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The normalized volume of a singularity is lower semicontinuous

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Abstract. We show that in any \mathbb{Q} -Gorenstein flat family of klt singularities, normalized volumes are lower semicontinuous with respect to the Zariski topology. A quick consequence is that smooth points have the largest normalized volume among all klt singularities. Using an alternative characterization of K-semistability developed by Li, Liu, and Xu, we show that K-semistability is a very generic or empty condition in any \mathbb{Q} -Gorenstein flat family of log Fano pairs.

Keywords. Singularities, normalized volume, K-stability

1. Introduction

Given an n -dimensional complex klt singularity $(x \in (X, D))$, Chi Li [Li18] introduced the *normalized volume* function on the space $\text{Val}_{x,X}$ of real valuations of $\mathbb{C}(X)$ centered at x . More precisely, for any such valuation v , its normalized volume is defined as $\widehat{\text{vol}}_{x,(X,D)}(v) := A_{X,D}(v)^n \text{vol}(v)$, where $A_{X,D}(v)$ is the log discrepancy of v with respect to (X, D) according to [JM12, BdF⁺15], and $\text{vol}(v)$ is the volume of v according to [ELS03]. Then we can define the *normalized volume of a klt singularity* $(x \in (X, D))$ by

$$\widehat{\text{vol}}(x, X, D) := \min_{v \in \text{Val}_{x,X}} \widehat{\text{vol}}_{x,(X,D)}(v)$$

where the existence of a minimizer of $\widehat{\text{vol}}$ was shown recently in [Blu18]. We also denote $\widehat{\text{vol}}(x, X) := \widehat{\text{vol}}(x, X, 0)$.

The normalized volume of a klt singularity $(x \in (X, D))$ carries interesting information of its geometry and topology. It was shown by the second author and Xu that $\widehat{\text{vol}}(x, X, D) \leq n^n$ and equality holds if and only if $(x \in X \setminus \text{Supp}(D))$ is smooth (see [LX19, Theorem A.4] or Theorem 32). By [Xu14] the local algebraic fundamental group $\hat{\pi}_1^{\text{loc}}(X, x)$ of a klt singularity $x \in X$ is always finite. Moreover, assuming the conjectural finite degree formula of normalized volumes [LX19, Conjecture 4.1], the size of

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$\widehat{\pi}_1^{\text{loc}}(X, x)$ is bounded from above by $n^n / \widehat{\text{vol}}(x, X)$ (see Remark 36). If X is a Gromov–Hausdorff limit of Kähler–Einstein Fano manifolds, then Li and Xu [LX18] showed that $\widehat{\text{vol}}(x, X) = n^n \cdot \Theta(x, X)$ where $\Theta(x, X)$ is the volume density of a closed point $x \in X$ (see [HS17, SS17] for background material).

In this article, it is shown that the normalized volume of a singularity is lower semicontinuous in families.

Theorem 1. *Let $\pi : (\mathcal{X}, \mathcal{D}) \rightarrow T$ together with a section $\sigma : T \rightarrow \mathcal{X}$ be a \mathbb{Q} -Gorenstein flat family of complex klt singularities over a normal variety T . Then the function $t \mapsto \widehat{\text{vol}}(\sigma(t), \mathcal{X}_t, \mathcal{D}_t)$ on $T(\mathbb{C})$ is lower semicontinuous with respect to the Zariski topology.*

One quick consequence of Theorem 1 is that smooth points have the largest normalized volumes among all klt singularities (see Theorem 32 or [LX19, Theorem A.4]). Another natural consequence is that if X is a Gromov–Hausdorff limit of Kähler–Einstein Fano manifolds, then the volume density function $x \mapsto \Theta(x, X)$ on $X(\mathbb{C})$ is lower semicontinuous in the Zariski topology, which is stronger than being lower semicontinuous in the Euclidean topology mentioned in [SS17] (see Corollary 34).

We also state the following natural conjecture on constructibility of normalized volumes of klt singularities (see also [Xu18, Conjecture 4.11]).

Conjecture 2. *Let $\pi : (\mathcal{X}, \mathcal{D}) \rightarrow T$ together with a section $\sigma : T \rightarrow \mathcal{X}$ be a \mathbb{Q} -Gorenstein flat family of complex klt singularities over a normal variety T . Then the function $t \mapsto \widehat{\text{vol}}(\sigma(t), \mathcal{X}_t, \mathcal{D}_t)$ on $T(\mathbb{C})$ is constructible.*

Verifying the Zariski openness of K-semistability is an important step in the construction of an algebraic moduli space of K-polystable \mathbb{Q} -Fano varieties. In a smooth family of Fano manifolds, Odaka [Oda13] and Donaldson [Don15] showed that the locus of fibers admitting Kähler–Einstein metrics (or equivalently, being K-polystable) with discrete automorphism groups is Zariski open. This was generalized by Li, Wang and Xu [LWX19] who proved the Zariski openness of K-semistability in a \mathbb{Q} -Gorenstein flat families of smoothable \mathbb{Q} -Fano varieties in their construction of the proper moduli space of smoothable K-polystable \mathbb{Q} -Fano varieties (see [SSY16, Oda15] for related results). A common feature is that analytic methods were used essentially in proving these results.

Using the alternative characterization of K-semistability by the affine cone construction developed by Li, the second author, and Xu [Li17a, LL19, LX20], we apply Theorem 1 to prove the following result on weak openness of K-semistability. Unlike the results described in the previous paragraph, our result is proved using purely algebraic methods and hence can be applied to \mathbb{Q} -Fano families with non-smoothable fibers (or more generally, families of log Fano pairs).

Theorem 3. *Let $\varphi : (\mathcal{Y}, \mathcal{E}) \rightarrow T$ be a \mathbb{Q} -Gorenstein flat family of complex log Fano pairs over a normal base T . If $(\mathcal{Y}_o, \mathcal{E}_o)$ is log K-semistable for some closed point $o \in T$, then the following statements hold:*

- (1) *There exists an intersection U of countably many Zariski open neighborhoods of o such that $(\mathcal{Y}_t, \mathcal{E}_t)$ is log K-semistable for any closed point $t \in U$. In particular, $(\mathcal{Y}_t, \mathcal{E}_t)$ is log K-semistable for a very general closed point $t \in T$.*

- (2) Denote by η the generic point of T . Then the geometric generic fiber $(\mathcal{Y}_{\bar{\eta}}, \mathcal{E}_{\bar{\eta}})$ is log K -semistable.
- (3) Assume Conjecture 2 is true. Then the U from (1) can be chosen as a genuine Zariski open neighborhood of o .

The following corollary generalizes [Li17b, Theorem 4] and follows easily from Theorem 3. Note that a similar result for Fano cones is proved by Li and Xu independently in [LX18, Proposition 2.36].

Corollary 4. *Suppose a complex log Fano pair (Y, E) specially degenerates to a log K -semistable log Fano pair (Y_0, E_0) . Then (Y, E) is also log K -semistable.*

Our strategy to prove Theorem 1 is to study invariants of ideals instead of invariants of valuations. From Liu’s characterization of normalized volume by normalized multiplicities of ideals (see [Liu18, Theorem 27] or Theorem 5), we know

$$\widehat{\text{vol}}(\sigma(t), \mathcal{X}_t, \mathcal{D}_t) = \inf_{\mathfrak{a}} \text{lct}(\mathcal{X}_t, \mathcal{D}_t; \mathfrak{a})^n \cdot e(\mathfrak{a})$$

where the infimum is taken over all ideals $\mathfrak{a} \subset \mathcal{O}_{\mathcal{X}_t}$ cosupported at $\sigma(t)$. These ideals are parametrized by a relative Hilbert scheme of \mathcal{X}/T with countably many components. Clearly $\mathfrak{a} \mapsto \text{lct}(\mathcal{X}_t, \mathcal{D}_t; \mathfrak{a})$ is lower semicontinuous on the Hilbert scheme, but $\mathfrak{a} \mapsto e(\mathfrak{a})$ may only be upper semicontinuous. Thus, it is unclear what semicontinuity properties $\mathfrak{a} \mapsto \text{lct}(\mathfrak{a})^n \cdot e(\mathfrak{a})$ may have.

To fix this issue, we introduce the *normalized colength of singularities* $\widehat{\ell}_{c,k}(\sigma(t), \mathcal{X}_t, \mathcal{D}_t)$ by taking the infimum of $\text{lct}(\mathcal{X}_t, \mathcal{D}_t; \mathfrak{a})^n \cdot \ell(\mathcal{O}_{\sigma(t), \mathcal{X}_t}/\mathfrak{a})$ for ideals \mathfrak{a} satisfying $\mathfrak{a} \supset \mathfrak{m}_{\sigma(t)}^k$ and $\ell(\mathcal{O}_{\sigma(t), \mathcal{X}_t}/\mathfrak{a}) \geq ck^n$. The normalized colength function behaves better in families since the colength function $\mathfrak{a} \mapsto \ell(\mathcal{O}_{\sigma(t), \mathcal{X}_t}/\mathfrak{a})$ is always locally constant in the Hilbert scheme, so $\mathfrak{a} \mapsto \text{lct}(\mathcal{X}_t, \mathcal{D}_t; \mathfrak{a})^n \cdot \ell(\mathcal{O}_{\sigma(t), \mathcal{X}_t}/\mathfrak{a})$ is constructibly lower semicontinuous on the Hilbert scheme. Thus, the properness of Hilbert schemes implies that $t \mapsto \widehat{\ell}_{c,k}(\sigma(t), \mathcal{X}_t, \mathcal{D}_t)$ is constructibly lower semicontinuous on T . Then we prove a key equality between the asymptotic normalized colength $\widehat{\ell}_{c,\infty}(\sigma(t), \mathcal{X}_t, \mathcal{D}_t)$ and the normalized volume $\widehat{\text{vol}}(\sigma(t), \mathcal{X}_t, \mathcal{D}_t)$ when c is small (see Theorem 12) using local Newton–Okounkov bodies following [Cut13, KK14] (see Lemma 13) and convex geometry (see Appendix A). Then by establishing a uniform approximation of volumes by colengths (see Theorem 16) and generalizing Li’s Izumi and properness estimates [Li18] to families (see Theorems 20 and 21), we show that the normalized colength functions uniformly approximate the normalized volume function from above (see Theorem 26). Putting these ingredients together, we get the proof of Theorem 1.

This paper is organized as follows. In Section 2, we give the preliminaries including notations, normalized volumes of singularities, and \mathbb{Q} -Gorenstein flat families of klt pairs. In Section 3.1, we introduce the concept of normalized colengths of singularities. We show in Theorem 12 that the normalized volume of a klt singularity is the same as its asymptotic normalized colength. The proof of Theorem 12 uses a comparison of colengths and multiplicities established in Lemma 13. In Section 3.2, we study the normalized volumes and normalized colength after algebraically closed field extensions. In Section 4, we establish a uniform approximation of volume of a valuation by colengths

of its valuation ideals. In Section 5, we generalize Li's Izumi and properness estimates to families. The results from Sections 4 and 5 enable us to prove the uniform approximation of normalized volumes by normalized colengths from above in families (see Section 6.1). The proofs of the main theorems are presented in Section 6.2. We give applications of our main theorems in Section 6.3. Theorem 32 generalizes the inequality part of [LX19, Theorem A.4]. We show that the volume density function on a Gromov–Hausdorff limit of Kähler–Einstein manifolds is lower semicontinuous in the Zariski topology (see Corollary 34). We give an effective upper bound on the degree of finite quasi-étale maps over klt singularities on Gromov–Hausdorff limits of Kähler–Einstein Fano manifolds (see Theorem 35). In Appendix A we provide certain convex geometric results on lattice points counting that are needed in proving Lemma 13. In Appendix B, we provide results on constructibility of Hilbert–Samuel functions that are needed in proving uniform approximation results in Section 4.

Postscript: After this document was first posted on the arXiv, the authors went on to show that the global log canonical threshold and the stability threshold are lower semicontinuous in families of polarized varieties [BL18]. The results in *loc. cit.* may be viewed as global analogues of Theorem 1 and their proofs are similar in spirit (though the technical details are quite different).

2. Preliminaries

2.1. Notations

In this paper, all varieties are assumed to be irreducible, reduced, and defined over a (not necessarily algebraically closed) field \mathbb{k} of characteristic 0. For a variety T over \mathbb{k} , we denote the residue field of any scheme-theoretic point $t \in T$ by $\kappa(t)$. Given a morphism $\pi : \mathcal{X} \rightarrow T$ between varieties over \mathbb{k} , we write $\mathcal{X}_t := \mathcal{X} \times_T \text{Spec}(\kappa(t))$ for the scheme-theoretic fiber over $t \in T$. We also denote the geometric fiber of π over $t \in T$ by $\mathcal{X}_{\bar{t}} := \mathcal{X} \times_T \text{Spec}(\overline{\kappa(t)})$. Suppose X is a variety over \mathbb{k} and $x \in X$ is a \mathbb{k} -rational point. Then for any field extension \mathbb{K}/\mathbb{k} , we denote $(x_{\mathbb{K}}, X_{\mathbb{K}}) := (x, X) \times_{\text{Spec}(\mathbb{k})} \text{Spec}(\mathbb{K})$.

Let X be a normal variety over \mathbb{k} and D be an effective \mathbb{Q} -divisor on X . We say that (X, D) is a *Kawamata log terminal (klt) pair* if $(K_X + D)$ is \mathbb{Q} -Cartier and $K_Y - f^*(K_X + D)$ has coefficients > -1 on some log resolution $f : Y \rightarrow (X, D)$. A klt pair (X, D) is called a *log Fano pair* if in addition X is proper and $-(K_X + D)$ is ample. A klt pair (X, D) together with a closed point $x \in X$ is called a *klt singularity* ($x \in (X, D)$).

Let (X, D) be a klt pair. For an ideal sheaf \mathfrak{a} on X , we define the *log canonical threshold of \mathfrak{a} with respect to (X, D)* by

$$\text{lct}(X, D; \mathfrak{a}) := \inf_E \frac{A_{X,D}(\text{ord}_E)}{\text{ord}_E(\mathfrak{a})},$$

where the infimum is taken over all prime divisors E on a log resolution $f : Y \rightarrow (X, D)$. We will often use the notation $\text{lct}(\mathfrak{a})$ to abbreviate $\text{lct}(X, D; \mathfrak{a})$ once the klt pair (X, D) is

specified. If \mathfrak{a} is cosupported at a single closed point $x \in X$, we define the *Hilbert–Samuel multiplicity* of \mathfrak{a} as

$$e(\mathfrak{a}) := \lim_{m \rightarrow \infty} \frac{\ell(\mathcal{O}_{x,X}/\mathfrak{a}^m)}{m^n/n!}$$

where $n := \dim(X)$ and $\ell(\mathcal{O}_{x,X}/\mathfrak{a}^m)$ denotes the length of $\mathcal{O}_{x,X}/\mathfrak{a}^m$ as an $\mathcal{O}_{x,X}$ -module.

2.2. Valuations

Let X be a variety defined over a field \mathbb{k} and $x \in X$ closed point. By a valuation of the function field $K(X)$, we mean a valuation $v: K(X)^\times \rightarrow \mathbb{R}$ that is trivial on \mathbb{k} . By convention, we set $v(0) := +\infty$. Such a valuation v has *center* x if $v \geq 0$ on $\mathcal{O}_{x,X}$ and > 0 on the maximal ideal of $\mathcal{O}_{x,X}$. We write $\text{Val}_{x,X}$ for the set of valuations of $K(X)$ with center x .

To any valuation $v \in \text{Val}_{x,X}$ and $m \in \mathbb{Z}_{>0}$ there is an associated *valuation ideal* defined locally by $\mathfrak{a}_m(v) := \{f \in \mathcal{O}_X \mid v(f) \geq m\}$. Note that $\mathfrak{a}_m(v)$ is \mathfrak{m}_x -primary for each $m \in \mathbb{Z}_{>0}$. For an ideal $\mathfrak{a} \subset \mathcal{O}_X$ and $v \in \text{Val}_{x,X}$, we set

$$v(\mathfrak{a}) := \min\{v(f) \mid f \in \mathfrak{a} \cdot \mathcal{O}_{x,X}\} \in [0, +\infty].$$

2.3. Normalized volumes of singularities

Let \mathbb{k} be an algebraically closed field of characteristic 0. For an n -dimensional klt singularity $x \in (X, D)$ over \mathbb{k} , C. Li [Li18] introduced the normalized volume function $\widehat{\text{vol}}_{x,(X,D)}: \text{Val}_{x,X} \rightarrow \mathbb{R}_{>0} \cup \{+\infty\}$. Recall that for $v \in \text{Val}_{x,X}$,

$$\widehat{\text{vol}}_{x,(X,D)}(v) := \begin{cases} A_{X,D}(v)^n \cdot \text{vol}(v) & \text{if } A_{X,D}(v) < +\infty, \\ +\infty & \text{if } A_{X,D}(v) = +\infty, \end{cases}$$

where $A_{X,D}(v)$ and $\text{vol}(v)$ denote the *log discrepancy* and *volume* of v . As defined in [ELS03], the *volume* of v is given by

$$\text{vol}(v) := \limsup_{m \rightarrow \infty} \frac{\ell(\mathcal{O}_{x,X}/\mathfrak{a}_m(v))}{m^n/n!}.$$

By [ELS03, Mus02, LM09, Cut13],

$$\text{vol}(v) = \lim_{m \rightarrow \infty} \frac{e(\mathfrak{a}_m(v))}{m^n}.$$

The *log discrepancy* of v , denoted $A_{X,D}(v)$, is defined in [JM12, BdF⁺15] (and [LL19] for the case of klt pairs).

The *normalized volume* (also known as *local volume*) of the singularity $x \in (X, D)$ is given by

$$\widehat{\text{vol}}(x, X, D) := \inf_{v \in \text{Val}_{x,X}} \widehat{\text{vol}}_{x,(X,D)}(v).$$

When \mathbb{k} is uncountable, the above infimum is a minimum [Blu18].

The following characterization of normalized volumes using log canonical thresholds and multiplicities of ideals is crucial in our study. Note that the right hand side of (2.1) was studied by de Fernex, Ein and Mustață [dFEM04] when $x \in X$ is smooth and $D = 0$.

Theorem 5 ([Liu18, Theorem 27]). *With the above notation, we have*

$$\widehat{\text{vol}}(x, X, D) = \inf_{\mathfrak{a}: m_x\text{-primary}} \text{lct}(X, D; \mathfrak{a})^n \cdot e(\mathfrak{a}). \tag{2.1}$$

The following theorem provides an alternative characterization of \mathbb{K} -semistability using the affine cone construction. Here we state the most general form, and special cases can be found in [Li17a, LL19].

Theorem 6 ([LX20, Proposition 4.6]). *Let (Y, E) be a log Fano pair of dimension $n - 1$ over an algebraically closed field \mathbb{k} of characteristic 0. For $r \in \mathbb{N}$ such that $L := -r(K_Y + E)$ is Cartier, the affine cone $X = C(Y, L)$ is defined by*

$$X := \text{Spec} \bigoplus_{m \geq 0} H^0(Y, L^{\otimes m}).$$

Let D be the \mathbb{Q} -divisor on X corresponding to E . Denote by x the cone vertex of X . Then

$$\widehat{\text{vol}}(x, X, D) \leq r^{-1}(-K_Y - E)^{n-1},$$

and equality holds if and only if (Y, E) is log \mathbb{K} -semistable.

2.4. \mathbb{Q} -Gorenstein flat families of klt pairs

In this section, the field \mathbb{k} is not assumed to be algebraically closed.

Definition 7. (a) Given a normal variety T , a \mathbb{Q} -Gorenstein flat family of klt pairs over T consists of a surjective flat morphism $\pi : \mathcal{X} \rightarrow T$ from a variety \mathcal{X} , and an effective \mathbb{Q} -divisor \mathcal{D} on \mathcal{X} avoiding codimension 1 singular points of \mathcal{X} and such that

- all fibers \mathcal{X}_t are connected, normal and not contained in $\text{Supp}(\mathcal{D})$;
- $K_{\mathcal{X}/T} + \mathcal{D}$ is \mathbb{Q} -Cartier;
- $(\mathcal{X}_t, \mathcal{D}_t)$ is a klt pair for any $t \in T$.

(b) A \mathbb{Q} -Gorenstein flat family of klt pairs $\pi : (\mathcal{X}, \mathcal{D}) \rightarrow T$ together with a section $\sigma : T \rightarrow \mathcal{X}$ is called a \mathbb{Q} -Gorenstein flat family of klt singularities. We denote by $\sigma(\bar{t})$ the unique closed point of $\mathcal{X}_{\bar{t}}$ lying over $\sigma(t) \in \mathcal{X}_t$.

Proposition 8. *Let $\pi : (\mathcal{X}, \mathcal{D}) \rightarrow T$ be a \mathbb{Q} -Gorenstein flat family of klt pairs over a normal variety T . Then:*

- (1) *There exists a closed subset \mathcal{Z} of \mathcal{X} of codimension at least 2 such that \mathcal{Z}_t has codimension at least 2 in \mathcal{X}_t for every $t \in T$, and $\pi : \mathcal{X} \setminus \mathcal{Z} \rightarrow T$ is a smooth morphism.*
- (2) *\mathcal{X} is normal.*

- (3) For any morphism $f : T' \rightarrow T$ from a normal variety T' to T , the base change $\pi_{T'} : (\mathcal{X}_{T'}, \mathcal{D}_{T'}) = (\mathcal{X}, \mathcal{D}) \times_T T' \rightarrow T'$ is a \mathbb{Q} -Gorenstein flat family of klt pairs over T' , and $K_{\mathcal{X}_{T'}/T'} + \mathcal{D}_{T'} = g^*(K_{\mathcal{X}/T} + \mathcal{D})$ where $g : \mathcal{X}_{T'} \rightarrow \mathcal{X}$ is the base change of f .

Proof. (1) Assume that π is of relative dimension n . We define $\mathcal{Z} := \{x \in \mathcal{X} \mid \dim_{\kappa(x)} \Omega_{\mathcal{X}/T} \otimes \kappa(x) > n\}$. It is clear that \mathcal{Z} is Zariski closed. Since \mathbb{k} is of characteristic 0, $\mathcal{Z}_t = \mathcal{Z} \cap \mathcal{X}_t$ is the singular locus of \mathcal{X}_t . Hence $\text{codim}_{\mathcal{X}_t} \mathcal{Z}_t \geq 2$ because \mathcal{X}_t is normal.

(2) From (1) we know that \mathcal{Z} is of codimension at least 2 in \mathcal{X} , and $\mathcal{X} \setminus \mathcal{Z}$ is smooth over T . Thus $\mathcal{X} \setminus (\mathcal{Z} \cup \pi^{-1}(T_{\text{sing}}))$ is regular, and $\mathcal{Z} \cup \pi^{-1}(T_{\text{sing}})$ has codimension at least 2 in \mathcal{X} . So \mathcal{X} satisfies property (R_1) . Since π is flat, for any point $x \in \mathcal{X}_t$ we have $\text{depth}(\mathcal{O}_{x,\mathcal{X}}) = \text{depth}(\mathcal{O}_{x,\mathcal{X}_t}) + \text{depth}(\mathcal{O}_{t,T})$ by [Mat80, (21.C) Corollary 1]. Hence it is easy to see that \mathcal{X} satisfies property (S_2) since both \mathcal{X}_t and T are normal (see [Mat80, p. 125] for properties (R_1) and (S_2)). Hence \mathcal{X} is normal.

(3) Let $\mathcal{Z}_{T'} := \mathcal{Z} \times_T T'$, and note that $\mathcal{X}_{T'} \setminus \mathcal{Z}_{T'}$ is smooth over T' . Since the fibers of $\pi_{T'}$ and T' are irreducible, $\mathcal{X}_{T'}$ is also irreducible. Thus the same argument of (2) implies that $\mathcal{X}_{T'}$ satisfies both (R_1) and (S_2) , which means $\mathcal{X}_{T'}$ is normal. Since $\pi|_{\mathcal{X} \setminus \mathcal{Z}}$ is smooth, we know that $K_{\mathcal{X}_{T'}/T'} + \mathcal{D}_{T'}$ and $g^*(K_{\mathcal{X}/T} + \mathcal{D})$ are \mathbb{Q} -linearly equivalent after restricting to $\mathcal{X}_{T'} \setminus \mathcal{Z}_{T'}$. Since $\mathcal{Z}_{T'}$ is of codimension at least 2 in $\mathcal{X}_{T'}$, the \mathbb{Q} -linear equivalence over $\mathcal{X}_{T'} \setminus \mathcal{Z}_{T'}$ extends to $\mathcal{X}_{T'}$. Thus we finish the proof. \square

- Definition 9.** (a) Let Y be a normal projective variety. Let E be an effective \mathbb{Q} -divisor on Y . We say that (Y, E) is a *log Fano pair* if (Y, E) is a klt pair and $-(K_Y + E)$ is \mathbb{Q} -Cartier and ample. We say Y is a *\mathbb{Q} -Fano variety* if $(Y, 0)$ is a log Fano pair.
 (b) Let T be a normal variety. A \mathbb{Q} -Gorenstein family of klt pairs $\varphi : (\mathcal{Y}, \mathcal{E}) \rightarrow T$ is called a *\mathbb{Q} -Gorenstein flat family of log Fano pairs* if φ is proper and $-(K_{\mathcal{Y}/T} + \mathcal{E})$ is φ -ample.

The following proposition states a well known result on the behavior of the log canonical threshold in families. See [Amb16, Corollary 1.10] for a similar statement. The proof is omitted because the arguments are similar to those in [Amb16].

Proposition 10. Let $\pi : (\mathcal{X}, \mathcal{D}) \rightarrow T$ be a \mathbb{Q} -Gorenstein flat family of klt pairs over a normal variety T . Let \mathfrak{a} be an ideal sheaf of \mathcal{X} . Then:

- (1) The function $t \mapsto \text{lct}(\mathcal{X}_t, \mathcal{D}_t; \mathfrak{a}_t)$ on T is constructible.
- (2) If in addition $V(\mathfrak{a})$ is proper over T , then the function $t \mapsto \text{lct}(\mathcal{X}_t, \mathcal{D}_t; \mathfrak{a}_t)$ is lower semicontinuous with respect to the Zariski topology on T .

3. Comparison of normalized volumes and normalized colengths

3.1. Normalized colengths of klt singularities

Definition 11. Let $x \in (X, D)$ be a klt singularity over an algebraically closed field \mathbb{k} of characteristic 0. Denote its local ring by $(R, \mathfrak{m}) := (\mathcal{O}_{x,X}, \mathfrak{m}_x)$.

- (a) Given constants $c \in \mathbb{R}_{>0}$ and $k \in \mathbb{N}$, we define the *normalized colength of $x \in (X, D)$ with respect to c, k* as

$$\widehat{\ell}_{c,k}(x, X, D) := n! \cdot \inf_{\substack{\mathfrak{m}^k \subset \mathfrak{a} \subset \mathfrak{m} \\ \ell(R/\mathfrak{a}) \geq ck^n}} \text{lct}(\mathfrak{a})^n \cdot \ell(R/\mathfrak{a}).$$

Note that the assumption $\mathfrak{m}^k \subset \mathfrak{a} \subset \mathfrak{m}$ implies \mathfrak{a} is an \mathfrak{m} -primary ideal.

- (b) Given a constant $c \in \mathbb{R}_{>0}$, we define the *asymptotic normalized colength function of $x \in (X, D)$ with respect to c* as

$$\widehat{\ell}_{c,\infty}(x, X, D) := \liminf_{k \rightarrow \infty} \widehat{\ell}_{c,k}(x, X, D).$$

It is clear that $\widehat{\ell}_{c,k}$ is an increasing function in c . The main result in this section is the following theorem.

Theorem 12. *For any klt singularity $x \in (X, D)$ over an algebraically closed field \mathbb{k} of characteristic 0, there exists $c_0 = c_0(x, X, D) > 0$ such that*

$$\widehat{\ell}_{c,\infty}(x, X, D) = \widehat{\text{vol}}(x, X, D) \quad \text{whenever } 0 < c \leq c_0. \tag{3.1}$$

Proof. We first show the “ \leq ” direction. Let us take a sequence $\{v_i\}_{i \in \mathbb{N}}$ of valuations such that $\lim_{i \rightarrow \infty} \widehat{\text{vol}}(v_i) = \widehat{\text{vol}}(x, X, D)$. We may rescale v_i so that $v_i(\mathfrak{m}) = 1$ for any i . Since $\{\widehat{\text{vol}}(v_i)\}_{i \in \mathbb{N}}$ are bounded from above, by [Li18, Theorem 1.1] we know that there exists $C_1 > 0$ such that $A_{X,D}(v_i) \leq C_1$ for any $i \in \mathbb{N}$. Then by Li’s Izumi type inequality [Li18, Theorem 3.1], there exists $C_2 > 0$ such that $\text{ord}_{\mathfrak{m}}(f) \leq v_i(f) \leq C_2 \text{ord}_{\mathfrak{m}}(f)$ for any $i \in \mathbb{N}$ and any $f \in R$. As a result, we have $\mathfrak{m}^k \subset \mathfrak{a}_k(v_i) \subset \mathfrak{m}^{\lceil k/C_2 \rceil}$ for any $i, k \in \mathbb{N}$. Thus $\ell(R/\mathfrak{a}_k(v_i)) \geq \ell(R/\mathfrak{m}^{\lceil k/C_2 \rceil}) \sim \frac{e(\mathfrak{m})}{n!C_2^n} k^n$. Let us take $c_0 = \frac{e(\mathfrak{m})}{2n!C_2^n}$. Then for $k \gg 1$ we have $\ell(R/\mathfrak{a}_k(v_i)) \geq c_0 k^n$ for any $i \in \mathbb{N}$. Therefore, for any $i \in \mathbb{N}$ we have

$$\widehat{\ell}_{c_0,\infty}(x, X, D) \leq n! \liminf_{k \rightarrow \infty} \text{lct}(\mathfrak{a}_k(v_i))^n \ell(R/\mathfrak{a}_k(v_i)) = \text{lct}(\mathfrak{a}_\bullet(v_i))^n \text{vol}(v_i) \leq \widehat{\text{vol}}(v_i).$$

In the last inequality we use $\text{lct}(\mathfrak{a}_\bullet(v_i)) \leq A_{X,D}(v_i)$ as in the proof of [Liu18, Theorem 27]. Thus $\widehat{\ell}_{c_0,\infty}(x, X, D) \leq \lim_{i \rightarrow \infty} \widehat{\text{vol}}(v_i) = \widehat{\text{vol}}(x, X, D)$. This finishes the proof of the “ \leq ” direction.

For the “ \geq ” direction, we will show that $\widehat{\ell}_{c,\infty}(x, X, D) \geq \widehat{\text{vol}}(x, X, D)$ for any $c > 0$. By a logarithmic version of the Izumi type estimate [Li18, Theorem 3.1], there exists a constant $c_1 = c_1(x, X, D) > 0$ such that $v(f) \leq c_1 A_{X,D}(v) \text{ord}_{\mathfrak{m}}(f)$ for any valuation $v \in \text{Val}_{x,X}$ and any function $f \in R$. For any \mathfrak{m} -primary ideal \mathfrak{a} , there exists a divisorial valuation $v_0 \in \text{Val}_{x,X}$ computing $\text{lct}(\mathfrak{a})$ by [Liu18, Lemma 26]. Hence we have the following Skoda type estimate:

$$\text{lct}(\mathfrak{a}) = \frac{A_{X,D}(v_0)}{v_0(\mathfrak{a})} \geq \frac{A_{X,D}(v_0)}{c_1 A_{X,D}(v_0) \text{ord}_{\mathfrak{m}}(\mathfrak{a})} = \frac{1}{c_1 \text{ord}_{\mathfrak{m}}(\mathfrak{a})}.$$

Let $0 < \delta < 1$. If $\mathfrak{a} \not\subset \mathfrak{m}^{\lceil \delta k \rceil}$ and $\ell(R/\mathfrak{a}) \geq ck^n$, then

$$\text{lct}(\mathfrak{a})^n \cdot \ell(R/\mathfrak{a}) \geq \frac{ck^n}{c_1^n (\lceil \delta k \rceil - 1)^n} \geq \frac{c}{c_1^n \delta^n}.$$

If we choose δ so small that $\delta^n \cdot c_1^n \widehat{\text{vol}}(x, X, D) \leq n!c$, then for any \mathfrak{m} -primary ideal \mathfrak{a} satisfying $\mathfrak{m}^k \subset \mathfrak{a} \not\subset \mathfrak{m}^{\lceil \delta k \rceil}$ and $\ell(R/\mathfrak{a}) \geq ck^n$ we have

$$n! \cdot \text{lct}(\mathfrak{a})^n \cdot \ell(R/\mathfrak{a}) \geq \widehat{\text{vol}}(x, X, D).$$

Thus it suffices to show

$$\widehat{\text{vol}}(x, X, D) \leq n! \cdot \liminf_{k \rightarrow \infty} \inf_{\substack{\mathfrak{m}^k \subset \mathfrak{a} \subset \mathfrak{m}^{\lceil \delta k \rceil} \\ \ell(R/\mathfrak{a}) \geq ck^n}} \text{lct}(\mathfrak{a})^n \ell(R/\mathfrak{a}).$$

By Lemma 13, we know that for any $\epsilon > 0$ there exists $k_0 = k_0(\delta, \epsilon, (R, \mathfrak{m}))$ such that for any $k \geq k_0$ we have

$$n! \cdot \inf_{\mathfrak{m}^k \subset \mathfrak{a} \subset \mathfrak{m}^{\lceil \delta k \rceil}} \text{lct}(\mathfrak{a})^n \ell(R/\mathfrak{a}) \geq (1 - \epsilon) \inf_{\mathfrak{m}^k \subset \mathfrak{a} \subset \mathfrak{m}^{\lceil \delta k \rceil}} \text{lct}(\mathfrak{a})^n e(\mathfrak{a}) \geq (1 - \epsilon) \widehat{\text{vol}}(x, X, D).$$

Hence the proof is finished. □

The following result on comparison between colengths and multiplicities is crucial in the proof of Theorem 12. Note that Lemma 13 is a special case of Lech’s inequality [Lec60, Theorem 3] when R is a regular local ring.

Lemma 13. *Let (R, \mathfrak{m}) be an n -dimensional analytically irreducible Noetherian local domain. Assume that the residue field R/\mathfrak{m} is algebraically closed. Then for any numbers $\delta, \epsilon \in (0, 1)$, there exists $k_0 = k_0(\delta, \epsilon, (R, \mathfrak{m}))$ such that for any $k \geq k_0$ and any ideal $\mathfrak{m}^k \subset \mathfrak{a} \subset \mathfrak{m}^{\lceil \delta k \rceil}$, we have*

$$n! \cdot \ell(R/\mathfrak{a}) \geq (1 - \epsilon)e(\mathfrak{a}).$$

Proof. By [KK14, 7.8] and [Cut13, Section 4], R admits a good valuation $\nu : R \rightarrow \mathbb{Z}^n$ for some total order on \mathbb{Z}^n . Let $\mathcal{S} := \nu(R \setminus \{0\}) \subset \mathbb{N}^n$ and $C(\mathcal{S})$ be the closed convex hull of \mathcal{S} . Then we know that

- $C(\mathcal{S})$ is a strongly convex cone;
- there exists a linear functional $\xi : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $C(\mathcal{S}) \setminus \{0\} \subset \xi_{>0}$;
- there exists $r_0 \geq 1$ such that for any $f \in R \setminus \{0\}$, we have

$$\text{ord}_{\mathfrak{m}}(f) \leq \xi(\nu(f)) \leq r_0 \text{ord}_{\mathfrak{m}}(f). \tag{3.2}$$

Suppose \mathfrak{a} is an ideal satisfying $\mathfrak{m}^k \subset \mathfrak{a} \subset \mathfrak{m}^{\lceil \delta k \rceil}$. Then $\nu(\mathfrak{m}^k) \subset \nu(\mathfrak{a}) \subset \nu(\mathfrak{m}^{\lceil \delta k \rceil})$. By (3.2), we know that

$$\mathcal{S} \cap \xi_{\geq r_0 k} \subset \nu(\mathfrak{a}) \subset \mathcal{S} \cap \xi_{\geq \delta k}.$$

Similarly, $\mathcal{S} \cap \xi_{\geq r_0 i k} \subset \nu(\mathfrak{a}^i) \subset \mathcal{S} \cap \xi_{\geq \delta i k}$ for any positive integer i .

Let us define a semigroup $\Gamma \subset \mathbb{N}^{n+1}$ as follows:

$$\Gamma := \{(\alpha, m) \in \mathbb{N}^n \times \mathbb{N} \mid x \in \mathcal{S} \cap \xi_{\leq 2r_0 m}\}.$$

For any $m \in \mathbb{N}$, denote by $\Gamma_m := \{\alpha \in \mathbb{N}^n \mid (\alpha, m) \in \Gamma\}$. It is easy to see Γ satisfies [LM09, (2.3-5)], thus [LM09, Proposition 2.1] implies

$$\lim_{m \rightarrow \infty} \frac{\#\Gamma_m}{m^n} = \text{vol}(\Delta),$$

where $\Delta := \Delta(\Gamma)$ is a convex body in \mathbb{R}^n defined in [LM09, Section 2.1]. It is easy to see that $\Delta = C(\mathcal{S}) \cap \xi_{\leq 2r_0}$.

Let us define $\Gamma^{(k)} := \{(\alpha, i) \in \mathbb{N}^n \times \mathbb{N} \mid (\alpha, ik) \in \Gamma\}$. Then we know that $\Delta^{(k)} := \Delta(\Gamma^{(k)}) = k\Delta$. For an ideal \mathfrak{a} and $k \in \mathbb{N}$ satisfying $\mathfrak{m}^k \subset \mathfrak{a} \subset \mathfrak{m}^{\lceil \delta k \rceil}$, we define

$$\Gamma_{\mathfrak{a}}^{(k)} := \{(\alpha, i) \in \Gamma^{(k)} \mid \alpha \in \nu(\mathfrak{a}^i)\}.$$

Then it is clear that $\Gamma_{\mathfrak{a}}^{(k)}$ also satisfies [LM09, (2.3-5)]. Since $\nu(\mathfrak{a}^i) = (\mathcal{S} \cap \xi_{>2r_0 ik}) \cup \Gamma_{\mathfrak{a},i}^{(k)}$ and R/\mathfrak{m} is algebraically closed, we have $\ell(R/\mathfrak{a}^i) = \#(\Gamma_i^{(k)} \setminus \Gamma_{\mathfrak{a},i}^{(k)})$ because ν has one-dimensional leaves. Again by [LM09, Proposition 2.11], we have

$$n!e(\mathfrak{a}) = \lim_{i \rightarrow \infty} \frac{\ell(R/\mathfrak{a}^i)}{i^n} = \lim_{i \rightarrow \infty} \frac{\#(\Gamma_i^{(k)} \setminus \Gamma_{\mathfrak{a},i}^{(k)})}{i^n} = \text{vol}(\Delta^{(k)}) - \text{vol}(\Delta_{\mathfrak{a}}^{(k)}),$$

where $\Delta_{\mathfrak{a}}^{(k)} := \Delta(\Gamma_{\mathfrak{a}}^{(k)})$. Since $\Gamma_{\mathfrak{a},i}^{(k)} \subset \nu(\mathfrak{a}^i) \subset \xi_{\geq \delta ik}$, we know that $\Delta_{\mathfrak{a}}^{(k)} \subset \xi_{\geq \delta k}$. Denote $\Delta' := C(\mathcal{S}) \cap \xi_{< \delta}$. Then it is clear that $\Delta_{\mathfrak{a}}^{(k)} \subset k(\Delta \setminus \Delta')$.

On the other hand,

$$\ell(R/\mathfrak{a}) = \#(\Gamma_1^{(k)} \setminus \Gamma_{\mathfrak{a},1}^{(k)}) \geq \#\Gamma_k - \#(\Delta_{\mathfrak{a}}^{(k)} \cap \mathbb{Z}^n).$$

Denote $\Delta_{\mathfrak{a},k} := \frac{1}{k}\Delta_{\mathfrak{a}}^{(k)}$. Then $\Delta_{\mathfrak{a},k} \subset \Delta \setminus \Delta'$. Since $\text{vol}(\Delta_{\mathfrak{a},k}) \leq \text{vol}(\Delta) - \text{vol}(\Delta')$, there exist positive numbers ϵ_1, ϵ_2 depending only on Δ and Δ' such that

$$\text{vol}(\Delta_{\mathfrak{a},k}) \leq \text{vol}(\Delta) - \text{vol}(\Delta') \leq (1 - \epsilon_1/\epsilon)\text{vol}(\Delta) - \epsilon_2/\epsilon. \tag{3.3}$$

Let us pick k_0 such that for any $k \geq k_0$ and any $\mathfrak{m}^k \subset \mathfrak{a} \subset \mathfrak{m}^{\lceil \delta k \rceil}$, we have

$$\frac{\#\Gamma_k}{k^n} \geq (1 - \epsilon_1)\text{vol}(\Delta), \quad \frac{\#(\Delta_{\mathfrak{a}}^{(k)} \cap \mathbb{Z}^n)}{k^n} \leq \text{vol}(\Delta_{\mathfrak{a},k}) + \epsilon_2.$$

Here the second inequality is guaranteed by applying Proposition 38 to $\Delta_{\mathfrak{a},k}$ as a convex body contained in a fixed convex body Δ . Thus

$$\begin{aligned} \frac{\ell(R/\mathfrak{a}) - (1 - \epsilon)n!e(\mathfrak{a})}{k^n} &\geq \frac{\#\Gamma_k}{k^n} - \frac{\#(\Delta_{\mathfrak{a}}^{(k)} \cap \mathbb{Z}^n)}{k^n} - (1 - \epsilon)(\text{vol}(\Delta) - \text{vol}(\Delta_{\mathfrak{a},k})) \\ &\geq (1 - \epsilon_1)\text{vol}(\Delta) - \text{vol}(\Delta_{\mathfrak{a},k}) - \epsilon_2 - (1 - \epsilon)(\text{vol}(\Delta) - \text{vol}(\Delta_{\mathfrak{a},k})) \\ &= (\epsilon - \epsilon_1)\text{vol}(\Delta) - \epsilon(\Delta_{\mathfrak{a},k}) - \epsilon_2 \geq 0. \end{aligned}$$

Here the last inequality follows from (3.3). Hence we finish the proof. □

3.2. Normalized volumes under field extensions

In the rest of this section, we use Hilbert schemes to describe normalized volumes of singularities after a field extension \mathbb{K}/\mathbb{k} . Let (X, D) be a klt pair over \mathbb{k} and $x \in X$ be a \mathbb{k} -rational point. Let $Z_k := \text{Spec}(\mathcal{O}_{x,X}/\mathfrak{m}_{x,X}^k)$ denote the k -th thickening of x . Consider the Hilbert scheme $H_{k,d} := \text{Hilb}_d(Z_k/\mathbb{k})$. For any field extension \mathbb{K}/\mathbb{k} we know that $H_{k,d}(\mathbb{K})$ parametrizes ideal sheaves \mathfrak{c} of $X_{\mathbb{K}}$ satisfying $\mathfrak{c} \supset \mathfrak{m}_{x_{\mathbb{K}},X_{\mathbb{K}}}^k$ and $\ell(\mathcal{O}_{X_{\mathbb{K}},X_{\mathbb{K}}}/\mathfrak{c}) = d$. In particular, any scheme-theoretic point $h \in H_{k,d}$ corresponds to an ideal \mathfrak{b} of $\mathcal{O}_{x_{\mathbb{K}}(h),X_{\mathbb{K}}(h)}$ satisfying those two conditions, and we denote $h = [\mathfrak{b}]$.

Proposition 14. *Let \mathbb{k} be a field of characteristic 0. Let (X, D) be a klt pair over \mathbb{k} . Let $x \in X$ be a \mathbb{k} -rational point. Then:*

(1) *For any field extension \mathbb{K}/\mathbb{k} with \mathbb{K} algebraically closed, we have*

$$\widehat{\ell}_{c,k}(x_{\mathbb{K}}, X_{\mathbb{K}}, D_{\mathbb{K}}) = n! \cdot \inf_{d \geq ck^n, [\mathfrak{b}] \in H_{k,d}} d \cdot \text{lct}(X_{\kappa([\mathfrak{b}])}, D_{\kappa([\mathfrak{b}])}; \mathfrak{b})^n.$$

(2) *With the assumption of (1), we have*

$$\widehat{\text{vol}}(x_{\mathbb{K}}, X_{\mathbb{K}}, D_{\mathbb{K}}) = \widehat{\text{vol}}(x_{\bar{\mathbb{k}}}, X_{\bar{\mathbb{k}}}, D_{\bar{\mathbb{k}}}).$$

Proof. (1) We first prove the “ \geq ” direction. By definition, $\widehat{\ell}_{c,k}(x_{\mathbb{K}}, X_{\mathbb{K}}, D_{\mathbb{K}})$ is the infimum of $n! \cdot \text{lct}(X_{\mathbb{K}}, D_{\mathbb{K}}; \mathfrak{c})^n \ell(\mathcal{O}_{X_{\mathbb{K}}}/\mathfrak{c})$ where \mathfrak{c} is an ideal on $X_{\mathbb{K}}$ satisfying $\mathfrak{m}_{x_{\mathbb{K}}}^k \subset \mathfrak{c} \subset \mathfrak{m}_{x_{\mathbb{K}}}$ and $\ell(\mathcal{O}_{X_{\mathbb{K}}}/\mathfrak{c}) =: d \geq ck^n$. Hence $[\mathfrak{c}]$ represents a point in $H_{k,d}(\mathbb{K})$. Suppose $[\mathfrak{c}]$ is lying over a scheme-theoretic point $[\mathfrak{b}] \in H_{k,d}$. Then it is clear that $(X_{\mathbb{K}}, D_{\mathbb{K}}, \mathfrak{c}) \cong (X_{\kappa([\mathfrak{b}])}, D_{\kappa([\mathfrak{b}])}, \mathfrak{b}) \times_{\text{Spec}(\kappa([\mathfrak{b}])), \text{Spec}(\mathbb{K})}$. Hence $\text{lct}(X_{\mathbb{K}}, D_{\mathbb{K}}; \mathfrak{c}) = \text{lct}(X_{\kappa([\mathfrak{b}])}, D_{\kappa([\mathfrak{b}])}; \mathfrak{b})$ by [JM12, Proposition 7.13], and the “ \geq ” direction is proved.

Next we prove the “ \leq ” direction. By Proposition 10, we know that the function $[\mathfrak{b}] \mapsto \text{lct}(X_{\kappa([\mathfrak{b}])}, D_{\kappa([\mathfrak{b}])}; \mathfrak{b})$ on $H_{k,d}$ is constructible and lower semicontinuous. Denote by $H_{k,d}^{\text{cl}}$ the set of closed points in $H_{k,d}$. Since the sets of closed points are dense in any stratum of $H_{k,d}$ with respect to the lct function, we have

$$\begin{aligned} n! \cdot \inf_{d \geq ck^n, [\mathfrak{b}] \in H_{k,d}} d \cdot \text{lct}(X_{\kappa([\mathfrak{b}])}, D_{\kappa([\mathfrak{b}])}; \mathfrak{b})^n \\ = n! \cdot \inf_{d \geq ck^n, [\mathfrak{b}] \in H_{k,d}^{\text{cl}}} d \cdot \text{lct}(X_{\kappa([\mathfrak{b}])}, D_{\kappa([\mathfrak{b}])}; \mathfrak{b})^n. \end{aligned}$$

For any $[\mathfrak{b}] \in H_{k,d}^{\text{cl}}$, $\kappa([\mathfrak{b}])$ is an algebraic extension of \mathbb{k} . Since \mathbb{K} is algebraically closed, $\kappa([\mathfrak{b}])$ can be embedded into \mathbb{K} as a subfield. Hence there exists a point $[\mathfrak{c}] \in H_{k,d}(\mathbb{K})$ lying over $[\mathfrak{b}]$. Thus similar arguments imply that $\text{lct}(X_{\mathbb{K}}, D_{\mathbb{K}}; \mathfrak{c}) = \text{lct}(X_{\kappa([\mathfrak{b}])}, D_{\kappa([\mathfrak{b}])}; \mathfrak{b})$, and the “ \leq ” direction is proved.

(2) From (1) we know that $\widehat{\ell}_{c,k}(x_{\mathbb{K}}, X_{\mathbb{K}}, D_{\mathbb{K}}) = \widehat{\ell}_{c,k}(x_{\bar{\mathbb{k}}}, X_{\bar{\mathbb{k}}}, D_{\bar{\mathbb{k}}})$ for any c, k . Hence (2) follows from Theorem 12. □

The following corollary is well-known to experts. We present a proof here using normalized volumes.

Corollary 15. *Let (Y, E) be a log Fano pair over a field \mathbb{k} of characteristic 0. The following are equivalent:*

- (i) $(Y_{\bar{\mathbb{k}}}, E_{\bar{\mathbb{k}}})$ is log K -semistable.
- (ii) $(Y_{\mathbb{K}}, E_{\mathbb{K}})$ is log K -semistable for some field extension \mathbb{K}/\mathbb{k} with $\mathbb{K} = \bar{\mathbb{K}}$.
- (iii) $(Y_{\mathbb{K}}, E_{\mathbb{K}})$ is log K -semistable for any field extension \mathbb{K}/\mathbb{k} with $\mathbb{K} = \bar{\mathbb{K}}$.

We say that (Y, E) is geometrically log K -semistable if one (or all) of these conditions holds.

Proof. Let us take the affine cone $X = C(Y, L)$ with $L = -r(K_Y + E)$ Cartier. Let D be the \mathbb{Q} -divisor on X corresponding to E . Denote by $x \in X$ the cone vertex of X . Let \mathbb{K}/\mathbb{k} be a field extension with $\mathbb{K} = \bar{\mathbb{K}}$. Then Theorem 6 implies that $(Y_{\mathbb{K}}, E_{\mathbb{K}})$ is log K -semistable if and only if $\widehat{\text{vol}}(x_{\mathbb{K}}, X_{\mathbb{K}}, D_{\mathbb{K}}) = r^{-1}(-K_Y - E)^{n-1}$. Hence the corollary is a consequence of Proposition 14(2). \square

We finish this section with a natural speculation. Suppose $x \in (X, D)$ is a klt singularity over a field \mathbb{k} of characteristic zero that is not necessarily algebraically closed. The definition of normalized volume of singularities extend verbatim to $x \in (X, D)$, which we also denote by $\widehat{\text{vol}}(x, X, D)$. Then we expect $\widehat{\text{vol}}(x, X, D) = \widehat{\text{vol}}(x_{\bar{\mathbb{k}}}, X_{\bar{\mathbb{k}}}, D_{\bar{\mathbb{k}}})$, i.e. normalized volumes are stable under base change to algebraic closures. Such a speculation should be a consequence of the *Stable Degeneration Conjecture* (SDC) stated in [Li18, Conjecture 7.1] and [LX18, Conjecture 1.2], which roughly says that a $\widehat{\text{vol}}$ -minimizing valuation v_{\min} over $x_{\bar{\mathbb{k}}} \in (X_{\bar{\mathbb{k}}}, D_{\bar{\mathbb{k}}})$ is unique and quasi-monomial, so v_{\min} is invariant under the action of $\text{Gal}(\bar{\mathbb{k}}/\mathbb{k})$ and hence has the same normalized volume as its restriction to $x \in (X, D)$.

4. Uniform approximation of volumes by colengths

In this section, we prove the following result that gives an approximation of the volume of valuation by the colengths of its valuation ideals. The result is a consequence of arguments in [Blu18, Section 3.4] (which in turn relies on ideas in [ELS03]) and properties of the Hilbert–Samuel function.

Theorem 16. *Let $\pi : (\mathcal{X}, \mathcal{D}) \rightarrow T$ together with a section $\sigma : T \rightarrow \mathcal{X}$ be a \mathbb{Q} -Gorenstein flat family of klt singularities. Set $n = \dim(\mathcal{X}) - \dim(T)$. For every $A \in \mathbb{R}_{>0}$ and $\epsilon > 0$, there exists a positive integer N such that the following holds: If $t \in T$ and $v \in \text{Val}_{\sigma(\bar{t}), \mathcal{X}_{\bar{t}}}$ satisfies $v(\mathfrak{m}_{\sigma(\bar{t})}) = 1$ and $A_{\mathcal{X}_{\bar{t}}, \mathcal{D}_{\bar{t}}}(v) \leq A$, then*

$$\frac{\ell(\mathcal{O}_{\sigma(\bar{t}), \mathcal{X}_{\bar{t}}}/\mathfrak{a}_m(v))}{m^n/n!} \leq \text{vol}(v) + \epsilon$$

for all positive integers m divisible by N .

We begin by approximating the volume of a valuation by the multiplicity of its valuation ideals.

Proposition 17. *Let $x \in (X, D)$ be a klt singularity defined over an algebraically closed field \mathbb{k} and r a positive integer such that $r(K_X + D)$ is Cartier. Fix $v \in \text{Val}_{x, X}$ satisfying $v(\mathfrak{m}_x) = 1$ and $A_{X, D}(v) < +\infty$.*

(a) *If $x \in X_{\text{sing}} \cup \text{Supp}(D)$, then for all $m \in \mathbb{Z}_{>0}$ we have*

$$\frac{e(\mathfrak{a}_m(v))^{1/n}}{m} \leq \text{vol}(v)^{1/n} + \frac{\lceil A_{X, D}(v) \rceil e(\mathfrak{m}_x)^{1/n}}{m} + \frac{e(\mathcal{O}_X(-rD) \cdot \text{Jac}_X + \mathfrak{m}_x^m)^{1/n}}{m}.$$

(b) *If $x \notin X_{\text{sing}} \cup \text{Supp}(D)$, then for all $m \in \mathbb{Z}_{>0}$ we have*

$$\frac{e(\mathfrak{a}_m(v))^{1/n}}{m} \leq \text{vol}(v)^{1/n} + \frac{\lceil A_{X, D}(v) \rceil e(\mathfrak{m}_x)^{1/n}}{m}.$$

Proof. Fix $v \in \text{Val}_{x, X}$ satisfying $v(\mathfrak{m}_x) = 1$ and $A_{X, D}(v) < +\infty$. To simplify notation, we set $\mathfrak{a}_\bullet := \mathfrak{a}_\bullet(v)$ and $A := \lceil A_{X, D}(v) \rceil$. By [Blu18, Theorem 7.2],

$$(\text{Jac}_X \cdot \mathcal{O}_X(-rD))^\ell \mathfrak{a}_{(m+A)\ell} \subset (\mathfrak{a}_m)^\ell \quad (4.1)$$

for all $m, \ell \in \mathbb{Z}_{>0}$. Since $v(\mathfrak{m}_x) = 1$, we see $\mathfrak{m}_x^m \subset \mathfrak{a}_m$ for all $m \in \mathbb{Z}_{>0}$. As in the proof of [Blu18, Proposition 3.7], it follows from the previous inclusion combined with (4.1) that

$$(\text{Jac}_X \cdot \mathcal{O}_X(-rD) + \mathfrak{m}_x^m)^\ell \mathfrak{a}_{(m+A)\ell} \subset \overline{(\mathfrak{a}_m)^\ell} \quad (4.2)$$

for all $m \in \mathbb{Z}_{>0}$. We now apply Teissier's Minkowski inequality [Laz04, Example 1.6.9] to the previous inclusion and find that

$$\ell e(\mathfrak{a}_m)^{1/n} \leq \ell \cdot e(\text{Jac}_X \cdot \mathcal{O}_X(-rD) + \mathfrak{m}_x^m)^{1/n} + e(\mathfrak{a}_{(m+A)\ell})^{1/n}. \quad (4.3)$$

Dividing both sides of (4.3) by $m \cdot \ell$ and taking the limit as $\ell \rightarrow \infty$ gives

$$\frac{e(\mathfrak{a}_m)^{1/n}}{m} \leq \frac{e(\text{Jac}_X \cdot \mathcal{O}_X(-rD) + \mathfrak{m}_x^m)^{1/n}}{m} + \left(\frac{m+A}{m}\right) \text{vol}(v)^{1/n}.$$

Since $\mathfrak{m}_x^m \subset \mathfrak{a}_m$ for all $m \in \mathbb{Z}_{>0}$, $\text{vol}(v) \leq e(\mathfrak{m}_x)$ and the desired inequality follows. In the case when $x \notin X_{\text{sing}} \cup \text{Supp}(D)$, the stronger inequality follows from a similar argument and the observation that $\text{Jac}_X \cdot \mathcal{O}_X(-rD)$ is trivial in a neighborhood of x . \square

Before proceeding, we recall the following definition of the Jacobian ideal. If X is a variety of dimension n , then the *Jacobian ideal* of X , denoted Jac_X , is the n -th Fitting ideal of Ω_X . More generally, if $\pi : \mathcal{X} \rightarrow T$ is flat morphism of varieties and $n = \dim(\mathcal{X}) - \dim(T)$, then the *Jacobian ideal* of π , denoted $\text{Jac}_{\mathcal{X}/T}$, is the n -th Fitting ideal of $\Omega_{\mathcal{X}/T}$.

Proposition 18. *With the same assumptions as in Theorem 16, fix a positive integer r such that $r(K_{\mathcal{X}/T} + \mathcal{D})$ is Cartier. Then, for every $\epsilon > 0$, there exists M such that if $t \in T$ satisfies $\sigma(t) \in V(\text{Jac}_{\mathcal{X}_t}) \cup \text{Supp}(\mathcal{D}_t)$, then*

$$\frac{e(\text{Jac}_{\mathcal{X}_t} \cdot \mathcal{O}_{\mathcal{X}_t}(-r\mathcal{D}_t) + \mathfrak{m}_{\sigma(t)}^m)}{m^n} \leq \epsilon$$

for all $m \geq M$.

Proof. To simplify notation, set $Z = \{t \in T \mid \sigma(t) \in V(\text{Jac}_{\mathcal{X}_t}) \cup \text{Supp}(\mathcal{D}_t)\}$. We will prove the following claim: for each $\epsilon > 0$, there exists a nonempty open set $U \subset T$ and a positive integer M such that if $t \in U \cap Z$, then

$$\frac{e(\text{Jac}_{\mathcal{X}_t} \cdot \mathcal{O}_{\mathcal{X}_t}(-r\mathcal{D}_t) + \mathfrak{m}_{\sigma(t)}^m)}{m^n} \leq \epsilon$$

for all $m \geq M$. By inducting on the dimension of T , the result will follow.

We proceed to prove the claim. It is enough to consider the case when \mathcal{X} and T are affine, since we may replace π with its restriction to a nonempty open subset of T and \mathcal{X} with an open subset containing $\sigma(T)$. Next, note that $\text{Jac}_{\mathcal{X}_t} = \text{Jac}_{\mathcal{X}/T} \cdot \mathcal{O}_{\mathcal{X}}$ for each $t \in T$, since the formation of Fitting ideals commutes with base change [SPA, Tag 0C3D]. Hence, $Z = \sigma^{-1}(V(\text{Jac}_{\mathcal{X}/T}) \cup \text{Supp}(\mathcal{D}))$ and is closed in T . Now, if $T \setminus Z \neq \emptyset$, then the above claim (trivially) holds with $U = T \setminus Z$. Therefore, we consider the case when $Z = T$.

Choose a nonempty affine open set $U \subset T$ and $g \in \text{Jac}_{\mathcal{X}/T} \cdot \mathcal{O}_{\mathcal{X}}(-r\mathcal{D})(\pi^{-1}(U))$ such that the restriction of g to $\mathcal{O}_{\sigma(t), \mathcal{X}_t}$, denoted g_t , is nonzero for all $t \in U$. Set $R_t := \mathcal{O}_{\sigma(t), \mathcal{X}_t}/(g_t)$ and $\tilde{\mathfrak{m}}_t = \mathfrak{m}_{\sigma(t)} \cdot R_t$ for each $t \in U$. Now,

$$\begin{aligned} e(\text{Jac}_{\mathcal{X}_t} \cdot \mathcal{O}_{\mathcal{X}_t}(-r\mathcal{D}_t) + \mathfrak{m}_{\sigma(t)}^m) &\leq e((g_t) + \mathfrak{m}_{\sigma(t)}^m) \\ &\leq n! \cdot \ell(\mathcal{O}_{\sigma(t), \mathcal{X}_t}/((g_t) + \mathfrak{m}_{\sigma(t)}^m)) \cdot e(\mathfrak{m}_{\sigma(t)}), \end{aligned}$$

where the first inequality follows from the inclusion $\text{Jac}_{\mathcal{X}_t} \cdot \mathcal{O}_{\mathcal{X}_t}(-r\mathcal{D}_t) + \mathfrak{m}_{\sigma(t)}^m \subset (g_t) + \mathfrak{m}_{\sigma(t)}^m$ and the second is precisely Lech’s inequality [Lec60, Theorem 3]. Thus,

$$\frac{e(\text{Jac}_{\mathcal{X}_t} \cdot \mathcal{O}_{\mathcal{X}_t}(-r\mathcal{D}_t) + \mathfrak{m}_{\sigma(t)}^m)}{m^n} \leq n! \left(\frac{\ell(R_t/\tilde{\mathfrak{m}}_t^m)}{m^n} \right) \cdot e(\mathfrak{m}_{\sigma(t)}) \tag{4.4}$$

for all $t \in U$. By Proposition 41, we may shrink U so that $U \ni t \mapsto e(\mathfrak{m}_{\sigma(t)})$ and $U \ni t \mapsto \ell(R_t/\tilde{\mathfrak{m}}_t^m)$, for all $m \geq 1$, are constant. Since $\dim R_t = n - 1$, we have $\ell(R_t/\tilde{\mathfrak{m}}_t^m) = O(m^{n-1})$. Therefore, there exists an integer M such that

$$n! \left(\frac{\ell(R_t/\tilde{\mathfrak{m}}_t^m)}{m^n} \right) \cdot e(\mathfrak{m}_{\sigma(t)}) \leq \epsilon$$

for all $m \geq M$ and $t \in U$. This proves the claim. □

The following proposition is a consequence of results in Appendix B.

Proposition 19. *Keep the assumptions and notation of Theorem 16, and fix $k \in \mathbb{Z}_{>0}$. Then, for any $\epsilon > 0$, there exists $M \in \mathbb{Z}_{>0}$ such that for any point $t \in T$ and ideal $\mathfrak{a} \subset \mathcal{O}_{\sigma(\bar{t}), \mathcal{X}_{\bar{t}}}$ satisfying $\mathfrak{m}_{\sigma(\bar{t})}^k \subset \mathfrak{a} \subset \mathfrak{m}_{\sigma(\bar{t})}$,*

$$\frac{\ell(\mathcal{O}_{\sigma(\bar{t}), \mathcal{X}_{\bar{t}}}/\mathfrak{a}^m)}{m^n/n!} \leq e(\mathfrak{a}) + \epsilon$$

for all $m \geq M$.

Proof. Set $d := \max\{\ell(\mathcal{O}_{\sigma(\bar{t})}, \mathcal{X}_{\bar{t}}/\mathfrak{m}_{\sigma(\bar{t})}^k) \mid t \in T\}$, and consider the union of Hilbert schemes $\mathcal{H} := \bigcup_{m=1}^d \text{Hilb}_m(\mathcal{Z}_k/T)$, where $\mathcal{Z}_k = \text{Spec}(\mathcal{O}_{\mathcal{X}}/\mathcal{I}_{\sigma(T)}^k)$. Let τ denote the morphism $\mathcal{H} \rightarrow T$. A point $h \in H$ corresponds to the ideal $\mathfrak{b}_h = \mathfrak{b} \cdot \mathcal{O}_{\mathcal{X} \times_T \kappa(h)}$, where \mathfrak{b} is the universal ideal sheaf on $\mathcal{X} \times_T \mathcal{H}$. By applying Proposition 41 to the irreducible components of \mathcal{H} endowed with reduced scheme structure, we see that the set of functions $\{H_{\mathfrak{b}_h} \mid h \in H\}$ is finite.

Next, fix $\epsilon > 0$. By the previous paragraph, there exists $M \in \mathbb{Z}_{>0}$ such that

$$\frac{H_{\mathfrak{b}_h}(m)}{m^n/n!} \leq e(\mathfrak{b}_h) + \epsilon \tag{4.5}$$

for all $m \geq M$. Now, consider a point $t \in T$ and an ideal $\mathfrak{a} \subset \mathcal{O}_{\mathcal{X}_{\bar{t}}}$ satisfying $\mathfrak{m}_{\sigma(\bar{t})}^k \subset \mathfrak{a} \subset \mathfrak{m}_{\sigma(\bar{t})}$. Since

$$\ell(\mathcal{O}_{\sigma(\bar{t})}, \mathcal{X}_{\bar{t}}/\mathfrak{a}) \leq \ell(\mathcal{O}_{\sigma(\bar{t})}, \mathcal{X}_{\bar{t}}/\mathfrak{m}_{\sigma(\bar{t})}^k) = \ell(\mathcal{O}_{\sigma(t)}, \mathcal{X}_t/\mathfrak{m}_{\sigma(t)}^k) \leq d,$$

there is a map $\rho : \text{Spec}(\overline{\kappa(t)}) \rightarrow \mathcal{H}$ such that $\mathfrak{a} = \mathfrak{b}_{\rho(0)} \cdot \mathcal{O}_{\mathcal{X} \times_T \overline{\kappa(t)}}$. Therefore, $H_{\mathfrak{a}} = H_{\mathfrak{b}_{\rho(0)}}$ and (4.5) implies

$$\frac{\ell(\mathcal{O}_{\sigma(\bar{t})}, \mathcal{X}_{\bar{t}}/\mathfrak{a}^m)}{m^n/n!} \leq e(\mathfrak{a}) + \epsilon$$

for all $m \geq M$. □

We will now deduce Theorem 16 from Propositions 17–19.

Proof of Theorem 16. To simplify notation, we set

$$W_t = \{v \in \text{Val}_{\sigma(\bar{t})}, \mathcal{X}_{\bar{t}} \mid v(\mathfrak{m}_{\sigma(\bar{t})}) = 1 \text{ and } A_{\mathcal{X}_{\bar{t}}, \mathcal{D}_{\bar{t}}}(v) \leq A\}$$

for each $t \in T$. It suffices to prove the following claim: For every $\epsilon > 0$, there exists an integer N such that if $t \in T$, then

$$\left(\frac{\ell(\mathcal{O}_{\sigma(\bar{t})}, \mathcal{X}_{\bar{t}}/\mathfrak{a}_m(v))}{m^n/n!} \right)^{1/n} \leq \text{vol}(v)^{1/n} + \epsilon$$

for all $v \in W_t$ and $m \in \mathbb{Z}_{>0}$ divisible by N . Indeed, if $v \in W_t$, then $\text{vol}(v) \leq e(\mathfrak{m}_{\sigma(\bar{t})})$. Since the set $\{e(\mathfrak{m}_{\sigma(\bar{t})}) \mid t \in T\}$ is bounded from above by Proposition 41, the claim implies the conclusion of the theorem.

We now fix $\epsilon > 0$ and proceed to bound the latter two terms in Proposition 17(a). First, we apply Proposition 19 to find a positive integer M_1 such that

$$\frac{A \cdot e(\mathfrak{m}_{\sigma(\bar{t})})^{1/n}}{M_1} \leq \epsilon/4$$

for all $t \in T$. Next, we apply Proposition 18 to find a positive integer M_2 such that if $t \in T$ and $\sigma(\bar{t}) \in V(\text{Jac}_{\mathcal{X}_{\bar{t}}}) \cup \text{Supp}(\mathcal{D}_{\bar{t}})$, then

$$\frac{e(\text{Jac}_{\mathcal{X}_{\bar{t}}} \cdot \mathcal{O}_{\mathcal{X}_{\bar{t}}}(-r\mathcal{D}_{\bar{t}}) + \mathfrak{m}_{\sigma(\bar{t})}^{m'})^{1/n}}{m'} = \frac{e(\text{Jac}_{\mathcal{X}_t} \cdot \mathcal{O}_{\mathcal{X}_t}(-r\mathcal{D}_t) + \mathfrak{m}_{\sigma(t)}^{m'})^{1/n}}{m'} < \epsilon/4$$

for all $m' \geq M_2$. Now, set $m' := \max\{M_1, M_2\}$. Proposition 17 implies that if $t \in T$, then

$$\frac{e(\mathfrak{a}_{m'}(v))^{1/n}}{m'} \leq \text{vol}(v)^{1/n} + \epsilon/2 \tag{4.6}$$

for all $v \in W_t$.

Next, note that if $t \in T$ and $v \in W_t$, then $\mathfrak{m}_{\sigma(\bar{t})}^{m'} \subset \mathfrak{a}_{m'}(v)$. Therefore, we may apply Proposition 19 to find an integer M such that if $t \in T$ and $v \in W_t$, then

$$\left(\frac{\ell(\mathcal{O}_{\sigma(\bar{t})}, \mathcal{X}_{\bar{t}} / (\mathfrak{a}_{m'}(v))^\ell)}{(m' \cdot \ell)^n / n!} \right)^{1/n} \leq \frac{e(\mathfrak{a}_{m'}(v))^{1/n}}{m'} + \epsilon/2 \tag{4.7}$$

for all $\ell \geq M$. Thus, if $t \in T$ and $v \in W_t$, then

$$\left(\frac{\ell(\mathcal{O}_{\sigma(\bar{t})}, \mathcal{X}_{\bar{t}} / (\mathfrak{a}_{m' \cdot \ell}(v)))}{(m' \cdot \ell)^n / n!} \right)^{1/n} \leq \frac{e(\mathfrak{a}_{m'}(v))^{1/n}}{m'} + \epsilon/2 \leq \text{vol}(v) + \epsilon$$

for all $\ell \geq M$, where the first inequality follows from (4.7) combined with the inclusion $\mathfrak{a}_{m'}(v)^\ell \subset \mathfrak{a}_{m' \cdot \ell}(v)$, and the second inequality from (4.6). Therefore, setting $N := m' \cdot M$ completes the proof of the claim. \square

5. Li’s Izumi and properness estimates in families

In this section, we generalize results of [Li18] to families of klt singularities. These results will be used to prove Theorem 26.

Theorem 20 (Izumi type estimate). *Let $\pi : (\mathcal{X}, \mathcal{D}) \rightarrow T$ together with a section $\sigma : T \rightarrow \mathcal{X}$ be a \mathbb{Q} -Gorenstein flat family of klt singularities over a variety T . There exists a constant $K_0 > 0$ such that the following holds: If $t \in T$ and $v \in \text{Val}_{\sigma(\bar{t}), \mathcal{X}_{\bar{t}}}$ satisfies $A_{\mathcal{X}_{\bar{t}}, \mathcal{D}_{\bar{t}}}(v) < +\infty$, then*

$$v(g) \leq K_0 \cdot A_{\mathcal{X}_{\bar{t}}, \mathcal{D}_{\bar{t}}}(v) \cdot \text{ord}_{\sigma(\bar{t})}(g)$$

for all $g \in \mathcal{O}_{\sigma(\bar{t}), \mathcal{X}_{\bar{t}}}$.

Theorem 21 (Properness estimate). *Let $\pi : (\mathcal{X}, \mathcal{D}) \rightarrow T$ together with a section $\sigma : T \rightarrow \mathcal{X}$ be a \mathbb{Q} -Gorenstein flat family of klt singularities over a variety T . There exists a constant $K_1 > 0$ such that the following holds: If $t \in T$ and $v \in \text{Val}_{\sigma(\bar{t}), \mathcal{X}_{\bar{t}}}$ satisfies $A_{\mathcal{X}_{\bar{t}}, \mathcal{D}_{\bar{t}}}(v) < +\infty$, then*

$$\frac{K_1 \cdot A_{\mathcal{X}_{\bar{t}}, \mathcal{D}_{\bar{t}}}(v)}{v(\mathfrak{m}_{\sigma(\bar{t})})} \leq A_{\mathcal{X}_{\bar{t}}, \mathcal{D}_{\bar{t}}}(v)^n \cdot \text{vol}(v).$$

The proofs of these theorems rely primarily on the result and techniques found in [Li18]. The main new ingredient can be found in Proposition 24, which is proved using arguments of [BFJ14] and [Li18, Appendix II].

5.1. Order functions

Let X be a normal variety defined over an algebraically closed field \mathbb{k} and $x \in X$ a closed point. For $g \in \mathcal{O}_{x,X}$ the order of vanishing of g at x is defined as

$$\text{ord}_x(g) := \max\{j \geq 0 \mid g \in \mathfrak{m}_x^j\}.$$

If X is smooth at x , then ord_x is a valuation of the function field of X . In the singular case, ord_x may fail to be a valuation. For example, the inequality

$$\text{ord}_x(g^{n+n'}) \geq \text{ord}_x(g^n) + \text{ord}_x(g^{n'})$$

may be strict. Following [BFJ14], we consider an alternative function $\widehat{\text{ord}}_x$, which is defined by

$$\widehat{\text{ord}}_x(g) := \lim_{n \rightarrow \infty} \frac{1}{n} \text{ord}_x(g^n) = \sup_n \frac{1}{n} \text{ord}_x(g^n).$$

Let $\nu : X^+ \rightarrow X$ denote the normalized blowup of \mathfrak{m}_x and write

$$\mathfrak{m}_x \cdot \mathcal{O}_{X^+} = \mathcal{O}_X \left(- \sum_{i=1}^r a_i E_i \right),$$

where the E_i are prime divisors on X and each $a_i \in \mathbb{Z}_{>0}$. The following statement, which was proved in [BFJ14, Theorem 4.3], gives an interpretation of $\widehat{\text{ord}}_x$ in terms of the exceptional divisors of ν .

Proposition 22. For any function $g \in \mathcal{O}_{x,X}$ and $m \in \mathbb{Z}_{>0}$,

- (1) $\widehat{\text{ord}}_x(g) = \min_{i=1, \dots, r} \text{ord}_{E_i}(g)/a_i$;
- (2) $\widehat{\text{ord}}_x(g) \geq m$ if and only if $g \in \overline{\mathfrak{m}_x^m}$.

Building upon results in [BFJ14, Section 4.1], we show a comparison between ord_x and $\widehat{\text{ord}}_x$.

Proposition 23. If there exists a \mathbb{Q} -divisor D such that (X, D) is klt pair, then

$$\text{ord}_x(g) \leq \widehat{\text{ord}}_x(g) \leq (n + 1) \text{ord}_x(g)$$

for all $g \in \mathcal{O}_{x,X}$ and $n = \dim(X)$.

Proof. The first inequality follows from the definition of $\widehat{\text{ord}}_x(g)$ as a supremum. For the second inequality, assume $m := \text{ord}_x(g) > 0$ and note that $g \notin \mathfrak{m}_x^{m+1}$. Since

$$\overline{\mathfrak{m}_x^{m+n}} \subset \mathcal{J}((X, D), \mathfrak{m}_x^{m+n}) = \mathfrak{m}_x^{m+1} \cdot \mathcal{J}((X, D), \mathfrak{m}_x^{n-1}) \subset \mathfrak{m}_x^{m+1},$$

where the first inclusion follows from the fact that (X, D) is klt and the second from Skoda’s Theorem [Laz04, 9.6.39], we see $g \notin \overline{\mathfrak{m}_x^{m+n}}$. Therefore, $\widehat{\text{ord}}_x(g) < m + n$, and the proof is complete. □

5.2. Izumi type estimates

The propositions in the section concern the following setup, which will arise in the proof of Theorem 20. Let $x \in (X, D)$ be an affine klt singularity over an algebraically closed field \mathbb{k} . Fix a projective compactification $X \subset \bar{X}$ and a resolution of singularities $\bar{\pi} : \bar{Y} \rightarrow \bar{X}$. Assume there exists a very ample line bundle L on \bar{Y} and the restriction of $\bar{\pi}$ to X , denoted $\pi : Y \rightarrow X$, is a log resolution of (X, D, \mathfrak{m}_x) .

Proposition 24. *There exists a constant C_0 such that the following holds: For any closed point $y \in \pi^{-1}(x)$ and $g \in \mathcal{O}_{x,X}$, we have*

$$\text{ord}_y(\pi^*g) \leq C_0 \cdot \text{ord}_x(g).$$

Furthermore, if we write $\mathfrak{m}_x \cdot \mathcal{O}_Y = \mathcal{O}_Y(-\sum_{i=1}^r a_i E_i)$ where each E_i is a prime divisor on Y and $a_i \in \mathbb{Z}_{>0}$, then there is a formula for such a constant C_0 given in terms of the coefficients of $\sum a_i E_i$, the intersection numbers $(E_i \cdot E_j \cdot L^{n-2})$ for $1 \leq i, j \leq r$, and the dimension of X .

The proposition is a refined version of [Li18, Theorem 3.2]. Its proof relies on ideas in [BFJ14] and [Li18, Appendix II].

Proof. Fix a closed point $y \in \pi^{-1}(x)$ and an element $g \in \mathcal{O}_{x,X}$. Let $\rho : B_y \bar{Y} \rightarrow \bar{Y}$ denote the blowup of \bar{Y} at y with exceptional divisor F_0 . We write $\mu := \pi \circ \rho$ and F_i for the strict transform of E_i . Consider the divisor G given by the closure of $\{\mu^*g = 0\}$ and write

$$G = \sum_{i=0}^r b_i F_i + \tilde{G},$$

where no F_i lies in the support of \tilde{G} . Note that $b_0 = \text{ord}_y(\pi^*g)$ and $b_i := \text{ord}_{E_i}(g)$ for $1 \leq i \leq r$. Since $\bar{Y} \rightarrow X$ factors through the normalized blowup of X along \mathfrak{m}_x , Proposition 22 implies

$$\min_{1 \leq i \leq r} \frac{b_i}{a_i} = \min_{1 \leq i \leq r} \frac{\text{ord}_{E_i}(g)}{a_i} \leq \widehat{\text{ord}}_x(g).$$

To simplify notation, we set $a := \max\{a_i\}$.

Our goal will be to find a constant C such that $b_0 \leq C \cdot b_i$ for all $i \in \{1, \dots, r\}$. After finding such a C , we will have

$$\text{ord}_y(\pi^*g) = b_0 \leq (C/a) \widehat{\text{ord}}_x(g) \leq (C/a)(n + 1) \text{ord}_x(g),$$

where the last inequality follows from Proposition 23. Thus, the desired inequality will hold with $C_0 = (C/a)(n + 1)$.

We now proceed to find such a constant C . Set $M = \rho^*L - (1/2)F_0$ and note that M is ample [Laz04, Example 5.1.6]. For each $i \in \{1, \dots, r\}$, we consider

$$\sum_{j=0}^r b_j (F_i \cdot F_j \cdot M^{n-2}) = (G \cdot F_i \cdot M^{n-2}) - (\tilde{G} \cdot F_i \cdot M^{n-2}) \leq (G \cdot F_i \cdot M^{n-2}) = 0,$$

where the last equality follows from the fact that G is a principal divisor in a neighborhood of $\pi^{-1}(x)$. Now, we set

$$c_{ij} := (F_i \cdot F_j \cdot M^{n-2}),$$

and see

$$\sum_{j \neq i} b_j c_{ij} \leq -b_i c_{ii} \leq b_i |c_{ii}|.$$

Note that if $i \neq j$, then $c_{ij} \neq 0$ if and only if $F_i \cap F_j \neq \emptyset$. When that is the case,

$$b_j \leq \frac{|c_{ii}|}{c_{ij}} b_i. \tag{5.1}$$

Computing the c_{ij} in terms of intersection numbers on \bar{Y} , we find that for $i, j \in \{1, \dots, r\}$,

$$c_{ij} = \begin{cases} (E_i \cdot E_j \cdot L^{n-2}) - (1/2)^{n-2} & \text{if } y \in E_i \cap E_j, \\ (E_i \cdot E_j \cdot L^{n-2}) & \text{otherwise.} \end{cases}$$

Additionally,

$$c_{0i} = \begin{cases} (1/2)^{n-2} & \text{if } y \in E_i, \\ 0 & \text{otherwise.} \end{cases}$$

Now, for each $i, j \in \{1, \dots, r\}$ such that $i \neq j$ and $E_i \cap E_j \neq \emptyset$, we set

$$C_{ij} = \frac{|(E_i \cdot E_i \cdot L^{n-2})| + (1/2)^{n-2}}{(E_i \cdot E_j \cdot L^{n-2}) - (1/2)^{n-2}}.$$

Note that $|c_{ii}|/c_{ij} \leq C_{ij}$. For each i , we set

$$C_{i0} = \frac{|(E_i \cdot E_i \cdot L^{n-2})| + (1/2)^{n-2}}{(1/2)^{n-2}}.$$

Similarly, note that $|c_{ii}|/c_{0i} \leq C_{i0}$ if $y \in E_i$.

Now, set $C' = \max\{1, C_{ij}, C_{i0}\}$. By our choice of C' , if $i, j \in \{0, 1, \dots, r\}$ are distinct and $F_i \cap F_j \neq \emptyset$, then $b_j \leq C' \cdot b_i$. Now, Zariski's Main Theorem implies $\bigcup F_i$ is connected. Therefore, we set $C = 1 + C' + C'^2 + \dots + C'^r$ and conclude $b_0 \leq C \cdot b_i$ for all $i \in \{1, \dots, r\}$. □

Proposition 25. *There exists a constant C_1 such that the following holds: If $v \in \text{Val}_{x,X}$ satisfies $A_{X,D}(v) < +\infty$ and $y \in c_Y(v)$, then*

$$v(g) \leq C_1 \cdot A_{X,D}(v) \text{ord}_y(\pi^*g)$$

for all $g \in \mathcal{O}_{x,X}$. Furthermore, if $K_Y - \pi^*(K_X + D)$ has coefficients $> -1 + \epsilon$ with $0 < \epsilon < 1$, then the condition holds with $C_1 := 1/\epsilon$.

Proof. A proof in the case when $D = 0$ can be found in [Lil18, proof of Theorem 3.1]. The more general statement follows from a similar argument. □

5.3. Proofs of Theorems 20 and 21

Proof of Theorem 20. It is sufficient to prove the theorem when both \mathcal{X} and T are affine. We will show that there exists a nonempty open set $U \subset T$ and a constant $K_0 > 0$ such that the conclusion holds for all $t \in U$. By induction on the dimension of T , the proof will be complete.

Fix a (relative) projective compactification $\bar{\pi} : \bar{\mathcal{X}} \rightarrow T$. Denote the ideal sheaf of $\sigma(T)$ in \mathcal{X} by $\mathcal{I}_{\sigma(T)}$. Fix a projective resolution of singularities $\bar{\rho} : \bar{\mathcal{Y}} \rightarrow \bar{\mathcal{X}}$ such that its restriction to \mathcal{X} , denoted $\rho : \mathcal{Y} \rightarrow \mathcal{X}$, is a log resolution of $(\mathcal{X}, \mathcal{D}, \mathcal{I}_{\sigma(T)})$. Set $\bar{\mu} = \bar{\pi} \circ \bar{\rho}$. We write

$$\mathcal{I}_{\sigma(T)} \cdot \mathcal{O}_{\mathcal{Y}} = \mathcal{O}_{\mathcal{Y}} \left(- \sum_{i=1}^k b_i \mathcal{E}_i \right) \quad \text{and} \quad K_{\mathcal{Y}} - \rho^*(K_{\mathcal{X}} + \mathcal{D}) = \sum_{i=1}^k a_i \mathcal{E}_i$$

where each \mathcal{E}_i is a prime divisor on \mathcal{Y} . We order these prime divisors so that each \mathcal{E}_i dominates T if and only if $1 \leq i \leq r$ for some positive integer $r \leq k$.

By generic smoothness, there exists a nonempty open set $U_1 \subset T$ such that $\mathcal{Y}_{\bar{t}} \rightarrow \mathcal{X}_{\bar{t}}$ is a log resolution of $(\mathcal{X}_{\bar{t}}, \mathcal{D}_{\bar{t}}, \mathfrak{m}_{\sigma(\bar{t})})$ for all $t \in U_1$ and $\bar{\mu}^{-1}(U_1) \rightarrow U_1$ is smooth. Further shrinking U_1 , we may assume $\mathcal{E}_{i,\bar{t}} \neq \emptyset$ if and only if $1 \leq i \leq r$. Let us assume $i \leq r$ for the rest of the proof. Now, we have

$$\mathfrak{m}_{\sigma(\bar{t})} \cdot \mathcal{O}_{\mathcal{Y}_{\sigma(\bar{t})}} = \mathcal{O}_{\mathcal{Y}_{\sigma(\bar{t})}} \left(- \sum_{i=1}^r b_i \mathcal{E}_{i,\bar{t}} \right) \quad \text{and} \quad K_{\mathcal{Y}_{\bar{t}}} - \rho_{\bar{t}}^*(K_{\mathcal{X}_{\bar{t}}} + \mathcal{D}_{\bar{t}}) = \sum_{i=1}^r a_i \mathcal{E}_{i,\bar{t}}$$

for each $t \in U_1$. Note that the divisors $\mathcal{E}_i|_{\bar{t}}$ may have multiple irreducible components.

Next, we apply [SPA, Tag 0551] to find an étale morphism $T' \rightarrow U_1$ with T' irreducible and such that all irreducible components of the generic fiber of $\mathcal{E}' = \mathcal{E}_i \times_T T' \rightarrow T'$ are geometrically irreducible. Denote $(\mathcal{X}', \mathcal{D}', \mathcal{Y}', \mathcal{E}'_i) := (\mathcal{X}, \mathcal{D}, \mathcal{Y}, \mathcal{E}_i) \times_T T'$, and write η' for the generic points of T' . Write

$$\mathcal{E}'_{i,\eta'} = \mathcal{E}'_{i,1,\eta'} \cup \dots \cup \mathcal{E}'_{i,m_i,\eta'}$$

for the decomposition of $\mathcal{E}'_{i,\eta'}$ into irreducible components, and set $\mathcal{E}_{i,j}$ equal to the closure of $\mathcal{E}'_{i,j,\eta'}$ in \mathcal{Y}' . Applying [SPA, Tag 0559], we may find an open subset $U' \subset T'$ such that each divisor $\mathcal{E}_{i,j,\bar{t}}$ is geometrically irreducible for all $t \in U'$. Further shrinking U' , we may assume that the divisors $\mathcal{E}_{i,j,\bar{t}}$ for $1 \leq i \leq r$ and $1 \leq j \leq m_i$ are distinct. We choose $U \subset T$ to be a nonempty open subset contained in the image of U' in T .

We seek to find a constant C_0 such that if $t \in U$, then

$$\text{ord}_{\sigma(\bar{t})}(g) \leq C_0 \cdot \text{ord}_y(\rho_{\bar{t}}^*(g)) \tag{5.2}$$

for all $g \in \mathcal{O}_{\sigma(\bar{t}),\mathcal{X}'_{\bar{t}}}$ and $y \in \rho_{\bar{t}}^{-1}(\sigma(\bar{t}))$. Since $(\mathcal{X}', \mathcal{D}', \mathcal{Y}')|_{\bar{t}} \cong (\mathcal{X}, \mathcal{D}, \mathcal{Y})|_{\bar{t}}$ if t is the image of t' , it suffices to establish an inequality of the form (5.2) on the singularities $\sigma(\bar{t}') \in (\mathcal{X}'_{\bar{t}'}, \mathcal{D}'_{\bar{t}'})$ for $t' \in U'$. Let \mathcal{L} be a line bundle on $\bar{\mathcal{Y}}$ such that $\mathcal{L}_{\bar{t}}$ is very ample for all $t \in T$, and write \mathcal{L}' for the pullback of \mathcal{L} to $\bar{\mathcal{Y}}'$. Now, fix $1 \leq i_1, i_2 \leq r$ such that

$b_{i_1}, b_{i_2} > 0$. For fixed $1 \leq j_1 \leq m_{i_1}$ and $1 \leq j_2 \leq m_{i_2}$, the function that sends $U' \ni t'$ to $(\mathcal{E}_{i_1, j_1, \bar{t}'} \cdot \mathcal{E}_{i_2, j_2, \bar{t}'} \cdot \mathcal{L}_{\bar{t}'}^{m-2})$ is constant [Kol96, Lemma VI.2.9]. From our choice of U' , we know that $\mathcal{E}_{i, j, \bar{t}'}$ is irreducible for any $t' \in U'$. Therefore, we may apply Proposition 24 to find a constant C_0 such that the desired inequality holds for all $t' \in U'$.

Next, choose $0 < \epsilon < 1$ so that $a_i < 1 - \epsilon$ for all $1 \leq i \leq r$. Set $C_1 := 1/\epsilon$. By Proposition 25, if $t \in U$ and $v \in \text{Val}_{\sigma(\bar{t}), \mathcal{X}_{\bar{t}}}$, then

$$v(g) \leq C_1 \cdot A_{\mathcal{X}_{\bar{t}}, \mathcal{D}_{\bar{t}}} \text{ord}_y(\rho_{\bar{t}}^* g) \tag{5.3}$$

for all $g \in \mathcal{O}_{\sigma(\bar{t}), \mathcal{X}_{\bar{t}}}$ and $y \in c_{\mathcal{Y}_{\bar{t}}}(v)$. Combining (5.2) and (5.3), we see that the desired inequality holds when $K_0 = C_0 \cdot C_1$. □

Proof of Theorem 21. The theorem follows immediately Theorem 20 and [Li18, Theorem 4.1], as in the proof of [Li18, Theorem 4.3]. □

6. Proofs and applications

6.1. A convergence result for normalized colengths

Theorem 26. *Let $\pi : (\mathcal{X}, \mathcal{D}) \rightarrow T$ together with a section $\sigma : T \rightarrow \mathcal{X}$ be a \mathbb{Q} -Gorenstein flat family of klt singularities. For every $\epsilon > 0$, there exists a constant $c_1 > 0$ and an integer N such that if $t \in T$, then*

$$\widehat{\ell}_{c, m}(\sigma(\bar{t}), \mathcal{X}_{\bar{t}}, \mathcal{D}_{\bar{t}}) \leq \widehat{\text{vol}}(\sigma(\bar{t}), \mathcal{X}_{\bar{t}}, \mathcal{D}_{\bar{t}}) + \epsilon$$

for all m divisible by N and $0 < c \leq c_1$.

Before beginning the proof of the previous theorem, we record the following statement.

Proposition 27. *Let $\pi : (\mathcal{X}, \mathcal{D}) \rightarrow T$ together with a section $\sigma : T \rightarrow \mathcal{X}$ be a \mathbb{Q} -Gorenstein flat family of klt singularities. There exists a constant A such that*

$$\widehat{\text{vol}}(\sigma(\bar{t}), \mathcal{X}_{\bar{t}}, \mathcal{D}_{\bar{t}}) = \inf\{\widehat{\text{vol}}(v) \mid v \in \text{Val}_{\sigma(\bar{t}), \mathcal{X}_{\bar{t}}} \text{ with } v(\mathfrak{m}_{\sigma(\bar{t})}) = 1 \text{ and } A_{\mathcal{X}_{\bar{t}}, \mathcal{D}_{\bar{t}}}(v) \leq A\}$$

for all $t \in T$.

Proof. We first note that there exists a real number B such that $\widehat{\text{vol}}(\sigma(\bar{t}), \mathcal{X}_{\bar{t}}, \mathcal{D}_{\bar{t}}) \leq B$ for all $t \in T$. Indeed, $\widehat{\text{vol}}(\sigma(\bar{t}), \mathcal{X}_{\bar{t}}, \mathcal{D}_{\bar{t}}) \leq \text{lct}(\mathfrak{m}_{\sigma(\bar{t})})^n e(\mathfrak{m}_{\sigma(\bar{t})}) = \text{lct}(\mathfrak{m}_{\sigma(t)})^n e(\mathfrak{m}_{\sigma(t)})$ and the function that sends $t \in T$ to $\text{lct}(\mathfrak{m}_{\sigma(t)})^n e(\mathfrak{m}_{\sigma(t)})$ takes finitely many values by Propositions 10 and 41. Thus,

$$\widehat{\text{vol}}(\sigma(\bar{t}), \mathcal{X}_{\bar{t}}, \mathcal{D}_{\bar{t}}) = \inf\{\widehat{\text{vol}}(v) \mid v \in \text{Val}_{\sigma(\bar{t}), \mathcal{X}_{\bar{t}}} \text{ with } v(\mathfrak{m}_{\sigma(\bar{t})}) = 1 \text{ and } \widehat{\text{vol}}(v) \leq B\}$$

for all $t \in T$.

Next, fix a constant $K_1 \in \mathbb{R}_{>0}$ satisfying the conclusion of Theorem 21. If $v \in \text{Val}_{\sigma(\bar{t}), \mathcal{X}_{\bar{t}}}$ satisfies $v(\mathfrak{m}_{\sigma(\bar{t})}) = 1$ and $\widehat{\text{vol}}(v) \leq B$, then $A_{\mathcal{X}_{\bar{t}}, \mathcal{D}_{\bar{t}}}(v) \leq B/K_2$. Therefore, the proposition holds with $A := B/K_2$. □

Proof of Theorem 26. Fix $\varepsilon > 0$ and a constant $A \in \mathbb{R}_{>0}$ satisfying the conclusion of the previous proposition. To simplify notation, set

$$W_t = \{v \in \text{Val}_{\sigma(\bar{t}), \mathcal{X}_{\bar{t}}} \mid v(\mathfrak{m}_{\sigma(\bar{t})}) = 1 \text{ and } A_{(\mathcal{X}_{\bar{t}}, \mathcal{D}_{\bar{t}})}(v) \leq A\}$$

for each $t \in T$. We proceed by proving the following two claims.

Claim 1. *There exist constants $c_1 \in \mathbb{R}_{>0}$ and $M_1 \in \mathbb{Z}_{>0}$ such that if $t \in T$, then*

$$\widehat{\ell}_{c,m}(\sigma(\bar{t}), \mathcal{X}_{\bar{t}}, \mathcal{D}_{\bar{t}}) \leq \inf_{v \in W_t} n! \cdot \text{lct}(\mathfrak{a}_m(v))^n \ell(\mathcal{O}_{\sigma(\bar{t}), \mathcal{X}_{\bar{t}}} / \mathfrak{a}_m(v))$$

for all $0 < c < c_1$ and $m \geq M_1$.

Proposition 41 implies there exist constants $c_1 > 0$ and $M_1 \in \mathbb{Z}_{>0}$ such that

$$\ell(\mathcal{O}_{\sigma(\bar{t}), \mathcal{X}_{\bar{t}}} / \mathfrak{m}_{\sigma(\bar{t})}^m) \geq m^n \cdot c_1$$

for all $t \in T$ and $m \geq M_1$. Now, consider $v \in W_t$ for some $t \in T$. Since $v(\mathfrak{m}_{\sigma(\bar{t})}) = 1$, $\mathfrak{m}_{\sigma(\bar{t})}^m \subset \mathfrak{a}_m(v)$ for all $m \in \mathbb{Z}_{>0}$. Therefore, $\ell(\mathcal{O}_{\sigma(\bar{t}), \mathcal{X}_{\bar{t}}} / \mathfrak{a}_m(v)) \geq \ell(\mathcal{O}_{\sigma(\bar{t}), \mathcal{X}_{\bar{t}}} / \mathfrak{m}_{\sigma(\bar{t})}^m)$. The claim now follows from the definition of $\widehat{\ell}_{c,m}(\sigma(\bar{t}), \mathcal{X}_{\bar{t}}, \mathcal{D}_{\bar{t}})$.

Claim 2. *There exists $M_2 \in \mathbb{Z}_{>0}$ such that if $t \in T$ and $v \in W_t$, then*

$$n! \cdot \text{lct}(\mathfrak{a}_m(v))^n \cdot \ell(\mathcal{O}_{\sigma(\bar{t}), \mathcal{X}_{\bar{t}}} / \mathfrak{a}_m(v)) \leq \widehat{\text{vol}}(v) + \varepsilon$$

for all integers m divisible by M_2 .

By Theorem 16, there exists $M_2 \in \mathbb{Z}_{>0}$ such that if $t \in T$ and $v \in W_t$, then

$$\frac{\ell(\mathcal{O}_{\sigma(\bar{t}), \mathcal{X}_{\bar{t}}} / \mathfrak{a}_m(v))}{m^n / n!} \leq \text{vol}(v) + \varepsilon / A^n \tag{6.1}$$

for all integers m divisible by M_2 . Note that

$$m \cdot \text{lct}(\mathfrak{a}_m(v)) \leq \text{lct}(\mathfrak{a}_\bullet(v)) \leq A_{\mathcal{X}_{\bar{t}}, \mathcal{D}_{\bar{t}}}(v).$$

Therefore, multiplying (6.1) by $(m \cdot \text{lct}(\mathfrak{a}_m(v)))^n$ yields the desired result.

We return to the proof of the corollary. Fix constants M_1, M_2 , and c_1 satisfying the conclusions of Claims 1 and 2. Set $M = M_1 \cdot M_2$. Now, if $t \in T$, m is a positive integer divisible by M , and c satisfies $0 < c < c_1$, then

$$\begin{aligned} \widehat{\ell}_{c,m}(\sigma(\bar{t}), \mathcal{X}_{\bar{t}}, \mathcal{D}_{\bar{t}}) &\leq \inf_{v \in W_t} n! \cdot \text{lct}(\mathfrak{a}_m(v))^n \ell(\mathcal{O}_{\sigma(\bar{t}), \mathcal{X}_{\bar{t}}} / \mathfrak{a}_m(v)) \\ &\leq \inf_{v \in W_t} \widehat{\text{vol}}(v) + \varepsilon = \widehat{\text{vol}}(\sigma(\bar{t}), \mathcal{X}_{\bar{t}}, \mathcal{D}_{\bar{t}}) + \varepsilon, \end{aligned}$$

where the first (in)equality follows from Claim 1, the second from Claim 2, and the third from our choice of A . □

6.2. Proofs

The following theorem is a stronger result that implies Theorem 1.

Theorem 28. *Let $\pi : (\mathcal{X}, \mathcal{D}) \rightarrow T$ together with a section $\sigma : T \rightarrow \mathcal{X}$ be a \mathbb{Q} -Gorenstein flat family of klt singularities over a field \mathbb{k} of characteristic 0. Then the function $t \mapsto \widehat{\text{vol}}(\sigma(\bar{t}), \mathcal{X}_{\bar{t}}, \mathcal{D}_{\bar{t}})$ on T is lower semicontinuous with respect to the Zariski topology.*

Proof. Let $\mathcal{Z}_k \rightarrow T$ be the k -th thickening of the section $\sigma(T)$, i.e. $\mathcal{Z}_k = \text{Spec}_T(\mathcal{O}_{\mathcal{X}}/\mathcal{I}_{\sigma(T)}^k)$. Let $d_k := \max_{t \in T} \ell(\mathcal{O}_{\sigma(t), \mathcal{X}_t}/\mathfrak{m}_{\sigma(t), \mathcal{X}_t}^k)$. For any $d \in \mathbb{N}$, denote $\mathcal{H}_{k,d} := \text{Hilb}_d(\mathcal{Z}_k/T)$. Since \mathcal{Z}_k is proper over T , we know that $\mathcal{H}_{k,d}$ is also proper over T . Let $\mathcal{H}_{k,d}^n$ be the normalization of $\mathcal{H}_{k,d}$. Denote by $\tau_{k,d}$ the map $\mathcal{H}_{k,d} \rightarrow T$. After pulling back the universal ideal sheaf on $\mathcal{X} \times_T \mathcal{H}_{k,d}$ over $\mathcal{H}_{k,d}$ to $\mathcal{H}_{k,d}^n$, we obtain an ideal sheaf $\mathfrak{b}_{k,d}$ on $\mathcal{X} \times_T \mathcal{H}_{k,d}^n$. Denote by $\pi_{k,d} : (\mathcal{X} \times_T \mathcal{H}_{k,d}^n, \mathcal{D} \times_T \mathcal{H}_{k,d}^n) \rightarrow \mathcal{H}_{k,d}^n$ the projection. Then $\pi_{k,d}$ provides a \mathbb{Q} -Gorenstein flat family of klt pairs.

Following the notation of Proposition 14, assume h is a scheme-theoretic point of $\mathcal{H}_{k,d}^n$ lying over $[\mathfrak{b}] \in \mathcal{H}_{k,d}$. Denote $t = \tau_{k,d}([\mathfrak{b}]) \in T$. By construction, the ideal sheaf $\mathfrak{b}_{k,d,h}$ on $\mathcal{X} \times_T \text{Spec}(\kappa(h))$ is the pullback of \mathfrak{b} under the flat base change $\text{Spec}(\kappa(h)) \rightarrow \text{Spec}(\kappa([\mathfrak{b}]))$. Hence

$$\text{lct}((\mathcal{X}, \mathcal{D}) \times_T \text{Spec}(\kappa(h)); \mathfrak{b}_{k,d,h}) = \text{lct}((\mathcal{X}, \mathcal{D}) \times_T \text{Spec}(\kappa([\mathfrak{b}])); \mathfrak{b}).$$

For simplicity, we abbreviate the above equation to $\text{lct}(\mathfrak{b}_{k,d,h}) = \text{lct}(\mathfrak{b})$. Applying Proposition 10 to the family $\pi_{k,d}$ and the ideal $\mathfrak{b}_{k,d}$ implies that the function $\Phi^n : \mathcal{H}_{k,d}^n \rightarrow \mathbb{R}_{>0}$ defined as $\Phi^n(h) := \text{lct}(\mathfrak{b}_{k,d,h})$ is constructible and lower semicontinuous with respect to the Zariski topology on $\mathcal{H}_{k,d}^n$. Since $\text{lct}(\mathfrak{b}_{k,d,h}) = \text{lct}(\mathfrak{b})$, Φ^n descends to a function Φ on $\mathcal{H}_{k,d}$ as $\Phi([\mathfrak{b}]) := \text{lct}(\mathfrak{b})$. Since $\mathcal{H}_{k,d}$ is proper over T , we know that the function $\phi : T \rightarrow \mathbb{R}_{>0}$ defined as

$$\phi(t) := n! \cdot \min_{\substack{ck^n \leq d \leq d_k \\ [\mathfrak{b}] \in \tau_{k,d}^{-1}(t)}} \Phi([\mathfrak{b}])^n$$

is constructible and lower semicontinuous with respect to the Zariski topology on T . Then Proposition 14 implies $\phi(t) = \widehat{\ell}_{c,k}(\sigma(\bar{t}), \mathcal{X}_{\bar{t}}, \mathcal{D}_{\bar{t}})$. Thus we conclude that $t \mapsto \widehat{\ell}_{c,k}(\sigma(\bar{t}), \mathcal{X}_{\bar{t}}, \mathcal{D}_{\bar{t}})$ is constructible and lower semicontinuous with respect to the Zariski topology on T .

Let us fix $\epsilon > 0$ and a scheme-theoretic point $o \in T$. By Theorem 26, there exist $c_1 > 0$ and $N \in \mathbb{N}$ such that

$$\widehat{\text{vol}}(\sigma(\bar{t}), \mathcal{X}_{\bar{t}}, \mathcal{D}_{\bar{t}}) \geq \widehat{\ell}_{c,k}(\sigma(\bar{t}), \mathcal{X}_{\bar{t}}, \mathcal{D}_{\bar{t}}) - \epsilon/2 \tag{6.2}$$

for any $t \in T$, k divisible by N and $0 < c \leq c_1$. Since $t \mapsto \widehat{\ell}_{c,k}(\sigma(\bar{t}), \mathcal{X}_{\bar{t}}, \mathcal{D}_{\bar{t}})$ is constructibly lower semicontinuous on T , there exists a Zariski open neighborhood U of o such that

$$\widehat{\ell}_{c,k}(\sigma(\bar{t}), \mathcal{X}_{\bar{t}}, \mathcal{D}_{\bar{t}}) \geq \widehat{\ell}_{c,k}(\sigma(\bar{o}), \mathcal{X}_{\bar{o}}, \mathcal{D}_{\bar{o}}) \quad \text{for any } t \in U. \tag{6.3}$$

By Theorem 12, there exist $c_0 > 0$ and $N_0 \in \mathbb{N}$ such that

$$\widehat{\ell}_{c,k}(\sigma(\bar{o}), \mathcal{X}_{\bar{o}}, \mathcal{D}_{\bar{o}}) \geq \widehat{\text{vol}}(\sigma(\bar{o}), \mathcal{X}_{\bar{o}}, \mathcal{D}_{\bar{o}}) - \epsilon/2 \tag{6.4}$$

for any $0 < c \leq c_0$ and any $k \geq N_0$. Let us choose $c = \min\{c_0, c_1\}$ and $k = N \cdot N_0$. Then combining (6.2)–(6.4) yields

$$\widehat{\text{vol}}(\sigma(\bar{t}), \mathcal{X}_{\bar{t}}, \mathcal{D}_{\bar{t}}) \geq \widehat{\text{vol}}(\sigma(\bar{o}), \mathcal{X}_{\bar{o}}, \mathcal{D}_{\bar{o}}) - \epsilon \quad \text{for any } t \in U.$$

The proof is finished. □

The following theorem is a stronger result that implies Theorem 3.

Theorem 29. *Let $\varphi : (\mathcal{Y}, \mathcal{E}) \rightarrow T$ be a \mathbb{Q} -Gorenstein flat family of log Fano pairs over a field \mathbb{k} of characteristic 0. Assume that some geometric fiber $(\mathcal{Y}_{\bar{o}}, \mathcal{E}_{\bar{o}})$ is log K -semistable for a point $o \in T$. Then:*

- (1) *There exists an intersection U of countably many Zariski open neighborhoods of o such that $(\mathcal{Y}_{\bar{t}}, \mathcal{E}_{\bar{t}})$ is log K -semistable for any point $t \in T$. If, in addition, $\mathbb{k} = \overline{\mathbb{k}}$ is uncountable, then $(\mathcal{Y}_t, \mathcal{E}_t)$ is log K -semistable for a very general closed point $t \in T$.*
- (2) *The geometrically log K -semistable locus*

$$T^{K\text{-ss}} := \{t \in T \mid (\mathcal{Y}_{\bar{t}}, \mathcal{E}_{\bar{t}}) \text{ is log } K\text{-semistable}\}$$

is stable under generalization.

Proof. (1) For $r \in \mathbb{N}$ satisfying $\mathcal{L} = -r(K_{\mathcal{Y}/T} + \mathcal{E})$ is Cartier, we define the relative affine cone \mathcal{X} of $(\mathcal{Y}, \mathcal{L})$ by

$$\mathcal{X} := \text{Spec}_T \bigoplus_{m \geq 0} \varphi_*(\mathcal{L}^{\otimes m}).$$

Assume r is sufficiently large. Then it is easy to see that $\varphi_*(\mathcal{L}^{\otimes m})$ is locally free on T for all $m \in \mathbb{N}$. Thus we have $\mathcal{X}_t \cong \text{Spec} \bigoplus_{m \geq 0} H^0(\mathcal{Y}_t, \mathcal{L}_t^{\otimes m}) := C(\mathcal{Y}_t, \mathcal{L}_t)$. Let \mathcal{D} be the \mathbb{Q} -divisor on \mathcal{X} corresponding to \mathcal{E} . By [Koll13, Section 3.1], the projection $\pi : (\mathcal{X}, \mathcal{D}) \rightarrow T$ together with the section of cone vertices $\sigma : T \rightarrow \mathcal{X}$ is a \mathbb{Q} -Gorenstein flat family of klt singularities.

Since $(\mathcal{Y}_{\bar{o}}, \mathcal{E}_{\bar{o}})$ is K -semistable, Theorem 6 implies

$$\widehat{\text{vol}}(\sigma(\bar{o}), \mathcal{X}_{\bar{o}}, \mathcal{D}_{\bar{o}}) = r^{-1}(-K_{\mathcal{Y}_{\bar{o}}} - \mathcal{E}_{\bar{o}})^{n-1}.$$

Then by Theorem 28, there exists an intersection U of countably many Zariski open neighborhoods of o such that $\widehat{\text{vol}}(\sigma(\bar{t}), \mathcal{X}_{\bar{t}}, \mathcal{D}_{\bar{t}}) \geq \widehat{\text{vol}}(\sigma(\bar{o}), \mathcal{X}_{\bar{o}}, \mathcal{D}_{\bar{o}})$ for any $t \in U$. Since the global volumes of log Fano pairs are constant in \mathbb{Q} -Gorenstein flat families, we have

$$\widehat{\text{vol}}(\sigma(\bar{t}), \mathcal{X}_{\bar{t}}, \mathcal{D}_{\bar{t}}) \geq \widehat{\text{vol}}(\sigma(\bar{o}), \mathcal{X}_{\bar{o}}, \mathcal{D}_{\bar{o}}) = r^{-1}(-K_{\mathcal{Y}_{\bar{o}}} - \mathcal{E}_{\bar{o}})^{n-1} = r^{-1}(-K_{\mathcal{Y}_{\bar{t}}} - \mathcal{E}_{\bar{t}})^{n-1}.$$

Then Theorem 6 implies that $(\mathcal{Y}_{\bar{t}}, \mathcal{E}_{\bar{t}})$ is K -semistable for any $t \in U$.

(2) Let $o \in T^{K\text{-ss}}$ be a scheme-theoretic point. Then by (1) there exist countably many Zariski open neighborhoods U_i of o such that $\bigcap_i U_i \subset T^{K\text{-ss}}$. If t is a generalization of o , then t belongs to all Zariski open neighborhoods of o , so $t \in T^{K\text{-ss}}$. \square

Proof of Theorem 3. It is clear that (1) and (2) follow from Theorem 29. For (3), the constructibility of normalized volumes implies that the set U in the proof of Theorem 29 (1) can be chosen as a Zariski open neighborhood of o . Then the same argument as in the proof of Theorem 29(1) works. \square

The following corollary is a stronger result that implies Corollary 4.

Corollary 30. *Let $\pi : (\mathcal{Y}, \mathcal{E}) \rightarrow T$ be a \mathbb{Q} -Gorenstein family of complex log Fano pairs. Assume that π is isotrivial over a Zariski open subset $U \subset T$, and $(\mathcal{Y}_o, \mathcal{E}_o)$ is log K -semistable for a closed point $o \in T \setminus U$. Then $(\mathcal{Y}_t, \mathcal{E}_t)$ is log K -semistable for any $t \in U$.*

Proof. Since $(\mathcal{Y}_o, \mathcal{E}_o)$ is log K -semistable, Theorem 29 implies that $(\mathcal{Y}_t, \mathcal{E}_t)$ is log K -semistable for a very general closed point $t \in T$. Hence for some (hence any) $t \in U$, $(\mathcal{Y}_t, \mathcal{E}_t)$ is log K -semistable. \square

Remark 31. If the ACC of normalized volumes (in bounded families) were true, then Conjecture 2 follows by applying Theorem 1. Moreover, we suspect that a much stronger result on discreteness of normalized volumes away from 0 (see also [LX19, Question 4.3]) might be true, but we do not have much evidence yet.

6.3. Applications

In this section we present applications of Theorem 1. The following theorem generalizes the inequality part of [LX19, Theorem A.4].

Theorem 32. *Let $x \in (X, D)$ be a complex klt singularity of dimension n . Let a be the largest coefficient of components of D containing x . Then $\widehat{\text{vol}}(x, X, D) \leq (1 - a)n^n$.*

Proof. Suppose D_i is the component of D containing x with coefficient D . Let D_i^n be the normalization of D_i . By applying Theorem 1 to $\text{pr}_2 : (X \times D_i^n, D \times D_i^n) \rightarrow D_i^n$ together with the natural diagonal section $\sigma : D_i^n \rightarrow X \times D_i^n$, we find that $\widehat{\text{vol}}(x, X, D) \leq \widehat{\text{vol}}(y, X, D)$ for a very general closed point $y \in D_i$. We may pick y to be a smooth point in both X and D . Then $\widehat{\text{vol}}(x, X, D) \leq \widehat{\text{vol}}(0, \mathbb{A}^n, a\mathbb{A}^{n-1})$ where \mathbb{A}^{n-1} is a coordinate hyperplane of \mathbb{A}^n . Let us take local coordinates (z_1, \dots, z_n) of \mathbb{A}^n such that $\mathbb{A}^{n-1} = V(z_1)$. Then the monomial valuation v_a on \mathbb{A}^n with weights $((1 - a)^{-1}, 1, \dots, 1)$ satisfies $A_{\mathbb{A}^n}(v) = \frac{1}{1-a} + (n - 1)$, $\text{ord}_{v_a}(\mathbb{A}^{n-1}) = \frac{1}{1-a}$ and $\text{vol}(v_a) = 1 - a$. Hence

$$\widehat{\text{vol}}(x, X, D) \leq \widehat{\text{vol}}_{0, (\mathbb{A}^n, a\mathbb{A}^{n-1})}(v_a) = (A_{\mathbb{A}^n}(v) - a \text{ord}_{v_a}(\mathbb{A}^{n-1}))^n \cdot \text{vol}(v_a) = (1 - a)n^n.$$

The proof is finished. \square

Theorem 33. *Let (X, D) be a klt pair over \mathbb{C} . Then:*

- (1) *The function $x \mapsto \widehat{\text{vol}}(x, X, D)$ on $X(\mathbb{C})$ is lower semicontinuous with respect to the Zariski topology.*

(2) Let Z be an irreducible subvariety of X . Then for a very general closed point $z \in Z$ we have

$$\widehat{\text{vol}}(z, X, D) = \sup_{x \in Z} \widehat{\text{vol}}(x, X, D).$$

In particular, there exists a countable intersection U of nonempty Zariski open subsets of Z such that $\widehat{\text{vol}}(\cdot, X, D)|_U$ is constant.

Proof. Part (1) follows quickly by applying Theorem 1 to $\text{pr}_2 : (X \times X, D \times X) \rightarrow X$ together with the diagonal section $\sigma : X \rightarrow X \times X$. For part (2), denote by Z^n the normalization of Z . Then we apply Theorem 1 to $\text{pr}_2 : (X \times Z^n, D \times Z^n) \rightarrow Z^n$ together with the natural diagonal section $\sigma : Z^n \rightarrow X \times Z^n$. \square

Next we study the case when X is a Gromov–Hausdorff limit of Kähler–Einstein Fano manifolds. Note that the function $x \mapsto \widehat{\text{vol}}(x, X) = n^n \cdot \Theta(x, X)$ is lower semicontinuous with respect to the Euclidean topology on X by [SS17, LX18]. The following corollary improves this result and follows easily from Theorem 33(1).

Corollary 34. *Let X be a Gromov–Hausdorff limit of Kähler–Einstein Fano manifolds. Then the function $x \mapsto \widehat{\text{vol}}(x, X) = n^n \cdot \Theta(x, X)$ on $X(\mathbb{C})$ is lower semicontinuous with respect to the Zariski topology.*

The following theorem partially generalizes [SS17, Lemma 3.3 and Proposition 3.10].

Theorem 35. *Let X be a Gromov–Hausdorff limit of Kähler–Einstein Fano manifolds. Let $x \in X$ be any closed point. Then for any finite quasi-étale morphism of singularities $\pi : (y \in Y) \rightarrow (x \in X)$, we have $\text{deg}(\pi) \leq \Theta(x, X)^{-1}$. In particular:*

- (1) $|\hat{\pi}_1^{\text{loc}}(X, x)| \leq \Theta(x, X)^{-1}$.
- (2) For any \mathbb{Q} -Cartier Weil divisor L on X , we have $\text{ind}(x, L) \leq \Theta(x, X)^{-1}$ where $\text{ind}(x, L)$ denotes the Cartier index of L at x .

Proof. By [LX18, Theorem 1.7], the finite degree formula holds for π , i.e. $\widehat{\text{vol}}(y, Y) = \text{deg}(\pi) \cdot \widehat{\text{vol}}(x, X)$. Since $\widehat{\text{vol}}(y, Y) \leq n^n$ by [LX19, Theorem A.4] or Theorem 32 and $\widehat{\text{vol}}(x, X) = n^n \cdot \Theta(x, X)$ by [LX18, Corollary 3.7], we have $\text{deg}(\pi) \leq n^n / \widehat{\text{vol}}(x, X) = \Theta(x, X)^{-1}$. \square

Remark 36. If the finite degree formula [LX19, Conjecture 4.1] were true for any klt singularity, then clearly $\text{deg}(\pi) \leq n^n / \widehat{\text{vol}}(x, X)$ for any finite quasi-étale morphism $\pi : (y, Y) \rightarrow (x, X)$ between n -dimensional klt singularities. In particular, we would get an effective upper bound $|\hat{\pi}_1^{\text{loc}}(X, x)| \leq n^n / \widehat{\text{vol}}(x, X)$ where $\hat{\pi}_1^{\text{loc}}(X, x)$ is known to be finite by [Xu14, BGO17] (see [LX19, Theorem 1.5] for a partial result in dimension 3).

Theorem 37. *Let V be a K -semistable complex \mathbb{Q} -Fano variety of dimension $n - 1$. Let q be the largest integer such that there exists a Weil divisor L satisfying $-K_V \sim_{\mathbb{Q}} qL$. Then*

$$q \cdot (-K_V)^{n-1} \leq n^n.$$

Proof. Consider the orbifold cone $X := C(V, L) = \text{Spec}(\bigoplus_{m \geq 0} H^0(V, \mathcal{O}_V([mL])))$ with the cone vertex $x \in X$. Let $\tilde{X} := \text{Spec}_V \bigoplus_{m \geq 0} \mathcal{O}_V([mL])$ be the partial resolution of X with exceptional divisor V_0 . Then by [Kol04, 40-42], $x \in X$ is a klt singularity, and $(V_0, 0) \cong (V, 0)$ is a K-semistable Kollár component over $x \in X$. Hence [LX20, Theorem A] implies that ord_{V_0} minimizes $\widehat{\text{vol}}_{x, X}$. By [Kol04, 40-42] we have $A_X(\text{ord}_{V_0}) = q$, $\text{vol}(\text{ord}_{V_0}) = (L^{n-1})$. Hence

$$\widehat{\text{vol}}(x, X) = A_X(\text{ord}_{V_0})^n \text{vol}(\text{ord}_{V_0}) = q^n (L^{n-1}) = q(-K_V)^{n-1},$$

and the proof is finished since $\widehat{\text{vol}}(x, X) \leq n^n$ by [LX19, Theorem A.4] or Theorem 32. □

Appendix A. Asymptotic lattice points counting in convex bodies

In this appendix, we will prove the following proposition.

Proposition 38. *For any positive number ϵ , there exists $k_0 = k_0(\epsilon, n)$ such that for any closed convex body $\Delta \subset [0, 1]^n$ and any integer $k \geq k_0$, we have*

$$\left| \frac{\#(k\Delta \cap \mathbb{Z}^n)}{k^n} - \text{vol}(\Delta) \right| \leq \epsilon. \tag{A.1}$$

Proof. We use induction on dimension. If $n = 1$, then $k\Delta$ is a closed interval of length $k\text{vol}(\Delta)$, hence

$$k\text{vol}(\Delta) - 1 \leq \#(k\Delta \cap \mathbb{Z}) \leq k\text{vol}(\Delta) + 1.$$

So (A.1) holds for $k_0 = \lceil 1/\epsilon \rceil$.

Next, assume that the conclusion is true for dimension $n - 1$. Denote by (x_1, \dots, x_n) the coordinates of \mathbb{R}^n . Let $\Delta_t := \Delta \cap \{x_n = t\}$ be the sectional convex body in $[0, 1]^{n-1}$. Let $[t_-, t_+]$ be the image of Δ under the projection onto the last coordinate. Then we know that $\text{vol}(\Delta) = \int_{t_-}^{t_+} \text{vol}(\Delta_t) dt$. By induction hypothesis, there exists $k_1 \in \mathbb{N}$ such that

$$\text{vol}(\Delta_t) - \epsilon/3 \leq \frac{\#(k\Delta_t \cap \mathbb{Z}^{n-1})}{k^{n-1}} \leq \text{vol}(\Delta_t) + \epsilon/3 \quad \text{for any } k \geq k_1.$$

It is clear that

$$\#(k\Delta \cap \mathbb{Z}^n) = \sum_{t \in [t_-, t_+] \cap \frac{1}{k}\mathbb{Z}} \#(k\Delta_t \cap \mathbb{Z}^{n-1}),$$

so for any $k \geq k_1$ we have

$$\left| \#(k\Delta \cap \mathbb{Z}^n) - k^n \cdot \sum_{t \in [t_-, t_+] \cap \frac{1}{k}\mathbb{Z}} \text{vol}(\Delta_t) \right| \leq \frac{\epsilon}{3} k^{n-1} \cdot \# \left([t_-, t_+] \cap \frac{1}{k}\mathbb{Z} \right) \leq \frac{2\epsilon}{3} k^n. \tag{A.2}$$

Next, we know that the function $t \mapsto \text{vol}(\Delta_t)^{1/(n-1)}$ is concave on $[t_-, t_+]$ by the Brunn–Minkowski theorem. In particular, we can find $t_0 \in [t_-, t_+]$ such that $g(t) :=$

$\text{vol}(\Delta_t)$ reaches its maximum at $t = t_0$. Hence g is increasing on $[t_-, t_0]$ and decreasing on $[t_0, t_+]$. Then applying Proposition 39 to $g|_{[t_-, t_0]}$ and $g|_{[t_0, t_+]}$ respectively yields

$$\left| \int_{t_-}^{t_0} \text{vol}(\Delta_t) dt - \frac{1}{k} \sum_{t \in [t_-, t_0] \cap \frac{1}{k}\mathbb{Z}} \text{vol}(\Delta_t) \right| \leq \frac{2}{k},$$

$$\left| \int_{t_0}^{t_+} \text{vol}(\Delta_t) dt - \frac{1}{k} \sum_{t \in [t_0, t_+] \cap \frac{1}{k}\mathbb{Z}} \text{vol}(\Delta_t) \right| \leq \frac{2}{k}.$$

Since $0 \leq \text{vol}(\Delta_{t_0}) \leq 1$, we have

$$\left| \int_{t_-}^{t_+} \text{vol}(\Delta_t) dt - \frac{1}{k} \sum_{t \in [t_+, t_-] \cap \frac{1}{k}\mathbb{Z}} \text{vol}(\Delta_t) \right| \leq \frac{5}{k}. \tag{A.3}$$

Therefore, by setting $k_0 = \max(k_1, \lceil 15/\epsilon \rceil)$, the inequality (A.1) follows easily by combining (A.2) and (A.3). \square

Proposition 39. *For any monotonic function $g : [a, b] \rightarrow [0, 1]$ and any $k \in \mathbb{N}$, we have*

$$\left| \int_a^b g(s) ds - \frac{1}{k} \sum_{t \in [a, b] \cap \frac{1}{k}\mathbb{Z}} g(t) \right| \leq \frac{2}{k}.$$

Proof. We may assume that g is an increasing function. Denote $a_k := \lceil ka \rceil / k$ and $b_k := \lfloor kb \rfloor / k$, so $[a, b] \cap \frac{1}{k}\mathbb{Z} = [a_k, b_k] \cap \frac{1}{k}\mathbb{Z}$. Since $\int_{t-1/k}^t g(s) ds \leq g(t)/k$ whenever $t \in [a_k + 1/k, b_k]$, we have

$$\int_{a_k}^{b_k} g(s) ds \leq \frac{1}{k} \sum_{t \in [a_k + 1/k, b_k] \cap \frac{1}{k}\mathbb{Z}} g(t) \leq \frac{1}{k} \sum_{t \in [a, b] \cap \frac{1}{k}\mathbb{Z}} g(t),$$

Similarly, $\int_t^{t+1/k} g(s) ds \geq g(t)/k$ for any $t \in [a_k, b_k - 1/k]$, we have

$$\int_{a_k}^{b_k} g(s) ds \geq \frac{1}{k} \sum_{t \in [a_k, b_k - 1/k] \cap \frac{1}{k}\mathbb{Z}} g(t) \geq \frac{1}{k} \sum_{t \in [a, b] \cap \frac{1}{k}\mathbb{Z}} g(t) - \frac{1}{k}.$$

It is clear that $a_k \in [a, a + 1/k]$ and $b_k \in [b - 1/k, b]$, so we have

$$\int_{a_k}^{b_k} g(s) ds \geq \int_a^b g(s) ds - \frac{2}{k}, \quad \int_{a_k}^{b_k} g(s) ds \leq \int_a^b g(s) ds.$$

As a result,

$$\frac{1}{k} \sum_{t \in [a, b] \cap \frac{1}{k}\mathbb{Z}} g(t) - \frac{1}{k} \leq \int_a^b g(s) ds \leq \frac{1}{k} \sum_{t \in [a, b] \cap \frac{1}{k}\mathbb{Z}} g(t) + \frac{2}{k}. \quad \square$$

Appendix B. Families of ideals and the Hilbert–Samuel function

The following proposition concerns the behavior of the Hilbert–Samuel function along a family of ideals. The statement is not new. The proof we give follows arguments found in [FM00].

Definition 40. If (R, \mathfrak{m}) is a local ring and I is an \mathfrak{m} -primary ideal, then the Hilbert–Samuel function of I , denoted $H_I : \mathbb{N} \rightarrow \mathbb{N}$, is given by $H_I(m) := \ell_R(R/I^m)$. Note that $e(I) = \lim_{n \rightarrow \infty} H_I(m)/m^n$, where $n = \dim(R)$.

Proposition 41. Let $\pi : \mathcal{X} \rightarrow T$ be a morphism of finite type \mathbb{k} -schemes. Assume T is integral and π has a section $\sigma : T \rightarrow \mathcal{X}$. If $\mathfrak{a} \subset \mathcal{O}_{\mathcal{X}}$ is an ideal and $\mathfrak{a}_t = \mathfrak{a} \cdot \mathcal{O}_{\mathcal{X}_{\sigma(t)}}$ is $\mathfrak{m}_{\sigma(t)}$ -primary for all $t \in T$, then T has a filtration

$$\emptyset = T_0 \subset T_1 \subset \dots \subset T_m = T$$

such that for every $1 \leq i \leq m$, T_i is closed in T and the function $T_i \setminus T_{i-1} \ni t \mapsto H_{\mathfrak{a}_t}$ is constant.

Proof. It is sufficient to show that there exists a nonempty open set $U \subset T$ such that $H_{\mathfrak{a}_t}$ is constant for all $t \in U$. We proceed to find such a set U .

For each $t \in T$, we have $H_{\mathfrak{a}_t}(m) = \sum_{i=0}^{m-1} \ell(\mathfrak{a}_t^i/\mathfrak{a}_t^{i+1})$. Therefore, we consider the finitely generated \mathcal{O}_X -algebra $\text{gr}_{\mathfrak{a}} := \bigoplus_{i \geq 0} \mathfrak{a}^i/\mathfrak{a}^{i+1}$. By generic flatness, we may choose a nonempty open set $U \subset T$ such that both $\mathcal{O}_X|_{\pi^{-1}(U)}$ and $\text{gr}_{\mathfrak{a}}|_{\pi^{-1}(U)}$ are flat over U .

For each $i \in \mathbb{N}$, the function $U \ni t \mapsto \dim_{\kappa(t)}(\mathfrak{a}^i/\mathfrak{a}^{i+1}|_t)$ is constant, since each $\mathfrak{a}^i/\mathfrak{a}^{i+1}$ is flat over U and $\mathfrak{a}^i/\mathfrak{a}^{i+1}|_t$ has zero-dimensional support for each $t \in U$. Since $\kappa(t) \simeq \mathcal{O}_{\sigma(t), \mathcal{X}}/\mathfrak{m}_{\sigma(t)}$, $\dim_{\kappa(t)}(\mathfrak{a}^i/\mathfrak{a}^{i+1}|_t) = \ell(\mathfrak{a}^i/\mathfrak{a}^{i+1}|_t)$ for all $t \in T$. Furthermore, Lemma 42 below implies $\mathfrak{a}^i/\mathfrak{a}^{i+1}|_t = \mathfrak{a}_t^i/\mathfrak{a}_t^{i+1}$ for all $t \in U$. Therefore, $U \ni t \mapsto \ell(\mathfrak{a}_t^i/\mathfrak{a}_t^{i+1})$ is constant, and the proof is complete. \square

Before stating the following lemma, we introduce some notation. Let A be a ring, $I \subset A$ an ideal, and M an A -module. We set

$$\text{gr}_I(M) := \bigoplus_{m \geq 0} \frac{I^m M}{I^{m+1} M}.$$

Lemma 42. Let $B \rightarrow A$ be a morphism of rings, $I \subset A$ an ideal, and $M \in \text{Mod}(A)$. If $\text{gr}_I M$ and M are both flat over B , then for any $N \in \text{Mod}(B)$,

$$(\text{gr}_I M) \otimes_B N \simeq \text{gr}_I(M \otimes_B N).$$

Proof. We follow the argument given in [FM00]. Consider the surjective map $\alpha_m : (I^m M) \otimes_B N \rightarrow I^m(M \otimes_B N)$. We claim that, for each $m \in \mathbb{Z}_{>0}$, α_m is injective and $I^m M$ is flat over B .

We induct on m . The claim holds when $m = 0$, since α_0 is clearly an isomorphism and M is flat over B by assumption. Next, consider the exact sequence

$$0 \rightarrow I^{m+1} M \rightarrow I^m M \rightarrow I^m M / I^{m+1} M \rightarrow 0$$

and assume the claim holds for a positive integer m . Since $I^m M$ and $I^m M/I^{m+1} M$ are flat over B , so is $I^{m+1} M$. By the flatness of $I^m M/I^{m+1} M$, we may tensor by N to get an exact sequence

$$0 \rightarrow I^{m+1} M \otimes_B N \rightarrow I^m M \otimes_B N \rightarrow I^m M/I^{m+1} M \otimes_B N \rightarrow 0.$$

By the above exact sequence, the injectivity of α_m implies the injectivity of α_{m+1} . Now that the claim has been proven, the lemma follows from applying the claim to the previous short exact sequence. \square

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