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Renormalization and Siegel disks for complex Hénon maps

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Abstract. We use hyperbolicity of golden-mean renormalization of dissipative Hénon-like maps to prove that the boundaries of Siegel disks of sufficiently dissipative quadratic complex Hénon maps with golden-mean rotation number are topological circles.

Conditionally on an appropriate renormalization hyperbolicity property, we derive the same result for Siegel disks of Hénon maps with all eventually periodic rotation numbers.

Keywords. Renormalization, Hénon map, Siegel disk

1. Introduction

Consider the complex quadratic Hénon map written as

$$H_{c,a}(x, y) = (x^2 + c + ay, ax) \quad \text{for } a \neq 0.$$

The maps $H_{c,a}$ and $H_{c,-a}$ are conjugate by the change of coordinates $(x, y) \mapsto (x, -y)$; and the pair of parameters (c, a^2) determines the Hénon map uniquely up to a biholomorphic conjugacy. In this parametrization the Jacobian is $-a^2$. Let $K^{\pm}$ be the sets of points that do not escape to infinity under forward, respectively backward iterations of the Hénon map. Their topological boundaries are $J^{\pm} = \partial K^{\pm}$. Let $K = K^+ \cap K^-$ and $J = J^- \cap J^+$. The sets $J^{\pm}$, $K^{\pm}$ are unbounded, connected subsets of $\mathbb{C}^2$ (see [BS1]). The sets J and K are compact (see [HOV1]). In analogy to one-dimensional dynamics, the set J is called the *Julia set* of the Hénon map.

In this paper we will always assume that the Hénon map is *dissipative*, $|a| < 1$. Note that for $a = 0$, the map $H_{c,a}$ degenerates to

$$(x, y) \mapsto (f_c(x), 0),$$

where $f_c(x) = x^2 + c$ is a one-dimensional quadratic polynomial. Thus for a fixed small value of a_0, the one-parameter family H_{c,a_0} is a small perturbation of the quadratic family.

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Note that an Hénon map $H_{c,a}$ is determined by the multipliers λ and μ at a fixed point uniquely up to changing the sign of a. In particular,

$$\lambda\mu = -a^2,$$

and the parameter c is a function of a^2 and λ:

$$c = (1-a^2)\left(\frac{\lambda}{2} - \frac{a^2}{2\lambda}\right) - \left(\frac{\lambda}{2} - \frac{a^2}{2\lambda}\right)^2.$$

Hence, we sometimes write $H_{\lambda,\mu}$ instead of $H_{c,a}$, when convenient. When $\mu = 0$, the Hénon map degenerates to

$$H_{\lambda,0}(x, y) = (P_\lambda(x), 0), \quad \text{where} \quad P_\lambda(x) = x^2 + \lambda/2 - \lambda^2/4. \tag{1}$$

We say that a dissipative Hénon map $H_{c,a}$ has a *semi-Siegel fixed point* (or simply that $H_{c,a}$ is semi-Siegel) if the eigenvalues of the linear part of $H_{c,a}$ at that fixed point are $\lambda = e^{2\pi i\theta}$ with $\theta \in (0, 1) \setminus \mathbb{Q}$ and μ with $|\mu| < 1$, and $H_{c,a}$ is locally biholomorphically conjugate to the linear map

$$L(x, y) = (\lambda x, \mu y).$$

The classical theorem of Siegel [Sie] states, in particular, that $H_{\lambda,\mu}$ is semi-Siegel whenever θ is Diophantine, that is, $q_{n+1} < cq_n^d$, where p_n/q_n are the continued fraction convergents of θ. The existence of a linearization is a local result, however, in this case there exists a linearizing biholomorphism $\phi : \mathbb{D} \times \mathbb{C} \to \mathbb{C}^2$ sending $(0, 0)$ to the semi-Siegel fixed point,

$$H_{\lambda,\mu} \circ \phi = \phi \circ L,$$

such that the image $\phi(\mathbb{D} \times \mathbb{C})$ is *maximal* (see [MNTU]). We call $\phi(\mathbb{D} \times \mathbb{C})$ the *Siegel cylinder*; it is a connected component of the interior of K^+ and its boundary coincides with J^+ (see [BS2]). We let

$$\Delta = \phi(\mathbb{D} \times \{0\}),$$

and by analogy with the one-dimensional case call it the *Siegel disk* of the Hénon map. Clearly, the Siegel cylinder is equal to the stable manifold $W^s(\Delta)$, and $\Delta \subset K$ (which is always bounded). Moreover, $\partial\Delta \subset J$, the Julia set of the Hénon map.

Remark 1.1. Let $\mathbf{q}$ be the semi-Siegel fixed point of the Hénon map. Then $\Delta \subset W^c(\mathbf{q})$, the center manifold of $\mathbf{q}$ (see e.g. [S] for the definition of W^c). The center manifold is not unique in general, but all center manifolds $W^c(\mathbf{q})$ coincide on the Siegel disk. This phenomenon is nicely illustrated in [O, Figure 5].

The main result of this paper is the following theorem:

Theorem A. *There exists $\delta > 0$ such that the following holds. Let $\theta_* = (\sqrt{5} - 1)/2$ be the inverse golden mean, $\lambda_* = e^{2\pi i\theta_*}$, and let $|\mu| < \delta$. Then the boundary of the Siegel disk of $H_{\lambda_*,\mu}$ is a homeomorphic image of the circle.*

By the Carathéodory Theorem, the linearizing map

$$\phi : \mathbb{D} \times \{0\} \to \Delta \tag{2}$$

extends continuously and injectively to the boundary. However, we note:

Theorem B. *The conjugacy*

$$\phi : S^1 \times \{0\} \to \partial\Delta$$

is not C^1-smooth.

It is worth mentioning that if we assume that $\lambda = e^{2\pi i\theta}$, $\mu = e^{2\pi i\theta'}$ and the pair (θ, θ') satisfies the two-dimensional Brjuno condition [Brj], then the conservative Hénon map $H_{\lambda,\mu}$ has a bounded maximal domain of linearization, called a *Siegel ball*. Herman [He] asked *whether the boundary of the Siegel ball is a topological or perhaps a C^∞ submanifold of $\mathbb{C}^2$*. We answer similar questions, in the dissipative setting, as outlined above.

The proofs of Theorems A and B are based on a renormalization theory for two-dimensional dissipative Hénon-like maps, developed by the first and third authors in [GaYa2]. An Hénon-like map (see [dCLM]) $H : \mathbb{C}^2 \to \mathbb{C}^2$ can be defined as $H(x, y) = (f(x) + \epsilon(x, y), ax)$ for some small ϵ. In this normalization, it has Jacobian $-a\partial\epsilon/\partial y$ and it reduces to the standard Hénon map when $f(x) = x^2 + c$ and $\epsilon(x, y) = ay$. In general, the Jacobian of an Hénon-like map is not constant. Following [LRT], we say that an Hénon-like map H has a semi-Siegel fixed point if there exists a local holomorphic change of variables ϕ such that $\tilde{H} = \phi \circ H \circ \phi^{-1}$ is a skew product of the form $\tilde{H}(x, y) = (\lambda x, \mu(x)y)$ for some holomorphic function $\mu(x) = \mu + O(x)$, where $\lambda = e^{2\pi i\theta}$ with $\theta \in (0, 1) \setminus \mathbb{Q}$, and $|\mu| < 1$. This condition is equivalent to the existence of a one-dimensional Siegel disk $\Delta = \phi(\mathbb{D} \times \{0\})$.

Below, we will be using several different renormalization operators. The first of them is the renormalization of pairs of two-dimensional dissipative maps introduced in [GaYa2]. We will recall its definition in §3.

In one complex dimension, it corresponds to the renormalization of *commuting pairs* $\mathcal{R}$ (cf. [Stir]). In particular, suppose that an analytic map f has a fixed Siegel disk Δ_f, with a rotation number $\theta \in (0, 1)$. Suppose furthermore that $\partial\Delta_f$ is a Jordan curve, and that there is a neighborhood of $\overline{\Delta_f}$ in which the only critical point of f is a simple critical point $c_f \in \partial\Delta_f$. The example to keep in mind is a polynomial P_λ, defined in (1) with $\lambda = e^{2\pi i\theta}$, such that the rotation number θ is of bounded type [Pet, Ya1].

Let $\theta \in (0, 1)$ and denote by $\theta_0 = \theta, \theta_1, \theta_2, \ldots$ its orbit under the Gauss map

$$G(x) = \{1/x\};$$

the orbit is finite if and only if $\theta \in \mathbb{Q}$. We denote by r_k the integer part $[1/\theta_k]$. Then the numbers r_k form a finite or infinite continued fraction expansion of θ, which we abbreviate as $\theta = [r_0, r_1, \ldots]$. As usual, the n-th continued fraction convergent of θ will be denoted by $p_n/q_n \equiv [r_0, \ldots, r_{n-1}]$.

The n-th *pre-renormalization* $p\mathcal{R}^n f$ is the restriction of the pair of iterates $(f^{q_{n+1}}, f^{q_n})$ to appropriate neighborhoods of the critical point c_f. Let $\kappa(z) = \bar{z}$ denote the complex conjugation, and set

$$\upsilon_n(z) \equiv (f^{q_n}(c_f) - c_f) \cdot \kappa^{\circ n}(z) + c_f;$$

this is a linear map if n is even, and an anti-linear map if n is odd. The n-th renormalization is obtained by rescaling $p\mathcal{R}^n f$ by υ_n:

$$\mathcal{R}^n f = (\upsilon_n^{-1} \circ f^{q_{n+1}} \circ \upsilon_n, \upsilon_n^{-1} \circ f^{q_n} \circ \upsilon_n).$$

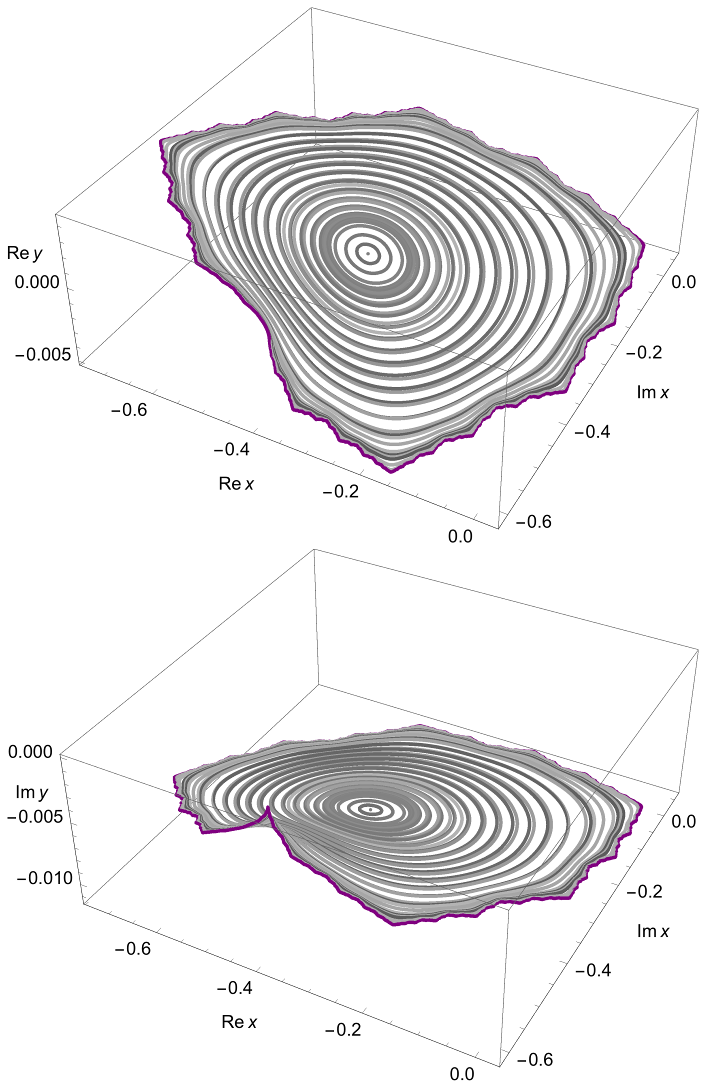

Fig. 1. A three-dimensional plot of the Siegel disk and its boundary for an Hénon map with a semi-Siegel fixed point with the golden mean rotation number. The parameter a is $0.01+0.01i$. The three axes are $\mathrm{Re}(x)$, $\mathrm{Im}(x)$ and $\mathrm{Re}(y)$ (top) and $\mathrm{Re}(x)$, $\mathrm{Im}(x)$ and $\mathrm{Im}(y)$ (bottom).

A different take on renormalization of one-dimensional analytic maps with Siegel disks was introduced by the third author [Ya2] based on the *cylinder renormalization operator* $\mathcal{R}_{\mathrm{cyl}}$. This operator acts on analytic maps defined in some neighborhood of a Siegel fixed point, rather than on pairs. The definition of cylinder renormalization involves a non-linear, rather than linear, rescaling of iterates. There exists a constant $s \in \mathbb{N}$ such that the following holds. Let f be a cylinder-renormalizable analytic map f, and denote

$(\eta, \xi) = \mathcal{R}^{s-1}(f)$. Then the cylinder renormalization $\mathcal{R}_{\mathrm{cyl}}(f)$ is obtained by a non-linear rescaling

$$\Phi \circ \eta \circ \Phi^{-1} = \mathcal{R}_{\mathrm{cyl}}(f) \tag{3}$$

of the map η by the uniformizing coordinate Φ of a particular fundamental domain (called a fundamental crescent in [Ya2]) of the map ξ. Furthermore (cf. [Ya2, Proposition 2.11]), the dependence $\xi \mapsto \Phi$ is locally analytic.

For a topological disk $Z \ni 0$ denote by $\mathcal{H}(Z)$ the Banach space of holomorphic functions f in Z with the uniform norm, and set $\mathcal{H}(Z, W) \equiv \mathcal{H}(Z) \times \mathcal{H}(W)$. We will typically use the notation $\zeta = (\eta, \xi)$ for an element of $\mathcal{H}(Z, W)$.

We let $\mathcal{C}(Z, W)$ denote the Banach subspace of $\mathcal{H}(Z, W)$ given by the linear conditions

$$\eta'(0) = \xi'(0) = 0.$$

We say that a pair $(\eta, \xi) \in \mathcal{C}(Z, W)$ is *almost commuting to order* $s \geq 0$ if

$$(\eta\circ\xi)^{(n)}(0) = (\xi\circ\eta)^{(n)}(0), \quad 0 \leq n \leq s; \quad \eta''(0) \neq 0; \quad \xi''(0) \neq 0; \quad \xi(0) = 1. \tag{4}$$

In the case $s = 2$, we will simply call the pair *almost commuting* (or *a.c.*). We denote by $\mathcal{B}(Z, W)$ the subset of $\mathcal{C}(Z, W)$ consisting of a.c. pairs. In [GaYa2], it is shown that there exists an open neighborhood $\mathcal{U}$ of $\mathcal{C}(Z, W)$ such that $\mathcal{B}(Z, W) \cap \mathcal{U}$ is a Banach submanifold of $\mathcal{H}(Z, W)$.

Let θ be periodic under the Gauss map with period p, and denote $r_l = [1/G^l(\theta)]$ (these are the digits in the continued fraction expansion of θ, and $q_{n+1} = r_n q_n + q_{n-1}$). Similarly to the above, for a pair $\zeta = (\eta, \xi)$, we define a sequence of *pre-renormalizations*

$$p\mathcal{R}^n\zeta = \zeta_n = (\eta_n, \xi_n)$$

by $\zeta_0 = \zeta$ and $\xi_{n+1} = \eta_n$, $\eta_{n+1} = \eta_n^{r_n} \circ \xi_n$. The renormalizations $\mathcal{R}^n(\zeta)$ are then defined as

$$\mathcal{R}^n(\zeta) = (\upsilon_n^{-1} \circ \eta_n \circ \upsilon_n, \upsilon_n^{-1} \circ \xi_n \circ \upsilon_n), \quad \text{where} \quad \upsilon_n(z) = \xi_n(0) \cdot \kappa(z).$$

McMullen [Mc] showed that there exists a pair of analytic maps ζ_λ which is periodic under the action of $\mathcal{R}$ with period p, and such that for every $\lambda_1 = e^{2\pi i\theta_1}$ where

$$G^m(\theta_1) = \theta \quad \text{for some } m \geq 0,$$

we have

$$\mathcal{R}^{np+m} P_{\lambda_1} \to \zeta_\lambda \quad \text{at a rate geometric in } n.$$

Let θ and p be as above. Set

$$k = \begin{cases} p & \text{if } p \text{ is even,} \\ 2p & \text{if } p \text{ is odd,} \end{cases} \tag{5}$$

to guarantee that the operator $\mathcal{R}^k$ is holomorphic (rather than anti-holomorphic). Let us say that the *renormalization hyperbolicity property* (**H**) holds for θ if the following is true:

(**H**) *There exist a pair of topological disks $\tilde{Z} \Supset Z$, $\tilde{W} \Supset W$ and $n = mk$, where $m \in \mathbb{N}$ and k is as in (5), such that:*

(i) *$\mathcal{R}^n$ is an analytic operator from an open neighborhood of its fixed point ζ_λ in $\mathcal{B}(Z, W)$ to $\mathcal{B}(\tilde{Z}, \tilde{W})$.*

(ii) *The differential $D\mathcal{R}^n|_{\zeta_\lambda}$ is a compact linear operator in $T_{\zeta_\lambda}\mathcal{B}(Z, W)$. Let $M \equiv D\mathcal{R}^n|_{\zeta_\lambda}$. Then M has a single simple eigenvalue outside the closed unit disk, and the rest of the spectrum of M lies inside the open unit disk.*

We prove a conditional theorem:

Theorem C. *Suppose the renormalization hyperbolicity property* (**H**) *holds for θ, and let θ_1 be such that $G^m(\theta_1) = \theta$ for some $m \in \mathbb{N}$. Set $\lambda_1 = e^{2\pi i\theta_1}$. Then the following statements hold:*

(I) *there exists $\delta > 0$ such that if $|\mu| < \delta$ then the map $H_{\lambda_1,\mu}$ lies in the stable set of ζ_λ;*
(II) *every Hénon-like map H in $W^s(\zeta_\lambda)$ has a Siegel disk Δ_H whose boundary is a topological circle;*
(III) *the Carathéodory extension of the linearizing coordinate ϕ as in equation (2) to a map $S^1 \times \{0\} \to \partial\Delta_H$ is not C^1-smooth.*

Our Theorems A and B will follow from Theorem C and the following statement proven in [GaYa2]:

Golden-mean renormalization hyperbolicity ([GaYa2]). *The renormalization hyperbolicity property* (**H**) *holds for $\theta_* = (\sqrt{5} - 1)/2$.*

2. Dynamical partitions and multi-indices

Consider the space $\mathcal{I}$ of multi-indices $\bar{s} = (a_1, b_1, a_2, b_2, \ldots, a_m, b_m)$ where $a_j \in \mathbb{N}$ for $2 \le m$, $a_1 \in \mathbb{N} \cup \{0\}$, $b_j \in \mathbb{N}$ for $1 \le j \le m-1$, and $b_m \in \mathbb{N} \cup \{0\}$. We introduce a partial ordering on multi-indices: $\bar{s} \succ \bar{t}$ if $\bar{s} = (a_1, b_1, a_2, b_2, \ldots, a_m, b_m)$, $\bar{t} = (a_1, b_1, \ldots, a_k, b_k, c, d)$, where $k < m$ and either $c < a_{k+1}$ and $d = 0$, or $c = a_{k+1}$ and $d < b_{k+1}$.

For a pair of maps $\zeta = (\eta, \xi)$ and $\bar{s}$ as above we will denote

$$\zeta^{\bar{s}} \equiv \xi^{b_m} \circ \eta^{a_m} \circ \cdots \circ \xi^{b_2} \circ \eta^{a_2} \circ \xi^{b_1} \circ \eta^{a_1}.$$

Similarly,

$$\zeta^{-\bar{s}} \equiv (\zeta^{\bar{s}})^{-1} = (\eta^{a_1})^{-1} \circ (\xi^{b_1})^{-1} \circ \cdots \circ (\eta^{a_m})^{-1} \circ (\xi^{b_m})^{-1}.$$

Consider the n-th pre-renormalization of ζ:

$$p\mathcal{R}^n\zeta = \zeta_n = (\eta_n|_{Z_n}, \xi_n|_{W_n}),$$

where $Z_n = \alpha_n(Z)$, $W_n = \alpha_n(W)$, and

$$\alpha_n(z) = \eta_n(0)z. \tag{6}$$

We define $\bar{s}_n, \bar{t}_n \in \mathcal{I}$ to be such that

$$\eta_n = \zeta^{\bar{s}_n}, \quad \xi_n = \zeta^{\bar{t}_n}.$$

A straightforward induction shows:

Lemma 2.1. *For $n \geq 1$, let $\bar{r} = \bar{s}_n$ or $\bar{t}_n$. Write $\bar{r} = (a_1, b_1, a_2, b_2, \ldots, a_{m_n}, b_{m_n})$. Then $b_{m_n} = 0$, and*

$$\text{either} \quad a_{m_n} \geq 2 \quad \text{or} \quad a_{m_n} = b_{m_n - 1} = 1.$$

Furthermore, if $\bar{s}_n$ ends in $\ldots, 1, 1, 0$ then so does $\bar{t}_n$.

Let $\tau_\theta : \mathbb{R} \to \mathbb{R}$ be the translation $x \mapsto x + \theta$ with $\theta \in (0, 1)$ and $\lambda = \exp(2\pi i\theta)$. Define

$$f(x) = \tau_\theta(x) \quad g(x) = x - 1,$$

and set

$$I = [-1, 0], \quad J = [0, \theta], \quad T = (f|_I, g|_J). \tag{7}$$

Define

$$T_n = (f_n, g_n) = (T^{\bar{s}_n}, T^{\bar{t}_n}),$$

and set

$$I_n = [0, g_n(0)], \quad J_n = [0, f_n(0)]$$

(the notation $[a, b]$ denotes the interval with endpoints a, b, not necessarily in that order).

Now consider the collection of intervals

$$\mathcal{P}_n \equiv \{T^{\bar{w}}(I_n) \text{ for all } \bar{w} \prec \bar{s}_n \text{ and } T^{\bar{w}}(J_n) \text{ for all } \bar{w} \prec \bar{t}_n\}. \tag{8}$$

It is easy to see that:

(a) $\bigcup_{H \in \mathcal{P}_n} H = I \cup J$;
(b) for any distinct $H_1, H_2 \in \mathcal{P}_n$, the interiors of H_1 and H_2 are disjoint.

In view of the above, we call $\mathcal{P}_n$ the n-th *dynamical partition of the segment $I \cup J$.*

Consider the sequence of domains

$$\mathcal{V}_n \equiv \{\zeta^{\bar{w}}(Z_n) \text{ for all } \bar{w} \prec \bar{s}_n \text{ and } \zeta^{\bar{w}}(W_n) \text{ for all } \bar{w} \prec \bar{t}_n\}.$$

By analogy with the above definition (and somewhat abusing the notation) we call $\mathcal{V}_n$ the n-th *dynamical partition of the pair ζ.*

Proposition 2.2. *Suppose that the renormalization hyperbolicity property holds for θ, and*

$$\zeta \in W^s(\zeta_\lambda), \quad \text{where} \quad \lambda = e^{2\pi i\theta}.$$

Then there exist $N = N(\zeta)$, $K > 0$, and $0 < \gamma < 1$ such that for every $n > N$ the following properties hold:

(1) *If $Q_n \in \mathcal{V}_n$ then* $\operatorname{diam}(Q_n) < \gamma^n$.
(2) *Any two neighboring domains $Q_n, Q'_n \in \mathcal{V}_n$ are K-commensurable.*
(3) *For every $\bar{w} \prec \bar{s}_n$ (or $\bar{w} \prec \bar{t}_n$) set $\psi^\zeta_{\bar{w}} = \zeta^{\bar{w}}\alpha_n$. Then $\|D\psi^\zeta_{\bar{w}}|_Z\|_\infty < \gamma^n$ ($\|D\psi^\zeta_{\bar{w}}|_W\|_\infty < \gamma^n$, respectively).*

Proof. By our assumption, there exists $N > 0$ and a pair of domains $\hat{Z} \Supset Z$ and $\hat{W} \Supset W$ such that for all $n \geq N$ the maps of the pair $\mathcal{R}^n\zeta$ are in $\mathcal{C}(\hat{Z}, \hat{W})$. By the Koebe Distortion Theorem, this implies that for all $\bar{w} \prec \bar{s}_n$ (or $\bar{w} \prec \bar{t}_n$) the branches $\zeta^{-\bar{w}}$ have bounded distortion. The domain $Z_n = \alpha_n(Z)$ has diameter $O(\gamma^n)$. The claims readily follow. □

3. Renormalization for pairs of two-dimensional dissipative maps

This section contains a summary of the extension of the renormalization operator from the space $\mathcal{B}(Z, W)$ of almost commuting pairs to an appropriately defined space of two-dimensional maps. The details of the procedure can be found in [GaYa3].

Let Ω, Γ be domains in $\mathbb{C}^2$. We denote by $O(\Omega, \Gamma)$ the Banach space of bounded analytic functions $F = (F_1(x, y), F_2(x, y))$ from Ω and Γ respectively to $\mathbb{C}^2$ equipped with the norm

$$\|F\| = \frac{1}{2}\Big(\sup_{(x,y)\in\Omega} |F_1(x, y)| + \sup_{(x,y)\in\Gamma} |F_2(x, y)|\Big). \tag{9}$$

We let $O(\Omega, \Gamma, \delta)$ stand for the δ-ball around the origin in this Banach space.

In what follows, we fix W, Z, $\tilde{Z}$, and $\tilde{W}$ as in (**H**), and $R > 0$ such that $\mathbb{D}_R \subset Z \cap W$, and let $\Omega = Z \times \mathbb{D}_R$, $\Gamma = W \times \mathbb{D}_R$. We select $\hat{Z}$ and $\hat{W}$ so that

$$Z \Subset \hat{Z} \Subset \tilde{Z}, \qquad W \Subset \hat{W} \Subset \tilde{W}.$$

We define an isometric embedding ι of the space $\mathcal{H}(Z, W)$ into $O(\Omega, \Gamma)$ which sends the pair $\zeta = (\eta, \xi)$ to the pair of functions $\iota(\zeta)$:

$$\left(\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \eta(x) \\ \eta(x) \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \xi(x) \\ \xi(x) \end{pmatrix} \right). \tag{10}$$

Let $\mathcal{U}$ be an open neighborhood of ζ_λ as in (**H**) in $\mathcal{C}(Z, W)$, and let Q be a neighborhood of 0 in $\mathbb{C}$. We will consider an open subset of $O(\Omega, \Gamma)$ of pairs of maps of the form

$$A(x, y) = (a(x, y), h(x, y)) = (a_y(x), h_y(x)), \tag{11}$$
$$B(x, y) = (b(x, y), g(x, y)) = (b_y(x), g_y(x)), \tag{12}$$

such that:

- the pair $(a(x, y), b(x, y))$ is in a δ-neighborhood of $\mathcal{U}$ in $O(\Omega, \Gamma)$;
- $(h, g) \in O(\Omega, \Gamma)$ are such that $|\partial_x h(x, 0)| > 0$ and $|\partial_x g(x, 0)| > 0$ whenever $x \notin \bar{Q}$, and
$$(h(x, y) - h(x, 0), g(x, y) - g(x, 0)) \in O(\Omega, \Gamma, \delta).$$

This open subset of $O(\Omega, \Gamma)$ will be denoted $\mathcal{A}(\mathcal{U}, Q, \delta)$ for brevity.

We say that a pair (A, B) is a *pre-renormalization* of a map H, written

$$(A, B) = p\mathcal{R}^n H,$$

if

$$A = H^{q_n} \text{ and } B = H^{q_{n+1}} \qquad \text{for some } n \geq 0.$$

3.1. Defining renormalization: coordinate transformations

Let $(\eta, \xi) \in \mathcal{B}(Z, W)$ be $n \geq 2$ times renormalizable, and consider its n-th pre-renormalization written as

$$p\mathcal{R}^n \zeta = (\zeta^{\bar{s}_n}, \zeta^{\bar{t}_n}).$$

Let $\bar{s}_n$ be given by $(a_1, b_1, a_2, b_2, \dots, a_{m_n}, 0)$ (recall Lemma 2.1). We denote

$$\hat{s}_n = \begin{cases} (a_1, b_1, a_2, b_2, \dots, a_{m_n} - 2, 0), & a_{m_n} \geq 2, \\ (a_1, b_1, a_2, b_2, \dots, 0, 0, 0), & a_{m_n} = 1, \end{cases}$$

$$\phi_0(x) = \begin{cases} \eta^2, & a_{m_n} \geq 2, \\ \eta \circ \xi & a_{m_n} = 1. \end{cases}$$

Define $\hat{t}_n$ in an identical way to $\hat{s}_n$ (see Lemma 2.1). Then $p\mathcal{R}^n \zeta$ can be written as

$$p\mathcal{R}^n \zeta = \phi_0 \circ (\zeta^{\hat{s}_n}, \zeta^{\hat{t}_n}).$$

For n sufficiently large, η^{-1} is a diffeomorphism of the neighborhood $\alpha_n(Z \cup W)$, and one can define the n-th pre-renormalization of ζ in $\eta^{-1}(\alpha_n(Z \cup W))$ as

$$\hat{p}\mathcal{R}^n \zeta = (\eta^{-1} \circ \zeta^{\bar{s}_n} \circ \eta, \eta^{-1} \circ \zeta^{\bar{t}_n} \circ \eta) = (f \circ \zeta^{\hat{s}_n} \circ \eta, f \circ \zeta^{\hat{t}_n} \circ \eta),$$

where $f = \eta$ if $a_n \geq 2$ and $f = \xi$ if $a_n = 1$.

Next, suppose $\Sigma = (A, B)$ lies in $\mathcal{A}(\mathcal{U}, Q, \delta)$ with $\mathcal{U}$ and δ sufficiently small, so that the following pre-renormalization is defined in a neighborhood of $\eta^{-1}(\alpha_n(Z \cup W)) \times \{0\}$:

$$\hat{p}\mathcal{R}^n \Sigma = (F \circ \Sigma^{\hat{s}_n} \circ A, F \circ \Sigma^{\hat{t}_n} \circ A),$$

where $F = A$ if $a_n \geq 2$ and $F = B$ if $a_n = 1$.

We will denote

$$\pi_1(x, y) = x, \qquad \pi_2(x, y) = y.$$

Set

$$\phi_y(x) = \phi(x, y) := \begin{cases} \pi_1 A^2(x, y), & a_n \geq 2, \\ \pi_1 A \circ B(x, y), & a_n = 1. \end{cases}$$

For sufficiently small δ, the map ϕ_z is close to ϕ_0 and is a diffeomorphism of a neighborhood of $\pi_1 \Sigma^{\hat{s}_n}(\alpha_n(Z), 0) \approx \zeta^{\hat{s}_n}(\alpha_n(Z))$ for all $z \in \mathbb{D}_R$ for some $R = R(\delta) > 0$. Similarly, g_z is a diffeomorphism of a neighborhood of $\pi_1 \Sigma^{\hat{s}_n}(\alpha_n(Z), 0)$ for all $z \in \mathbb{D}_R$ for some $R = R(\delta) > 0$.

Furthermore, set

$$q_z(x) \equiv q(x, z) = \pi_2 F(x, z) = \begin{cases} h_z(x), & a_n \geq 2, \\ g_z(x), & a_n = 1. \end{cases}$$

According to our definition of the class $\mathcal{A}(\mathcal{U}, Q, \delta)$, this is a diffeomorphism outside a neighborhood of zero. Also, set

$$w_z(x) \equiv w(x, z) := q_z(\phi_z^{-1}(x)),$$

a diffeomorphism of a neighborhood of $\pi_1 \phi_z \circ \Sigma^{\hat{s}_n}(\alpha_n(Z), 0)$ in $\mathbb{C}^2$ onto its image for all $z \in \mathbb{D}_R$ for some $R = R(\delta) > 0$. Notice that $\partial_z w_z(x)$ and $\partial_z w_z^{-1}(x)$ are functions whose uniform norms are $O(\delta)$.

Define

$$H_\Sigma(x, y) = (a_y(x), w^{-1}_{q_0^{-1}(y)}(y)). \tag{13}$$

This transformation is δ-close to $(\eta(x), \phi_0(q_0^{-1}(y)))$ in $O(\Omega, \Gamma)$, and therefore, for small δ, is a diffeomorphism of a neighborhood of $\pi_1 F \circ \Sigma^{\hat{s}_n}(\alpha_n(Z), 0) \approx f(\zeta^{\hat{s}_n}(\alpha_n(Z)))$ onto its image. In particular,

$$A \circ H_\Sigma^{-1}(x, y) = (x, h(\eta^{-1}(x), y)) + O(\delta). \tag{14}$$

We use $H_\Sigma(x, y)$ to pull back $\hat{p}\mathcal{R}^n \Sigma$ to a neighborhood of definition of the n-th pre-renormalization of a pair (η, ξ)—that is, a neighborhood of $\alpha_n(Z \cup W)$ in $\mathbb{C}^2$:

$$p\mathcal{R}^n \Sigma = (\bar{A}, \bar{B}) = H_\Sigma \circ F \circ (\Sigma^{\hat{s}_n}, \Sigma^{\hat{t}_n}) \circ A \circ H_\Sigma^{-1}(x, y).$$

The following has been proved in [GaYa2].

Lemma 3.1. *There is an $n \in \mathbb{N}$ and a choice of $\mathcal{U}$, Q, δ_0 and $C > 0$ such that the following holds. For every $\delta < \delta_0$ and every $\Sigma \in \mathcal{A}(\mathcal{U}, Q, \delta)$ the pair $p\mathcal{R}^n \Sigma$ is defined, lies in $O(\hat{\Omega}, \hat{\Gamma})$ where $\hat{\Omega} = \hat{Z} \times \mathbb{D}_R$, $\hat{\Gamma} = \hat{W} \times \mathbb{D}_R$, and*

$$\mathrm{dist}(p\mathcal{R}^n \Sigma, \iota(\mathcal{H}(\alpha_n(\hat{Z}), \alpha_n(\hat{W})))) < C\delta(\|\pi_1 \Sigma - \pi_2 \Sigma\| + \delta).$$

Let us write

$$\bar{A}(x, y) = \begin{pmatrix} \bar{\eta}_1(x) + \bar{\tau}_1(x, y) \\ \bar{\eta}_2(x) + \bar{\tau}_2(x, y) \end{pmatrix}, \tag{15}$$

where

$$\bar{\eta}_1(x) \equiv \pi_1 \bar{A}(x, 0), \quad \bar{\eta}_2(x) \equiv \pi_2 \bar{A}(x, 0)$$

are $O(\delta \|\pi_1 \Sigma - \pi_2 \Sigma\| + \delta^2)$-close to each other, and both are δ-close to $\pi_\eta p\mathcal{R}^n \zeta = \zeta^{\bar{s}_n}$, where π_η and π_ξ are the projections on, correspondingly, the first and the second map in a pair, and

$$\bar{\tau}_1(x, y) \equiv \pi_1 \bar{A}(x, y) - \pi_1 \bar{A}(x, 0), \quad \bar{\tau}_2(x, y) = \pi_2 \bar{A}(x, y) - \pi_2 \bar{A}(x, 0),$$

are functions whose norms are $O(\delta^2)$. Similarly,

$$\bar{B}(x, y) = \begin{pmatrix} \bar{\xi}_1(x) + \bar{\pi}_1(x, y) \\ \bar{\xi}_2(x) + \bar{\pi}_2(x, y) \end{pmatrix},$$

where

$$\bar{\xi}_1(x) \equiv \pi_1 \bar{B}(x, 0), \quad \bar{\xi}_2(x) \equiv \pi_2 \bar{B}(x, 0)$$

are $O(\delta \|\pi_1 \Sigma - \pi_2 \Sigma\| + \delta^2)$-close to each other, and both are δ-close to $\pi_\xi p\mathcal{R}^n \zeta = \zeta^{\bar{t}_n}$, and

$$\bar{\pi}_1(x, y) \equiv \pi_1 \bar{B}(x, y) - \pi_1 \bar{B}(x, 0), \quad \bar{\pi}_2(x, y) \equiv \pi_2 \bar{B}(x, y) - \pi_2 \bar{B}(x, 0)$$

are functions whose norms are $O(\delta^2)$.

3.2. *Defining renormalization: critical projection*

By the Argument Principle, if δ is sufficiently small, then the function $\pi_1 \bar{B} \circ \bar{A}(x, 0)$ has a unique critical point c_1 in a neighborhood of 0. Set $T_1(x, y) = (x + c_1, y)$. Then

$$\partial_x(\pi_1 T_1^{-1} \circ \bar{B} \circ \bar{A} \circ T_1)(0, 0) = 0.$$

Similarly, if δ is sufficiently small, the function $\pi_1 T_1^{-1} \circ \bar{A} \circ \bar{B} \circ T_1(x, 0)$ has a unique critical point c_2 in a neighborhood of 0. Set $T_2(x, y) = (x + c_2, y)$. Then

$$\partial_x(\pi_1 T_2^{-1} \circ T_1^{-1} \circ \bar{A} \circ \bar{B} \circ T_1 \circ T_2)(0, 0) = 0.$$

We now set

$$\begin{aligned}\Pi_1(\bar{A}, \bar{B}) = (\tilde{A}, \tilde{B}) &:= (T_2^{-1} \circ T_1^{-1} \circ \bar{A} \circ T_1, T_1^{-1} \circ \bar{B} \circ T_1 \circ T_2) \\ &= \left(\begin{pmatrix} \tilde{\eta}_1(x) + \tilde{\tau}_1(x, y) \\ \tilde{\eta}_2(x) + \tilde{\tau}_2(x, y) \end{pmatrix}, \begin{pmatrix} \tilde{\xi}_1(x) + \tilde{\pi}_1(x, y) \\ \tilde{\xi}_2(x) + \tilde{\pi}_2(x, y) \end{pmatrix} \right),\end{aligned}$$

where the norms of the functions $\tilde{\tau}_k, \tilde{\pi}_k$, $k = 1, 2$, are $O(\delta^2)$.

The critical points of the functions $\pi_1(\bar{A} \circ \bar{B})(x, 0)$ and $\pi_1(\bar{B} \circ \bar{A})(x, 0)$ are $O(\delta \|\pi_1 \Sigma - \pi_2 \Sigma\| + \delta^2)$-close to each other, and therefore

$$T_2 = \mathrm{Id} + O(\delta \|\pi_1 \Sigma - \pi_2 \Sigma\| + \delta^2). \tag{16}$$

Let us set

$$\tilde{\Sigma} = (\tilde{A}, \tilde{B}) = \Pi_1 p \mathcal{R}^n \Sigma.$$

We note that if the maps $\bar{A}$ and $\bar{B}$ commute, then the critical point of $\pi_1 T_1^{-1} \circ \bar{A} \circ \bar{B} \circ T_1(x, 0)$ is at 0. We therefore have

Proposition 3.2. *Suppose (A, B) is a pre-renormalization of a map H. Then $T_2 \equiv \mathrm{Id}$, and hence the projection Π_1 is the conjugacy by T_1.*

3.3. *Defining renormalization: commutation projection*

At the next step we will project the pair $(\tilde{A}, \tilde{B})$ onto the subset of pairs satisfying the following almost commutation conditions:

$$\partial_x^i \pi_1(\tilde{A} \circ \tilde{B}(x, 0) - \tilde{B} \circ \tilde{A}(x, 0))|_{x=0} = 0, \quad i = 0, 2, \tag{17}$$

$$\pi_1 \tilde{B}(0, 0) = 1. \tag{18}$$

To that end we set

$$\Pi_2(\tilde{A}, \tilde{B})(x, y) = \left(\begin{pmatrix} \tilde{\eta}_1(x) + ax^4 + bx^6 + \tilde{\tau}_1(x, y) \\ \tilde{\eta}_2(x) + ax^4 + bx^6 + \tilde{\tau}_2(x, y) \end{pmatrix}, \begin{pmatrix} \tilde{\xi}_1(x) + c + \tilde{\pi}_1(x, y) \\ \tilde{\xi}_2(x) + c + \tilde{\pi}_2(x, y) \end{pmatrix} \right),$$

and require that (17) and (18) are satisfied for maps in the pair $\Pi_2(\tilde{A}, \tilde{B})(x, y)$. The following proposition is proved in [GaYa2].

Proposition 3.3. *There exists* $\rho > 0$ *such that for all* $\tilde{\Sigma}$ *in the* ρ*-neighborhood of*

$$\iota(\mathcal{C}(\alpha_n(\hat{Z}), \alpha_n(\hat{W})))$$

there is a unique tuple (a, b, c, d) *such that the pair* $\Pi_2(\tilde{A}, \tilde{B})$ *satisfies* (17) *and* (18)*. Moreover, in this neighborhood, the dependence of* Π_2 *on* Σ *is analytic. Furthermore, if* $A \circ B = B \circ A$*, then* $\Pi_2 = \mathrm{Id}$.

Let us fix $n \in 2\mathbb{N}$, $\mathcal{U}$, Q, δ so that Lemma 3.1 holds, and furthermore, the image $\Pi_1 p\mathcal{R}^n \mathcal{A}(\mathcal{U}, Q, \delta)$ lies in the ρ-neighborhood of $\iota(\mathcal{C}(\alpha_n(\hat{Z}), \alpha_n(\hat{W})))$ as in Proposition 3.3. We then have

Proposition 3.4. *For every* $\Sigma \in \mathcal{A}(\mathcal{U}, Q, \delta)$,

$$\mathrm{dist}(\Pi_2 \Pi_1 p\mathcal{R}^n \Sigma, \iota(\mathcal{B}(\alpha_n(\hat{Z}), \alpha_n(\hat{W})))) < C\delta(\|\pi_1 \Sigma - \pi_2 \Sigma\| + \delta).$$

Let $\ell_n = \pi_1 \bar{B}(0, 0)$ and $\Lambda_n(x, y) = (\ell_n x, \ell_n y)$.

Definition 3.5. We define *the renormalization of depth n* of a pair $\Sigma \in \mathcal{A}(\mathcal{U}, Q, \delta)$ as

$$\mathcal{R}_n \Sigma = \Lambda_n^{-1} \circ \Pi_2 \circ \Pi_1 \circ p\mathcal{R}^n \Sigma \circ \Lambda_n. \tag{19}$$

Given a map H from a subset of $\mathbb{C}^2$ to $\mathbb{C}^2$ such that $(A, B) = p\mathcal{R}^n H = (H^{q_{n+1}}, H^{q_n}) \in \mathcal{A}(\mathcal{U}, Q, \delta)$ for some integer n, we will also use the shorthand notation

$$\mathcal{R}_n H \equiv \Lambda_n^{-1} \circ \Pi_2 \circ \Pi_1 \circ p\mathcal{R}^n H \circ \Lambda_n.$$

3.4. Hyperbolicity of renormalization of 2D dissipative maps

We conclude this section by formulating the following theorem:

Theorem 3.6. *Given a p-periodic* θ*, set* $\lambda = e^{2\pi i\theta}$*. Assume that* (**H**) *holds. Then there exists an even* $n = mk$*, where* $m \in \mathbb{N}$ *and k is as in* (5)*, such that* $\iota(\zeta_\lambda)$ *is a fixed point of* $\mathcal{R}_n$ *in* $O(\Omega, \Gamma)$*. The linear operator* $N = D\mathcal{R}_n|_{\iota(\zeta_\lambda)}$ *is compact. The spectrum of N coincides with the spectrum of M, where M is as in* (**H**)*. More specifically,* $\kappa \neq 0$ *is an eigenvalue of M, and h is a corresponding eigenvector if and only if* κ *is an eigenvalue of N, and* $D\iota(h)$ *is a corresponding eigenvector.*

Proof. Since ι is an immersion on $\mathcal{C}(Z, W)$, and

$$\iota \circ \mathcal{R}^k = \mathcal{R}_k \circ \iota,$$

the spectral decomposition of N splits into the direct sum $T_1 \oplus T_2$, where T_1 is the tangent subspace

$$T_1 = T_{\iota(\zeta_\lambda)} \iota(\mathcal{B}(Z, W)).$$

The restriction $N|_{T_1}$ is isomorphic to M. Further, by Proposition 3.4, the magnitude of a perturbation of $\iota(\zeta_\lambda)$ in the direction of a vector in T_2 is decreased quadratically by $(\mathcal{R}_n)^2$. Hence, in the spectral decomposition, the subspace T_2 corresponds to the zero eigenvalue. □

4. Proof of Theorem C

4.1. The Hénon family intersects $W^s(\zeta_\lambda)$

Let us fix θ, θ_1, λ, λ_1 as in Theorem C. As before, let k be as in (5), and let n be as in Theorem 3.6. For brevity, we set

$$\mathfrak{R} = \mathcal{R}_n. \tag{20}$$

We prove

Theorem 4.1. *There exists $\delta > 0$ such that if $|\mu| < \delta$ then the one-parameter family $l \mapsto H_{l,\mu}$ intersects the stable set of ζ_λ under $\mathfrak{R}$.*

Proof. Let $U \ni 0$ be a Jordan domain in $\mathbb{C}$ and let $\mathbf{C}_U$ denote the Banach space of bounded analytic maps f in U equipped with the uniform norm $\|\cdot\|_U$ and such that $f(0) = 0$. Let f_* be the periodic point of $\mathcal{R}_{\text{cyl}}$ with $f_*'(0) = e^{2\pi i\theta}$ constructed in [Ya2]. We denote the period of f_* under $\mathcal{R}_{\text{cyl}}$ by p. As shown in [Ya2], there exists a choice of domains $U_1 \Supset U$ such that

$$f_* \in \mathbf{C}_U, \quad \mathcal{R}_{\text{cyl}} f_* \in \mathbf{C}_{U_1}.$$

Let n be as in Theorem 3.6. For the quadratic polynomial P_{λ_1} there exists N such that the Nn-th cylinder renormalization of P_{λ_1} lies in the local stable set of f_* in $\mathbf{C}_U$.

As shown in [Ya2], the family $l \mapsto \mathcal{R}_{\text{cyl}}^N P_l$ lies in the unstable cone field of $\mathcal{R}_{\text{cyl}}$. Specifically, if $l_t = \lambda + t$, then

$$\|\mathcal{R}_{\text{cyl}}^{(i+N)n} P_{l_t} - \mathcal{R}_{\text{cyl}}^{(i+N)n} P_\lambda\|_U = a\beta^i t + o(t), \quad \text{where } \beta > 1 \text{ and } a > 0. \tag{21}$$

Let us select i large enough, so that $\mathcal{R}_{\text{cyl}}^{(i+N)n} P_\lambda \in \mathbf{C}_{U_2}$ with $U_2 \Supset U$. By the Koebe Distortion Theorem,

$$\|\mathcal{R}_{\text{cyl}}^{(i+N)n} P_{l_t} - \mathcal{R}_{\text{cyl}}^{(i+N)n} P_\lambda\|_U \sim |(\mathcal{R}_{\text{cyl}}^{(i+N)n} P_{l_t})(1) - (\mathcal{R}_{\text{cyl}}^{(i+N)n} P_\lambda)(1)|, \tag{22}$$

where 1 is the critical point, and $\sim$ denotes K-commensurability with a uniform K.

Let us turn to renormalization of commuting pairs. We recall that, according to (3), sn steps of $\mathcal{R}$ correspond to n steps of $\mathcal{R}_{\text{cyl}}$. Using the Koebe Distortion Theorem again, we see that

$$\|\mathcal{R}^{(i+N)sn} P_{l_t} - \mathcal{R}^{(i+N)sn} P_\lambda\| \sim |(\mathcal{R}^{(i+N)sn} P_{l_t})(0) - (\mathcal{R}^{(i+N)sn} P_\lambda)(0)|. \tag{23}$$

Denote

$$(\eta_l, \xi_l) = \mathcal{R}^{s-1}(\mathcal{R}_{\text{cyl}}^{(i+N)n-1} P_l).$$

Let Φ_t, Φ_0 denote the uniformizing coordinates of the fundamental crescents of ξ_{l_t}, ξ_λ respectively (3). Note that, by complex *a priori* bounds [Ya2] and the Koebe Distortion Theorem, Φ_t has universally bounded distortion and $\Phi_t' \simeq 1$. We have

$$\|\Phi_t - \Phi_0\| \sim \|\mathcal{R}^{(i+N)sn} P_{l_t} - \mathcal{R}^{(i+N)sn} P_\lambda\|. \tag{24}$$

The estimates (21)–(24) imply that

$$\|\mathcal{R}^{(i+N)sn} P_{l_t} - \mathcal{R}^{(i+N)sn} P_\lambda\| \sim \beta^i t.$$

Thus the family

$$l \mapsto g_l \equiv \iota\mathcal{R}^{Nsn} P_l$$

lies in the expanding cone field of ζ_λ under $\mathcal{R}$.

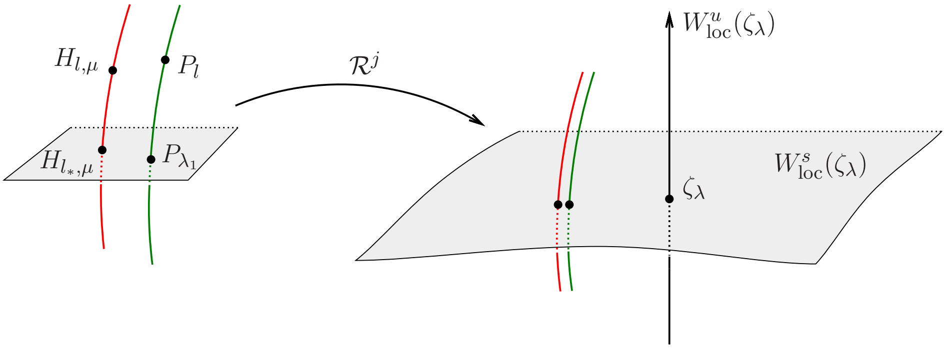

Fig. 2. An illustration to the proof of Theorem 4.1; $j = Nns$.

Since for μ small enough, the family $l \mapsto G_l \equiv \mathcal{R}^{Ns} H_{l,\mu}$ is a C^1-small perturbation of g_l, it is transverse to $W^s_{\rm loc}(\zeta_\lambda)$ and hence intersects it (see Fig. 2). □

4.2. Construction of an invariant curve

In this section we prove the following statement:

Proposition 4.2. *There exists $\epsilon > 0$ such that the following holds. Let $|\mu| < \epsilon$, and*

$$H_{l_*,\mu} \in W^s(\zeta_\lambda) \quad \text{where } \lambda = e^{2\pi i\theta}.$$

Denote by Ω_n, Γ_n the domains of definition of the n-th pre-renormalization $p\mathcal{R}^n H_{l_*,\mu}$. Then there exists a curve $\gamma_* \subset \mathbb{C}^2$ such that the following properties hold:

- *γ_* is a homeomorphic image of the circle;*
- *$\gamma_* \cap \Omega_n \neq \emptyset$ and $\gamma_* \cap \Gamma_n \neq \emptyset$ for all $n \geq 0$;*
- *there exists a topological conjugacy $\varphi_* : \mathbb{T} \to \gamma_*$ between the rigid rotation $x \mapsto x + \theta_1 \bmod \mathbb{Z}$ and $H_{l_*,\mu}|_{\gamma_*}$;*
- *there exists m such that $G^m(\theta_1) = \theta$;*
- *the conjugacy φ_* is not C^1-smooth.*

Before proving the above proposition, we need to introduce some further notation. Below, for brevity, we will denote $\Upsilon^1 = \Omega$, $\Upsilon^2 = \Gamma$.

We set $n = km$ as in Theorem 3.6 for some $m \geq 1$ (to be fixed later).

To differentiate between transformations for different pairs we will use the folllowing notation. Denote

$$\bar{s}_n = (a_1, b_1, \ldots, a_{m_n}, 0), \qquad \bar{t}_n = (c_1, d_1, \ldots, c_{l_n}, 0).$$

Given a pair Σ, denote by Λ_Σ the rescaling that corresponds to the first renormalization $\mathcal{R}$, and by H_Σ the transformation constructed for Σ in (13), that is,

$$\mathcal{R}\Sigma = \Lambda_\Sigma^{-1} \circ T_\Sigma^{-1} \circ H_\Sigma \circ (\Sigma^{\tilde{s}_n}, \Sigma^{\tilde{t}_n}) \circ H_\Sigma^{-1} \circ T_\Sigma \circ \Lambda_\Sigma = L_\Sigma^{-1} \circ \hat{p}\mathcal{R}^n \Sigma \circ L_\Sigma,$$

where

$$\tilde{s}_n = (1, 0, a_1, b_1, \ldots, a_{m_n} - 1, 0), \qquad \tilde{t}_n = (1, 0, c_1, d_1, \ldots, c_{l_n} - 1, 0), \tag{25}$$

and

$$L_\Sigma = H_\Sigma^{-1} \circ T_\Sigma \circ \Lambda_\Sigma.$$

Note that since the elements of Σ pairwise commute, the projection Π_2 is Id and Π_1 is the conjugation by the translation $T_\Sigma := T_1$.

Let $\bar{s}_n^l$ and $\bar{t}_n^l$ be defined by

$$(\hat{p}\mathcal{R}^n)^l \zeta = (\zeta^{\bar{s}_n^l}, \zeta^{\bar{t}_n^l}).$$

For each multi-index

$$\bar{w} = (a_0, b_0, a_1, b_1, \ldots, a_k, b_k) \prec \bar{s}_n^l \quad \text{or} \quad \bar{w} = (a_1, b_1, \ldots, a_k, b_k) \prec \bar{t}_n^l$$

we define a domain

$$Q_{\bar{w}}^i = \Sigma^{\bar{w}} \circ L_\Sigma \circ L_{\mathcal{R}\Sigma} \circ \ldots \circ L_{\mathcal{R}^{l-1}\Sigma}(\Upsilon^i), \qquad i = 1 \text{ for } \bar{w} \prec \bar{s}_n^l,\ i = 2 \text{ for } \bar{w} \prec \bar{t}_n^l. \tag{26}$$

By analogy with a dynamical partition of a commuting pair from Section 2, the collection

$$\mathcal{Q}_{ln} \equiv \{Q_{\bar{w}}^i\}$$

will be referred to as the ln-th *partition* for the two-dimensional pair Σ.

Given $\Sigma \in W_{\text{loc}}^s(\zeta_\lambda)$, consider the following collection of functions defined on $\Omega \cup \Gamma$:

$$\Psi_{\bar{w}}^\Sigma = \Sigma^{\bar{w}} \circ L_\Sigma.$$

Given a collection of index sets $\{\bar{w}^i\}$, $\bar{w}^i \prec \bar{s}_n$ or $\bar{w}^i \prec \bar{t}_n$, consider the following *renormalization microscope*:

$$\Phi^j_{\bar{w}^0, \bar{w}^1, \ldots, \bar{w}^{j-1}, \Sigma} = \Psi^\Sigma_{\bar{w}^0} \circ \Psi^{\mathcal{R}\Sigma}_{\bar{w}^1} \circ \cdots \circ \Psi^{\mathcal{R}^{(j-1)}\Sigma}_{\bar{w}^{j-1}},$$

which we will also denote $\Phi^j_{\hat{w}_0^{j-1}, \Sigma}$, where $\hat{w}_0^{j-1} = \{\bar{w}^0, \bar{w}^1, \ldots, \bar{w}^{j-1}\}$, for brevity.

Lemma 4.3. *The renormalization microscope maps each set Υ^i onto an element of the partition $\mathcal{Q}_{jn}$ for Σ.*

Proof. The claim holds for $j = 1$ by the definition (26) of the elements of the partition. Assume that $\Phi^j_{\hat{w}^j_0,\Sigma}(\Upsilon^i)$ is an element of the partition $\mathcal{Q}_{jn}$ for Σ, and consider

$$\Phi^{j+1}_{\hat{w}^j_0,\Sigma}(\Upsilon^i) = \Psi^{\Sigma}_{\bar{w}^0} \circ \Psi^{\mathcal{R}\Sigma}_{\bar{w}^1} \circ \cdots \circ \Psi^{\mathcal{R}^j\Sigma}_{\bar{w}^j}(\Upsilon^i).$$

By assumption,

$$\Phi^j_{\hat{w}^j_1,\mathcal{R}\Sigma}(\Upsilon^i) \equiv \Psi^{\mathcal{R}\Sigma}_{\bar{w}^1} \circ \cdots \circ \Psi^{\mathcal{R}^j\Sigma}_{\bar{w}^j}(\Upsilon^i)$$

is an element of the partition of level jn for the pair $\mathcal{R}\Sigma$, that is, by (26),

$$\Phi^j_{\hat{w}^j_1,\mathcal{R}\Sigma}(\Upsilon^i) = (\mathcal{R}\Sigma)^{\bar{v}} \circ L_{\mathcal{R}\Sigma} \circ L_{\mathcal{R}^2\Sigma} \circ \cdots \circ L_{\mathcal{R}^j\Sigma}(\Upsilon^i)$$

for some admissible $\bar{v} = (\alpha_0, \beta_0, \alpha_1, \beta_1, \ldots, \alpha_m, \beta_m)$. Therefore, using the shorthand

$$\mathcal{R}\Sigma = (A_1, B_1),$$

we have

$$\begin{aligned}
\Phi^{j+1}_{\hat{w}^j_0,\Sigma}(\Upsilon^i) &= \Psi^{\Sigma}_{\bar{w}^0} \circ \Phi^j_{\hat{w}^j_1,\mathcal{R}\Sigma}(\Upsilon^i) = \Sigma^{\bar{w}^0} \circ L_\Sigma \circ (\mathcal{R}\Sigma)^{\bar{v}} \circ L_{\mathcal{R}\Sigma} \circ \cdots \circ L_{\mathcal{R}^j\Sigma}(\Upsilon^i)\\
&= \Sigma^{\bar{w}^0} \circ L_\Sigma \circ (B_1^{\beta_m} \circ A_1^{\alpha_m} \circ \cdots \circ B_1^{\beta_0} \circ A_1^{\alpha_0}) \circ L_{\mathcal{R}\Sigma} \circ \cdots \circ L_{\mathcal{R}^j\Sigma}(\Upsilon^i)\\
&= \Sigma^{\bar{w}^0} \circ L_\Sigma \circ \Lambda_\Sigma^{-1} \circ H_\Sigma \circ T_\Sigma^{-1} \circ \left((\Sigma^{\tilde{t}_n})^{\beta_m} \circ (\Sigma^{\tilde{s}_n})^{\alpha_m} \circ \ldots \circ (\Sigma^{\tilde{t}_n})^{\beta_0} \circ (\Sigma^{\tilde{s}_n})^{\alpha_0}\right)\\
&\quad \circ T_\Sigma \circ H_\Sigma^{-1} \circ \Lambda_\Sigma \circ L_{\mathcal{R}\Sigma} \circ \cdots \circ L_{\mathcal{R}^j\Sigma}(\Upsilon^i)\\
&= \Sigma^{\bar{u}} \circ L_\Sigma \circ \cdots \circ L_{\mathcal{R}^j\Sigma}(\Upsilon^i),
\end{aligned}$$

for some index $\bar{u}$. By (26), the latter is an element of the partititon $\mathcal{Q}_{(j+1)n}$. □

Since $\mathcal{R}^l\Sigma$ converges to ζ_λ at a geometric rate, the function $\Psi^{\mathcal{R}^l\Sigma}_{\bar{w}}$ converges to the function $(\psi^{\zeta_*}_{\bar{w}}, \psi^{\zeta_*}_{\bar{w}})$, defined in Proposition 2.2, at a geometric rate in C^1-metric. Therefore, by Proposition 2.2, there exists a neighborhood $\mathcal{S}$ in $W^s_{\text{loc}}(\zeta_\lambda)$, and a sufficiently large l, such that

$$\|D\Psi^{\mathcal{R}^l\Sigma}_{\bar{w}}|_{\Upsilon^i}\|_\infty < 1/2 \quad \text{whenever } \mathcal{R}^l\Sigma \in \mathcal{S}.$$

For every $\Sigma \in W^s_{\text{loc}}(\zeta_\lambda)$, there exists $i_0 \in \mathbb{N}$ such that $\mathcal{R}^i\Sigma \in \mathcal{S}$ for $i \geq i_0$. Hence, there exists $C = C(\Sigma)$ such that

$$\|D\Phi^j_{\hat{w},\Sigma}|_{\Upsilon^i}\|_\infty < C/2^j, \tag{27}$$

and thus the renormalization microscope is a uniform metric contraction.

We are now ready to prove Proposition 4.2.

Proof of Proposition 4.2. Let

$$\mathcal{R}^r(H_{l_*,\mu}) \equiv \Sigma = (A, B) \in W^s(\zeta_\lambda)$$

for some $r \in \mathbb{N}$. Select a distinct point $(x_{\bar{w}}, y_{\bar{w}})$ in each of the sets $Q^i_{\bar{w}} \in \mathcal{Q}_{ln}$. Consider the ln-th dynamical partition $\mathcal{P}_{ln}$ for the pair T as defined in (7). Consider a piecewise constant map φ_l sending the element of the partition with a multi-index $\bar{w}$ to $(x_{\bar{w}}, y_{\bar{w}})$. According to (27), the diameters of the sets $Q^i_{\bar{w}}$ decrease at a geometric rate. Thus, the maps φ_l converge uniformly to a continuous map φ of the interval $[-1, \theta]$ which is a homeomorphism onto the image. Set

$$\varphi([-1, \theta]) \equiv \gamma.$$

By construction,

$$\varphi \circ T = \Sigma \circ \varphi.$$

Let $\gamma_1 \subset K^+(H_{l_*,\mu})$ be the preimage of γ under renormalization rescaling, and set

$$\gamma_* \equiv \bigcup_{n \in \mathbb{N}} H_{l_*,\mu}(\gamma_1).$$

The conjugacy φ induces a conjugacy $\varphi_* : \mathbb{T} \to \gamma_*$ between a rigid rotation and $H_{l_*,\mu}|_{\gamma_*}$. Hence, setting $l_* = e^{2\pi i \theta_1}$, we have $G^r(\theta_1) = \theta$ for some $r \geq 0$.

Finally, since the limiting pair ζ_λ has a critical point at $z = 0$, the conjugacies φ and φ_* cannot be C^1-smooth. Indeed, assume the contrary. This would imply that there exists $K > 1$ such that for every arc $J \subset \gamma_*$ and every $n \in \mathbb{N}$, we have

$$\frac{1}{K} \operatorname{diam}(J) < \operatorname{diam}(H^n_{l_*,\mu}(J)) < K \operatorname{diam}(J). \tag{28}$$

However, let Ω_n, Γ_n denote the domains of the pair $p\mathcal{R}^n H_{l_*,\mu}$. Let $z \in \gamma_* \cap \Omega_n$ and $z' = H^{q_n}_{l_*,\mu}(z)$, and denote by J_n the smaller subarc of γ_* bounded by these two points. Since $\mathcal{R}H_{l_*,\mu} \approx \zeta_\lambda$ for large n, we have

$$\operatorname{diam}(H^{q_{n+1}}_{l_*,\mu}(J_n)) \sim (\operatorname{diam}(J_n))^2.$$

This clearly contradicts (28). □

4.3. *The curve γ_* bounds a Siegel disk*

Let us define a *ϱ-vertical cone field* in the tangent bundle $T\Omega$ where Ω is a subdomain of $\mathbb{C}^2$ as

$$C^{\text{vert},\varrho}_{(x,y)} = \{(u, v) \in T_{(x,y)}\Omega : |u| < \varrho|v|\}.$$

Let $f, g : U \to \mathbb{C}$ be holomorphic maps. We consider two-dimensional perturbations $F : \Omega \to \mathbb{C}^2$ of the map (f, g) of the form

$$F(x, y) = (w(x, y), h(x, y)) = (f(x) + \tau(x, y), g(x) + \chi(x, y)). \tag{29}$$

We note:

Proposition 4.4. *Suppose $|f'(x)|, |g'(x)| > \kappa$ and $|f'(x)|, |g'(x)| < K$ on the domain U for some $\kappa > 0$. Let F^{-1} be defined on $\Delta = F(\Omega)$. Then there exist $\epsilon, \varrho > 0$ such that the following holds. Suppose the uniform norms of τ and χ in (29) on Ω are bounded by ϵ. Given $\hat{\Delta} \Subset \Delta$, for every $(x, y) \in \hat{\Delta}$, denoting $(x_1, y_1) = F(x, y)$, we have*

$$DF^{-1}|_{(x_1,y_1)}(C^{\mathrm{vert},\varrho}_{(x_1,y_1)}) \subset C^{\mathrm{vert},\varrho}_{(x,y)},$$

and $\|DF^{-1}\| > O\left(\frac{\kappa}{K\epsilon}\right)$ in $C^{\mathrm{vert},\varrho}$.

Proof. Let $w_i(x, y) = \partial_i w(x, y)$ denote the i-th partial derivative of $w(x, y)$, $i = 1, 2$, and similarly for $h_i(x, y)$. A simple computation shows that

$$DF^{-1}(x_1, y_1)\begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{D(x,y)}\begin{bmatrix} h_2(x,y) - w_2(x,y) \\ -h_1(x,y) \quad w_1(x,y) \end{bmatrix} \cdot \begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{D(x,y)}\begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix}, \quad (30)$$

where $D(x, y) = w_1(x, y)h_2(x, y) - w_2(x, y)h_1(x, y)$, and

$$|\tilde{u}| < C\epsilon(|u| + |v|) < C\epsilon(\varrho + 1)|v|,$$
$$|\tilde{v}| > (\kappa - C\epsilon)|v| - (|g_1(x_1)| + C\epsilon)|u| > (\kappa - C(1 + \varrho)\epsilon - \varrho K)|v|,$$

and $|\tilde{u}| < \varrho|\tilde{v}|$ if $\varrho(\kappa - \varrho K) > C\epsilon(\varrho + 1)^2$. Furthermore, $|D(x, y)| < 2(K + C\epsilon)C\epsilon$ for some $C > 0$ and all $(x, y) \in \hat{\Delta}$. The lower bound on the operator norm $\|DF^{-1}\|$ on the vertical cone field follows. □

As before, for $H_{l_*,\mu} \in W^s(\zeta_\lambda)$, we let Ω_n, Γ_n be the domains of the pair

$$\mathfrak{Z}_n = (\mathfrak{A}_n, \mathfrak{B}_n) \equiv p\mathcal{R}^n H_{l_*,\mu}.$$

For brevity, let us also write

$$\Delta_n \equiv \Omega_n \cup \Gamma_n, \qquad \Delta'_n \equiv \mathfrak{Z}_n(\Delta_n) \equiv \mathfrak{A}_n(\Omega_n) \cup \mathfrak{B}_n(\Gamma_n).$$

Let α_* be the scaling factor α_n (see definition (6)) for the pair ζ_λ.

Proposition 4.5. *There exists $N \in \mathbb{N}$ such that for any $n \geq N$ we can select $\delta_0 > 0$, $k \in \mathbb{N}$ and $\varrho > 0$ so that the following holds. Let $|\mu| < \delta < \delta_0$ and $H_{l_*,\mu} \in W^s(\zeta_\lambda)$. Then the derivatives of the inverse branches of the restriction of the pair $\mathfrak{Z}_n$ to the domains $\Delta_n \setminus \Delta_{n+k}$ preserve the vertical cone field $C^{\mathrm{vert},\varrho}$ and expand vectors in $C^{\mathrm{vert},\varrho}$ at a rate $O(|\alpha_*|^k\delta^{-2})$.*

Proof. Let $\mathfrak{Z}_n = (f_n(x) + \tau_n(x, y), g_n(x) + \chi_n(x, y))$. By Lemma 3.1, the uniform norms of τ_n and χ_n on Δ_n are bounded from above by $O(\delta^2)$.

Notice that Δ_{n+k} is the image of Δ_n under a linear map which converges to $(\alpha_*^k, 0)$ as $n \to \infty$. Therefore, if $(x, 0) \in (\Delta_n \setminus \Delta_{n+k}) \cap \{y = 0\}$, then

$$C_2|\alpha_*|^n > |x| > C_1|\alpha_*|^{n+k}$$

for some C_1 and C_2, which gives

$$C_4|\alpha_*|^n > |f'_n(x)|, |g'_n(x)| > C_3|\alpha_*|^{n+k}$$

for some C_3 and C_4. The result follows from Proposition 4.4 with $\epsilon = O(\delta^2)$, $\kappa = O(|\alpha_*|^{n+k})$ and $K = O(|\alpha_*|^n)$. □

The following result will be used in the proof of Proposition 4.7.

Lemma 4.6 (Löwner [Löw]). *Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic with $f(0) = 0$. If f extends to a homeomorphism of $\partial\mathbb{D}$ itself, then f is a rotation.*

We can now complete the proof of Theorem C:

Proposition 4.7. *There exists $\delta > 0$ such that the following holds. Let $H_{l_*,\mu} \in W^s(\zeta_\lambda)$ with $|\mu| < \delta$ and let γ_* be the invariant curve constructed in Proposition 4.2. Then γ_* bounds a Siegel disk for $H_{l_*,\mu}$. The eigenvalue l_* is equal to λ_1,*

$$\lambda_1 = e^{2\pi i\theta_1} \quad \text{with } \theta = G^m(\theta_1) \text{ for some } m \geq 0. \tag{31}$$

Finally, there exists $\epsilon_1 > 0$ such that for all $|\mu| < \epsilon_1$ and for all λ_1 satisfying (31), we have $H_{\lambda_1,\mu} \in W^s(\zeta_\lambda)$.

Proof. Let us select k, N, and ϱ as in Proposition 4.5. Let $i \geq N$. Fix an open subdomain $\hat{\Delta}_i \Subset \Delta_i \cap \Delta_i'$. Since $H_{l_*,\mu}$ is a δ-small perturbation of the Siegel quadratic polynomial P_{l_*}, we can select $\delta > 0$ small enough so that the map $H_{l_*,\mu}$ is normally hyperbolic in a sufficiently large neighborhood of the α-fixed point of P_{l_*}. In particular, by Proposition 4.5, it is normally hyperbolic in the set $\hat{\Delta}_i \setminus \Delta_{i+k}$. Let $\mathbf{q}$ be the fixed point of $H_{l_*,\mu}$ which is closest to the α-fixed point of P_{l_*}. By the Graph Transform, the map $H_{l_*,\mu}$ has a weak stable/unstable/center manifold W of $\mathbf{q}$ which is δ-close to the slice $\{y = 0\}$ (see [HPS]), and therefore $W \cap \hat{\Delta}_i \neq \emptyset$ if δ is sufficiently small.

Let us begin with the case when $\mathbf{q}$ is attracting. By Proposition 4.5 the inverse branches of $\mathfrak{Z}_{i+mk}$, $m \geq 0$, are normally hyperbolic in $\Delta_{i+mk} \setminus \Delta_{i+(m+1)k}$. Therefore, the weak attracting submanifold W interesects Δ_{i+mk} for all $m \in \mathbb{N}$. We conclude that the invariant curve γ_* lies in the closure of W. Applying Löwner's Lemma 4.6, we arrive at a contradiction.

Suppose $\mathbf{q}$ is hyperbolic. Then $W = W^u(\mathbf{q})$, the unstable manifold of $\mathbf{q}$, and successive applications of Proposition 4.5 as above imply that W extends to the invariant curve γ_*, which is then its boundary. This, again, contradicts Löwners Lemma 4.6.

Finally, suppose that $\mathbf{q}$ is *semi-neutral* (that is, the linear part of the Hénon map at $\mathbf{q}$ has a neutral eigenvalue of absolute value 1 and a dissipative eigenvalue of absolute value smaller than 1). In this case $W = W^c(\mathbf{q})$: it is only smooth, and *a priori*, not uniquely defined. The restriction $H_{l_*,\mu}|_W$ is not necessarily holomorphic.

By density of the irrationals of bounded type in the circle, we can choose a sequence $H_{l_j,\mu}$ of maps whose neutral eigenvalue l_j equals $e^{2\pi i\vartheta_j}$ for some angle $\vartheta_j \in \mathbb{R}\setminus\mathbb{Q}$ of bounded type, converging to $H_{l_*,\mu}$. By continuity of the renormalization operator, for every $M \in \mathbb{N}$, there exists $J = J(M)$ such that for all $j > J(M)$, $H_{l_j,\mu}$ is $i + Mk$ times renormalizable with the height of the renormalizations coinciding with those for the map $H_{l_*,\mu}$. The Siegel disk W_j of $H_{l_j,\mu}$ is an analytic submanifold of $\mathbb{C}^2$. Applying Proposition 4.5 to the inverse branches of $\mathfrak{Z}^j_{i+mk}$, $0 \leq m \leq M$ of $H_{l_j,\mu}$, and using considerations of dominated splitting, we can extend W_j for large j to intersect each $\hat{\Delta}_{i+km}$, $0 \leq m \leq M$. The rotation numbers of the orbits of points in $W_j \cap \hat{\Delta}_{i+km}$, whose continued fraction expansion is given by the renormalization heights, approach θ_1. Since the rotation number of the orbits of $H_{l_j,\mu}|_{W_j}$ is constant, $\vartheta_j \mapsto \theta_1$, $DH_{l_*,\mu}(\mathbf{q}) = \lim_{j\to\infty} DH_{l_j,\mu}(\mathbf{q}_j)$, and $l_* = e^{2\pi i\theta_1}$. Therefore, W is a Siegel disk for $H_{l_*,\mu}$, and $H_{l_*,\mu}|_W$ is holomorphic. By Proposition 4.5 the submanifold W interesects Δ_{i+mk} for all $m \in \mathbb{N}$, and therefore γ_* lies in the closure of W. By Proposition 4.2, the restriction $H_{l_*,\mu}|_{\gamma_*}$ is homeomorphically, but not smoothly conjugate to the rigid rotation, so γ_* cannot lie in W.

Conversely, let $\lambda_1 = e^{2\pi i\theta_1}$ satisfy (31). As shown in Theorem 4.1, if μ is small enough, then the family $l \mapsto H_{l,\mu}$ intersects the stable set of ζ_λ near P_{λ_1}. Denote by $l = \lambda_2$ the parameter of the intersection. As shown above, if $|\mu| < \epsilon$, then $\lambda_2 = e^{2\pi i\theta_2}$, where $\theta = G^j(\theta_2)$. The digits in the continued fraction expansion of θ_2 correspond to the periods of renormalizations of $H_{\lambda_2,\mu}$. By considerations of continuity, if μ is small enough, then the digits in the continued fractions of θ_2 and θ_1 coincide, and hence $\lambda_2 = \lambda_1$. □

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