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Refined global Gross–Prasad conjecture on special Bessel periods and Böcherer's conjecture

To the memory of Joseph Shalika

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Abstract. In this paper we pursue the refined global Gross–Prasad conjecture for Bessel periods formulated by Yifeng Liu in the case of special Bessel periods for $\mathrm{SO}(2n+1)\times\mathrm{SO}(2)$. Recall that a Bessel period for $\mathrm{SO}(2n+1)\times\mathrm{SO}(2)$ is called *special* when the representation of $\mathrm{SO}(2)$ is trivial. Let π be an irreducible cuspidal tempered automorphic representation of a special orthogonal group of an odd-dimensional quadratic space over a totally real number field F whose local component π_v at any archimedean place v of F is a discrete series representation. Let E be a quadratic extension of F and suppose that the special Bessel period corresponding to E does not vanish identically on π. Then we prove the Ichino–Ikeda type explicit formula conjectured by Liu for the central value $L(1/2,\pi)L(1/2,\pi\times\chi_E)$, where χ_E denotes the quadratic character corresponding to E. Our result yields a proof of Böcherer's conjecture on holomorphic Siegel cusp forms of degree two which are Hecke eigenforms.

Keywords. Böcherer's conjecture, central L-values, Gross–Prasad conjecture, periods of automorphic forms

1. Introduction

Research on special values of arithmetic L-functions is one of the pivotal subjects in number theory. The central values are of particular interest because of the Birch and Swinnerton-Dyer conjecture and its natural generalizations.

In the early 1990s, Gross and Prasad [23, 24] proclaimed a conjecture concerning a relationship between non-vanishing of certain period integrals on special orthogonal groups and non-vanishing of central values of certain tensor product L-functions, together with the local counterpart conjecture. Recently Gan, Gross and Prasad [17] extended the conjecture to classical groups and metaplectic groups. On the other hand, Ichino and Ikeda, in their very influential paper [27], refined the Gross–Prasad conjecture and formulated

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a conjectural precise formula for the central L-value in terms of the period integral for a tempered cuspidal automorphic representation in the $\mathrm{SO}(n+1) \times \mathrm{SO}(n)$ case, i.e. for codimension 1. Inspired by [27], Harris [25] formulated a similar conjectural formula in the codimension 1 unitary group case. Recently Liu [37] extended the work of Ichino–Ikeda and Harris to Bessel periods for orthogonal and unitary groups and formulated a conjectural precise formula expressing the central L-values in terms of the Bessel periods in the arbitrary codimension case.

In our previous paper [15], we investigated the Gross–Prasad conjecture for the special Bessel periods on $\mathrm{SO}(2n+1) \times \mathrm{SO}(2)$ and proved that the non-vanishing of the period implies the non-vanishing of the corresponding central L-value. In this paper, we refine the results in [15] and prove the Ichino–Ikeda type precise L-value formula conjectured by Liu [37] in the aforementioned case. As a corollary, we also obtain a proof of the long-standing conjecture by Böcherer in [7], concerning central critical values of imaginary quadratic twists of spinor L-functions for holomorphic Siegel cusp forms of degree two which are Hecke eigenforms, thanks to the beautiful work by Dickson, Pitale, Saha and Schmidt [10].

In order to state our main results, let us introduce some notation. For the convenience of the reader, we shall use as much as possible the notation in [15], to which this paper is a sequel.

1.1. Notation

Let F be a number field and $\mathbb{A}_F$ its ring of adeles. We shall often abbreviate $\mathbb{A}_F$ as $\mathbb{A}$ for simplicity. Let ψ be a non-trivial character of $\mathbb{A}$ which is trivial on F. For $a \in F^\times$, we denote by ψ^a the character of $\mathbb{A}$ defined by $\psi^a(x) = \psi(ax)$. For a place v of F, let F_v be the completion of F at v and ψ_v the character of F_v induced by ψ. When v is non-archimedean, we denote by $\mathcal{O}_v$ and ϖ_v the ring of integers in F_v and a prime element of F_v, respectively.

Let E be a quadratic extension field of F and χ_E the quadratic character of $\mathbb{A}_F^\times / F^\times$ corresponding to E. Throughout the paper, we fix E. We simply write χ for χ_E when there is no fear of confusion.

For a positive integer $n \geq 2$, let $\mathcal{G}_n = \mathcal{G}_{n,E}$ denote a certain set of F-isomorphism classes of special orthogonal groups defined as follows. Let $(V, (\ ,\ ))$ be a quadratic space over F, i.e. a finite-dimensional vector space over F equipped with a non-degenerate symmetric bilinear form $(\ ,\ )$. We suppose that $\dim V = 2n+1$, the Witt index of V is at least $n-1$ and V has an orthogonal direct sum decomposition $V = \mathbb{H}^{n-1} \oplus L$ where $\mathbb{H}$ denotes the hyperbolic plane over F and L is a three-dimensional quadratic space containing $(E, c \cdot \mathrm{N}_{E/F})$ for some $c \in F^\times$. Then we define $\mathcal{G}_n$ as the set of F-isomorphism classes of the special orthogonal groups $\mathrm{SO}(V)$ for such V. Let $\mathrm{disc}(V)$ denote the discriminant of $(V, (\ ,\ ))$ which takes a value in $F^\times/(F^\times)^2$. We often denote the quadratic space $(V, b\cdot(\ ,\ ))$ simply as $b\,V$. We note that then $\mathrm{disc}(b\,V) = b \cdot \mathrm{disc}(V) \in F^\times/(F^\times)^2$ and $\mathrm{SO}(b\,V) = \mathrm{SO}(V)$. Thus from now on we shall assume $\mathrm{disc}(V) = (-1)^n$, i.e. $\mathrm{disc}(L) = -1$, without loss of generality. We shall often identify the group $\mathrm{SO}(V)$ with its isomorphism class in $\mathcal{G}_n$ by abuse of notation. Let us denote by $\mathbb{V} = \mathbb{V}_n$

such a quadratic space with $\dim \mathbb{V} = 2n+1$ and the Witt index n, which is uniquely determined up to scalar multiplication, and we denote by $\mathbb{G} = \mathbb{G}_n$ its special orthogonal group $\mathrm{SO}(\mathbb{V})$ (and its F-isomorphism class). We note that $\mathbb{G}$ splits over F.

Throughout the paper, for an algebraic group $\mathbf{G}$ defined over F, we write $\mathbf{G}_v$ for $\mathbf{G}(F_v)$ and we always take the measure dg on $\mathbf{G}(\mathbb{A})$ to be the Tamagawa measure, unless specified otherwise. For each v, we take the self-dual measure with respect to ψ_v on F_v. Recall that the product measure on $\mathbb{A}$ is the self-dual measure with respect to ψ and is also the Tamagawa measure since $\mathrm{Vol}(\mathbb{A}/F) = 1$. For a unipotent algebraic group $\mathbf{U}$ defined over F, we also specify the local measure du_v on $\mathbf{U}_v$ to be the measure corresponding to the gauge form defined over F, together with our choice of the measure on F_v, at each place v of F. Then for $du = \prod_v du_v$, we have $\mathrm{Vol}(\mathbf{U}(F)\backslash\mathbf{U}(\mathbb{A}), du) = 1$ and du is the Tamagawa measure on $\mathbf{U}(\mathbb{A})$.

1.2. Special Bessel periods

Let $G = \mathrm{SO}(V) \in \mathcal{G}$. First we decompose V as a direct sum $V = X^+ \oplus L \oplus X^-$ where $X^\pm$ are totally isotropic $(n-1)$-dimensional subspaces of V which are dual to each other and orthogonal to L. When $G = \mathbb{G}$, i.e. $V = \mathbb{V}$, we extend X^+ to V^+ and X^- to V^- respectively so that $V^\pm$ are totally isotropic n-dimensional subspaces of $\mathbb{V}$ which are dual to each other. We take a basis $\{e_1, \dots, e_{n-1}\}$ of X^+ and a basis $\{e_{-1}, \dots, e_{-n+1}\}$ of X^- respectively so that

$$(e_i, e_{-j}) = \delta_{i,j} \quad \text{for } 1 \le i, j \le n-1, \tag{1.1}$$

where $\delta_{i,j}$ denotes Kronecker's delta. When $V = \mathbb{V}$, we take $e_n \in V^+$ and $e_{-n} \in V^-$ respectively so that (1.1) holds for $1 \le i, j \le n$. We also fix a basis of L. When $V = \mathbb{V}$, we take it to be of the form $\{e_{-n}, e, e_n\}$ where e is a vector in L orthogonal to e_{-n} and e_n with $(e, e) = 1$. Then for a matrix representation of elements of G, as a basis of V, we employ

$$e_{-1}, \dots, e_{-n+1}, \text{ basis of } L, \ e_{n-1}, \dots, e_1.$$

We denote by P' the maximal parabolic subgroup of G defined as the stabilizer of the isotropic subspace X^-. Let

$$P' = M'S' \tag{1.2}$$

be the Levi decomposition where M' and S' denote the Levi part and the unipotent part of P' respectively. Let us take $\lambda \in F^\times$ so that $E = F(\sqrt{\lambda})$. Since L contains the quadratic space $(E, c \cdot \mathrm{N}_{E/F})$ and $\mathrm{disc}(L) = -1$, we may take $e_\lambda \in L(F)$ such that $(e_\lambda, e_\lambda) = \lambda$ and we fix it once and for all. Then there is a homomorphism from S' to $\mathbb{G}_a$ defined by

$$\begin{pmatrix} 1_{n-1} & A & B \\ 0 & 1_3 & A' \\ 0 & 0 & 1_{n-1} \end{pmatrix} \mapsto (Ae_\lambda, e_{n-1}),$$

where we regard A as an element of $\mathrm{Hom}(L, X^-)$ and $(\ ,\ )$ is the symmetric bilinear form on V, and its stabilizer in the Levi component M' is given by

$$\left\{ \begin{pmatrix} p & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & p^* \end{pmatrix} : p \in \mathcal{P}_{n-1},\ h \in \mathrm{SO}(L),\ he_\lambda = e_\lambda \right\}$$

where $\mathcal{P}_{n-1}$ denotes the mirabolic subgroup of GL_{n-1}, i.e.

$$\mathcal{P}_{n-1} = \left\{ \begin{pmatrix} \alpha & u \\ 0 & 1 \end{pmatrix} : \alpha \in \mathrm{GL}_{n-2},\ u \in \mathbb{G}_a^{n-2} \right\},$$

and $p^* = J_{n-1}{}^t p^{-1} J_{n-1}$. Here J_r denotes the $r \times r$ matrix with ones on the sinister diagonal, zeros elsewhere. Let U_{n-1} denote the group of upper unipotent matrices in GL_{n-1}. We define $\check{u} \in M'$ for $u \in U_{n-1}$ by

$$\check{u} = \begin{pmatrix} u & 0 & 0 \\ 0 & 1_3 & 0 \\ 0 & 0 & u^* \end{pmatrix} \tag{1.3}$$

and let S be a unipotent subgroup of P' defined by

$$S := S'S'' \quad \text{where} \quad S'' = \{\check{u} : u \in U_{n-1}\}. \tag{1.4}$$

Let us define a subgroup D_λ of M' by

$$D_\lambda := \left\{ \begin{pmatrix} 1_{n-1} & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & 1_{n-1} \end{pmatrix} : h \in \mathrm{SO}(L),\ he_\lambda = e_\lambda \right\} \simeq \mathrm{SO}(E) \simeq E^\times / F^\times.$$

Definition 1. The Bessel subgroup R_λ of G is defined by

$$R_\lambda := D_\lambda\, S$$

and we define a character χ_λ of $R_\lambda(\mathbb{A})$ by setting $\chi_\lambda(t) := 1$ for $t \in D_\lambda(\mathbb{A})$ and

$$\chi_\lambda(s'\check{u}) = \psi((Ae_\lambda, e_{n-1}))\psi(u_{1,2} + \cdots + u_{n-2,n-1}) \tag{1.5}$$

for

$$s' = \begin{pmatrix} 1_{n-1} & A & B \\ 0 & 1_3 & A' \\ 0 & 0 & 1_{n-1} \end{pmatrix} \in S'(\mathbb{A}) \quad \text{and} \quad u = (u_{i,j}) \in U_{n-1}(\mathbb{A}).$$

Then for an automorphic form ϕ on $G(\mathbb{A})$, its special Bessel period of type E is defined by

$$B_{\lambda,\psi}(\phi) = \int_{R_\lambda(F)\backslash R_\lambda(\mathbb{A})} \phi(r)\chi_\lambda(r)^{-1}\, dr.$$

We refer to [15, (5)] for the dependency of $B_{\lambda,\psi}$ on the choice of λ and e_λ.

1.3. Refined Gan–Gross–Prasad conjecture by Liu in our case

Let π be an irreducible *tempered* cuspidal automorphic representation of $G(\mathbb{A})$ for $G \in \mathcal{G}$ and V_π its space of automorphic forms.

Let $\langle\ ,\ \rangle$ denote the $G(\mathbb{A})$-invariant Hermitian inner product on V_π given by the Petersson inner product, i.e.

$$\langle \phi_1, \phi_2 \rangle = \int_{G(F)\backslash G(\mathbb{A})} \phi_1(g)\,\overline{\phi_2(g)}\,dg \quad \text{for } \phi_1, \phi_2 \in V_\pi.$$

Since $\pi = \otimes_v \pi_v$ where π_v is unitary, we may also choose a G_v-invariant Hermitian inner product $\langle\ ,\ \rangle_v$ on the space V_{π_v} of π_v for each place v so that

$$\langle \phi_1, \phi_2 \rangle = \prod_v \langle \phi_{1,v}, \phi_{2,v} \rangle_v$$

for any decomposable vectors $\phi_1 = \otimes_v \phi_{1,v}$ and $\phi_2 = \otimes_v \phi_{2,v} \in V_\pi$.

We choose a local Haar measure dg_v on G_v for each place v of F so that $\mathrm{Vol}(K_{G,v}, dg_v) = 1$ at almost all v, where $K_{G,v}$ is a maximal compact subgroup of G_v. Let us also choose a local Haar measure dt_v on $D_{\lambda,v} = D_\lambda(F_v)$ at each place v of F so that $\mathrm{Vol}(K_{\lambda,v}, dt_v) = 1$ at almost all v, where $K_{\lambda,v}$ is a maximal compact subgroup of $D_{\lambda,v}$. We define positive constants C_G and C_λ, called Haar measure constants in [27], by

$$dg = C_G \cdot \prod_v dg_v \quad \text{and} \quad dt = C_\lambda \cdot \prod_v dt_v \tag{1.6}$$

respectively. Here we recall that dg and dt are the Tamagawa measures on $G(\mathbb{A})$ and $D_\lambda(\mathbb{A})$, respectively.

1.3.1. Local integral. At each place v of F, a local integral $\alpha_v(\phi_v, \phi_v')$ for $\phi_v, \phi_v' \in V_{\pi_v}$ is defined as follows.

First suppose that v is non-archimedean. Let us first recall the definition of a *stable integral* by Lapid and Mao [33, Definition 2.1, Remark 2.2].

Definition 2. Let U be a unipotent group over F_v and f a locally constant function on U. We say that f *has a stable integral over* U if there exists a compact open subgroup N of U such that for any compact open subgroup N' of U containing N we have

$$\int_{N'} f(u)\,du = \int_N f(u)\,du.$$

Then we denote this common value by $\int_U^{\mathrm{st}} f(u)\,du$ and say that the integral *stabilizes at* N.

Remark 1. Note that if $f \in L^1(U)$ and f has a stable integral over U, then we have

$$\int_U f(u)\,du = \int_U^{\mathrm{st}} f(u)\,du.$$

Definition 3. For a non-archimedean place v, we define $\alpha_v(\phi_v, \phi_v')$ for $\phi_v, \phi_v' \in V_{\pi_v}$ by

$$\alpha_v(\phi_v, \phi_v') := \int_{D_{\lambda,v}} \int_{S_v}^{\mathrm{st}} \langle \pi_v(s_v t_v)\phi_v, \phi_v' \rangle_v\, \chi_\lambda(s_v)^{-1}\,ds_v\,dt_v. \tag{1.7}$$

Indeed, it is shown in Liu [37] that for any $t_v \in D_{\lambda,v}$ the inner integral of (1.7) stabilizes at a certain open compact subgroup of S_v [37, Proposition 3.1] and the outer integral of (1.7) converges [37, Theorem 2.1]. We note that the well-definedness of (1.7) is also shown in Waldspurger [48, Section 5.1, Lemme].

Now suppose that v is archimedean.

Definition 4. For an archimedean place v, we define $\alpha_v(\phi_v, \phi'_v)$ by a regularized integral whose regularization is achieved using the Fourier transform. We refer to Liu [37, 3.4] for the details.

Remark 2. It is shown in Liu [37, Proposition 3.5] that for *any* place v where π_v is square integrable, the local integral

$$\int_{D_{\lambda,v}} \int_{S_v} \langle \pi_v(s_v t_v)\phi_v, \phi'_v \rangle_v \, \chi_\lambda(s_v)^{-1} \, ds_v \, dt_v \tag{1.8}$$

does converge absolutely and is equal to $\alpha_v(\phi_v, \phi'_v)$ defined as above. We note that later we are only concerned with the case when π_v is a discrete series representation at any archimedean place v.

We recall that the multiplicity one property, i.e.

$$\dim_{\mathbb{C}} \mathrm{Hom}_{R_{\lambda,v}}(\pi_v, \chi_{\lambda,v}) \leq 1, \tag{1.9}$$

holds at *any* place v. For the proof, we refer to Gan, Gross and Prasad [17, Corollary 15.3] and Jiang, Sun and Zhu [29, Theorem A] for the non-archimedean case and the archimedean case, respectively.

Moreover when v is non-archimedean, it is shown that

$$\begin{aligned} &\dim_{\mathbb{C}} \mathrm{Hom}_{R_{\lambda,v}}(\pi_v, \chi_{\lambda,v}) = 1 \\ &\qquad \Longleftrightarrow \ \alpha_v(\phi_v, \phi'_v) \neq 0 \text{ for some } \phi_v, \ \phi'_v \in V_v \text{ which are } K_{G,v}\text{-finite}, \end{aligned} \tag{1.10}$$

by Waldspurger [48, Proposition 5.7]. It is expected that the equivalence (1.10) holds also when v is archimedean. Indeed, Beuzart–Plessis [5] proved the corresponding assertion in the unitary group case for tempered representations. We also note that the condition on the right hand side of (1.10) is equivalent to

$$\alpha_v(\phi_v, \phi_v) \neq 0 \text{ for some } K_{G,v}\text{-finite vector } \phi_v \in V_v. \tag{1.11}$$

Indeed, $\alpha_v(\phi_v, \phi'_v) \neq 0$ implies that the two linear forms L and L' on V_{π_v} defined by $L(\varphi_v) = \alpha_v(\varphi_v, \phi'_v)$ and $L'(\varphi_v) = \overline{\alpha_v(\phi_v, \varphi_v)}$ for $\varphi_v \in V_{\pi_v}$, respectively, are non-zero elements of $\mathrm{Hom}_{R_{\lambda,v}}(\pi_v, \chi_{\lambda,v})$. By (1.9) there exists $c \in \mathbb{C}^\times$ such that $L' = c \cdot L$. Thus $L'(\phi_v) = c \cdot L(\phi_v) = c \cdot \alpha_v(\phi_v, \phi'_v) \neq 0$ and hence $\alpha_v(\phi_v, \phi_v) \neq 0$.

1.3.2. Normalization of local integrals. We fix maximal compact subgroups $K_G = \prod_v K_{G,v}$ of $G(\mathbb{A})$ and $K_\lambda = \prod_v K_{\lambda,v}$ of $D_\lambda(\mathbb{A})$.

A place v is called *good* (with respect to π and a decomposable vector $\phi = \otimes_v \phi_v \in V_\pi = \otimes_v V_{\pi_v}$) if:

$$v \text{ is non-archimedean and is not lying over 2;} \tag{1.12a}$$
$$K_{G,v} \text{ is a hyperspecial maximal compact subgroup of } G_v; \tag{1.12b}$$
$$E_v \text{ is an unramified quadratic extension of } F_v \text{ or } E_v = F_v \oplus F_v; \tag{1.12c}$$
$$\pi_v \text{ is an unramified representation of } G_v; \tag{1.12d}$$
$$\phi_v \text{ is a } K_{G,v}\text{-fixed vector such that } \langle \phi_v, \phi_v \rangle_v = 1 \text{ and } \chi_{\lambda,v} \text{ is } K_{\lambda,v}\text{-fixed;} \tag{1.12e}$$
$$K_{\lambda,v} \subset K_{G,v} \text{ and } \mathrm{Vol}(K_{G,v}, dg_v) = \mathrm{Vol}(K_{\lambda,v}, dt_v) = 1. \tag{1.12f}$$

Then Liu's theorem [37, Theorem 2.2] states that when v is good, one has

$$\alpha_v(\phi_v, \phi_v) = \frac{L(1/2, \pi_v) L(1/2, \pi_v \times \chi_{E,v}) \prod_{j=1}^{n} \zeta_{F_v}(2j)}{L(1, \pi_v, \mathrm{Ad}) L(1, \chi_{E,v})}. \tag{1.13}$$

Definition 5. We define the normalized local integral $\alpha_v^\natural(\phi_v, \phi_v')$ at each place v of F by

$$\alpha_v^\natural(\phi_v, \phi_v') := \frac{L(1, \pi_v, \mathrm{Ad}) L(1, \chi_{E,v})}{L(1/2, \pi_v) L(1/2, \pi_v \times \chi_{E,v}) \prod_{j=1}^{n} \zeta_{F_v}(2j)} \cdot \alpha_v(\phi_v, \phi_v'). \tag{1.14}$$

We shall often use the notation

$$\alpha_v(\phi_v) := \alpha_v(\phi_v, \phi_v) \quad \text{and} \quad \alpha_v^\natural(\phi_v) := \alpha_v^\natural(\phi_v, \phi_v). \tag{1.15}$$

Remark 3. Recall that $\zeta_{\mathbb{R}}(s) = \pi^{-s/2}\,\Gamma(s/2)$ and $\zeta_{\mathbb{C}}(s) = (2\pi)^{1-s}\,\Gamma(s)$. Here we note that $L(s, \pi)$ and $L(s, \pi \times \chi_E)$ are defined by the doubling method as in Lapid and Rallis [36] and are holomorphic for $\mathrm{Re}(s) > 0$ by Yamana [52] since π is tempered. It is believed that $L(s, \pi, \mathrm{Ad})$ can be analytically continued to the whole s-plane, is holomorphic for $\mathrm{Re}(s) > 0$ and $L(1, \pi, \mathrm{Ad})$ is non-zero when π is tempered.

1.3.3. Refined global Gross–Prasad conjecture on $B_{\lambda,\psi}$. As in Ichino and Ikeda [27], we say that $\pi = \otimes_v \pi_v$ is *almost locally generic* if the local representation π_v is generic at almost all places v of F. Then as explained in [27, Section 2], such a π is conjectured to come from an elliptic Arthur parameter

$$\Psi(\pi) : \mathcal{L}_F \to {}^L G := \hat{G} \rtimes W_F.$$

Here $\mathcal{L}_F$ denotes the conjectural Langlands group of F and ${}^L G$ is the Langlands dual group of G. The local representation π_v is expected to be tempered at every v by the generalized Ramanujan conjecture. Let $\mathcal{S}(\Psi(\pi))$ be the centralizer of the image of the Arthur parameter $\Psi(\pi)$ in the complex dual group $\hat{G}$. For $G \in \mathcal{G}$, $\mathcal{S}(\Psi(\pi))$ is a finite elementary 2-group. We refer to [27, 2.5] for the details.

The conjecture formulated by Liu [37, Conjecture 2.5] reads as follows in our case.

Conjecture. *Let $\pi = \otimes_v \pi_v$ be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$ for $G \in \mathcal{G}$. Suppose that π is almost locally generic.*

(1) *We have* $\dim_{\mathbb{C}} \mathrm{Hom}_{R_{\lambda,v}}(\pi_v, \chi_{\lambda,v}) = 1$ *if and only if* $\alpha_v(\phi_v') \neq 0$ *for some $K_{G,v}$-finite vector $\phi_v' \in V_{\pi_v}$.*

(2) *For any non-zero decomposable cusp form $\phi = \otimes_v \phi_v \in V_\pi$, we have*

$$\frac{|B_{\lambda,\psi}(\phi)|^2}{\langle \phi, \phi \rangle} = \frac{C_\lambda}{|\mathcal{S}(\Psi(\pi))|} \cdot \Big(\prod_{j=1}^{n} \zeta_F(2j) \Big) \times \frac{L(1/2, \pi) L(1/2, \pi \times \chi_E)}{L(1, \pi, \mathrm{Ad}) L(1, \chi_E)} \cdot \prod_v \frac{\alpha_v^\natural(\phi_v)}{\langle \phi_v, \phi_v \rangle_v} \tag{1.16}$$

where the product is indeed over the finite set of places v of F which are not good in the sense of (1.12)*. Here all L-functions in* (1.16) *denote the completed L-functions. In particular $\zeta_F(s)$ denotes the completed Dedekind zeta function of F, i.e.*

$$\zeta_F(s) = \prod_{v:\, \text{place of } F} \zeta_{F_v}(s). \tag{1.17}$$

Remark 4. When $n = 2$ and $G = \mathbb{G}$, Liu [37], inspired by Prasad and Takloo-Bighash [40], proved (1.16) for endoscopic Yoshida lifts, and Corbett [9] recently proved it for non-endoscopic Yoshida lifts. We mention that Qiu [41] considered a non-tempered case when $n = 2$, namely the Saito–Kurokawa lifting case. We also mention that Murase and Narita [39] proved an explicit formula for the central L-values in terms of the Bessel periods for Arakawa lifts when $n = 2$ and G is not split.

1.4. Main Theorem

We say that an irreducible cuspidal tempered automorphic representation $\pi = \otimes_v \pi_v$ of $G(\mathbb{A})$ for $G \in \mathcal{G}$ has a *weak lift* to $\mathrm{GL}_{2n}(\mathbb{A})$ if there exists an irreducible automorphic representation $\Pi = \otimes_v \Pi_v$ of $\mathrm{GL}_{2n}(\mathbb{A})$ such that Π_v is a local Langlands lift of π_v at almost all non-archimedean places and all archimedean places. If such a Π exists, it is unique by the classification theorem of Jacquet and Shalika [28, (4.4)] and is written as an isobaric sum

$$\Pi = \boxplus_{i=1}^{l} \pi_i \tag{1.18}$$

where π_i is an irreducible cuspidal automorphic representation of $\mathrm{GL}_{2n_i}(\mathbb{A})$ such that

$$L(s, \pi_i, \wedge^2) \text{ has a pole at } s = 1, \qquad \sum_{i=1}^{l} n_i = n, \qquad \pi_i \not\simeq \pi_j \text{ for } i \neq j.$$

When $G = \mathbb{G}$, the existence of a weak lift is guaranteed by Arthur [3, Theorem 1.5.2].

Our aim in this paper is to prove the following theorem.

Theorem 1. *Let F be a totally real number field and $\pi = \otimes_v \pi_v$ an irreducible cuspidal tempered automorphic representation of $G(\mathbb{A})$ for $G \in \mathcal{G}$ such that π_v is a discrete series representation at any archimedean place v of F.*

Suppose that the special Bessel period $B_{\lambda,\psi}$ of type E does not vanish identically on the space of cusp forms V_π for π. Let Π be a weak lift of π to $\mathrm{GL}_{2n}(\mathbb{A})$, which is written of the form (1.18).

Then the following assertions hold.

(1) *At each place v, there exists a $K_{G,v}$-finite vector $\phi_v' \in V_{\pi_v}$ such that $\alpha_v(\phi_v') \neq 0$.*
(2) *For any non-zero decomposable cusp form $\phi = \otimes_v \phi_v \in V_\pi$, we have*

$$\frac{|B_{\lambda,\psi}(\phi)|^2}{\langle \phi, \phi \rangle} = 2^{-l}\, C_\lambda \cdot \Big(\prod_{j=1}^{n} \zeta_F(2j) \Big) \times \frac{L(1/2,\pi) L(1/2, \pi \times \chi_E)}{L(1,\pi,\mathrm{Ad}) L(1,\chi_E)} \cdot \prod_v \frac{\alpha_v^\natural(\phi_v)}{\langle \phi_v, \phi_v \rangle_v}. \tag{1.19}$$

Here $L(s,\pi,\mathrm{Ad})$ is defined as $L(s,\pi,\mathrm{Ad}) = \prod_v L(s,\pi_v,\mathrm{Ad})$ where

$$L(s,\pi_v,\mathrm{Ad}) := L(s,\Pi_v,\mathrm{Sym}^2) \tag{1.20}$$

for each place v and Sym^2 denotes the symmetric square representation of $\mathrm{GL}_{2n}(\mathbb{C})$.

Remark 5. The existence of a weak lift Π readily follows from our previous paper [15, Theorem 1], as explained in the beginning of 2.1.

Remark 6. When π has a weak lift Π to $\mathrm{GL}_{2n}(\mathbb{A})$ of the form (1.18), it is clear from the definition of the Arthur parameter that $2^l = |\mathcal{S}(\Psi(\pi))|$.

Remark 7. Suppose that π_v is unramified. Then we may define $L(s,\pi_v,\mathrm{Ad})$ in terms of the Satake parameter of π_v. This coincides with the one defined by (1.20).

The following corollary is proved in Section 4.

Corollary 1. *Keep the assumption in Theorem 1 except for $B_{\lambda,\psi} \not\equiv 0$ on V_π. If we assume that Arthur's conjectures* [3, *Conjecture* 9.4.2, *Conjecture* 9.5.4] *hold for any $G' \in \mathcal{G}$, the equality* (1.19) *holds for any non-zero decomposable cusp form $\phi = \otimes_v \phi_v \in V_\pi$.*

We refer to Ichino and Ikeda [27, 2.5] for the relevance of Arthur's conjectures to the Gross–Prasad conjecture.

For the sake of the reader, here we explain the skeleton of our proof of (1.19). As in our previous paper [15], the theta correspondence between $G \in \mathcal{G}_n$ and $\widetilde{\mathrm{Sp}}_n$, i.e. rank n metaplectic group, plays a pivotal role.

Suppose that $B_{\lambda,\psi}(\phi) \neq 0$ for $\phi \in V_\pi$. The computation in [11] of the pull-back of the ψ_λ-Whittaker period $W(\theta_\psi^\varphi(\phi); \psi_\lambda)$, which is defined by (2.8), of $\theta_\psi^\varphi(\phi)$, the theta lift of ϕ to $\widetilde{\mathrm{Sp}}_n(\mathbb{A})$ with respect to the additive character ψ and the test function φ, yields

$$B_{\lambda,\psi}(\phi) \underset{\text{a.a.}}{=} C_G^{-1} C_\lambda \cdot W(\theta_\psi^\varphi(\phi); \psi_\lambda). \tag{1.21}$$

Here we use the symbol "$\underset{\text{a.a.}}{=}$" to imply that the two sides are equal up to multiplication by a product of finitely many local factors. Then the remarkable formula obtained by Lapid and Mao [35] implies that we have

$$\frac{|W(\theta_\psi^\varphi(\phi);\psi_\lambda)|^2}{(\theta_\psi^\varphi(\phi),\theta_\psi^\varphi(\phi))} \underset{\text{a.a.}}{=} 2^{-l}\cdot\frac{L(1/2,\pi\times\chi_E)\prod_{j=1}^n\zeta_F(2j)}{L(1,\pi,\mathrm{Ad})} \tag{1.22}$$

where $(\theta_\psi^\varphi(\phi),\theta_\psi^\varphi(\phi))$ is the square of the Petersson norm of $\theta_\psi^\varphi(\phi)$. On the other hand, some proper adjustments to the proof of the *precise Rallis inner product formula* in Gan and Takeda [19] yield one in our case, namely

$$\frac{(\theta_\psi^\varphi(\phi),\theta_\psi^\varphi(\phi))}{\langle\phi,\phi\rangle} \underset{\text{a.a.}}{=} C_G\cdot\frac{L(1/2,\pi)}{\prod_{j=1}^n\zeta_F(2j)}. \tag{1.23}$$

Since we have $C_G\cdot\prod_{j=1}^n\zeta_F(2j)=C_\lambda\cdot L(1,\chi_E)$, the combination of (1.21), (1.22) and (1.23) yields

$$\frac{|B_{\lambda,\psi}(\phi)|^2}{\langle\phi,\phi\rangle} \underset{\text{a.a.}}{=} 2^{-l}\cdot C_\lambda\cdot\frac{L(1/2,\pi)L(1/2,\pi\times\chi_E)\prod_{j=1}^n\zeta_F(2j)}{L(1,\pi,\mathrm{Ad})L(1,\chi_E)}. \tag{1.24}$$

Thus our task is to elaborate (1.24) to the precise equality (1.19) by executing the above idea rigorously and proving a certain local equality by some intricate arguments. It is done in Section 3, which is the heart of the matter of this paper.

1.5. Böcherer's conjecture

By considering the case when $n=2$, $F=\mathbb{Q}$ and $G=\mathbb{G}_2\simeq\mathrm{PGSp}_2$, Theorem 1 yields a proof of the long-standing conjecture by Böcherer [7] concerning central critical values of imaginary quadratic twists of spinor L-functions for holomorphic Siegel cusp forms of degree two which are Hecke eigenforms, thanks to the recent work of Dickson, Pitale, Saha and Schmidt [10]. Namely Böcherer's conjecture holds in the following refined form.

Theorem 2. *Let Φ be a holomorphic Siegel cusp form of degree two and weight k with respect to $\mathrm{Sp}_2(\mathbb{Z})$ which is a Hecke eigenform and $\pi(\Phi)$ the associated automorphic representation of $\mathrm{PGSp}_2(\mathbb{A})\simeq\mathbb{G}_2(\mathbb{A})$. Suppose that Φ is not a Saito–Kurokawa lift. Let*

$$\Phi(Z)=\sum_{T>0}a(T,\Phi)\exp[2\pi\sqrt{-1}\,\mathrm{tr}(TZ)],\qquad Z\in\mathfrak{H}_2,$$

be the Fourier expansion where T runs over semi-integral positive definite two by two symmetric matrices and $\mathfrak{H}_2$ denotes the Siegel upper half space of degree two.

For an imaginary quadratic field E with discriminant $-D_E$, define

$$B(\Phi;E):=w(E)^{-1}\sum_{\{T:\det T=D_E/4\}/\sim}a(T,\Phi)$$

where $\sim$ *denotes the equivalence relation defined by* $T_1 \sim T_2$ *if there exists an element* γ *of* $\mathrm{SL}_2(\mathbb{Z})$ *such that* ${}^t\gamma T_1 \gamma = T_2$ *and* $w(E)$ *is the number of roots of unity in* E. *We recall that when* $\det T = D_E/4$, *the number of elements in* $\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : {}^t\gamma T \gamma = T\}$ *is equal to* $w(E)$.

Then

$$\frac{|B(\Phi; E)|^2}{\langle \Phi, \Phi \rangle} = 2^{2k-4} \cdot D_E^{k-1} \cdot \frac{L(1/2, \pi(\Phi))L(1/2, \pi(\Phi) \times \chi_E)}{L(1, \pi(\Phi), \mathrm{Ad})}. \tag{1.25}$$

Here

$$\langle \Phi, \Phi \rangle = \int_{\mathrm{Sp}_2(\mathbb{Z})\backslash \mathfrak{H}_2} |\Phi(Z)|^2 \det(Y)^{k-3}\, dX\, dY \quad \textit{where} \quad Z = X + \sqrt{-1}\, Y.$$

Proof. First we note that $\pi(\Phi)$ is tempered by Weissauer [51] since Φ is not a Saito–Kurokawa lift, as explained in the proof of [15, Theorem 4].

Recall that it is shown in our previous paper [15, Theorem 5] that

$$B(\Phi; E) \neq 0 \iff L(1/2, \pi(\Phi))L(1/2, \pi(\Phi) \times \chi_E) \neq 0. \tag{1.26}$$

When k is odd, we have $L(1/2, \pi(\Phi)) = 0$ by Andrianov [2, Theorem 3.1.1(II)]. Hence (1.25) holds by (1.26). We mention that $B(\Phi; E) = 0$ also follows in a more elementary way from ${}^t\gamma\{T : \det T = D_E/4\}\gamma = \{T : \det T = D_E/4\}$ and $a({}^t\gamma T \gamma, \Phi) = (\det \gamma)^k \cdot a(T, \Phi)$ for $\gamma \in \mathrm{GL}_2(\mathbb{Z})$, as remarked in Böcherer [7, p. 31].

Suppose that k is even. If $B(\Phi; E) = 0$, (1.25) holds by (1.26). If $B(\Phi; E) \neq 0$, (1.25) follows from (1.19) by Dickson et al. [10, 1.13 Theorem]. □

Remark 8. In [7], Böcherer conjectured that there exists a constant c_Φ which depends only on Φ such that we have

$$|B(\Phi; E)|^2 = c_\Phi \cdot D_E^{k-1} \cdot L(1/2, \pi(\Phi) \times \chi_E)$$

for any imaginary quadratic field E. As far as we know, Böcherer did not speculate on the constant c_Φ except when Φ is a Saito–Kurokawa lift. For the exact formula for the left hand side of (1.25) when Φ is a Saito–Kurokawa lift, we refer to Dickson et al. [10, 3.13 Theorem].

Remark 9. For brevity, only the full modular case is stated in Theorem 2. In fact, Theorem 1 yields [10, 1.13 Theorem] unconditionally when $\Lambda = 1$, i.e. the refined form of Böcherer's conjecture has an extension to the odd square free level case.

We expect Theorem 2 and its extension to have a broad spectrum of interesting consequences, e.g. [10, Section 3.6]. We also mention Blomer [6] and Kowalski, Saha and Tsimerman [31]. It is expected that the extension of Theorem 2 holds also in the case when $k = 2$, which we wish to consider in the near future.

Remark 10. In our next paper, we shall pursue the generalization of Theorem 2 and its extension to the case when the character of the ideal class group is not necessarily trivial. We shall also pursue the case when the Siegel cusp form in question is vector valued.

1.6. Organization of the paper

This paper is organized as follows. In Section 2, first we review the precise Rallis inner product formula by Gan and Takeda [19] and the explicit formula for the Whittaker periods on the metaplectic group by Lapid and Mao [35]. Both formulas play decisive roles as explained in Section 1.4. After the review, we turn to the proof of Theorem 1. By combining these two formulas with the pull-back formula for the Whittaker periods on the metaplectic group in [11], the proof of Theorem 1 is reduced to verifying a certain local equality, which we prove in Section 3. Then in Section 4 we deduce Corollary 1 from Theorem 1, assuming Arthur's conjectures.

2. Reduction to a local equality

2.1. Set up

Throughout this section, $\pi = \otimes_v \pi_v$ is an irreducible cuspidal tempered automorphic representation of $G(\mathbb{A})$ for $G = \mathrm{SO}(V) \in \mathcal{G}_n$ over a totally real number field F such that

$$\pi_v \text{ is a discrete series representation at any real place } v; \tag{2.1a}$$

$$B_{\lambda,\psi}, \text{ the Bessel model of type } E, \text{ does not vanish identically on } V_\pi. \tag{2.1b}$$

Since π is tempered, by Remark 2 in [15], Theorem 1 in [15] and the arguments in the course of its proof are all applicable to π. Hence we have

$$L(1/2, \pi) \cdot L(1/2, \pi \times \chi_E) \neq 0 \tag{2.1c}$$

and there exists a globally generic irreducible cuspidal automorphic representation π° of $\mathbb{G}(\mathbb{A})$ which is nearly equivalent to π. Thus

$$\pi \text{ is almost locally generic.} \tag{2.1d}$$

We note that when $n = 2$, $F = \mathbb{Q}$ and $G = \mathbb{G}$, the existence of such π° also follows from Weissauer [50].

Since π° has a weak lift to $\mathrm{GL}_{2n}(\mathbb{A})$ by Arthur [3, Theorem 1.5.2], we may say

$$\pi \text{ has a weak lift } \Pi \text{ to } \mathrm{GL}_{2n}(\mathbb{A}) \text{ of the form (1.18).} \tag{2.1e}$$

We note that the existence of a weak lift of π° to $\mathrm{GL}_{2n}(\mathbb{A})$ also follows from Cogdell, Kim, Piatetski-Shapiro and Shahidi [8] since π° is generic.

For a positive integer n, let Y_n be the space of $2n$-dimensional row vectors equipped with the alternating form

$$\langle w_1, w_2 \rangle = w_1 \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} {}^t w_2 \quad \text{for } w_1, w_2 \in Y_n.$$

Let $Y_n = Y_n^+ \oplus Y_n^-$ be the polarization where

$$Y_n^+ := \{(y_1, \ldots, y_{2n}) : y_i = 0 \, (1 \leq i \leq n)\}$$

and

$$Y_n^- := \{(y_1, \dots, y_{2n}) : y_i = 0\,(n+1 \le i \le 2n)\}.$$

Let Sp_n denote the rank n symplectic group defined by

$$\mathrm{Sp}_n := \left\{ g \in \mathrm{GL}_{2n} : g \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} {}^t g = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} \right\},$$

which acts on Y_n from the right. We recall that $\widetilde{\mathrm{Sp}}_n(\mathbb{A})$, the rank n metaplectic group over $\mathbb{A}$, is a certain twofold central extension of $\mathrm{Sp}_n(\mathbb{A})$. The theta correspondence of automorphic forms between $\widetilde{\mathrm{Sp}}_n(\mathbb{A})$ and $G(\mathbb{A})$ for $G \in \mathcal{G}_n$ plays the essential role as in our previous paper [15].

Let us realize $\omega_\psi = \omega_{\psi, V, Y_n}$, the Weil representation of $G(\mathbb{A}) \times \widetilde{\mathrm{Sp}}_n(\mathbb{A})$ with respect to ψ, on $\mathcal{S}((V \otimes Y_n^+)(\mathbb{A}))$, the Schwartz–Bruhat space on $(V \otimes Y_n^+)(\mathbb{A})$, by taking $V \otimes Y_n = (V \otimes Y_n^+) \oplus (V \otimes Y_n^-)$ as a polarization of the symplectic space $V \otimes Y_n$. For $\phi \in V_\pi$ and $\varphi \in \mathcal{S}((V \otimes Y_n^+)(\mathbb{A}))$, the theta lift $\theta_\psi^\varphi(\phi)$ of ϕ to $\widetilde{\mathrm{Sp}}_n(\mathbb{A})$ with respect to the additive character ψ and the test function φ is defined by

$$\theta_\psi^\varphi(\phi)(h) := \int_{G(F)\backslash G(\mathbb{A})} \Big(\sum_{z \in (V \otimes Y_n^+)(F)} (\omega_\psi(g,h)\varphi)(z) \Big) \phi(g)\, dg \quad \text{for } h \in \widetilde{\mathrm{Sp}}_n(\mathbb{A}).$$

Let $\Theta_n(\pi, \psi)$ denote the automorphic representation of $\widetilde{\mathrm{Sp}}_n(\mathbb{A})$ generated by $\theta_\psi^\varphi(\phi)$ where ϕ and φ vary in V_π and $\mathcal{S}((V \otimes Y_n^+)(\mathbb{A}))$, respectively. For the sake of simplicity, we shall write σ for $\Theta_n(\pi, \psi)$ and V_σ for its space of automorphic forms.

Then by the proof of Theorem 1 in [15], we have

$$\sigma = \Theta_n(\pi, \psi) \text{ is } \psi_\lambda\text{-generic, irreducible and cuspidal.} \tag{2.1f}$$

We refer to Section 2.3 for the definition of ψ_λ-genericity.

2.2. *Rallis inner product formula*

Gan and Takeda [19] proved the precise Rallis inner product formula in the $(\mathrm{O}_{2n+1}, \widetilde{\mathrm{Sp}}_n)$ setting. On the other hand, the one we need for our purpose is in the $(\mathrm{SO}_{2n+1}, \widetilde{\mathrm{Sp}}_n)$ setting. Here we recall the Rallis inner product formula, with some explanations of adjustments to the proof in [19] necessary to deduce the one in our setting.

Let $\mathbb{W}$ be the quadratic space $V \oplus (-V)$, i.e. as a vector space $\mathbb{W}$ is a direct sum $V \oplus V$ and its symmetric bilinear form $(\ ,\ )_\mathbb{W}$ on $\mathbb{W}$ is defined by

$$(v_1 \oplus v_2, v_1' \oplus v_2')_\mathbb{W} := (v_1, v_1')_V - (v_2, v_2')_V.$$

Let $\mathbb{W}^+$ be a maximal isotropic subspace of $\mathbb{W}$ defined by

$$\mathbb{W}^+ := \{v \oplus v \in \mathbb{W} : v \in V\}.$$

We note that there is a natural embedding

$$\iota : \mathrm{SO}(V) \times \mathrm{SO}(-V) \hookrightarrow \mathrm{SO}(\mathbb{W}) \quad \text{such that} \quad \iota(g_1, g_2)(v_1 \oplus v_2) = g_1 v_1 \oplus g_2 v_2.$$

Also there exists an $\mathrm{SO}(V,\mathbb{A})\times\mathrm{SO}(-V,\mathbb{A})$-intertwining map

$$\tau : \mathcal{S}((V\otimes Y_n^+)(\mathbb{A}))\hat{\otimes}\mathcal{S}(((-V)\otimes Y_n^+)(\mathbb{A}))\to\mathcal{S}((\mathbb{W}^+\otimes Y_n)(\mathbb{A}))$$

with respect to the Weil representations, obtained by composing the natural map

$$\mathcal{S}((V\otimes Y_n^+)(\mathbb{A}))\hat{\otimes}\mathcal{S}(((-V)\otimes Y_n^+)(\mathbb{A}))\to\mathcal{S}((\mathbb{W}\otimes Y_n^+)(\mathbb{A}))$$

with the partial Fourier transform

$$\mathcal{S}((\mathbb{W}\otimes Y_n^+)(\mathbb{A}))\xrightarrow{\sim}\mathcal{S}((\mathbb{W}^+\otimes Y_n)(\mathbb{A})).$$

Namely we have

$$\tau(\omega_{\psi,V,Y_n}(g_1)\varphi_+\otimes\omega_{\psi,-V,Y_n}(g_2)\varphi_-)=\omega_{\psi,\mathbb{W},Y_n}(\iota(g_1,g_2))\tau(\varphi_+\otimes\varphi_-)$$

for $(g_1,g_2)\in\mathrm{SO}(V,\mathbb{A})\times\mathrm{SO}(-V,\mathbb{A})$ and $\varphi_\pm\in\mathcal{S}(((\pm V)\otimes Y_n^+)(\mathbb{A}))$.

We also consider the local counterparts of τ.

2.2.1. Local doubling zeta integrals. Let P be the maximal parabolic subgroup of $\mathrm{SO}(\mathbb{W})$ defined as the stabilizer of the isotropic subspace $V\oplus\{0\}$. Then the Levi subgroup of P is isomorphic to $\mathrm{GL}(V)$.

At each place v of F, we consider the degenerate principal series representation

$$I_v(s):=\mathrm{Ind}_{P(F_v)}^{\mathrm{SO}(\mathbb{W},F_v)}|\ |_v^s\quad\text{for } s\in\mathbb{C}.$$

Here the induction is normalized and $|\ |_v^s$ denotes the character of $P(F_v)$ which is given by $|\det|_v^s$ on its Levi subgroup $\mathrm{GL}(V,F_v)$ and is trivial on its unipotent radical.

For $\phi_v,\phi_v'\in V_{\pi_v}$ and $\Phi_v\in I_v(s)$, the local doubling zeta integral is defined by

$$Z_v(s,\phi_v,\phi_v',\Phi_v,\pi_v):=\int_{G_v}\langle\pi_v(g_v)\phi_v,\phi_v'\rangle_v\,\Phi_v(\iota(g_v,e_v))\,dg_v \tag{2.2}$$

where e_v denotes the unit element of G_v. We recall that the integral (2.2) converges absolutely when $\mathrm{Re}(s)>-\frac{1}{2}$ by Yamana [52, Lemma 7.2] since π_v is tempered.

For $\varphi_v\in\mathcal{S}((\mathbb{W}^+\otimes Y_n)(F_v))$, we define $\Phi_{\varphi_v}\in I_v(0)$ by

$$\Phi_{\varphi_v}(g_v)=(\omega_{\psi_v}(g_v)\varphi_v)(0)\quad\text{for } g_v\in\mathrm{SO}(\mathbb{W},F_v). \tag{2.3}$$

Definition 6. We define $Z_v^\circ(\phi_v,\varphi_v,\pi_v)$ for $\phi_v\in V_{\pi_v}$ and $\varphi_v\in\mathcal{S}((V\otimes Y_n^+)(F_v))$ by

$$Z_v^\circ(\phi_v,\varphi_v,\pi_v):=\frac{\prod_{j=1}^n\zeta_{F_v}(2j)}{L(1/2,\pi_v)}\cdot\frac{1}{\langle\phi_v,\phi_v\rangle_v}\cdot Z_v(0,\phi_v,\phi_v,\Phi_{\tau_v(\varphi_v\otimes\varphi_v)},\pi_v). \tag{2.4}$$

2.2.2. Rallis inner product formula. For automorphic forms $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ on $\widetilde{\mathrm{Sp}}_n(\mathbb{A})$, we define the Petersson inner product $(\tilde{\varphi}_1, \tilde{\varphi}_2)$ by

$$(\tilde{\varphi}_1, \tilde{\varphi}_2) := \int_{\mathrm{Sp}_n(F)\backslash \mathrm{Sp}_n(\mathbb{A})} \tilde{\varphi}_1(g)\,\overline{\tilde{\varphi}_2(g)}\,dg$$

when the integral converges absolutely. We recall that dg is the Tamagawa measure.

Let us first recall the Siegel–Weil formula. Let Φ be a standard section of $\mathrm{Ind}_{P(\mathbb{A})}^{\mathrm{SO}(\mathbb{W})(\mathbb{A})} |\ |^s$. Then we form the Siegel Eisenstein series by

$$E(g, s; \Phi) = \sum_{\gamma \in P(F)\backslash \mathrm{SO}(\mathbb{W})(F)} \Phi(\gamma g, s).$$

This sum converges absolutely when $\mathrm{Re}(s) > n$ and it has a meromorphic continuation to $\mathbb{C}$. We note that our Eisenstein series slightly differs from the Eisenstein series $E^{(m,m)}$ in [19, p. 183] for $m = 2n+1$ since P is also the Siegel parabolic subgroup of $\mathrm{O}(\mathbb{W})$ and $P\backslash \mathrm{O}(\mathbb{W}) \neq P\backslash \mathrm{SO}(\mathbb{W})$.

Let

$$E(g, s; \Phi) = \sum_{d \geq -1} A_d(\Phi)(g) s^d$$

be the Laurent expansion of $E(g, s; \Phi)$ at $s = 0$ where $A_d(\Phi)$ is an automorphic form on $\mathrm{SO}(\mathbb{W})(\mathbb{A})$. For $\varphi \in \mathcal{S}((\mathbb{W}^+ \otimes Y_n)(\mathbb{A}))$, we define the section $\Phi_\varphi \in \mathrm{Ind}_{P(\mathbb{A})}^{\mathrm{SO}(\mathbb{W})(\mathbb{A})} |\ |^s$ as in the local situation (2.3) and we simply write $A_d(\varphi)$ for $A_d(\Phi_\varphi)$.

On the other hand, let $I^{(2n+1,n)}(g, s; \varphi)$ be the regularized theta integral defined in [19, p. 185]. Then we have an equality

$$I^{(2n+1,n)}(g, s; \omega(z)\varphi) = P_z(s)\mathcal{E}^{(2n+1,n)}(g, s; \varphi)$$

as in [19, p. 186]. Here z is a regularizing element given in [19, p. 185], $P_z(s)$ is a certain holomorphic function of s depending on z and $\mathcal{E}^{(2n+1,n)}(g, s; \varphi)$ is a certain Eisenstein series defined in [19, p. 186]. Let us write the Laurent expansion of $\mathcal{E}^{(2n+1,n)}(g, s; \varphi)$ at $s = \frac{n+1}{2}$ as

$$\mathcal{E}^{(2n+1,n)}(g, s; \varphi) = \sum_{d \geq -2} B_d^{(2n+1,n)}(\varphi)\left(s - \frac{n+1}{2}\right)^d.$$

Let φ^0 denote the spherical Schwartz function defined in [19, p. 181]. We denote by $\mathcal{S}^\circ((\mathbb{W}^+ \otimes Y_n)(\mathbb{A}))$ the $\mathrm{SO}(\mathbb{W})(\mathbb{A})$-span of φ^0. Then we have the following Siegel–Weil formula.

Proposition 1. *With the above notation, we have*

$$A_0(\varphi) = B_{-1}^{(2n+1,n)}(\varphi)$$

for any $\varphi \in \mathcal{S}^\circ((\mathbb{W}^+ \otimes Y_n)(\mathbb{A}))$.

Proof. The argument for the proof of [19, Proposition 5.3] for $(\mathrm{O}_{2n+1}, \widetilde{\mathrm{Sp}}_n)$ works, mutatis mutandis, for $(\mathrm{SO}_{2n+1}, \widetilde{\mathrm{Sp}}_n)$. It is easily seen that the constant λ_2 which appears in the proof of [19, Lemma 4.3] is given by $\frac{\xi_F(1)}{2\xi_F(2)}$ in our case. Here $\xi_F(s)$ denotes the completed *normalized* zeta function of F given by

$$\xi_F(s) = |D_F|^{s/2}\, \zeta_F(s),$$

where D_F is the discriminant of F, ζ_F is the completed Dedekind zeta function of F defined by (1.17) and we define $\xi_F(1) := \mathrm{Res}_{s=1}\, \xi_F(s)$ as in [19, p. 180]. Thus the constant 2 does not appear in the Siegel–Weil formula above unlike [19, Proposition 5.3]. □

We obtain the following Rallis inner product formula from Proposition 1 as in [19].

Theorem 3. *For any non-zero decomposable vectors $\phi = \otimes_v \phi_v \in V_\pi$ and $\varphi = \otimes_v \varphi_v \in \mathcal{S}((V \otimes Y_n^+)(\mathbb{A}))$, we have*

$$\frac{(\theta_\psi^\varphi(\phi), \theta_\psi^\varphi(\phi))}{\langle \phi, \phi \rangle} = C_G \cdot \frac{L(1/2, \pi)}{\prod_{j=1}^n \zeta_F(2j)} \cdot \prod_v Z_v^\circ(\phi_v, \varphi_v, \pi_v) \tag{2.5}$$

where $Z_v^\circ(\phi_v, \varphi_v, \pi_v) = 1$ for almost all v.

Proof. Since $\sigma = \Theta_n(\pi, \psi)$ is cuspidal, by a similar computation to the proof of [19, Proposition 6.1], it is shown that

$$\sum_i (\theta_\psi^{\varphi_{1,i}}(\phi_1), \theta_\psi^{\varphi_{2,i}}(\phi_2)) = \int_{(G\times G)(F)\backslash (G\times G)(\mathbb{A})} \phi_1(g_1)\, \overline{\phi_2(g_2)} \cdot B_{-1}^{(2n+1,n)}\Big(\sum_i \tau(\varphi_{1,i} \otimes \overline{\varphi_{2,i}})\Big)(\iota(g_1, g_2))\, dg_1\, dg_2$$

for $\phi_i \in V_\pi$ and $\varphi = \sum_i \tau(\varphi_{1,i} \otimes \overline{\varphi_{2,i}}) \in \mathcal{S}^\circ((\mathbb{W}^+ \otimes Y_n)(\mathbb{A}))$ such that $\Phi_\varphi = \otimes \Phi_v$ is factorizable. Then by Proposition 1 and the doubling method, we obtain

$$\frac{\sum_i (\theta_\psi^{\varphi_{1,i}}(\phi), \theta_\psi^{\varphi_{2,i}}(\phi))}{\langle \phi_1, \phi_2 \rangle} = C_G \cdot \frac{L(1/2, \pi)}{\prod_{j=1}^n \zeta_F(2j)} \cdot \prod_v Z_v^\natural(0, \phi_{1,v}, \phi_{2,v}, \Phi_v, \pi_v)$$

where we define $Z_v^\natural(0, \phi_{1,v}, \phi_{2,v}, \Phi_v, \pi_v)$ by

$$Z_v^\natural(0, \phi_{1,v}, \phi_{2,v}, \Phi_v, \pi_v) = \frac{\prod_{j=1}^n \zeta_{F_v}(2j)}{L(1/2, \pi_v)} \cdot \frac{1}{\langle \phi_{1,v}, \phi_{2,v} \rangle_v} \cdot Z_v(0, \phi_{1,v}, \phi_{2,v}, \Phi_v, \pi_v),$$

in the same manner as (2.4). Further, we may extend the formula above to the whole space $\mathcal{S}((V \otimes Y_n^+)(\mathbb{A}))$ by a simple argument as remarked in [19, p. 243] and (2.5) holds. □

Remark 11. We note that there is a typo in the Rallis inner product formula stated in [19, Theorem 6.6]. It needs to be remedied as follows. There the Petersson inner product of the theta lifts is essentially equal to 2 times a certain L-value. However the Siegel–Weil formula [19, Proposition 5.3] implies that it is essentially equal to 2^{-1} times the L-value instead.

2.3. Whittaker periods of cusp forms on the metaplectic groups

We recall the Ichino–Ikeda type formula proved by Lapid and Mao [35] for the Whittaker periods of cusp forms on the metaplectic groups.

Since $\widetilde{\mathrm{Sp}}_n$ splits trivially over unipotent subgroups of Sp_n both locally and globally, we regard these subgroups as subgroups of $\widetilde{\mathrm{Sp}}_n$. For $a \in \mathrm{GL}_n$ and a symmetric $n \times n$ matrix S, we denote by $m(a)$ and $v(S)$ the elements of Sp_n given by

$$m(a) = \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix}, \qquad v(S) = \begin{pmatrix} 1_n & S \\ 0 & 1_n \end{pmatrix}.$$

Let $U_{\mathrm{Sp}} = \{v(S) : {}^t S = S\}$ and U_n the group of upper unipotent matrices in GL_n. Then the standard maximal unipotent subgroup $N = N_n$ of Sp_n is given by

$$N = m(U_n)U_{\mathrm{Sp}}. \tag{2.6}$$

We define a character ψ_λ of $N(\mathbb{A})$ by

$$\psi_\lambda(m(u)v(S)) = \psi\left(u_{1,2} + \cdots + u_{n-1,n} + \frac{\lambda}{2}s_{n,n}\right) \tag{2.7}$$

where $u_{i,j}$ denotes the (i, j)-entry of u and $s_{n,n}$ the (n, n)-entry of S.

Definition 7. For an automorphic form $\tilde{\phi}$ on $\widetilde{\mathrm{Sp}}_n(\mathbb{A})$, its ψ_λ-Whittaker period $W(\tilde{\phi}; \psi_\lambda)$ is defined by

$$W(\tilde{\phi}; \psi_\lambda) := \int_{N(F)\backslash N(\mathbb{A})} \tilde{\phi}(n)\psi_\lambda(n)^{-1}\, dn. \tag{2.8}$$

An automorphic representation of $\widetilde{\mathrm{Sp}}_n(\mathbb{A})$ is called *ψ_λ-generic* when $W(\ ; \psi_\lambda)$ does not vanish identically on its space of automorphic forms.

As we noted in (2.1f),

$$\sigma = \Theta_n(\pi, \psi) \text{ is } \psi_\lambda\text{-generic, irreducible and cuspidal.} \tag{2.9}$$

Let $\sigma = \otimes_v \sigma_v$. Then by Adams and Barbasch [1],

$$\sigma_v \text{ is a discrete series representation at any archimedean place } v \tag{2.10}$$

since so is π_v. Let π° be the theta lift of σ to $\mathbb{G}(\mathbb{A})$ with respect to $\psi^{-\lambda}$, which is globally generic by [15, Proposition 1, 3]. Let $\Sigma = \Pi \otimes \chi_E$ where Π is a weak lift of π to $\mathrm{GL}_{2n}(\mathbb{A})$. Then by [15, Lemma 1] and its proof, we have

$$L(s, \Sigma_v) = L(s, \pi_v^\circ) = L(s, \pi_v \times \chi_{E,v})$$

at every place v. Thus we may say that Σ is a weak lift of π° to $\mathrm{GL}_{2n}(\mathbb{A})$.

At each place v, we choose a $\widetilde{\mathrm{Sp}}_n(F_v)$-invariant Hermitian inner product $(\ ,\ )_v$ on V_{σ_v} so that we have $(\tilde{\varphi}_1, \tilde{\varphi}_2) = \prod_v (\tilde{\varphi}_{1,v}, \tilde{\varphi}_{2,v})_v$ for any decomposable vectors $\tilde{\varphi}_1 = \otimes_v \tilde{\varphi}_{1,v}$, $\tilde{\varphi}_2 = \otimes_v \tilde{\varphi}_{2,v} \in V_\sigma$.

Then by Lapid and Mao [35, Corollary 1.4], we have the following theorem.

Theorem 4. *For any non-zero decomposable cusp form $\tilde{\varphi} = \otimes_v \tilde{\varphi}_v \in V_\sigma$, we have*

$$\frac{|W(\tilde{\varphi}; \psi_\lambda)|^2}{(\tilde{\varphi}, \tilde{\varphi})} = 2^{-l} \cdot \frac{L(1/2, \pi \times \chi_E) \prod_{j=1}^n \zeta_F(2j)}{L(1, \pi, \mathrm{Ad})} \cdot \prod_v I_v(\tilde{\varphi}_v) \tag{2.11}$$

where $I_v(\tilde{\varphi}_v)$ is the stable integral defined by

$$I_v(\tilde{\varphi}_v) := \frac{L(1, \pi_v, \mathrm{Ad})}{L(1/2, \pi_v \times \chi_{E,v}) \prod_{j=1}^n \zeta_{F_v}(2j)} \int_{N_v}^{\mathrm{st}} \frac{(\sigma_v(n_v)\tilde{\varphi}_v, \tilde{\varphi}_v)_v}{(\tilde{\varphi}_v, \tilde{\varphi}_v)_v} \cdot \psi_\lambda^{-1}(n_v)\, dn_v \tag{2.12}$$

and we have

$$I_v(\tilde{\varphi}_v) = 1 \quad \textit{for almost all places } v \textit{ of } F. \tag{2.13}$$

Remark 12. When v is non-archimedean, the integrand of (2.12) does have a stable integral over N_v by [33, Proposition 2.3]. When v is archimedean, by (2.10) the integrand of (2.12) is integrable over N_v as explained in [33, p. 455]. Thus $I_v(\tilde{\varphi}_v)$ at an archimedean place v is indeed given by the absolutely convergent integral over N_v.

The assertion (2.13) was actually proved by Ginzburg, Rallis and Soudry [21] prior to [33]. Also we recall that for any place v, we have

$$\dim_{\mathbb{C}} \mathrm{Hom}_{N_v}(\sigma_v, \psi_{\lambda,v}) \leq 1 \tag{2.14}$$

by Szpruch [44, Theorem 3.1] and Liu and Sun [38, Theorem A] for the non-archimedean case and the archimedean case, respectively.

2.4. Pull-back of the ψ_λ-Whittaker period

Since $B_{\lambda,\psi} \not\equiv 0$ on V_π, we have $\mathrm{Hom}_{R_{\lambda,v}}(\pi_v, \chi_{\lambda,v}) \neq \{0\}$ for any place v of F. Hence when v is non-archimedean, $\alpha_v \not\equiv 0$ by (1.10). *Here we proceed further assuming the statement* (1) *of Theorem 1, i.e. $\alpha_v \not\equiv 0$ at* any *place v of F, which we shall prove later in Section 3.5.*

By the multiplicity one property (1.9) of the special Bessel model, there exists $C \in \mathbb{C}^\times$ such that

$$B_{\lambda,\psi}(\phi) \cdot \overline{B_{\lambda,\psi}(\phi')} = C \cdot \prod_v \alpha_v^\natural(\phi_v, \phi'_v) \tag{2.15}$$

for any non-zero decomposable cusp forms $\phi = \otimes_v \phi_v$, $\phi' = \otimes_v \phi'_v \in V_\pi$. Here we note that $\alpha_v^\natural(\phi_v, \phi'_v) = 1$ for almost all v. In particular when $\phi = \phi'$, we have

$$B_{\lambda,\psi}(\phi) \cdot \overline{B_{\lambda,\psi}(\phi)} = C \cdot \prod_v \alpha_v^\natural(\phi_v, \phi_v). \tag{2.16}$$

When $B_{\lambda,\psi}(\phi) = 0$, we have $\alpha_v^\natural(\phi_v, \phi_v) = 0$ at some place v by (2.16). Hence both sides of (1.19) vanish and the equality (1.19) holds. Thus from now on we suppose that $B_{\lambda,\psi}(\phi) \neq 0$. Since $\alpha_v^\natural(\phi_v, \phi_v) \neq 0$ for any v by (2.16), we may define

$$\alpha_v^\circ(g_v; \phi_v) := \alpha_v^\natural(\pi_v(g_v)\phi_v, \phi_v)/\alpha_v^\natural(\phi_v, \phi_v) \quad \text{for } g_v \in G_v \tag{2.17}$$

for every place v of F. Then from (2.15), for $g = (g_v) \in G(\mathbb{A})$, we have

$$\begin{aligned} B_{\lambda,\psi}(\pi(g)\phi) \cdot \overline{B_{\lambda,\psi}(\phi)} &= C \cdot \prod_v \alpha_v^\natural(\pi_v(g_v)\phi_v, \phi_v) \\ &= B_{\lambda,\psi}(\phi) \cdot \overline{B_{\lambda,\psi}(\phi)} \cdot \prod_v \alpha_v^\circ(g_v; \phi_v) \end{aligned}$$

where $\alpha_v^\circ(g_v; \phi_v) = 1$ for almost all v. Hence

$$B_{\lambda,\psi}(\pi(g)\phi) = B_{\lambda,\psi}(\phi) \cdot \prod_v \alpha_v^\circ(g_v; \phi_v). \tag{2.18}$$

Now we recall the pull-back formula for the ψ_λ-Whittaker period. We identify $V \otimes Y_n^+$ with V^n. The group G acts on V^n from the left by

$$g \cdot x = (gv_1, \dots, gv_n) \quad \text{for } x = (v_1, \dots, v_n) \in V^n. \tag{2.19}$$

Then by [11, (9)], for $\varphi \in \mathcal{S}(V(\mathbb{A})^n)$, we have

$$W(\theta_\psi^\varphi(\phi); \psi_\lambda) = \int_{R'_\lambda(\mathbb{A})\backslash G(\mathbb{A})} \varphi(g^{-1} \cdot (e_{-1}, \dots, e_{-n+1}, e_\lambda)) B_{\lambda,\psi}(\pi(g)\phi)\, dg \tag{2.20}$$

where

$$R'_\lambda := \{g \in G : ge_{-j} = e_{-j} \text{ for } 1 \le j \le n-1,\ ge_\lambda = e_\lambda\}. \tag{2.21}$$

The integral (2.20) is well-defined since χ_λ is trivial on $R'_\lambda(\mathbb{A})$. Here we note that

$$R'_\lambda = D_\lambda S'_\lambda \quad \text{where} \quad S'_\lambda = \left\{ \begin{pmatrix} 1_{n-1} & A & B \\ 0 & 1_3 & A' \\ 0 & 0 & 1_{n-1} \end{pmatrix} \in S' : Ae_\lambda = 0 \right\}. \tag{2.22}$$

For each j $(1 \le j \le n-1)$, let L_j be the subspace of L spanned by e_{-j}, e_λ and e_j. Then for $a \in F$, let $s_j(a)$ denote the element of G such that

$$s_j(a)|_{L_j^\perp} = 1_{L_j^\perp} \quad \text{and} \quad s_j(a)|_{L_j} = \begin{pmatrix} 1 & a & -\lambda^{-1}a^2/2 \\ 0 & 1 & -\lambda^{-1}a \\ 0 & 0 & 1 \end{pmatrix} \tag{2.23}$$

with respect to the basis $\{e_{-j}, e_\lambda, e_j\}$ of L_j. Also for each j $(1 \le j \le n-1)$, let us define a subgroup S'_j of S' by

$$S'_j := \left\{ \begin{pmatrix} 1_{n-1} & A & B \\ 0 & 1_3 & A' \\ 0 & 0 & 1_{n-1} \end{pmatrix} : Ae_\lambda \in Fe_{-1} + \cdots + Fe_{-j} \right\}$$

and $S'_0 := S'_\lambda$. We recall that S' has a filtration

$$S'_\lambda = S'_0 \lhd S'_1 \lhd \cdots \lhd S'_{n-1} = S' \tag{2.24}$$

and we have

$$S'_{j-1}\backslash S'_j \simeq \{s_j(a) : a \in F\}. \tag{2.25}$$

We also note the induced filtration of R_λ, namely

$$R'_\lambda = D_\lambda S'_\lambda = D_\lambda S'_0 \lhd D_\lambda S'_1 \lhd \cdots \lhd D_\lambda S'_{n-1} = D_\lambda S' \lhd D_\lambda S' S'' = R_\lambda.$$

Let $\varphi \in \mathcal{S}(V(\mathbb{A})^n)$ be of the form $\varphi = \otimes_v \varphi_v$ where $\varphi_v \in \mathcal{S}(V(F_v)^n)$ and suppose that the local integral

$$\mathcal{L}_v(\varphi_v; \phi_v) := \int_{R'_{\lambda,v}\backslash G_v} \varphi_v(g_v^{-1} \cdot (e_{-1}, \ldots, e_{-n+1}, e_\lambda))\alpha_v^\circ(g_v; \phi_v)\, dg_v \tag{2.26}$$

converges absolutely and $\mathcal{L}_v(\varphi_v; \phi_v) \neq 0$ at each place v. Then $\mathcal{L}_v(\varphi_v; \phi_v) = 1$ for almost all v and we may rewrite (2.20) as

$$W(\theta_\psi^\varphi(\phi); \psi_\lambda) = (C_G\, C_\lambda^{-1}) \cdot B_{\lambda,\psi}(\phi) \cdot \prod_v \mathcal{L}_v(\varphi_v; \phi_v) \tag{2.27}$$

and we have $W(\theta_\psi^\varphi(\phi); \psi_\lambda) \neq 0$.

Let $\Theta(\pi_v, \psi_v) := \mathrm{Hom}_{G_v}(\omega_{\psi_v}, \bar{\pi}_v)$ where ω_{ψ_v} is the local Weil representation of $G_v \times \widetilde{\mathrm{Sp}}_n(F_v)$ realized on $\mathcal{S}(V(F_v)^n)$, the space of Schwartz–Bruhat functions on $V(F_v)^n$. As in the global case (e.g. see [11, p. 94]), the action of $G_v \times \widetilde{\mathrm{Sp}}_n(F_v)$ via ω_{ψ_v} on $\varphi \in \mathcal{S}(V(F_v)^n)$ is given by the following formulas:

$$\omega_{\psi_v}(g, 1)\varphi(x) = \varphi(g^{-1} \cdot x), \qquad g \in G_v, \tag{2.28a}$$

$$\omega_{\psi_v}(1, (m(a), \varepsilon))\varphi(x) = \varepsilon\, \frac{\gamma_\psi(1)}{\gamma_\psi((\det a)^{2n+1})} |\det a|^{n+1/2} \varphi(xa), \qquad a \in \mathrm{GL}_n(F_v), \tag{2.28b}$$

$$\omega_{\psi_v}(1, v(S))\varphi(x) = \psi_v\big(\tfrac{1}{2}\mathrm{tr}(\mathrm{Gr}(x)S)\big)\varphi(x), \tag{2.28c}$$

where γ_ψ denotes the Weil constant and $\mathrm{Gr}(x)$ denotes the Gram matrix $((x_i, x_j))$ for $x = (x_1, \ldots, x_n) \in V(F_v)^n$. We recall that for $\sigma = \Theta_n(\pi, \psi)$, we have $\sigma = \otimes_v \sigma_v$ where $\sigma_v = \theta(\pi_v, \psi_v)$, the unique irreducible quotient of $\Theta(\pi_v, \psi_v)$ determined by the Howe duality. The Howe duality was proved by Howe [26] at archimedean places, by Waldspurger [46] at odd non-archimedean places and finally by Gan and Takeda [20] at all non-archimedean places. Let

$$\theta_v : \mathcal{S}(V(F_v)^n) \otimes V_{\pi_v} \to V_{\sigma_v}$$

be the $G_v \times \widetilde{\mathrm{Sp}}_n(F_v)$-equivariant linear map, which is unique up to multiplication by a scalar. Since the mapping

$$\mathcal{S}(V(\mathbb{A})^n) \otimes V_\pi \ni (\varphi', \phi') \mapsto \theta_\psi^{\varphi'}(\phi') \in V_\sigma$$

is $G_v \times \widetilde{\mathrm{Sp}}_n(F_v)$-equivariant at any place v, by the uniqueness of θ_v, we may adjust $\{\theta_v\}_v$ so that

$$\theta_\psi^{\varphi'}(\phi') = \otimes_v \theta_v(\varphi'_v \otimes \phi'_v) \quad \text{for } \phi' = \otimes_v \phi'_v \in V_\pi \text{ and } \varphi' = \otimes_v \varphi'_v \in \mathcal{S}(V(\mathbb{A})^n).$$

Hence by combining (2.5), (2.11) and (2.27), we have

$$\frac{|B_{\lambda,\psi}(\phi)|^2}{\langle\phi,\phi\rangle} = C_G^{-1}\, C_\lambda^2 \cdot 2^{-l} \cdot \frac{L(1/2,\pi)\, L(1/2,\pi\times\chi_E)}{L(1,\pi,\mathrm{Ad})} \times \prod_v \frac{Z_v^\circ(\phi_v,\varphi_v,\pi_v)\, I_v(\theta_v(\varphi_v\otimes\phi_v))}{|\mathcal{L}_v(\varphi_v;\phi_v)|^2} \tag{2.29}$$

where

$$\frac{Z_v^\circ(\phi_v,\varphi_v,\pi_v)\, I_v(\theta_v(\varphi_v\otimes\phi_v))}{|\mathcal{L}_v(\varphi_v;\phi_v)|^2} = 1$$

for almost all v.

Since the right hand side of (1.19) does not depend on the decompositions of the global Tamagawa measures (1.6), we may take specific local measures which are suitable for our further considerations on the local integrals appearing on the right hand side of (2.29). In Section 3, we shall specify local measures and show that we have

$$C_G^{-1}\, C_\lambda = \frac{\prod_{j=1}^n \zeta_F(2j)}{L(1,\chi_E)} \tag{2.30}$$

and the following proposition holds.

Proposition 2. *Let v be an arbitrary place of F. For a given $\phi_v \in V_{\pi_v}$ satisfying $\alpha_v(\phi_v,\phi_v) \neq 0$, there exists $\varphi_v \in \mathcal{S}(V(F_v)^n)$ such that the local integral $\mathcal{L}_v(\varphi_v;\phi_v)$ converges absolutely, $\mathcal{L}_v(\varphi_v;\phi_v) \neq 0$ and the equality*

$$\frac{Z_v^\circ(\phi_v,\varphi_v,\pi_v)\, I_v(\theta_v(\varphi_v\otimes\phi_v))}{|\mathcal{L}_v(\varphi_v;\phi_v)|^2} = \frac{\alpha_v^\natural(\phi_v,\phi_v)}{\langle\phi_v,\phi_v\rangle_v} \tag{2.31}$$

holds with respect to the specified local measures.

Then for $\phi = \otimes_v \phi_v \in V_\pi$ such that $B_{\lambda,\psi}(\phi) \neq 0$, it is clearly seen that the main identity (1.19) holds by taking $\varphi_v \in \mathcal{S}(V(F_v)^n)$ as in Proposition 2 for each place v and by combining (2.29)–(2.31).

3. Proof of the local equality

3.1. *Specification of local measures*

Recall that the group G acts on V^n from the left by (2.19). Let

$$x_0 := (e_{-1},\dots,e_{-n+1},e_\lambda) \in V^n \tag{3.1}$$

and $\mathcal{X}_\lambda := G\cdot x_0 \subset V^n$. Since R'_λ defined by (2.21) is the stabilizer of x_0, $R'_\lambda\backslash G \ni g \mapsto g^{-1}\cdot x_0 \in \mathcal{X}_\lambda$ is a G-homogeneous space isomorphism with the right action of G on $\mathcal{X}_\lambda$ given by $\mathcal{X}_\lambda \ni x \mapsto g^{-1}\cdot x \in \mathcal{X}_\lambda$. We note that $\mathcal{X}_\lambda$ is a locally closed subvariety of V^n since $\mathcal{X}_\lambda$ is a set of $x = (x_1,\dots,x_n) \in V^n$ such that $\mathrm{Gr}(x) = \mathrm{Gr}(x_0)$ and $x_1,\dots,x_n$ are linearly independent, by Witt's theorem.

Let ω and ω_G be non-zero gauge forms on V^n and G, respectively. Let ω_0 be the gauge form on $\mathcal{X}_\lambda$ given by pulling back ω via the inclusion $\mathcal{X}_\lambda \hookrightarrow V^n$. We choose a gauge form ω_λ on R'_λ such that ω_G, ω_0 and ω_λ match algebraically in the sense of Weil [49, p. 24], i.e. $\omega_G = \omega_0\,\omega_\lambda$. Also, we denote by ω_D the gauge form on D given by pulling back ω_λ via $D \hookrightarrow R'_\lambda$.

In [22], Gross associated a motive of Artin–Tate type to a connected reductive algebraic group over F. Thus let M_G be the motive associated to G and $M_G^\vee(1)$ its twisted dual motive. The local Tamagawa measure dg_v on G_v corresponding to ω_G is given by $dg_v = L_v(M_G^\vee(1)) \cdot |\omega_G|_v$ at each place v of F. We refer to Gross [22] and Rogawski [42, 1.7] for the details concerning the definition of local Tamagawa measures. Then the Haar measure constant C_G defined by $dg = C_G \prod_v dg_v$, where dg is the Tamagawa measure, is given by

$$C_G = \Big(\prod_{j=1}^{n} \zeta_F(2j)\Big)^{-1}. \tag{3.2}$$

Similarly we specify the measure dt_v on $D_{\lambda,v}$ to be the local Tamagawa measure corresponding to ω_D at each place v. Then the Haar measure constant C_λ defined by $dt = C_\lambda \prod_v dt_v$, where dt is the Tamagawa measure, is given by

$$C_\lambda = \frac{1}{L(1, \chi_E)}. \tag{3.3}$$

For the unipotent group S'_λ, let the measure ds'_v on $S'_{\lambda,v}$ at each place v be the measure specified in Section 1.1 for unipotent groups. We define the measure dr'_v on $R'_{\lambda,v} = D_{\lambda,v} S'_{\lambda,v}$ by $dr'_v = dt_v\, ds'_{\lambda,v}$. Finally, we take the quotient measure dh_v on $R'_{\lambda,v}\backslash G_v$ so that

$$dg_v = dh_v\, dr'_v. \tag{3.4}$$

It is clear from (3.2) and (3.3) that the equality (2.30) holds with these choices of the local measures.

3.2. Two sesquilinear forms on V_σ

Since our discussion is purely local, from now on till the end of Section 3.4 we suppress the subscript v expressing a place from the notation, e.g. F now denotes a local field.

We construct two sesquilinear forms on V_σ which satisfy the same transformation property with respect to the subgroup N of $\widetilde{\mathrm{Sp}}_n(F)$.

3.2.1. Sesquilinear form $\mathcal{W}$. First we define a Hermitian inner product $\mathcal{B}_\omega$ on $\mathcal{S}(V^n)$ by

$$\mathcal{B}_\omega(\varphi, \varphi') := \int_{V^n} \varphi(x)\,\overline{\varphi'(x)}\,dx \quad \text{for } \varphi, \varphi' \in \mathcal{S}(V^n) \tag{3.5}$$

where dx denotes the measure corresponding to the gauge form ω on V^n in Section 3.1. Then Liu [37, Lemma 3.19] proved that the integral

$$Z^\flat(\phi, \phi'; \varphi, \varphi') = \int_G \langle \pi(g)\phi, \phi' \rangle \mathcal{B}_\omega(\omega_\psi(g)\varphi, \varphi')\,dg \tag{3.6}$$

converges absolutely for $\phi, \phi' \in V_\pi$ and $\varphi, \varphi' \in \mathcal{S}(V^n)$. We note that our setting belongs to Case 2 in the proof of [37, Lemma 3.19].

As in Gan and Ichino [18, 16.5], there exists a unique $\widetilde{\mathrm{Sp}}_n(F)$-invariant Hermitian inner product $\mathcal{B}_\sigma : V_\sigma \times V_\sigma \to \mathbb{C}$ satisfying

$$\mathcal{B}_\sigma(\theta(\varphi \otimes \phi), \theta(\varphi' \otimes \phi')) = Z^\flat(\phi, \phi'; \varphi, \varphi')$$

for $\phi, \phi' \in V_\pi$ and $\varphi, \varphi' \in \mathcal{S}(V^n)$. Here we note that for $h \in \widetilde{\mathrm{Sp}}_n(F)$ we have

$$\mathcal{B}_\sigma(\sigma(h)\theta(\varphi \otimes \phi), \theta(\varphi' \otimes \phi')) = \mathcal{B}_\sigma(\theta(\omega_\psi(h)\varphi \otimes \phi), \theta(\varphi' \otimes \phi')). \tag{3.7}$$

Definition 8. We define a sesquilinear form $\mathcal{W} = \mathcal{W}_{\varphi,\varphi'} : V_\sigma \times V_\sigma \to \mathbb{C}$ by

$$\mathcal{W}(\tilde{\phi}_1, \tilde{\phi}_2) := \int_N^{\mathrm{st}} \mathcal{B}_\sigma(\sigma(n)\tilde{\phi}_1, \tilde{\phi}_2)\psi_\lambda(n)^{-1}\, dn \tag{3.8}$$

for $\tilde{\phi}_1, \tilde{\phi}_2 \in V_\sigma$.

Recall that, by Remark 12, the integrand of (3.8) has a stable integral over N when F is non-archimedean and it is integrable when F is archimedean.

We note that for $n_1, n_2 \in N$ and $\tilde{\phi}_1, \tilde{\phi}_2 \in V_\sigma$, we have

$$\mathcal{W}(\sigma(n_1)\tilde{\phi}_1, \sigma(n_2)\tilde{\phi}_2) = \psi_\lambda(n_1)\psi_\lambda(n_2)^{-1} \cdot \mathcal{W}(\tilde{\phi}_1, \tilde{\phi}_2). \tag{3.9}$$

3.2.2. Sesquilinear form $\mathcal{W}^\circ$. For $\phi, \phi' \in V_\pi$ and $\varphi \in C_c^\infty(V^n)$, let

$$\mathcal{V}(\phi, \phi'; \varphi) := \int_{R'_\lambda \backslash G} (\omega_\psi(g, 1)\varphi)(x_0) \cdot \alpha(\pi(g)\phi, \phi')\, dg. \tag{3.10}$$

Recall that

$$\alpha(\phi, \phi') = \int_{D_\lambda} \int_S^{\mathrm{st}} \langle \pi(st)\phi, \phi' \rangle \chi_\lambda(s)^{-1}\, ds\, dt.$$

For $\varphi \in C_c^\infty(V^n)$, the support of $R'_\lambda \backslash G \ni g \mapsto \varphi(g^{-1} \cdot x_0)$ is compact since $\mathcal{X}_\lambda$ is locally closed in V^n. Hence the integral (3.10) indeed converges absolutely.

We note that when $\alpha(\phi, \phi) \neq 0$, we have

$$\mathcal{V}(\phi, \phi; \varphi) = \alpha(\phi, \phi) \cdot \mathcal{L}(\varphi; \phi). \tag{3.11}$$

Recall that $\mathcal{L}(\varphi; \phi)$ is defined by (2.26). We also note that

$$\mathcal{V}(\pi(g)\phi, \phi'; \omega_\psi(g, 1)\varphi) = \mathcal{V}(\phi, \phi'; \varphi) \quad \text{for } g \in G. \tag{3.12}$$

By (3.12) there exists a unique linear form $\ell_{\phi',\varphi} : V_\sigma \to \mathbb{C}$ such that

$$\ell_{\phi',\varphi}(\theta(\varphi \otimes \phi)) = \mathcal{V}(\phi, \phi'; \varphi) \quad \text{for } \phi \in V_\pi \text{ and } \varphi \in C_c^\infty(V^n). \tag{3.13}$$

Then, for $n \in N$ and $\tilde{\phi} \in V_\sigma$, we have

$$\ell_{\phi',\varphi}(\sigma(n)\tilde{\phi}) = \psi_\lambda(n)\ell_{\phi',\varphi}(\tilde{\phi}). \tag{3.14}$$

Definition 9. For $\phi, \phi' \in V_\pi$ and $\varphi, \varphi' \in C_c^\infty(V^n)$, we define a sesquilinear form $\mathcal{W}^\circ = \mathcal{W}^\circ_{\phi,\phi',\varphi,\varphi'} : V_\sigma \times V_\sigma \to \mathbb{C}$ by

$$\mathcal{W}^\circ(\tilde{\phi}_1, \tilde{\phi}_2) := \ell_{\phi',\varphi}(\tilde{\phi}_1) \cdot \overline{\ell_{\phi,\varphi'}(\tilde{\phi}_2)} \tag{3.15}$$

for $\tilde{\phi}_1, \tilde{\phi}_2 \in V_\sigma$.

It is clear from (3.14) that for $n_1, n_2 \in N$ and $\tilde{\phi}_1, \tilde{\phi}_2 \in V_\sigma$,

$$\mathcal{W}^\circ(\sigma(n_1)\tilde{\phi}_1, \sigma(n_2)\tilde{\phi}_2) = \psi_\lambda(n_1)\psi_\lambda(n_2)^{-1} \cdot \mathcal{W}^\circ(\tilde{\phi}_1, \tilde{\phi}_2). \tag{3.16}$$

3.2.3. Comparison between $\mathcal{W}$ and $\mathcal{W}^\circ$. First we note the following lemma whose proof is clear since $\mathcal{X}_\lambda$ is locally closed in V^n.

Lemma 1. *Suppose that $\alpha(\phi, \phi') \neq 0$. Then for any open neighborhood O_{x_0} of x_0 in V^n, there exists $\varphi \in C_c^\infty(V^n)$ such that* Supp(φ)*, the support of φ, is contained in O_{x_0} and $\mathcal{V}(\phi, \phi'; \varphi) \neq 0$. In particular the linear form $\ell_{\phi',\varphi}$ on V_σ defined by (3.13) is non-zero for such φ.*

By the uniqueness of the ψ_λ-Whittaker model (2.14), the equalities (3.9) and (3.16) imply that $\mathcal{W}$ is a scalar multiple of $\mathcal{W}^\circ$ when $\mathcal{W}^\circ$ is non-zero. The following proposition states that the constant of proportionality is given explicitly.

Proposition 3. *Suppose that $\phi, \phi' \in V_\pi$ satisfy $\alpha(\phi, \phi') \neq 0$. Then for any $\varphi, \varphi' \in C_c^\infty(V^n)$ satisfying $\ell_{\phi',\varphi}(\phi') \neq 0$ and $\ell_{\phi,\varphi'}(\phi) \neq 0$, we have*

$$\mathcal{W}_{\varphi,\varphi'} = \frac{C_{E/F}}{\alpha(\phi, \phi')} \cdot \mathcal{W}^\circ_{\phi,\phi',\varphi,\varphi'} \tag{3.17}$$

where

$$C_{E/F} = \frac{L(1, \chi_{E/F})}{\prod_{j=1}^n \zeta_F(2j)}. \tag{3.18}$$

Let us show that Proposition 2 follows from Proposition 3 before proceeding to a proof of Proposition 3.

Proof of Proposition 2. Suppose that $\alpha(\phi, \phi) \neq 0$. By Lemma 1, we may take $\varphi \in C_c^\infty(V^n)$ so that $\mathcal{V}(\phi, \phi; \varphi) = \alpha(\phi, \phi) \cdot \mathcal{L}(\varphi; \phi) \neq 0$. Then $\mathcal{W}^\circ = \mathcal{W}^\circ_{\phi,\phi,\varphi,\varphi}$ is non-zero. Hence by (3.17), we have

$$\mathcal{W}(\theta(\varphi \otimes \phi), \theta(\varphi \otimes \phi)) = \frac{C_{E/F}}{\alpha(\phi, \phi)} \cdot \mathcal{W}^\circ(\theta(\varphi \otimes \phi), \theta(\varphi \otimes \phi)),$$

i.e.

$$\mathcal{W}(\theta(\varphi \otimes \phi), \theta(\varphi \otimes \phi)) = C_{E/F} \cdot \alpha(\phi, \phi) \cdot |\mathcal{L}(\varphi; \phi)|^2 \tag{3.19}$$

by (3.11) and (3.13).

On the other hand, by using the $\widetilde{\mathrm{Sp}}_n(F)$-invariant Hermitian inner product $\mathcal{B}_\sigma(\ ,\ )$ in the definition (2.12) for $I(\theta(\varphi\otimes\phi))$, we have

$$\begin{aligned}\mathcal{B}_\sigma(\theta(\varphi\otimes\phi),\theta(\varphi\otimes\phi))\cdot I(\theta(\varphi\otimes\phi))\\ = \frac{L(1,\pi,\mathrm{Ad})}{L(1/2,\pi\times\chi_E)\prod_{j=1}^n\zeta_F(2j)}\cdot\mathcal{W}(\theta(\varphi\otimes\phi),\theta(\varphi\otimes\phi))\end{aligned}\tag{3.20}$$

by (3.8). Here by Gan and Ichino [18, 16.3], we have

$$\mathcal{B}_\sigma(\theta(\varphi\otimes\phi),\theta(\varphi\otimes\phi)) = Z(0,\phi,\phi,\Phi_{\tau(\varphi\otimes\varphi)},\pi)\tag{3.21}$$

where the right hand side is the local doubling integral defined by (2.2). Hence by rewriting (3.21) in terms of $Z^\circ(\phi,\varphi,\pi)$ defined by (2.4), we have

$$\frac{\prod_{j=1}^n\zeta_F(2j)}{L(1/2,\pi)}\cdot\mathcal{B}_\sigma(\theta(\varphi\otimes\phi),\theta(\varphi\otimes\phi)) = \langle\phi,\phi\rangle\cdot Z^\circ(\phi,\varphi,\pi).\tag{3.22}$$

Thus by combining (3.20) and (3.22), we have

$$\begin{aligned}\frac{L(1,\pi,\mathrm{Ad})}{L(1/2,\pi\times\chi_E)L(1/2,\pi)}\cdot\mathcal{W}(\theta(\varphi\otimes\phi),\theta(\varphi\otimes\phi))\\ = \langle\phi,\phi\rangle\cdot Z^\circ(\phi,\varphi,\pi)\cdot I(\theta(\varphi\otimes\phi)).\end{aligned}\tag{3.23}$$

Substituting (3.19) into (3.23) yields (2.31) and thus Proposition 2 holds. □

3.3. *Reduction to another local equality*

Here we shall observe that Proposition 3 follows from a local equality (3.26) below.

Since

$$\mathcal{W}^\circ(\theta(\varphi\otimes\phi),\theta(\varphi'\otimes\phi')) = \mathcal{V}(\phi,\phi';\varphi)\cdot\overline{\mathcal{V}(\phi',\phi;\varphi')}\neq 0$$

by (3.13) and (3.15), the equality (3.17) follows from

$$\mathcal{W}(\theta(\varphi\otimes\phi),\theta(\varphi'\otimes\phi')) = \frac{C_{E/F}}{\alpha(\phi,\phi')}\cdot\mathcal{V}(\phi,\phi';\varphi)\cdot\overline{\mathcal{V}(\phi',\phi;\varphi')}.\tag{3.24}$$

Here

$$\begin{aligned}\mathcal{V}(\phi,\phi';\varphi)\cdot\overline{\mathcal{V}(\phi',\phi;\varphi')} = \int_{R'_\lambda\backslash G}\int_{R'_\lambda\backslash G}\alpha(\pi(h)\phi,\phi')\,\overline{\alpha(\pi(h')\phi',\phi)}\\ \times(\omega_\psi(h,1)\varphi)(x_0)\,\overline{(\omega_\psi(h',1)\varphi')(x_0)}\,dh\,dh'.\end{aligned}\tag{3.25}$$

We observe that a sesquilinear form $\mathcal{A}$ on V_π defined by

$$\mathcal{A}(\phi_1,\phi_1') := \alpha(\phi_1,\phi')\,\overline{\alpha(\phi_1',\phi)}$$

satisfies

$$\mathcal{A}(\pi(r)\phi_1,\pi(r')\phi_1') = \chi_\lambda(r)\,\chi_\lambda(r')^{-1}\cdot\mathcal{A}(\phi_1,\phi_1')$$

for $r, r' \in R_\lambda$. Hence the uniqueness of the special Bessel model (1.9) implies that there exists a constant c' such that $\mathcal{A} = c' \cdot \alpha$. Since $\alpha(\phi, \phi') \neq 0$, we have

$$c' = \mathcal{A}(\phi, \phi')/\alpha(\phi, \phi') = \alpha(\phi, \phi').$$

Hence in the integrand of (3.25), we have

$$\alpha(\pi(h)\phi, \phi')\,\overline{\alpha(\pi(h')\phi', \phi)} = \mathcal{A}(\pi(h)\phi, \pi(h')\phi') = \alpha(\phi, \phi')\alpha(\pi(h)\phi, \pi(h')\phi').$$

Thus the equality (3.24) follows from the following proposition.

Proposition 4. *For any $\phi, \phi' \in V_\pi$ and any $\varphi, \varphi' \in C_c^\infty(V^n)$, we have*

$$\mathcal{W}(\theta(\varphi \otimes \phi), \theta(\varphi' \otimes \phi')) = C_{E/F}$$
$$\times \int_{R'_\lambda \backslash G} \int_{R'_\lambda \backslash G} \alpha(\pi(h)\phi, \pi(h')\phi')(\omega_\psi(h, 1)\varphi)(x_0)\,\overline{(\omega_\psi(h', 1)\varphi')(x_0)}\,dh\,dh'. \tag{3.26}$$

Remark 13. Note that we shall show (3.26) in more generality than just necessary to prove (3.24) because of its later use in the proof of Corollary 2. In particular, we do not assume $\alpha(\phi, \phi') \neq 0$ in Proposition 4.

Remark 14. The equality (3.26) may be naturally regarded as a local pull-back formula for the ψ_λ-Whittaker pairing.

3.4. Proof of Proposition 4

By the definition in Section 3.2.1, we have

$$\mathcal{W}(\theta(\varphi \otimes \phi), \theta(\varphi' \otimes \phi')) = \int_{U_n}^{\mathrm{st}} \int_{U_{\mathrm{Sp}}}^{\mathrm{st}} \int_G \int_{V^n}$$
$$(\omega_\psi(1, m(u)v)\varphi)(g^{-1} \cdot x)\,\overline{\varphi'(x)}\langle \pi(g)\phi, \phi' \rangle \psi_\lambda(m(u)v)^{-1}\,dx\,dg\,dv\,du. \tag{3.27}$$

Here we use the decomposition (2.6) of N. We shall show (3.26) by modifying the right hand side of (3.27) in several steps.

3.4.1. Inner triple integral. We shall take care of the inner triple integral of (3.27) by adapting Liu's computations in [37, 3.5] to our setting.

Let $V_\circ^n$ be a subset of V^n consisting of $(v_1, \ldots, v_n) \in V^n$ such that $v_1, \ldots, v_n$ are linearly independent and the inner product (v_n, v_n) is non-zero. Then $V_\circ^n$ is open in V^n and $\mathrm{Vol}(V^n \setminus V_\circ^n, dx) = 0$.

Let Sym^n denote the set of $n \times n$ symmetric matrices with entries in F and

$$\mathrm{Sym}_\circ^n := \{S = (s_{i,j}) \in \mathrm{Sym}^n \mid s_{n,n} \neq 0\}.$$

We consider a mapping $\mathrm{Gr} : V_\circ^n \to \mathrm{Sym}_\circ^n$ given by the Gram matrix $\mathrm{Gr}(x)$ for $x \in V_\circ^n$. It is clear that Gr is surjective. For each $S \in \mathrm{Sym}_\circ^n$, we fix $x_S \in V_\circ^n$ such that $\mathrm{Gr}(x_S) = S$.

Then by Witt's theorem, the fiber $\mathrm{Gr}^{-1}(S)$ of S is given by

$$\mathrm{Gr}^{-1}(S) = \{g^{-1} \cdot x_S \mid g \in G\}.$$

Let R'_S denotes the stabilizer of x_S in G. Then we may identify $\mathrm{Gr}^{-1}(S)$ with $R'_S\backslash G$ as G-homogeneous spaces. We have the following integration formula.

Lemma 2. *For each $S \in \mathrm{Sym}^n_\circ$, there exists a Haar measure dr'_S on R'_S such that*

$$\int_{V^n} \Phi(x)\,dx = \int_{\mathrm{Sym}^n_\circ} \int_{R'_S\backslash G} \Phi(h^{-1} \cdot x_S)\,dh_S\,dS \tag{3.28}$$

for any $\Phi \in L^1(V^n)$. Here dh_S denotes the quotient measure $dr'_S\backslash dg$ on $R'_S\backslash G$.

Proof. Since $\mathrm{Vol}(V^n \setminus V^n_\circ, dx) = 0$, we have

$$\int_{V^n} \Phi(x)\,dx = \int_{V^n_\circ} \Phi(x)\,dx$$

for $\Phi \in L^1(V^n)$. Then the lemma readily follows from the observation above. □

Remark 15. Let $S_\circ := \mathrm{Gr}(x_0) \in \mathrm{Sym}^n_\circ$ where x_0 is given by (3.1). We take x_S to be x_0 when $S = S_0$. Then $R'_{S_\circ} = R'_\lambda$ defined by (2.21), which has a decomposition $R'_\lambda = D_\lambda S'_\lambda$ as (2.22). Recall that we take the local Tamagawa measure as explained in Section 3.1. Recall also that the measure on the unipotent group S'_λ is taken as explained in Section 1.1. On the other hand, the quotient measure $dh_{S_\circ}$ on $R'_\lambda\backslash G$ used in (3.28) is the quotient measure of the local measures corresponding to the gauge forms ω_λ and ω_G, which are not normalized as local Tamagawa measures by the local L-factors. Hence the relationship between the two quotient measures on $R'_\lambda\backslash G$, $dh_{S_\circ}$ in (3.28) and dh defined by (3.4), is given by

$$dh_{S_\circ} = C_{E/F} \cdot dh \tag{3.29}$$

where $C_{E/F}$ is as in (3.17).

Before proceeding further, we note the following lemma, which is proved by an argument similar to the one for [37, Lemma 3.20] when F is non-archimedean and to the one for [37, Proposition 3.22] when F is archimedean, respectively.

Lemma 3. *For $\varphi_1, \varphi_2 \in C^\infty_c(V^n)$ and $\phi_1, \phi_2 \in V_\pi$, let*

$$\mathcal{G}_{\varphi_1,\varphi_2,\phi_1,\phi_2}(S) = \int_G \int_{R'_S\backslash G} \varphi_1((hg')^{-1} \cdot x_S)\varphi_2(h^{-1} \cdot x_S)\langle \pi(g')\phi_1, \phi_2\rangle\,dh\,dg'$$

for $S \in \mathrm{Sym}^n_\circ$.

(1) *When F is non-archimedean, the integral is absolutely convergent and is locally constant.*
(2) *When F is archimedean, the integral is absolutely convergent and is a function in $L^1(\mathrm{Sym}^n)$ which is continuous on $\mathrm{Sym}^n_\circ$.*

Now for $\varphi, \varphi' \in C_c^\infty(V^n)$ and $\phi, \phi' \in V_\pi$, let

$$f_{\varphi,\varphi',\phi,\phi'}(n) := \int_G \int_{V^n} (\omega_\psi(1,n)\varphi)(g^{-1}\cdot x)\,\overline{\varphi'(x)}\,\langle \pi(g)\phi, \phi'\rangle\, dx\, dg \tag{3.30}$$

for $n \in N$. Then

$$\mathcal{W}(\theta(\varphi\otimes\phi), \theta(\varphi'\otimes\phi')) = \int_{U_n}^{\mathrm{st}} \int_{U_{\mathrm{Sp}}}^{\mathrm{st}} f_{\varphi,\varphi',\phi,\phi'}(m(u)v)\psi_\lambda(m(u)v)^{-1}\, dv\, du. \tag{3.31}$$

Since

$$U_{\mathrm{Sp}} = \left\{ v(S) = \begin{pmatrix} 1_n & S \\ 0 & 1_n \end{pmatrix} : S \in \mathrm{Sym}^n \right\},$$

we may regard $\int_{U_{\mathrm{Sp}}}^{\mathrm{st}}$ as $\int_{\mathrm{Sym}^n}^{\mathrm{st}}$. Then by rewriting the integration over V^n in (3.30) using the integration formula (3.28), we have the following lemma.

Lemma 4. *We have*

$$\begin{aligned}&\int_{U_{\mathrm{Sp}}}^{\mathrm{st}} f_{\varphi,\varphi',\phi,\phi'}(v)\psi_\lambda(v)^{-1}\, dv \\ &\quad = C_{E/F}\cdot \int_G \int_{R'_\lambda\backslash G} (\omega_\psi(hg,1)\varphi)(x_0)\,\overline{\varphi'(h^{-1}\cdot x_0)}\,\langle \pi(g)\phi, \phi'\rangle\, dh\, dg.\end{aligned} \tag{3.32}$$

Proof. The argument using the Fourier inversion for the proof of [37, Proposition 3.21] in the non-archimedean case and the one for [37, Corollary 3.23] in the archimedean case work mutatis mutandis, since Lemma 3 holds. Thus we obtain (3.32) by taking into account (3.29) also. □

By Lemma 4, we have

$$\begin{aligned}\mathcal{W}(\theta(\varphi\otimes\phi), \theta(\varphi'\otimes\phi')) = C_{E/F}\cdot \int_{U_n}^{\mathrm{st}} \int_G \int_{R'_\lambda\backslash G} (\omega_\psi(hg, m(u))\varphi)(x_0)\,\overline{\varphi'(h^{-1}\cdot x_0)} \\ \times \langle \pi(g)\phi, \phi'\rangle\, \psi_\lambda(m(u))^{-1}\, dh\, dg\, du.\end{aligned} \tag{3.33}$$

Then by a change of variable $g \mapsto h^{-1}g$ and also noting that we have $\langle \pi(h^{-1}g)\phi, \phi'\rangle = \langle \pi(g)\phi, \pi(h)\phi'\rangle$, we may write (3.33) as

$$\begin{aligned}\mathcal{W}(\theta(\varphi\otimes\phi), \theta(\varphi'\otimes\phi')) = C_{E/F}\cdot \int_{U_n}^{\mathrm{st}} \int_G \int_{R'_\lambda\backslash G} (\omega_\psi(g, m(u))\varphi)(x_0)\,\overline{\varphi'(h^{-1}\cdot x_0)} \\ \times \langle \pi(g)\phi, \pi(h)\phi'\rangle \psi_\lambda(m(u))^{-1}\, dh\, dg\, du.\end{aligned} \tag{3.34}$$

Here the inner double integral on the right hand side of (3.34) converges absolutely by Lemma 3. Hence we may change the order of integration and we have

$$\begin{aligned}\mathcal{W}(\theta(\varphi\otimes\phi), \theta(\varphi'\otimes\phi')) = C_{E/F}\cdot \int_{U_n}^{\mathrm{st}} \int_{R'_\lambda\backslash G} \int_G (\omega_\psi(g, m(u))\varphi)(x_0)\,\overline{\varphi'(h^{-1}\cdot x_0)} \\ \times \langle \pi(g)\phi, \pi(h)\phi'\rangle \psi_\lambda(m(u))^{-1}\, dg\, dh\, du.\end{aligned}$$

Moreover, since the innermost integral converges absolutely, we may telescope the G-integration to obtain

$$\begin{aligned}\mathcal{W}(\theta(\varphi\otimes\phi),\theta(\varphi'\otimes\phi'))&\\ &= C_{E/F}\cdot\int_{U_n}^{\mathrm{st}}\int_{R'_\lambda\backslash G}\int_{R'_\lambda\backslash G}\int_{R'_\lambda}(\omega_\psi(g,m(u))\varphi)(x_0)\,\overline{\varphi'(h^{-1}\cdot x_0)}\\ &\quad\times\langle\pi(r'g)\phi,\pi(h)\phi'\rangle\psi_\lambda(m(u))^{-1}\,dr'\,dg\,dh\,du.\end{aligned}\tag{3.35}$$

Remark 16. As we have seen, because of Lemma 3,

$$\int_{R'_\lambda}\langle\pi(r'g)\phi,\pi(h)\phi'\rangle\,dr',\tag{3.36}$$

the innermost integral of (3.35), converges absolutely. This R'_λ-integration appears as an inner integral of the definition (1.7) for $\alpha(\pi(g)\phi,\pi(h)\phi')$ since $R'_\lambda = D_\lambda S'_\lambda \subset R_\lambda$ and $\chi_\lambda(r') = 1$ for $r'\in R'_\lambda$.

3.4.2. Stable integration over U_n. Suppose that F is non-archimedean. We shall transform the stable integration over U_n as a subgroup of $\widetilde{\mathrm{Sp}}_n(F)$ in (3.35) into an integration over a subgroup of R_λ by adapting the global argument in [11, pp. 97–98] to our local setting and shall reduce Proposition 4 to Lemma 6 below.

Recall that U_n is the group of upper unipotent matrices in $\mathrm{GL}_n(F)$. Let us identify U_{n-1} with the subgroup $\left\{\left(\begin{smallmatrix}u&0\\0&1\end{smallmatrix}\right) : u\in U_{n-1}\right\}$ of U_n. Let U_0 be the subgroup of U_n defined by

$$U_0 = \left\{\begin{pmatrix}1_{n-1}&a\\0&1\end{pmatrix}\in U_n\right\}.$$

Thus we have $U_n = U_0\rtimes U_{n-1}$. We note that for

$$u' = \begin{pmatrix}1_{n-1}&a\\0&1\end{pmatrix}\in U_0\quad\text{with}\quad a=\begin{pmatrix}a_1\\\vdots\\a_{n-1}\end{pmatrix}\tag{3.37}$$

and $u_1\in U_{n-1}$, we have

$$\omega_\psi(g,m(u'u_1))\varphi(x_0) = \omega_\psi(g,m(u_1))\varphi\Big(e_{-1},\dots,e_{-n+1},e_\lambda+\sum_{j=1}^{n-1}a_je_{-j}\Big)$$

by (2.28b). For $u'\in U_0$ of the form (3.37), let

$$s(u') := s_n(a_{n-1})\cdots s_1(a_1).$$

We recall that $s_j(a)$ is defined by (2.23). Then by (2.28a), we have

$$\omega_\psi(g,m(u_1))\varphi\Big(e_{-1},\dots,e_{-n+1},e_\lambda+\sum_{j=1}^{n-1}a_je_{-j}\Big) = \omega_\psi(s(u')^{-1}g,m(u_1))\varphi(x_0).$$

Further we note that by (2.28a) and (2.28b), we have

$$\omega_\psi(s(u')^{-1}g, m(u_1))\varphi(x_0) = \omega_\psi(\check{u}_1^{-1}s(u')^{-1}g, 1)\varphi(x_0)$$

where $\check{u}_1$ is defined by (1.3) for $u_1 \in U_{n-1}$. We also note that

$$\psi_\lambda(m(u'u_1)) = \chi_\lambda(s(u')\check{u}_1)$$

by (1.5). Hence the integral of the right hand side of (3.35) is equal to

$$\int_{U_n}^{\mathrm{st}} \int_{R'_\lambda\backslash G} \int_{R'_\lambda\backslash G} \int_{R'_\lambda} \omega_\psi(\check{u}_1^{-1}s(u')^{-1}g, 1)\varphi(x_0)\,\overline{\varphi'(h^{-1}\cdot x_0)} \times \langle \pi(r'g)\phi, \pi(h)\phi'\rangle \chi_\lambda(s(u')\check{u}_1)^{-1}\,dr'\,dg\,dh\,du \tag{3.38}$$

where $u = u'u_1$, $u' \in U_0$ and $u_1 \in U_{n-1}$. We note the following elementary lemma.

Lemma 5. (1) *For a given compact open subgroup U°_{n-1} of U_{n-1} and a compact open subgroup U'_0 of U_0, there exists a compact open subgroup U°_0 of U_0 such that $U^\circ_{n-1}U^\circ_0$ is a subgroup of U_n and $U^\circ_0 \supset U'_0$.*

(2) *For a given compact open subgroup U° of U_n, there exist a compact open subgroup U°_{n-1} of U_{n-1} and a compact open subgroup U°_0 of U_0 such that $U^\circ_{n-1}U^\circ_0$ is a subgroup of U_n containing U°_n.*

Proof. For a positive integer r, let

$$U_0^{(r)} = \left\{ \begin{pmatrix} 1_{n-1} & a \\ 0 & 1 \end{pmatrix} : a \in \varpi^{-r} \begin{pmatrix} \mathcal{O} \\ \vdots \\ \mathcal{O} \end{pmatrix} \right\}.$$

(1) Take r sufficiently large so that $U_0^{(r)} \supset U'_0$. Since U°_{n-1} is compact, there exists an integer s such that all entries of elements of U°_{n-1} are in $\varpi^{-s}\mathcal{O}$. Let us take integers $r_1, \dots, r_{n-1}$ inductively so that $r_{n-1} = r$ and $r_{n-k} \geq \max\{r, sr_{n-1}, \dots, sr_{n-k+1}\}$ for $2 \leq k \leq n-1$. Let

$$U^\circ_0 = \left\{ \begin{pmatrix} 1_{n-1} & a \\ 0 & 1 \end{pmatrix} : a \in \begin{pmatrix} \varpi^{-r_1}\mathcal{O} \\ \vdots \\ \varpi^{-r_{n-1}}\mathcal{O} \end{pmatrix} \right\}. \tag{3.39}$$

Then $U^\circ_{n-1}U^\circ_0$ is a subgroup of U_n and $U^\circ_0 \supset U'_0$.

(2) Let $U^\circ_{n-1} = U^\circ \cap U_{n-1}$. Then U°_{n-1} is a compact open subgroup of U_{n-1}. Since $U^\circ \subset \bigcup_{r\geq 1} U^\circ_{n-1}U_0^{(r)}$ and U° is compact, we have $U^\circ \subset U^\circ_{n-1}U_0^{(r)}$ for r sufficiently large. By (1), we may take a compact open subgroup of U°_0 of U_0 such that $U^\circ_{n-1}U^\circ_0$ is a subgroup U_n and $U^\circ_0 \supset U_0^{(r)}$. □

By the definition of stable integration, for any sufficiently large compact open subgroup U° of U_n, the integral (3.38) is equal to

$$\int_{U_n}\int_{R'_\lambda\backslash G}\int_{R'_\lambda\backslash G}\int_{R'_\lambda}\chi_{U^\circ}(u'u_1)\cdot\omega_\psi(\check{u}_1^{-1}s(u')^{-1}g,1)\varphi(x_0)\,\overline{\varphi'(h^{-1}\cdot x_0)}\\ \times\langle\pi(r'g)\phi,\pi(h)\phi'\rangle\chi_\lambda(s(u')\check{u}_1)^{-1}\,dr'\,dg\,dh\,du'\,du_1 \tag{3.40}$$

where χ_{U° is the characteristic function of U°. By Lemma 5, we may take a compact open subgroup U°_{n-1} of U_{n-1} and a compact open subgroup U°_0 of U_0 of the form (3.39) such that $U^\circ_{n-1}U^\circ_0$ is a compact open subgroup of U_n and $U^\circ_{n-1}U^\circ_0\supset U^\circ$. Then (3.40) is equal to

$$\int_{U^\circ_{n-1}}\int_{U^\circ_0}\int_{R'_\lambda\backslash G}\int_{R'_\lambda\backslash G}\int_{R'_\lambda}\omega_\psi(\check{u}_1^{-1}s(u')^{-1}g,1)\varphi(x_0)\,\overline{\varphi'(h^{-1}\cdot x_0)}\\ \times\langle\pi(r'g)\phi,\pi(h)\phi'\rangle\chi_\lambda(s(u')\check{u}_1)^{-1}\,dr'\,dg\,dh\,du'\,du_1. \tag{3.41}$$

Since the argument to obtain (3.35) ensures the absolute convergence of the inner triple integral and the outer double integral is over a compact group $U^\circ_{n-1}U^\circ_0$, the integral (3.41) converges absolutely. Hence we may change the order of integration and we obtain

$$\begin{aligned}&\mathcal{W}(\theta(\varphi\otimes\phi),\theta(\varphi'\otimes\phi'))\\ &\quad=C_{E/F}\cdot\int_{R'_\lambda\backslash G}\int_{R'_\lambda\backslash G}\int_{R'_\lambda}\int_{U^\circ_0}\int_{U^\circ_{n-1}}(\omega_\psi(g)\varphi)(x_0)\,\overline{(\omega_\psi(h)\varphi')(x_0)}\\ &\qquad\times\langle\pi(r's(u')\check{u}_1g)\phi,\pi(h)\phi'\rangle\chi_\lambda(s(u')\check{u}_1)^{-1}\,du_1\,du'\,dr'\,dg\,dh.\end{aligned} \tag{3.42}$$

Then Proposition 4 is reduced to the following lemma.

Lemma 6. *Keep the above notation. Then*

$$\int_{R'_\lambda}\int_{U^\circ_0}\int_{U^\circ_{n-1}}\langle\pi(r's(u')\check{u}_1g)\phi,\pi(h)\phi'\rangle\chi_\lambda(s(u')\check{u}_1)^{-1}\,du_1\,du'\,dr'\\ =\alpha(\pi(g)\phi,\pi(h)\phi'). \tag{3.43}$$

Assume that (3.43) holds. Then by replacing the the most inner triple integral of (3.42) by $\alpha(\pi(g)\phi,\pi(h)\phi')$, we obtain the equality (3.26) in Proposition 4.

3.4.3. Proof of Lemma 6. Let us prove Lemma 6 and complete the proof of Proposition 2 in the non-archimedean case.

Since U°_0 and U°_{n-1} are compact, the integral on the left hand side of (3.43) converges absolutely by Remark 16. Hence by noting that $R'_\lambda=D_\lambda S'_\lambda$ where S'_λ is as given in (2.22)

and by changing the order of integration, we have

$$\int_{R'_\lambda}\int_{U_0^\circ}\int_{U_{n-1}^\circ}\langle\pi(r's(u')\check{u}_1 g)\phi,\pi(h)\phi'\rangle\chi_\lambda(s(u')\check{u}_1)^{-1}\,du_1\,du'\,dr'$$
$$=\int_{D_\lambda}\int_{U_{n-1}^\circ}\int_{U_0^\circ}\int_{S'_\lambda}\langle\pi(s_0s(u')\check{u}_1 tg)\phi,\pi(h)\phi'\rangle\chi_\lambda(s_0s(u')\check{u}_1)^{-1}\,ds_0\,du'\,du_1\,dt. \tag{3.44}$$

Let us define an open subgroup $S^\sharp$ of S' by

$$S^\sharp:=\left\{\begin{pmatrix}1_{n-1}&A&B\\0&1_3&A'\\0&0&1_{n-1}\end{pmatrix}:Ae_\lambda\in\varpi^{-r_1}\mathcal{O}e_{-1}+\cdots+\varpi^{-r_{n-1}}\mathcal{O}e_{-n+1}\right\}$$

with r_i given in (3.39). Then by considering a filtration of $S^\sharp$ given by

$$S'_\lambda\lhd(S'_1\cap S^\sharp)\lhd\cdots\lhd(S'_{n-1}\cap S^\sharp)=S^\sharp$$

induced from (2.24) and by taking (2.25) into account, the integral (3.44) is equal to

$$\int_{D_\lambda}\int_{S^\star}\langle\pi(stg)\phi,\pi(h)\phi'\rangle\chi_\lambda(s)^{-1}\,ds\,dt. \tag{3.45}$$

Here $S^\star$ is an open subgroup of S given by $S^\star=\check{U}_{n-1}^\circ S^\sharp$ where $\check{U}_{n-1}^\circ$ is a subgroup $\{\check{u}:u\in U_{n-1}^\circ\}$ of S''. Hence, by the definition (1.7) of $\alpha(\pi(g)\phi,\pi(h)\phi')$, it suffices for us to show

$$\int_{S^\star}\langle\pi(stg)\phi,\pi(h)\phi'\rangle\chi_\lambda(s)^{-1}\,ds=\int_S{}^{\mathrm{st}}\langle\pi(stg)\phi,\pi(h)\phi'\rangle\chi_\lambda(s)^{-1}\,ds \tag{3.46}$$

in order to prove Lemma 6.

We recall that the integrand of (3.45) has a stable integral over S by [37, Proposition 3.1]. Hence there exists a compact open subgroup $S^\flat$ of S such that for any compact open subgroups S° of S containing $S^\flat$, we have

$$\int_S{}^{\mathrm{st}}\langle\pi(stg)\phi,\pi(h)\phi'\rangle\chi_\lambda(s)^{-1}\,ds=\int_{S^\circ}\langle\pi(stg)\phi,\pi(h)\phi'\rangle\chi_\lambda(s)^{-1}\,ds.$$

By taking U_{n-1}° and r_i sufficiently large, we may suppose that $S^\flat\subset S^\star$. Then for any compact open subgroups S° of S containing $S^\flat$, we have

$$\int_{S^\circ\cap S^\star}\langle\pi(stg)\phi,\pi(h)\phi'\rangle\chi_\lambda(s)^{-1}\,ds=\int_S{}^{\mathrm{st}}\langle\pi(stg)\phi,\pi(h)\phi'\rangle\chi_\lambda(s)^{-1}\,ds$$

since $S^\circ\cap S^\star$ is a compact open subgroup of S containing $S^\flat$. Let f denote a function on S defined by

$$f(s):=\chi_{S^\star}(s)\cdot\langle\pi(stg)\phi,\pi(h)\phi'\rangle\chi_\lambda(s)^{-1}$$

for $s \in S$, where $\chi_{S^\star}$ denotes the characteristic function of $S^\star$. Then we have

$$\int_{S^\circ} f(s)\,ds = \int_{S^\circ \cap S^\star} \langle \pi(stg)\phi, \pi(h)\phi' \rangle \chi_\lambda(s)^{-1}\,ds = \int_S{}^{\mathrm{st}} \langle \pi(stg)\phi, \pi(h)\phi' \rangle \chi_\lambda(s)^{-1}\,ds.$$

This implies that f has a stable integral over S and we have

$$\int_S{}^{\mathrm{st}} f(s)\,ds = \int_S{}^{\mathrm{st}} \langle \pi(stg)\phi, \pi(h)\phi' \rangle \chi_\lambda(s)^{-1}\,ds.$$

Hence by applying Remark 1 to f, we have (3.46). This completes the proof of Lemma 6 and the proof of Proposition 2 in the non-archimedean case.

3.4.4. Archimedean case. Suppose that F is archimedean. Since $\mathcal{X}_\lambda$ is locally closed in V^n, the function $R'_\lambda \backslash G \ni g \mapsto \varphi(g^{-1} \cdot x_0)$ is compactly supported for any $\varphi \in C_c^\infty(V^n)$. Therefore, by Liu [37, Proposition 3.5], the integral

$$\int_{R'_\lambda \backslash G} \int_{R'_\lambda \backslash G} \int_{D_\lambda} \int_S \langle \pi(stg)\phi, \pi(h)\phi' \rangle \chi_\lambda(s)^{-1}\,ds\,dt\,dg\,dh \tag{3.47}$$

converges absolutely. Then we may change the order of integration in (3.47) and, by an argument similar to the one in Sections 3.4.2 and 3.4.3 in the non-archimedean case, we may show that the integral (3.47) is equal to the right hand side of (3.35). Then the equality (3.26) readily follows and Proposition 4 is proved also in the archimedean case.

3.4.5. Corollary of Proposition 4. We note the following, which is a local counterpart of [15, Proposition 2 and 3], as a corollary of Proposition 4.

Corollary 2. *Let π be an irreducible unitary representation of G. Suppose that π is tempered when F is non-archimedean and π is a discrete series representation when F is archimedean. Then for $\sigma = \theta(\pi, \psi)$, we have*

$$\mathrm{Hom}_N(\sigma, \psi_\lambda) \neq \{0\} \iff \alpha \not\equiv 0. \tag{3.48}$$

Proof. By Lapid and Mao [33, Proposition 2.10], $\mathrm{Hom}_N(\sigma, \psi_\lambda) \neq \{0\}$ implies that $\mathcal{W}$ defined by (3.8) is not identically zero. Then (3.26) clearly implies that α is not identically zero. Conversely suppose that there exist $\phi, \phi' \in V_\pi$ such that $\alpha(\phi, \phi') \neq 0$. Then by Lemma 1 and Proposition 3, $\mathcal{W}$ is not identically zero and it clearly implies that σ is ψ_λ-generic, i.e. $\mathrm{Hom}_N(\sigma, \psi_\lambda) \neq \{0\}$, by (3.9). □

3.5. Proof of the statement (1) of Theorem 1

We return to the global setting. As we noted in (2.1f), $\sigma = \Theta_n(\pi, \psi)$ is ψ_λ-generic when $B_{\lambda,\psi} \not\equiv 0$. Hence its local component σ_v is $\psi_{\lambda,v}$-generic at every place v of F. Thus at any place v of F, α_v does not vanish identically by Corollary 2.

4. Proof of Corollary 1

By Theorem 1, it is enough to show that the right hand side of (1.19) vanishes identically when $B_{\lambda,\psi} \equiv 0$. Suppose the contrary. Then in particular $L(1/2, \pi)L(1/2, \pi \times \chi_E) \neq 0$. By the assumption that Conjecture 9.5.4 in Arthur [3] holds for any group in $\mathcal{G}$, π has a weak lift to $\mathrm{GL}_{2n}(\mathbb{A})$. Then the global descent method by Ginzburg, Rallis and Soudry [21] gives an irreducible cuspidal globally generic automorphic representation π° of $\mathbb{G}(\mathbb{A})$ which is nearly equivalent to π. Thus Proposition 5 in [15] is applicable to π. Hence there exist $G' = \mathrm{SO}(V') \in \mathcal{G}$ where $\mathrm{disc}(V') = (-1)^n$ and an irreducible cuspidal automorphic representation π' of $G'(\mathbb{A})$ having the special Bessel model of type E, which is nearly equivalent to π. We shall reach a contradiction by showing that $G = G'$ and $\pi = \pi'$.

Since $B_{\lambda,\psi} \not\equiv 0$ on $V_{\pi'}$, $\Theta_n(\pi', \psi)$, the theta lift of π' to $\widetilde{\mathrm{Sp}}_n(\mathbb{A})$ with respect to ψ is ψ_λ-generic by [15, Proposition 2]. In particular $\theta(\pi'_v, \psi_v)$ is $\psi_{\lambda,v}$-generic for any v. On the other hand, we have $\alpha_v \not\equiv 0$ on V_{π_v} since the right hand side of (1.19) is not identically zero. Hence $\theta(\pi_v, \psi_v)$ is also $\psi_{\lambda,v}$-generic for any v by Corollary 2.

Suppose that v is finite. Since π and π' are nearly equivalent, it is readily shown that they have the same A-parameter by an argument similar to the one in Atobe and Gan [4]. Further the temperedness of π implies that π and π' share the same local L-parameter at each finite place. Here we recall the assumption that the local Langlands correspondence [3, Conjecture 9.4.2] holds for any element of $\mathcal{G}$. Since π_v and π'_v both have the special Bessel model of type E_v, we have $G_v \simeq G'_v$ and $\pi_v \simeq \pi'_v$ by Waldspurger [47, 48].

When v is real, $\theta(\pi_v, \psi_v)$ and $\theta(\pi'_v, \psi_v)$ have the same L-parameter by Adams and Barbasch [1]. Then we have $\theta(\pi_v, \psi_v) \simeq \theta(\pi'_v, \psi_v)$ by the uniqueness of generic element in tempered L-packets (see Kostant [30], Shelstad [43] and Vogan [45]). Since V and V' have the same discriminant, we have $G_v \simeq G'_v$ by [1]. Hence by the Howe duality, we have $\pi_v \simeq \pi'_v$.

Thus we have shown that $G_v \simeq G'_v$ and $\pi_v \simeq \pi'_v$ for any place v of F. Hence we have $G = G'$ and $\pi \simeq \pi'$. The latter actually implies that $\pi = \pi'$ since the multiplicity of π is one by Arthur [3, Conjecture 9.5.4].

Acknowledgments. It is evident that this paper owes its existence to the Ichino–Ikeda type formula for Whittaker periods on metaplectic groups proved by Lapid and Mao, as the culmination of their profound papers [32, 33, 34, 35]. The authors would like to express their admiration for Lapid and Mao. Indeed in [33] they formulate the Ichino–Ikeda type conjectural formula for Whittaker periods on general quasi-split reductive groups, besides metaplectic groups. We expect their conjectural formula to have broad and deep influence on future research in the field of automorphic forms. It is also evident that, as in our previous paper [15], we take full advantage of the recent significant contributions to theta correspondence. The most notable ones in this paper are [20, 19] by Gan and Takeda. The former [20] allows us to use freely the global-local machinery and the latter [19] is one of the main ingredients for the proof of (1.19), as explained above. We also mention that not only the formulation of the conjecture by Liu [37] gave us the starting point of this work but also our arguments in Section 3 are very much inspired by his arguments in [37, 3.5].

The first author has been pursuing the relative trace formula approach towards Böcherer's conjecture with Martin and Shalika [12, 13, 14, 16]. Although some substantial progress has been made, one must admit that there still lies ahead a long way to go by this approach. The remarkable result by Lapid and Mao enabled us to present a proof of Böcherer's conjecture here in the special Bessel

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