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Level spacing and Poisson statistics for continuum random Schrödinger operators

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Abstract. We prove a probabilistic level-spacing estimate at the bottom of the spectrum for continuum alloy-type random Schrödinger operators, assuming sign-definiteness of a single-site bump function and absolutely continuous randomness. More precisely, given a finite-volume restriction of the random operator onto a box of linear size L, we prove that with high probability the eigenvalues below some threshold energy E_{sp} keep a distance of at least $e^{-(\log L)^\beta}$ for sufficiently large $\beta > 1$. This implies simplicity of the spectrum of the infinite-volume operator below E_{sp}. Under the additional assumption of Lipschitz-continuity of the single-site probability density we also prove a Minami-type estimate and Poisson statistics for the point process given by the unfolded eigenvalues around a reference energy E.

Keywords. Anderson localization, Poisson statistics of eigenvalues, Minami estimate, level statistics

1. Introduction

This work deals with spectral properties of random Schrödinger operators (RSO) $H_\omega = H_o + V_\omega$ acting on the Hilbert space $L^2(\mathbb{R}^d)$. Here H_o is a fixed self-adjoint and non-random operator, for instance the Laplacian $-\Delta$, and V_ω is a real-valued multiplication operator whose spatial profile depends on a random variable ω from a probability space $(\Omega, \mathbb{P})$. The interest in studying the properties of such operators was sparked by the seminal work of P. W. Anderson [A], who proposed the lattice counterpart of H_ω as a prototypical model for a metal-insulator transition. Specifically, he considered the operator $H^A_\omega := -\Delta + V_\omega$ on $\ell^2(\mathbb{Z}^d)$, with random potential $V_\omega(x) = \lambda\omega_x$, $x \in \mathbb{Z}^d$. Here, the $(\omega_x)_{x\in\mathbb{Z}^d}$ are a family of independent random variables distributed according to the uniform distribution on an interval.

For 'typical' configurations ω Anderson gave a semi-empirical argument supporting existence of a localized and a delocalized spectral regime for H^A_ω if $d \geq 3$. The localized

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spectral regime consists of pure point spectrum with exponentially localized eigenfunctions which cannot spread spatially under the dynamical evolution. Conversely, the delocalized spectral regime consists of wide-spread eigenfunctions which can carry diffusive transport.

This model and its various extensions have since become focus of intensive research in both physics and mathematics. The effect of spectral localization due to disorder is relatively well understood by now on a mathematical level, by virtue of two known robust approaches to this phenomenon. In [FS] Fröchlich and Spencer developed a KAM-type method known as the multiscale analysis, and in [AM] Aizenman and Molchanov introduced the fractional moment method. We do not attempt to give an exhaustive bibliography on the various extensions of those seminal works here but refer to the recent monograph [AW].

The folk wisdom in physics, and a frequently used litmus test for disordered systems, is that the spectral structure at energy E is characterized by the limiting behavior of the point process of the appropriately rescaled eigenvalues around E. More precisely, for a large but finite box $\Lambda_L := [-L/2, L/2]^d$ we consider the point process $\xi^L_{E,\omega} = \sum_n \delta_{L^d(E^L_{n,\omega}-E)}$, where $E^L_{n,\omega}$ are the eigenvalues of the finite-volume restriction of the disordered system $H_{\omega,L}$.

If the energy E is within an exponentially localized spectral region, the eigenvalues localized in disjoint regions of space are almost independent. The point process mentioned above is then expected to converge to a Poisson point process as the system's volume grows. Conversely, extended states imply that distant regions have mutual influence, leading to completely different eigenvalue statistics, such as the Gaussian orthogonal ensemble. This duality is known as the spectral statistics conjecture. It plays an important role in the analysis of disordered systems (see, e.g., [Mir, ABF, EY]).

Poisson statistics were proved rigorously in the localization regime for the classical Anderson model H^A_ω in [Min] and for a one-dimensional model in [M]. The method from [Min] is based on a probabilistic estimate on the event that two or more eigenvalues of $H_{\omega,L}$ are located in a small energy window. Such estimates are referred to as *Minami estimates* and have been further developed in [BHS, GV, CGK1, B2, TV, HK]. However, with the exception of the one-dimensional case [Klo], these techniques heavily rely on the concrete structure of the random potential V_ω in H^A_ω. In particular, they do not use the specific structure of kinetic energy and are only applicable for single-site potentials that are, or can be transformed to, rank-1 potentials (cf. the discussion in Section 2.3 for more details). Our approach circumvents this difficulty by exploiting the kinetic energy term to find a sufficiently rich subset of the configuration space where the eigenvalues of H_ω are well spaced. We then invoke analytic estimates of Cartan type, developed earlier by Bourgain [B1] for an alternative approach towards Wegner's estimate, the key technical input of multiscale analysis. A similar analytic estimate was employed in the related paper [IM], where localization and level spacing for a specific lattice model with non-monotone rank-2 random potential has been considered. This is however the only commonality of the two ([IM] and ours) approaches.

One of our results is a Minami-type estimate for continuum random Schrödinger operators $H_\omega = -\Delta + V_\omega$ near the bottom ($= 0$ without loss of generality) of the spectrum.

Although this bound is much weaker than the usual Minami estimate known for H_ω^A, it is sufficient to yield Poisson statistics for the point process of rescaled eigenvalues of H_ω. We now present an informal version of this estimate (its precise statement will be formulated in Section 2). There exists $E_{\mathrm{M}} > 0$ such that for all $K > 0$ and sufficiently large $L \gg 1$,

$$\mathbb{P}(\operatorname{tr} \mathbb{1}_{[E-\delta,E+\delta]}(H_{\omega,L}) \geq 2) \leq C_K L^{4d} \delta |\log \delta|^{-K}, \tag{1.1}$$

provided that $\delta < 1$. This bound in turn is a consequence of our main technical result, a probabilistic estimate on the level spacing, i.e. the minimal distance between distinct eigenvalues (counting their multiplicities) of a self-adjoint operator in some spectral range. Informally, there exists $E_{\mathrm{sp}} > 0$ such that

$$\mathbb{P}\Big(\sup_{E \leq E_{\mathrm{sp}}} \operatorname{tr} \mathbb{1}_{[E-\delta,E+\delta]}(H_{\omega,L}) \geq 2 \Big) \leq C L^{2d} \exp(-|\log \delta|^{1/(9d)}) \tag{1.2}$$

for $L \gg 1$ and $\delta < 1$. Beside the application to level statistics discussed above, the bound (1.2) is also of independent interest. For instance, it allows one to deduce simplicity of point spectrum below the energy E_{sp} (via the method in [KM]). The level spacing is also expected to play an important role in the localization studies of an interacting electron gas in a random environment—a subject of growing importance in theoretical and mathematical physics. In this context, the limited evidence from perturbative [FA, AGKL, GMP, BAA, I] approaches supports the persistence of a many-body localized phase for one-dimensional spin systems in the presence of weak interactions.

The paper is organized as follows: In Section 2 we first introduce the model, a standard continuum random alloy-type Schrödinger operator, and discuss our technical assumptions. We then present the main results and outline their proofs. In Section 3 we formulate and prove some preparatory lemmas on clusters of eigenvalues. Sections 4 and 5 contain the proofs of our two main results, Theorems 2.1 and 2.4, that correspond to the informal estimates (1.1)–(1.2) above. These bounds yield statements on simplicity of spectrum and Poisson statistics for H_ω by known techniques [CGK1]; we outline these arguments in Section 6.

2. Model and results

2.1. Model

We consider a standard continuum alloy-type RSO

$$H_\omega := -\mu\Delta + V_\omega = -\mu\Delta + \sum_{k\in\mathbb{Z}^d} \omega_k V_k \tag{2.1}$$

for $\mu > 0$, acting on the Hilbert space $L^2(\mathbb{R}^d)$. Here V_ω is a random alloy-type potential with random coupling constants $\Omega \ni \omega = (\omega_k)_{k\in\mathbb{Z}^d}$ taken from a probability space $(\Omega, \mathbb{P})$ specified below. We now introduce technical assumptions on our model which we assume to hold for the rest of the section.

(V_1) The single-site bump functions V_k are translates of a function V_0, $V_k(u) = V_0(u-k)$ for $u \in \mathbb{R}^d$ and $k \in \mathbb{Z}^d$. There exist constants $v_-, v_+ \in (0,1]$ and $r, R \in (0,\infty)$ such that

$$v_-\chi_{B_r(0)} \le V_0 \le v_+\chi_{B_R(0)}. \tag{2.2}$$

(V_2) The random potential satisfies a covering condition: For constants $V_-, V_+ \in (0,1]$ we have

$$V_- \le \sum_{k\in\mathbb{Z}^d} V_k \le V_+. \tag{2.3}$$

(V_3) The random couplings $\omega = (\omega_k)_{k\in\mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d}$ distribution is given by $\mathbb{P} := \bigotimes_{\mathbb{Z}^d} P_0$. The single-site probability measure P_0 is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$. Its Lebesgue density $\rho \in L^\infty(\mathbb{R})$ satisfies $\mathrm{supp}(\rho) \subseteq [0,1]$.

The assumptions $v_+, V_+ \le 1$ and $\mathrm{supp}(\rho) \subset [0,1]$ are made for convenience. The covering condition from (V_2) is necessary for Theorems 2.4 and 2.5 below, but not for the level spacing estimate, Theorem 2.1. One could also include more general background operators H_o instead of $-\mu\Delta$. However, in contrast to the situation for the classical Anderson model H_A, the choice of H_o is not arbitrary. For further comments we refer to the discussion in Section 2.3. On the other hand, the regularity assumption on P_0 in (V_3) is the principal technical assumption here.

Before we state detailed versions of our results, we introduce notation and review some well-known properties of the random operator introduced above. For a Borel-measurable set $A \subset \mathbb{R}^d$ let χ_A be the L^2-projection onto A. The finite-volume restriction of H_ω to an open set $U \subset \mathbb{R}^d$ is defined as

$$H_{\omega,U} := -\mu\Delta_U + \sum_{k\in\mathbb{Z}^d} \omega_k V_k^U, \qquad V_k^U := \chi_U V_k, \tag{2.4}$$

where $-\Delta_U$ is endowed with Dirichlet boundary conditions. Hence the random potential $V_\omega^U = \sum_{k\in\mathbb{Z}^d} \omega_k V_k^U$ may depend on random variables from an R-neighborhood of U and the random operators $H_{\omega,U_1}, H_{\omega,U_2}$ are independent if $\mathrm{dist}(U_1,U_2) > 2R$. Here, $\mathrm{dist}(A,B) := \inf\{|a-b| : a \in A, b \in B\}$ for $A, B \subset \mathbb{R}^d$ and $|x| := \max_i |x_i|$ for $x \in \mathbb{R}^d$. This choice of the finite-volume random potential matters to some extent in the proof of Theorem 2.4. By $\Lambda_L := [-L/2, L/2]^d$ and $\Lambda_L(x) := x + \Lambda_L$ we denote the box of side length L centered at $0 \in \mathbb{R}^d$ ($x \in \mathbb{R}^d$, respectively), and abbreviate $H_{\omega,L} := H_{\omega,\Lambda_L}$. In the same vein we set $V_k^L := V_k^{\Lambda_L}$ etc.

The first property we need is a bound on the probability of spectrum of $H_{\omega,L}$ in an interval I, known as Wegner's estimate. It was first proved for the classical Anderson model H_A in [W] and later generalized substantially due to its central role in multiscale analysis. For further references and more recent developments we refer to [CGK2, RMV, Kle].

(W) For fixed $E > 0$ there exists a constant $C_W = C_{W,E}$ such that

$$\mathbb{P}(\mathrm{tr}\, \mathbb{1}_I(H_{\omega,L}) \ge 1) \le C_W L^d |I| \tag{2.5}$$

for intervals $I \subset [0,E]$.

This estimate in particular implies regularity of the integrated density of states. Due to ergodicity of H_ω, almost surely (with respect to $\mathbb{P}$) the function

$$\mathcal{N}(E) := \lim_{L\to\infty} L^{-d} \operatorname{tr} \mathbb{1}_{(-\infty,E]}(H_{\omega,L}) \tag{2.6}$$

is well-defined for all $E \in \mathbb{R}$ and is non-random [CL, PF]. Wegner's estimate ensures that $\mathcal{N}$ is Lipschitz continuous and possesses a Lebesgue density $n := \mathcal{N}'$, the density of states of H_ω.

The second property that we employ is exponential spectral localization, which for the model considered here is known to hold at the bottom of the spectrum. Both methods to study this phenomenon that were mentioned in the introduction have been extended to continuum RSO, initially in [CH, AE+]. For recent developments and further references we refer to [BK, EK, GHK]. We will work with the technically slightly stronger output generated by fractional moment analysis. For $x \in \mathbb{R}^d$ let $\chi_x := \chi_{x+\Lambda_1}$.

(Loc) There exist $E_{\mathrm{loc}} > 0$, $1 > s > 0$ and constants $C_{\mathrm{loc}}, m > 0$ such that for all $E < E_{\mathrm{loc}}$ and all $x, y \in \mathbb{R}^d$,

$$\sup_{U\subset\mathbb{R}^d} \mathbb{E}[\|\chi_x R_E(H_{\omega,U})\chi_y\|^s] \le C_{\mathrm{loc}} e^{-m|x-y|} \tag{2.7}$$

for all $x, y \in \mathbb{R}^d$. Here the supremum in U is over open and bounded sets and $R_z(A) := (A - z)^{-1}$ denotes the resolvent of an operator A for $z \in \mathbb{C} \setminus \sigma(A)$.

In [AE+] the bound (2.7) is proved with a boundary-adapted distance function in the exponent. As noted there, for Hamiltonians without magnetic potentials, (2.7) also holds true with the usual distance $|\cdot|$; see also [BNSS].

2.2. Results

Let $E^\omega_{i,L}$, $i \in \mathbb{N}$, denote the eigenvalues of $H_{\omega,L}$ in ascending order. Here and in the following, the eigenvalues are counted according to their multiplicity. To quantify the level spacing of the operator $H_{\omega,L}$ in an interval $I \subset \mathbb{R}$ we set

$$\operatorname{spac}_I(H_{\omega,L}) := \inf\{|E^\omega_{i,L} - E^\omega_{j,L}| : i \neq j,\ E^\omega_{i,L}, E^\omega_{j,L} \in I\} \tag{2.8}$$

and denote $\operatorname{spac}_E(H_{\omega,L}) := \operatorname{spac}_{(-\infty,E]}(H_{\omega,L})$ for any $E \in \mathbb{R}$. The function $\operatorname{spac}_I(H_{\omega,L})$ is, by Weyl's inequality [K, Ch. 4, Thm. 3.17], continuous for an appropriate topology on Ω and therefore measurable. The first result of this paper is a probabilistic bound on the minimal spacing of eigenvalues below the energy

$$E_{\mathrm{sp}} := \frac{\mu\pi^2 V_-}{2R^2(2R+1)^d v_+}. \tag{2.9}$$

As far as dependence on V_ω is concerned, this threshold is certainly suboptimal. But, regardless of the choice of random potential, the method below is limited to $E_{\mathrm{sp}} \le \lambda_2^{(N)}/2$, where $\lambda_2^{(N)}$ is the second eigenvalue of the Neumann Laplacian on $\operatorname{supp}(V_0)$ (provided that the boundary is sufficiently regular). This is related to the fact that the spectral projection of this operator onto $[0, \lambda_2^{(N)})$ is rank-1, which we use explicitly in our reduction scheme, Lemmas 4.1–4.2 below. However, one can still partially carry out this reduction for an arbitrary fixed interval $[0, E]$. In the discrete setting, this output is sufficient to establish a weaker result, namely compound Poisson statistics, [HK]. We expect that an adaptation of the method to our context will show compound Poisson statistics for energies above E_{sp}.

We state two versions of the level spacing estimate. The first—stronger—estimate relies on localization but does not require any additional assumptions besides (V_1)–(V_3) above.

Theorem 2.1 (Probabilistic level-spacing estimate, Version 1). *For a fixed energy $E < \min\{E_{\mathrm{sp}}, E_{\mathrm{loc}}\}$ there exist $\mathcal{L}_{\mathrm{sp}} = \mathcal{L}_{\mathrm{sp},E}$ and $C_{\mathrm{sp}} = C_{\mathrm{sp},E}$ such that*

$$\mathbb{P}(\mathrm{spac}_E(H_{\omega,L}) < \delta) \leq C_{\mathrm{sp}} L^{2d} \exp(-|\log\delta|^{1/(9d)}) \tag{2.10}$$

for $L \geq \mathcal{L}_{\mathrm{sp}}$ and $\delta < 1$.

An estimate such as (2.10) is typically used (as in this paper) to derive spectral properties of systems that exhibit localization. However, it is reasonable to expect that the estimate itself should not rely on localization per se, as long as some disorder is present. This is the case for the classical Anderson model H^A, where the Minami estimate holds irrespective of localization. We corroborate this intuition in our second version of the level-spacing estimate. To this end, we will use the following additional assumption:

(V_4) The single-site probability density ρ is Lipschitz-continuous and bounded below,

$$\mathcal{K} := \sup_{\substack{x,y\in[0,1]\\ x\neq y}} \frac{|\rho(x)-\rho(y)|}{|x-y|} < \infty \quad \text{and} \quad \rho_- := \min_{x\in[0,1]} \rho(x) > 0. \tag{2.11}$$

Theorem 2.2 (Probabilistic level-spacing estimate, Version 2). *Assume that (V_4) holds. For fixed $E \in (0, E_{\mathrm{sp}})$ and $K > 0$ there exist $\mathcal{L}_{\mathrm{sp}} = \mathcal{L}_{\mathrm{sp},E,K}$ and $C_{\mathrm{sp}} = C_{\mathrm{sp},E,K}$ such that*

$$\mathbb{P}(\mathrm{spac}_E(H_{\omega,L}) < \delta) \leq C_{\mathrm{sp}} L^{2d} |\log\delta|^{-K} \tag{2.12}$$

for $L \geq \mathcal{L}_{\mathrm{sp}}$ and $\delta < 1$.

In Section 4.2 the probabilistic level-spacing estimate (2.12) is in fact proved for the larger class of deformed random Schrödinger operators $H_\omega = H_o + V_\omega$, where $H_o = -\mu G\Delta G + V_o$. Here, G, V_o are sufficiently nice periodic potentials where V_o is small in norm and $G \geq G_- > 0$ for a constant G_-. This enlargement of the model, which does not alter the arguments but complicates notation, is necessitated by the proof of the Minami-type estimate, Theorem 2.4 below. There we use deformed operators with $G = V^{-1/2}$ and $V_o = EV^{-1}$ as auxiliary operators. For a short description of this step we refer to Section 2.3.

Degenerate eigenvalues of Schrödinger operators are typically caused by symmetry. Randomness tends to break symmetry and accidental degeneracies in generic random models are expected to occur with probability zero. The first result on simplicity of RSO goes back to Simon [S1], who proved almost sure simplicity of the eigenvalues of the standard Anderson model H^A. In [JL] the almost sure simplicity was extended to the singular spectrum of H_A. The simplicity of pure point spectrum was also derived for some other forms of random potential in the discrete case in [NNS].

Here, we use a different route to establish this assertion which goes back to Klein and Molchanov [KM]. Namely, the level spacing estimate, together with the argument from [KM, CGK1], yields simplicity of the pure-point spectrum of the infinite-volume operator H_ω below $\min\{E_{\rm sp}, E_{\rm loc}\}$.

Corollary 2.3 (Eigenvalue simplicity). *The spectrum in* $[0, \min\{E_{\rm sp}, E_{\rm loc}\}] \cap \sigma(H_\omega)$ *almost surely only consists of simple eigenvalues.*

We continue with the Minami-type estimate, which we prove for energies below

$$E_{\rm M} := \frac{E_{\rm sp} V_-}{V_+} = \frac{\mu \pi^2 V_-^2}{2R^2(2R+1)^d V_+ v_+}. \tag{2.13}$$

For its proof we employ Theorem 2.2 although a similar result could be deduced by working with Theorem 2.1. This would result in a faster δ-decay in (2.14) below but possibly (depending on the size of μ) restrict the energy range from $E_{\rm M}$ to $\min\{E_{\rm M}, E_{\rm loc}\}$. We note that Assumption (V_4) is required in the proof of Theorem 2.4 below even if Theorem 2.1 is used.

Theorem 2.4 (Minami-type estimate). *Assume that* (V_4) *holds. For fixed* $E_0 < E_{\rm M}$ *and* $K > 0$ *there exist* $\mathcal{L}_{\rm M} = \mathcal{L}_{{\rm M}, E_0, K}$ *and* $C_{\rm M} = C_{{\rm M}, E_0, K} > 0$ *such that the following holds. For* $E \le E_0$,

$$\mathbb{P}(\operatorname{tr} \mathbb{1}_{[E-\delta, E+\delta]}(H_{\omega,L}) \ge 2) \le C_{\rm M} L^{4d} \delta |\log \delta|^{-K} \tag{2.14}$$

for all $L \ge \mathcal{L}_{\rm M}$ *and* $\delta < 1$.

Theorem 2.4 is sufficient to prove, with the method from [Min, M, CGK1], that the point process given by the properly rescaled eigenvalues around some small energy E weakly converges to a Poisson point process as $L \to \infty$. The point process of the rescaled eigenvalues of $H_{\omega,L}$ around a fixed reference energy $E \in \mathbb{R}$ is given by

$$\xi_{E,\omega}^L(B) := \operatorname{tr}(\mathbb{1}_{E+L^{-d}B}(H_{\omega,L})) \tag{2.15}$$

for bounded, Borel-measurable sets $B \subset \mathbb{R}$.

Theorem 2.5 (Poisson statistics). *Assume that* (V_4) *holds. Let* $E < \min\{E_{\rm M}, E_{\rm loc}\}$ *be such that the integrated density of states* $\mathcal{N}$ *is differentiable at* E*, with derivative* $\mathcal{N}'(E) = n(E) > 0$*. Then, as* $L \to \infty$*, the point process* $\xi_{E,\omega}^L$ *converges weakly to the Poisson point process on* $\mathbb{R}$ *with intensity measure* $n(E)dx$.

Under assumption (V_4) it follows from [DG+] that $n(E) > 0$ for (Lebesgue-) almost every $E \in (0, \min\{E_{\rm loc}, V_-\})$. Hence the conclusion of the theorem holds for almost every energy $E \in [0, E_{\rm M}]$.

2.3. Outline of the proofs

In this section we comment on the arguments pertaining to the proof of Theorem 2.1. The principal ideas used to establish Theorem 2.2 are similar to the ones discussed below. We also address the derivation of Theorem 2.4 from Theorem 2.2. We will not comment on the proofs of the applications, Corollary 2.3 and Theorem 2.5, as they follow via the strategy developed earlier in [KM, Min, CGK1].

The known strategies to obtain a Minami estimate rely on the fact that the random potential itself, i.e. the operator V_ω, readily satisfies this bound. Combined with the rank-1 structure of single-site bump functions in H_A, this feature allows one to prove a Minami estimate for an arbitrary choice of the non-random local operator H_o in H_A. Therefore it does not come as a surprise that the method already breaks down for the dimer potential, where the single-site bump functions are translates of $u = \mathbb{1}_{\{0,1\}}$, a rank-2 operator. Consequently, the effect of the kinetic energy term H_o has to be taken into account in order to prove a Minami-type estimate for, say, the dimer model.

Typically, degenerate eigenvalues are a manifestation of symmetry within the system. A 'typical' kinetic energy term on a generic domain, say the Laplace operator on a box, only possesses—if any—global symmetries. In contrast, "independence at a distance" property of the random potential ensures that the symmetries of the random potential—if any—are local. The idea now is to harness the random potential to destroy global symmetries of the kinetic energy and, in turn, to use the repulsion of the kinetic energy to destroy local symmetries. A qualitative implementation of this observation was employed in the works [S1, NNS] and [JL] to prove simplicity of the point spectrum, respectively the singular spectrum.

Utilizing Wegner's estimate and localization we first reduce the level-spacing estimate (2.10) to the analysis of small clusters of at most ℓ^d eigenvalues, $\ell \sim |\log \delta|^\gamma \ll L$, for some $\gamma < 1$, which are separated from the rest of the spectrum by a small spectral gap of size $\delta \ll \varepsilon \ll |\log \delta|^{-1}$.

For such a cluster we apply a Feynman–Hellmann type estimate, Lemma 3.1. The Feynman–Hellmann theorem states that for self-adjoint operators A, B and a one-parameter spectral family $s \mapsto A + sB$ we have $\operatorname{tr} P_s B = \partial_s \bar{E}^s \operatorname{tr} P_s$, where P_s denotes the projection onto a cluster of eigenvalues and $\bar{E}^s$ denotes the central energy, i.e. the arithmetic mean of the eigenvalues in the cluster. In Lemma 3.1 we show that a stronger statement holds under the assumption that the cluster is tightly concentrated around $\bar{E}^s$, namely that $P_s(B - \partial_s \bar{E}^s)P_s$ is small in operator norm.

We next argue that low lying eigenvalues cannot cluster everywhere in the configuration space. Let us assume we have bad luck and the cluster of at most ℓ^d eigenvalues is tightly concentrated around its central energy for configurations in a small neighborhood of some $\omega_0 \in \Omega$. We then apply Lemma 3.1 for every $k \in \Gamma_L$ to the spectral family $s \mapsto H_{\omega_0,L} + sV_k$. As an output, we find that the tight concentration of the cluster originates from high amount of local symmetry. More precisely, for every $k \in \Gamma_L$ one of the following two scenarios applies: Either all eigenfunctions of the cluster have almost no mass on $\operatorname{supp}(V_k)$ or they form an almost orthogonal family when restricted to $\operatorname{supp}(V_k)$. Via a bracketing argument we utilize this to conclude that the central energy $\bar{E}^{\omega_0}$ of the cluster has to be $\gtrsim \lambda_2^{(N)}$, the second eigenvalue of the kinetic energy H_o restricted to $\mathrm{supp}(V_k)$ with Neumann boundary.

After iterating this argument, we find that for a cluster of eigenvalues $\lesssim \lambda_2^{(N)}$ there exists a quite rich set of configurations for which the eigenvalues of the cluster are rather far apart from each other. Let ω_0 be such a configuration. The spectral gap surrounding the cluster ensures that quantities such as the central energy and the local discriminant of the cluster, defined in (3.19), can be extended to complex analytic functions in a vicinity of ω_0 which is roughly of linear size ε. We can now use a version of Cartan's Lemma, Lemma 3.4, to show that in a neighborhood of the good configuration the eigenvalues of the cluster are still spaced with high probability. After collecting all the probabilistic estimates performed along this argument one obtains Theorem 2.1.

For the proof of Theorem 2.4, let us for the moment assume that $\sum_{k\in\mathbb{Z}^d} V_k = 1$. The principal idea leading from Theorem 2.1 to a local estimate is to clone the interval $J := J_0 := [E-\delta, E+\delta]$ for which we want to prove (2.14). Let $\{J_k\}_{k=1}^K$ be K disjoint intervals of length 2δ and such that $\mathrm{dist}(J_k, J_0) \lesssim K\delta \ll 1$. We now utilize that (in view of $\sum_k V_k = 1$) a shift $\{\omega_k\}_k \to \{\omega_k + \varepsilon\}_k$ in the configuration space results in an energy shift by ε. Together with the homogeneity of the single-site probability measures—which is where the additional assumption (V_4) enters—it implies that

$$\mathbb{P}(\mathrm{spac}_{J_0}(H_{\omega,L}) < \delta) \sim \mathbb{P}(\mathrm{spac}_{J_k}(H_{\omega,L}) < \delta). \tag{2.16}$$

Summing both sides over $1 \le k \le K$ then yields

$$\mathbb{P}(\mathrm{spac}_{J_0}(H_{\omega,L}) < \delta) \lesssim K^{-1}\mathbb{P}(\mathrm{spac}_{E_{\mathrm{sp}}}(H_{\omega,L}) < \delta), \tag{2.17}$$

by arguing that the events on the right hand side of (2.16) are nearly disjoint. By choosing $K = (L^d\delta)^{-1}$ we ensure that $\mathrm{dist}(J_k, J_0) \lesssim L^{-d}$, which turns out to be a sufficient condition for (2.16) to hold. On the other hand, this yields the additional factor of δ on the right hand side of (2.17) and allows us to apply Theorem 2.2 to finish the argument.

In order to remove the constraint $V = \sum_{k\in\mathbb{Z}^d} V_k = 1$ we consider the auxiliary operator $\widetilde{H}_\omega^E := V^{-1/2}(H_\omega - E)V^{-1/2}$. This motivates the introduction of the larger class of deformed random Schrödinger operators in Section 4 for which we prove Theorem 2.2—see Theorem 4.3. We then repeat the line of arguments above to conclude that (2.14) holds for the operator $\widetilde{H}_\omega^E$ at energy zero. Exploiting that the spectrum of H_ω around energy E and the spectrum of $\widetilde{H}_\omega^E$ around energy zero are in good agreement (see Lemma A.1 for details), we finally obtain the same estimate for H_ω around energy E.

3. Clusters of eigenvalues

For the whole section let A be a self-adjoint operator on a separable Hilbert space $\mathcal{H}$ Moreover we denote by $I \subset \mathbb{R}$ the interval which contains the cluster of eigenvalues and by ε the size of a spectral gap around I, with

$$|I| \le 1/2 \quad \text{and} \quad 0 < \varepsilon < 1/12. \tag{3.1}$$

Throughout the section we also assume that

$$n := \operatorname{tr}(\mathbb{1}_I(A)) < \infty \quad \text{and} \quad \operatorname{dist}(I, \sigma(A) \setminus I) \geq 6\varepsilon. \tag{3.2}$$

The explicit choice of numerical values in (3.1) and (3.2) is not particularly important.

3.1. *A Feynman–Hellmann type estimate*

In this subsection we consider the one-parameter operator family

$$(-\varepsilon, \varepsilon) \ni s \mapsto A_s := A + sB, \tag{3.3}$$

where B is a bounded and self-adjoint operator with $\|B\| \leq 1$. For the enlarged interval $I_\varepsilon := I + (-\varepsilon, \varepsilon)$ the properties (3.2) yield

$$n = \operatorname{tr}(\mathbb{1}_{I_\varepsilon}(A_s)) \quad \text{and} \quad \operatorname{dist}(I_\varepsilon, \sigma(A_s) \setminus I_\varepsilon) \geq 4\varepsilon \tag{3.4}$$

for all $s \in (-\varepsilon, \varepsilon)$. For such s let $E_1^s, \dots, E_n^s$ denote the eigenvalues of A_s in I_ε counted with multiplicities.

For the arithmetic mean $\bar{E}^s := n^{-1} \sum_i E_i^s$ of the eigenvalues of A_s in I_ε the Feynman–Hellmann formula gives $\operatorname{tr} \mathbb{1}_{I_\varepsilon}(A_s)B = n\partial_s \bar{E}_s$. The next lemma provides additional information under the assumption that the n eigenvalues in I_ε are moving as a small (in comparison to ε) cluster in the coupling parameter s. For the rest of the section we use the notation $P_s := \mathbb{1}_{I_\varepsilon}(A_s)$ for $s \in (-\varepsilon, \varepsilon)$.

Lemma 3.1. *Let* $0 < \delta < \varepsilon$. *If*

$$\sup_{s \in (-\varepsilon, \varepsilon)} \sup_{i=1,\dots,n} |E_i^s - \bar{E}^s| \leq \delta, \tag{3.5}$$

then

$$\sup_{s \in (-\varepsilon, \varepsilon)} \|P_s(B - \partial_s \bar{E}^s)P_s\| \leq 9\sqrt{\delta/\varepsilon}. \tag{3.6}$$

In the proof of Lemma 3.1 we apply the following bounds which are, for convenience, proven at the end of this section.

Lemma 3.2. *For* $s \in (-\varepsilon, \varepsilon)$ *we have*

$$\|\partial_s P_s\| \leq \frac{1}{2\varepsilon} \quad \textit{and} \quad \|\partial_s^2 P_s\| \leq \frac{1}{\pi \varepsilon^2}. \tag{3.7}$$

If moreover (3.5) *holds for* $0 < \delta < \varepsilon$, *then also*

$$\|\partial_s^2 (P_s(A_s - \bar{E}^s)P_s)\| \leq 7/\varepsilon. \tag{3.8}$$

Proof of Lemma 3.1. Assumption (3.5) gives

$$\|(A_s - \bar{E}^s)P_s\| \leq \delta. \tag{3.9}$$

Let $T_s = P_s(A_s - \bar{E}^s)P_s$. Then differentiation of T_s, together with (3.7) from Lemma 3.2 and (3.9), yields

$$\begin{aligned}\|P_s(B - \partial_s \bar{E}^s)P_s\| &\le 2\|\partial_s P_s\|\,\|(A_s - \bar{E}^s)P_s\| + \|\partial_s T_s\| \\ &\le \delta/\varepsilon + \|\partial_s T_s\|. \end{aligned} \tag{3.10}$$

The lemma follows if $\|\partial_s T_s\| = \max_{\phi\in\mathcal{H}} |\langle\phi, (\partial_s T_s)(s_0)\phi\rangle| \le 8\sqrt{\delta/\varepsilon}$ for all $s \in (-\varepsilon, \varepsilon)$. Assume by way of contradiction that there exists $s_0 \in (-\varepsilon, \varepsilon)$ and a normalized $\psi \in \mathcal{H}$ such that $|\langle\psi, (\partial_s T_s)(s_0)\psi\rangle| > 8\sqrt{\delta/\varepsilon}$. Set $T_{s,\psi} := \langle\psi, T_s\psi\rangle$. Then either $(\partial_s T_{s,\psi})(s_0) > 8\sqrt{\delta/\varepsilon}$ or $(\partial_s T_{s,\psi})(s_0) < -8\sqrt{\delta/\varepsilon}$, and without loss of generality we can assume the former relation. Using the bound (3.8) from Lemma 3.2 we find that for $s_1 \in (-\varepsilon, \varepsilon)$,

$$(\partial_s T_{s,\psi})(s_1) \ge (\partial_s T_{s,\psi})(s_0) - \frac{7}{\varepsilon}|s_1 - s_0| \ge 8\sqrt{\frac{\delta}{\varepsilon}} - \frac{7}{\varepsilon}|s_1 - s_0| \tag{3.11}$$

by the fundamental theorem of calculus. Hence for any s in

$$\mathcal{S} := \{s \in (-\varepsilon, \varepsilon) :\ |s - s_0| \le \sqrt{\delta\varepsilon}/2\}$$

we have $(\partial_s T_{s,\psi})(s) > 9\sqrt{\delta}/(2\sqrt{\varepsilon})$. It implies the existence of $s_2 \in \mathcal{S}$ such that

$$\delta \ge |T_{s_2,\psi}| \ge \frac{\sqrt{\delta\varepsilon}}{2}\,\frac{9\sqrt{\delta}}{2\sqrt{\varepsilon}} - |T_{s_0,\psi}| \ge \frac{5}{4}\delta, \tag{3.12}$$

a contradiction. □

Proof of Lemma 3.2. Let $I_+ = \sup I$ and $I_- = \inf I$. We denote by $\gamma_{I,\varepsilon}$ the contour consisting of the oriented line segments $[I_- - 3\varepsilon + i\infty, I_- - 3\varepsilon - i\infty]$ and $[I_+ + 3\varepsilon - i\infty, I_+ + 3\varepsilon + i\infty]$. On $\mathrm{ran}(\gamma_{I,\varepsilon})$ the resolvent of A_s can be estimated as $\|R_{x+iy}(A_s)\| \le ((2\varepsilon)^2 + y^2)^{-1/2}$ and hence

$$\begin{aligned}\|\partial_s P_s\| &= \frac{1}{2\pi}\left\|\int_{\gamma_{I,\varepsilon}} dz\, R_z(A_s) B R_z(A_s)\right\| \\ &\le \frac{1}{\pi}\int_{\mathbb{R}} dy\, \frac{1}{(2\varepsilon)^2 + y^2} = \frac{1}{2\varepsilon},\end{aligned} \tag{3.13}$$

$$\begin{aligned}\|\partial_s^2 P_s\| &= \frac{1}{\pi}\left\|\int_{\gamma_{I,\varepsilon}} dz\, R_z(A_s) B R_z(A_s) B R_z(A_s)\right\| \\ &\le \frac{2}{\pi}\int_{\mathbb{R}} dy\, \frac{1}{((2\varepsilon)^2 + y^2)^{3/2}} = \frac{1}{\pi\varepsilon^2}.\end{aligned} \tag{3.14}$$

We next turn to estimating (3.8). For the rest of the proof we set $P := P_s$, $\dot{P} := \partial_s P_s$ and $\ddot{P} := \partial_s^2 P_s$ as well as $\bar{E} := \bar{E}^s$. We have

$$\partial_s(P(A_s - \bar{E})P) = \dot{P}P(A_s - \bar{E})P + P(A_s - \bar{E})P\dot{P} + P(B - \dot{\bar{E}})P. \tag{3.15}$$

Taking the second derivative, we get

$$\partial_s^2(P(A_s-\bar{E})P) = \big\{\big(\ddot{P}(A_s-\bar{E})P + \dot{P}^2(A_s-\bar{E})P + \dot{P}P(B-\dot{\bar{E}})P + \dot{P}P(A_s-\bar{E})\dot{P}\big) + \text{h.c.}\big\} + \{\dot{P}(B-\dot{\bar{E}})P + \text{h.c.}\} - P\ddot{\bar{E}}P \tag{3.16}$$

This yields

$$\|\partial_s^2(P(A_s-\bar{E})P)\| \le 2\|\ddot{P}\|\,\|(A_s-\bar{E})P\| + 4\|\dot{P}\|^2\|(A_s-\bar{E})P\| + 8\|\dot{P}\| + |\ddot{\bar{E}}|, \tag{3.17}$$

where we used $\|P\| = 1$, $\|B\| \le 1$, and the fact that the first derivative of $\bar{E} = n^{-1}\operatorname{tr}(PA_s)$ satisfies

$$-1 \le \dot{\bar{E}} = \frac{1}{n}\big(2\operatorname{tr}(P\dot{P}A_s) + \operatorname{tr}(PB)\big) = \frac{1}{n}\operatorname{tr}(PB) \le 1. \tag{3.18}$$

Using now the estimates (3.7), (3.9), and $\ddot{\bar{E}} = n^{-1}\operatorname{tr}(\dot{P}B)$, we obtain

$$\|\partial_s^2(P(A_s-\bar{E})P)\| \le 2\frac{\delta}{\pi\varepsilon^2} + 4\frac{\delta}{4\varepsilon^2} + \frac{4}{\varepsilon} + \frac{1}{2\varepsilon} \le \frac{2\delta}{\varepsilon^2} + \frac{5}{\varepsilon}.$$ □

3.2. The local discriminant and a Cartan estimate

With the notation from the preceding section, if at least two eigenvalues of A are inside I and $n \ge 2$, then we define the *local discriminant* of A_s on I_ε as

$$\operatorname{disc}_{I_\varepsilon}(A_s) := \prod_{1\le i<j\le n}(E_i^s - E_j^s)^2 \tag{3.19}$$

for $s \in (-\varepsilon, \varepsilon)$.

Lemma 3.3. *The local discriminant, interpreted as a function* $(-\varepsilon,\varepsilon) \ni s \mapsto \operatorname{disc}_{I_\varepsilon}(A_s)$, *has an extension to a complex analytic function on* $B_{3\varepsilon}^{\mathbb{C}} := \{z \in \mathbb{C} : |z| < 3\varepsilon\}$ *which is bounded by* 1.

Let now $N \in \mathbb{N}$ and $0 \le B_k \le 1$ be self-adjoint operators for $k = 1,\dots,N$ such that $\sum_k B_k \le 1$. We consider the N-parameter spectral family

$$(-\varepsilon,\varepsilon)^N \ni \mathbf{s} := (s_1,\dots,s_N) \mapsto A + \sum_{k=1}^N s_k B_k. \tag{3.20}$$

Then the following version of Cartan's lemma holds for the local discriminant.

Lemma 3.4. *If for fixed* $0 < \delta_0 < \varepsilon$ *there exists* $\mathbf{s_0} \in (-\varepsilon,\varepsilon)^N$ *such that*

$$\operatorname{spac}_{I_\varepsilon}(A_{\mathbf{s_0}}) > \delta_0, \tag{3.21}$$

then there exist constants C_1, C_2 (*independent of all the relevant parameters above*) *such that*

$$|\{\mathbf{s} \in (-\varepsilon,\varepsilon)^N : \operatorname{spac}_{I_\varepsilon}(A_{\mathbf{s}}) < \delta\}| \le C_1 N (2\varepsilon)^N \exp\left(-\frac{C_2}{n^2}\left|\frac{\log\delta}{\log\delta_0}\right|\right) \tag{3.22}$$

for all $\delta \in (0, 1)$.

Proof of Lemma 3.3. Thanks to (3.2), we have $\mathbb{1}_{I_\varepsilon}(A_s) = \mathbb{1}_{I_{3\varepsilon}+i\mathbb{R}}(A_s)$ and $\mathbb{1}_{I_\varepsilon^c}(A_s) = \mathbb{1}_{I_{3\varepsilon}^c+i\mathbb{R}}(A_s)$ for $s \in (-\varepsilon, \varepsilon)$. That is, the two projections can be extended to the complex analytic operators

$$B_{3\varepsilon}^{\mathbb{C}} \ni s \mapsto \mathbb{1}_{I_{3\varepsilon}+i\mathbb{R}}(A_s), \qquad B_{3\varepsilon}^{\mathbb{C}} \ni s \mapsto \mathbb{1}_{I_{3\varepsilon}^c+i\mathbb{R}}(A_s),$$

defined via holomorphic functional calculus [K]. Define

$$z \mapsto p_s(z) = \det(\mathbb{1}_{I_{3\varepsilon}}(A_s)(A_s - z) + \mathbb{1}_{I_{3\varepsilon}^c}(A_s)) = \prod_{i=1}^{n}(E_i^s - z), \tag{3.23}$$

which is a polynomial of degree n in z. Here the $E_{i,s}$, $i = 1, \ldots, n$, are the eigenvalues of A_s for $s \in (-3\varepsilon, 3\varepsilon)$ counted with multiplicities. For fixed $z \in \mathbb{C}$ the function $s \mapsto p_s(z)$ can be extended to a complex analytic function $\widetilde{p}_s(z)$ on $B_{3\varepsilon}^{\mathbb{C}}$, given by

$$B_{3\varepsilon}^{\mathbb{C}} \ni s \mapsto \widetilde{p}_s(z) = \det\big(\mathbb{1}_{I_{3\varepsilon}+i\mathbb{R}}(A_s)(A_s - z) + \mathbb{1}_{I_{3\varepsilon}^c+i\mathbb{R}}(A_s)\big). \tag{3.24}$$

If we write the polynomial as $\widetilde{p}_s(z) = \sum_{k=0}^{n} a_k(s) z^k$, then the coefficients $a_k(s)$ are also complex analytic on $B_{3\varepsilon}^{\mathbb{C}}$ since they can be expressed via evaluations of $\widetilde{p}_s(z)$ at different values of z, for instance via Lagrange polynomials. For $s \in B_{3\varepsilon}^{\mathbb{C}}$ the resultant of $\widetilde{p}_s$ and $\widetilde{p}_s'$, which is a polynomial of degree $n(n-1)$ in each of the coefficients $a_n(s)$, is then

$$\operatorname{res}(p_s, p_s') = (-1)^{n(n-1)/2} \prod_{i<j} (\lambda_i(s) - \lambda_j(s))^2, \tag{3.25}$$

where the $\lambda_i(s)$ are an arbitrary enumeration of the zeros of $\widetilde{p}_s$. For $s \in (-\varepsilon, \varepsilon)$ this agrees, up to the prefactor ± 1 in (3.25), with the local discriminant $\operatorname{disc}_{I_\varepsilon}(A_s)$ for A_s defined above. This proves the first part of the lemma.

For the second part we note that the $\lambda_i(s)$ in (3.25) are the eigenvalues of A_s in $B_{3\varepsilon}^{\mathbb{C}}$. Because $\sigma(A_s) \subset \sigma(A) + B_{3\varepsilon}^{\mathbb{C}}$ for $s \in B_{3\varepsilon}^{\mathbb{C}}$, and because $|I| \le 1/2$ and $\varepsilon < 1/12$, this shows that $|\lambda_i(s) - \lambda_j(s)| \le 1$ for $s \in B_{3\varepsilon}^{\mathbb{C}}$. □

Proof of Lemma 3.4. We define the map

$$(-\varepsilon, \varepsilon)^N \ni \mathbf{z} := (z_1, \ldots, z_N) \mapsto F(z) := \operatorname{disc}_{I_\varepsilon}\Big(A + \sum_{k=1}^{N} z_k B_k\Big). \tag{3.26}$$

Lemma 3.3 implies that for $\xi = (\xi_i)_i \in [-1, 1]^N$ the map

$$(-\varepsilon, \varepsilon) \ni s \mapsto F(s\xi_1, \ldots, s\xi_N) \tag{3.27}$$

can be extended to a complex analytic map on $B_{3\varepsilon}^{\mathbb{C}}$. If we set $F_\varepsilon(z) := F(2\varepsilon z)$ for $z \in [-1/2, 1/2]^N$ then $[-1/2, 1/2] \ni s \mapsto F_\varepsilon(s\xi_1, \ldots, s\xi_N)$ is real analytic and can be extended to a complex analytic map on $B_{3/2}^{\mathbb{C}}$ with $|F_\varepsilon| \le 1$. Since by assumption there exists $z_0 \in [-1/2, 1/2]^N$ such that $|F_\varepsilon(z_0)| > \delta_0^{n^2}$, Lemma 1 from [B1] is applicable and yields

$$|\{z \in [-1/2, 1/2]^N : |F_\varepsilon(z)| < \delta\}| \le C_1 N \exp\left(-\frac{C_2}{n^2}\left|\frac{\log \delta}{\log \delta_0}\right|\right) \tag{3.28}$$

for $\delta \in (0,1)$ and constants C_1, C_2 that are uniform in all relevant parameters. Estimate (3.22) now follows from (3.28) and

$$
\begin{aligned}
|\{\mathbf{s} \in (-\varepsilon,\varepsilon)^N : \mathrm{spac}_{I_\varepsilon}(A_{\mathbf{s}}) < \delta\}| &\leq |\{\mathbf{s} \in (-\varepsilon,\varepsilon)^N : \mathrm{disc}_{I_\varepsilon}(A_{\mathbf{s}}) < \delta\}| \\
&= |\{\mathbf{s} \in (-\varepsilon,\varepsilon)^N : |F(\mathbf{s})| < \delta\}| \\
&= (2\varepsilon)^N |\{z \in [-1/2,1/2]^N : |F_\varepsilon(z)| < \delta\}|. \qquad \square
\end{aligned}
$$

4. Proof of the level spacing estimates

In this section we prove Theorems 2.1 and 2.2. In the proof of Theorem 2.4 we have to apply Theorem 2.2 for the auxiliary operators $\widetilde{H}_\omega^E$ described in Section 2.3. In order to prove Theorem 2.2 and simultaneously establish the same result for the auxiliary operators, we prove a variant of Theorem 2.1 for the deformed random Schrödinger operators $-\mu G\Delta G + V_o + V_\omega$, where G, V_o are bounded $\mathbb{Z}^d$-periodic potentials.

In the course of this section we denote both, the standard RSO and the deformed RSO, by H_ω. To absorb this ambiguity of notation we specify the setup for each subsection separately.

4.1. *Existence of good configurations*

In this section we work with the deformed random Schrödinger operators

$$
H_\omega := -\mu G\Delta G + V_o + V_\omega. \tag{4.1}
$$

Here G, V_o are bounded and $\mathbb{Z}^d$-periodic potentials and $V_\omega = \sum_{k\in\mathbb{Z}^d} \omega_k V_k$ is as introduced in Section 2. In particular, the properties (V_1)–(V_3) still hold. Moreover, we assume that G satisfies $G_- \leq G \leq G_+$ with constants $G_-, G_+ \in (0,\infty)$.

The first step towards Theorems 2.1 and 2.2 is to prove that the configuration space Ω contains a sufficiently rich set of configurations for which the energy levels are well-spaced. More precisely, let $\omega_0 \in \Omega$ and assume that a cluster of eigenvalues is isolated from the rest of the spectrum by a gap. Then the lemma below shows that there exists at least one configuration close to ω_0 such that the cluster literally separates into clusters consisting of single eigenvalues. The lemma states that if localization for the cluster of eigenvalues is known then the amount of random variables that is needed to obtain such a 'good configuration' can be reduced to $\ell^d \ll L^d$. If localization is not known then the lemma can still be applied for $\ell = L$ (see Lemma 4.5 below).

We first introduce some additional notation. For $L > 0$ let $\Gamma_L := \Lambda_{L+R} \cap \mathbb{Z}^d$ be the index set of relevant couplings for the operator $H_{\omega,L}$ and for $x \in \Lambda_L$ let $\Gamma_{\ell,x} := \Gamma_L \cap \Lambda_\ell(x)$, where the dependence on L is suppressed in notation. In the same vein we denote by $\omega_{0,\Lambda_\ell(x)}$ and $\omega_{0,\Lambda_\ell^c(x)}$ the restrictions of $\omega_0 \in [0,1]^{\Gamma_L}$ to the index sets $\Gamma_{\ell,x}$, respectively $\Gamma_L \setminus \Gamma_{\ell,x}$. We also define the local subcubes $Q_\varepsilon^{\Lambda_\ell(x)}(\omega_0) := \omega_{0,\Lambda_\ell(x)} + [-\varepsilon,\varepsilon]^{\Gamma_{\ell,x}}$ for $\varepsilon > 0$. Moreover, for $\omega_1 \in [0,1]^{\Gamma_L}$ we set

$$Q_\varepsilon^{(x,\ell)}(\omega_1,\omega_0) := \omega_{1,\Lambda_\ell^c(x)} \times Q_\varepsilon^{\Lambda_\ell(x)}(\omega_0)$$
$$:= \{\omega = (\omega_{1,\Lambda_\ell^c(x)}, \omega_{\Lambda_\ell(x)}) \in [0,1]^{\Gamma_L} : \omega_{\Lambda_\ell(x)} \in Q_\varepsilon^{\Lambda_\ell(x)}(\omega_0)\}. \tag{4.2}$$

For $n \in \mathbb{N}$, $L \geq \ell > 0$ and $r > 0$ we define

$$\xi_{L,\ell,n,r} := \frac{\mu\pi^2 G_-^2}{2R^2(2R+1)^d v_+}(V_- - v_+ L^d e^{-m\ell} - 26\sqrt{n}\,\ell^{-r}) - \|V_o\|. \tag{4.3}$$

Lemma 4.1. *Let $0 < \varepsilon < 1/12$, $r > 0$ and $m > 0$ be fixed. Moreover, let $L \geq \ell \geq (8n)^{1/(2d+2r)}$ and $\omega_0, \omega_1 \in [0,1]^{\Gamma_L}$ be such that the following hold:*

(i) $\omega_{1,\Lambda_\ell(x)} \in Q_\varepsilon^{\Lambda_\ell(x)}(\omega_0)$.

(ii) *There exist eigenvalues $E_1^{\omega_1} \leq \cdots \leq E_n^{\omega_1} \leq \xi_{L,\ell,n,r}$ of $H_{\omega_1,L}$ which are separated from the rest of the spectrum: For the cluster $\mathcal{C}_n^{\omega_1} := \{E_1^{\omega_1}, \ldots, E_n^{\omega_1}\}$ we have*

$$\operatorname{dist}(\mathcal{C}_n^{\omega_1}, \sigma(H_{\omega_1,L}) \setminus \mathcal{C}_n^{\omega_1}) \geq 8\varepsilon. \tag{4.4}$$

(iii) *The spectral projection P_{ω_1} of $H_{\omega_1,L}$ onto the cluster $\mathcal{C}_n^{\omega_1}$ is localized with localization center $x \in \Lambda_L$, i.e.*

$$\|P_{\omega_1}\mathbb{1}_{\Lambda_1(y)}\| \leq e^{-m\ell} \tag{4.5}$$

for all $y \in \Lambda_L$ that satisfy $|x-y| > \ell$.

Then there exists $\widehat{\omega} \in Q_\varepsilon^{(x,\ell)}(\omega_1,\omega_0)$ such that

$$\min_{i=1,\ldots,n-1} |E_{i+1}^{\widehat{\omega}} - E_i^{\widehat{\omega}}| > 8\varepsilon\ell^{-(n-1)(2d+2r)}. \tag{4.6}$$

Here, $E_1^\omega \leq \cdots \leq E_n^\omega$ for $\omega \in Q_\varepsilon^{(x,\ell)}(\omega_1,\omega_0)$ denote the ascendingly ordered eigenvalues of $H_{\omega,L}$ in the interval $[E_1^{\omega_1} - 2\varepsilon, E_n^{\omega_1} + 2\varepsilon]$.

Up to an iterative step, this lemma is a consequence of the following assertion.

Lemma 4.2. *Assume that the assumptions of Lemma 4.1 hold. Then there exist $\widehat{\omega} \in Q_{\varepsilon-\varepsilon\ell^{-(2d+2)}}^{(x,\ell)}(\omega_1,\omega_0)$ and $1 \leq k \leq n-1$ such that*

$$E_{k+1}^{\widehat{\omega}} - E_k^{\widehat{\omega}} > 8\varepsilon\ell^{-(2d+2r)}. \tag{4.7}$$

Proof. We set $I := [E_1^{\omega_1} - \varepsilon, E_n^{\omega_1} + \varepsilon]$, where the dependence of I on ε is suppressed in notation. By Weyl's inequality on the movement of eigenvalues and assumption (4.4) we can without loss of generality assume that

$$\operatorname{dist}(I, \sigma(H_{\omega_0,L}) \setminus I) \geq 6\varepsilon. \tag{4.8}$$

If this was not true, then (4.7) would readily hold. Another application of Weyl's inequality yields $\operatorname{tr} \mathbb{1}_{I_\varepsilon}(H_{\omega,L}) = n$ for $\omega \in Q_\varepsilon^{(x,\ell)}(\omega_1,\omega_0)$, where $I_\varepsilon := I + [-\varepsilon,\varepsilon] = [E_1^{\omega_1} - 2\varepsilon, E_n^{\omega_1} + 2\varepsilon]$. This justifies the notation $E_1^\omega \leq \cdots \leq E_n^\omega$ for the ascendingly ordered eigenvalues of $H_{\omega,L}$ in the interval I_ε. For such ω we also define $\bar{E}^\omega := n^{-1}\sum_{i=1}^n E_i^\omega$. For notational convenience we set $Q := Q_\varepsilon^{(x,\ell)}(\omega_1,\omega_0)$. We now assume that

$$\max_{\omega\in Q}\max_{i=1,\ldots,n}|E_i^\omega - \bar{E}^\omega| \le 8n\varepsilon\ell^{-(2d+2)}. \tag{4.9}$$

For fixed $k\in\Gamma_{\ell,x}$ there exists $-\varepsilon < a_k < \varepsilon$ such that $\omega_1 + e_k(a_k + (-\varepsilon,\varepsilon)) \subset Q$. Here e_k is the unit vector onto $k\in\Gamma_{\ell,x}$. Hence Lemma 3.1 can be applied to the operator family

$$(-\varepsilon,\varepsilon)\ni s\mapsto H_{\omega_1+e_k a_k,L} + sV_k^L \tag{4.10}$$

for $\delta = 8n\varepsilon\ell^{-(2d+2)}$. For $P_\omega := \mathbb{1}_{I_\varepsilon}(H_{\omega,L})$ let

$$\alpha_k^{\omega_1} := (\partial_{\omega_k}\bar{E}^\omega)(\omega_1) = \frac{1}{n}\operatorname{tr} P_{\omega_1}V_k^L \ge 0, \tag{4.11}$$

where we have used the Feynman–Hellmann theorem. Evaluation of (3.6) at $s=-a_k$ yields the bound

$$\|P_{\omega_1}(V_k^L - \alpha_k^{\omega_1})P_{\omega_1}\| \le 26\sqrt{n}\,\ell^{-d-r} \tag{4.12}$$

for every $k\in\Gamma_{\ell,x}$. We next decompose $\Gamma_{\ell,x}$ into disjoint subsets $(U_t)_{t\in\mathcal{T}}$ such that $|k-l|>2R$ for $k,l\in U_t$, $k\neq l$, and such that $|\mathcal{T}|\le(2R+1)^d$. For the sets $\Lambda_R^L(k) := \Lambda_R(k)\cap\Lambda_L$, $k\in\Gamma_L$, we then have $\Lambda_R^L(k)\cap\Lambda_R^L(k')=\emptyset$ for $k,k'\in U_t$ with $k\neq k'$. For fixed $t\in\mathcal{T}$ Neumann decoupling yields

$$\operatorname{tr} P_{\omega_1}H_{\omega_1,L} \ge \sum_{k\in U_t}\operatorname{tr} P_{\omega_1}G(-\mu\Delta^{(N)}_{\Lambda_R^L(k)})G - n\|V_o\|, \tag{4.13}$$

where we have also used $V_k\omega_{1,k}\ge 0$ for all $k\in U_t\subset\Gamma_{\ell,x}$. After summing (4.13) over $t\in\mathcal{T}$, we obtain

$$\operatorname{tr} P_{\omega_1}H_{\omega_1,L} \ge (2R+1)^{-d}\sum_{k\in\Gamma_{\ell,x}}\operatorname{tr} P_{\omega_1}G(-\mu\Delta^{(N)}_{\Lambda_R^L(k)})G - n\|V_o\|. \tag{4.14}$$

Since $\Lambda_R^L(k)$ is a hyperrectangle with side lengths bounded by R, we have

$$-\Delta^{(N)}_{\Lambda_R^L(k)} \ge \frac{\pi^2}{R^2}R_k, \tag{4.15}$$

where R_k is the projection onto $\operatorname{ran}(\Delta^{(N)}_{\Lambda_R^L(k)})$. With the shorthand notation

$$C_{\omega_1,k} := G\chi_{\Lambda_R^L(k)}P_{\omega_1}\chi_{\Lambda_R^L(k)}G$$

we conclude that

$$(4.14) \ge \frac{\mu\pi^2}{R^2(2R+1)^d}\sum_{k\in\Gamma_{\ell,x}}\operatorname{tr} C_{\omega_1,k}R_k - n\|V_o\|. \tag{4.16}$$

Next, we bound the trace on the right hand side as

$$\operatorname{tr} C_{\omega_1,k} R_k = \operatorname{tr} C_{\omega_1,k} - \operatorname{tr} C_{\omega_1,k}(\chi_{\Lambda_R^L(k)} - R_k) \geq \operatorname{tr} C_{\omega_1,k} - \|C_{\omega_1,k}\| = \sum{}' \nu_j, \tag{4.17}$$

where $(\nu_j)_j$ are the eigenvalues of $C_{\omega_1,k}$ counted with multiplicity and $\sum'$ stands for the sum of all but the largest eigenvalue of $C_{\omega_1,k}$. Here we have also used $\operatorname{rank}(\chi_{\Lambda_R^L(k)} - R_k) = 1$. Since $\sigma(C_{\omega_1,k}) \setminus \{0\} = \sigma(P_{\omega_1}\chi_{\Lambda_R^L(k)} G^2 \chi_{\Lambda_R^L(k)} P_{\omega_1}) \setminus \{0\}$ and, by (4.12),

$$P_{\omega_1}\chi_{\Lambda_R^L(k)} G^2 \chi_{\Lambda_R^L(k)} P_{\omega_1} \geq \frac{G_-^2}{v_+} P_{\omega_1} V_k^L P_{\omega_1} \geq \frac{G_-^2}{v_+}(\alpha_k^{\omega_1} - 26\sqrt{n}\,\ell^{-d-r}) P_{\omega_1}, \tag{4.18}$$

we deduce by the min-max principle that

$$\begin{aligned} \operatorname{tr} C_{\omega_1,k} R_k &\geq \sum{}' \nu_j \geq \frac{1}{v_+}(\alpha_k^{\omega_1} - 26\sqrt{n}\,\ell^{-d-r})(\operatorname{tr} P_{\omega_1} - 1) \\ &= \frac{(n-1)G_-^2}{v_+}(\alpha_k^{\omega_1} - 26\sqrt{n}\,\ell^{-d-r}). \end{aligned} \tag{4.19}$$

This implies that

$$\operatorname{tr} P_{\omega_1} H_{\omega_1,L} \geq \frac{\mu\pi^2 G_-^2 (n-1)}{R^2(2R+1)^d v_+} \sum_{k\in\Gamma_{\ell,x}} (\alpha_k^{\omega_1} - 26\sqrt{n}\,\ell^{-d-r}) - n\|V_o\|. \tag{4.20}$$

Moreover, (4.5) and (4.11) yield

$$\sum_{k\in\Gamma_{\ell,x}} \alpha_k^{\omega_1} = \frac{1}{n}\sum_{k\in\Gamma_{\ell,x}} \operatorname{tr} P_{\omega_1} V_k^L \geq \frac{1}{n}\sum_{k\in\Gamma_L} \operatorname{tr} P_{\omega_1} V_k^L - v_+ L^d e^{-m\ell}. \tag{4.21}$$

Now we can use $\sum_{k\in\Gamma_L} \operatorname{tr} P_{\omega_1} V_k^L \geq nV_-$. Putting all bounds together, we get

$$\begin{aligned} \bar{E}^{\omega_1} = \frac{1}{n}\operatorname{tr} P_{\omega_1} H_{\omega_1,L} &\geq \frac{\mu\pi^2 G_-^2}{2R^2(2R+1)^d v_+}(V_- - v_+ L^d e^{-m\ell} - 26\sqrt{n}\,\ell^{-r}) - \|V_o\| \\ &= \xi_{L,\ell,n,r}. \end{aligned}$$

□

Proof of Lemma 4.1. First, we apply Lemma 4.2 to the cluster $\mathcal{C}_n^{\omega_0} = \{E_1^{\omega_0}, \ldots, E_n^{\omega_0}\}$ and the set $Q_0 := Q_\varepsilon^{(x,\ell)}(\omega_1, \omega_0)$ in configuration space. We conclude that there exists $\omega_{0,2} \in Q_1 := Q^{(x,\ell)}_{\varepsilon - \varepsilon\ell^{-(2d+2r)}}(\omega_1, \omega_0)$ and $1 \leq k_1 \leq n-1$ such that

$$E_{k_1+1}^{\omega_{0,2}} - E_{k_1}^{\omega_{0,2}} > 8\varepsilon\ell^{-(2d+2r)}. \tag{4.22}$$

If $k_1 = 1$ or $k_1 = n-1$ then we isolate one eigenvalue from the rest of the eigenvalues and only proceed with one cluster of eigenvalues. In the other cases we obtain two sets of eigenvalues $E_1^{\omega_1} \leq \cdots \leq E_{k_1}^{\omega_{0,2}}$ and $E_{k_1+1}^{\omega_{0,2}} \leq \cdots \leq E_n^{\omega_{0,2}}$ which both satisfy (4.4) for $\varepsilon_1 := \varepsilon\ell^{-(2d+2r)}$. We then apply Lemma 4.2 to the set of eigenvalues $E_1^{\omega_{0,2}} \le \cdots \le E_{k_1}^{\omega_{0,2}}$. This yields $\omega_{0,3} \in Q_2 := Q^{(x,\ell)}_{\varepsilon_1-\varepsilon_1\ell^{-(2d+2r)}}(\omega_1, \omega_{0,2})$ and $1 \le k_2 \le k_1 - 1$ such that

$$E_{k_2+1}^{\omega_{0,3}} - E_{k_2}^{\omega_{0,3}} > 8\varepsilon_1\ell^{-(2d+2r)}. \tag{4.23}$$

Set $\varepsilon_2 := \varepsilon_1\ell^{-(2d+2r)}$. Then since $|\omega_2 - \omega_1|_\infty \le \varepsilon_1 - \varepsilon_2$ we have

$$E_{k_1+1}^{\omega_{0,3}} - E_{k_1}^{\omega_{0,3}} > 8\varepsilon_1 - 2(\varepsilon_1 - \varepsilon_2) \ge 8\varepsilon_2 \tag{4.24}$$

by Weyl's inequality and we can apply Lemma 4.2 to the set $E_{k_1+1}^{\omega_{0,3}} \le \cdots \le E_n^{\omega_{0,3}}$ of eigenvalues. Overall we find $\omega_{0,4} \in Q_3 := Q_{\varepsilon_2-\varepsilon_2\ell^{-(2d+2r)}}(\omega_1, \omega_{0,3})$ and up to four clusters of eigenvalues which are separated from each other (and the rest of the spectrum of H_L) by $8\varepsilon_3 := 8\varepsilon_2\ell^{-(2d+2r)}$. We repeat this procedure at most $n-1$ times until each cluster consists of exactly one eigenvalue. □

4.2. Proof of Theorem 2.2

The setup is as in Section 4.1, i.e.

$$H_\omega := -\mu G\Delta G + V_o + V_\omega \tag{4.25}$$

and G, V_o, V_ω satisfy the conditions specified there. Let

$$E_{\mathrm{sp}} := \frac{\mu\pi^2 V_- G_-^2}{2R^2(2R+1)^d v_+} - \|V_o\|. \tag{4.26}$$

Next is this section's main result, which for $G = \mathbb{1}_{L^2(\mathbb{R}^d)}$ gives Theorem 2.2.

Theorem 4.3. *Assume that* (V_4) *holds. Then for fixed* $E \in (0, E_{\mathrm{sp}})$ *and* $K > 0$ *there exist constants* $\mathcal{L}_{\mathrm{sp}} = \mathcal{L}_{\mathrm{sp},E,K}$ *and* $C_{\mathrm{sp}} = C_{\mathrm{sp},E,K}$ *such that*

$$\mathbb{P}(\mathrm{spac}_E(H_{\omega,L}) < \delta) \le C_{\mathrm{sp}} L^{2d} |\log \delta|^{-K} \tag{4.27}$$

for $L \ge \mathcal{L}_{\mathrm{sp}}$ *and* $\delta < 1$.

In order to extract (2.14) at energy E from (4.27) we have to apply the estimate multiple times for the E-dependent potential $V_o = EV^{-1}$ and for a set of slightly varying L-dependent coupling constants μ_L. This is why we will occasionally comment on the stability of constants as functions of V_o and μ variables.

Besides the existence of good configurations for clusters of eigenvalues established above, the second ingredient for the proof of Theorem 2.2 is a probabilistic estimate on the maximal size of generic clusters of eigenvalues. For lattice models, such estimates follow from an adaption of the method developed in [CGK1] (see [HK]). The following assertion extends this idea.

Lemma 4.4. *For fixed* $E > 0$ *and* $\theta, \vartheta \in (0,1)$ *there exist constants* $c_\theta = c_{\theta,E}$ *and* $C_\vartheta = C_{\vartheta,E} > 0$ *such that*

$$\mathbb{P}(\operatorname{tr} \mathbb{1}_I(H_{\omega,L}) > c_\theta |I|^{-\theta}) \le C_\vartheta L^{2d} |I|^{2-\vartheta} \tag{4.28}$$

for all intervals $I \subset (-\infty, E]$.

Proof. As in the proof of Lemma A.2, we apply Lemma A.1 to estimate, for a fixed interval $I := E_0 + [-\delta G_-^{-1}, \delta G_-^{-1}] \subset (-\infty, E]$,

$$\operatorname{tr} \mathbb{1}_I(H_{\omega,L}) \le \operatorname{tr} \mathbb{1}_{[-\delta,\delta]}(\widetilde{H}_{\omega,L}), \tag{4.29}$$

where $\widetilde{H}_\omega := -\mu\Delta + G^{-2}(V_o - E_0) + G^{-2}V_\omega$. Then (4.29) implies

$$\mathbb{P}(\operatorname{tr} \mathbb{1}_I(H_{\omega,L}) > C) \le \mathbb{P}(\operatorname{tr} \mathbb{1}_{[-\delta,\delta]}(\widetilde{H}_{\omega,L}) > C) \tag{4.30}$$

for any $C > 0$. We denote by $\xi(\mathcal{E}, \widetilde{H}^{\omega_x=0}_{\omega,L}, \widetilde{H}^{\omega_x=1}_{\omega,L}) \ge 0$ the spectral shift function at energy $\mathcal{E}$ of the operators

$$\widetilde{H}^{\omega_x=0}_{\omega,L} := \widetilde{H}_{\omega,L} - \omega_x G^{-2} V_x \quad \text{and} \quad \widetilde{H}^{\omega_x=1}_{\omega,L} := \widetilde{H}_{\omega,L} + (1-\omega_x) G^{-2} V_x. \tag{4.31}$$

We then define the random variable

$$X_\omega := \sup_{x\in\Gamma_L} \operatorname*{ess\,inf}_{\mathcal{E}\in[-\delta,\delta]} \xi(\mathcal{E}, \widetilde{H}^{\omega_x=0}_{\omega,L}, \widetilde{H}^{\omega_x=1}_{\omega,L}) \ge 0, \tag{4.32}$$

where $\Gamma_L := \Lambda_{L+R} \cap \mathbb{Z}^d$. Because X_ω is integer valued, we have

$$\begin{aligned} &\mathbb{P}(\operatorname{tr} \mathbb{1}_{[-\delta,\delta]}(\widetilde{H}_{\omega,L}) > X_\omega) \\ &\quad \le \mathbb{E}\big[\operatorname{tr} \mathbb{1}_{[-\delta,\delta]}(\widetilde{H}_{\omega,L})(\operatorname{tr} \mathbb{1}_{[-\delta,\delta]}(\widetilde{H}_{\omega,L}) - X_\omega)\mathbb{1}_{\{\operatorname{tr} \mathbb{1}_{[-\delta,\delta]}(\widetilde{H}_{\omega,L}) > X_\omega\}}\big]. \end{aligned} \tag{4.33}$$

Omitting the ω, L-subscripts for the moment, we get, for $\mathcal{E} \in [-\delta, \delta]$ and $x \in \Gamma_L$,

$$\begin{aligned} \operatorname{tr} \mathbb{1}_{[-\delta,\delta]}(\widetilde{H}) &= \operatorname{tr}\big(\mathbb{1}_{(-\infty,\delta]}(\widetilde{H}) - \mathbb{1}_{(-\infty,\mathcal{E}]}(\widetilde{H})\big) + \operatorname{tr}\big(\mathbb{1}_{(-\infty,\mathcal{E}]}(\widetilde{H}) - \mathbb{1}_{(-\infty,-\delta]}(\widetilde{H})\big) \\ &\le \operatorname{tr}\big(\mathbb{1}_{(-\infty,\delta]}(\widetilde{H}^{\omega_x=0}) - \mathbb{1}_{(-\infty,\mathcal{E}]}(\widetilde{H}^{\omega_x=0})\big) \\ &\quad + \operatorname{tr}\big(\mathbb{1}_{(-\infty,\mathcal{E}]}(\widetilde{H}^{\omega_x=0}) - \mathbb{1}_{(-\infty,\mathcal{E}]}(\widetilde{H})\big) \\ &\quad + \operatorname{tr}\big(\mathbb{1}_{(-\infty,\mathcal{E}]}(\widetilde{H}^{\omega_x=1}) - \mathbb{1}_{(-\infty,-\delta]}(\widetilde{H}^{\omega_x=1})\big) \\ &\quad + \operatorname{tr}\big(\mathbb{1}_{(-\infty,\mathcal{E}]}(\widetilde{H}) - \mathbb{1}_{(-\infty,\mathcal{E}]}(\widetilde{H}^{\omega_x=1})\big) \\ &\le \operatorname{tr} \mathbb{1}_{[-\delta,\delta]}(\widetilde{H}^{\omega_x=0}) + \operatorname{tr} \mathbb{1}_{[-\delta,\delta]}(\widetilde{H}^{\omega_x=1}) \\ &\quad + \xi(\mathcal{E}, \widetilde{H}^{\omega_x=0}, \widetilde{H}^{\omega_x=1}). \end{aligned} \tag{4.34}$$

Since the inequality holds for all $\mathcal{E} \in [-\delta, \delta]$, we obtain

$$\operatorname{tr} \mathbb{1}_{[-\delta,\delta]}(\widetilde{H}) \le \operatorname{tr} \mathbb{1}_{[-\delta,\delta]}(\widetilde{H}^{\omega_x=0}) + \operatorname{tr} \mathbb{1}_{[-\delta,\delta]}(\widetilde{H}^{\omega_x=1}) + X. \tag{4.35}$$

Next we use (4.35) to estimate (4.33). We first note that for a constant C'_{W} the Wegner estimate

$$\mathbb{E}[\operatorname{tr} \mathbb{1}_{[-\delta,\delta]}(\widetilde{H}^{\omega_x=1}_{\omega,L})] \le C'_{\mathrm{W}} L^d \delta \tag{4.36}$$

holds, for instance via [CGK2] or [Kle]. With (4.36) at hand we obtain

$$\begin{aligned}(4.33) &\le V_- G_+^2 \sum_{x\in\Gamma_L} \mathbb{E}\big[\operatorname{tr} G^{-2} V_x \mathbb{1}_{[-\delta,\delta]}(\widetilde{H}_{\omega,L}) \operatorname{tr} \mathbb{1}_{[-\delta,\delta]}(\widetilde{H}^{\omega_x=0}_{\omega,L})\big] \\ &\quad + V_- G_+^2 \sum_{x\in\Gamma_L} \mathbb{E}\big[\operatorname{tr} G^{-2} V_x \mathbb{1}_{[-\delta,\delta]}(\widetilde{H}_{\omega,L}) \operatorname{tr} \mathbb{1}_{[-\delta,\delta]}(\widetilde{H}^{\omega_x=1}_{\omega,L})\big] \\ &\le C_\vartheta (2\delta)^{2-\vartheta} L^{2d}. \end{aligned} \tag{4.37}$$

In the last inequality we have applied the Birman–Solomyak formula [BS] to obtain

$$\int_{[0,1]} d\omega_x \operatorname{tr} G^{-2} V_x \mathbb{1}_{[-\delta,\delta]}(\widetilde{H}_{\omega,L}) = \int_{[-\delta,\delta]} d\mathcal{E}\, \xi(\mathcal{E}, \widetilde{H}^{\omega_x=0}_{\omega,L}, \widetilde{H}^{\omega_x=1}_{\omega,L}). \tag{4.38}$$

The estimate then follows from the local L^p-boundedness of the spectral shift function as a function of energy [CHN], applied for $p = \vartheta^{-1}$.

We finish the argument by proving the upper bound $X_\omega \le c_\theta |I|^{-\theta}$, where c_θ does not depend on ω. After estimating X_ω as

$$\begin{aligned} X_\omega &\le \sup_{x\in\Gamma_L} \frac{1}{2\delta} \int_{[-\delta,\delta]} d\mathcal{E}\, \xi(\mathcal{E}, \widetilde{H}^{\omega_x=0}_{\omega,L}, \widetilde{H}^{\omega_x=1}_{\omega,L}) \\ &\le \sup_{x\in\Gamma_L} (2\delta)^{-\theta} \bigg(\int_{[-\delta,\delta]} d\mathcal{E}\, \xi(\mathcal{E}, \widetilde{H}^{\omega_x=0}_{\omega,L}, \widetilde{H}^{\omega_x=1}_{\omega,L})^{1/\theta}\bigg)^{\theta} \end{aligned} \tag{4.39}$$

we can again apply the local L^p-boundedness of the spectral shift function, this time for $p = 1/\theta$, to obtain $X_\omega \le c_\theta |I|^{-\theta}$. □

Before we start proving Theorem 2.2 we state a version of the 'good configurations Lemma' 4.1 which is adapted to the present situation, i.e. $L = \ell$ and $r = d/2 + 1$. Let

$$\xi_{L,n} := \frac{\mu\pi^2 G_-^2}{2R^2(2R+1)^d v_+}(V_- - 26\sqrt{n}\, L^{-d-1}), \tag{4.40}$$

where we have omitted the term $v_+ L^d e^{-mL}$, which does not appear in (4.21) in the $\ell = L$ case. The choice $r = d/2+1$ ensures that for $E_{\mathrm{sp}} - \xi_{L,n} \sim \sqrt{n}\, L^{-d/2-1} \le C_1 L^{-1}$, with C_1 as in Lemma A.2.

Lemma 4.5 (Lemma 4.1 for $\ell = L$, $r = d/2+1$). *Let* $0 < \varepsilon < 1/12$*,* $L \ge 1$ *and* $\omega_0, \omega_1 \in [0,1]^{\Gamma_L}$ *be such that the following hold:*

(i) $\omega_1 \in Q_\varepsilon(\omega_0)$.
(ii) *There exist eigenvalues* $E_1^{\omega_1} \le \cdots \le E_n^{\omega_1} \le \xi_{L,n}$ *of* $H_{\omega_1,L}$ *which are separated from the rest of the spectrum: For the cluster* $\mathcal{C}_n^{\omega_1} := \{E_1^{\omega_1}, \ldots, E_n^{\omega_1}\}$ *we have*

$$\operatorname{dist}(\mathcal{C}_n^{\omega_1}, \sigma(H_{\omega_1,L}) \setminus \mathcal{C}_n^{\omega_1}) \ge 8\varepsilon. \tag{4.41}$$

Then there exists $\widehat{\omega} \in Q_\varepsilon(\omega_0)$ *such that*

$$\min_{i=1,\dots,n-1} |E_{i+1}^{\widehat{\omega}} - E_i^{\widehat{\omega}}| > 8\varepsilon L^{-(n-1)(3d+2)}. \tag{4.42}$$

Here, $E_1^\omega \leq \dots \leq E_n^\omega$ *for* $\omega \in Q_\varepsilon(\omega_0)$ *denote the ascendingly ordered eigenvalues of* $H_{\omega,L}$ *in the interval* $[E_1^{\omega_1} - 2\varepsilon, E_n^{\omega_1} + 2\varepsilon]$.

Proof of Theorem 4.3. For fixed $E \in (0, E_{\mathrm{sp}})$ we first decompose $[-\|V_o\|, E]$ into a family $(K_i)_{i\in\mathcal{I}}$ of intervals with length $|K_i| = \kappa < E_{\mathrm{sp}}$, with $|K_{i+1} \cap K_i| \geq \kappa/2$, and such that $|\mathcal{I}| \leq 2(E_{\mathrm{sp}} + \|V_o\|)\kappa^{-1} + 1$. Let $i \in \mathcal{I}$ and define $K_{i,8\varepsilon} := K_i + [-8\varepsilon, 8\varepsilon]$ for $\varepsilon \in (0, 1/12)$. Let $\theta \in (0,1)$. Then the probability of the event

$$\Omega_{i,\varepsilon} := \{\operatorname{tr} \mathbb{1}_{K_i}(H_{\omega,L}) \leq c_\theta |K_i|^{-\theta} \text{ and } \operatorname{tr} \mathbb{1}_{K_{i,8\varepsilon}\setminus K_i}(H_{\omega,L}) = 0\} \tag{4.43}$$

can be estimated by Wegner's estimate and Lemma 4.4 with $\vartheta = 1/2$ as

$$\mathbb{P}(\Omega_{i,\varepsilon}) \geq 1 - 16C_{\mathrm{W}} L^d \varepsilon - C L^{2d} \kappa^{3/2}. \tag{4.44}$$

For $0 < \delta < \kappa/2$ this yields

$$\begin{aligned} &\mathbb{P}(\operatorname{spac}_E(H_{\omega,L}) < \delta) \\ &\quad \leq \sum_{i\in\mathcal{I}} \mathbb{P}(\{\operatorname{spac}_{K_i}(H_{\omega,L}) < \delta\} \cap \Omega_{i,\varepsilon}) + 16C_{\mathrm{W}}|\mathcal{I}|L^d\varepsilon + C|\mathcal{I}|L^{2d}\kappa^{3/2}. \end{aligned} \tag{4.45}$$

We next partition the configuration space $[0,1]^{\Gamma_L}$ into (not necessarily disjoint) cubes Q_j, $j \in \mathcal{J}$, of side length 2ε, i.e. $|Q_j| = (2\varepsilon)^{|\Gamma_L|}$, such that

$$|\mathcal{J}| \leq ((2\varepsilon)^{-1} + 1)^{|\Gamma_L|} \quad \text{and} \quad \sum_{j\in\mathcal{J}} \mathbb{P}(Q_j) \leq 1 + 4\varepsilon|\Gamma_L|\rho_+. \tag{4.46}$$

Now, fix $i \in \mathcal{I}$ and $j \in \mathcal{J}$ such that $Q_j \cap \Omega_{i,\varepsilon} \neq \emptyset$, and let $\omega_{i,j} \in Q_j \cap \Omega_{i,\varepsilon}$. This configuration satisfies

$$n_{i,j} := \operatorname{tr} \mathbb{1}_{K_i}(H_{\omega_{i,j},L}) \leq c_\theta \kappa^{-\theta} \quad \text{and} \quad \operatorname{dist}(K_i, \sigma(H_{\omega_{i,j},L}) \setminus K_i) \geq 8\varepsilon. \tag{4.47}$$

Due to the choice $r = d/2 + 1$ in Lemma 4.5, we have $E < \xi_{L,L^d}$. Hence the lemma is applicable for sufficiently large L and yields $\widehat{\omega}_{i,j} \in Q_j$ such that

$$\operatorname{spac}_{K_{i,\varepsilon}}(H_{\widehat{\omega}_{i,j},L}) \geq 8\varepsilon L^{-(n_{i,j}-1)(3d+2)}. \tag{4.48}$$

This in turn can be used as an input for Lemma 3.4 with $\delta_0 := 8\varepsilon L^{-(n_{i,j}-1)(3d+2)}$. For $Q_j =: \bigtimes_{k\in\Gamma_L} [a_{j,k}, b_{j,k}]$ we obtain

$$\begin{aligned} &\mathbb{P}(Q_j \cap \{\operatorname{spac}_{K_{i,2\varepsilon}}(H_{\omega,L}) < \delta\}) \\ &\qquad \leq \Big(\prod_{k\in\Gamma_L} \sup_{x\in[a_{j,k},b_{j,k}]} \rho(x)\Big) |\{\omega \in Q_j : \operatorname{spac}_{K_{i,2\varepsilon}}(H_{\omega,L}) < \delta\}| \\ &\qquad \leq C_1 \Big(1 + \frac{\mathcal{K}2\varepsilon}{\rho_-}\Big)^{|\Gamma_L|} L^d \mathbb{P}(Q_j) \exp\Big(\frac{-c'_\theta \kappa^{2\theta} |\log\delta|}{|\log 8\varepsilon| + c''_\theta \kappa^{-\theta} \log L}\Big). \end{aligned} \tag{4.49}$$

Here we have used $n_{i,j} \le c_\theta \kappa^{-\theta}$ and the fact that ρ satisfies (V_4), which for $k \in \Gamma_L$ gives

$$\sup_{x\in[a_{j,k},b_{j,k}]} \rho(x) \le \inf_{x\in[a_{j,k},b_{j,k}]} \rho(x) + \mathcal{K}2\varepsilon \le \inf_{x\in[a_{j,k},b_{j,k}]} \rho(x)\Big(1 + \frac{\mathcal{K}2\varepsilon}{\rho_-}\Big). \tag{4.50}$$

The above estimate (4.49) holds for all pairs $i \in \mathcal{I}$, $j \in \mathcal{J}$ such that $Q_j \cap \Omega_{i,\varepsilon} \neq \emptyset$. So far we assumed that $0 < \varepsilon < 1/12$ and $0 < \delta < \kappa/2 < E_{\mathrm{sp}}/2$. If we set $\mathcal{J}_i := \{j \in \mathcal{J} : Q_j \cap \Omega_{i,\varepsilon} \neq \emptyset\}$ for $i \in \mathcal{I}$, then

$$\begin{aligned}(4.45) &\le \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}_i} \mathbb{P}(\{\mathrm{spac}_{K_i}(H_{\omega,L}) < \delta\} \cap Q_j) + 16\mathrm{w}|\mathcal{I}|L^d\varepsilon + C|\mathcal{I}|L^{2d}\kappa^{3/2}\\ &\le C'_{\mathrm{W}} L^d \kappa^{-1}\varepsilon + C' L^{2d}\kappa^{1/2}\\ &\quad + C'_1 L^d \Big(1 + \frac{\mathcal{K}2\varepsilon}{\rho_-}\Big)^{|\Gamma_L|} (1 + 4\varepsilon|\Gamma_L|\rho_+)\kappa^{-1} \exp\Big(\frac{-c'_\theta \kappa^{2\theta}|\log\delta|}{|\log 8\varepsilon| + c''_\theta \kappa^{-\theta}\log L}\Big).\end{aligned}$$

For $0 < \delta \le \exp(-(\log L)^5)$ we now choose

$$\kappa := |\log\delta|^{-1/(4\theta)} \quad \text{and} \quad \varepsilon := \exp(-|\log\delta|^{1/4}). \tag{4.51}$$

Those choices in particular imply $\delta < \kappa/2$ for sufficiently large L. Because $\varepsilon|\Gamma_L| \le 1$ for sufficiently large L we end up with

$$\begin{aligned}\mathbb{P}(\mathrm{spac}_E(H_{\omega,L}) < \delta) &\le C''_\theta L^{2d}|\log\delta|^{-1/(8\theta)} + C''_1 L^d |\log\delta|^{1/(4\theta)} \exp(-\tilde{c}_\theta |\log\delta|^{1/20})\\ &\le C_{\mathrm{sp}} L^{2d} |\log\delta|^{-1/(8\theta)}\end{aligned} \tag{4.52}$$

for a suitable constant C_{sp} and for $L \ge \mathcal{L}_{\mathrm{sp}}$, where $\mathcal{L}_{\mathrm{sp}}$ is sufficiently large. □

4.3. *Proof of Theorem 2.1*

For this section $H_\omega := -\mu\Delta + V_\omega$ denotes the standard random Schrödinger operator specified in Section 2.

For the proof of Theorem 2.1 we apply Lemma 4.1 with two length scales $\ell \ll L$. The smaller scale ℓ serves two purposes. Together with localization it establishes a bound on the maximal size of clusters of eigenvalues that is stronger than the corresponding bound from Lemma 4.4. This is the reason why (2.10) is stronger than (2.12). Secondly, we use the smaller scale ℓ to suppress the impact of the absolutely continuous density. This way we avoid the additional regularity assumption (V_4) from Theorem 2.2.

For the scale $\mathcal{L}_{\mathrm{loc}}$, m' as in Lemma B.3 and $L \ge \ell \ge \mathcal{L}_{\mathrm{loc}}$ we denote by Ω^{loc} the set of $\omega \in \Omega$ that satisfy the following properties: For all eigenpairs (λ, ψ) of $H_{\omega,L}$ with $\lambda \in (-\infty, E_{\mathrm{loc}}]$ there exists $x \in \Lambda_L$ such that

(i) $\|\psi\|_y \le e^{-m'\ell}$ for all $y \in \Lambda_L$ with $|x - y| \ge \ell + 2R$,
(ii) $\mathrm{dist}(\sigma(H_{\Lambda^L_{2\ell+4R}(x)}), \lambda) \le e^{-m'\ell}$,

where we again use the notation $\Lambda_\ell^L(x) := \Lambda_\ell(x) \cap \Lambda_L$. According to the same lemma we have $\mathbb{P}(\Omega^{\text{loc}}) \geq 1 - L^{2d} e^{-m'\ell}$. Moreover, for $\kappa > 0$ we define

$$\Omega_\kappa^{\text{W}} := \bigcap_{\substack{x,y \in \Lambda_L \\ |x-y| > 2\ell+6R}} \left\{ \operatorname{dist} \begin{pmatrix} \sigma(H_{\omega,\Lambda^L_{2\ell+4R}(x)}) \cap (-\infty, E_{\text{loc}}] \\ \text{and} \\ \sigma(H_{\omega,\Lambda^L_{2\ell+4R}(y)}) \cap (-\infty, E_{\text{loc}}] \end{pmatrix} > 3\kappa \right\},$$
$$\Omega_\kappa^g := \Omega_\kappa^{\text{W}} \cap \Omega^{\text{loc}}. \tag{4.53}$$

If the Wegner estimate (2.5) is applied to 'boxes' $\Lambda^L_{2\ell+4R}(x_1)$ and $\Lambda^L_{2\ell+4R}(x_2)$ with $\operatorname{dist}(\Lambda^L_{2\ell+4R}(x_1), \Lambda^L_{2\ell+4R}(x_2)) > 2R$, then the independence of the corresponding operators $H_{\omega,\Lambda^L_{2\ell+4R}(x_1)}$ and $H_{\omega,\Lambda^L_{2\ell+4R}(x_2)}$ yields

$$\mathbb{P}\big(\operatorname{tr} \mathbb{1}_I(H_{\omega,\Lambda^L_{2\ell+4R}(x_1)}) \geq 1 \text{ and } \operatorname{tr} \mathbb{1}_I(H_{\omega,\Lambda^L_{2\ell+4R}(x_2)}) \geq 1\big) \leq C_W'^2 \ell^{2d} |I|^2 \tag{4.54}$$

for a slightly enlarged constant C'_{W}. Together with Lemma B.3 the probability of the event Ω_κ^g can be bounded from below by

$$\mathbb{P}(\Omega_\kappa^g) \geq 1 - 6C_{\text{W}}'^2 L^{2d} \ell^{2d} \kappa - L^{2d} e^{-m'\ell} \tag{4.55}$$

for $L \geq \mathcal{L}_{\text{loc}}$, with $\mathcal{L}_{\text{loc}}$ as in Lemma B.3.

Lemma 4.6. *Let $\mathcal{L}_{\text{loc}}$, m' be as in Lemma B.3. Then, for $L \geq \ell \geq \mathcal{L}_{\text{loc}}$ and $\kappa > e^{-m'\ell}$ with $L^{2d} \leq e^{m'\ell}$ the following holds. If $\omega \in \Omega_\kappa^g$ and $I \subset (-\infty, E_{\text{loc}}]$ is an interval with $|I| \leq \kappa$, then*

(i) *there exists $x = x_\omega \in \Lambda_L$ such that $\operatorname{tr} \mathbb{1}_I(H_{\omega,L})\chi_y \leq e^{-m'\ell}$ for all $y \in \Lambda_L$ such that $|x - y| > 3\ell + 8R =: \ell'$,*
(ii) *$\operatorname{tr} \mathbb{1}_I(H_{\omega,L}) \leq C_1' \ell^d$, with constant C_1' specified in (4.57).*

Proof of Lemma 4.6. Let I and ω as in the lemma's statement and let $(\psi_i, \lambda_i)_{i \in \mathcal{I}}$ be the collection of eigenpairs of $H_{\omega,L}$ with $\lambda_i \in I$. For now we denote the localization centers of ψ_i, i.e. the points specified by Lemma B.3, by x_i. Since $\omega \in \Omega_\kappa^{\text{W}}$ we thus have $\operatorname{dist}(\sigma(H_{\omega,\Lambda^L_{2\ell+4R}(z)}), I) > \kappa$ for all $z \in \Lambda_L$ with $|z - x_1| \geq 2\ell + 6R$. Since by assumption $\kappa > e^{-m'\ell}$ this implies that $|x_i - x_1| < 2\ell + 6R$ for all $i \in \mathcal{I}$. For the first statement let $x := x_1$. Because $|\mathcal{I}| = \operatorname{tr} \mathbb{1}_I(H_{\omega,L}) \leq C_1 L^d$ with C_1 as in Lemma A.2, it follows that

$$\operatorname{tr} \mathbb{1}_I(H_{\omega,L})\chi_y \leq C_1 L^d e^{-m'\ell} \tag{4.56}$$

for all $y \in \Lambda_L$ such that $|x - y| > 3\ell + 8R$. As $L^{2d} \leq e^{m'\ell}$ this proves (i). For the second assertion, we use

$$\operatorname{tr} \chi_{\Lambda_L \setminus \Lambda^L_{6\ell+16R}(x)} \mathbb{1}_I(H_{\omega,L}) \leq \sum_{\substack{y \in \mathbb{Z}^d \\ |y-x| > 3\ell+8R}} \operatorname{tr} \mathbb{1}_I(H_{\omega,L})\chi_y \leq C_1 L^{2d} e^{-m'\ell} \leq C_1$$

by (4.56). This gives the estimate

$$\operatorname{tr} \mathbb{1}_I(H_{\omega,L}) \leq C_1 + \operatorname{tr} \chi_{\Lambda^L_{6\ell+16R}(x)} \mathbb{1}_I(H_{\omega,L}) \leq C_1(1 + (6\ell + 16R)^d) \leq C_1' \ell^d. \tag{4.57}$$

□

Proof of Theorem 2.1. The proof is similar to the one of Theorem 2.2. First, let $L \geq \ell \geq \mathcal{L}_{\mathrm{loc}}$ and $\min\{E_{\mathrm{loc}}, E_{\mathrm{sp}}\} > \kappa > 0$ be such that $\kappa > e^{-m'\ell}$ and $L^{2d} \leq e^{m'\ell}$. We again start by choosing a fixed $E \in (0, \min\{E_{\mathrm{loc}}, E_{\mathrm{sp}}\})$ and decompose $[0, E]$ into a family $(K_i)_{i\in\mathcal{I}}$ of intervals of length $|K_i| = \kappa$, with $|K_{i+1} \cap K_i| \geq \kappa/2$ and such that $|\mathcal{I}| \leq 4E_{\mathrm{sp}}\kappa^{-1} + 1$. We also set $K_{i,8\varepsilon} := K_i + [-8\varepsilon, 8\varepsilon]$ for $\varepsilon \in (0, 1/12)$. By Wegner's estimate,

$$\mathbb{P}(\operatorname{tr} \mathbb{1}_{K_{i,8\varepsilon}\setminus K_i}(H_{\omega,L}) = 0) \geq 1 - 16C_{\mathrm{W}} L^d \varepsilon. \tag{4.58}$$

If we define the event

$$\Omega^g_{i,\kappa} := \Omega^g_\kappa \cap \{\operatorname{tr} \mathbb{1}_{K_{i,8\varepsilon}\setminus K_i}(H_{\omega,L}) = 0\}, \tag{4.59}$$

then for $0 < \delta < \kappa/2$ we obtain from (4.58) and (4.55) the bound

$$\begin{aligned}\mathbb{P}(\operatorname{spac}_E(H_{\omega,L}) < \delta) &\leq \mathbb{P}(\{\operatorname{spac}_E(H_{\omega,L}) < \delta\} \cap \Omega^g_\kappa) + 6C'^2_{\mathrm{W}} L^{2d} \ell^{2d} \kappa + L^{2d} e^{-m'\ell} \\ &\leq \sum_{i\in\mathcal{I}} \mathbb{P}(\{\operatorname{spac}_{K_i}(H_{\omega,L}) < \delta\} \cap \Omega^g_{i,\kappa}) + \Xi_{L,\ell,\kappa,\varepsilon}.\end{aligned} \tag{4.60}$$

Here we have also abbreviated $\Xi_{L,\ell,\kappa,\varepsilon} := C''_{\mathrm{W}} L^d \kappa^{-1} \varepsilon + C''_{\mathrm{W}} L^{2d} \ell^{2d} \kappa + L^{2d} e^{-m'\ell}$ for a suitable constant C''_{W}. Lemma 4.6 implies that for fixed $i \in \mathcal{I}$ and $\omega \in \Omega^g_{i,\kappa}$ there exists $x_{i,\omega} \in \Lambda_L$ (which we can assume without loss of generality is in $\Lambda^\#_L := \Lambda_L \cap \mathbb{Z}^d$) such that $P_{i,\omega} := \mathbb{1}_{K_{i,\varepsilon}}(H_{\omega,L})$ is localized with localization center $x_{i,\omega}$:

$$\operatorname{tr} \chi_x P_{i,\omega} \leq e^{-m'\ell} \leq e^{-m''\ell'} \tag{4.61}$$

for all $x \in \Lambda_L$ with $|x - x_{i,\omega}| \geq 3\ell + 8R = \ell'$ and a suitable $0 < m'' < m'$. If we define

$$\Omega^{\mathrm{loc}}_{i,x} := \{P_{i,\omega} \text{ is localized with localization center } x\}, \tag{4.62}$$

$$\Omega^{\mathrm{sp}}_i := \{\operatorname{spac}_{K_{i,\varepsilon}}(H_{\omega,L}) < \delta\}, \tag{4.63}$$

then we arrive at

$$(4.60) \leq \sum_{i\in\mathcal{I}} \sum_{x\in\Lambda^\#_L} \mathbb{P}(\Omega^{\mathrm{sp}}_i \cap \Omega^{\mathrm{loc}}_{i,x} \cap \Omega^g_{i,\kappa}) + \Xi_{L,\ell,\kappa,\varepsilon}. \tag{4.64}$$

Next we again partition the configuration space into subcubes, but now only in a spacial neighborhood of the localization center x. More precisely, we partition $[0, 1]^{\Gamma_{\ell',x}}$ into (not necessarily disjoint) cubes $Q_{j,x} \subset [0, 1]^{\Gamma_{\ell',x}}$, $j \in \mathcal{J}$, of side length 2ε and such that

$$|\mathcal{J}| \leq ((2\varepsilon)^{-1} + 1)^{|\Gamma_{\ell',x}|} \quad \text{and} \quad \sum_{j\in\mathcal{J}} \mathbb{P}(Q_j) \leq 1 + 4\varepsilon|\Gamma_{\ell',x}|\rho_+. \tag{4.65}$$

We denote the centers of $Q_{j,x}$ by $\omega_{0,j,x} \in [0,1]^{\Gamma_{\ell',x}}$, i.e. $Q_{j,x} = \omega_{0,j,x} + [-\varepsilon, \varepsilon]^{\Gamma_{\ell',x}}$. So far we have estimated

$$(4.60) \leq \sum_{i \in \mathcal{I}} \sum_{x \in \Lambda_L^{\#}} \sum_{j \in \mathcal{J}} \mathbb{P}\big((Q_{j,x} \times [0,1]^{\Gamma_L \setminus \Gamma_{\ell',x}}) \cap \Omega_i^{\mathrm{sp}} \cap \Omega_{i,\kappa}^{g} \cap \Omega_{i,x}^{\mathrm{loc}}\big) + \Xi_{L,\ell,\kappa,\varepsilon}. \tag{4.66}$$

Let $i \in \mathcal{I}$, $x \in \Lambda_L^{\#}$ and $j \in \mathcal{J}$ be fixed and such that the probability on the right hand side of (4.66) is non-zero. For a set $A \subset [0,1]^{\Gamma_L}$ let

$$\mathrm{pr}^{Q_{j,x}}_{\Lambda^c_{\ell'}(x)}(A) := \{\omega|_{\Lambda^c_{\ell'}(x)} : \omega \in A \text{ and } \omega|_{\Lambda_{\ell'}(x)} \in Q_{j,x}\} \subseteq [0,1]^{\Gamma_L \setminus \Gamma_{\ell',x}}. \tag{4.67}$$

We now estimate the probability in (4.66) by

$$\mathbb{P}\big((Q_{j,x} \times \mathrm{pr}^{Q_{j,x}}_{\Lambda^c_{\ell'}(x)}(\Omega_i^{g} \cap \Omega_{i,x}^{\mathrm{loc}})) \cap \Omega_i^{\mathrm{sp}}\big) \tag{4.68}$$

and choose a fixed

$$\omega_{1,\Lambda^c_{\ell'}(x)} \in \mathrm{pr}^{Q_{j,x}}_{\Lambda^c_{\ell'}(x)}(\Omega_i^{g} \cap \Omega_{i,x}^{\mathrm{loc}}) \neq \emptyset. \tag{4.69}$$

Here the dependence on i and j is suppressed in notation. By construction, there exists $\omega_{1,\Lambda_{\ell'}(x)} \in Q_{j,x}$ such that $\omega_1 := (\omega_{1,\Lambda_{\ell'}(x)}, \omega_{1,\Lambda^c_{\ell'}(x)}) \in \Omega_i^{g} \cap \Omega_{i,x}^{\mathrm{loc}}$, where also the dependence on x is suppressed in notation. Hence, Lemma 4.1 can be applied for ℓ' as small scale, m'' as inverse localization length in (4.5), $n \leq C_1' \ell^d$ and $r = d+1$. This yields a configuration $\widehat{\omega} \in Q_\varepsilon^{(x,\ell')}(\omega_1, \omega_{0,j})$ such that

$$\mathrm{spac}_{I_{i,\varepsilon}}(H_{\widehat{\omega},L}) \geq 8\varepsilon \ell'^{-\ell'^d 2C_1'(2d+2r)}. \tag{4.70}$$

Lemma 3.4 is now applicable for $n \leq C_1' \ell^d$, $\delta_0 = 8\varepsilon \ell'^{-\ell'^d 2C_1'(2d+2r)}$ and the family $(\omega_j)_{j \in \Gamma_{\ell',x}}$ of random variables. This yields

$$\begin{aligned} &|\{\omega \in Q_\varepsilon^{(x,\ell')}(\omega_1, \omega_{0,j}) : \mathrm{spac}_{K_{i,\varepsilon}}(H_{\omega,L}) < \delta\}|_{\Lambda_{\ell'}(x)} \\ &\qquad \leq c_1' \ell'^d (2\varepsilon)^{|\Gamma_{x,\ell'}|} \exp\left(\frac{-c_2' |\log \delta|}{\ell'^{2d}(|\log \varepsilon| + \ell'^{d+1})}\right). \end{aligned}$$

Here $|A|_{\Lambda_{\ell'}(x)}$ stands for the $|\Gamma_{\ell',x}|$-dimensional Lebesgue measure of a set A. Because this bound is independent of the $\omega_{1,\Lambda^c_{\ell'}(x)}$ chosen in (4.69), we can use (4.68) to estimate

$$\begin{aligned} &\mathbb{P}((Q_{j,x} \times [0,1]^{\Gamma_L \setminus \Gamma_{\ell',x}}) \cap \Omega_i^{\mathrm{sp}} \cap \Omega_{i,\kappa}^{g} \cap \Omega_{i,x}^{\mathrm{loc}}) \\ &\qquad \leq c_1' \ell'^d (2\varepsilon)^{|\Gamma_{\ell',x}|} \exp\left(\ell'^d \log \rho_+ - \frac{c_2' |\log \delta|}{\ell'^{2d}(|\log \varepsilon| + \ell'^{d+1})}\right). \end{aligned} \tag{4.71}$$

Overall, we arrive at

$$\begin{aligned} (4.66) \leq\; & c_1'' L^d \kappa^{-1} \exp\left(\ell'^d \log \rho_+ - \frac{c_2' |\log \delta|}{\ell'^{2d}(|\log \varepsilon| + \ell'^{d+1})}\right) \\ & + C_{\mathrm{W}}'' L^d \kappa^{-1} \varepsilon + C_{\mathrm{W}}'' L^{2d} \ell^{2d} \kappa + L^{2d} e^{-m'\ell}. \end{aligned} \tag{4.72}$$

We now choose $\varepsilon := \exp(-|\log\delta|^{1/4})$, $\kappa := \exp(-|\log\delta|^{1/8})$ and $\ell = |\log\delta|^{1/(8d)}$, which yields

$$\begin{aligned}\mathbb{P}(\mathrm{spac}_E(H_{\omega,L}) < \delta) &\le C'_{\mathrm{sp}} L^{2d}\big(e^{-m''|\log\delta|^{1/(8d)}} + e^{|\log\delta|^{1/8}(1+\rho_+)-c'_2|\log\delta|^{1/2}}\big)\\ &\le C_{\mathrm{sp}} L^{2d} e^{-|\log\delta|^{1/(9d)}}\end{aligned} \tag{4.73}$$

for $\delta \le \delta_0$, where $\delta_0 > 0$ is sufficiently small. Finally, the condition $\kappa > e^{-m'\ell}$ is satisfied for sufficiently large L and the conditions $L \ge \ell$ and $L^{2d} \le e^{m'\ell}$ are satisfied for

$$\exp(-L^{8d}) \le \delta \le \exp(-(\log L)^{9d}). \tag{4.74}$$

If $\delta < \exp(-L^{8d})$ we can omit the introduction of a second scale $\ell \ll L$ and directly carry out the argument on the whole box Λ_L, in a similar fashion to the proof of Theorem 2.2. □

5. Proof of the Minami-type estimate

Before we start with the proof of Theorem 2.4 we make some preliminary remarks. Let $H^\mu_\omega = -\mu\Delta + V_\omega$ be the standard random Schrödinger operator from Section 2. The random operator

$$\widetilde{H}^{\mu,E}_\omega := V^{-1/2}(H_\omega - E)V^{-1/2} = -\mu V^{-1/2}\Delta V^{-1/2} + \widetilde{V}^E_o + \widetilde{V}_\omega \tag{5.1}$$

is a deformed random Schrödinger operator with periodic potential $\widetilde{V}^E_o := -EV^{-1}$ and random potential $\widetilde{V}_\omega := \sum_{k\in\mathbb{Z}^d}\omega_k\widetilde{V}_k$, where $\widetilde{V}_k := V^{-1}V_k$. We stress the dependence on μ in notation because, as mentioned earlier, we will have to work with L-dependent couplings μ_L in some small neighborhood of a fixed μ.

Tracking constants in Section 4.2 shows the following. For fixed $E_0 \in (0, E_{\mathrm{M}})$, with E_{M} as defined in (2.13), and $K > 0$ there exists $\varepsilon > 0$ and constants $\mathcal{L}_{\mathrm{sp}}, C_{\mathrm{sp}} > 0$ such that for all $\mu' \in [\mu-\varepsilon, \mu+\varepsilon]$ and all $E \in [0, E_0]$,

$$\mathbb{P}(\mathrm{spac}_{[-\varepsilon,\varepsilon]}(\widetilde{H}^{\mu',E}_{\omega,L}) < \delta) \le C_{\mathrm{sp}} L^{2d}|\log\delta|^{-K} \tag{5.2}$$

for all $L \ge \mathcal{L}_{\mathrm{sp}}$ and $\delta < 1$.

Proof of Theorem 2.4. For fixed $E_0 \in (0, E_{\mathrm{M}})$ and $K > 0$ we denote by $\varepsilon, \mathcal{L}_{\mathrm{sp}}, C_{\mathrm{sp}}$ the constants above. After possibly enlarging $\mathcal{L}_{\mathrm{sp}}$ we have $\delta \le \mathcal{L}_{\mathrm{sp}}^{-d} \le \varepsilon/2$ and $4\delta L^d \le 1$ for L, δ which satisfy $L \ge \mathcal{L}_{\mathrm{sp}}$ and $\delta \le \exp(-(\log L)^{5d})$.

Let now $E \in [0, E_0]$, $L \ge \mathcal{L}_{\mathrm{sp}}$ and $0 < \delta \le \exp(-(\log L)^{5d})$ be fixed. Our starting point is Lemma A.1, which, applied for $A = H^\mu_{\omega,L} - E$, $S = V_-^{1/2}V^{-1/2}$ and $\varepsilon = \delta V_-/2$, yields

$$\begin{aligned}\mathrm{tr}\,\mathbb{1}_{[E-\delta V_-,E+\delta V_-]}(H^\mu_{\omega,L}) &= \mathrm{tr}\,\mathbb{1}_{[-\delta V_-,\delta V_-]}(H^\mu_{\omega,L} - E)\\ &\le \mathrm{tr}\,\mathbb{1}_{[-\delta,\delta]}(\widetilde{H}^{\mu,E}_{\omega,L}).\end{aligned} \tag{5.3}$$

We denote by $\widetilde{E}^{\mu,E}_{\omega,j}$, $j \in \mathbb{N}$, the eigenvalues of $\widetilde{H}^{\mu,E}_{\omega,L}$ in ascending order. If C_1 denotes the constant from Lemma A.2, then

$$\mathbb{P}(\operatorname{tr} \mathbb{1}_{[-\delta,\delta]}(\widetilde{H}^{\mu,E}_{\omega,L}) \geq 2) \leq \sum_{j=1}^{C_1 L^d} \mathbb{P}\big(\operatorname{spac}_{[-\varepsilon/2,\varepsilon/2]}(\widetilde{H}^{\mu,E}_{\omega,L}) < 2\delta,\ \widetilde{E}^{\mu,E}_{\omega,j} \in [-\delta,\delta]\big), \quad (5.4)$$

where we have used $\delta \leq \varepsilon/2$. Each term on the right hand side will be estimated separately. Let us first introduce some notation. Let $N \in \mathbb{N}$ be such that $(2L^d\delta)^{-1} - 1 < N \leq (2L^d\delta)^{-1}$ and

$$I_i := [-\delta,\delta] + (i-1)2\delta \qquad \text{for } i \in \{1,\dots,N\}. \quad (5.5)$$

Moreover, for $i \in \{1,\dots,N\}$, $j \in \mathbb{N}$ and $\theta > 0$ we define

$$\Omega^{\theta}_{i,j} := \{\operatorname{spac}_{[-\theta,\theta]}(\widetilde{H}^{\mu,E}_{\omega,L}) < 2\delta\} \cap \{\widetilde{E}^{\mu,E}_{\omega,j} \in I_i\}. \quad (5.6)$$

Let $\kappa := (1 + L^{-d})^{-1}$. Then we claim that for some constant C_ρ, that only depends on the single-site density ρ,

$$\mathbb{P}(\Omega^{\varepsilon/2}_{1,j}) \leq C_\rho \mathbb{P}\big(\operatorname{spac}_{[-\varepsilon,\varepsilon]}(\widetilde{H}^{\kappa\mu,\kappa E}_{\omega,L}) < 2\delta,\ \widetilde{E}^{\kappa\mu,\kappa E}_{\omega,j} \in \kappa I_i\big). \quad (5.7)$$

In this case, summation of (5.7) over $i \in \{1,\dots,N\}$ yields

$$\mathbb{P}(\Omega^{\varepsilon/2}_{1,j})\, 4 \leq C_\rho L^d \delta\, \mathbb{P}(\operatorname{spac}_{[-\varepsilon,\varepsilon]}(\widetilde{H}^{\kappa\mu,\kappa E}_{\omega,L}) < 2\delta), \quad (5.8)$$

where we have used $N^{-1} \leq 4L^d\delta$ and the fact that for $i_1 \neq i_2$,

$$\{\widetilde{E}^{\kappa\mu,\kappa E}_{\omega,j} \in \kappa I_{i_1}\} \cap \{\widetilde{E}^{\kappa\mu,\kappa E}_{\omega,j} \in \kappa I_{i_2}\} = \emptyset. \quad (5.9)$$

The statement now follows from an application of (5.2) to the right hand side of (5.8).

We are left with proving (5.7). For the operator $\widetilde{H}^{\mu,E}_{\omega,L}$ a shift of random couplings results in an energy shift. If we denote $\boldsymbol{\tau} = (\tau,\dots,\tau) \in \Gamma_L$ for fixed $\tau \in \mathbb{R}$, then

$$\widetilde{H}^{\mu,E}_{\omega+\boldsymbol{\tau},L} = \widetilde{H}^{\mu,E}_{\omega,L} + \tau \chi_{\Lambda_L} V V^{-1} \chi_{\Lambda_L} = \widetilde{H}^{\mu,E}_{\omega,L} + \tau \quad (5.10)$$

as operators on $L^2(\Lambda_L)$. This implies that

$$\operatorname{spac}_K(\widetilde{H}^{\mu,E}_{\omega,L}) = \operatorname{spac}_{K+\tau}(\widetilde{H}^{\mu,E}_{\omega+\boldsymbol{\tau},L}) \quad (5.11)$$

for any interval $K \subset \mathbb{R}$. Let $\eta_i := (i-1)2\delta$ denote the centers of the intervals I_i. The change of variables $\omega_k \mapsto \omega_k + \eta_i$ and (5.11) give

$$\mathbb{P}(\Omega^{\varepsilon/2}_{1,j}) \leq \int_{[\eta_i, 1+\eta_i]^{\Gamma_L}} \mathbb{1}_{\Omega^{\varepsilon}_{i,j}}(\omega) \prod_{k\in\Gamma_L} \rho(\omega_k - \eta_i)\, d\omega_k, \quad (5.12)$$

where we have also used $\eta_i \leq L^{-d} \leq \varepsilon/2$ and (5.11). Another change of variables $\omega_k \mapsto \kappa\omega_k$ yields

$$(5.12) \leq \kappa^{-|\Gamma_L|} \int_{[a_i,b_i]^{\Gamma_L}} \mathbb{1}_{\Omega^{\varepsilon}_{i,j}}(\kappa^{-1}\omega) \prod_{k\in\Gamma_L} \rho(\kappa^{-1}\omega_k - \eta_i) d\omega_k, \quad (5.13)$$

where $a_i := \kappa\eta_i$ and $b_i := \kappa(1+\eta_i)$ (which both depend on L through κ). Note that

$$\widetilde{H}^{\mu,E}_{\kappa^{-1}\omega,L} = \kappa^{-1}\widetilde{H}^{\kappa\mu,\kappa E}_{\omega,L}, \tag{5.14}$$

and hence by definition of the events $\Omega^{\varepsilon}_{i,j}$,

$$\kappa^{-1}\omega \in \Omega^{\varepsilon}_{i,j} \iff \omega \in \kappa\Omega^{\varepsilon}_{i,j} \iff \begin{cases} \mathrm{spac}_{\kappa\varepsilon}(\widetilde{H}^{\kappa\mu,\kappa E}_{\omega,L}) < \kappa 2\delta, \\ \widetilde{E}^{\kappa\mu,\kappa E}_{\omega,j} \in \kappa I_i. \end{cases} \tag{5.15}$$

Because $\kappa < 1$ the relation (5.15) yields

$$\kappa\Omega^{\varepsilon}_{i,j} \subset \{\mathrm{spac}_{\varepsilon}(\widetilde{H}^{\kappa\mu,\kappa E}_{\omega,L}) < 2\delta,\ \widetilde{E}^{\kappa\mu,\kappa E}_{\omega,j} \in \kappa I_i\}. \tag{5.16}$$

Moreover, since ρ satisfies (V_4), for $x \in (a_i, b_i) \subset (0,1)$ we have $\kappa^{-1}x - \eta_i \in (0,1)$ and

$$\rho(\kappa^{-1}x - \eta_i) \le \rho(x) + 2\mathcal{K}L^{-d} \le \rho(x)\left(1 + \frac{2\mathcal{K}}{L^d\rho_-}\right). \tag{5.17}$$

Estimating (5.13) via (5.16) and (5.17) yields

$$(5.13) \le C_\rho \mathbb{P}\big(\mathrm{spac}_{\varepsilon}(\widetilde{H}^{\kappa\mu,\kappa E}_{\omega,L}) < 2\delta,\ \widetilde{E}^{\kappa\mu,\kappa E}_{\omega,j} \in \kappa I_i\big). \qquad \square$$

6. Simplicity of spectrum and Poisson statistics

As mentioned in Section 2, both statements follow from Theorem 2.1 respectively Theorem 2.4 and the techniques from [KM, CGK1] respectively [Min, M, CGK1]. For convenience we recap the arguments here, closely following the above references.

For the proof of Corollary 2.3 we apply the following consequence of (2.7): With probability 1, for any normalized eigenpair (ψ, λ) of H_ω with $\lambda < E_{\mathrm{loc}}$ there exists a constant C_ψ such that for all $x \in \mathbb{R}^d$,

$$\|\psi\|_x \le C_\psi e^{-m|x|}. \tag{6.1}$$

Here, the localization center has been absorbed into the (ω-dependent) constant C_ψ.

Proof of Corollary 2.3. Let $E < \min\{E_{\mathrm{sp}}, E_{\mathrm{loc}}\}$ be fixed. First we note that by Theorem 2.2 there exists $\mathcal{L}_0$ such that for $L \ge \mathcal{L}_0$,

$$\mathbb{P}\big(\mathrm{spac}_E(H_{\omega,L}) < 3e^{-\sqrt{L}}\big) \le L^{-2}. \tag{6.2}$$

Since the right hand side is summable over $L \in \mathbb{N}$, the Borel–Cantelli lemma shows that the set

$$\Omega_\infty := \{\mathrm{spac}_E(H_{\omega,L}) < 3e^{-\sqrt{L}} \text{ for infinitely many } L \in \mathbb{N}\} \tag{6.3}$$

is of measure zero with respect to $\mathbb{P}$. Let Ω_{loc} be the set of measure 1 such that (6.1) holds for all $\omega \in \Omega_{\mathrm{loc}}$. We now choose a fixed

$$\omega \in \Omega_{\mathrm{loc}} \cap \{\exists E' \le E : \mathrm{tr}\,\mathbb{1}_{\{E'\}}(H_\omega) \ge 2\} =: \Omega_{\mathrm{loc}} \cap \Omega_{\ge 2}; \tag{6.4}$$

i.e. for the configuration ω there exists $E' \leq E$ such that E' is an eigenvalue of H_ω with two linearly independent, normalized and exponentially decaying eigenfunctions ϕ, ψ. We now apply [KM, Lemma 1] with the slightly modified choice $\varepsilon_L = L^d e^{-mL/2} \ll e^{-\sqrt{L}}$. The lemma is formulated for the lattice but generalizes to the continuum as remarked in [CGK1]. This implies that for $I_L := [E - e^{-\sqrt{L}}, E + e^{-\sqrt{L}}]$ and all sufficiently large $L \in \mathbb{N}$,

$$\operatorname{tr} \mathbb{1}_{I_L}(H_{\omega,L}) \geq 2, \tag{6.5}$$

and consequently $\Omega_{\text{loc}} \cap \Omega_{\geq 2} \subset \Omega_\infty$. The latter set is of $\mathbb{P}$-measure zero, and the result follows from $\mathbb{P}(\Omega_{\text{loc}} \cap \Omega_{\geq 2}) = 0$. □

Proof of Theorem 2.5. The proof closely follows the one in [CGK1, Section 6]. Let $E \in [0, \min\{E_{\mathrm{M}}, E_{\text{loc}}\}]$ be fixed and such that $n(E) > 0$. The starting point is to construct a triangular array of point processes which approximate $\xi_\omega^L := \xi_{E,\omega}^L$ sufficiently well. To this end, let L be fixed and $\ell := (\log L)^2$. Then we define point processes $\xi_\omega^{L,m}$ for $m \in \Upsilon_L := (\ell + 2\lceil R \rceil)\mathbb{Z}^d \cap \Lambda_{L-\ell}$ via $\xi_\omega^{L,m}(B) := \operatorname{tr} \mathbb{1}_{E+L^{-d}B}(H_{\omega,\Lambda_\ell(m)})$ ($B \subset \mathbb{R}$ Borel measurable). This definition ensures that for $m, n \in \Upsilon_L$, $m \neq n$, the processes $\xi_\omega^{L,m}$ and $\xi_\omega^{L,n}$ are independent.

The proof now consists of two parts. In the first part one shows that the superposition $\widetilde{\xi}_\omega^L := \sum_{m \in \Upsilon_L} \xi_\omega^{L,m}$ is a good approximation of the process ξ_ω^L in the sense that, if one of them converges weakly, then they share the same weak limit. This is a consequence of spectral localization, and the arguments are very similar to [CGK1]. However, slight adaptions are in place since we work with different finite-volume restrictions of H_ω. We comment on this below. In the second part one then proves that the process $\widetilde{\xi}_\omega^L$ weakly converges towards the Poisson point process with intensity measure $n(E)dx$. This is the case if and only if for all bounded intervals $I \subset \mathbb{R}$ the three properties

$$\lim_{L\to\infty} \max_{m \in \Upsilon_L} \mathbb{P}(\xi_\omega^{L,m}(I) \geq 1) = 0, \tag{6.6}$$

$$\lim_{L\to\infty} \sum_{m \in \Upsilon_L} \mathbb{P}(\xi_\omega^{L,m} \geq 1) = |I| n(E), \tag{6.7}$$

$$\lim_{L\to\infty} \sum_{m \in \Upsilon_L} \mathbb{P}(\xi_\omega^{L,m}(I) \geq 2) = 0 \tag{6.8}$$

hold. We assume for convenience that $|I| \leq 1$ and note that (6.6) follows from Wegner's estimate. Let L be sufficiently large such that $\ell \geq \mathcal{L}_{\mathrm{M}}$, where $\mathcal{L}_{\mathrm{M}}$ is the initial scale from Theorem 2.4. We can then apply the theorem for $K = 12d$ to estimate

$$\mathbb{P}(\xi_\omega^{L,m}(I) \geq 2) \leq C'_{\mathrm{M}} \ell^{-2d} L^{-d} \tag{6.9}$$

for all $m \in \Upsilon_L$, which ensures (6.8). Moreover, for $n > C_1 \ell^d$ (with C_1 as in Lemma A.2) we have $\mathbb{P}(\xi_\omega^{L,m} \geq n) = 0$. The estimate

$$\sum_{m \in \Upsilon_L} \sum_{n=2}^{\infty} \mathbb{P}(\xi_\omega^{L,m}(I) \geq n) \leq C_1 \ell^d |\Upsilon_L| \sup_{m \in \Upsilon_L} \mathbb{P}(\xi_\omega^{L,m}(I) \geq 2) \leq C''_{\mathrm{M}} \ell^{-d} \tag{6.10}$$

then readily yields (6.11). Moreover, it also shows that (6.7) would follow from

$$\lim_{L\to\infty}\sum_{m\in\Upsilon_L}\mathbb{E}[\xi_\omega^{L,m}(I)] = n(E)|I|. \tag{6.11}$$

To verify (6.11), we will use the following lemma, which is a slight variant of [CGK1, Lemma 6.1].

Lemma 6.1. *For bounded intervals $J \subset \mathbb{R}$ we have*

$$\lim_{L\to\infty}\mathbb{E}\big[|\widetilde{\xi}_\omega^L(J) - \xi_\omega^L(J)|\big] = 0, \tag{6.12}$$

$$\lim_{L\to\infty}\mathbb{E}\big[|\Theta_\omega^L - \xi_\omega^L(J)|\big] = 0, \tag{6.13}$$

where $\Theta_\omega^L(J) := \operatorname{tr}\chi_{\Lambda_L}\mathbb{1}_{E+L^{-d}J}(H_\omega)$.

A sketch of proof for the lemma is given below. By combining (6.12) and (6.13) we obtain

$$\lim_{L\to\infty}\sum_{m\in\Upsilon_L}\mathbb{E}[\xi_\omega^{L,m}(I)] = \lim_{L\to\infty}\mathbb{E}[\Theta_\omega^L] = n(E)|I| \tag{6.14}$$

for the interval I above. Hence (6.9)–(6.11) hold and $\widetilde{\xi}_\omega^L$ converges weakly to the Poisson process with intensity measure $n(E)dx$. As argued in [CGK1], the convergence (6.12) and the density of step functions in L^1 are sufficient to prove that ξ_ω^L weakly converges to the same limit as $\widetilde{\xi}_\omega^L$. □

Proof of Lemma 6.1. We first note that for our model a local Wegner estimate holds, i.e. there exists C'_{W} such that

$$\sup_{x\in\mathbb{R}^d\cap\Lambda_L}\mathbb{E}[\chi_x\mathbb{1}_J(H_{\omega,L})] \le C'_{\mathrm{W}}|J| \tag{6.15}$$

for all intervals $J \subset (-\infty, E_{\mathrm{M}}]$. This is proved in [CGK2, Theorem 2.4] for periodic boundary conditions, but the argument also applies for Dirichlet boundary conditions. The second ingredient of the proof is the following consequence of spectral localization [DGM, Theorem 3.2]. There exist constants $C'_{\mathrm{loc}}, m' > 0$ such that the following holds: For open sets $G \subset G' \subset \mathbb{R}^d$ with $\operatorname{dist}(\partial G', \partial G) \ge 1$ and $a \in G$ we have

$$\mathbb{E}[\|\chi_a(\mathbb{1}_J(H_{\omega,G}) - \mathbb{1}_J(H_{\omega,G'}))\chi_a\|_1] \le C'_{\mathrm{loc}}e^{-m'\operatorname{dist}(a,\partial G)} \tag{6.16}$$

for all intervals $J \subset (-\infty, E_{\mathrm{M}}]$. We now establish (6.12); the proof of (6.13) is similar. To this end, we split each $\Lambda_\ell(m)$, $m \in \Upsilon_L$, into a bulk part $\Lambda_\ell^{(i)}(m) := \Lambda_{\ell-\ell^{2/3}}(m)$ and a boundary part $\Lambda_\ell^{(o)}(m) := \Lambda_\ell(m) \setminus \Lambda_\ell^{(i)}(m)$. If we abbreviate $J_{E,L} := E + L^{-d}J$ then this splitting yields

$$\mathbb{E}\big[|\widetilde{\xi}_\omega^L(J) - \xi_\omega^{m,L}(J)|\big] = \sum_{m\in\Upsilon_L} \mathbb{E}\big[|\operatorname{tr}\chi_{\Lambda_\ell^{(i)}(m)}(\mathbb{1}_{J_{E,L}}(H_{\omega,\Lambda_\ell(m)}) - \mathbb{1}_{J_{E,L}}(H_{\omega,L}))|\big]$$
$$+ \sum_{m\in\Upsilon_L} \mathbb{E}\big[|\operatorname{tr}\chi_{\Lambda_\ell^{(o)}(m)}(\mathbb{1}_{J_{E,L}}(H_{\omega,\Lambda_\ell(m)}) - \mathbb{1}_{J_{E,L}}(H_{\omega,L}))|\big]$$
$$+ \mathbb{E}\Big[\operatorname{tr}\Big(\chi_{\Lambda_L} - \sum_{m\in\Upsilon_L}\chi_{\Lambda_\ell(m)}\Big)\mathbb{1}_{J_{E,L}}(H_{\omega,L})\Big]$$
$$=: (\text{bulk}) + (\text{boundary}) + (\text{rest}). \tag{6.17}$$

For the last two terms we apply the local Wegner estimate from (6.15) to get

$$(\text{boundary}) \le |\Upsilon_L| C'_{\mathrm{W}} L^{-d} d\ell^{d-1}(\sqrt{\ell} + 2R) \le C''_{\mathrm{W}}\ell^{-1/2}, \tag{6.18}$$
$$(\text{rest}) \le C'_{\mathrm{W}} L^{-d} |\Upsilon_L| \ell^{d-1}(2R+2) \le C'''_{\mathrm{W}}\ell^{-1}. \tag{6.19}$$

For the bulk contribution we in turn apply localization via (6.16) to get

$$(\text{bulk}) \le |\Upsilon_L| C'_{\mathrm{loc}} \ell^d e^{-m'\ell^{2/3}} = C''_{\mathrm{loc}} L^d e^{-m'\ell^{3/2}}. \tag{6.20}$$

Because $L = e^{\sqrt{\ell}}$, all three terms (6.18)–(6.20) converge to zero as $L \to \infty$. □

Appendix A. Properties of deformed Schrödinger operators

In this appendix we consider random deformed operators $H_\omega := -\mu G\Delta G + V_o + V_\omega$. The assumptions on G, V_o and V_ω are the same as in Section 4. Lemmas A.2 and A.3 below establish two technical properties of deformed RSO which enter the proof of Theorem 2.2, an a priori trace bound and Wegner's estimate.

Both of them are proven by rewriting the respective estimates in terms of a standard RSO via the following lemma.

Lemma A.1. *Let A be a self-adjoint operator on a separable Hilbert space $\mathcal{H}$, let S be an invertible contraction on $\mathcal{H}$ (i.e. $\|S\| \le 1$), and let $C_\varepsilon(A) := \operatorname{tr}\mathbb{1}_{[-\varepsilon,\varepsilon]}(A)$. Then*

$$C_\varepsilon(A) \le C_\varepsilon(SAS^*). \tag{A.1}$$

Proof. Consider $B := \mathbb{1}_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]}(A)A$. Then $C_0(B) = C_\varepsilon(A)$ and, by Sylvester's law of inertia, we have $C_0(SBS^*) = C_0(B)$. But

$$SAS^* = SBS^* + S\mathbb{1}_{[-\varepsilon,\varepsilon]}(A)AS^* \quad \text{and} \quad \|S\mathbb{1}_{[-\varepsilon,\varepsilon]}(A)AS^*\| \le \varepsilon,$$

so Weyl's inequality implies that $C_0(SBS^*) \le C_\varepsilon(SAS^*)$. □

Lemma A.2 (A priori bound). *For every $E < \infty$ we have, for (almost) every ω and $L > 0$,*

$$\operatorname{tr}\mathbb{1}_{(-\infty,E]}(H_{\omega,L}) \le C_E L^d. \tag{A.2}$$

Proof. With the constant $c := \operatorname{ess\,inf}_{x\in\mathbb{R}^d} V_o(x)$ we have

$$H_{\omega,L} \geq -\mu G\Delta_L G - c.$$

Hence by the min-max principle,

$$\operatorname{tr} \mathbb{1}_{(-\infty,E]}(H_{\omega,L}) \leq \operatorname{tr} \mathbb{1}_{(-\infty,E+c]}(-\mu G\Delta_L G) = \operatorname{tr} \mathbb{1}_{[-\kappa,\kappa]}(-\mu U\Delta_L U^*)$$

for $E < \infty$, where $U = U^* := G_-^{-1}G$ and $\kappa := (E+c)G_-^{-1}$. Since $S := U^{-1}$ satisfies $\|S\| \leq 1$, we can conclude via Lemma A.1 that

$$\operatorname{tr} \mathbb{1}_{(-\infty,E]}(H_{\omega,L}) \leq \operatorname{tr} \mathbb{1}_{[-\kappa,\kappa]}(-\mu\Delta_L) \leq C_{E,\mu}L^d,$$

where the latter bound is well known [S2]. □

Lemma A.3 (Wegner estimate). *For every $E > 0$ there exists $C_W = C_{W,E}$ such that for all $I \subset (-\infty, E]$,*

$$\mathbb{P}(\operatorname{tr} \mathbb{1}_I(H_{\omega,L}) \geq 1) \leq C_W L^d |I|. \tag{A.3}$$

Proof. Let $I = \mathcal{E} + [-\delta,\delta]$ for suitable $\mathcal{E} < E$ and $\delta > 0$. Using $\operatorname{tr} \mathbb{1}_I(H_{\omega,L}) = \operatorname{tr} \mathbb{1}_{[-\delta,\delta]}(H_{\omega,L} - \mathcal{E})$ and Lemma A.1 we get

$$\operatorname{tr} \mathbb{1}_{[-\delta,\delta]}(H_{\omega,L} - \mathcal{E}) \leq \operatorname{tr} \mathbb{1}_{[-\delta,\delta]}(S(H_{\omega,L} - \mathcal{E})S^*), \tag{A.4}$$

where $S = G_-G^{-1}$. If we introduce the auxiliary periodic potential $\widetilde{V}_{o,\mathcal{E}} := G_-^2G^{-2}V_o - \mathcal{E}G_-^2G^{-2}$ and the random potential $\widetilde{V}_\omega := G_-^2G^{-2}V_\omega$, then

$$\widetilde{H}_{\omega,L} := S(H_{\omega,L} - \mathcal{E})S^* = -\mu G_-^2\Delta + \widetilde{V}_{o,\mathcal{E}} + \widetilde{V}_\omega$$

is a standard ergodic RSO for which the Wegner estimate is known. The statement follows since the constant for Wegner's estimate at energy zero can be chosen to be stable in the norm of the periodic background potential. This can for instance be seen from [CGK2, Theorem 2.4]. As mentioned in the proof of Theorem 2.5, the proof from [CGK2] extends to Dirichlet boundary conditions. □

Appendix B. Eigenfunction decay for localized energies

For standard RSO $H_\omega := -\mu\Delta + V_\omega$ as in Section 2 we briefly sketch the proof of Lemma B.3. The exponential decay of eigenfunctions in the localized regime that it describes is a direct consequence of the bound (2.7) and the Wegner estimate.

As before, we denote $\Lambda_\ell^L(x) := \Lambda_\ell(x) \cap \Lambda_L$ for $L \geq \ell$ and $x \in \Lambda_L$. For a set $S \subset \mathbb{R}^d$, we will use the notation ∂S for its topological boundary. For $U \subset \Lambda$ we set $\partial_1^L U := \{u \in U : \operatorname{dist}(u, \partial U \setminus \partial\Lambda_L) \leq 1\}$.

Lemma B.1. *Let $J \subset \mathbb{R}$ be an interval and assume that H_ω satisfies (2.7) for all $E \in J$. Then there exist $\tilde{m}, \mathcal{L}_{\mathrm{loc}} > 0$ such that for $L \geq \ell \geq \mathcal{L}_{\mathrm{loc}}$, with probability $\geq 1 - L^{2d} e^{-\tilde{m}\ell}$ the following holds: For all λ in J and all $x, y \in \Lambda_L$ that satisfy $|x - y| \geq \ell + 2R$,*

$$\text{either} \quad \|\chi_y (H_{\omega,\Lambda_\ell^L(y)} - \lambda)^{-1} \chi_{\partial_1^L \Lambda_\ell^L(y)}\| \leq e^{-\tilde{m}\ell}, \tag{B.1}$$

$$\text{or} \quad \|\chi_x (H_{\omega,\Lambda_\ell^L(x)} - \lambda)^{-1} \chi_{\partial_1^L \Lambda_\ell^L(x)}\| \leq e^{-\tilde{m}\ell}. \tag{B.2}$$

Proof. For the lattice case this assertion has been proven in [ETV, Proposition 5.1]. The proof immediately extends to the continuum case, because, in addition to (2.7), it only relies on the Wegner estimate, Lemma A.3. □

Lemma B.2. *Let ω be a configuration for which the conclusion of Lemma B.1 holds. Then for all $\lambda \in J$ there exists $x = x_\lambda \in \Lambda_L$ such that for all $y \in \Lambda_L \setminus \Lambda_{2\ell+4R}^L(x)$,*

$$\|\chi_y (H_{\omega,\Lambda_\ell^L(y)} - \lambda)^{-1} \chi_{\partial_1^L \Lambda_\ell^L(y)}\| \leq e^{-\tilde{m}\ell}. \tag{B.3}$$

Proof of Lemma B.2. We have two possibilities: Either we can find some $x \in \Lambda_L$ such that (B.2) does not hold, or there is no such x. In the first one the assertion (with the same choice of x) immediately follows from (B.1); in the second case we can choose x arbitrary. □

The next assertion is used in the proof of Lemma 2.1.

Lemma B.3. *Let ω be a configuration for which the conclusion of Lemma B.1 holds. Then, given an eigenpair (λ, ψ) of $H_{\omega,L}$ with $\lambda \in J$, there exists $x = x_\lambda \in \Lambda_L$ such that, with $m' := \tilde{m}/2$,*

(i) $\|\psi\|_y \leq e^{-m'\ell}$ *for all $y \in \Lambda_L$ with $|x - y| \geq \ell + 2R$,*
(ii) $\operatorname{dist}(\sigma(H_{\Lambda_{2\ell+4R}^L(x)}), \lambda) \leq e^{-m'\ell}$.

Proof. (i) Let x be as in Lemma B.2 and let $y \in \Lambda_L \setminus \Lambda_{2\ell+4R}^L(x)$. We will denote by σ_ℓ a smooth characteristic function of $\Lambda_\ell^L(y)$, i.e. a smooth function with $\chi_{\Lambda_{\ell-1}^L(y)} \leq \sigma_\ell \leq \chi_{\Lambda_\ell^L(y)}$ and $\|\partial_i \sigma_\ell\|_\infty, \|\partial_{i,j} \sigma_\ell\|_\infty \leq 4$ for $i, j \in \{1, \ldots, d\}$. Since

$$[H_{\omega,L}, \sigma_\ell] = H_{\omega,\Lambda_\ell^L(y)} \sigma_\ell - \sigma_\ell H_{\omega,L}, \tag{B.4}$$

we obtain the identity

$$\chi_y (H_{\omega,\Lambda_\ell^L(y)} - \lambda)^{-1} [H_{\omega,L}, \sigma_\ell] \psi = \chi_y \psi. \tag{B.5}$$

Together with $[H_{\omega,L}, \sigma_\ell] = \chi_{\partial_1^L \Lambda_\ell(y)} [H_{\omega,L}, \sigma_\ell]$ this implies

$$\|\psi\|_y = \|\chi_y \psi\| \leq \|\chi_y (H_{\omega,\Lambda_\ell^L(y)} - \lambda)^{-1} \chi_{\partial_1^L \Lambda_\ell^L(y)}\| \cdot \|[H_{\omega,L}, \sigma_\ell] \psi\|. \tag{B.6}$$

To bound the first factor on the right hand side, we use (B.3). For the second term in (B.6) we express

$$[H_{\omega,L}, \sigma_\ell] \psi = -[\Delta_L, \sigma_\ell] \psi = -(\lambda - \lambda_0) [\Delta_L, \sigma_\ell] (H_{\omega,L} - \lambda_0)^{-1} \psi \tag{B.7}$$

with

$$\lambda_0 = \inf \sigma(H_{\omega,L}) - 1. \tag{B.8}$$

The statement now follows from the bound

$$\|[\Delta_L, \sigma_\ell](H_{\omega,L} - \lambda_0)^{-1}\| \le C \tag{B.9}$$

(see, e.g., [S2]).

(ii) For the proof we abbreviate $\tilde{\ell} := 2\ell + 4R$. We will denote by $\sigma_{\tilde{\ell}}$ a smooth characteristic function of $\Lambda^L_{\tilde{\ell}}$. Applying the analogue of (B.4) to the eigenfunction ψ, we get

$$[H_{\omega,L}, \sigma_{\tilde{\ell}}]\psi = (H_{\omega,\Lambda^L_{\tilde{\ell}}(x)} - \lambda)\sigma_{\tilde{\ell}}\psi. \tag{B.10}$$

We claim that the left hand side is bounded in norm by $e^{-\tilde{m}\ell/2}$. This implies that the function $\sigma_{\tilde{\ell}}\psi$ is an approximate solution of $(H_{\omega,\Lambda^L_{\tilde{\ell}}(x)} - \lambda)f = 0$. Combining this observation with the bound $1 \ge \|\sigma_{\tilde{\ell}}\psi\| \ge 1 - L^d e^{-\tilde{m}\ell}$ that follows from (i), we deduce (ii) (cf. [EK, Lemma 3.4] and its proof).

Let $\widetilde{\sigma}_{\tilde{\ell}}$ be a smooth function such that $\chi_{\operatorname{supp}\nabla\sigma_{\tilde{\ell}}} \le \widetilde{\sigma}_{\tilde{\ell}} \le \chi_{\Lambda^L_{\tilde{\ell}+1}(y)\setminus\Lambda^L_{\tilde{\ell}-2}(y)}$ and such that $\|\partial_i\widetilde{\sigma}_\ell\|_\infty, \|\partial_{i,j}\widetilde{\sigma}_\ell\|_\infty \le 4$ for $i, j \in \{1,\dots,d\}$.

To establish the claim, we first express (a multiple of) the left hand side of (B.10) as

$$\begin{aligned}(\lambda-\lambda_0)^{-1}[H_{\omega,L}, \sigma_{\tilde{\ell}}]\psi &= [H_{\omega,L}, \sigma_{\tilde{\ell}}]\widetilde{\sigma}_{\tilde{\ell}}(H_{\omega,L}-\lambda_0)^{-1}\psi \\ &= [H_{\omega,L}, \sigma_{\tilde{\ell}}](H_{\omega,L}-\lambda_0)^{-1}\widetilde{\sigma}_{\tilde{\ell}}\psi + (\lambda-\lambda_0)^{-1}[H_{\omega,L}, \sigma_{\tilde{\ell}}](H_{\omega,L}-\lambda_0)^{-1}[H_{\omega,L}, \widetilde{\sigma}_{\tilde{\ell}}]\psi,\end{aligned}$$

with λ_0 is given in (B.8). We can bound the first term on the right hand side by

$$\|[H_{\omega,L}, \sigma_{\tilde{\ell}}](H_{\omega,L}-\lambda_0)^{-1}\|\,\|\chi_{\Lambda^L_{\tilde{\ell}+1}(y)\setminus\Lambda^L_{\tilde{\ell}-2}(y)}\psi\| \le C(\tilde{\ell}+1)^d e^{-\tilde{m}\ell} \le \frac{e^{-\tilde{m}\ell/2}}{2}$$

by (i). The second term can be bounded by

$$(\lambda-\lambda_0)^{-1}\|[H_{\omega,L}, \sigma_{\tilde{\ell}}](H_{\omega,L}-\lambda_0)^{-1}[H_{\omega,L}, \widetilde{\sigma}_{\tilde{\ell}}]\|\,\|\chi_{\Lambda^L_{\tilde{\ell}+1}(y)\setminus\Lambda^L_{\tilde{\ell}-2}(y)}\psi\| \le \frac{e^{-\tilde{m}\ell/2}}{2}$$

as well, and the result follows. □

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References

[AE+] Aizenman, M., Elgart, A., Naboko, S., Schenker, J. H., Stolz, G.: Moment analysis for localization in random Schrödinger operators. Invent. Math. **163**, 343–413 (2006) Zbl 1090.81026 MR 2207021

[AM] Aizenman, M., Molchanov, S.: Localization at large disorder and at extreme energies: An elementary derivation. Comm. Math. Phys. **157**, 245–278 (1993) Zbl 0782.60044 MR 1244867

[AW] Aizenman, M., Warzel, S.: Random Operators: Disorder Effects on Quantum Spectra and Dynamics. Grad. Stud. Math. 168, Amer. Math. Soc., Providence, RI (2015) Zbl 1333.82001

[ABF] Akemann, G., Baik, J., Di Francesco, P.: The Oxford Handbook of Random Matrix Theory. Oxford Univ. Press (2015) Zbl 1321.15005 MR 2920518

[AGKL] Altshuler, B., Gefen, Y., Kamenev, A. Levitov, L. S.: Quasiparticle lifetime in a finite system: A nonperturbative approach. Phys. Rev. Lett. **78**, art. 2803, 4 pp. (1997)

[A] Anderson, P. W.: Absence of diffusion in certain random lattices. Phys. Rev. **109**, 1492–1505 (1958)

[BAA] Basko, D., Aleiner, I., Altshuler, B.: Metal–insulator transition in a weakly interacting many-electron system with localized single-particle states. Ann. Phys. **321**, 1126–1205 (2006) Zbl 1091.82014

[BHS] Bellissard, J. V., Hislop, P. D., Stolz, G.: Correlation estimates in the Anderson model. J. Statist. Phys. **129**, 649–662 (2007) Zbl 1131.82016 MR 2360226

[BS] Birman, M. Sh., Solomyak, M. Z.: Remarks on the spectral shift function. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **27**, 33–46 (1972) (in Russian) Zbl 0329.47009 MR 0315482

[B1] Bourgain, J.: An approach to Wegner's estimate using subharmonicity. J. Statist. Phys. **134**, 969–978 (2009) Zbl 1193.82018 MR 2518977

[B2] Bourgain, J.: On the local eigenvalue spacings for certain Anderson–Bernoulli Hamiltonians. In: Geometric Aspects of Functional Analysis, Springer, 85–96 (2014) Zbl 1319.35126 MR 3364681

[BK] Bourgain, J., Kenig, C. E.: On localization in the continuous Anderson–Bernoulli model in higher dimension. Invent. Math. **161**, 389–426 (2005) Zbl 1084.82005 MR 2180453

[BNSS] Boutet de Monvel, A., Naboko, S., Stollmann, P., Stolz, G.: Localization near fluctuation boundaries via fractional moments and applications. J. Anal. Math. **100**, 83–116 (2006) Zbl 1173.82331

[CL] Carmona, R., Lacroix, J.: Spectral Theory of Random Schrödinger Operators. Springer (2012)

[CGK1] Combes, J.-M., Germinet, F., Klein, A.: Poisson statistics for eigenvalues of continuum random Schrödinger operators. Anal. PDE **3**, 49–80 (2010); Erratum: Anal. PDE **7**, 1235–1236 (2014) Zbl 1298.82031 MR 3265964

[CGK2] Combes, J.-M., Germinet, F., Klein, A.: Local Wegner and Lifshitz tails estimates for the density of states for continuous random Schrödinger operators. J. Math. Phys. **55**, art. 083523, 15 pp. (2014) Zbl 1295.81064 MR 3390763

[CH] Combes, J.-M., Hislop, P. D.: Localization for some continuous, random Hamiltonians in d-dimensions. J. Funct. Anal. **124**, 149–180 (1994) Zbl 0801.60054 MR 1284608

[CHN] Combes, J.-M., Hislop, P. D., Nakamura, S.: The L^p-theory of the spectral shift function, the Wegner estimate, and the integrated density of states for some random operators. Comm. Math. Phys. **218**, 113–130 (2001) Zbl 1042.82024 MR 1824200

[DG$^+$] Dietlein, A., Gebert, M., Hislop, P. D., Klein, A., Müller, P.: A bound on the averaged spectral shift function and a lower bound on the density of states for random Schrödinger operators on $\mathbb{R}^d$. Int. Math. Res. Notices **2018**, 6673–6697 (2018) Zbl 07013565 MR 3873541

[DGM] Dietlein, A., Gebert, M., Müller, P.: Bounds on the effect of perturbations of continuum random Schrödinger operators and applications. J. Spectral Theory, to appear; arXiv:1701.02956

[EK] Elgart, A., Klein, A.: An eigensystem approach to Anderson localization. J. Funct. Anal. **271**, 3465–3512 (2016) Zbl 1358.82019 MR 3558248

[ETV] Elgart, A., Tautenhahn, M., Veselić, I.: Anderson localization for a class of models with a sign-indefinite single-site potential via fractional moment method. Ann. Henri Poincaré **12**, 1571–1599 (2011) Zbl 1237.82026 MR 2855178

[EY] Erdős, L., Yau, H.-T.: A Dynamical Approach to Random Matrix Theory. Courant Lecture Notes in Math. 28, Courant Inst. Math. Sci., New York (2017) Zbl 1379.15003 MR 3699468

[FA] Fleishman, L., Anderson, P. W.: Interactions and the Anderson transition. Phys. Rev. B **21**, art. 2366, 12 pp. (1980)

[FS] Fröhlich, J., Spencer, T.: Absence of diffusion in the Anderson tight binding model for large disorder or low energy. Comm. Math. Phys. **88**, 151–184 (1983) Zbl 0519.60066 MR 0696803

[GHK] Germinet, F., Hislop, P. D., Klein, A.: Localization for Schrödinger operators with Poisson random potential. J. Eur. Math. Soc. **9**, 577–607 (2007) Zbl 1214.82053 MR 2314108

[GK] Germinet, F., Klein, A.: Explicit finite volume criteria for localization in continuous random media and applications. Geom. Funct. Anal. **13**, 1201–1238 (2003) Zbl 1086.82008 MR 2033837

[GMP] Gornyi, I., Mirlin, A., Polyakov, D.: Interacting electrons in disordered wires: Anderson localization and low-T transport. Phys. Rev. Lett. **95**, art. 206603 (2005)

[GV] Graf, G. M., Vaghi, A.: A remark on the estimate of a determinant by Minami. Lett. Math. Phys. **79**, 17–22 (2007) Zbl 1104.82008 MR 2290333

[HK] Hislop, P. D., Krishna, M.: Eigenvalue statistics for random Schrödinger operators with non rank one perturbations. Comm. Math. Phys. **340**, 125–143 (2015) Zbl 1331.35342 MR 3395149

[I] Imbrie, J. Z.: Diagonalization and many-body localization for a disordered quantum spin chain. Phys. Rev. Lett. **117**, art. 027201 (2016)

[IM] Imbrie, J. Z., Mavi, R.: Level spacing for non-monotone Anderson models. J. Statist. Phys. **162**, 1451–1484 (2016) Zbl 1347.82014 MR 3463783

[JL] Jaksic, V., Last, Y.: Simplicity of singular spectrum in Anderson-type Hamiltonians. Duke Math. J. **133**, 185–204 (2006) Zbl 1107.47027 MR 2219273

[K] Kato, T.: Perturbation Theory for Linear Operators. Springer, Berlin (1995) Zbl 0836.47009 MR 1335452

[Kle] Klein, A.: Unique continuation principle for spectral projections of Schrödinger operators and optimal Wegner estimates for non-ergodic random Schrödinger operators. Comm. Math. Phys. **323**, 1229–1246 (2013) Zbl 1281.47026 MR 3106507

[KM] Klein, A., Molchanov, S.: Simplicity of eigenvalues in the Anderson model. J. Statist. Phys. **122**, 95–99 (2006) Zbl 1152.82010 MR 2203783

[Klo] Klopp, F.: Inverse tunneling estimates and applications to the study of spectral statistics of random operators on the real line. J. Reine Angew. Math. **2014**, 79–113 (2014) Zbl 1293.35337 MR 3200336

[Min] Minami, N.: Local fluctuation of the spectrum of a multidimensional Anderson tight binding model. Comm. Math. Phys. **177**, 709–725 (1996) Zbl 0851.60100 MR 1385082

[Mir] Mirlin, A. D.: Statistics of energy levels and eigenfunctions in disordered systems. Phys. Rep. **326**, 259–382 (2000) MR 1745139

[M] Molchanov, S.: The local structure of the spectrum of the one-dimensional Schrödinger operator. Comm. Math. Phys. **78**, 429–446 (1981) Zbl 0449.60049 MR 0603503

[NNS] Naboko, S., Nichols, R., Stolz, G.: Simplicity of eigenvalues in Anderson-type models. Ark. Mat. **51**, 157–183 (2013) Zbl 1269.82030 MR 3029341

[PF] Pastur, L., Figotin, A.: Spectra of Random and Almost-Periodic Operators. Springer, Berlin (1992) Zbl 0752.47002 MR 1223779

[RMV] Rojas-Molina, C., Veselić, I.: Scale-free unique continuation estimates and applications to random Schrödinger operators. Comm. Math. Phys. **320**, 245–274 (2013) Zbl 1276.47051 MR 3046996

[S1] Simon, B.: Cyclic vectors in the Anderson model. Rev. Math. Phys. **6**, 1183–1185 (1994) Zbl 0841.60081 MR 1301372

[S2] Simon, B.: Trace Ideals and Their Applications. 2nd ed., Math. Surveys Monogr. 120, Amer. Math. Soc., Providence, RI, (2005) Zbl 1074.47001 MR 2154153

[TV] Tautenhahn, M., Veselić, I.: Minami's estimate: beyond rank one perturbation and monotonicity. Ann. Henri Poincaré **15**, 737–754 (2014) Zbl 1287.81049 MR 3177919

[W] Wegner, F.: Bounds on the density of states in disordered systems. Z. Phys. B **44**, 9–15 (1981) MR 0639135