mathrm{Er}_{\mathrm{cts,low}} \ll k^{6.5} T^{7/4} |m|^{-1/28+\epsilon} |X|^{1+\epsilon} = O(|m|^{-1/28} k^{10} X^{1+\epsilon} |D|^{\epsilon}).$$

Therefore,

$$\#_{w_X} A_{d_1,d_2;m} = \sigma_{\infty,w_X} \prod_p \sigma_p(V_{m,k}) + \mathrm{Er} + O(1 + |m|^{-1/28} k^{10} X^{1+\epsilon} |D|^{\epsilon}).$$

This completes the proof of equation (3.4). If $k^{308+\delta} \leq D$ then

$$m^{-1/28} k^{10} = O\left(\frac{|D|^{-\delta/28}}{k}\right).$$

Moreover if $X \leq |D|^{1/2} \log(|D|)^B$, then $X^{1+\epsilon} = O(X|D|^{\epsilon})$. By the above inequalities and choosing $\epsilon$ small enough compared to $\delta$, we deduce (3.5) and our theorem. □

### 3.1. Main term

We define the generalized class number $h(k, m)$ to be the number of $\Gamma_k$ orbits of $V_{m,k}(\mathbb{Z})$ weighted by their representation number,

$$h(k, m) := \sum_{\Gamma_k \mathbf{h} \in H_k(m)} \frac{1}{|\Gamma_{k,\mathbf{h}}|}.$$

We cite the following theorem from [CS99, Chapter 15, Theorem 19].

**Theorem 3.3** (due to Kneser, Earnest and Hsia). *If $Q$ is an indefinite integral quadratic form with at least 3 variables and the genus of $Q$ contains more than one class, then for some prime number $p$, $Q$ can be $p$-adically diagonalized and the diagonal entries all involve distinct powers of $p$.*

**Lemma 3.4.** *The genus of $q_k(x, y, z)$ contains only one class for every $k \in \mathbb{Z}$.*

*Proof.* We show this by computing the local spinor norms; see [CS99, Chapter 15]. By the work of Kneser [Kne56] on the computation of the local spinor norms for odd primes $p$ and its improvement by Earnest and Hsia [EH75, EH84] for prime 2, we have the following theorem that implies the genus of an indefinite quadratic forms contains only one class. We can diagonalize the quadratic form $q_k(x, y, z)$ over every local ring $\mathbb{Z}_p$ where $p \neq 2$ by changing the variables to $x_1 = z$ , $x_2 = x - y$ and $x_3 = x + y$ and obtain

$$q_k(x_1, x_2, x_3) = x_1^2 + kx_2^2 - kx_3^2.$$

It is easy to check that $q_k$ does not satisfy the conditions of the above theorem and as a result the genus class of $q_k$ contains only one element. This completes the proof of our lemma. □

**Proposition 3.5.** *We have*

$$h(k, m) := \sigma_\infty \prod_p \sigma_p(V_{m,k}),$$

*where $\sigma_p(V_{m,k})$ was defined in (1.17) and*

$$\sigma_\infty := \lim_{\epsilon \to 0} \frac{\mathrm{vol}(\Gamma_k \backslash (|q_k(x, y, z) - m| < \epsilon))}{2\epsilon} = \int_{\Gamma_k \backslash V_{m,k}} d\sigma_\infty.$$

*Proof.* By Lemma 3.4, the genus class of $q_k$ contains only one class. Next, we apply the Siegel Mass formula to the indefinite ternary quadratic from $q_k$, and obtain

$$\frac{1}{\int_{\Gamma_k \backslash V_{m,k}} d\sigma_\infty} \sum_{\Gamma_k \mathbf{h} \in H_k(m)} \frac{1}{|\Gamma_{k,\mathbf{h}}|} = \prod_p \sigma_p(V_{m,k}).$$

This completes the proof of our proposition. □

### *3.2. Bounding* $\mathrm{Er}_{\mathrm{high}}$

In this section we give an upper bound on $\mathrm{Er}_{\mathrm{high}}$. Note that $q_k(\mathbf{v}) = \mathbf{v}^{\intercal} A_k \mathbf{v}$, where

$$A_k := \begin{bmatrix} 0 & -2k & 0 \\ -2k & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let

$$C_k := \begin{bmatrix} 1/(2\sqrt{k}) & 1/(2\sqrt{k}) & 0 \\ 1/(2\sqrt{k}) & -1/(2\sqrt{k}) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then

$$C_k^{\intercal} A_k C_k = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We proceed by defining the Casimir operator of the orthogonal group $SO_{q_k}$ which induces $\Delta$ on $V_{m,k}$. Let

$$X_1 := \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad X_2 := \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad X_3 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

By the definition of the Casimir operator of $SO_{q_k}$,

$$\Omega := Y_1^2 + Y_2^2 - Y_3^3,$$

where $Y_1 := C_k X_1 C_k^{-1}$, $Y_2 := C_k X_2 C_k^{-1}$ and $Y_3 := C_k X_3 C_k^{-1}$. In the following lemma, we give a formula for $\Omega$ in terms of the $(x, y, z)$ coordinates of the quartic $V_{m,k}$.

**Lemma 3.6.** *The restriction of $\Omega$ to $V_{m,k}$ is given by*

$$\begin{aligned} \Omega = {} & x^2 \frac{\partial^2}{\partial x^2} + 2x \frac{\partial}{\partial x} + \frac{4kxy + 2m}{2k} \frac{\partial^2}{\partial x \partial y} + 2xz \frac{\partial^2}{\partial x \partial z} \\ & + y^2 \frac{\partial^2}{\partial y^2} + 2y \frac{\partial}{\partial y} + 2yz \frac{\partial^2}{\partial y \partial z} + (z^2 - m) \frac{\partial^2}{\partial z^2} + 2z \frac{\partial}{\partial z}. \end{aligned} \tag{3.6}$$

*Proof.* We compute the induced first order differential operators associated to $Y_1$, $Y_2$ and $Y_3$. Note that

$$Y_1 := C_k X_1 C_k^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This vector is associated to the first order differential operator

$$Z_1 := x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}.$$

Similarly $Y_2 := C_k X_2 C_k^{-1} = \begin{bmatrix} 0 & 0 & 1/(2\sqrt{k}) \\ 0 & 0 & -1/(2\sqrt{k}) \\ \sqrt{k} & \sqrt{k} & 0 \end{bmatrix}$ is associated to

$$Z_2 := \frac{z}{2\sqrt{k}}\frac{\partial}{\partial x} - \frac{y}{2\sqrt{k}}\frac{\partial}{\partial y} + \sqrt{k}\,(x+y)\frac{\partial}{\partial z},$$

and $Y_3 := C_k X_3 C_k^{-1} = \begin{bmatrix} 0 & 0 & 1/(2\sqrt{k}) \\ 0 & 0 & -1/(2\sqrt{k}) \\ -\sqrt{k} & \sqrt{k} & 0 \end{bmatrix}$ is associated to

$$Z_3 := \frac{z}{2\sqrt{k}}\frac{\partial}{\partial x} - \frac{z}{2\sqrt{k}}\frac{\partial}{\partial y} + (y-x)\sqrt{k}\,\frac{\partial}{\partial z}.$$

The induced Casimir operator is given by

$$Z_1^2 + Z_2^2 - Z_3^2.$$

We have

$$Z_1^2 = \left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)^2 = x^2\frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial x} - 2xy\frac{\partial^2}{\partial x\partial y} + y^2\frac{\partial^2}{\partial y^2} + y\frac{\partial}{\partial y}, \tag{3.7}$$

$$\begin{aligned} Z_2^2 &= \left(\frac{z}{2\sqrt{k}}\frac{\partial}{\partial x} + \frac{z}{2\sqrt{k}}\frac{\partial}{\partial y} + \sqrt{k}\,(x+y)\frac{\partial}{\partial z}\right)^2 \\ &= \frac{z^2}{4k}\frac{\partial^2}{\partial x^2} + \frac{z^2}{2k}\frac{\partial^2}{\partial x\partial y} + z(x+y)\frac{\partial}{\partial x\partial z} + \frac{z}{2}\frac{\partial}{\partial z} + \frac{x+y}{2}\frac{\partial}{\partial x} \\ &\quad + \frac{z^2}{4k}\frac{\partial^2}{\partial y^2} + z(x+y)\frac{\partial^2}{\partial y\partial z} + \frac{z}{2}\frac{\partial}{\partial z} + \frac{x+y}{2}\frac{\partial}{\partial y} + k(x+y)^2\frac{\partial^2}{\partial z^2}, \end{aligned} \tag{3.8}$$

$$\begin{aligned} Z_3^2 &= \left(\frac{z}{2\sqrt{k}}\frac{\partial}{\partial x} - \frac{z}{2\sqrt{k}}\frac{\partial}{\partial y} + (y-x)\sqrt{k}\frac{\partial}{\partial z}\right)^2 \\ &= \frac{z^2}{4k}\frac{\partial^2}{\partial x^2} - \frac{z^2}{2k}\frac{\partial^2}{\partial x\partial y} + z(y-x)\frac{\partial^2}{\partial x\partial z} - \frac{z}{2}\frac{\partial}{\partial z} + \frac{y-x}{2}\frac{\partial}{\partial x} \\ &\quad + \frac{z^2}{4k}\frac{\partial^2}{\partial y^2} - z(y-x)\frac{\partial^2}{\partial y\partial z} - \frac{z}{2}\frac{\partial}{\partial z} - \frac{y-x}{2}\frac{\partial}{\partial y} + (y-x)^2k\frac{\partial^2}{\partial z^2}. \end{aligned} \tag{3.9}$$

By using the formulas in (3.7)–(3.9), we have the following formula for the induced Casimir operator on $V_{m,k}$:

$$\begin{aligned} \Omega &= x^2\frac{\partial^2}{\partial x^2} + 2x\frac{\partial}{\partial x} + \frac{4kxy+2m}{2k}\frac{\partial^2}{\partial x\partial y} + 2xz\frac{\partial^2}{\partial x\partial z} \\ &\quad + y^2\frac{\partial^2}{\partial y^2} + 2y\frac{\partial}{\partial y} + 2yz\frac{\partial^2}{\partial y\partial z} + (z^2-m)\frac{\partial^2}{\partial z^2} + 2z\frac{\partial}{\partial z}. \end{aligned}$$

□

In the following lemma, we prove an upper bound on the $L^2$ norm of $W$.

**Lemma 3.7.** *Let $W$, $X$ and $k$ be as above. Then*

$$|W|_2 \ll \frac{X^{1+\epsilon}}{\sqrt{\max(d_1, d_2)m}}.$$

*Proof.* We have

$$|W|_2^2 = \int_{\Gamma_k\backslash V_{m,k}} |W|^2\, d\mu \le \sup|W| \int_{\Gamma_k\backslash V_{m,k}} |W|\, d\mu. \tag{3.10}$$

First, we give an upper bound on $\int_{\Gamma_k\backslash V_{m,k}} |W|\, d\mu$. Recall that $W(\Gamma(x,y,z)) := \sum_{\gamma\in\Gamma_k} w(\gamma_k(x,y,z))$, where $w(x,y,x) := w_{X_1}(x)w_{X_2}(y)$, and $X_1 = X/d_1$, $X_2 = X/d_2$, and $w_X(u) := w(u/X)$. Note that $d\mu = \frac{1}{\sqrt{m}}d\sigma_\infty$. Hence, by Lemma 2.1,

$$\int_{\Gamma_k\backslash V_{m,k}} |W|\, d\mu \le \int_{V_{m,k}} |w|\, d\mu \ll \frac{X}{d_1 d_2\sqrt{m}} = \frac{X}{k\sqrt{m}}. \tag{3.11}$$

Next, we give an upper bound on $\sup|W|$. Let

$$B(X_1, X_2) := \{(x,y,z) \in V_{m,k}(\mathbb{R}) : X_1 \le x \le 2X_1 \text{ and } X_2 \le y \le 2X_2\}.$$

For $\mathbf{h} \in V_{m,k}$, define $N(X_1, X_2, \mathbf{h}) := \#\{\gamma \in \Gamma_k : \Gamma_k\mathbf{h} \in B(X_1, X_2)\}$. Then

$$W(\mathbf{h}) = \sum_{\gamma\in\Gamma_k} w(\gamma\mathbf{h}) \ll N(X_1, X_2, \mathbf{h}). \tag{3.12}$$

We give an upper bound on $N(X_1, X_2, \mathbf{h})$ by applying results in hyperbolic geometry. Let $\operatorname{diam}(B(X_1, X_2))$ be the diameter of $B(X_1, X_2)$ with respect to the hyperbolic metric on $V_{m,k}$. For $\mathbf{h} \in V_{m,k}$ define the invariant height of $\mathbf{h}$ by $y_\Gamma(\mathbf{h}) = \max_{\mathfrak{a}}(y_{\mathfrak{a}}(\mathbf{h}))$.

**Lemma 3.8.** *We have*

$$\operatorname{diam}(B(X_1, X_2)) \ll 1 + \log\left(\frac{X}{\sqrt{m}}\right)$$

*and*

$$\sup_{\mathbf{h}\in B(X_1,X_2)} y_\Gamma(\mathbf{h}) \ll \min(d_1, d_2)\frac{X}{\sqrt{m}}.$$

*Proof.* Without loss of generality, suppose that $d_2 = \min(d_1, d_2)$. Let $\mathfrak{c} := \begin{bmatrix} 0\\1\\0 \end{bmatrix}$, which is a cusp for $\Gamma_k\backslash V_{m,k}$. Consider the following change of coordinates:

$$u_1 := \frac{d_1x_1}{\sqrt{|m|}}, \qquad u_2 := \frac{d_2x_2}{\sqrt{|m|}}, \qquad u_3 := \frac{x_3}{\sqrt{m}}.$$

Then $V_{m,k}$ maps to $u_3^2 - 4u_1u_2 = -1$, and $B(X_1, X_2)$ maps to

$$B(X,m) := \left\{(u_1,u_2,u_3) : u_3^2 - 4u_1u_2 = -1,\right.$$
$$\left.\frac{X}{\sqrt{m}} \le u_1 \le 2\frac{X}{\sqrt{m}} \text{ and } \frac{X}{\sqrt{m}} \le u_2 \le 2\frac{X}{\sqrt{m}}\right\}.$$

The quartic $u_3^2 - 4u_1u_2 = -1$ with its induced metric $(du_3)^2 - 4du_1du_2$ is isomorphic to the hyperbolic plane by $\beta : (u_1, u_2, u_3) \mapsto \frac{-u_3+i}{2u_1}$. This maps $\mathfrak{c}$ to $\infty$. Since $X \gg \sqrt{m}$, we have $|u_3/u_1| \ll 1$. It follows that

$$\operatorname{diam}(B(X_1, X_2)) = \operatorname{diam}(B(X, m)) \ll 1 + \log(|X/\sqrt{m}|).$$

It follows that the stabilizer of $\mathfrak{c}$ in $\Gamma_k$ is

$$\Gamma_{\mathfrak{c}} := \left\{ \begin{bmatrix} 1 & 0 & 0 \\ n^2k & 1 & n \\ 2kn & 0 & 1 \end{bmatrix} : n \in \mathbb{Z} \right\}.$$

For $a \in \mathbb{R}$, let

$$\alpha_a := \begin{bmatrix} 1 & 0 & 0 \\ a^2\frac{d_1}{d_2} & 1 & a/d_2 \\ 2ad_1 & 0 & 1 \end{bmatrix}.$$

Then $\beta$ identifies $V_{m,k}$ with the upper-half plane and maps $\alpha_a$ to $\left[\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right]$. Hence, it identifies $\Gamma_{\mathfrak{c}}$ with $\beta(\Gamma_{\mathfrak{c}}) = \left\{\left[\begin{smallmatrix} 1 & d_2n \\ 0 & 1 \end{smallmatrix}\right] : n \in \mathbb{Z}\right\}$. Hence, $y_{\mathfrak{c}}(\mathbf{h}) = \frac{1}{d_2}\operatorname{Im}(\beta(\mathbf{h})) = \frac{1}{2d_2u_1}$. Since $\frac{X}{\sqrt{m}} \leq u_1 \leq 2\frac{X}{\sqrt{m}}$, we have $\frac{\sqrt{m}}{2Xd_2} \leq y_{\mathfrak{c}}(\mathbf{h}) \leq \frac{\sqrt{m}}{Xd_2}$ for every $\mathbf{h} \in B(X_1, X_2)$. By Margulis' lemma and decomposing $\Gamma_k \backslash V_{m,k}$ into thin and thick parts, it follows that $y_{\mathfrak{c}}(\mathbf{h})y_{\mathfrak{a}}(\mathbf{h}) \ll 1$ for every $\mathfrak{a} \neq \mathfrak{c}$. Therefore, $y_\Gamma(\mathbf{h}) \ll d_2X/\sqrt{m}$. This completes the proof of Lemma 3.8. □

By [Iwa02, Corollary 2.12, p. 52], we have

$$N(X_1, X_2, \mathbf{h}) \ll \operatorname{diam}(B(X_1, X_2)) \sup_{\mathbf{h} \in B(X_1, X_2)} y_\Gamma(\mathbf{h}).$$

Therefore, by Lemma 3.8,

$$N(X_1, X_2, \mathbf{h}) \ll \min(d_1, d_2)\frac{X^{1+\epsilon}}{\sqrt{m}}. \tag{3.13}$$

By the above inequality and inequalities (3.12), (3.15) and (3.10), we obtain

$$|W|_2^2 \ll \frac{X}{k\sqrt{m}} \min(d_1, d_2)\frac{X^{1+\epsilon}}{\sqrt{m}} \ll \frac{X^{2+\epsilon}}{\max(d_1, d_2)m}.$$

This concludes the proof of Lemma 3.7. □

Next, by applying integration by parts, we give an upper bound on the inner product of $W$ with the Maass forms and also the Eisenstein series.

**Lemma 3.9.** *Let $A$ be any positive integer. Then*

$$\left[\sum_{\lambda \geq T} |\langle W, f_\lambda\rangle|^2 + \sum_{\mathfrak{a} \in \mathcal{E}_k} \int_{|1/4+t^2|>T} |\langle W, E_{\mathfrak{a}}(\cdot, 1/2+it)\rangle|^2\, dt\right]^{1/2} = O_A\left(\frac{X^{1+\epsilon}}{\sqrt{md_1}T^A}\right),$$

*where the implicit constant depends only on $\sup_{1 \leq n \leq A} d^{(n)}w$.*

*Proof.* By the Plancherel theorem and integration by parts we have

$$\begin{aligned}
&\sum_{\lambda\geq T}|\langle W, f_\lambda\rangle|^2+\sum_{\mathfrak{a}\in\mathcal{E}_k}\int_{|1/4+t^2|>T}|\langle W, E_{\mathfrak{a}}(\cdot,1/2+it)\rangle|^2\,dt\\
&=\sum_{\lambda\geq T}\frac{1}{\lambda^{2A}}|\langle W,\Omega^A f_\lambda\rangle|^2+\sum_{\mathfrak{a}\in\mathcal{E}_k}\int_{|1/4+t^2|>T}\frac{1}{(1/4+t^2)^{2A}}\langle W,\Omega^A E_{\mathfrak{a}}(\cdot,1/2+it)\rangle^2\,dt\\
&\leq\frac{1}{T^{2A}}\left[\sum_{\lambda\geq T}|\langle\Omega^A W, f_\lambda\rangle|^2+\sum_{\mathfrak{a}\in\mathcal{E}_k}\int_{|1/4+t^2|>T}|\langle\Omega^A W, E_{\mathfrak{a}}(\cdot,1/2+it)\rangle|^2\,dt\right]\\
&\leq\frac{1}{T^{2A}}\int_{\Gamma_k\backslash V_{m,k}}|\Omega^n W|^2\,d\mu.
\end{aligned}\tag{3.14}$$

By a similar argument to that in Lemma 3.7, we give an upper bound on $\int_{\Gamma_k\backslash V_{m,k}}|\Omega^n W|^2\,d\mu$. We have

$$\int_{\Gamma_k\backslash V_{m,k}}|\Omega^n W|^2\,d\mu\leq\sup|\Omega^n W|\int_{\Gamma_k\backslash V_{m,k}}|\Omega^n W|\,d\mu$$

and

$$\begin{aligned}
\int_{\Gamma_k\backslash V_{m,k}}|\Omega^n W|\,d\mu&\leq\int_{V_{m,k}}|\Omega^n w|\,d\mu\leq\sup|\Omega^n w|\int_{X_1\leq x\leq 2X_1}\int_{X_2\leq y\leq 2X_2}d\mu\\
&\ll\sup|\Omega^n w|\frac{X}{d_1d_2\sqrt{m}}.
\end{aligned}\tag{3.15}$$

We show that $\sup\Omega^n w=O_n(1)$. Note that $w(x,y,z):=w_{X_1}(x)w_{X_2}(y)$ is independent of the $z$ variable. Therefore, all the partial derivatives that include $\frac{\partial}{\partial z}$ in formula (3.6) vanishes on $w$ and we obtain

$$\begin{aligned}
&\Omega^n w(x,y,z)\\
&\quad=\left(x^2\frac{\partial^2}{\partial x^2}+2x\frac{\partial}{\partial x}+(2xy+m/D)\frac{\partial^2}{\partial x\partial y}+y^2\frac{\partial^2}{\partial y^2}+2y\frac{\partial}{\partial y}\right)^n w_{X_1}(x)w_{X_2}(y).
\end{aligned}$$

For $n=1$, we check that $\Omega w$ is bounded by a constant. We have

$$\begin{aligned}
\Omega w=&\frac{x^2}{X_1^2}w''\left(\frac{x}{X_1}\right)w\left(\frac{y}{X_2}\right)+\frac{2x}{X_1}w'\left(\frac{x}{X_1}\right)w\left(\frac{y}{X_2}\right)+2\frac{x}{X_1}\frac{y}{X_2}w'\left(\frac{x}{X_1}\right)w'\left(\frac{y}{X_2}\right)\\
&+\frac{m}{DX_1X_2}w'\left(\frac{x}{X_1}\right)w'\left(\frac{y}{X_2}\right)+\frac{y^2}{X_2^2}w\left(\frac{x}{X_1}\right)w''\left(\frac{y}{X_2}\right)+2\frac{y}{X_2}w\left(\frac{x}{X_1}\right)w'\left(\frac{y}{X_2}\right).
\end{aligned}\tag{3.16}$$

We assume that for every $n\geq 0$ all the derivatives $\frac{d^k w}{dt^k}$ for $0\leq k\leq n$ are bounded by a constant $|w|_{\infty,n}$. Since $w$ is supported inside $[1,2]$ we have $1\leq\frac{x}{X_1},\frac{y}{X_2}\leq 2$, otherwise $\Omega w=0$. Since $m<0$ and $z^2-4kxy=m$ we have $m\leq 4kX_1X_2$, otherwise $V_{m,k}$ does not have any point where $|x|<2X_1$ and $|y|<2X_2$. By these assumptions we can bound each term in (3.16) and obtain $|\Omega w| \ll 1$. Similarly, for every $n$, it follows that $\sup|\Omega^n w| \ll 1$. We have

$$\Omega^n W(\mathbf{h}) = \sum_{\gamma\in\Gamma} \Omega^n w(\Gamma_k \mathbf{h}) \leq N(X_1, X_2, \mathbf{h}) \sup \Omega^n w.$$

By inequality (3.13), we have $N(X_1, X_2, \mathbf{h}) \ll d_2 \frac{X^{1+\epsilon}}{\sqrt{m}}$, and hence

$$\sup|\Omega^n W(\mathbf{h})| \ll d_2 \frac{X^{1+\epsilon}}{\sqrt{m}}.$$

Therefore, by the above and inequality (3.15), we obtain

$$\int_{\Gamma_k\backslash V_{m,k}} |\Omega^n W|^2 \, d\mu \ll \frac{X^{2+\epsilon}}{d_1 m}.$$

Finally, by the above and (3.14), we conclude the proof. □

Finally, we show that the contribution of the high frequency spectrum is bounded. Recall that

$$\begin{aligned}\mathrm{Er}_{\text{high}} := &\sum_{\lambda\geq T} \langle W, f_\lambda\rangle R(m, f_\lambda)\\ &+ \sum_{\mathfrak{a}\in\mathcal{E}_k} \int_{|1/4+t^2|>T} \langle W, E_{\mathfrak{a}}(\cdot, 1/2+it)\rangle R(m, E_{\mathfrak{a}}(\mathbf{v}, 1/2+it)) \, dt.\end{aligned}$$

**Proposition 3.10.** *Suppose that $T = D^\delta$ for some $\delta > 0$. Then*

$$\mathrm{Er}_{\text{high}} = O(1),$$

*where the implicit constant in $O$ depends on $\sup_{1\leq n\leq 10/\delta} d^{(n)} w$ and $\delta > 0$.*

*Proof.* First, we give an upper bound on Weyl sums $R(m, f_\lambda)$ and $R(m, E_{\mathfrak{a}}(\mathbf{v}, 1/2+it))$. We have

$$R(m, f_\lambda) = \sum_{\Gamma_k\mathbf{h}\in H_k(m)} \frac{1}{|\Gamma_{k,\mathbf{h}}|} f_\lambda(\Gamma_k\mathbf{h}) \leq |f_\lambda|_\infty \sum_{\Gamma_k\mathbf{h}\in H_k(m)} \frac{1}{|\Gamma_{k,\mathbf{h}}|} \leq |f_\lambda|_\infty h(k, m),$$

where $k = d_1 d_2$. By the Weyl law we have the following trivial upper bound on the $L^\infty$ norm of an eigenfunction (see the recent work of Templier [Tem15] for a sharper upper bound):

$$|f_\lambda|_\infty \ll \lambda^{1/4} k^{1/2}.$$

By Proposition 3.5, Lemma 2.1 and Lemma 2.2, we have $h(k, m) \ll X^{1+\epsilon}/k$. Therefore,

$$|R(m, f_\lambda)| \ll \frac{\lambda^{1/4} X^{1+\epsilon}}{k^{1/2}}.$$

Similarly,

$$|R(m, E_{\mathfrak{a}}(\mathbf{v}, 1/2+it))| \ll \frac{(1/4+t^2)^{1/4} X^{1+\epsilon}}{k^{1/2}}.$$

By Lemma 3.9 and the above inequalities,

$$\mathrm{Er}_{\mathrm{high}} \ll \sum_{\lambda>T} \frac{X^{1+\epsilon}}{\sqrt{md_1}\,\lambda^A} \frac{\lambda^{1/4} X^{1+\epsilon}}{k^{1/2}} \ll \frac{X^{2+\epsilon}}{k^{1/2}} \sum_{\lambda>T} \lambda^{1/4-A}.$$

By the Weyl law for $\Gamma_k\backslash V_{m,k}$,

$$\sum_{\lambda>T} \lambda^{1/4-A} \ll k|D|^{\delta(1+1/4-A)}.$$

Recall that $X \ll |D|^{1/2+\epsilon}$, $k = d_1 d_2 \leq |D|^{1/10}$. Therefore, by choosing $A$ large enough, we obtain $\mathrm{Er}_{\mathrm{high}} = O(1)$. This completes the proof of the proposition. □

### 3.3. *The Maass identity and the Siegel theta kernel*

Let $L \in C(\Gamma_k\backslash V_{m,k})$ be any continuous function which decays with an exponential rate at cusps, e.g. $L$ is a truncated Eisenstein series, a cusp form or any function with a compact support. In this section, we write $R(m, L)$ in terms of the asymptotic of the $m$-th Fourier coefficient of the theta transfer of $L$. We begin by introducing Siegel's theta kernel associated to the indefinite quadratic form $q_k := z^2 - kxy$. Let $H_{A_k}$ denote the *majorant space* of the symmetric matrix $A_k$ (see [Sie67]):

$$H_{A_k} := \{P : P^\intercal = P,\ P > 0 \text{ and } PA_k^{-1}P = A_k\}.$$

For $P \in H_{A_k}$ and $z = x+iy \in \mathbb{C}$ with $y > 0$, define $R(z) := xA + iyP$. The *Siegel theta function* is defined for $\boldsymbol{\alpha} \in \mathbb{Q}^3$ with $2A_k\boldsymbol{\alpha} \in \mathbb{Z}^3$ by

$$\Theta_{\boldsymbol{\alpha}}(z, P) := y^{3/4} \sum_{\mathbf{h}\in\mathbb{Z}^3} e(R(z)[\mathbf{h}+\boldsymbol{\alpha}]), \tag{3.17}$$

where $R(z)[\mathbf{h}+\boldsymbol{\alpha}] := (\mathbf{h}+\boldsymbol{\alpha})^\intercal R(z)(\mathbf{h}+\boldsymbol{\alpha})$. We write $\Theta(z, P)$ when $\boldsymbol{\alpha} = 0$. More generally, let $A[B] := B^\intercal AB$ for matrices $A$ and $B$. Note that $\Theta_{\boldsymbol{\alpha}}(z, P)$ is absolutely convergent, since $y > 0$ and $P > 0$. The orthogonal group $SO_{q_k}$ acts transitively on the majorant space $H_{A_k}$ by sending $P \in H_{A_k}$ to $P[g] := g^\intercal Pg$ for $g \in SO_{q_k}$. We define $P_0 \in H_{A_k}$ to be

$$P_0 := \begin{bmatrix} 2k & 0 & 0 \\ 0 & 2k & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We extend the definition of the theta kernel from $H_{A_k}$ to $SO_{q_k}$ by defining

$$\Theta_{\boldsymbol{\alpha}}(z, g) := \Theta_{\boldsymbol{\alpha}}(z, P_0[g^{-1}]). \tag{3.18}$$

Note that we used $g^{-1}$ to transform $P_0$. Next, we cite a theorem that gives the transformation properties of the theta kernel $\Theta_\alpha(z, P_0[g^{-1}])$ in the $z$ variable. This theorem is essentially due to Siegel [Sie51] and is stated in this form in [Duk88, Theorem 3]. It is a consequence of the properties of the Weil representation; see [KS93, Proposition 2.2].

**Theorem 3.11** ([Duk88], [KS93]). *For* $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = \gamma \in \Gamma_0(4k)$ *we have*

$$\Theta(\gamma z, g) = j(\gamma, z)\Theta(z, g), \qquad \Omega\Theta(z, g) = 4\Delta_{z,1/2}\Theta(z, g) + \tfrac{3}{4}\Theta(z, g),$$

*where* $j(\gamma, z) = \frac{\theta(\gamma z)}{\theta(z)}$ *is the theta multiplier for* $\theta(z) = y^{1/4}\sum_{n\in\mathbb{Z}} e(n^2 z)$*, and* $\Delta_{z,1/2}$ *is the Laplacian operator defined on weight* $1/2$ *modular forms and* $\Omega$ *is the Casimir operator.*

**Remark 3.12.** By the above theorem it follows that if $f_\lambda$ is a cusp form with eigenvalues $\lambda = 1/4 + (2r)^2$, then $\Theta * f_\lambda(z) := \int \Theta(z, g)\overline{f_\lambda}(g)\, d\mu(g)$ is a weight $1/2$ modular form defined on $\Gamma_0(4k)\backslash\mathbb{H}$ with eigenvalues $\lambda' = 1/4 + r^2$.

Note that $SO_{q_k}$ also acts transitively on $V_{m,k}$. We define

$$\mathbf{x}_0 := \begin{bmatrix} \frac{1}{2}\sqrt{|m|/k} \\ \frac{1}{2}\sqrt{|m|/k} \\ 0 \end{bmatrix},$$

and extend the definition of $L$ from $\Gamma_k\backslash V_{m,k}$ to $\Gamma_K\backslash SO_{q_k}$ by $L(g) := L(g\mathbf{x}_0)$. It is easy to check that the stabilizer of $\mathbf{x}_0 \in V_{m,k}$ is the same as $P_0 \in H_A$, and it is a maximal compact subgroup of $SO_{q_k}$. We denote this maximal compact subgroup by $K$. Let

$$F(z) := \Theta * L = \int_{\Gamma_K\backslash SO_{q_k}} \Theta(z, g)\bar{L}(g)\, d\mu(g).$$

Theorem 3.11 implies that $F(z)$ is a weight 1/2 modular form of level $4k$ and has moderate growth. Let

$$F(u + iv) = c_{F,\infty}(v) + \sum_{n\neq 0} \rho_{F,\infty}(n, v)e(nu)$$

be the Fourier expansion of $F$ at $\infty$. Define the $m$-th Fourier coefficient of $F$ to be

$$\rho_{F,\infty}(m) := \lim_{v\to\infty} \rho_{F,\infty}(m, v)e^{2\pi|m|v}(4\pi|m|v)^{-\mathrm{sgn}(m)/4}. \tag{3.19}$$

Next, we prove an identity that relates $\rho_{F,\infty}(m)$ to $R(m, L)$. Originally, Maass [Maa71] proved a version of this identity for the eigenfunctions of co-compact lattices. As noted by Duke [Duk88, Theorem 6] Maass' proof extends easily to cusp forms, since the theta integral is convergent for cusp forms. However, the theta integral is not absolutely convergent for Eisenstein series. For our application, we need to extend this identity to Eisenstein series. In Section 3.5, we prove the analogue of Maass' identity for Eisenstein series by using the center of the enveloping algebra.

**Lemma 3.13.** *We have* $\rho_{F,\infty}(m) = \frac{\pi^{1/4}}{\sqrt{2}}|m|^{-3/4}\overline{R(m, L)}$.

*Proof.* Note that $\rho_{F,\infty}(m,v)=\int_0^1 F(u+iv)e(-mu)\,du$. We have

$$\begin{aligned}\rho_{F,\infty}(m,v)&=\int_0^1\int_{\Gamma_K\backslash SO_{q_k}}\Theta(u+iv,\,P_0[g^{-1}])\bar{L}(g)e(-mu)\,d\mu(g)\,du\\&=v^{3/4}\int_{\Gamma_K\backslash SO_{q_k}}\int_0^1\sum_{\mathbf{h}\in\mathbb{Z}^3}e(uq_k(\mathbf{h})+ivP_0[g^{-1}\mathbf{h}])\bar{L}(g)e(-mu)\,du\,d\mu(g)\\&=v^{3/4}\int_{\Gamma_K\backslash SO_{q_k}}\sum_{\substack{\mathbf{h}\in\mathbb{Z}^3\\ q_k(\mathbf{h})=m}}e(ivP_0[g^{-1}\mathbf{h}])\bar{L}(g)\,d\mu(g).\end{aligned}$$

We unfold the above integral and write it as a finite sum over the integral orbits. Then

$$\rho_{F,\infty}(m,v)=\sum_{\mathbf{l}\in H_k(m)}\frac{v^{3/4}}{|\Gamma_{k,\mathbf{l}}|}\int_{SO_{q_k}}e(ivP_0[g^{-1}\mathbf{l}])\bar{L}(g)\,d\mu(g),$$

where $\mathbf{l}=\Gamma_k\mathbf{h}$ is an orbit for some $\mathbf{h}\in V_{m,k}(\mathbb{Z})$, and $|\Gamma_{k,\mathbf{l}}|=|\Gamma_{k,\mathbf{h}}|$ is the size of the stabilizer of any representative $\mathbf{h}\in\mathbf{l}$, which is independent of the choice of $\mathbf{h}\in\mathbf{l}$. Next, we use Fubini's theorem and write the above integral over the ternary quadric $V_{m,k}$ with its invariant measure induced from the transitive action of $SO_{q_k}$ on $V_{m,k}$. Recall that

$$\mathbf{x}_0:=\begin{bmatrix}\sqrt{|m|/(4k)}\\ \sqrt{|m|/(4k)}\\ 0\end{bmatrix}.$$

Since $SO_{q_k}$ acts transitively on $V_{m,k}$, for any $\mathbf{l}\in V_{m,k}$ there exist $l\in SO_{q_k}$ such that $l\mathbf{x}_0=\mathbf{l}$. In fact if $l\mathbf{x}_0=\mathbf{l}$ then $lk\mathbf{x}_0=\mathbf{l}$ for any $k\in K$. We write every element $g\in SO_{q_k}$ as $g=lkt$ for some $t\in K\backslash SO_{q_k}$ and $k\in K$. Since $d\mu$ is a Haar measure, $d\mu(lg)=d\mu(g)$. Note that $K$ is a compact group, so we normalize the Haar measure so that $\int_K dk=1$. We have

$$\begin{aligned}\rho_{F,\infty}(m,v)&=\sum_{\mathbf{l}\in H_k(m)}\frac{v^{3/4}}{|\Gamma_{k,\mathbf{l}}|}\int_{SO_{q_k}}e(ivP_0[g^{-1}\mathbf{l}])\bar{L}(g)\,d\mu(g)\\&=\sum_{\mathbf{l}\in H_k(m)}\frac{v^{3/4}}{|\Gamma_{k,\mathbf{l}}|}\int_{K\backslash SO_{q_k}}\int_K e(ivP_0[(lkt)^{-1}\mathbf{l}])\bar{L}(lkt)\,dk\,dt\\&=\sum_{\mathbf{l}\in H_k(m)}\frac{v^{3/4}}{|\Gamma_{k,\mathbf{l}}|}\int_{K\backslash SO_{q_k}}\int_K e(ivP_0[t^{-1}k^{-1}l^{-1}\mathbf{l}])\bar{L}(lkt)\,dk\,dt\\&=\sum_{\mathbf{l}\in H_k(m)}\frac{v^{3/4}}{|\Gamma_{k,\mathbf{l}}|}\int_{K\backslash SO_{q_k}}\int_K e(ivP_0[t^{-1}\mathbf{x}_0])\bar{L}(lkt)\,dk\,dt\\&=\sum_{\mathbf{l}\in H_k(m)}\frac{v^{3/4}}{|\Gamma_{k,\mathbf{l}}|}\int_{K\backslash SO_{q_k}}e(ivP_0[t^{-1}\mathbf{x}_0])\int_K\bar{L}(lkt)\,dk\,dt.\end{aligned}$$

Recall that $L(lkt) = L(lkt\mathbf{x}_0)$. We take the integral over the compact group $K$ and obtain

$$\rho_{F,\infty}(m,v) = \sum_{\mathbf{l}\in H_k(m)} \frac{v^{3/4}}{|\Gamma_{k,\mathbf{l}}|} \int_{K\backslash SO_{q_k}} e(ivP_0[t^{-1}\mathbf{x}_0])V_l(t)\,dt, \tag{3.20}$$

where $V_l(t) := \int_K \bar{L}(lktx_0)\,dk$. By our normalization of the Haar measure of $K$ we obtain

$$\sup_{t\in K\backslash SO_{q_k}} |V(t)| \leq \sup_{\mathbf{x}\in V_{m,k}} |L(\mathbf{x})|.$$

So $V_l$ is a bounded function on $K\backslash SO_{q_k}$. We note that the quotient space $K\backslash SO_{q_k}$ is identified with $V_{m,k}$ by sending $t \in K\backslash SO_{q_k}$ to $\mathbf{h} := t^{-1}\mathbf{x}_0 \in V_m$ and we write

$$\mathbf{h} := \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}.$$

The measure $dt$ is identified with the invariant measure defined over $V_{m,k}$ that is the hyperbolic measure. We denote this measure by $d\mu$. Next, we change the variables and write the integral (3.20) that is over the quotient space $K\backslash SO_{q_k}$ in terms of an integral over $V_{m,k}$ and its hyperbolic measure. We also consider the smooth weight function $V_l(t)$ as a function on $V_{m,k}$ by our identification $t \mapsto t^{-1}\mathbf{x}_0 \in V_{m,k}$. Hence, we obtain

$$\rho_{F,\infty}(m,v) = \sum_{\mathbf{l}\in H_k(m)} \frac{v^{3/4}}{|\Gamma_{k,\mathbf{l}}|} \int_{V_{m,k}} e(ivP_0[h])V_l(\mathbf{h})\,d\mu.$$

Let $I(l,v) := v^{3/4}\int_{V_{m,k}} e(ivP_0[h])V_l(\mathbf{h})\,d\mu$. Then

$$\rho_{F,\infty}(m,v) = \sum_{\mathbf{l}\in H_k(m)} \frac{1}{|\Gamma_{k,\mathbf{l}}|} I(l,v). \tag{3.21}$$

Next, we give an asymptotic formula for $I(l,v)$ as $v \to \infty$. We note that $P_0[\mathbf{h}] = 2kh_1^2 + 2kh_2^2 + h_3^2$. Then

$$I(l,v) = v^{3/4}\int_{V_{m,k}} \exp(-2\pi v(2kh_1^2 + 2kh_2^2 + h_3^2))V_l(\mathbf{h})\,d\mu.$$

Since $\mathbf{h} \in V_m$, we have $h_3^2 - 4kh_1h_2 = m$, and we obtain

$$I(l,v) = \exp(-2\pi v|m|)v^{3/4}\int_{V_{m,k}} \exp(-2\pi v(2k(h_1-h_2)^2 + 2h_3^2))V_l(\mathbf{h})\,d\mu.$$

We change the variables to $u_1 := \frac{h_1\sqrt{2k}}{\sqrt{|m|}}$, $u_2 := \frac{h_2\sqrt{2k}}{\sqrt{|m|}}$ and $u_3 := \frac{h_3}{\sqrt{|m|}}$. Hence,

$$I(l,v) = \exp(-2\pi v|m|)v^{3/4}\int_{u_3^2-u_1u_2=-1} \exp(-2\pi vm((u_1-u_2)^2 + 2u_3^2))V_l(\mathbf{u})\,d\mu.$$

We note that as $v \to \infty$ the above integral localizes around $\mathbf{u}_0 = (1, 1, 0)$. By stationary phase,

$$\lim_{v\to\infty} \int_{u_3^2-u_1u_2=-1} \exp(-2\pi vm((u_1-u_2)^2+2u_3^2))V_l(\mathbf{u})\,d\mu = \left(1/2+O\left(\frac{1}{\sqrt{v}}\right)\right)\frac{V_l(\mathbf{x}_0)}{v|m|},$$

where $\mathbf{x}_0 = \begin{bmatrix} \frac{1}{2}\sqrt{|m|/k} \\ \frac{1}{2}\sqrt{|m|/k} \\ 0 \end{bmatrix}$ is the minimum of the quadratic form $2k(h_1-h_2)^2+2h_3^2$ on $V_{m,k}$. Note that

$$V_l(\mathbf{x}_0) := \int_K \bar{L}(lk\mathbf{x}_0)\,dk = \bar{L}(l\mathbf{x}_0) = \bar{L}(\mathbf{l}).$$

Therefore,

$$I(l, v) = \exp(-2\pi v|m|)(4\pi|m|v)^{-1/4}\bar{L}(\mathbf{l})\frac{|m|^{-3/4}\pi^{1/4}}{\sqrt{2}}(1 + O(1/\sqrt{v})).$$

We insert the above identity for $I(l, v)$ into (3.21) and obtain

$$\begin{aligned}\rho_{F,\infty}(m, v)& \\ &= \exp(-2\pi v|m|)(4\pi|m|v)^{-1/4}\frac{|m|^{-3/4}\pi^{1/4}}{\sqrt{2}}\sum_{\mathbf{l}\in H_k(m)}\frac{1}{|\Gamma_{k,\mathbf{l}}|}\bar{L}(\mathbf{l})(1 + O(1/\sqrt{v}))\\ &= \exp(-2\pi v|m|)(4\pi|m|v)^{-1/4}\frac{|m|^{-3/4}\pi^{1/4}}{\sqrt{2}}\overline{R(m, L)}(1 + O(1/\sqrt{v})).\end{aligned}$$

By (3.19), we have

$$\rho_{F,\infty}(m) = \frac{|m|^{-3/4}\pi^{1/4}}{\sqrt{2}}\overline{R(m, L)}.$$

This completes the proof of the Maass identity. □

### 3.4. *Bounding* $\mathrm{Er}_{\mathrm{low}}$

Recall that $\mathrm{Er}_{\mathrm{low}} := \sum_{\lambda<T}\langle W, f_\lambda\rangle R(m, f_\lambda)$, where $T = |D|^\delta$ for some fixed power $\delta > 0$. In this section, we give an upper bound on $\mathrm{Er}_{\mathrm{low}}$. Shimura [Shi73, p. 450] defined Hecke operators $T_{v^2}$ for all integers $v$ (including $v = 2$ and $v \mid 4k$) for holomorphic half-integral weight modular forms. It is standard to define the analogue of them for the space of half-integral weight Maass forms with level $\Gamma_0(4k)$ and check that the $T_{v^2}$ form a commutative set of linear transformations which commute with $\Delta_{1/2}$; see [KS93, Proposition 1.4]. Moreover, $T_v^2$ is self-adjoint with respect to the Petersson inner product if $\gcd(v, 4k) = 1$. Therefore, there exists an orthonormal basis of simultaneous eigenfunctions of $\Delta_{1/2}$ and all $T_{v^2}$, where $\gcd(v, 4k) = 1$ for weight $1/2$ Maass cusp forms with level $\Gamma_0(4k)$. Let

$$B_T := \{\psi_{\lambda'} : \psi_{\lambda'} \text{ is an eigenfunction of } \Delta_{1/2} \text{ and all } T_{p^2}, \text{ where } \gcd(p, 4k) = 1\} \tag{3.22}$$

be an orthonormal basis of weight 1/2 cusp forms of level $\Gamma_0(4k)$ and $\Delta_{1/2}$ eigenvalue $\lambda' < T/4 + 3/16$, and $\psi_{\lambda'}$ is not an elementary theta series, which means $\lambda' \neq 3/16$. It is known that $\psi_{\lambda'}(z)$ has a Fourier development at $\infty$ of the form

$$\psi_{\lambda'}(u+iv) = c_{\psi_{\lambda'},\infty}(v) + \sum_{n\neq 0} \rho_{\psi_{\lambda'},\infty}(n) W_{(1/4)\mathrm{sgn}(n),it}(4\pi|n|v)e(nu),$$

where $1/4+t^2 = \lambda'$, $c_{\psi_{\lambda'},\infty}(v)$ is a linear combination of $v^{1/2+it}$ and $v^{1/2-it}$ and $W_{\beta,\mu}(v)$ is the Whittaker function normalized so that $W_{\beta,\mu}(v) \approx e^{-v/2}v^{\beta}$ as $v \to \infty$. We note that the asymptotic of the Whittaker function is independent of the spectral parameter $\lambda$. In the following lemma, we apply the Maass identity proved in Lemma 3.13 and write $\mathrm{Er}_{\mathrm{low}}$ in terms of the Fourier coefficients of $\psi_{\lambda'} \in B_T$.

**Lemma 3.14.** *We have*

$$\overline{\mathrm{Er}_{\mathrm{low}}} = |m|^{3/4}\pi^{-1/4}\sqrt{2} \sum_{\psi_{\lambda'}\in B_T} \langle \Theta * W, \psi_{\lambda'}\rangle \rho_{\psi_{\lambda'},\infty}(m).$$

*Proof.* Let $W_T = \sum_{0\leq\lambda\leq T}\langle W, f_\lambda\rangle f_\lambda$, where $\{f_\lambda\}$ is an orthonormal basis of the cusp forms with the $\Omega$ eigenvalue less than $T$. Since $W_T$ is a finite linear combination of Maass cusp forms, it decays rapidly at cusps. By Lemma 3.13, we have

$$\overline{\mathrm{Er}_{\mathrm{low}}} = \overline{R(m, W_T)} = |m|^{3/4}\pi^{-1/4}\sqrt{2}\, \rho_{\Theta * W_T,\infty}(m). \tag{3.23}$$

Note that $\Theta * W$ is orthogonal to the elementary theta series; see [KS93, last line of p. 79]. By Theorem 3.11 (see Remark 3.12), $\Theta * W_T$ is spanned by the orthonormal basis $B_T$. Hence,

$$\Theta * W_T = \sum_{\psi_{\lambda'}\in B_T} \langle \Theta * W, \psi_{\lambda'}\rangle \psi_{\lambda'}.$$

By computing the $m$-th Fourier coefficient of both sides of the above identity and using the asymptotic of the Whittaker function, we have

$$\rho_{\Theta * W_T,\infty}(m) = \sum_{\psi_{\lambda'}\in B_T} \langle \Theta * W, \psi_{\lambda'}\rangle \rho_{\psi_{\lambda'},\infty}(m).$$

By the above and (3.23), it follows that

$$\overline{\mathrm{Er}_{\mathrm{low}}} = |m|^{3/4}\pi^{-1/4}\sqrt{2} \sum_{\psi_{\lambda'}\in B_T} \langle \Theta * W, \psi_{\lambda'}\rangle \rho_{\psi_{\lambda'},\infty}(m).$$

This completes the proof of the lemma. □

Finally, we bound the contribution of $\mathrm{Er}_{\mathrm{low}}$.

**Proposition 3.15.** *We have*

$$|\mathrm{Er}_{\mathrm{low}}| \ll |m|^{-1/28} k^{10} X^{1+\epsilon} T^7.$$

*Proof.* By Lemma 3.14, we have

$$|\mathrm{Er}_{\mathrm{low}}| \leq |m|^{3/4}\pi^{-1/4}\sqrt{2} \sum_{\lambda' < T/4+3/16} |\langle \Theta * W, \psi_{\lambda'}\rangle|\,|\rho_{\psi_{\lambda'},\infty}(m)|. \tag{3.24}$$

Recall that $m = Dv_0^2$ where $D$ is a fundamental discriminant, and $\gcd(k, v_0) = 1$. By Theorem 5.1, we have

$$|\rho_{\psi_{\lambda'},\infty}(m)| \ll_\varepsilon |\lambda'|^{3/2}\cosh(\pi t/2)|m|^{-2/7+\varepsilon}. \tag{3.25}$$

Duke stated the above inequality only for the square-free integers $m$, since this bound does not hold for the Fourier coefficients of elementary theta series at squares $m$. Note that $\Theta * W$ is orthogonal to the elementary theta series; see [KS93, last line of p. 79]. Following Sarnak [Sar90, p. 100], we extend Duke's upper bound in Theorem 5.1 to all $m = Dv_0^2$, where $\gcd(v_0, k) = 1$.

Next, we give an upper bound on $\langle \Theta * W, \psi_{\lambda'}\rangle$. We have

$$\begin{aligned}\langle \Theta * W, \psi_{\lambda'}\rangle &= \int_{\Gamma_0(4k)\backslash\mathbb{H}} \overline{\psi_{\lambda'}(z)} \int_{\Gamma_k\backslash V_{m,k}} \Theta(z, \mathbf{h}) w(\mathbf{h})\, d\mu(\mathbf{h})\, d\eta(z)\\ &= \int_{\Gamma_k\backslash V_{m,k}} w(\mathbf{h}) \int_{\Gamma_0(4k)\backslash\mathbb{H}} \overline{\psi_{\lambda'}(z)}\Theta(z, \mathbf{h})\, d\eta(z)\, d\mu(\mathbf{h}),\end{aligned} \tag{3.26}$$

where $d\eta(z)$ and $d\mu(\mathbf{h})$ are invariant measures on $\Gamma_0(4k)\backslash\mathbb{H}$ and $\Gamma_k\backslash V_{m,k}$, respectively. Let

$$\varphi_\lambda(h) := \int_{\Gamma_0(4k)\backslash\mathbb{H}} \Theta(z, \mathbf{h})\overline{\psi_{\lambda'}(z)}\, d\eta(z).$$

It follows from Theorem 3.11 that $\varphi_\lambda$ is a Maass form of weight zero and eigenvalue $\lambda = 4\lambda' - 3/4$. We say $\varphi_\lambda$ is the *theta lift* of the weight $1/2$ modular form $\psi_{\lambda'}$. By equation (3.26), we have

$$\langle \Theta * W, \psi_{\lambda'}\rangle = \int_{\Gamma_k\backslash V_{m,k}} \varphi_\lambda(\mathbf{h}) W(\mathbf{h})\, d\mu(\mathbf{h}) = \langle \varphi_\lambda, W\rangle.$$

By the Cauchy–Schwarz inequality

$$|\langle \Theta * W, \psi_{\lambda'}\rangle| \leq |W|_2 |\varphi_\lambda|_2,$$

where $|W|_2$ and $|\varphi_\lambda|_2$ are the $L^2$ norm of $W$ and $\varphi_\lambda$. By Lemma 3.7, we have $|W|_2 \ll \frac{X^{1+\epsilon}}{\sqrt{d_1 m}}$. By Theorem 4.9, we have $|\varphi_\lambda|_2 \ll \cosh(-\pi t/2)k^9\lambda^{9/2}$. Therefore,

$$|\langle \Theta * W, \psi_{\lambda'}\rangle| \ll \cosh(-\pi t/2)k^9\lambda^{9/2}\frac{X^{1+\epsilon}}{\sqrt{m}}.$$

By applying the above and the inequality (3.25) in equation (3.24), we obtain

$$|\mathrm{Er}_{\mathrm{low}}| \ll |m|^{3/4}\left(\sum_{\lambda' < T/4+3/16} |\lambda|^{3/2}\cosh(\pi t/2)|m|^{-2/7+\varepsilon}\cosh(-\pi t/2)k^9\lambda^{9/2}\frac{X^{1+\epsilon}}{\sqrt{m}}\right).$$

By the Weyl law the number of eigenvalues $\lambda' \leq T$ is bounded by $kT$. Therefore,

$$|\mathrm{Er}_{\mathrm{low}}| \ll |m|^{-1/28} k^{10} X^{1+\epsilon} T^7.$$

We choose $T = |D|^{\delta}$ for a small fixed $\delta > 0$. □

### 3.5. *Bounding* $\mathrm{Er}_{\mathrm{cts,low}}$

We briefly explain our method for bounding $\mathrm{Er}_{\mathrm{cts,low}}$. Recall that

$$\mathrm{Er}_{\mathrm{cts,low}} := \sum_{\mathfrak{a} \in \mathcal{E}_k} \int_{|1/4+t^2| \leq T} \langle W, E_{\mathfrak{a}}(\cdot, 1/2 + it) \rangle R(m, E_{\mathfrak{a}}(\mathbf{v}, 1/2 + it))\, dt.$$

We wish to apply Lemma 3.13 to $R(m, E_{\mathfrak{a}}(\mathbf{v}, 1/2 + it))$, which relates this quantity to the $m$-th Fourier coefficient of $\Theta * E_{\mathfrak{a}}(\mathbf{v}, 1/2 + it)$. However, we note that the theta integral $\Theta * E_{\mathfrak{a}}(\mathbf{v}, 1/2 + it)$ is not absolutely convergent and we need to regularize it. To this end, we use the center of the enveloping algebra (Casimir operator). This method has been used in the work of Maass [Maa71], Deitmar and Krieg [DK91] and Kudla and Rallis [KR94, Section 5]. We begin by proving an auxiliary lemma.

**Lemma 3.16.** *We have*

$$\frac{1}{k^{2.5}} \ll y_{\mathfrak{a}}(\mathbf{x}_0) \ll 1 \quad \textit{for every } \mathfrak{a} \in \mathcal{E}_k.$$

*Proof.* Let $\mathbf{v}_0 := \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \in V_{-4k,k}$. By scaling $\Gamma_k \backslash V_{-4k,k}$ maps isometrically to $\Gamma_k \backslash V_{m,k}$. This maps $\mathbf{v}_0$ to $\mathbf{x}_0$. Hence $y_{\Gamma_k}(\mathbf{x}_0) = y_{\Gamma_k}(\mathbf{v}_0)$. It follows that

$$y_{\Gamma_k}(\mathbf{v}_0) \ll \max_{n_{\mathfrak{a}}} \left(1, \frac{1}{\mathrm{dist}(\mathbf{v}_0, n_{\mathfrak{a}} \mathbf{v}_0)}\right),$$

where $n_{\mathfrak{a}}$ is a parabolic element of $\Gamma_k$. Since, $n_{\mathfrak{a}}$ is parabolic, $\mathbf{v}_0 \neq n_{\mathfrak{a}} \mathbf{v}_0$. Let $n_{\mathfrak{a}} \mathbf{v}_0 = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$ for some $n_1, n_2, n_3 \in \mathbb{Z}$, where $n_3^2 - 4kn_1n_2 = -4k$. By integrality of $n_i$ and $\mathbf{v}_0 \neq n_{\mathfrak{a}} \mathbf{v}_0$, we have $|n_1 n_2| \geq 2$. Hence, $|n_3| \geq 2\sqrt{k}$. We define the following isometry from $V_{-4k,k}$ to the upper half-plane $\mathbb{H}$:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in V_{-4k,k} \mapsto \begin{bmatrix} ka_1/(2\sqrt{k}) \\ a_2/(2\sqrt{k}) \\ a_3/(2\sqrt{k}) \end{bmatrix} \in V_{-1,1} \mapsto z_a = \frac{-a_3 + i2\sqrt{k}}{2ka_1} \in H. \tag{3.27}$$

We note that $\mathbf{v}_0$ maps to $\frac{i}{\sqrt{k}}$ and $n_{\mathfrak{a}} \mathbf{v}_0$ maps to $\frac{-n_3}{2kn_1} + \frac{i}{n_1\sqrt{k}}$. Hence,

$$\mathrm{dist}(\mathbf{v}_0, n_{\mathfrak{a}} \mathbf{v}_0) = \mathrm{dist}\left(i, \frac{-n_3}{2\sqrt{k}\, n_1} + \frac{i}{n_1}\right) \gg 1.$$

This completes the proof of our upper bound. To prove the lower bound, we identify $SO_{q_k}$ with $PSL_2(\mathbb{R})$, so that $\Gamma_k$ is identified with $\Gamma' \subset SO_{q_k}$, where $\Gamma'$ contains the congruence subgroup $\Gamma_0(k)$. Then we parametrize the cusps of $\Gamma_0(k)$ with $1/w$ for $1 \le w \le k$ and show $1/k^{2.5} \le y_{1/w}(i/\sqrt{k})$. Since $\Gamma'\backslash\mathbb{H}$ is a covering of $\Gamma_0(k)\backslash\mathbb{H}$ and $\mathbf{x}_0$ maps to $i/\sqrt{k}$, we have $1/k^{2.5} \le y_{1/w}(i/\sqrt{k}) \le y_{\mathfrak{a}}(\mathbf{x}_0)$.

We give the details of our argument. $PSL_2(\mathbb{R})$ acts on the space of binary quadratic forms $Q := \{ax^2+bxy+cy^2 : a, b, c \in \mathbb{R}\}$ by linear change of variables and it preserves the discriminant of the binary quadratic forms

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.F(x, y) = F(ax + by, cx + dy).$$

This identifies $PSL_2(\mathbb{R})$ with $SO_{q_1}$ where $q_1(x, y, z) = z^2 - 4xy$ through the map

$$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto g_\gamma = \begin{bmatrix} a^2 & c^2 & ac \\ b^2 & d^2 & bd \\ 2ab & 2cd & ad+bc \end{bmatrix}.$$

Let $B_k := \left[\begin{smallmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right]$. Then

$$B_k^{\intercal} \begin{bmatrix} 0 & -2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} B_k = \begin{bmatrix} 0 & -2k & 0 \\ -2k & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We note that if $g \in SO_{q_1}$ then $B_k^{-1} g B_k \in SO_{q_k}$. This identifies $PSL_2(\mathbb{R})$ with $SO_{q_k}$,

$$\gamma \in PSL_2(\mathbb{R}) \mapsto g_\gamma \in SO_{q_1} \mapsto B_k^{-1} g_\gamma B_k \in SO_{q_k}. \tag{3.28}$$

By the above isomorphism the lattice $\Gamma_k \subset SO_{q_k}$ is identified with $\Gamma' \subset PSL_2(\mathbb{R})$, where

$$\Gamma' := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PSL_2(\mathbb{R}) : \begin{bmatrix} a^2 & k^{-1}c^2 & k^{-1}ac \\ kb^2 & d^2 & bd \\ 2kab & 2cd & ad+bc \end{bmatrix} \in M_{3\times 3}(\mathbb{Z}) \right\}.$$

It is easy to check that $\Gamma'$ contains the congruence subgroup

$$\Gamma_0(k) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z} \text{ and } k \mid c \right\}.$$

By Propositions 4.2 and 4.3, the cusps of $\Gamma_0(k)$ are parametrized (not uniquely) with $1/w$ for $1 \le w \le k$ and its scaling matrix is

$$\sigma_{1/w} = \begin{bmatrix} 1 & 0 \\ w & 1 \end{bmatrix} \begin{bmatrix} \sqrt{k''} & 0 \\ 0 & 1/\sqrt{k''} \end{bmatrix},$$

where $k'' = \frac{k'}{\gcd(k', w)}$ and $k' = \frac{k}{\gcd(k, w)}$. Since $\mathbf{x}_0$ maps to $\frac{i}{\sqrt{k}}$, we have

$$y_{1/w}\left(\frac{i}{\sqrt{k}}\right) = \operatorname{Im}\left(\sigma_{1/w}^{-1}\left(\frac{i}{\sqrt{k}}\right)\right) = \frac{1}{k''}\frac{\sqrt{k}}{k+w^2} \ge \frac{1}{k^{2.5}}.$$

This completes the proof of our lemma. □

Let $\theta(z,g) := \Omega\Theta(z,g)$. Since $\Omega$ is inside the center of enveloping algebra, $\theta(z,g)$ remains a theta kernel. The following theorem follows from the work of Kudla and Rallis [KR94, Proposition 5.3.1]; see also [TG, Lemma 7.7].

**Proposition 3.17.** *$\theta(z,g) = O_C(|y_{\Gamma_0(4k)}(z)|^{-A})$ for every $A > 0$ and $g \in C$, where $C \subset \Gamma_k \backslash SO_{q_k}$ is a fixed compact subset. Similarly $\theta(z,g) = O_{C'}(|y_{\Gamma_k}(g)|^{-A})$ for every $A > 0$ and $g \in K$, where $C' \subset \Gamma_0(4k)\backslash\mathbb{H}$ is a fixed compact subset. Moreover, for every $z$,*

$$\int_{\Gamma_k \backslash SO_{q_k}} \theta(z,g)\, d\mu(g) = 0.$$

Let $\mathcal{E}'_{4k} := \{\mathfrak{a}' : \mathfrak{a}' \text{ ranges over all inequivalent cusps of } \Gamma_0(4k)\backslash\mathbb{H}\}$. For $z \in \Gamma_0(4k)\backslash\mathbb{H}$ and $s \in \mathbb{C}$, let $E_{\mathfrak{a}'}(z,s)$ be the Eisenstein series of weight $1/2$ whose constant Fourier coefficient at cusp $\mathfrak{b}'$ is $\delta_{\mathfrak{a}'\mathfrak{b}'} y_{\mathfrak{b}'}^{s} + \varphi_{\mathfrak{a}'\mathfrak{b}'}(s) y_{\mathfrak{b}'}^{1-s}$; see [Duk88, Section 2].

**Proposition 3.18.** *We have*

$$\overline{\mathrm{Er}_{\mathrm{cts,low}}} = \frac{\sqrt{2}}{\pi^{1/4}}|m|^{3/4} \sum_{\mathfrak{a}' \in \mathcal{E}'_{4k}} \int_{|1/4+t^2| \le T} \frac{1}{1/4+t^2} \rho_{E_{\mathfrak{a}'}(\cdot,1/2+it/2),\infty}(m) \langle \theta * W, E_{\mathfrak{a}'}(\cdot,1/2+it/2)\rangle\, dt.$$

*Proof.* Since $W$ is compactly supported and $\theta$ is rapidly decreasing uniformly on compact sets, $\theta * W(z)$ is also rapidly decreasing on $\Gamma_0(4k)\backslash\mathbb{H}$. Since the Eisenstein series $E_{\mathfrak{a}}(\cdot, 1/2+it)$ has moderate growth on $\Gamma_k\backslash SO_{q_k}$ and $\theta$ is rapidly decreasing on $\Gamma_k\backslash SO_{q_k}$,

$$\theta(z,\cdot) * E_{\mathfrak{a}}(\cdot,1/2+it) := \int_{\Gamma_K \backslash SO_{q_k}} \theta(z,g) \overline{E_{\mathfrak{a}}(g\mathbf{x}_0, 1/2+it)}\, d\mu(g)$$

is absolutely convergent. By the Siegel–Weil formula, we have

$$\theta(z,\cdot) * E_{\mathfrak{a}}(\cdot,1/2+it) = \sum_{\mathfrak{a}' \in \mathcal{E}'_{4k}} \alpha_{\mathfrak{a}\mathfrak{a}'}(t) E_{\mathfrak{a}'}(z, 1/2+it/2),$$

where $\alpha_{\mathfrak{a}\mathfrak{a}'}(t)$ is an analytic function for every pair $\mathfrak{a}\mathfrak{a}'$ . Since $\theta$ transfers the cusp forms on $\Gamma_k\backslash SO_{q_k}$ to the weight $1/2$ cusp forms on $\Gamma_0(4k)\backslash\mathbb{H}$ and $\theta * W(z)$ is rapidly decreasing, by the Plancherel theorem

$$\sum_{\mathfrak{a} \in \mathcal{E}_k} \overline{\langle W, E_{\mathfrak{a}}(\cdot,1/2+it)\rangle} \theta(z,\cdot) * E_{\mathfrak{a}}(\cdot,1/2+it) = \sum_{\mathfrak{a}' \in \mathcal{E}'_{4k}} \langle \theta * W, E_{\mathfrak{a}'}(\cdot,1/2+it/2)\rangle E_{\mathfrak{a}'}(z,1/2+it/2)$$

for every $t \in \mathbb{R}$. Hence, the $m$-th Fourier coefficients of both sides are equal, and we obtain

$$\sum_{\mathfrak{a} \in \mathcal{E}_k} \overline{\langle W, E_{\mathfrak{a}}(\cdot,1/2+it)\rangle} \rho_{\theta * E_{\mathfrak{a}}(\cdot,1/2+it)}(m) = \sum_{\mathfrak{a}' \in \mathcal{E}'_{4k}} \langle \theta * W, E_{\mathfrak{a}'}(\cdot,1/2+it/2)\rangle \rho_{E_{\mathfrak{a}'}(\cdot,1/2+it/2),\infty}(m)$$

for every $t \in \mathbb{R}$. By a similar computation to that in Lemma 3.13, and by the identity

$\Omega E_{\mathfrak{a}}(\mathbf{v}, s) = s(1-s)E_{\mathfrak{a}}(\mathbf{v}, s)$, it follows that

$$\frac{\sqrt{2}}{\pi^{1/4}}|m|^{3/4}\rho_{\theta * E_{\mathfrak{a}}(\cdot,s),\infty}(m) = \overline{s(s-1)R(m, E_{\mathfrak{a}}(\cdot, s))}.$$

Finally,

$$\begin{aligned}
\overline{\mathrm{Er}_{\mathrm{cts,low}}} &= \sum_{\mathfrak{a}\in\mathcal{E}_k}\int_{|1/4+t^2|\leq T}\overline{\langle W, E_{\mathfrak{a}}(\cdot, 1/2+it)\rangle R(m, E_{\mathfrak{a}}(\cdot, 1/2+it))}\,dt \\
&= \frac{\sqrt{2}}{\pi^{1/4}}|m|^{3/4}\sum_{\mathfrak{a}\in\mathcal{E}_k}\int_{|1/4+t^2|\leq T}\frac{1}{1/4+t^2}\langle E_{\mathfrak{a}}(\cdot, 1/2+it), W\rangle\rho_{\theta * E_{\mathfrak{a}}(\cdot,1/2+it),\infty}(m)\,dt \\
&= \frac{\sqrt{2}}{\pi^{1/4}}|m|^{3/4} \\
&\quad\times\sum_{\mathfrak{a}'\in\mathcal{E}'_{4k}}\int_{|1/4+t^2|\leq T}\frac{1}{1/4+t^2}\rho_{E_{\mathfrak{a}'}(\cdot,1/2+it/2),\infty}(m)\langle\theta * W, E_{\mathfrak{a}'}(\cdot, 1/2+it/2)\rangle\,dt. \qquad \square
\end{aligned}$$

**Proposition 3.19.** *We have*

$$\begin{aligned}
&|\langle\theta * W, E_{\mathfrak{a}'}(\cdot, 1/2+it/2)\rangle| \\
&\qquad\ll k^5(1+|t|)^{2.25}|\Gamma(1/4+it/2)|\sum_{\mathfrak{a}\in\mathcal{E}_k}|\langle W, E_{\mathfrak{a}}(\cdot, 1/2+it)\rangle|\,.
\end{aligned}$$

*Proof.* For simplicity, we give the proof for $\mathfrak{a}' = \infty$. We briefly explain how to prove this proposition for every cusp $\mathfrak{a}' \in \mathcal{E}'_{4k}$. By Proposition 4.2, $\mathcal{E}'_{4k}$ is parametrized (not uniquely) with $1/w$ for $1 \leq w \leq 4k$. We use the scaling matrices $\sigma_{1/w}$ that are introduced in Proposition 4.3 and the transformation properties of the theta series (see [Duk88, equation 4.4]), to reduce the problem to the cases $\mathfrak{a}' = \infty$.

We write $E(\cdot, 1/2+it/2)$ for $E_\infty(\cdot, 1/2+it/2)$. Suppose that $\mathrm{Re}(s) > 1$. Then by unfolding $\int_{\Gamma_0(4k)\backslash\mathbb{H}}$ against the Eisenstein series and unfolding $\int_{\Gamma_k\backslash SO_{q_k}}$ against $W(g)$, we have

$$\begin{aligned}
\langle\theta * W, E(\cdot, s)\rangle &= \int_{\Gamma_k\backslash SO_{q_k}}\int_{\Gamma_0(4k)\backslash\mathbb{H}}\theta(z, g)\bar{E}(z, s)W(g)\,\frac{dx\,dy}{y^2}\,d\mu(g) \\
&= \int_{SO_{q_k}}\int_0^1\int_0^\infty\theta(z, g)y^{\bar{s}}w(g)\,\frac{dy}{y^2}\,dx\,d\mu(g) \\
&= \int_{SO_{q_k}}\int_0^1\int_1^\infty\Big(y^{3/4}\sum_{\mathbf{h}\neq 0\in\mathbb{Z}^3}e(xq_k(\mathbf{h}))e(iy\mathbf{h}^\intercal g^{-\intercal}P_0g^{-1}\mathbf{h})\Big)y^{\bar{s}}\Omega w(g)\,\frac{dy}{y^2}\,dx\,d\mu(g) \\
&= \int_{SO_{q_k}}\int_0^\infty\Big(y^{3/4}\sum_{\substack{\mathbf{h}\neq 0\in\mathbb{Z}^3\\ q_k(\mathbf{h})=0}}e(iy\mathbf{h}^\intercal g^{-\intercal}P_0g^{-1}\mathbf{h})\Big)y^{\bar{s}}\Omega w(g)\,\frac{dy}{y^2}\,d\mu(g) \\
&= \overline{\Gamma(s-1/4)}\int_{SO_{q_k}}\overline{\sum_{\substack{\mathbf{h}\neq 0\in\mathbb{Z}^3\\ q_k(\mathbf{h})=0}}\frac{1}{(\mathbf{h}^\intercal g^{-\intercal}P_0g^{-1}\mathbf{h})^{s-1/4}}}\Omega w(g)\,d\mu(g).
\end{aligned}$$

Let

$$\psi(g,s) := \sum_{\substack{\mathbf{h}\neq 0\in\mathbb{Z}^3\\ q_k(\mathbf{h})=0}} \frac{1}{(\mathbf{h}^\intercal g^{-\intercal} P_0 g^{-1}\mathbf{h})^{s-1/4}}.$$

It follows that $\psi(g,s)$ is absolutely convergent for $\mathrm{Re}(s) > 3/4$. Note that $\Gamma_k$ acts on the set of projective vectors $[\mathbf{h}] \in P^2(\mathbb{Q})$, where $q_k(\mathbf{h}) = 0$, and the inequivalent classes are in one-to-one correspondence with $\mathcal{E}_k$. For every $\mathfrak{a} \in \mathcal{E}_k$, pick a primitive representative $\mathbf{h}_\mathfrak{a} \neq 0 \in \mathbb{Z}^3$. We have

$$\psi(g,s) = 2\zeta(2s-1/2) \sum_{\mathfrak{a}\in\mathcal{E}_k} \sum_{\gamma\in\Gamma_{k,\mathbf{h}_\mathfrak{a}}\backslash\Gamma_k} \frac{1}{(\mathbf{h}_\mathfrak{a}^\intercal(\gamma g)^{-\intercal} P_0(\gamma g)^{-1}\mathbf{h}_\mathfrak{a})^{s-1/4}},$$

where $\Gamma_{k,\mathbf{h}_\mathfrak{a}} \subset \Gamma_k$ is the unipotent subgroup which fixes $\mathbf{h}_\mathfrak{a}$. By Iwasawa decomposition, we write $\gamma g = n_\mathfrak{a}(\gamma g)a_\mathfrak{a}(\gamma g)k(\gamma g)$, where $k(\gamma g) \in K$, $n_\mathfrak{a}(\gamma g) \in N_\mathfrak{a}$, and $a_\mathfrak{a}(\gamma g)$ is a symmetric matrix where $a_\mathfrak{a}(\gamma g)\mathbf{h}_\mathfrak{a} = t\mathbf{h}_\mathfrak{a}$ for some $t_\mathfrak{a}(\gamma g) \in \mathbb{R}^+$. Hence,

$$\begin{aligned}
\frac{\psi(g,s)}{2\zeta(2s-1/2)} &= \sum_{\mathfrak{a}\in\mathcal{E}_k} \sum_{\gamma\in\Gamma_{k,\mathbf{h}_\mathfrak{a}}\backslash\Gamma_k} \frac{1}{(\mathbf{h}_\mathfrak{a}^\intercal(\gamma g)^{-\intercal} P_0(\gamma g)^{-1}\mathbf{h}_\mathfrak{a})^{s-1/4}}\\
&= \sum_{\mathfrak{a}\in\mathcal{E}_k} \frac{1}{(\mathbf{h}_\mathfrak{a}^\intercal P_0\mathbf{h}_\mathfrak{a})^{s-1/4}} \sum_{\gamma\in\Gamma_{k,\mathbf{h}_\mathfrak{a}}\backslash\Gamma_k} t_\mathfrak{a}(\gamma g)^{2s-1/2}\\
&= \sum_{\mathfrak{a}\in\mathcal{E}_k} \frac{1}{(\mathbf{h}_\mathfrak{a}^\intercal P_0\mathbf{h}_\mathfrak{a})^{s-1/4} y_\mathfrak{a}(\mathbf{x}_0)^{2s-1/2}} \sum_{\gamma\in\Gamma_{k,\mathbf{h}_\mathfrak{a}}\backslash\Gamma_k} y_\mathfrak{a}(\gamma g\mathbf{x}_0)^{2s-1/2}\\
&= \sum_{\mathfrak{a}\in\mathcal{E}_k} \frac{1}{(\mathbf{h}_\mathfrak{a}^\intercal P_0\mathbf{h}_\mathfrak{a})^{s-1/4} y_\mathfrak{a}(\mathbf{x}_0)^{2s-1/2}} E_\mathfrak{a}(g, 2s-1/2).
\end{aligned}$$

Note that both sides have analytic continuation to the whole complex plane. Hence

$$\psi(g,s) = \sum_{\mathfrak{a}\in\mathcal{E}_k} \frac{2\zeta(2s-1/2)}{(\mathbf{h}_\mathfrak{a}^\intercal P_0\mathbf{h}_\mathfrak{a})^{s-1/4} y_\mathfrak{a}(\mathbf{x}_0)^{2s-1/2}} E_\mathfrak{a}(g, 2s-1/2).$$

By Lemma 3.16, integrality of $\mathbf{h}_\mathfrak{a}^\intercal P_0\mathbf{h}_\mathfrak{a}$ and the convexity bound on the zeta function, for every $\mathfrak{a} \in \mathcal{E}_k$, we have

$$\left|\frac{2\zeta(1/2+it)}{(\mathbf{h}_\mathfrak{a}^\intercal P_0\mathbf{h}_\mathfrak{a})^{1/4+it/2} y_\mathfrak{a}(\mathbf{x}_0)^{1/2+it}}\right| \ll t^{1/4}k^{1.25}.$$

Therefore,

$$\begin{aligned}
&|\langle\theta * W, E(\cdot, 1/2+it/2)\rangle|\\
&\qquad \ll |\Gamma(1/4+it/2)|t^{1/4}k^{1.25}(1/4+t^2) \sum_{\mathfrak{a}\in\mathcal{E}_k} |\langle W, E_\mathfrak{a}(\cdot, 1/2+it)\rangle|.
\end{aligned}$$

This completes the proof of our proposition. □

Finally, we give an upper bound on the contribution of $\mathrm{Er}_{\mathrm{cts,low}}$.

**Proposition 3.20.** *We have* $|\mathrm{Er}_{\mathrm{cts,low}}| \ll k^{6.5} T^{7/4} |m|^{-1/28+\epsilon} |X|^{1+\epsilon}$.

*Proof.* By Proposition 3.18, we have

$$|\mathrm{Er}_{\mathrm{cts,low}}| \ll |m|^{3/4} \sum_{\mathfrak{a}' \in \mathcal{E}'_{4k}} \int_{|1/4+t^2| \le T} \frac{1}{1/4+t^2} |\rho_{E_{\mathfrak{a}'}(\cdot,1/2+it/2),\infty}(m)| \, |\langle \theta * W, E_{\mathfrak{a}'}(\cdot, 1/2+it/2)\rangle| \, dt.$$

Recall that $m = Dv_0^2$, where $\gcd(v_0, k) = 1$. By Theorem 5.1,

$$\left|\rho_{E_{\mathfrak{a}'}(\cdot,1/2+it/2),\infty}(m)\right| \ll (1+|t|)^3 \cosh(\pi t/4) |m|^{-2/7+\epsilon}.$$

By Proposition 3.19,

$$|\langle \theta * W, E_{\mathfrak{a}'}(\cdot, 1/2+it/2)\rangle| \ll k^5 (1+|t|)^{2.25} |\Gamma(1/4+it/2)| \sum_{\mathfrak{a} \in \mathcal{E}_k} |\langle W, E_{\mathfrak{a}}(\cdot, 1/2+it)\rangle| \,.$$

Proposition 4.2 implies that $|\mathcal{E}'_{4k}| \le 4k$. Hence,

$$\begin{aligned}
|\mathrm{Er}_{\mathrm{cts,low}}| &\ll |m|^{3/4} \sum_{\mathfrak{a}' \in \mathcal{E}'_{4k}} \int_{|1/4+t^2| \le T} \frac{1}{1/4+t^2} |\rho_{E_{\mathfrak{a}'}(\cdot,1/2+it/2),\infty}(m)| |\langle \theta * W, E_{\mathfrak{a}'}(\cdot, 1/2+it/2)\rangle| \, dt \\
&\ll k^6 |m|^{13/28+\epsilon} \sum_{\mathfrak{a} \in \mathcal{E}_k} \int_{|1/4+t^2| \le T} (1+|t|)^{3.25} \cosh(\pi t/4) |\Gamma(1/4+it/2)| \\
&\qquad \times |\langle W, E_{\mathfrak{a}}(\cdot, 1/2+it)\rangle| \, dt.
\end{aligned}$$

By Stirling's formula for the Gamma function, we have

$$|\Gamma(1/4+it/2)| \sim \sqrt{2\pi}\, |t/2|^{-1/4} e^{-\pi |t|/4}.$$

Hence,

$$|\mathrm{Er}_{\mathrm{cts,low}}| \ll k^6 |m|^{13/28+\epsilon} \sum_{\mathfrak{a} \in \mathcal{E}_k} \int_{|1/4+t^2| \le T} (1+|t|)^3 \, |\langle W, E_{\mathfrak{a}}(\cdot, 1/2+it)\rangle| \, dt.$$

By Plancherel's theorem and Lemma 3.7, we have

$$\sum_{\mathfrak{a} \in \mathcal{E}_k} \int_{|1/4+t^2| \le T} |\langle W, E_{\mathfrak{a}}(\cdot, 1/2+it)\rangle|^2 \le \int_{\Gamma_k \backslash V_{m,k}} |W|^2 \, d\mu \ll \frac{X^{2+\epsilon}}{d_1 m}.$$

Since $\Gamma_0(k)$ is a subgroup of $\Gamma_k$, $|\mathcal{E}_k| \le k$. By the Cauchy–Schwarz inequality, we have

$$\sum_{\mathfrak{a}\in\mathcal{E}_k} \int_{|1/4+t^2|\le T} (1+|t|)^3 \, |\langle W, E_{\mathfrak{a}}(\cdot, 1/2+it)\rangle| \, dt$$

$$\left[\sum_{\mathfrak{a}\in\mathcal{E}_k} \int_{|1/4+t^2|\le T} (1+|t|)^6 \, dt\right]^{1/2} \left[\sum_{\mathfrak{a}\in\mathcal{E}_k} \int_{|1/4+t^2|\le T} \left|\langle W, E_{\mathfrak{a}}(\cdot, 1/2+it)\rangle\right|^2 dt\right]^{1/2}$$

$$\ll k^{1/2} T^{7/4} \frac{X^{1+\epsilon}}{\sqrt{d_1 m}}.$$

Therefore, $\mathrm{Er}_{\mathrm{cts,low}} \ll k^6 |m|^{13/28+\epsilon} k^{1/2} T^{7/4} \frac{X^{1+\epsilon}}{\sqrt{d_1 m}} \ll k^{6.5} T^{7/4} |m|^{-1/28+\epsilon} |X|^{1+\epsilon}$. □

## 4. Bounding the $L^2$ norm of the Siegel theta transfer

### *4.1. The Mellin transform of the theta transfer*

We follow the same notations as in the previous sections. Let $f(z)$ be a weight 1/2 modular form on $\Gamma_0(4k)\backslash\mathbb{H}$ with $L^2$ norm 1 and eigenvalue $\lambda'$. Recall the Fourier expansion of $f(z)$ at $\infty$,

$$f(z) = c_{f,\infty}(y) + \sum_{n\neq 0} b_{f,\infty}(n) W_{(1/4)\mathrm{sgn}(n), it}(4\pi|n|y) e(nx),$$

where $1/4+t^2 = \lambda'$, $c_{f,\infty}(y)$ is a linear combination of $y^{1/2+it}$ and $y^{1/2-it}$ and $W_{\beta,\mu}(y)$ is the Whittaker function normalized so that $W_{\beta,\mu}(y) \approx e^{-y/2} y^{\beta}$ as $y \to \infty$. For $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in SL_2(\mathbb{R})$, we define

$$f_g(z) := \left(\frac{cz+d}{|cz+d|}\right)^{-1/2} f(gz),$$

where for $z \neq 0$ and $\nu \in \mathbb{R}$ we define $z^{\nu} = |z| \exp(i\nu \arg(z))$, where $\arg z \in (-\pi, \pi]$. Since $f$ is an eigenfunction of $\Delta_{1/2}$ with eigenvalue $\lambda'$ and invariant under $\Gamma_0(4k)$ with a multiplier of weight 1/2, it follows that $f_g$ is an eigenfunction of $\Delta_{1/2}$ with eigenvalue $\lambda'$ and is invariant under $g^{-1}\Gamma_0(4k)g$ with a multiplier of weight 1/2. Let

$$\varphi(g) := \int_{\Gamma_0(4k)\backslash\mathbb{H}} \Theta(x+iy, g) \overline{f(x+iy)} \frac{dx\,dy}{y^2}. \tag{4.1}$$

Recall that $\Theta(z, g)$ is $\Gamma_k$ invariant from the left and $K$ invariant from the right in the $g$ variable. It follows from Theorem 3.11 that $\varphi$ is a Maass form of weight zero and eigenvalue $\lambda = 4\lambda' - 3/4$ on $\Gamma_k\backslash V_{m,k}$. We consider the following torus $\mathbb{G}_m$ inside $SO_{q_k}$:

$$t \in \mathbb{G}_m \mapsto g_t := \begin{bmatrix} t & 0 & 0 \\ 0 & t^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \in SO_{q_k}.$$

In the following lemma, we compute the Mellin transfom of $\varphi$ along the above embedding of $\mathbb{G}_m$. Let

$$\Omega(s) := \int_0^\infty \varphi(g_t)t^s \frac{dt}{t}, \tag{4.2}$$
$$\theta(z) := y^{1/4}\sum_{h\in\mathbb{Z}} e((x+iy)h^2),$$
$$E(s,z) := {\sum_{h_1,h_2}}' \left(\frac{y}{|h_1+4h_2zD|^2}\right)^s, \tag{4.3}$$

where $\sum'_{h_1,h_2}$ is the sum over pairs of co-prime integers.

**Lemma 4.1.** *We have*

$$\Omega(s) = k^{s/2}2^{s-1}\Gamma\left(\frac{s+1}{2}\right)\pi^{-\frac{s+1}{2}} \int_{\Gamma_0(4k)\backslash\mathbb{H}} \overline{f(z)}\theta(z)E\left(\frac{s+1}{2}, z\right)\frac{dx\,dy}{y^2}.$$

*Proof.* We use the integral representation of $\varphi$ in equation (4.1) and obtain:

$$\begin{aligned}\Omega(s) &= \int_0^\infty\left(\int_{\Gamma_0(N)\backslash\mathbb{H}} \overline{f(x+iy)}\Theta(x+iy,g_t)\frac{dx\,dy}{y^2}\right)t^s\frac{dt}{t}\\ &= \int_{\Gamma_0(4k)\backslash\mathbb{H}}\overline{f(x+iy)}\left(\int_0^\infty\Theta(x+iy,g_t)t^s\frac{dt}{t}\right)\frac{dx\,dy}{y^2}.\end{aligned} \tag{4.4}$$

Next, we split $\Theta(z,g_t)$ into a product of two theta series. By the definition (3.18), we have

$$\begin{aligned}\Theta(x+iy,g_t) &:= y^{3/4}\sum_{h_1,h_2,h_3\in\mathbb{Z}} e(x(h_3^2-4kh_1h_2))e(iy(2kt^{-2}h_1^2+2kt^2h_2^2+h_3^2))\\ &= \left(y^{1/4}\sum_{h\in\mathbb{Z}}e((x+iy)h^2)\right)\left(y^{1/2}\sum_{h_1,h_2\in\mathbb{Z}}e((-4kxh_1h_2)e(iy(2kt^{-2}h_1^2+2kt^2h_2^2))\right).\end{aligned} \tag{4.5}$$

We note that the first term on the right hand side is the elementary theta series in one variable:

$$\theta(z) := y^{1/4}\sum_{h\in\mathbb{Z}}e((x+iy)h^2).$$

We denote the second term by $\theta_2(z,t)$. By the symmetry between $h_1$ and $h_2$ we have $\theta_2(z,t)=\theta_2(z,t^{-1})$. By (4.5), the Siegel theta kernel $\Theta(z,g_t)$ splits into the product of two theta series of dimensions 1 and 2:

$$\Theta(z,g_t) := \theta(z)\theta_2(z,t).$$

Let

$$M(s,z) := \int_0^\infty \theta_2(x+iy,t)t^s\frac{dt}{t}, \tag{4.6}$$

which is the Mellin transform of $\theta_2(z,t)$. By the definition of $\Omega(s)$ in (4.4), we obtain

$$\Omega(s)=\int_{\Gamma_0(4k)\backslash\mathbb{H}}\overline{f(z)}\theta(z)M(s,z)\,\frac{dx\,dy}{y^2}.$$

Next, we show that $M(s,z)$ is an Eisenstein series of weight zero and level $4k$. We show this by explicit computation. Let

$$\begin{aligned}Q_{z,t}(h_1,h_2)&:=8\pi kixh_1h_2+4\pi kyt^2h_1^2+4\pi kyt^{-2}h_2^2\\&=4\pi\left(\left(\sqrt{ky}\,th_1+\frac{ix\sqrt{k}}{t\sqrt{y}}h_2\right)^2+\frac{k|z|^2h_2^2}{yt^2}\right).\end{aligned}$$

Then

$$\theta_2(z,t)=\theta_2(z,t^{-1})=y^{1/2}\sum_{h_1,h_2\in\mathbb{Z}}\exp(-Q_{z,t}(h_1,h_2)).$$

Next, we apply the Poisson summation formula in the $h_1$ variable. Let $\hat{\exp}(\xi_1,h_2)$ be the Fourier transform of $\exp(-Q_z(h_1,h_2))$ in the $h_1$ variable. Then

$$\hat{\exp}(\xi_1,h_2):=\int_{-\infty}^{\infty}\exp(-Q_{z,t}(u,h_2)-2\pi iu\xi_1)\,du.$$

By applying the Poisson summation formula in the $h_1$ variable, we obtain

$$y^{1/2}\sum_{h_1,h_2\in\mathbb{Z}}\exp(-Q_z(h_1,h_2))=y^{1/2}\sum_{\xi_1,h_2\in\mathbb{Z}}\hat{\exp}(\xi_1,h_2). \tag{4.7}$$

Next, we compute $\hat{\exp}(\xi_1,h_2)$:

$$\begin{aligned}\hat{\exp}(\xi_1,h_2)&=\int_{-\infty}^{\infty}\exp\left(-4\pi\left(\left(\sqrt{ky}\,tu+\frac{ix\sqrt{k}}{t\sqrt{y}}h_2\right)^2+\frac{k|z|^2h_2^2}{yt^2}\right)-2\pi iu\xi_1\right)du\\&=\frac{1}{2t\sqrt{ky}}\exp\left(-\frac{4\pi}{yt^2}\left|\sqrt{k}\,zh_2+\frac{\xi_1}{4\sqrt{k}}\right|^2\right).\end{aligned}$$

We use the above formula and equation (4.7) to obtain

$$\theta_2(z,t^{-1})=\frac{1}{2t\sqrt{k}}\sum_{h_1,h_2\in\mathbb{Z}}\exp\left(-\frac{4\pi}{yt^2}\left|\frac{h_1}{4\sqrt{k}}+\sqrt{k}\,zh_2\right|^2\right).$$

Next, we use the above formula in order to simplify $M(s,z)$ that is defined in (4.6). We have

$$\begin{aligned}M(s,z)&=\int_0^{\infty}\theta_2(z,t)t^s\,\frac{dt}{t}\\&=\frac{1}{2\sqrt{k}}\int_0^{\infty}\sum_{h_1,h_2\in\mathbb{Z}}\exp\left(-\frac{4\pi t^2}{y}\left|\frac{h_1}{4\sqrt{k}}+\sqrt{k}\,zh_2\right|^2\right)t^{s+1}\,\frac{dt}{t}.\end{aligned}$$

Therefore,

$$\begin{aligned}\Omega(s) &= \int_{\Gamma_0(4k)\backslash\mathbb{H}} \overline{f(z)}\theta(z)M(s,z)\,\frac{dx\,dy}{y^2}\\ &= \frac{1}{2\sqrt{k}}\int_0^\infty \int_{\Gamma_0(4k)\backslash\mathbb{H}} \overline{f(z)}\theta(z)\sum_{h_1,h_2\in\mathbb{Z}}\exp\biggl(-\frac{4\pi t^2}{y}\biggl|\frac{h_1}{4\sqrt{k}}+\sqrt{k}\,zh_2\biggr|^2\biggr)t^{s+1}\,\frac{dt}{t}.\end{aligned}$$

Since $\int_{\Gamma_0(4k)\backslash\mathbb{H}} f(z)\bar{\theta}(z)\,\frac{dx\,dy}{y^2}=0$, we have

$$\Omega(s) = \frac{1}{2\sqrt{k}}\int_0^\infty \int_{\Gamma_0(4k)\backslash\mathbb{H}} \overline{f(z)}\theta(z)\sum_{h_1,h_2}{}'\exp\biggl(-\frac{4\pi t^2}{y}\biggl|\frac{h_1}{4\sqrt{k}}+\sqrt{k}\,zh_2\biggr|^2\biggr)t^{s+1}\,\frac{dt}{t}.$$

where $\sum'_{h_1,h_2}$ is the sum over integers $h_1,h_2\in\mathbb{Z}$ excluding $h_1=h_2=0$. Next, we change the variable to $\tau := \frac{2t\sqrt{\pi}}{\sqrt{y}}\bigl|\frac{h_1}{4\sqrt{k}}+\sqrt{k}\,h_2z\bigr|$. Then $t=\frac{\tau\sqrt{y}}{2\sqrt{\pi}|\frac{h_1}{4\sqrt{k}}+\sqrt{k}\,h_2z|}$ and $d\tau/\tau = dt/t$. Therefore,

$$\begin{aligned}&\int_0^\infty \sum_{h_1,h_2}{}'\exp\biggl(-\frac{4\pi t^2}{y}\biggl|\frac{h_1}{4\sqrt{k}}+\sqrt{k}\,zh_2\biggr|^2\biggr)t^{s+1}\,\frac{dt}{t}\\ &\qquad= \biggl(\int_0^\infty \exp(-\tau^2)\tau^{s+1}\frac{d\tau}{\tau}\biggr)\sum_{h_1,h_2}{}'\biggl(\frac{\sqrt{y}}{2\sqrt{\pi}|\frac{h_1}{4\sqrt{k}}+\sqrt{k}\,h_2z|}\biggr)^{s+1}\\ &\qquad= 2^s k^{\frac{s+1}{2}}\pi^{-\frac{s+1}{2}}\Gamma\biggl(\frac{s+1}{2}\biggr)\sum_{h_1,h_2}{}'\biggl(\frac{y}{|h_1+4h_2zk|^2}\biggr)^{\frac{s+1}{2}}.\end{aligned}$$

We define

$$E(s,z) := \sum_{h_1,h_2}{}'\biggl(\frac{y}{|h_1+4h_2zk|^2}\biggr)^s, \tag{4.8}$$

Therefore,

$$\Omega(s) = k^{s/2}2^{s-1}\Gamma\biggl(\frac{s+1}{2}\biggr)\pi^{-\frac{s+1}{2}}\int_{\Gamma_0(4k)\backslash\mathbb{H}} \overline{f(z)}\theta(z)E\biggl(\frac{s+1}{2},z\biggr)\,\frac{dx\,dy}{y^2}. \qquad \square$$

Let

$$I(s) := \int_{\Gamma_0(4k)\backslash\mathbb{H}} \overline{f(z)}\theta(z)E\biggl(\frac{s+1}{2},z\biggr)\,\frac{dx\,dy}{y^2}. \tag{4.9}$$

Hence,

$$\Omega(s) = k^{s/2}2^{s-1}\Gamma\biggl(\frac{s+1}{2}\biggr)\pi^{-\frac{s+1}{2}}I(s).$$

Next, we give an explicit formula for $I(s)$ in terms of the Fourier coefficients of $f$. We begin by writing $E(s,z)$ as a linear combination of Eisenstein series associated to the cusps of $\Gamma_0(4k)$. Then by the unfolding method we write the integral $I(s)$ as a Dirichlet series with coefficients associated to the Fourier coefficients of $f(z)\bar{\theta}(z)$. First we parametrize the cusps of $\Gamma_0(4k)$.

**Proposition 4.2** ([KY19, Proposition 3.1]). *Every cusp of $\Gamma_0(N)$ is equivalent to one of the form $1/w$ with $1 \le w \le N$. Two cusps of the form $1/w$ and $1/v$ with $1 \le v, w \le N$ are equivalent to each other if and only if*

$$(v, N) = (w, N) \quad \textit{and} \quad \frac{v}{(v, N)} \equiv \frac{w}{(w, N)} \bmod \left((w, N), \frac{N}{(w, N)}\right).$$

*A cusp of the form $p/q$ is equivalent to one of the form $1/w$ with $w \equiv p'q \bmod N$ where $p' \equiv p \bmod (q, N)$ and $(p', N) = 1$. In particular, the cusp at $\infty$ is associated to $w = N$.*

For each cusp $\mathfrak{a} \in \mathbb{Q} \cup \{\infty\}$ of a finite covolume discrete subgroup $\Gamma$ of $SL_2(\mathbb{R})$, we call $\sigma_{\mathfrak{a}} \in SL_2(\mathbb{R})$ a *scaling matrix* for cusp $\mathfrak{a}$ if $\sigma_{\mathfrak{a}}\infty = \mathfrak{a}$ and $\sigma_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}} = \left\{\left[\begin{smallmatrix} 1 & n \\ 0 & 1\end{smallmatrix}\right] : n \in \mathbb{Z}\right\}$, where $\Gamma_{\mathfrak{a}}$ is the centralizer of the cusp $\mathfrak{a}$. Note that scaling matrices are not unique. If $\sigma_{\mathfrak{a}}$ is a scaling matrix for $\mathfrak{a}$, so is $\sigma_{\mathfrak{a}}\left[\begin{smallmatrix} 1 & \alpha \\ 0 & 1\end{smallmatrix}\right]$. We use [KY19, Proposition 3.3], where the authors give a representative for the scaling matrix $\sigma_{1/w}$ of each cusp $1/w$ of $\Gamma_0(N)$.

**Proposition 4.3** ([KY19, Proposition 3.3]). *Let $1/w$ be a cusp of $\Gamma = \Gamma_0(N)$, and set*

$$N = (N, w)N'_w, \qquad w = (N, w)w' = (N'_w, w)w'', \qquad N' = (N'_w, w)N''_w.$$

*The stabilizer of $1/w$ is given as*

$$\Gamma_{1/w} = \left\{\pm\begin{bmatrix} 1 - w''N't & N''t \\ -w'w''Nt & 1 + w''N't\end{bmatrix} : t \in \mathbb{Z}\right\},$$

*and one may choose the scaling matrix as*

$$\sigma_{1/w} = \begin{bmatrix} 1 & 0 \\ w & 1\end{bmatrix}\begin{bmatrix} \sqrt{N''} & 0 \\ 0 & 1/\sqrt{N''}\end{bmatrix}.$$

For each cusp $1/w$ of $\Gamma_0(4k)$, we define the Eisenstein series

$$E_{1/w,4k}(s, z) := \sum_{\gamma \in \Gamma_{1/w}\backslash\Gamma_0(4k)} \mathrm{Im}(\sigma_{1/w}^{-1}\gamma z)^s.$$

By the spectral theory of $\Gamma_0(4k)\backslash\mathbb{H}$, the continuous spectrum of the Laplacian operator on $\Gamma_0(4k)\backslash\mathbb{H}$ is spanned by the Eisenstein series associated to the cusps of $\Gamma_0(4k)$. In the following lemma, we write $E(s, z)$ that is defined in (4.8) as a linear combination of $E_{1/w,4k}(s, z)$.

**Lemma 4.4.** *Let $E(s, z)$ and $E_{1/w,4k}(s, z)$ be the Eisenstein series as above. Then*

$$E(s, z) = \sum_{1/w} \phi_{1/w}(s)E_{1/w,4k}(s, z),$$

*where $\phi_{1/w}(s) := 2\zeta(2s)(N''_w/N'^2_w)^s$ with $N'_w$ and $N''_w$ defined in Proposition 4.1.*

*Proof.* We note that the Eisenstein series $E_{1/w,4k}(s,z)$ is zero asymptotically at every cusp for $\mathrm{Re}(s) > 1$ except the cusp $1/w$, where

$$\lim_{\mathrm{Im} z\to\infty} E_{1/w,4k}(s,\sigma_{1/w}z) = y^s + o(1).$$

Hence, the asymptotic of $E(s,z)$ at the cusp $1/w$ gives the coefficient of the associated Eisenstein series $E_{1/w,4k}(s,z)$ in the basis of $\{E_{1/w,4k}(s,z) : w \in \text{cusps of } \Gamma_0(4k)\}$ for the continuous spectrum of $\Gamma_0(4k)$. Next, we give the asymptotic of $E(s,z)$ at the cusp $1/w$. By (4.8), we have

$$E(s,z) = {\sum_{h_1,h_2}}' \frac{y^s}{|4kh_1z + h_2|^{2s}}.$$

We use the scaling matrix

$$\sigma_{1/w} = \begin{bmatrix} 1 & 0 \\ w & 1 \end{bmatrix} \begin{bmatrix} \sqrt{N''} & 0 \\ 0 & 1/\sqrt{N''} \end{bmatrix},$$

that is given in Proposition 4.1 in order to compute the asymptotic of $E(s,z)$ at the cusp $1/w$. We have

$$\begin{aligned} E(s,\sigma_{1/w}z) &= \sum_{h_1,h_2\in\mathbb{Z}} \frac{\mathrm{Im}(\sigma_{1/w}z)^s}{|4kh_1\sigma_{1/w}z + h_2|^{2s}} \\ &= \sum_{h_1,h_2\in\mathbb{Z}} \frac{N_w''^{s} y^s}{|wN_w''z+1|^{2s}\left|4kh_1\frac{N_w''z}{wN_w''z+1} + h_2\right|^{2s}} \\ &= \sum_{h_1,h_2\in\mathbb{Z}} \frac{N_w''^{s} y^s}{|4kh_1N_w''z + h_2(wN_w''z+1)|^{2s}} \\ &= \zeta(2s) \sum_{\gcd(h_1,h_2)=1} \frac{N_w''^{s} y^s}{|4kh_1N_w''z + h_2(wN_w''z+1)|^{2s}}. \end{aligned}$$

We note that as $\mathrm{Im}(z) \to \infty$ then all the terms in the above sum go to zero except $h_1$ and $h_2$ such that the coefficient of $z$ in the denominator is zero, that is,

$$4kh_1N_w'' + h_2wN_w'' = 0.$$

Since $\gcd(h_1,h_2) = 1$ we have $h_2 = \pm\frac{4k}{\gcd(w,4k)} = N_w'$ in the notation of Proposition 4.1. Therefore,

$$\lim_{\mathrm{Im}(z)\to\infty} E(s,\sigma_{1/w}z) = 2\zeta(2s)\frac{N_w''^{s}}{N_w'^{2s}}.$$

As a corollary,

$$E(s,z) = \sum_{1/w\in\text{cusp of }\Gamma_0(4k)} 2\zeta(2s)\left(\frac{N_w''}{N_w'^2}\right)^s E_{1/w,4k}(s,z).$$

This completes the proof of our lemma. □

*4.1.1. Fourier expansion of the Jacobi function at every cusp of* $\Gamma_0(4k)$. In this section we give the Fourier expansion of the classical Jacobi theta series at each cusp of $\Gamma_0(4k)$. We note that the Fourier expansion of the Jacobi theta series at $\infty$ is

$$\theta(z) := y^{1/4}\sum_{n\in\mathbb{Z}} e(n^2 z).$$

$\theta(z)$ is a weight $1/2$ modular form invariant by $\Gamma_0(4)$ that has three inequivalent cusps $\infty$, 0 and $1/2$. Hence, it suffices to give the Fourier expansion of $\theta(z)$ at $1/2$ and 0. We use the following scaling matrices for $\Gamma_0(4)$. We let

$$\tau_0 := \begin{bmatrix} 0 & -1/2 \\ 2 & 0 \end{bmatrix}, \qquad \tau_{1/2} := \begin{bmatrix} 1 & -1/2 \\ 2 & 0 \end{bmatrix},$$

where $\tau_0$ and $\tau_{1/2}$ are scaling matrices for the cusps 0 and $1/2$ of $\Gamma_0(4)$. The Fourier expansion of $\theta(z)$ at the cusp 0 is given by expanding $\theta_{\tau_0}$, that is,

$$\theta_{\tau_0}(z) = \left(\frac{z}{|z|}\right)^{-1/2}\theta(-1/4z)$$

at $\infty$. We use the following formula from [KS93, equation (2.4)]:

$$\theta(z)_{\tau_0} = e^{-i\pi/4}\theta(z). \tag{4.10}$$

Next, we give the Fourier expansion of $\theta(z)$ at the cusp $1/2$. We have

$$\begin{aligned}
\theta(\tau_{1/2}z) &= \operatorname{Im}(\tau_{1/2}z)^{1/4}\sum_{n\in\mathbb{Z}} e(n^2(\tau_{1/2}z)) \\
&= \frac{y^{1/4}}{|2z|^{1/2}}\sum_{n\in\mathbb{Z}} e(n^2(1/2 - 1/(4z))) \\
&= \frac{y^{1/4}}{|2z|^{1/2}}\sum_{n\in\mathbb{Z}} (-1)^n e(-n^2/(4z)) \\
&= \frac{y^{1/4}}{|2z|^{1/2}}\Big(2\sum_{n \text{ even}} e(-n^2/(4z)) - \sum_{n\in\mathbb{Z}} e(-n^2/(4z))\Big) \\
&= \frac{y^{1/4}}{|2z|^{1/2}}\Big(2\sum_{n\in\mathbb{Z}} e(-n^2/z) - \sum_{n\in\mathbb{Z}} e(-n^2/(4z))\Big) \\
&= \sqrt{2}\,\theta(-1/z) - \theta(-1/4z).
\end{aligned}$$

We use the transformation formula of $\theta(z)$ under $\gamma_2 := \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ (see [KS93, p. 202])

$$\theta(-1/z) = i^{-1/2}\left(\frac{z}{|z|}\right)^{1/2}\frac{\theta(z)+\theta(z+1/2)}{\sqrt{2}}. \tag{4.11}$$

By equations (4.10) and (4.11), we have

$$\theta(\tau_{1/2}z) = e^{-\pi/4}\left(\frac{z}{|z|}\right)^{1/2}(\theta(z)+\theta(z+1/2)-\theta(z)) = e^{-\pi/4}\left(\frac{z}{|z|}\right)^{1/2}\theta(z+1/2). \tag{4.12}$$

We note that $\theta_{\sigma_{1/w}}$ is invariant under $\Gamma_\infty$. Hence,

$$\theta_{\sigma_{1/w}}(z) := y^{1/4} \sum_{n\in\mathbb{Z}} b_{\theta,1/w}(n) e(nz),$$

where $b_{\theta,1/w}(n)$ is the $n$-th Fourier coefficient of $\theta(z)$ at the cusp $1/w$ associated to the scaling matrices $\sigma_{1/w}$. In the following lemma, we give the Fourier coefficients of $\theta(z)$.

**Lemma 4.5.** *Let $\theta(z) = y^{1/4}\sum_{n\in\mathbb{Z}} e(n^2 z)$ and $\sigma_{1/w}$ be the scaling matrices introduced above. Then $\theta(z)$ has the following Fourier coefficients for each cusp $1/w$ of $\Gamma_0(4k)$. If $w \equiv 0 \bmod 4$ then*

$$\theta_{\sigma_{1/w}} = \theta(N''_{1/w} z),$$
$$|b_{\theta,1/w}(n)| := \begin{cases} (N''_{1/w})^{1/4} & \text{if } n = m^2 N''_{1/w} \text{ for some } m \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

*If $w \equiv \pm 1 \bmod 4$ then $N''_{1/w} = 4\alpha$ and*

$$\theta_{\sigma_{1/w}}(z) = \theta(\alpha z \pm 1/4),$$
$$|b_{\theta,1/w}(n)| := \begin{cases} \alpha^{1/4} & \text{if } n = m^2 \alpha \text{ for some } m \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

*Finally, if $w \equiv 2 \bmod 4$ then*

$$\theta_{1/w}(z) = \theta(N''_{1/w} z),$$
$$|b_{\theta,1/w}(n)| := \begin{cases} (N''_{1/w})^{1/4} & \text{if } n = m^2 N''_{1/w} \text{ for some } m \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* We note that $\theta(z)$ is invariant under $\Gamma_0(4)$ and $\Gamma_0(4)$ has three cusps, $\{0, 1/2, \infty\}$. If $w \equiv 0 \bmod 4$ then the cusp $1/w$ is equivalent to $\infty$ in $\Gamma_0(4)$ and the Fourier expansion of $\theta_{\sigma_{1/w}}$ is given by the identity

$$\theta_{\sigma_{1/w}}(z) = \theta(N''_{1/w} z).$$

If $w = 4\alpha + 2$ then $1/w$ is equivalent to $1/2$ in $\Gamma_0(4)$ and we have

$$\sigma_{1/w} = \begin{bmatrix} 1 & 0 \\ 4\alpha & 1 \end{bmatrix} \tau_{1/2} \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{N''_{1/w}} & 0 \\ 0 & 1/\sqrt{N''_{1/w}} \end{bmatrix}.$$

By the above decomposition and equation (4.12), we have

$$\theta_{1/w}(z) = \theta(N''_{1/w} z).$$

If $w = 4\alpha + 1$ then $1/w$ is equivalent to $0$ in $\Gamma_0(4)$ and we have

$$\sigma_{1/w} = \begin{bmatrix} 1 & 1 \\ 4\alpha & 4\alpha + 1 \end{bmatrix} \tau_0 \begin{bmatrix} 1 & 1/4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{N''/4} & 0 \\ 0 & 1/\sqrt{N''/4} \end{bmatrix}.$$

By the above decomposition and equation (4.10), we have

$$\theta_{1/w}(z) = \theta(N''z/4 + 1/4).$$

Finally if $w = 4\alpha + 3$ then $1/w$ is equivalent to 0 in $\Gamma_0(4)$ and we have

$$\sigma_{1/w} = \begin{bmatrix} -1 & 1 \\ -4(\alpha+1) & 4\alpha+3 \end{bmatrix} \tau_0 \begin{bmatrix} 1 & -1/4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{N''/4} & 0 \\ 0 & 1/\sqrt{N''/4} \end{bmatrix}.$$

By the above decomposition and equation (4.10), we have

$$\theta_{1/w}(z) = \theta(N''z/4 - 1/4).$$

This completes the proof of our lemma. □

Note that $f_{\sigma_{1/w}}$ is invariant under $\Gamma_\infty = \left\{\left[\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right] : n \in \mathbb{Z}\right\}$. So, we can write the Fourier expansion of $f_{\sigma_{1/w}}$ at $\infty$ and obtain

$$f_{\sigma_{1/w}} := \sum_{n \neq 0} b_{f,1/w}(n) W_{(1/4)\mathrm{sgn}(n),ir}(4\pi|n|y)e(nx). \tag{4.13}$$

Next, we apply Hardy's method in order to give the trivial bound on $b_{f,1/w}(n)$ the $n$-th Fourier coefficient of $f$ at the cusp $1/w$. This method was implemented by Matthes for real analytic cusp forms [Mat92, p. 157].

**Lemma 4.6.** *Let $f$ be a weight $1/2$ modular form defined on $\Gamma_0(4k)$ with Laplacian eigenvalue $1/4 + r^2$ and $|f|_2 = 1$. Then*

$$|b_{f,1/w}(m)| \ll r^{\frac{1-(1/4)sgn(m)}{2}} e^{\frac{\pi r}{2}} N''^{1/2}_{1/w}(1 + O(|r|^{-1})),$$

*Proof.* Let $\Lambda_{y_0} := \{z = x + iy : |x| < 1/2 \text{ and } y \geq y_0\}$. For each $z \in H$, we denote by $N(z, 1/w, y_0)$ the number of elements of the orbit of $z$ under the discrete group $\sigma_{1/w}^{-1}\Gamma_0(4k)\sigma_{1/w}$ that lies inside $\Lambda_{y_0}$. For each cusp $1/w$ of $\Gamma_0(4k)$, let

$$c_{1/w} := \min\left\{c > 0 : \begin{bmatrix} * & * \\ c & * \end{bmatrix} \in \sigma_{1/w}^{-1}\Gamma_0(4k)\sigma_{1/w}\right\}.$$

By definition of $\sigma_{1/w}$ in Proposition 4.1, it is easy to check that $c_{1/w} \in (1/N''_{1/w})\mathbb{Z}$. Hence, $c_{1/w} \geq 1/N''_{1/w}$. By [Iwa02, Lemma 2.10], we have

$$N(z, 1/w, y_0) \leq 1 + \frac{10}{c_{1/w}y_0} \leq 1 + \frac{10N''_{1/w}}{y_0}. \tag{4.14}$$

By inequality (4.14) and $|f|^2 = 1$, we have

$$\begin{aligned} \int_{\Lambda_{y_0}} |f(\sigma_{1/w}z)|^2 \, d\mu(z) &= \int_{\sigma_{1/w}^{-1}\Gamma_0(4k)\sigma_{1/w}\backslash\mathbb{H}} N(z, 1/w, y_0)|f(\sigma_{1/w}z)|^2 \, d\mu(z) \\ &\leq 1 + \frac{10N''_{1/w}}{y_0}. \end{aligned} \tag{4.15}$$

Next, for each $m \in \mathbb{Z}$, we give an upper bound on $|b_{f,1/w}(m)|$, the $m$th Fourier coefficient of $f$ at cusp $1/w$ defined in equation (4.13)

$$\begin{aligned}\int_{\Lambda_{y_0}} |f(\sigma_{1/w}z)|^2\, d\mu(z) &= \sum_{n\neq 0} |b_{f,1/w}(n)|^2 \int_{y_0}^{\infty} |W_{(1/4)\mathrm{sgn}(n),ir}(4\pi|n|y)|^2\, dy/y^2 \\ &= \sum_{n\neq 0} |b_{f,1/w}(n)|^2 4\pi|n| \int_{4\pi|n|y_0}^{\infty} |W_{(1/4)\mathrm{sgn}(n),ir}(u)|^2\, du/u^2 \\ &\geq |b_{f,1/w}(m)|^2 4\pi|m| \int_{4\pi|m|y_0}^{\infty} |W_{(1/4)\mathrm{sgn}(m),ir}(u)|^2\, du/u^2. \end{aligned} \tag{4.16}$$

We take $y_0 := (4\pi|m|)^{-1}$ then by inequalities (4.15) and (4.16) we have

$$|b_{f,1/w}(m)|^2 \int_1^2 |W_{(1/4)\mathrm{sgn}(m),ir}(u)|^2\, du/u^2 \ll N''_{1/w}. \tag{4.17}$$

For $t \to \infty$ and bounded $y$, we have

$$\begin{aligned} &W_{\mathrm{sgn}(m)/4,ir}(y) \\ &= \left( \frac{\Gamma(-2ir)}{\Gamma(1/2-\mu-\mathrm{sgn}(m)/4)} y^{1/2+ir} + \frac{\Gamma(2ir)}{\Gamma(1/2+2ir-\mathrm{sgn}(m)/4)} y^{1/2-ir} \right)(1+O(t^{-1})). \end{aligned} \tag{4.18}$$

By the Stirling formula, we have

$$\Gamma(x+iy) = \sqrt{2\pi}\, y^{x-1/2} e^{-\pi|y|/2}(1+O(|y|^{-1})), \quad x \text{ bounded.}$$

By using the above asymptotic formula, equation (4.18) and (4.17), we have

$$|b_{f,1/w}(m)|^2 \ll r^{1-(1/4)\mathrm{sgn}(m)} e^{\pi r} N''_{1/w}(1+O(|r|^{-1})),$$

with an absolute constant. This completes the proof of our lemma. □

Finally, we compute the integral $I(s)$ defined in equation (4.9). By Lemma 4.4 and the unfolding method we simplify the right hand side.

**Lemma 4.7.** *We have*

$$I(s) = \psi(s) \sum_{n\geq 1} \frac{\rho(n)}{n^{s-1/2}},$$

*where*

$$\rho(n) := \frac{1}{\sqrt{2}} \sum_{w \text{ odd}} \frac{N''}{N'^{3/2}} \overline{b_{f,1/w}\left(\left(\frac{2n}{N'_w}\right)^2 N''_w/4\right)} + \sum_{w \text{ even}} \frac{N''}{N'^{3/2}} \overline{b_{f,1/w}\left(\left(\frac{n}{N'_w}\right)^2 N''\right)},$$

*and*

$$\psi(s) := 2\zeta(s+1)(4\pi)^{-(s/2-1/4)} \frac{\Gamma(s/2+1/4+ir)\Gamma(s/2+1/4-ir)}{\Gamma\left(\frac{s+1}{2}\right)}.$$

*Proof.* We have

$$
\begin{aligned}
I(s) &:= \int_{\Gamma_0(4k)\backslash\mathbb{H}} \overline{f(z)}\theta(z)E\left(\frac{s+1}{2}, z\right) d\mu(z)\\
&= \int_{\Gamma_0(4k)\backslash\mathbb{H}} \overline{f(z)}\theta(z) \sum_{1/w\in\text{cusps}} \phi_{1/w}\left(\frac{s+1}{2}\right)E_{1/w,4k}\left(\frac{s+1}{2}, z\right) d\mu(z)\\
&= \sum_{1/w\in\text{cusps}} \phi_{1/w}\left(\frac{s+1}{2}\right)\int_{\Gamma_0(4k)\backslash\mathbb{H}} \overline{f(z)}\theta(z)E_{1/w,4k}\left(\frac{s+1}{2}, z\right) d\mu(z)\\
&= \sum_{1/w\in\text{cusps}} \phi_{1/w}\left(\frac{s+1}{2}\right)\int_{\Gamma_0(4k)\backslash\mathbb{H}} \overline{f(z)}\theta(z) \sum_{\gamma\in\Gamma_{1/w}\backslash\Gamma_0(4k)} \operatorname{Im}(\sigma_{1/w}^{-1}\gamma z)^{\frac{s+1}{2}} d\mu(z)\\
&= \sum_{1/w\in\text{cusps}} \phi_{1/w}\left(\frac{s+1}{2}\right)\int_{\Gamma_{1/w}\backslash\mathbb{H}} \overline{f(z)}\theta(z)\operatorname{Im}(\sigma_{1/w}^{-1}z)^{\frac{s+1}{2}} d\mu(z)\\
&= \sum_{1/w\in\text{cusps}} \phi_{1/w}\left(\frac{s+1}{2}\right)\int_{\Gamma_\infty\backslash\mathbb{H}} \overline{f(\sigma_{1/w}z)}\theta(\sigma_{1/w}z)y^{\frac{s+1}{2}} d\mu(z)\\
&= \sum_{1/w\in\text{cusps}} \phi_{1/w}\left(\frac{s+1}{2}\right)\int_{\Gamma_\infty\backslash\mathbb{H}} \overline{f(z)}_{\sigma_{1/w}}\theta(z)_{\sigma_{1/w}}y^{\frac{s+1}{2}} d\mu(z).
\end{aligned}
$$

By Lemmas 4.5 and 4.6, we write $I(s)$ as a Dirichlet series

$$
\begin{aligned}
I(s) = &\sum_{1/w\in\text{cusps}} \phi_{1/w}\left(\frac{s+1}{2}\right)\sum_{n>0}\overline{b_{f,1/w}(n)}b_{\theta,1/w}(n)\\
&\times \int_0^\infty \overline{W_{1/4,ir}(4\pi|n|y)}\exp(-2\pi ny)y^{s/2-1/4}dy/y\\
= &\sum_{1/w\in\text{cusps}} \phi_{1/w}\left(\frac{s+1}{2}\right)\sum_{n>0}\frac{\overline{b_{f,1/w}(n)}b_{\theta,1/w}(n)}{n^{s/2-1/4}}\\
&\times \int_0^\infty \overline{W_{1/4,ir}(4\pi u)}\exp(-2\pi u)u^{s/2-1/4}\,du/u\\
= &(4\pi)^{-(s/2-1/4)}\frac{\Gamma(s/2+1/4+ir)\Gamma(s/2+1/4-ir)}{\Gamma\left(\frac{s+1}{2}\right)}\\
&\times \sum_{1/w\in\text{cusps}} \phi_{1/w}\left(\frac{s+1}{2}\right)\sum_{n>0}\frac{\overline{b_{f,1/w}(n)}b_{\theta,1/w}(n)}{n^{s/2-1/4}}\\
= &\psi(s)\sum_{n\geq 1}\frac{\rho(n)}{n^{s-1/2}}.
\end{aligned}
$$

□

Lemmas 4.7 and 4.1 imply

**Corollary 4.8.** *We have*

$$
\Omega(s) = \sqrt{2}\pi^{-s-1/4}\zeta(s+1)\Gamma(s/2+1/4+ir)\Gamma(s/2+1/4-ir)k^{s/2}\sum_{n\geq 1}\frac{\rho(n)}{n^{s-1/2}}. \tag{4.19}
$$

### 4.2. *Bounding the $L^2$ norm of the theta transfer*

Recall that $f$ is a weight 1/2 modular form on $\Gamma_0(4k)\backslash\mathbb{H}$ with eigenvalue $1/4 + r^2$ and $|f|_2 = 1$, and

$$\varphi(g) := \int_{\Gamma_0(4k)\backslash\mathbb{H}} \Theta(x+iy, g)\overline{f(x+iy)}\,\frac{dx\,dy}{y^2}.$$

In the following theorem, we give an upper bound on the $L^2$ norm of $\varphi$.

**Theorem 4.9.** *Let $f$, $\varphi$ and $r$ be as above. Then $\varphi$ can be realized as a Maass form of weight* 0 *on $\Gamma_0(k)\backslash\mathbb{H}$. Moreover*

$$|\varphi|_2 \ll \cosh(-\pi r/2)(kr)^9,$$

*where the implied constant is absolute.*

*Proof.* Recall that $\Theta(z, g)$ is $\Gamma$ invariant from the left and $K$ invariant from the right in the $g$ variable. By Theorem 3.11, $\varphi(g)$ is a Maass form of weight 0 on $\Gamma_k\backslash V_{m,k}$ by $\varphi(\mathbf{v}) := \varphi(g_v)$, where $\mathbf{v} \in V_{m,k}$ and $g_v \in SO_{q_k}$ is an element such that $g_v\mathbf{x}_0 = \mathbf{v}$. Define the involution $\tau : SO_{q_k} \to SO_{q_k}$ by

$$\tau(g) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} g \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

By definition of theta series at (3.17), it is easy to check that $\Theta(z, g) = \Theta(z, \tau(g))$. As a result $\varphi(g) = \varphi(\tau(g))$ and this means that $\varphi$ is an even Maass form on $\Gamma_k\backslash V_{m,k}$. Recall the isomorphism between $PSL_2(\mathbb{R})$ and $SO_{q_k}$ that we introduced in (3.28):

$$\gamma \in PSL_2(\mathbb{R}) \mapsto g_\gamma \in SO_{q_1} \mapsto B_k^{-1} g_\gamma B_k \in SO_{q_k},$$

where $B_k := \left[\begin{smallmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right]$. Recall that we introduced an isomorphism between $\Gamma_k\backslash V_{m,k}$ and $\Gamma'\backslash\mathbb{H}$ in (3.27), where $\Gamma_0(k) \subset \Gamma'$, by

$$\mathbf{a} := \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in V_{m,k} \mapsto z_{\mathbf{a}} = \frac{-a_3 + i\sqrt{|m|}}{2ka_1} \in H.$$

As a result, we define the even Maass form $u(z)$ with the Laplacian eigenvalue $1/4 + (2r)^2$ on the congruence curve $\Gamma_0(k)\backslash\mathbb{H}$ by

$$u(z_{\mathbf{a}}) := \varphi(\mathbf{a}).$$

Next, we relate the coefficients of $\Omega(s)$ defined in (4.2) to the Fourier coefficients of $u(z)$ at the cusp $\infty$ of $\Gamma_0(k)$. Recall that

$$\Omega(s) := \int_0^\infty \varphi(g_t)t^s\,\frac{dt}{t},$$

where $g_t = \begin{bmatrix} t & 0 & 0 \\ 0 & t^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \in G$. By (3.27), $z_{\mathbf{x}_0} = i/\sqrt{k}$. Moreover, by the isomorphism (3.28),

$$g_t = \begin{bmatrix} t & 0 & 0 \\ 0 & t^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} \sqrt{t} & 0 \\ 0 & \sqrt{t}^{-1} \end{bmatrix} \in SL_2(\mathbb{R}).$$

Hence, $\varphi(g_t) = u(it/\sqrt{k})$ and as a result

$$\Omega(s) = \int_0^\infty u(it/\sqrt{k})t^s \frac{dt}{t}.$$

$u(z)$ is an even Maass form with eigenvalue $1/4 + (2r)^2$ on $\Gamma_0(k)$. We write the Fourier expansion of $u$ at $\infty$ and obtain

$$u(x+iy) = 2\sum_{n=1}^{\infty} a_u(n)n^{-1/2}\cos(2\pi nx)W_{0,2ir}(4\pi ny),$$

where $W_{0,2ir}(y)$ is the usual Whittaker function which is normalized so that $W_{\beta,\mu}(y) \approx e^{-y/2}y^{\beta}$ as $y \to \infty$. By using the above expansion, we have

$$\begin{aligned}
\Omega(s) &= 2\int_0^\infty \sum_{n=1}^{\infty} a_u(n)n^{-1/2}W_{2ir}(4\pi nt/\sqrt{k})t^s \frac{dt}{t} \\
&= 2k^{s/2}\pi^{-s}\sum_{n=0}^{\infty}\frac{a_u(n)}{n^{s+1/2}}\int_{t=0}^{\infty} W_{2ir}(4t)t^s \frac{dt}{t} \\
&= k^{s/2}\pi^{-s-1/2}\Gamma\left(\frac{s+1/2+2ir}{2}\right)\Gamma\left(\frac{s+1/2-2ir}{2}\right)\sum_{n=1}^{\infty}\frac{a_u(n)}{n^{s+1/2}}. \qquad (4.20)
\end{aligned}$$

where we used

$$\int_0^\infty W_{2ir}(4u)u^s \frac{du}{u} = \frac{\pi^{-1/2}}{2}\Gamma\left(\frac{s+1/2+2ir}{2}\right)\Gamma\left(\frac{s+1/2-2ir}{2}\right),$$

from [GR15].

By (4.19) and (4.20), we obtain

$$\begin{aligned}
&\sqrt{2}\pi^{-s-1/4}\zeta(s+1)\Gamma(s/2+1/4+ir)\Gamma(s/2+1/4-ir)k^{s/2}\sum_{n\geq 1}\frac{\rho(n)}{n^{s-1/2}} \\
&\qquad = k^{s/2}\pi^{-s-1/2}\Gamma\left(\frac{s+1/2+2ir}{2}\right)\Gamma\left(\frac{s+1/2-2ir}{2}\right)\sum_{n=1}^{\infty}\frac{a_u(n)}{n^{s+1/2}}.
\end{aligned}$$

Hence,

$$\sqrt{2}\pi^{1/4}\zeta(s+1)\sum_{n\geq 1}\frac{\rho(n)}{n^{s-1/2}} = \sum_{n=1}^{\infty}\frac{a_u(n)}{n^{s+1/2}}.$$

Therefore,

$$a_u(n) = n^{1/2}\sqrt{2}\pi^{1/4} \sum_{lm=n} l^{-1}m^{1/2}\rho(m),$$

where

$$\rho(m) := \frac{1}{\sqrt{2}} \sum_{w \text{ odd}} \frac{N''}{N'^{3/2}} \overline{b_{f,1/w}\left(\left(\frac{2m}{N'_w}\right)^2 N''_w/4\right)} + \sum_{w \text{ even}} \frac{N''}{N'^{3/2}} \overline{b_{f,1/w}\left(\left(\frac{m}{N'_w}\right)^2 N''\right)}.$$

By Lemma 4.6 and Proposition 4.1, we have

$$\begin{aligned}\rho(m) &\leq \frac{1}{\sqrt{2}} \sum_{w \text{ odd}} \frac{N''}{N'^{3/2}} \left| b_{f,1/w}\left(\left(\frac{2m}{N'_w}\right)^2 N''_w/4\right)\right| + \sum_{w \text{ even}} \frac{N''}{N'^{3/2}} \left| b_{f,1/w}\left(\left(\frac{m}{N'_w}\right)^2 N''\right)\right| \\ &\ll r^{5/8}e^{\pi r/2} \sum_{w \text{ cusp of } \Gamma_0(4k)} \frac{N''^{3/2}}{N'^{3/2}} \ll r^{5/8}e^{\pi r/2}k^{\epsilon}.\end{aligned}$$

Therefore,

$$\begin{aligned}|a_u(n)| &\ll n^{1/2} \sum_{lm=n} l^{-1}m^{1/2}|\rho(m)| \ll n^{1+\epsilon} \max_{1\leq m\leq n} |\rho(m)| \\ &\ll n^{1+\epsilon}k^{\epsilon}r^{5/8}e^{\pi r/2}. \end{aligned} \tag{4.21}$$

Recall that $u$ is a Maass form of weight 0 on the congruence group $\Gamma_0(k)$. We use [Iwa02, (8.17)], and obtain

$$\sum_{|n|\leq X} |\nu_u(n)|^2 = 8[SL_2(\mathbb{Z}) : \Gamma_0(k)]^{-1}X|\varphi|_2^2 + O(rX^{7/8}|\varphi|_2^2),$$

where $\nu_u(n) = \left(\frac{4\pi}{\cosh 2\pi r}\right)^{1/2} a_u(n)$. We have

$$[SL_2(\mathbb{Z}) : \Gamma_0(k)] = k\prod_{p|k}(1 + 1/p) \leq k\log(k).$$

Let $(kr)^{8+\epsilon} < X$. Then the main term $8[SL_2(\mathbb{Z}) : \Gamma_0(k)]^{-1}X|\varphi|_2^2$ dominates the error term $O(rX^{7/8}|\varphi|_2^2)$ and we obtain

$$|\varphi|_2^2 \ll \frac{k^{1+\epsilon}}{X} \sum_{|n|\leq X} |\nu_u(n)|^2. \tag{4.22}$$

By inequality (4.21), we have

$$|\nu_u(n)|^2 = \left(\frac{4\pi}{\cosh 2\pi r}\right)|a_u(n)|^2 \ll \cosh(-\pi r)n^{2+\epsilon}k^{\epsilon}r^{5/4}.$$

We apply the above inequality in (4.22) and obtain

$$|\varphi|_2^2 \ll \cosh(-\pi r)k^{1+\epsilon}r^{5/4}\frac{1}{X} \sum_{1\leq n\leq X} n^{2+\epsilon} \ll \cosh(-\pi r)k^{1+\epsilon}r^{5/4}X^{2+\epsilon}.$$

By choosing $X = (kr)^{8+\epsilon}$, we deduce that

$$|\varphi|_2^2 \ll \cosh(-\pi r) k^{17+\epsilon} r^{18}.$$

This completes the proof of our theorem. $\square$

## 5. Appendix: Extending Duke's result (following Sarnak)

Recall $B_T$ from (3.22), and $E_{\mathfrak{a}'}(z,s)$ the Eisenstein series of weight $1/2$ for every $\mathfrak{a}' \in \mathcal{E}'_{4k}$ from the paragraph before Proposition 3.18. Following Duke [Duk88, p. 79], we say $F(z)$ is a *spectral Maass form* if $F(z) \in B_T$ or $F(z) = E_{\mathfrak{a}'}(1/2+it, s)$ for some $\mathfrak{a}' \in \mathcal{E}'_{4k}$.

In this section, we follow Sarnak's argument [Sar90, p. 96] for holomorphic modular forms, and extend Duke's Theorem 5 in [Duk88] to the $m$-th Fourier coefficient of weight $1/2$ spectral Maass forms with level $\Gamma_0(4k)$ when $m = Dv_0^2$, where $D$ is square-free and $\gcd(4k, v_0) = 1$.

**Theorem 5.1.** *Suppose that $m = Dv_0^2$, where $D$ is square-free and* $\gcd(4k, v_0) = 1$, *and $F(z)$ is a spectral mass form with $\Delta_{1/2}$ eigenvalue $\lambda = 1/4 + t^2$. We have*

$$|\rho_{\psi_{\lambda',\infty}}(m)| \ll_\varepsilon |\lambda|^{3/2} \cosh(\pi t/2) |m|^{-2/7+\varepsilon}.$$

*Proof.* We follow [Sar90, p. 96]. First, suppose that $F \in B_T$, and let

$$F(z) = \sum_{n \neq 0} \rho(n) W_{(1/4)\,\mathrm{sgn}(n), it}(4\pi |n| y) e(nx)$$

be the Fourier expansion of $F$. Recall that $F$ is a simultaneous eigenfunction of all Hecke operators $T_{v^2}$ for $\gcd(v, 4k) = 1$. Since $F$ is not an elementary theta series, the Shimura correspondence implies that this system of eigenvalues is associated to a system of eigenvalues of a weight zero Maass newform that is the Shimura lift of $F$; see [TG]. Let $\varphi := \mathrm{Shim}(F)$ be the Shimura lift of $F$ with the Fourier expansion

$$\varphi(z) := \sum_{n \neq 0} a(n) W_{0,2it}(4\pi |n| y) e(nx),$$

where $a(1) = 1$. By the Shimura correspondence (see [Shi73, Theorem 1.9] and [KS93] for the Maass forms), we have

$$\rho(m) = \rho(Dv_0^2) = v_0^{-1/2} \rho(D) \sum_{d | v_0} \mu(d) \chi_D(d) d^{-1} a(v_0/d),$$

where $\gcd(v_0, 4k) = 1$ and $\mu$ is the Möbius function. By Kim and Sarnak's upper bound [Kim03], we have

$$|a(v_0/d)| \ll (v_0/d)^{-1/2+\theta+\epsilon},$$

where $\ll$ only depends on $\epsilon$ and $\theta \geq 7/64$ is any real number. Therefore,

$$|\rho(m)| \ll |\rho(D)| v_0^{-1+7/64+\epsilon}.$$

By Duke's upper bound [Duk88, Theorem 5],

$$|\rho(D)| \ll |\lambda|^{3/2}\cosh(\pi t/2)|D|^{-2/7+\varepsilon},$$

where $\ll$ also only depends on $\epsilon$. Therefore,

$$|\rho(m)| \ll |\rho(D)|v_0^{-1+7/64+\epsilon} \ll |\lambda|^{3/2}\cosh(\pi t/2)|m|^{-2/7+\varepsilon}.$$

This completes the proof of the theorem for the cusp newforms.

Next, suppose that $F(z) = E_{\mathfrak{a}'}(z, 1/2+it)$ for some $\mathfrak{a}' \in \mathcal{E}'_{4k}$ and let $\rho(m)$ be its $m$-th Fourier coefficient. Similarly, the Shimura correspondence implies that

$$\rho(m) = v_0^{-1/2}\rho(D)\sum_{d|v_0}\mu(d)\chi_D(d)d^{-1}a_t(v_0/d),$$

where $a_t(n) = n^{-1/2}\sum_{ab=n}\left(\frac{a}{b}\right)^{it}$ is the Hecke eigenvalue of the unitary Eisenstein series; see [Iwa95, p. 128]. Note that we have the Ramanujan bound on the Fourier coefficient of the unitary Eisenstein series,

$$|a_t(n)| \ll n^{-1/2+\epsilon}.$$

This implies the theorem for $F(z) = E_{\mathfrak{a}'}(z, 1/2+it)$. □

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