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Olga Balkanova · Dmitry Frolenkov

Moments of L-functions and the Liouville–Green method

To the memory of Professor N. V. Kuznetsov

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Abstract. We show that for at least 20% of primitive forms of level 1 and weight $4k \to \infty$, $k \in \mathbb{N}$, the associated L-function at the central point is no less than $(\log k)^{-2}$. The key ingredients of our proof are the Kuznetsov convolution formula and the Liouville–Green method.

Keywords. Central values of L-functions, non-vanishing, Liouville–Green method, weight aspect, WKB approximation

1. Introduction

Non-vanishing results for central values of L-functions in families have numerous applications, discovered, for example, in [7, 15, 20, 22, 35]. In particular, this paper is inspired by the work of Iwaniec and Sarnak [15], where the problem of non-existence of Landau–Siegel zeros was approached by studying the non-vanishing of automorphic L-functions at the critical point.

In the weight aspect, Iwaniec and Sarnak proved the following result. Let $H_{2k}(1)$ be the space of primitive forms of level 1 and weight $2k \geq 12$, $k \in \mathbb{N}$. For $f \in H_{2k}(1)$, let $L_f(1/2)$ be the associated L-function at the critical point. Then for any $\epsilon > 0$ one has

$$\lim_{K\to\infty} \frac{1}{K} \sum_{k\leq K} \frac{\#\{f \in H_{4k}(1) : L_f(1/2) \geq 1/(\log k)^2\}}{\#\{f \in H_{4k}(1)\}} \geq \frac{1}{2} - \epsilon. \tag{1.1}$$

Moreover, it was shown in [15] that the non-existence of Landau–Siegel zeros for Dirichlet L-functions of real primitive characters would follow if (1.1) is established with a proportion strictly greater than $1/2$.

The problem of non-vanishing for the individual weight was first studied by Iwaniec, Luo and Sarnak [14]. More precisely, [14, Corollary 1.6] states that under the generalized Riemann Hypothesis the percentage of non-vanishing central L-values in this case is at

O. Balkanova: Steklov Mathematical Institute of Russian Academy of Sciences,
8 Gubkina St., Moscow, 119991, Russia; e-mail: olgabalkanova@gmail.com

D. Frolenkov: HSE University and Steklov Mathematical Institute of Russian Academy of Sciences,
8 Gubkina St., Moscow, 119991, Russia; e-mail: frolenkov@mi-ras.ru

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least 25%. The first unconditional results for the individual weight were obtained independently by Fomenko [8] and by Lau & Tsang [24] who proved that the proportion of non-vanishing is at least $1/\log k$ by computing the asymptotics of pure unmollified first and second moments. Recently, Luo [26] showed that there is a strictly positive proportion of non-vanishing. However, his approach does not allow finding the exact proportion. The reason is that only an upper bound for the mollified second moment was proved in [26], while the full asymptotic expansion is required to make this result quantitative.

The aim of the present paper is to obtain an effective and strictly positive lower bound on the proportion of non-vanishing central L-values.

Theorem 1.1. *For any $\epsilon > 0$ there exists $k_0 = k_0(\epsilon)$ such that for any $k \geq k_0$ and $k \equiv 0 \pmod 2$ we have*

$$\frac{1}{|H_{2k}(1)|} \sum_{\substack{f \in H_{2k}(1) \\ L_f(1/2) \geq (\log k)^{-2}}} 1 \geq \frac{1}{5} - \epsilon. \tag{1.2}$$

The proof of Theorem 1.1 relies in most part on methods developed by Kuznetsov in 1990's. More precisely, we use Theorem 4.2, which gives an exact formula for the second moment of cusp form L-functions in the critical strip. Off-diagonal terms in this formula are given by two shifted convolution sums:

$$\frac{1}{2\sqrt{l}} \sum_{n=1}^{l-1} \tau(n)\tau(l-n)\phi_k\left(\frac{n}{l}\right) + \frac{1}{\sqrt{l}} \sum_{n=1}^{\infty} \tau(n)\tau(n+l)\Phi_k\left(\frac{l}{n+l}\right),$$

where $\tau(n)$ is the number of divisors function and $\phi_k(x)$, $\Phi_k(x)$ are certain special functions that can be expressed in terms of the Gauss hypergeometric function $F(a,b,c;x)$.

The most challenging problem is to prove sharp estimates for $\phi_k(x)$ and $\Phi_k(x)$ that are uniform in both x and k. The key observation is that these functions are solutions of second-order differential equations, and therefore we can choose the Liouville–Green method as our main tool. This method, also called the WKB approximation or the Liouville–Steklov method, is one of the oldest approximation techniques widely applied for example in quantum mechanics. The idea of using it in analytic number theory belongs to Kuznetsov. The method is based on the observation that "close" differential equations have "close" solutions. Accordingly, in Section 5 we find differential equations satisfied by the functions $\phi_k(x)$ and $\Phi_k(x)$. The equations can be approximated by other differential equations which have "simpler" functions as solutions. This allows approximating the off-diagonal terms uniformly in k with any power of precision (see Theorems 5.14 and 5.17).

More precisely, the Liouville–Green approximation models the behaviour of the functions $\phi_k(x)$ and $\Phi_k(x)$ using J-, Y- and K-Bessel functions with the large parameter k in the argument. Accordingly, we conclude that in the required ranges $\Phi_k(x)$ decays exponentially and $\phi_k(x)$ is oscillatory. To smooth out the oscillations of $\phi_k(x)$ one can average the off-diagonal terms over weight with a suitable test function, recovering the result of Iwaniec and Sarnak (1.1), as shown in Theorem 8.8.

Asymptotic formulas for twisted moments have several other applications. For example, Hough [11] considered zero-density estimates for L-functions in the weight aspect. His proof is based on the asymptotic evaluation of the second moment near the critical line with the error term estimated as $O(l^{3/4}k^{-1/2+\epsilon})$ at the central point. The same error bound was obtained by Ng [30] using a different approach. Our method (see Theorem 6.4) yields $O(l^{1/2}k^{-1/2+\epsilon})$.

Finally, the techniques developed in the present paper can be beneficial in solving other problems in analytic number theory which involve analysis of special functions. In particular, our approach yields new results for moments of symmetric square L-functions in the weight aspect. See [2] for details.

2. Notation and technical lemmas

For $v \in \mathbb{C}$ let

$$\tau_v(n) = \sum_{n_1 n_2 = n} \left(\frac{n_1}{n_2}\right)^v = n^{-v}\sigma_{2v}(n), \tag{2.1}$$

where

$$\sigma_v(n) = \sum_{d|n} d^v. \tag{2.2}$$

Note that $\tau_v(n) = \tau_{-v}(n)$.

Let $e(x) = \exp(2\pi i x)$. The classical Kloosterman sum

$$S(m, n; c) = \sum_{\substack{a \,(\mathrm{mod}\, c)\\ (a,c)=1}} e\left(\frac{am + a^* n}{c}\right), \qquad aa^* \equiv 1 \pmod{c},$$

satisfies Weil's bound (see [12, Theorem 4.5])

$$|S(m, n; c)| \le \tau_0(c)\sqrt{(m, n, c)}\sqrt{c}. \tag{2.3}$$

For $\Re s > 1$, $m \ge 1$ (see [34, Eq. 1.5.4])

$$\sum_{c=1}^{\infty} \frac{S(0, m; c)}{c^s} = \frac{\sigma_{1-s}(m)}{\zeta(s)}, \tag{2.4}$$

where $\zeta(s)$ is the Riemann zeta function.

Let $H_{2k}(1)$ be the set of primitive forms of level 1 and weight $2k \ge 12$. Every $f \in H_{2k}(1)$ has a Fourier expansion of the form

$$f(z) = \sum_{n \ge 1} \lambda_f(n) n^{(2k-1)/2} e(nz). \tag{2.5}$$

The Fourier coefficients of primitive forms are multiplicative:

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \lambda_f\left(\frac{mn}{d^2}\right). \tag{2.6}$$

For each $f \in H_{2k}(1)$, the associated L-function is defined by

$$L_f(s) = \sum_{n\geq 1} \frac{\lambda_f(n)}{n^s}, \quad \Re s > 1, \tag{2.7}$$

and the associated symmetric square L-function is given by

$$L(\mathrm{sym}^2 f, s) = \zeta(2s) \sum_{n=1}^{\infty} \frac{\lambda_f(n^2)}{n^s} = \sum_{n=1}^{\infty} \frac{\rho_f(n)}{n^s}, \quad \Re s > 1. \tag{2.8}$$

As a consequence of (2.6), for $\Re u > 1/2$, $\Re v = 0$ we have

$$\sum_{n=1}^{\infty} \frac{\tau_v(n)\lambda_f(n)}{n^{1/2+u}} = \frac{1}{\zeta(1+2u)} L_f(1/2+u+v) L_f(1/2+u-v). \tag{2.9}$$

Let $\Gamma(s)$ be the Gamma function. The completed L-function

$$\Lambda_f(s) = \left(\frac{1}{2\pi}\right)^s \Gamma\left(s + \frac{2k-1}{2}\right) L_f(s) \tag{2.10}$$

satisfies the functional equation

$$\Lambda_f(s) = \epsilon_f \Lambda_f(1-s), \quad \epsilon_f = i^{2k}, \tag{2.11}$$

and can be analytically continued onto the whole complex plane. It follows from (2.11) that $L_f(1/2) = 0$ for odd k.

The harmonic weight is defined by (see [14, Lemma 2.5])

$$\omega_f = \frac{\Gamma(2k-1)}{(4\pi)^{2k-1}\langle f, f\rangle_1} = \frac{12\zeta(2)}{(2k-1)L(\mathrm{sym}^2 f, 1)}, \tag{2.12}$$

where $\langle f, f\rangle_1$ is the Petersson inner product on the space of level 1 holomorphic modular forms. Then the *harmonic summation* can be written as

$$\sum_{f\in H_{2k}(1)}^{h} \alpha_f := \sum_{f\in H_{2k}(1)} \omega_f \alpha_f. \tag{2.13}$$

We denote by $J_v(x)$, $Y_v(x)$, $K_v(x)$ the Bessel functions.

Theorem 2.1 (Petersson's trace formula, [12, Theorem 4.5]). *For $2k \geq 12$ and integers $l, n \geq 1$ the following formula holds:*

$$\sum_{f\in H_{2k}(1)}^{h} \lambda_f(l)\lambda_f(n) = \delta_{l,n} + 2\pi i^{2k} \sum_{c=1}^{\infty} \frac{S(l,n;c)}{c} J_{2k-1}\left(\frac{4\pi\sqrt{ln}}{c}\right). \tag{2.14}$$

Consider the Bessel kernels

$$k_0(x, v) = \frac{1}{2\cos(\pi(1/2+v))}(J_{2v}(x) - J_{-2v}(x)), \tag{2.15}$$

$$k_1(x, v) = \frac{2}{\pi}\sin(\pi(1/2+v))K_{2v}(x). \tag{2.16}$$

Lemma 2.2. *Let*

$$\gamma(u,v)=\frac{2^{2u-1}}{\pi}\Gamma(u+v)\Gamma(u-v). \tag{2.17}$$

For $3/2>\Re w>2|\Re v|$ *we have*

$$\int_0^\infty k_0(x,v)x^{w-1}\,dx=\gamma(w/2,v)\cos(\pi w/2), \tag{2.18}$$

and for $\Re w>2|\Re v|$,

$$\int_0^\infty k_1(x,v)x^{w-1}\,dx=\gamma(w/2,v)\sin(\pi(1/2+v)). \tag{2.19}$$

Proof. (2.19) follows from [9, Eq. 6.561.16]. Let us prove (2.18). Applying [9, Eq. 6.561.14] we obtain, for $3/2>\Re w>2|\Re v|$,

$$\int_0^\infty k_0(x,v)x^{w-1}\,dx=\frac{2^{w-1}}{2\cos(\pi(1/2+v))}\times\left(\frac{\Gamma(v+w/2)}{\Gamma(1+v-w/2)}-\frac{\Gamma(-v+w/2)}{\Gamma(1-v-w/2)}\right). \tag{2.20}$$

Then [29, Eq. 5.5.3] yields

$$\int_0^\infty k_0(x,v)x^{w-1}\,dx=\frac{2^{w-1}}{2\pi\cos(\pi(1/2+v))}\Gamma(v+w/2)\Gamma(w/2-v)\times[\sin(\pi(w/2-v))-\sin(\pi(w/2+v))]. \tag{2.21}$$

Applying [29, Eq. 4.21.7] we obtain (2.18). □

Consider the series

$$D_v(s,x):=\sum_{n\ge1}\frac{\tau_v(n)}{n^s}e(nx),\quad \Re s>1. \tag{2.22}$$

Let x be a rational number d/c with $(d,c)=1$, $c\ge1$. Then the function $D_v(s,x)$ of two complex parameters s and v is meromorphic on the whole complex plane. If we fix v such that $\Re v=0$ and $v\neq0$, then $D_v(s,d/c)$, as a function of s, has two simple poles at $s=1+v$ and $s=1-v$ with residues $c^{-1-2v}\zeta(1+2v)$ and $c^{-1+2v}\zeta(1-2v)$, respectively, and it is regular elsewhere. Also it satisfies the functional equation (see [27, Lemma 3.7])

$$D_v(s,d/c)=(4\pi/c)^{2s-1}\gamma(1-s,v)\times\{-\cos(\pi s)D_v(1-s,-d^*/c)+\sin(\pi(1/2+v))D_v(1-s,d^*/c)\}, \tag{2.23}$$

where $dd^*\equiv1 \pmod c$ and $\gamma(u,v)$ is defined by (2.17). For $\Re s<0$ the following estimate is satisfied (see [27, Eq. 3.3.24]):

$$|D_v(s,d/c)|\ll(c|s|)^{1-2\Re s}(\log|s|)^2. \tag{2.24}$$

Note that

$$D_v(s,d/c)=\sum_{n=1}^{\infty}\frac{\tau_v(n)}{n^s}e\left(n\frac{d}{c}\right)=\frac{1}{c^{2s}}\sum_{a,b=1}^{c}e\left(ab\frac{d}{c}\right)\zeta(a/c;s-v)\zeta(b/c;s+v), \tag{2.25}$$

where

$$\zeta(\alpha;s)=\sum_{n+\alpha>0}\frac{1}{(n+\alpha)^s},\qquad \Re s>1, \tag{2.26}$$

is the Lerch zeta function. Applying the Euler–Maclaurin formula, we have (see [18, Lemma 3, p. 16])

$$\zeta(\alpha;s)=\sum_{n=0}^{N}\frac{1}{(n+\alpha)^s}+\frac{1}{s-1}(N+1/2+\alpha)^{1-s}+s\int_{N+1/2}^{\infty}\frac{1/2-\{u\}}{(u+\alpha)^{s+1}}\,du, \tag{2.27}$$

where $\{u\}$ is the fractional part of u. For any $\epsilon>0$ we estimate the absolute value of (2.27):

$$|\zeta(\alpha;\sigma+iT)|\ll_\epsilon\begin{cases}1, & \sigma\geq 1+\epsilon,\\ \log T, & 1\leq\sigma\leq 1+\epsilon,\\ T^{1-\sigma}, & \epsilon\leq\sigma<1.\end{cases} \tag{2.28}$$

The Mellin transform of a function f is defined by

$$\hat{f}(z)=\int_0^{\infty}t^{z-1}f(t)\,dt. \tag{2.29}$$

Lemma 2.3 (Parseval's inequality). *Assume that for some $a\in\mathbb{R}$,*

$$\int_0^{\infty}|g(x)|x^{-a}\,dx<\infty,\qquad \int_{(a)}|\hat{\phi}(z)|\,dz<\infty. \tag{2.30}$$

Then

$$\frac{1}{2\pi i}\int_{(a)}\hat{\phi}(z)\hat{g}(1-z)\,dz=\int_0^{\infty}\phi(x)g(x)\,dx. \tag{2.31}$$

Proof. See, for example, [31, Section 3.1.3]. □

3. The first moment

In this section we derive the following asymptotic formula for the first moment of automorphic L-functions at the critical point.

Theorem 3.1. *For $2k\geq 12$ and $l<k/(4\pi e)$ we have*

$$\sum_{f\in H_{2k}(1)}^{h}\lambda_f(l)L_f(1/2)=l^{-1/2}(1+i^{2k})+O\left(\frac{1}{\sqrt{l}}\left(2\pi e\frac{l}{k}\right)^k\right). \tag{3.1}$$

Proof. This is a consequence of [3, Theorem 3.1] and [1, Theorem 1.7]. □

4. The second moment

4.1. Voronoi's summation formula

Standard Voronoi's summation formulas (see [16]) are stated for a function ϕ with compact support. However, weaker conditions on ϕ are required to prove a convolution formula for the second moment.

Lemma 4.1 (Kuznetsov, 1981). *Assume that the Mellin transform $\hat{\phi}(s)$ of $\phi : [0, \infty) \to \mathbb{C}$ satisfies the following conditions:*

(1) *$\hat{\phi}(2s)$ is regular in the region $\sigma_0 < \Re s < \sigma_1$ for some $\sigma_1 > 1$ and $\sigma_0 < 0$;*
(2) *for some $\epsilon > 0$ and for $\sigma_0 < \sigma < \sigma_1$ the function*

$$\big((1+|t|)^{1-2\sigma+\epsilon}+1\big)|\hat{\phi}(2\sigma+2it)|$$

is integrable on $(-\infty, \infty)$.

Then for any v with $\Re v = 0$, $v \neq 0$ and for any coprime integers $c, d \geq 1$ we have

$$\frac{4\pi}{c}\sum_{n=1}^{\infty}\tau_v(n)e\left(\frac{nd}{c}\right)\phi\left(\frac{4\pi\sqrt{n}}{c}\right) = 2\frac{\zeta(1+2v)}{(4\pi)^{1+2v}}\hat{\phi}(2+2v) + 2\frac{\zeta(1-2v)}{(4\pi)^{1-2v}}\hat{\phi}(2-2v)$$
$$+\sum_{n=1}^{\infty}\tau_v(n)\int_0^{\infty}\left(e\left(-\frac{nd^*}{c}\right)k_0(x\sqrt{n}, v) + e\left(\frac{nd^*}{c}\right)k_1(x\sqrt{n}, v)\right)\phi(x)x\,dx, \quad (4.1)$$

where $dd^* \equiv 1 \pmod{c}$.

Originally, Lemma 4.1 was proved by Kuznetsov in his doctoral thesis (1981) and published in [23]. Unfortunately, the book [23] is hard to find, so we provide all the details here.

The proof of Lemma 4.1 is based on the properties of the series $D_v(s, x)$ defined by (2.22). Applying the inverse Mellin transform, we have

$$\phi\left(\frac{4\pi\sqrt{n}}{c}\right) = \frac{1}{i\pi}\int_{(b)}\hat{\phi}(2s)\left(\frac{c}{4\pi}\right)^{2s}\frac{1}{n^s}\,ds, \qquad 1 < b < \sigma_1.$$

Therefore,

$$\frac{4\pi}{c}\sum_{m=1}^{\infty}e\left(\frac{md}{c}\right)\tau_v(m)\phi\left(\frac{4\pi\sqrt{m}}{c}\right) = \frac{1}{i\pi}\int_{(b)}\left(\frac{c}{4\pi}\right)^{2s-1}D_v\left(s, \frac{d}{c}\right)\hat{\phi}(2s)\,ds.$$

The change of the order of integration and summation in the formula above is allowed since $\int_{(b)}|\hat{\phi}(2s)|\,ds < \infty$ and the series $D_v(s, d/c)$ converges absolutely for $\Re s = b > 1$.

Moving the contour of integration to $\Re s = \delta$ with $\sigma_0 < \delta < 0$, we encounter two simple poles of $D_v(s, d/c)$ at $1+v$ and $1-v$. Computation of the residues gives the first two summands on the right-hand side of (4.1). To justify this contour shift, we show that for

$$f(s) := \left(\frac{c}{4\pi}\right)^{2s-1} D_v\left(s, \frac{d}{c}\right)\hat{\phi}(2s)$$

one has

$$\int_{\Re s=b,\, |\Im s|>T} f(s)\, ds \to 0 \quad \text{as } T \to \infty, \tag{4.2}$$

$$\int_{\Re s=\delta,\, |\Im s|>T} f(s)\, ds \to 0 \quad \text{as } T \to \infty, \tag{4.3}$$

$$\int_{\delta}^{b} f(\sigma \pm iT)\, d\sigma \to 0 \quad \text{as } T \to \infty. \tag{4.4}$$

First, (4.2) is satisfied since

$$\int_T^{\infty} |\hat{\phi}(2b+2iy)|\, dy \to 0 \quad \text{as } T \to \infty.$$

The property (4.3) follows from the inequality (2.24) and

$$\int_{(\delta)} (|s|+1)^{1-2\delta+\epsilon} |\phi(\hat{2}s)|\, ds < \infty.$$

We split the integral in (4.4) as $\int_{\delta}^{\epsilon} + \int_{\epsilon}^{b}$. For the first part we apply the functional equation (2.23) and estimate everything by the absolute value using (2.25) and (2.28). The second part is also evaluated using (2.25) and (2.28). This implies (4.4).

Finally, we compute

$$\frac{1}{i\pi} \int_{(\delta)} \left(\frac{c}{4\pi}\right)^{2s-1} D_v\left(s, \frac{d}{c}\right)\hat{\phi}(2s)\, ds$$

by applying the functional equation (2.23). Since $\Re(1-s) > 1$, we switch the order of summation and integration, obtaining

$$\sum_{m\geq 1} \tau_v(m) \frac{1}{i\pi} \int_{(\delta)} \gamma(1-s, v)\hat{\phi}(2s) m^{s-1} \\ \times \big(-e(-md^*/c)\cos(\pi s) + e(md^*/c)\sin(\pi(1/2+v))\big)\, ds.$$

Now the contour of integration can be moved to $\Re s = \alpha$ so that $3/4 < \alpha < 1$. Then the result follows from Lemmas 2.3 and 2.2, as we now show. Let

$$g_1(x) := x k_0(x\sqrt{m}, v).$$

Then

$$\hat{g}_1(1-2s) = -\gamma(1-s, v)\cos(\pi s) m^{s-1}$$

and

$$-\frac{1}{i\pi}\int_{(\alpha)}\gamma(1-s,v)\cos(\pi s)\hat{\phi}(2s)m^{s-1}\,ds=\int_0^\infty k_0(x\sqrt{m},v)\phi(x)x\,dx.$$

The parameter α is chosen so that the condition (2.30) is satisfied for $g_1(x)$, i.e.

$$\begin{cases}1-2\alpha>-1 & \text{as } x\to 0,\\ 1/2-2\alpha<-1 & \text{as } x\to\infty.\end{cases}$$

Similarly,

$$\frac{1}{i\pi}\int_{(\alpha)}\gamma(1-s,v)\sin(\pi(1/2+v))\hat{\phi}(2s)m^{s-1}\,ds=\int_0^\infty k_1(x\sqrt{m},v)\phi(x)x\,dx.$$

This concludes the proof of Lemma 4.1.

4.2. *Convolution formula for the second moment*

Exact formulas for moments reveal the structure of the mean values and allow obtaining asymptotic expansions. Here we use an exact formula for the second twisted moment proved by Kuznetsov. A similar formula was also independently obtained by Iwaniec and Sarnak [15, Theorem 17].

Theorem 4.2 (Kuznetsov, preprint 1994). *For $\Re v=0$, $\Im v\neq 0$, $|\Re u|<k-1$ we have*

$$\begin{aligned}M_2(l;u,v):=&\sum_{f\in H_{2k}(1)}^{h}\lambda_f(l)L_f(1/2+u+v)L_f(1/2+u-v)\\ &=\tau_v(l)\left(\frac{\zeta(1+2u)}{l^{1/2+u}}+\frac{(2\pi)^{4u}}{l^{1/2-u}}\zeta(1-2u)\frac{\Gamma(k-u+v)\Gamma(k-u-v)}{\Gamma(k+u+v)\Gamma(k+u-v)}\right)\\ &\quad+(-1)^k\tau_u(l)\frac{\zeta(1+2v)}{(2\pi)^{-2u+2v}l^{1/2+v}}\frac{\Gamma(k-u+v)}{\Gamma(k+u-v)}\\ &\quad+(-1)^k\tau_u(l)\frac{\zeta(1-2v)}{(2\pi)^{-2u-2v}l^{1/2-v}}\frac{\Gamma(k-u-v)}{\Gamma(k+v+u)}+E(l;u,v).\end{aligned}\tag{4.5}$$

The summand $E(l;u,v)$ can be expressed in terms of hypergeometric functions:

$$\begin{aligned}E(l;u,v)=&\frac{(-1)^k}{\sqrt{l}}\sum_{1\le n\le l-1}\tau_v(n)\tau_u(l-n)\phi_k\left(\frac{n}{l};u,v\right)\\ &+\frac{1}{\sqrt{l}}\sum_{n\ge l+1}\tau_v(n)\tau_u(n-l)\Phi_k\left(\frac{l}{n};u,v\right)+\frac{(-1)^k}{\sqrt{l}}\sum_{n\ge 1}\tau_v(n)\tau_u(n+l)\psi_k\left(\frac{l}{n};u,v\right),\end{aligned}\tag{4.6}$$

where

$$\phi_k(x;u,v)=\tilde{\phi}_k(x;u,v)+\tilde{\phi}_k(x;u,-v), \tag{4.7}$$

$$\tilde{\phi}_k(x;u,v)=\frac{(2\pi)^{2u+1}}{2\cos(\pi(1/2+v))}\frac{\Gamma(k-u+v)}{\Gamma(2v+1)\Gamma(k+u-v)} \times x^v(1-x)^{-u}F(k-u+v,1-k-u+v,1+2v;x), \tag{4.8}$$

$$\Phi_k(x;u,v)=2(2\pi)^{2u}\frac{\Gamma(k-u+v)\Gamma(k-u-v)}{\Gamma(2k)} \times \sin(\pi(1/2+u))x^k(1-x)^{-u}F(k-u+v,k-u-v,2k;x), \tag{4.9}$$

$$\psi_k(x;u,v)=2(2\pi)^{2u}\frac{\Gamma(k-u+v)\Gamma(k-u-v)}{\Gamma(2k)} \times \sin(\pi(1/2+v))x^k(1+x)^{-u}F(k-u+v,k-u-v,2k;-x). \tag{4.10}$$

Proof. Assume that $k-1>\Re u>3/4$. We multiply both sides of the Petersson trace formula (2.14) by

$$n^{-1/2-u}\tau_v(n)\zeta(1+2u)$$

and sum over $n\geq 1$. Using (2.9), we obtain the first summand on the right-hand side of (4.5) plus the non-diagonal contribution,

$$M_2(l;u,v)=\zeta(1+2u)\tau_v(l)l^{-1/2-u}+M^{\text{ND}},$$
$$M^{\text{ND}}=2\pi i^{2k}\zeta(1+2u)\sum_{n\geq1}\sum_{c\geq1}\frac{S(l,n;c)}{c}\frac{\tau_v(n)}{n^{1/2+u}}J_{2k-1}\left(\frac{4\pi\sqrt{ln}}{c}\right).$$

Applying Weil's bound for Kloosterman sums and standard estimates for the J-Bessel function, we can bound the double series as follows:

$$l^{\epsilon}\sum_{n\geq1}\sum_{c\geq1}c^{-1/2+\epsilon}n^{-\Re u-1/2+\epsilon}\min\left(\left(\frac{\sqrt{ln}}{c}\right)^{2k-1},\frac{\sqrt{c}}{(ln)^{1/4}}\right)\ll l^{1/4+\epsilon}\sum_{n\geq1}n^{-\Re u-1/4+\epsilon},$$

where ϵ is an arbitrarily small positive number. Thus for $\Re u>3/4$ the series is absolutely convergent and we can change the order of summation in M^{ND}. Opening the Kloosterman sum, we obtain

$$M^{\text{ND}}=2\pi i^{2k}\zeta(1+2u)(4\pi)^{2u}\sum_{c\geq1}\frac{1}{c^{1+2u}}\sum_{\substack{a \,(\text{mod}\, c)\\(a,c)=1}}e\left(\frac{al}{c}\right) \times\frac{4\pi}{c}\sum_{n\geq1}\tau_v(n)e\left(\frac{a^*n}{c}\right)\phi\left(\frac{4\pi\sqrt{n}}{c}\right),$$

where

$$\phi(x)=x^{-1-2u}J_{2k-1}(x\sqrt{l}). \tag{4.11}$$

It follows from [9, Eq. 6.561.14] that for $1-k+\Re u<\Re s<5/4+\Re u$,

$$\hat{\phi}(2s)=2^{2s-2-2u}l^{-s+1/2+u}\frac{\Gamma(k+s-1-u)}{\Gamma(k-s+1+u)}.$$

Applying [29, Eq. 5.11.9] to estimate the gamma factors, we obtain

$$|\hat{\phi}(2\sigma+2it)|\ll(1+|t|)^{-2\sigma-2-2\Re u},\qquad |t|\to\infty.$$

Thus $\phi(x)$ satisfies the conditions of Lemma 4.1, which yields

$$M^{\mathrm{ND}}=2\pi i^{2k}(4\pi)^{2u}\zeta(1+2u)\sum_{c=1}^{\infty}\frac{1}{c^{1+2u}}\Bigg(2S(0,l;c)\frac{\zeta(1+2v)}{(4\pi)^{1+2v}}\hat{\phi}(2+2v)$$
$$+2S(0,l;c)\frac{\zeta(1-2v)}{(4\pi)^{1-2v}}\hat{\phi}(2-2v)+\sum_{n=1}^{\infty}\tau_v(n)S(0,l-n;c)\int_0^{\infty}k_0(x\sqrt{n},v)\phi(x)x\,dx$$
$$+\sum_{n=1}^{\infty}\tau_v(n)S(0,l+n;c)\int_0^{\infty}k_1(x\sqrt{n},v)\phi(x)x\,dx\Bigg).$$

Ramanujan's identity (2.4) and [9, Eq. 6.561(14)] allow us to express the first two terms in M^{ND} for $-1/4<\Re u<k$ as the third and the fourth summands on the right-hand side of (4.5). The second summand in (4.5) comes from the third term in M^{ND} when $n=l$ by applying [9, Eq. 6.574(2)] for $0<\Re u<k$.

Consider

$$2\pi i^{2k}(4\pi)^{2u}\zeta(1+2u)\sum_{c=1}^{\infty}\frac{1}{c^{1+2u}}\sum_{\substack{n=1\\ n\neq l}}^{\infty}\tau_v(n)S(0,l-n;c)\int_0^{\infty}k_0(x\sqrt{n},v)\phi(x)x\,dx. \tag{4.12}$$

According to (2.15) and (4.11) we have

$$\int_0^{\infty}k_0(x\sqrt{n},v)\phi(x)x\,dx=\frac{1}{2\cos(\pi(1/2+v))}$$
$$\times\left(\int_0^{\infty}J_{2v}(x\sqrt{n})J_{2k-1}(x\sqrt{l})x^{-2u}\,dx-\int_0^{\infty}J_{-2v}(x\sqrt{n})J_{2k-1}(x\sqrt{l})x^{-2u}\,dx\right). \tag{4.13}$$

If $n<l$ we apply [9, Eq. 6.574(1)] (with $\alpha=\sqrt{n},\ \beta=\sqrt{l},\ \nu=2v,\ \mu=2k-1$), and infer for $-1/2<\Re u<k$ that

$$\int_0^{\infty}J_{2v}(x\sqrt{n})J_{2k-1}(x\sqrt{l})x^{-2u}\,dx=\frac{n^v}{2^{2u}l^{v-u+1/2}}\frac{\Gamma(k+v-u)}{\Gamma(k-v+u)\Gamma(1+2v)}$$
$$\times F(k+v-u,1-k+v-u,1+2v;n/l)=\frac{2\cos(\pi(1/2+v))\tilde{\phi}_k(n/l;u,v)}{2^{2u}(2\pi)^{2u+1}(1-n/l)^{-u}l^{1/2-u}}, \tag{4.14}$$

where we use (4.8). Substituting (4.14) to (4.13) and applying (4.7), we show that

$$\int_0^\infty k_0(x\sqrt{n}, v)\phi(x)x\,dx = \frac{\phi_k(n/l; u, v)}{2^{2u}(2\pi)^{2u+1}(1-n/l)^{-u}l^{1/2-u}}. \tag{4.15}$$

Recall that we consider the part of (4.12) with $n < l$. Substituting (4.15) into (4.12) and using Ramanujan's identity (2.4) to compute the sum over c, we recover the first term in (4.6).

Analogously, for $n > l$ we obtain the second term in (4.6) using [9, Eq. 6.574(1)] with $\alpha = \sqrt{l}$, $\beta = \sqrt{n}$, $\nu = 2k-1$, $\mu = 2v$.

Finally, the third term in (4.6) comes from

$$2\pi i^{2k}(4\pi)^{2u}\zeta(1+2u)\sum_{c=1}^{\infty}\frac{1}{c^{1+2u}}\sum_{n=1}^{\infty}\tau_v(n)S(0, l+n; c)\int_0^\infty k_1(x\sqrt{n}, v)\phi(x)x\,dx$$

by applying (2.4) and [9, Eq. 6.576(3)] for $\Re u < k$.

Thus we have proved (4.5) for $k-1 > \Re u > 3/4$. To extend the range of validity of (4.5), we note that the left-hand side of the convolution formula (4.5) is an entire function of u and v. Since

$$|\tau_u(n \pm l)| \ll n^{|\Re u|+\epsilon}, \qquad |\Phi_k(x; u, v)|, |\psi_k(x; u, v)| \ll x^k \quad \text{as } x \to 0,$$

the right-hand side of (4.5) is a regular function for $|\Re v| + |\Re u| < k-1$. □

5. The Liouville–Green method

Our main references are the paper [5] and the book [28] . In particular, Chapters 6 and 10–12 of [28] are devoted to the Liouville–Green method. Note that the case we are interested in is mainly covered by Chapter 12.

Throughout this section we assume that k is an even positive integer.

5.1. Some properties of ϕ_k

Let $0 < x < 1$ be a real number. Consider the function

$$\phi_k(x) = \lim_{\substack{u\to 0\\ v\to 0}} \phi_k(x; u, v),$$

where $\phi_k(x; u, v)$ is defined by (4.7). Letting $u = 0$ and computing the limit as $v \to 0$ by L'Hospital's rule, we obtain

$$\phi_k(x) = \frac{\partial}{\partial v}\left[\frac{-2\Gamma(k+v)x^v}{\Gamma(1+2v)\Gamma(k-v)}F(k+v, 1-k+v, 1+2v; x)\right]\Bigg|_{v=0}. \tag{5.1}$$

Differentiation with respect to v gives

$$\phi_k(x) = 2\Big(-\log x - 2\frac{\Gamma'}{\Gamma}(k) + 2\frac{\Gamma'}{\Gamma}(1)\Big)F(k, 1-k, 1; x) - 2\Big(\frac{\partial}{\partial a} + \frac{\partial}{\partial b} + 2\frac{\partial}{\partial c}\Big)F(a, b, c; x)\Big|_{\substack{a=k\\ b=1-k\\ c=1}}. \tag{5.2}$$

Lemma 5.1. *The following property holds:*

$$\phi_k(x) = (-1)^k \phi_k(1-x). \tag{5.3}$$

Proof. Recall that

$$\phi_k(x; u, v) = \tilde{\phi}_k(x; u, v) + \tilde{\phi}_k(x; u, -v),$$

where $\tilde{\phi}_k(x; u, v)$ is defined by (4.8). Applying [4, Eq. 33, p. 107] and Euler's reflection formula, we obtain

$$\tilde{\phi}_k(x; u, v) = (-1)^k \frac{(2\pi)^{2u}\pi}{\sin(\pi v)}\Bigg(\frac{\Gamma(k+v-u)\Gamma(k-v-u)}{\Gamma(k+v+u)\Gamma(k-v+u)} \times \frac{\sin(\pi(v+u))}{\Gamma(1-2u)\sin(\pi u)} x^v(1-x)^{-u}F(k+v-u, 1-k+v-u, 1-2u; 1-x) + \frac{\sin(\pi(v-u))}{\Gamma(1+2u)\sin(-2\pi u)} x^v(1-x)^u F(k+v+u, 1-k+v+u, 1+2u; 1-x)\Bigg).$$

Let

$$F(v, u) := x^v(1-x)^u F(k+v+u, 1-k+v+u, 1+2u; 1-x).$$

Then

$$\tilde{\phi}_k(x; u, v) = (-1)^k \frac{(2\pi)^{2u}\pi}{2\sin(\pi u)}\Bigg(\frac{\Gamma(k+v-u)\Gamma(k-v-u)}{\Gamma(k+v+u)\Gamma(k-v+u)}\frac{F(v,-u)}{\Gamma(1-u)} - \frac{F(v,u)}{\Gamma(1+2u)}\Bigg) + (-1)^k \frac{(2\pi)^{2u}\pi}{2\sin(\pi v)}\frac{\cos(\pi v)}{\cos(\pi u)}\Bigg(\frac{\Gamma(k+v-u)\Gamma(k-v-u)}{\Gamma(k+v+u)\Gamma(k-v+u)}\frac{F(v,-u)}{\Gamma(1-2u)} + \frac{F(v,u)}{\Gamma(1+2u)}\Bigg).$$

Computing the limit as $u \to 0$ by L'Hospital rule, we have

$$\lim_{u\to 0}\tilde{\phi}_k(x; u, v) = (-1)^k \frac{\pi\cos(\pi v)}{\sin(\pi v)}F(v, 0) + \frac{(-1)^k}{2}\frac{\partial}{\partial u}\Bigg(\frac{\Gamma(k+v-u)\Gamma(k-v-u)}{\Gamma(k+v+u)\Gamma(k-v+u)}\frac{F(v,-u)}{\Gamma(1-2u)} - \frac{F(v,u)}{\Gamma(1+2u)}\Bigg)\Bigg|_{u=0}.$$

By [4, Eq. 1-2, p. 105] it follows that $F(-v,0)=F(v,0)$. Thus

$$\begin{aligned}
&\lim_{u\to 0}(\tilde{\phi}_k(x;u,v)+\tilde{\phi}_k(x;u,-v))\\
&\quad=\frac{(-1)^k}{2}\left[\frac{\partial}{\partial u}\left(\frac{\Gamma(k+v-u)\Gamma(k-v-u)}{\Gamma(k+v+u)\Gamma(k-v+u)}\frac{F(v,-u)}{\Gamma(1-2u)}-\frac{F(v,u)}{\Gamma(1+2u)}\right)\Bigg|_{u=0}\right.\\
&\qquad\left.+\frac{\partial}{\partial u}\left(\frac{\Gamma(k-v-u)\Gamma(k+v-u)}{\Gamma(k-v+u)\Gamma(k+v+u)}\frac{F(-v,-u)}{\Gamma(1-2u)}-\frac{F(-v,u)}{\Gamma(1+2u)}\right)\Bigg|_{u=0}\right].
\end{aligned}$$

Letting $v=0$, we have

$$\phi_k(x)=(-1)^k4\left(\frac{\Gamma'}{\Gamma}(1)-\frac{\Gamma'}{\Gamma}(k)\right)F(0,0)-(-1)^k2F'_u(0,0),$$

where

$$F'_u(0,0)=\log(1-x)F(k,1-k,1;1-x)+\left(\frac{\partial}{\partial a}+\frac{\partial}{\partial b}+2\frac{\partial}{\partial c}\right)F(a,b,c;1-x)\Bigg|_{\substack{a=k\\ b=1-k\\ c=1}}.$$

Then (5.2) implies that $\phi_k(x)=(-1)^k\phi_k(1-x)$. □

Corollary 5.2. *For any positive even integer k we have $\phi'_k(1/2)=0$.*

Lemma 5.3. *The following series representation holds:*

$$\begin{aligned}
\phi_k(x)&=-F(k,1-k,1;x)2\log x+2(-1)^k\sum_{n=k}^{\infty}\frac{\Gamma(n+k)\Gamma(n-k+1)x^n}{\Gamma^2(n+1)}\\
&\quad-2\sum_{n=0}^{k-1}\frac{(-1)^n\Gamma(k+n)}{\Gamma(k-n)\Gamma^2(n+1)}\left(-2\frac{\Gamma'}{\Gamma}(n+1)+\frac{\Gamma'}{\Gamma}(k+n)+\frac{\Gamma'}{\Gamma}(k-n)\right).
\end{aligned}\tag{5.4}$$

Proof. By (5.1) and Euler's reflection formula,

$$\phi_k(x)=-\frac{2}{\pi}\frac{\partial}{\partial v}\left(x^v\sum_{n=0}^{\infty}\frac{\sin(\pi(k-v))\Gamma(1-k+v+n)\Gamma(k+v+n)}{\Gamma(1+2v+n)}\frac{x^n}{n!}\right)\Bigg|_{v=0}.$$

Using

$$\Gamma(1-k+v+n)=\begin{cases}\Gamma(1-k+v+n), & n\geq k,\\ \frac{\pi}{\sin(\pi(k-v-n))\Gamma(k-n-v)}, & n\leq k-1,\end{cases}$$

we have

$$\begin{aligned}
\phi_k(x)&=2(-1)^k\sum_{n=k}^{\infty}\frac{\Gamma(n+k)\Gamma(n-k+1)x^n}{\Gamma(n+1)n!}+\sum_{n=0}^{k-1}\frac{(-1)^nx^n}{n!}\frac{\Gamma(k+n)}{\Gamma(k-n)\Gamma(n+1)}\\
&\quad\times\left(-2\log x+4\frac{\Gamma'}{\Gamma}(n+1)-2\frac{\Gamma'}{\Gamma}(k+n)-2\frac{\Gamma'}{\Gamma}(k-n)\right).
\end{aligned}\tag{5.5}$$

It follows from [29, Eq. 15.2.4] that

$$F(-m, b, c; z) = \frac{\Gamma(m+1)\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{m} \frac{(-1)^n z^n}{n!} \frac{\Gamma(b+n)}{\Gamma(m-n+1)\Gamma(c+n)},$$

and therefore

$$F(1-k, k, 1; x) = \sum_{n=0}^{k-1} \frac{(-1)^n x^n}{n!} \frac{\Gamma(k+n)}{\Gamma(k-n)\Gamma(n+1)}. \tag{5.6}$$

Substituting (5.6) into (5.5) we obtain (5.4). □

Lemma 5.4. *The function $\phi_k(x)$ satisfies the differential equation*

$$(x - x^2)\phi_k''(x) + (1-2x)\phi_k' + k(k-1)\phi_k(x) = 0. \tag{5.7}$$

Proof. Note that $F(k, 1-k, 1; x)$ is a solution of (5.7). Using (5.4) we can write

$$\phi_k(x) = -2\alpha_1 - 2\alpha_2 + 2(-1)^k \alpha_3,$$

where

$$\alpha_1 := F(k, 1-k, 1; x)\log x, \qquad \alpha_2 := \sum_{n=0}^{k-1} A(n)B(n)x^n, \qquad \alpha_3 := \sum_{n=k}^{\infty} C(n)x^n.$$

The coefficients $A(n)$, $B(n)$, $C(n)$ are defined by

$$A(n) := (-1)^n \frac{\Gamma(k+n)}{\Gamma^2(n+1)\Gamma(k-n)},$$

$$B(n) := \frac{\Gamma'}{\Gamma}(n+k) + \frac{\Gamma'}{\Gamma}(n-k) - 2\frac{\Gamma'}{\Gamma}(n+1),$$

$$C(n) := \frac{\Gamma(n+k)\Gamma(n-k+1)}{\Gamma^2(n+1)}.$$

They satisfy the recurrence relations

$$A(n+1) = -\frac{(k+n)(k-n-1)}{(n+1)^2} A(n),$$

$$B(n+1) = B(n) + \frac{1}{k+n} - \frac{1}{k-n-1} - \frac{2}{n+1},$$

$$C(n+1) = \frac{(n+k)(n-k+1)}{(n+1)^2} C(n).$$

Let us denote

$$D(f) := (x - x^2)f'' + (1-2x)f' + k(k-1)f.$$

Using the recurrence relations above, we compute $D(\alpha_1)$, $D(\alpha_2)$, $D(\alpha_3)$ and prove the lemma by showing that $D(\alpha_1) + D(\alpha_2) = (-1)^k D(\alpha_3)$. □

Lemma 5.5. *Let $y = y(x)$ be a solution of the differential equation*

$$A(x)y''(x) + B(x)y'(x) + C(x)y(x) = 0. \tag{5.8}$$

Then $z(x) = y(x)/\alpha(x)$ satisfies the equation

$$A_1(x)z''(x) + B_1(x)z'(x) + C_1(x)z(x) = 0, \tag{5.9}$$

where

$$A_1(x) = A(x)\alpha(x), \qquad B_1(x) = 2A(x)\alpha'(x) + B(x)\alpha(x), \tag{5.10}$$

$$C_1(x) = A(x)\alpha''(x) + B(x)\alpha'(x) + C(x)\alpha(x). \tag{5.11}$$

Corollary 5.6. *For $0 < x < 1$ the function $Y(x) := \sqrt{x(1-x)}\,\phi_k(x)$ is a solution of the differential equation*

$$Y''(x) + \left(\frac{1}{4x^2(1-x)^2} + \frac{k(k-1)}{x(1-x)}\right)Y(x) = 0. \tag{5.12}$$

Proof. Apply Lemma 5.5 with $\alpha(x) = 1/\sqrt{x(1-x)}$. □

Lemma 5.7. *Assume that k is an even positive integer. Then*

$$F(k, 1-k, 1; 1/2) = 0, \tag{5.13}$$

$$\frac{d}{dx}(F(k, 1-k, 1; x))\bigg|_{x=1/2} = (-1)^{k/2}\frac{4\Gamma(1/2)\Gamma((k+1)/2)}{\pi\Gamma(k/2)}. \tag{5.14}$$

Proof. Using [29, Eq. 15.8.25] and Euler's reflection formula, we have

$$F(a, 1-a, 1; x) = \frac{\Gamma(1/2)}{\pi}\frac{\Gamma(a/2)\sin(\pi a/2)}{\Gamma((a+1)/2)}F\left(\frac{a}{2}, \frac{1-a}{2}, \frac{1}{2}; (1-2x)^2\right)$$
$$-(1-2x)\frac{2\Gamma(1/2)}{\pi}\frac{\Gamma((a+1)/2)\sin(\pi(a+1)/2)}{\Gamma(a/2)}F\left(\frac{a+1}{2}, 1-\frac{a}{2}, \frac{3}{2}; (1-2x)^2\right)$$

for some complex a. Setting $a = k$ we obtain

$$F(k, 1-k, 1; x) = -(-1)^{k/2}\frac{2\Gamma(1/2)\Gamma((k+1)/2)}{\pi\Gamma(k/2)}(1-2x)$$
$$\times F\left(\frac{k+1}{2}, 1-\frac{k}{2}, \frac{3}{2}; (1-2x)^2\right),$$

and (5.13) follows by taking $x = 1/2$.

Further, (5.14) is obtained by differentiation of $F(k, 1-k, 1; x)$ with respect to x using the series representation

$$F\left(\frac{k+1}{2}, 1-\frac{k}{2}, \frac{3}{2}; (1-2x)^2\right)$$
$$= \sum_{n=0}^{k/2-1}(-1)^n\binom{k/2-1}{n}\frac{\Gamma(k/2+1/2+n)\Gamma(3/2)}{\Gamma(k/2+1/2)\Gamma(3/2+n)}(1-2x)^{2n}.$$

This representation is a consequence of [29, Eq. 15.2.4] for even positive integer k. □

Lemma 5.8. *For even positive integer k we have*

$$\phi_k(1/2) = 2\sqrt{\pi}\,(-1)^{k/2}\frac{\Gamma(k/2)}{\Gamma((k+1)/2)}. \tag{5.15}$$

Proof. By (5.2) and (5.13) we have

$$\phi_k(1/2) = -2\left(\frac{\partial}{\partial a}+\frac{\partial}{\partial b}+2\frac{\partial}{\partial c}\right)F(a,b,c;1/2)\Big|_{\substack{a=k\\ b=1-k\\ c=1}}.$$

Define

$$G_1 := F(k+2\epsilon, 1-k, 1+\epsilon; z), \qquad G_2 := F(k-\epsilon, 1-k+\epsilon, 1+\epsilon; z).$$

According to [29, Eq. 15.8.26] we have

$$\begin{aligned} G_1 &= \frac{\Gamma(1/2)}{\pi}(1-z)^{\epsilon}\frac{\Gamma(1+\epsilon)\Gamma(k/2-\epsilon)}{\Gamma(k/2+1/2)}\sin\left(\frac{\pi}{2}(k-2\epsilon)\right)g_{1,1}\\ &\quad+\frac{\Gamma(-1/2)}{\pi}(1-2z)(1-z)^{\epsilon}\frac{\Gamma(1+\epsilon)\Gamma(k/2+1/2-\epsilon)}{\Gamma(k/2)}\sin\left(\frac{\pi}{2}(k+1-2\epsilon)\right)g_{1,2}, \end{aligned}$$

where

$$\begin{aligned} g_{1,1} &= F(1/2-k/2+\epsilon, k/2, 1/2; (1-2z)^2),\\ g_{1,2} &= F(1-k/2+\epsilon, k/2+1/2, 3/2; (1-2z)^2). \end{aligned}$$

By [29, Eq. 15.8.25],

$$G_2 = -\frac{2(-1)^{k/2}\Gamma(1/2)}{\pi}(1-2z)\frac{\Gamma(1+\epsilon)\Gamma(k/2+1/2)}{\Gamma(k/2+\epsilon)}g_{2,2},$$

where

$$g_{2,2} = F(k/2+1/2+\epsilon, 1-k/2, 3/2; (1-2z)^2).$$

Differentiating G_1 and G_2 with respect to ϵ at $\epsilon=0$ and summing the results, we obtain

$$\begin{aligned} \left(\frac{\partial}{\partial a}+\frac{\partial}{\partial b}+2\frac{\partial}{\partial c}\right)F(a,b,c;z)\Big|_{\substack{a=k\\ b=1-k\\ c=1}} &= -(1-2z)\log(1-z)\\ &\times(-1)^{k/2}F(k/2,1/2-k/2,1/2;(1-2z)^2)\frac{2\Gamma(k/2+1/2)\Gamma(1/2)}{\pi\Gamma(k/2)}\\ &-(-1)^{k/2}F(k/2,1/2-k/2,1/2;(1-2z)^2)\frac{\Gamma(1/2)\Gamma(k/2)}{\Gamma(k/2+1/2)}\\ &-\frac{\Gamma(k/2+1/2)\Gamma(1/2)}{\pi\Gamma(k/2)}(1-2z)\Bigg(F(k/2+1/2,1-k/2,3/2;(1-2z)^2)\\ &\times(-1)^{k/2}\left(4\frac{\Gamma'}{\Gamma}(1)-2\frac{\Gamma'}{\Gamma}(k/2+1/2)-2\frac{\Gamma'}{\Gamma}(k/2)\right)\\ &\qquad+2(-1)^{k/2}\left(\frac{\partial}{\partial a}+\frac{\partial}{\partial b}\right)F(a,b,3/2;(1-2z)^2)\Big|_{\substack{a=(k+1)/2\\ b=1-k/2}}\Bigg). \end{aligned}$$

Setting $z=1/2$ we prove the lemma. □

5.2. Asymptotic approximation of ϕ_k

We apply the Liouville–Green method to find a uniform approximation of the function $\phi_k(x)$ in terms of the J- and Y-Bessel functions. Some preliminary work is required in order to introduce various functions and constants in the Liouville–Green approximation. Therefore, the main result, Theorem 5.14, is stated at the end of this subsection.

In Corollary 5.6 we showed that $y(x)=\sqrt{x(1-x)}\phi_k(x)$ is a solution of (5.12). This equation is a particular type of [5, Eq. 1.1] when $\alpha=0$. Let

$$u := k-1/2, \qquad f(x) := -\frac{1}{x(1-x)}, \tag{5.16}$$

$$g(x) := -\frac{1}{4x^2(1-x)^2}+\frac{1}{4x(1-x)}. \tag{5.17}$$

Then (5.12) can be written as

$$y''(x)=(u^2f(x)+g(x))y(x). \tag{5.18}$$

Note that $x^2f(x)\to 0$ and $x^2g(x)\to -1/4$ as $x\to 0$. The same conditions on $f(x)$ and $g(x)$ are also assumed in [5, p. 1]. We would like to transform (5.18) into the shape

$$\frac{d^2Z}{d\xi^2}+\left[\frac{u^2}{4\xi}+\frac{1}{4\xi^2}-\frac{\psi(\xi)}{\xi}\right]Z=0, \tag{5.19}$$

which corresponds to [5, Eq. 2.4, 2.6].

Let $\alpha(x)$, $\eta(x)$ be suitable functions (to be chosen later). We make the change of variable

$$Z(x) := y(x)/\alpha(x) \tag{5.20}$$

in (5.18) and apply Lemma 5.5. Then the substitution

$$\xi := \int \eta(x)\,dx \tag{5.21}$$

gives

$$\alpha(x)\eta^2(x)\frac{d^2Z}{d\xi^2}+(\alpha(x)\eta'(x)+2\alpha'(x)\eta(x))\frac{dZ}{d\xi} + \big(\alpha''(x)-\alpha(x)(u^2f(x)+g(x))\big)Z(\xi)=0. \tag{5.22}$$

In order to obtain (5.19) we make the coefficient before $\frac{dZ}{d\xi}$ vanish by requiring

$$\alpha^2(x)\eta(x)=1. \tag{5.23}$$

Next, we assume that $-\alpha^4(x)f(x)=1/(4\xi)$. This implies

$$\xi=4\arcsin^2(\sqrt{x}), \qquad \alpha(x)=\frac{(x-x^2)^{1/4}}{2(\arcsin(\sqrt{x}))^{1/2}}. \tag{5.24}$$

With the use of (5.23) and (5.24), the equation (5.22) can be transformed into (5.19) with

$$\psi(\xi) := -\frac{1}{16\sin^2(\sqrt{\xi})} + \frac{1}{16\xi}. \tag{5.25}$$

Note that $\psi(\xi)$ is regular in the ξ-plane apart from the poles at $\xi = (\pi m)^2$, $m \neq 0$. Since

$$\psi(\xi) = -\frac{1}{48} + O(\xi) \quad \text{as } \xi \to 0,$$

the function $\psi(\xi)$ is smooth on $[0, \delta]$ for any $0 < \delta < \pi^2$.

Removing the summand with $\psi(\xi)/\xi$ in (5.19), we have

$$\frac{d^2 Z}{d\xi^2} + \left[\frac{u^2}{4\xi} + \frac{1}{4\xi^2}\right] Z = 0. \tag{5.26}$$

The following functions are solutions of (5.26):

$$Z_C = \sqrt{\xi}\,\Upsilon_0(u\sqrt{\xi}), \tag{5.27}$$

where Υ_i is either the J- or Y-Bessel function of index i. Note that

$$\frac{d}{dz}\Upsilon_0(z) = -\Upsilon_1(z). \tag{5.28}$$

Therefore, following [28, Chapter 12] we are searching for a solution of the differential equation (5.19) in the form

$$Z_C(\xi) = \sqrt{\xi}\,\Upsilon_0(u\sqrt{\xi}) \sum_{n=0}^{\infty} \frac{A(n;\xi)}{u^{2n}} - \frac{\xi}{u}\Upsilon_1(u\sqrt{\xi}) \sum_{n=0}^{\infty} \frac{B(n;\xi)}{u^{2n}}, \tag{5.29}$$

and our problem reduces to finding the coefficients $A(n;\xi)$, $B(n;\xi)$. Let us denote

$$W(\xi) := \sqrt{\xi}\,\Upsilon_0(u\sqrt{\xi}), \qquad V(\xi) := \xi\Upsilon_1(u\sqrt{\xi}). \tag{5.30}$$

These functions satisfy the differential equations (see [9, Eq. 8.491(3)])

$$W''(\xi) + \left(\frac{u^2}{4\xi} + \frac{1}{4\xi^2}\right) W(x) = 0, \tag{5.31}$$

$$V''(\xi) - \frac{1}{\xi}V'(\xi) + \left(\frac{u^2}{4\xi} + \frac{3}{4\xi^2}\right) V(x) = 0. \tag{5.32}$$

Note that

$$W'(\xi) = \frac{1}{2\xi}W(\xi) - \frac{u}{2\xi}V(\xi), \tag{5.33}$$

$$V'(\xi) = \frac{1}{2\xi}V(\xi) + \frac{u}{2}W(\xi). \tag{5.34}$$

Substituting (5.29) in (5.19), we find

$$W(\xi)\sum_{n=0}^{\infty}\frac{C_n(\xi)}{u^{2n}}-V(\xi)\sum_{n=0}^{\infty}\frac{D_n(\xi)}{u^{2n-1}}=0, \tag{5.35}$$

where

$$C_n(\xi):=A''(n;\xi)+\frac{1}{\xi}A'(n;\xi)-\frac{\psi(\xi)}{\xi}A(n;\xi)-B'(n;\xi)-\frac{B(n;\xi)}{2\xi},$$
$$D_n(\xi):=B''(n-1;\xi)+\frac{1}{\xi}B'(n-1;\xi)-\frac{\psi(\xi)}{\xi}B(n-1;\xi)+\frac{1}{\xi}A'(n;\xi).$$

Assuming that $C_n(\xi)=D_n(\xi)=0$, we have

$$\sqrt{\xi}(\sqrt{\xi}\,B(n;\xi))'=\xi A''(n;\xi)+A'(n;\xi)-\psi(\xi)A(n;\xi),$$
$$A'(n;\xi)=-(\xi B'(n-1;\xi))'+\psi(\xi)B(n-1;\xi).$$

This yields the following recurrence relations:

$$A(n;\xi)=-\xi B'(n-1;\xi)+\int_0^{\xi}\psi(x)B(n-1;x)\,dx+\lambda_n, \tag{5.36}$$

$$\sqrt{\xi}\,B(n;\xi)=\int_0^{\xi}\frac{1}{\sqrt{x}}(xA''(n;x)+A'(n;x)-\psi(x)A(n;x))\,dx \tag{5.37}$$

for some real integration constants λ_n (to be chosen later). Letting $A(0;\xi)=1$, we have

$$B(0;\xi)=-\frac{1}{8\sqrt{\xi}}\left(\cot(\sqrt{\xi})-\frac{1}{\sqrt{\xi}}\right), \tag{5.38}$$

$$A(1;\xi)=\frac{1}{8}\left(\frac{1}{\xi}-\frac{\cot(\sqrt{\xi})}{2\sqrt{\xi}}-\frac{1}{2\sin^2(\sqrt{\xi})}\right)-\frac{1}{128}\left(\cot(\sqrt{\xi})-\frac{1}{\sqrt{\xi}}\right)^2+\lambda_1. \tag{5.39}$$

The next step is to apply [28, Theorem 4.1, p. 444] or [5, Theorem 1]. This allows approximating ϕ_k by a finite series plus an error term.

Theorem 5.9. *Let $\xi_2=\pi^2/4$. For each u and each non-negative integer N, the equation (5.19) has solutions $Z_Y(\xi)$, $Z_J(\xi)$ which are infinitely differentiable in ξ on $(0,\xi_2)$ and are given by*

$$Z_Y(\xi)=\sqrt{\xi}\,Y_0(u\sqrt{\xi})\sum_{n=0}^{N}\frac{A_Y(n;\xi)}{u^{2n}}-\frac{\xi}{u}Y_1(u\sqrt{\xi})\sum_{n=0}^{N-1}\frac{B_Y(n;\xi)}{u^{2n}}+\epsilon_{2N+1,1}(u,\xi), \tag{5.40}$$

$$Z_J(\xi)=\sqrt{\xi}\,J_0(u\sqrt{\xi})\sum_{n=0}^{N}\frac{A_J(n;\xi)}{u^{2n}}-\frac{\xi}{u}J_1(u\sqrt{\xi})\sum_{n=0}^{N-1}\frac{B_J(n;\xi)}{u^{2n}}+\epsilon_{2N+1,2}(u,\xi), \tag{5.41}$$

where

$$\epsilon_{2N+1,1}(u,\xi) \ll \frac{\sqrt{\xi}\,|Y_0(u\sqrt{\xi})|}{u^{2N+1}}\sqrt{\xi_2-\xi}, \tag{5.42}$$

$$\epsilon_{2N+1,2}(u,\xi) \ll \frac{\sqrt{\xi}\,|J_0(u\sqrt{\xi})|}{u^{2N+1}}\min(\sqrt{\xi},1) \tag{5.43}$$

and the coefficients $(A_Y(n;\xi), B_Y(n;\xi))$, $(A_J(n;\xi), B_J(n;\xi))$ *satisfy the recurrence relations* (5.36)–(5.37).

Proof. The only difference between Theorem 5.9 and [28, Theorem 4.1, p. 444] (or [5, Theorem 1]) is that we simplify the estimates of the error terms $\epsilon_{2N+1,i}(u,\xi)$. To this end, we apply [28, (1.22)–(1.24), p. 437]. Furthermore, we use the fact (see (5.38)) that

$$\mathrm{Var}_{0,\xi}(\sqrt{x}\,B(0;x)) \ll 1, \qquad \mathrm{Var}_{\xi,\xi_2}(\sqrt{x}\,B(0;x)) \ll 1,$$

and the following estimates for variations:

$$\mathrm{Var}_{0,\xi}(\sqrt{x}\,B(n;x)) = \int_0^{\xi} |(\sqrt{x}\,B(n;x))'|\,dx \ll \min(\sqrt{\xi},1),$$

$$\mathrm{Var}_{\xi,\xi_2}(\sqrt{x}\,B(n;x)) = \int_{\xi}^{\xi_2} |(\sqrt{x}\,B(n;x))'|\,dx \ll \sqrt{\xi_2-\xi},$$

which can be proved by induction. □

Note that in Theorem 5.9 it is possible to choose ξ_2 to be any number less than π^2 since the function $\psi(\xi)$ is smooth on $[0,\delta]$ for any $0<\delta<\pi^2$. However, we let $\xi_2=\pi^2/4$ because this point corresponds to $x=1/2$. Consequently, Theorem 5.9 allows us to approximate the function $\phi_k(x)$ for $0<x<1/2$, which is sufficient for our purposes in view of the functional equation (5.3).

Theorem 5.10. *There are* $C_Y = C_Y(u)$ *and* $C_J = C_J(u)$ *such that*

$$\xi^{1/4}(\sin(\sqrt{\xi}))^{1/2}\phi_k(\sin^2(\sqrt{\xi}/2)) = C_Y Z_Y(\xi) + C_J Z_J(\xi). \tag{5.44}$$

Now our goal is not only to prove Theorem 5.10 but also to determine explicitly C_Y and C_J.

As $\xi\to 0$, both $F(k,1-k,1;\sin^2(\sqrt{\xi}/2))$ and $Z_J(\xi)$ are recessive solutions and $\phi_k(\sin^2(\sqrt{\xi}/2))$ and $Z_Y(\xi)$ are dominant. This is because

$$\phi_k(\xi)\sim\log\xi, \qquad F(k,1-k,1;\xi)\sim 1 \quad \text{as } \xi\to 0, \tag{5.45}$$

$$Z_Y(\xi)\xi^{-1/2}\sim\log\xi, \qquad Z_J(\xi)\xi^{-1/2}\sim 1 \quad \text{as } \xi\to 0. \tag{5.46}$$

For the definition and the theory of recessive and dominant solutions see [28, §5.7].

Thus there is a constant c_0 such that

$$\xi^{1/4}(\sin(\sqrt{\xi}))^{1/2}F(k,1-k,1;\sin^2(\sqrt{\xi}/2)) = c_0 Z_J(\xi). \tag{5.47}$$

Note that

$$\lim_{\xi\to 0} F(k,1-k,1;\sin^2(\sqrt{\xi}/2)) = 1. \tag{5.48}$$

By [29, Eq. 10.2.2] we have $J_0(x) = 1 + O(x^2)$ and $J_1(x) = x/2 + O(x^3)$. Therefore,

$$Z_J(\xi) = \sqrt{\xi}\sum_{n=0}^{N}\frac{A_J(n;\xi)}{u^{2n}} + O(\xi) \quad \text{as } \xi \to 0. \tag{5.49}$$

Choosing the constants of integration λ_n in (5.36) such that $A_J(0;0) = 1$ and $A_J(n;0) = 0$ for $n \geq 1$, we find that $\lim_{\xi\to 0} Z_J(\xi) = \sqrt{\xi}$ and $c_0 = 1$.

Since $\phi_k(\sin^2(\sqrt{\xi}/2))$ and $Z_Y(\xi)$ are dominant for any $0 < \xi < \xi_2$, it is not possible to find a proportionality relation between them analogous to (5.47). To solve this problem, we apply the method described in [28, §12.5].

The differential equation (5.19) has two solutions $\phi_k(\xi)$ and $F(k,1-k,1;\xi)$, which are linearly independent in view of (5.45). Therefore, Z_Y can be written as a linear combination

$$Z_Y(\xi) = \xi^{1/4}(\sin(\sqrt{\xi}))^{1/2}\left(\phi_k\left(\sin^2\left(\frac{\sqrt{\xi}}{2}\right)\right)c_1 + F\left(k,1-k,1;\sin^2\left(\frac{\sqrt{\xi}}{2}\right)\right)c_2\right) \tag{5.50}$$

for some constants c_1, c_2.

Substituting (5.47) with $c_0 = 1$ into (5.50) we have

$$\xi^{1/4}(\sin(\sqrt{\xi}))^{1/2}\phi_k\left(\sin^2\left(\frac{\sqrt{\xi}}{2}\right)\right) = \frac{1}{c_1}Z_Y(\xi) - \frac{c_2}{c_1}Z_J(\xi), \tag{5.51}$$

provided that $c_1 \neq 0$.

In order to determine the constants c_1, c_2, we compute $Z_Y(\xi)$ and its derivative at $\xi_2 = \pi^2/4$. Applying Lemmas 5.7, 5.8 and Corollary 5.2, we have

$$Z_Y(\xi_2) = \xi_2^{1/4}c_1\phi_k(1/2), \tag{5.52}$$

$$Z_Y'(\xi_2) = \frac{Z_Y(\xi_2)}{4\xi_2} + \frac{c_2}{4\xi_2^{1/4}}\frac{\partial}{\partial x}F(k,1-k,1;x)\Big|_{x=1/2}. \tag{5.53}$$

Using Lemma 5.8 we find

$$c_1 = (-1)^{k/2}\frac{\Gamma(k/2+1/2)}{2\Gamma(1/2)\Gamma(k/2)}\frac{Z_Y(\xi_2)}{\xi_2^{1/4}}, \tag{5.54}$$

$$c_2 = (-1)^{k/2}\frac{\pi\Gamma(k/2)}{\Gamma(1/2)\Gamma(k/2+1/2)}\xi_2^{1/4}\left(Z_Y'(\xi_2) - \frac{Z_Y(\xi_2)}{4\xi_2}\right). \tag{5.55}$$

The final step is to compute $Z_Y(\xi_2)$ and $Z_Y'(\xi_2)$.

Lemma 5.11. *For $\xi_2 = \pi^2/4$ the following asymptotic formulas hold:*

$$Z_Y(\xi_2) = \frac{(-1)^{k/2+1}}{\sqrt{u}}\left(1 + \frac{1}{u^2}(\lambda_1 - 1/16) + O(u^{-4})\right), \tag{5.56}$$

$$Z'_Y(\xi_2) = \frac{(-1)^{k/2+1}}{\sqrt{u}\pi^2}\left(1 + \frac{1}{u^2}\left[\frac{5\lambda_1}{4} - \frac{5}{64} - \frac{405}{128\pi^2}\right]\right) + O(u^{-5/2}). \tag{5.57}$$

Proof. It follows from [5, Eq. 3.9] that

$$\epsilon_{2N+1,1}(u;\xi_2) \to 0 \quad \text{and} \quad \frac{\partial}{\partial \xi}\epsilon_{2N+1,1}(u;\xi) \to 0 \quad \text{as } \xi \to \xi_2.$$

Such properties of error terms in the Liouville–Green approximation are well known. For example, for a simpler differential equation the same property is stated in [28, Eq. 2.19, p. 196]. Therefore,

$$Z_Y(\xi_2) = \sqrt{\xi_2}\, Y_0(u\sqrt{\xi_2}) \sum_{n=0}^{N} \frac{A_Y(n;\xi_2)}{u^{2n}} - \frac{\xi_2}{u} Y_1(u\sqrt{\xi_2}) \sum_{n=0}^{N-1} \frac{B_Y(n;\xi_2)}{u^{2n}}.$$

Using (5.33) and (5.34), we obtain

$$\begin{aligned} Z'_Y(\xi_2) = {} & \sqrt{\xi_2} Y_0(u\sqrt{\xi_2})\left(\frac{1}{2\xi_2}\sum_{n=0}^{N}\frac{A_Y(n;\xi_2)}{u^{2n}} + \sum_{n=0}^{N}\frac{A'_Y(n;\xi_2)}{u^{2n}} - \frac{1}{2}\sum_{n=0}^{N-1}\frac{B_Y(n;\xi_2)}{u^{2n}}\right) \\ & - \xi_2 Y_1(u\sqrt{\xi_2})\left(\frac{u}{2\xi_2}\sum_{n=0}^{N}\frac{A_Y(n;\xi_2)}{u^{2n}} + \frac{1}{2\xi_2 u}\sum_{n=0}^{N-1}\frac{B_Y(n;\xi_2)}{u^{2n}} + \frac{1}{u}\sum_{n=0}^{N-1}\frac{B'_Y(n;\xi_2)}{u^{2n}}\right). \end{aligned}$$

For our purposes it is sufficient to take $N = 1$. Applying [29, Eq. 10.17.1, 10.17.4] and [9, Eq. 8.451(1,7,8)], we can write the Hankel expansions for the Bessel functions:

$$\sqrt{\xi_2}\, Y_0(u\sqrt{\xi_2}) = \frac{(-1)^{k/2+1}}{\sqrt{u}} \sum_{j=0}^{\infty} (-1)^j \frac{a_{2j}(0)}{(\pi u/2)^{2j}},$$

$$\xi_2 Y_1(u\sqrt{\xi_2}) = \frac{\pi}{2}\frac{(-1)^{k/2+1}}{\sqrt{u}} \sum_{j=0}^{\infty} (-1)^j \frac{a_{2j+1}(1)}{(\pi u/2)^{2j+1}},$$

where

$$a_j(v) = \frac{\Gamma(v + j + 1/2)}{2^j j! \Gamma(v - j + 1/2)}.$$

Thus

$$\sqrt{\xi_2}\, Y_0(u\sqrt{\xi_2}) = \frac{(-1)^{k/2+1}}{\sqrt{u}}\left(1 - \frac{a_2(0)}{(\pi u/2)^2} + O(u^{-4})\right),$$

$$\xi_2 Y_1(u\sqrt{\xi_2}) = \frac{\pi}{2}\frac{(-1)^{k/2+1}}{\sqrt{u}}\left(\frac{a_1(1)}{\pi u/2} + O(u^{-3})\right).$$

Consequently, taking $N = 1$ we have

$$Z_Y(\xi_2) = \frac{(-1)^{k/2+1}}{\sqrt{u}}\Bigg(A_Y(0;\xi_2) + \frac{1}{u^2}(A_Y(1;\xi_2) - \frac{A_Y(0;\xi_2)a_2(0)}{(\pi/2)^2} - B_Y(0;\xi_2)a_1(1)) + O(u^{-4})\Bigg).$$

According to (5.38) and (5.39),

$$A_Y(0;\xi_2) = 1, \quad A_Y(1;\xi_2) = -\frac{1}{16} + \frac{15}{32\pi^2} + \lambda_1, \quad B_Y(0;\xi_2) = \frac{1}{2\pi^2}.$$

Substituting $a_2(0) = 9/128$, $a_1(1) = 3/8$, we prove the equation (5.56).

Next we find an asymptotic expansion for the derivative of $Z_Y(\xi)$ at $\xi = \xi_2 = \pi^2/4$. From the recurrence relations (5.36) and (5.37), it follows that

$$A'_Y(1;\xi_2) = -\frac{15}{8\pi^4} + \frac{3}{32\pi^2}.$$

Taking $N = 1$, using the Hankel expansions for the Bessel functions and computing $a_3(1) = -105/1024$, we obtain the formula (5.57). □

Lemma 5.12. *For $\lambda_1 = \frac{1}{16} + \frac{405}{32\pi^2}$ we have*

$$C_J = -\pi^2 \frac{\Gamma^2(k/2)}{\Gamma^2(k/2+1/2)} \frac{1}{Z_Y(\xi_2)}\left(Z'_Y(\xi_2) - \frac{Z_Y(\xi_2)}{\pi^2}\right), \tag{5.58}$$

$$C_J = O(k^{-5}). \tag{5.59}$$

Proof. Note that with our choice of λ_1 the value of c_1 is non-zero. Therefore, by (5.44), (5.51), (5.54), (5.55) we have

$$C_J = -\frac{c_2}{c_1} = -\pi^2 \frac{\Gamma^2(k/2)}{\Gamma^2(k/2+1/2)} \frac{1}{Z_Y(\xi_2)}\left(Z'_Y(\xi_2) - \frac{Z_Y(\xi_2)}{\pi^2}\right).$$

By Lemma 5.11,

$$Z'_Y(\xi_2) - \frac{Z_Y(\xi_2)}{\pi^2} = \frac{(-1)^{k/2+1}}{u^{5/2}}\left(\frac{\lambda_1}{4\pi^2} - \frac{405}{128\pi^4} - \frac{1}{64\pi^2}\right) + O(u^{-9/2}).$$

Since $\lambda_1 = \frac{1}{16} + \frac{405}{32\pi^2}$ the first summand in the formula above is zero, so that

$$Z'_Y(\xi_2) - \frac{Z_Y(\xi_2)}{\pi^2} = O(u^{-9/2}).$$

Stirling's formula and [29, Eq. 5.5.5] yield

$$\frac{\Gamma^2(k/2)}{\Gamma^2(k/2+1/2)} = \left(\frac{2^{k-1}\Gamma^2(k/2)}{\sqrt{\pi}\Gamma(k)}\right)^2 = O(k^{-1}).$$

Combining all the above results we find that $C_J = O(u^{-5}) = O(k^{-5})$. □

Lemma 5.13. *For* $\lambda_1 = \frac{1}{16} + \frac{405}{32\pi^2}$ *and* $u = k - 1/2$ *we have*

$$C_Y = (-1)^{k/2} \frac{2\Gamma(1/2)\Gamma(k/2)}{\Gamma(k/2+1/2)} \frac{\xi_2^{1/4}}{Z_Y(\xi_2)}. \tag{5.60}$$

For any $n \geq 1$ *there exist constants* $d_1, \dots, d_n$ *such that*

$$C_Y = -2\pi \left(1 + \frac{d_1}{u} + \frac{d_2}{u^2} + \dots + \frac{d_n}{u^n} + O(u^{-n-1}) \right). \tag{5.61}$$

Proof. Applying the formulas (5.44), (5.51), (5.54), we prove (5.60). The asymptotics (5.61) follows from the equation (5.56), Stirling's formula and [29, Eq. 5.5.5]. □

Finally, we deduce the main result of this subsection.

Theorem 5.14. *Let* $\xi_2 = \pi^2/4$. *Then for any* $\xi \in (0, \xi_2)$ *the equality* (5.44) *holds with* Z_Y, Z_J, C_Y, C_J *given by* (5.40), (5.41), (5.60), (5.58), *respectively.*

5.3. *Asymptotic approximation of* ψ_k *and* Φ_k

In this section we study the functions defined by (4.9) and (4.10) when $u = v = 0$, namely

$$\psi_k(x) = \psi_k(x; 0; 0) = 2\frac{\Gamma^2(k)}{\Gamma(2k)} x^k F(k, k, 2k; -x), \tag{5.62}$$

$$\Phi_k(x) = \Phi_k(x; 0; 0) = 2\frac{\Gamma^2(k)}{\Gamma(2k)} x^k F(k, k, 2k; x). \tag{5.63}$$

The main difference from the previous subsection is that now the Liouville–Green approximation is based on K-Bessel functions. See Theorem 5.17 for the precise statement.

By [29, Eq. 15.8.1],

$$\psi_k(x) = \frac{2\Gamma^2(k)}{\Gamma(2k)} \left(\frac{x}{1+x} \right)^k F\left(k, k, 2k; \frac{x}{1+x} \right) = \Phi_k\left(\frac{x}{1+x} \right).$$

Hence

$$\psi_k\left(\frac{l}{n} \right) = \Phi_k\left(\frac{l}{n+l} \right), \tag{5.64}$$

and it is sufficient to consider only the function Φ_k.

Let $u := k - 1/2$ and

$$f(x) := \frac{1}{x^2(1-x)}, \qquad g(x) := -\frac{1}{4x^2(1-x)^2} + \frac{1}{4x(1-x)}. \tag{5.65}$$

Lemma 5.15. *The function* $y(x) = F(k, k, 2k; x) x^k \sqrt{1-x}$ *is a solution of the differential equation*

$$y''(x) - (u^2 f(x) + g(x)) y(x) = 0. \tag{5.66}$$

Proof. The hypergeometric function $F(x) = F(k, k, 2k; x)$ satisfies

$$x(1-x)F''(x) + (2k - (2k+1)x)F'(x) - k^2F(x) = 0.$$

Applying Lemma 5.5 with $\alpha(x) = x^{-k}(1-x)^{-1/2}$, we have

$$y''(x) + \left(\frac{1-(2k-1)^2}{4x^2} + \frac{1}{4(1-x)^2} + \frac{1-(2k-1)^2}{4x(1-x)}\right)y(x) = 0.$$

The assertion follows by rearranging the expression in brackets. □

Note that the differential equations (5.66) and (5.18) are almost identical except that the functions $f(x)$ defined by (5.65) and (5.16) differ in sign. According to [28, Chapter 12] this means that in the current case I and K Bessel functions (instead of Y and J) should be chosen as approximation functions. Consequently, we transform (5.66) to the type

$$\frac{d^2Z}{d\xi^2} + \left[-\frac{u^2}{4\xi} + \frac{1}{4\xi^2} - \frac{\psi(\xi)}{\xi}\right]Z = 0, \tag{5.67}$$

where

$$\psi(\xi) = \frac{1}{16}\left(\frac{1}{\xi} - \frac{1}{\sinh^2(\sqrt{\xi})}\right). \tag{5.68}$$

This can be done similarly to the previous case by making the change

$$Z(x) := \frac{y(x)}{\alpha(x)}, \qquad \alpha(x) := \frac{(x^2-x^3)^{1/4}}{2(\operatorname{atanh}(\sqrt{1-x}))^{1/2}} \tag{5.69}$$

and the substitution

$$\xi := 4\operatorname{atanh}^2(\sqrt{1-x}). \tag{5.70}$$

Note that as $\xi \to 0$ the function $\psi(\xi)$ is analytic. Removing the term with $\psi(\xi)/\xi$ in (5.67) we obtain

$$Z'' + \left(-\frac{u^2}{4\xi} + \frac{1}{4\xi}\right)Z = 0. \tag{5.71}$$

The following functions are solutions of this equation (see [29, Eq. 10.13.2]):

$$Z_L = \sqrt{\xi}\,L_0(u\sqrt{\xi}), \tag{5.72}$$

where L_0 is either the K_0- or I_0-Bessel function. In general,

$$L_v := \begin{cases} I_v, \\ e^{\pi i v}K_v. \end{cases} \tag{5.73}$$

Therefore, according to [28, Eq. 2.09, Chapter 12], a solution of the differential equation (5.67) can be found in the form

$$Z_L = \sqrt{\xi}\,L_0(u\sqrt{\xi})\sum_{n=0}^{\infty}\frac{A(n;\xi)}{u^{2n}} + \frac{\xi}{u}L_1(u\sqrt{\xi})\sum_{n=0}^{\infty}\frac{B(n;\xi)}{u^{2n}}. \tag{5.74}$$

To determine the coefficients $A(n;\xi)$ and $B(n;\xi)$ we introduce two functions:

$$W(\xi) := \sqrt{\xi}\, L_0(u\sqrt{\xi}), \qquad V(\xi) := \xi L_1(u\sqrt{\xi}), \tag{5.75}$$

which satisfy the following differential equations (see [29, Eq. 10.13.2, 10.13.5, 10.36]):

$$W'' + \left(-\frac{u^2}{4\xi} + \frac{1}{4\xi^2}\right) W = 0, \tag{5.76}$$

$$V'' - \frac{1}{\xi} V' + \left(-\frac{u^2}{4\xi} + \frac{3}{4\xi^2}\right) V = 0. \tag{5.77}$$

Furthermore, using [29, Eq. 10.29.2,10.29.3] we prove that

$$V' = \frac{1}{2\xi} V + \frac{u}{2} W, \qquad W' = \frac{1}{2\xi} W + \frac{u}{2\xi} V. \tag{5.78}$$

Substituting (5.74) into (5.67) we have

$$W(\xi) \sum_{n=0}^{\infty} \frac{C(n;\xi)}{u^{2n}} + V(\xi) \sum_{n=0}^{\infty} \frac{D(n;\xi)}{u^{2n+1}} = 0, \tag{5.79}$$

where

$$C(n;\xi) = A''(n;\xi) + \frac{A'(n;\xi)}{\xi} - \frac{\psi(\xi)}{\xi} A(n;\xi) + B'(n;\xi) + \frac{B(n;\xi)}{2\xi},$$

$$D(n;\xi) = B''(n-1;\xi) + \frac{B'(n-1;\xi)}{\xi} - \frac{\psi(\xi)}{\xi} B(n-1;\xi) + \frac{A'(n;\xi)}{\xi}.$$

Setting $C(n;\xi) = D(n;\xi) = 0$ we find the recurrence relations

$$\sqrt{\xi}\, B(n;\xi) = -\sqrt{\xi} A'(n;\xi) + \int_0^{\xi} \left(\psi(x) A(n;x) - \frac{1}{2} A'(n;x)\right) \frac{dx}{\sqrt{x}}, \tag{5.80}$$

$$A(n;\xi) = -\xi B'(n-1;\xi) + \int_0^{\xi} \psi(x) B(n-1;x)\, dx + \lambda_n \tag{5.81}$$

for some real constants of integration λ_n.

Let $A(0;\xi) = 1$. Then

$$B(0;\xi) = \frac{1}{8}\left(\frac{\coth(\sqrt{\xi})}{\sqrt{\xi}} - \frac{1}{\xi}\right), \tag{5.82}$$

$$A(1;\xi) = -\frac{1}{8}\left(\frac{1}{\xi} - \frac{\coth(\sqrt{\xi})}{2\sqrt{\xi}} - \frac{1}{2\sinh^2(\sqrt{\xi})}\right) + \frac{1}{128}\left(\coth(\sqrt{\xi}) - \frac{1}{\sqrt{\xi}}\right)^2 + \lambda_1. \tag{5.83}$$

Note that

$$B(0;\xi) = \frac{1}{24} + O(\xi), \qquad A(1;\xi) = \lambda_1 + O(\xi) \quad \text{as } \xi \to 0, \tag{5.84}$$

$$\lim_{\xi\to\infty} \sqrt{\xi}\, B(0;\xi) = \frac{1}{8}, \qquad \lim_{\xi\to\infty} A(1;\xi) = \frac{1}{128} + \lambda_1. \tag{5.85}$$

Theorem 5.16. *For each value of u and each non-negative integer N, the equation* (5.67) *has the solution $Z_K(\xi)$ which is infinitely differentiable in ξ on the interval $(0,\infty)$ and is given by*

$$Z_K(\xi)=\sqrt{\xi}\,K_0(u\sqrt{\xi})\sum_{n=0}^{N}\frac{A_K(n;\xi)}{u^{2n}}-\frac{\xi}{u}K_1(u\sqrt{\xi})\sum_{n=0}^{N-1}\frac{B_K(n;\xi)}{u^{2n}}+\epsilon_{2N+1,3}(u,\xi),\tag{5.86}$$

where

$$|\epsilon_{2N+1,3}(u,\xi)|\le\frac{\sqrt{\xi}\,K_0(u\sqrt{\xi})}{u^{2N+1}}V_{\xi,\infty}(\sqrt{\xi}\,B_K(N;\xi))\exp\left(\frac{1}{u}V_{\xi,\infty}(\sqrt{\xi}\,B_K(0;\xi))\right).$$

In particular, for $N=1$,

$$\epsilon_{3,3}(u,\xi)\ll\frac{\sqrt{\xi}\,K_0(u\sqrt{\xi})}{u^3}\min\left(\sqrt{\xi},\frac{1}{\xi}\right).\tag{5.87}$$

Proof. Our arguments require only minor changes compared to [28, Theorem 3.1, p. 441]. The first difference is that we take the parameter $\beta=\infty$ in [28, Theorem 3.1, p. 441]. As mentioned on [28, p. 442], this is possible as long as the variations of $\sqrt{x}\,B(0;x)$ and $\sqrt{x}\,B(n;x)$ converge at infinity. The convergence of the variations of $\sqrt{x}\,B(0;x)$ follows directly from (5.82). Furthermore, note that

$$\psi^{(s)}(\xi)=O(1/|\xi|^{s+1}).$$

Hence for $n>1$ the variation

$$\mathrm{Var}_{\xi,\infty}(\sqrt{x}\,B(n;x))=\int_{\xi}^{\infty}|(\sqrt{x}\,B(n;x))'|\,dx$$

converges by [28, Exercise 4.2, p. 445]. To prove (5.87) we need to estimate the variation of $\sqrt{x}\,B(1;x)$ at infinity. Using the recurrence relation (5.80), we find

$$(\sqrt{x}\,B(1;x))'=-\frac{1}{\sqrt{x}}(xA''(1;x)+A'(1;x)-\psi(x)A(1;x)).\tag{5.88}$$

Therefore,

$$(\sqrt{x}\,B(1;x))'=O(x^{-1/2})\quad\text{as } x\to0,\tag{5.89}$$

$$(\sqrt{x}\,B(1;x))'=O(x^{-2})\quad\text{as } x\to\infty.\tag{5.90}$$

Thus

$$\mathrm{Var}_{\xi,\infty}(\sqrt{x}\,B(1;x))=\int_{\xi}^{\infty}|(\sqrt{x}\,B(1;x))'|\,dx\ll\min(\sqrt{\xi},1/\xi).\qquad\square$$

The solution of the differential equation (5.67),

$$Z(\xi) = \left(\Phi_k(x) \frac{\sqrt{1-x}}{\alpha(x)} \right)\Bigg|_{x=1/\cosh^2(\sqrt{\xi}/2)}$$
$$= \Phi_k\left(\frac{1}{\cosh^2(\sqrt{\xi}/2)} \right) (\xi \sinh^2(\sqrt{\xi}))^{1/4} \tag{5.91}$$

is recessive as $\xi \to \infty$. Another recessive solution is $Z_K(\xi)$ defined by (5.74) with $L_v = e^{\pi i v} K_v$. Therefore, there exists $C_K = C_K(u)$ such that

$$\Phi_k\left(\frac{1}{\cosh^2(\sqrt{\xi}/2)} \right) (\xi \sinh^2(\sqrt{\xi}))^{1/4} = C_K Z_K(\xi). \tag{5.92}$$

Computing the limit as $\xi \to \infty$ of the left- and right-hand sides of (5.92), we find

$$C_K = 2\frac{\Gamma^2(k)}{\Gamma(2k)} \frac{2^{2k}\sqrt{u}}{\sqrt{\pi}} \left[\sum_{n=0}^{N} \frac{a_n}{u^{2n}} - \sum_{n=0}^{N-1} \frac{b_n}{u^{2n+1}} \right]^{-1}, \tag{5.93}$$

where

$$a_n = \lim_{\xi\to\infty} A(n;\xi), \qquad b_n = \lim_{\xi\to\infty} B(n;\xi)\sqrt{\xi}, \tag{5.94}$$

$$a_0 = 1, \qquad a_1 = \frac{1}{128} + \lambda_1, \qquad b_0 = \frac{1}{8}. \tag{5.95}$$

Since

$$\frac{\Gamma^2(k)}{\Gamma(2k)} = \frac{2\sqrt{\pi}}{\sqrt{k}\, 2^{2k}}(1 + O(k^{-1})),$$

we have

$$C_K = 4 + O(k^{-1}). \tag{5.96}$$

To sum up, we have proved the following result.

Theorem 5.17. *For $\xi \in (0,\infty)$ the equality (5.92) holds with Z_K, C_K given by (5.86), (5.93) , respectively.*

6. Error terms for the individual weight

Lemma 6.1. *The following exact formula holds:*

$$M_2(l;0,0)$$
$$= \sum_{f\in H_{2k}(1)}^{h} \lambda_f(l) L_f^2(1/2) = (1+(-1)^k)\Bigg(\frac{\tau(l)}{\sqrt{l}} \left[2\frac{\Gamma'}{\Gamma}(k) - \log l - 2\log(2\pi) + 2\gamma \right]$$
$$+ \frac{1}{2\sqrt{l}} \sum_{n=1}^{l-1} \tau(n)\tau(l-n)\phi_k\left(\frac{n}{l}\right) + \frac{1}{\sqrt{l}} \sum_{n=1}^{\infty} \tau(n)\tau(n+l)\Phi_k\left(\frac{l}{n+l}\right)\Bigg). \tag{6.1}$$

Proof. The assertion follows from Theorem 4.2 by computing the limit as $u, v \to 0$. The off-diagonal terms can be simplified as follows. First, using (5.3) we find that

$$\sum_{n=1}^{l-1} \tau(n)\tau(l-n)\phi_k\left(\frac{n}{l}\right) = (-1)^k \sum_{n=1}^{l-1} \tau(n)\tau(l-n)\phi_k\left(\frac{n}{l}\right).$$

Second, by (5.64) we have

$$\psi_k\left(\frac{l}{n}\right) = \Phi_k\left(\frac{l}{n+l}\right).$$

Therefore,

$$\frac{1}{\sqrt{l}} \sum_{n=l+1}^{\infty} \tau(n)\tau(n-l)\Phi_k\left(\frac{l}{n}\right) + \frac{(-1)^k}{\sqrt{l}} \sum_{n=0}^{\infty} \tau(n)\tau(n+l)\psi_k\left(\frac{l}{n}\right)$$
$$= \frac{1+(-1)^k}{\sqrt{l}} \sum_{n=1}^{\infty} \tau(n)\tau(n+l)\Phi_k\left(\frac{l}{n+l}\right). \qquad \square$$

Lemma 6.2. *For any* $\epsilon > 0$, $l \ll k^2$ *we have*

$$\frac{1}{\sqrt{l}} \sum_{n=1}^{\infty} \tau(n)\tau(n+l)\Phi_{2k}\left(\frac{l}{n+l}\right) \ll \exp\left(-\frac{ck}{\sqrt{l}}\right)\left(\frac{l^{\epsilon}}{l^{1/4}k^{1/2}} + \frac{l^{1/2+\epsilon}}{k^{3/2}}\right) \tag{6.2}$$

for some absolute constant $c > 0$.

Proof. Consider

$$\frac{1}{\sqrt{l}} \sum_{n=1}^{\infty} \tau(n)\tau(n+l)\Phi_{2k}\left(\frac{l}{n+l}\right) \ll \frac{l^{\epsilon}}{\sqrt{l}} \sum_{n=1}^{\infty} n^{\epsilon}\left|\Phi_{2k}\left(\frac{l}{n+l}\right)\right|$$
$$\ll \frac{l^{\epsilon}}{\sqrt{l}} \int_1^{\infty} x^{\epsilon}\left|\Phi_{2k}\left(\frac{l}{x+l}\right)\right| dx + O\left(\frac{l^{\epsilon}}{\sqrt{l}}\Phi_{2k}\left(\frac{l}{1+l}\right)\right).$$

First, we estimate the integral

$$I := \frac{l^{\epsilon}}{\sqrt{l}} \int_1^{\infty} x^{\epsilon}\left|\Phi_{2k}\left(\frac{l}{x+l}\right)\right| dx.$$

Letting

$$x := l\sinh^2(\sqrt{\xi}/2), \qquad a := 2\,\mathrm{asinh}(1/\sqrt{l}),$$

we have

$$I = \frac{l^{\epsilon}}{\sqrt{l}} l \int_{a^2}^{\infty} \sinh^{\epsilon}\left(\frac{\sqrt{\xi}}{2}\right)\left|\Phi_{2k}\left(\frac{1}{\cosh^2(\sqrt{\xi}/2)}\right)\right| \frac{\sinh(\sqrt{\xi})}{\sqrt{\xi}}\, d\xi.$$

Applying the Liouville–Green method (see (5.92)) the integral is equal to

$$I = l^{1/2+\epsilon} C_K \int_{a^2}^{\infty} \sinh^{\epsilon}(\sqrt{\xi}/2)|Z_K(\xi)| \frac{(\sinh(\sqrt{\xi}))^{1/2}}{\xi^{3/4}}\, d\xi,$$

where C_K satisfies the asymptotic formula (5.96). Note that

$$\sqrt{\xi} \geq 2\,\mathrm{asinh}(l^{-1/2}) \gg 1/\sqrt{l}.$$

Thus

$$(2k-1/2)\sqrt{\xi} \gg k/\sqrt{l} \gg 1 \text{ for any } l \ll k^2.$$

Applying Theorem 5.16 with $N = 0$ we estimate

$$Z_K(\xi) \ll \sqrt{\xi}\, K_0((2k-1/2)\sqrt{\xi}).$$

The last inequality and [29, Eq. 10.40.2] yield

$$\begin{aligned} I &\ll l^{1/2+\epsilon} \int_{a^2}^{\infty} |K_0((2k-1/2)\sqrt{\xi})| \frac{(\sinh(\sqrt{\xi}))^{1/2+\epsilon}}{\xi^{1/4}}\, d\xi \\ &\ll \frac{l^{1/2+\epsilon}}{\sqrt{2k-1/2}} \int_{a^2}^{\infty} \exp(-(2k-1/2)\sqrt{\xi}) \frac{(\sinh(\sqrt{\xi}))^{1/2+\epsilon}}{\sqrt{\xi}}\, d\xi. \end{aligned}$$

Making the change of variables $\xi = x^2$ and splitting the integral into two parts

$$\begin{aligned} I &\ll \frac{l^{1/2+\epsilon}}{\sqrt{k}} \int_{a}^{\infty} \exp(-(2k-1/2)x)(\sinh(x))^{1/2+\epsilon}\, dx \\ &\ll \frac{l^{1/2+\epsilon}}{\sqrt{k}} \biggl(\int_{a}^{1} \exp(-(2k-1/2)x) x^{1/2+\epsilon}\, dx \\ &\quad + \int_{1}^{\infty} \exp(-(2k-1/2)x) \exp(x(1/2+\epsilon))\, dx \biggr) \ll \frac{l^{1/2+\epsilon}}{k^{3/2}} \exp\left(-\frac{ck}{\sqrt{l}}\right) \end{aligned}$$

for some $c > 0$.

Now we estimate the second term

$$E := \frac{l^{\epsilon}}{\sqrt{l}} \Phi_{2k}\left(\frac{l}{n+l}\right).$$

Let

$$\xi_0 := \left(2\,\mathrm{atanh}\left(\frac{1}{\sqrt{l+1}}\right)\right)^2.$$

Then by equation (5.92),

$$E = \frac{l^{\epsilon}}{\sqrt{l}} C_K \frac{Z_K(\xi_0)}{(\xi_0 \sinh^2(\sqrt{\xi_0}))^{1/4}}.$$

Note that $\xi_0 \sim l^{-1}$. Then for $l \ll k^2$ we have

$$Z_K(\xi_0) \ll \sqrt{\xi_0}\, K_0((2k-1/2)\sqrt{\xi_0}) \ll \frac{\xi_0^{1/4}}{\sqrt{k}} \exp(-(2k-1/2)\sqrt{\xi_0}).$$

Finally, for some constant $c > 0$ independent of k,

$$E \ll \frac{l^{\epsilon}}{\sqrt{l}} \frac{\exp(-(2k-1/2)\sqrt{\xi_0})}{\sqrt{k}(\sinh(\sqrt{\xi_0}))^{1/2}} \ll \frac{l^{\epsilon}}{l^{1/4}k^{1/2}} \exp\left(-\frac{ck}{\sqrt{l}}\right). \qquad \square$$

Lemma 6.3. *For any* $\epsilon > 0$ *and* $l \ll k^2$ *we have*

$$\frac{1}{\sqrt{l}}\sum_{n=1}^{l-1}\tau(n)\tau(n-l)\phi_{2k}\left(\frac{n}{l}\right) \ll \frac{l^{1/2+\epsilon}}{\sqrt{k}}. \tag{6.3}$$

Proof. We apply (5.44) with $n/l =: \sin^2(\sqrt{\xi}/2)$, so that

$$\phi_{2k}\left(\frac{n}{l}\right) = \frac{C_Y Z_Y(\xi) + C_J Z_J(\xi)}{(2\arcsin(\sqrt{n/l}))^{1/2}(2n/l)^{1/4}(1-n/l)^{1/4}}.$$

Using the property (5.3) we find

$$\frac{1}{\sqrt{l}}\sum_{n=1}^{l-1}\tau(n)\tau(n-l)\phi_{2k}\left(\frac{n}{l}\right) \ll \frac{l^{\epsilon}}{\sqrt{l}}\sum_{n\le l/2}\left|\phi_{2k}\left(\frac{n}{l}\right)\right|.$$

Thus $n/l \le 1/2$ and $\xi \le \pi^2/4$. Since $n/l \ge 1/l$ we have $\xi \gg 1/l$. Hence

$$(2k-1/2)\sqrt{\xi} \gg k/\sqrt{l} \gg 1 \quad \text{when } l \ll k^2.$$

Applying Theorem 5.9 with $N = 0$, Theorem 5.10 and standard estimates for the Bessel functions, we find

$$C_Y Z_Y(\xi) \ll \sqrt{\xi}\, Y_0((2k-1/2)\sqrt{\xi}) \ll \frac{\xi^{1/4}}{k^{1/2}},$$
$$C_J Z_J(\xi) \ll C_J J_0((2k-1/2)\sqrt{\xi}) \ll \frac{\xi^{1/4}}{k^{11/2}}$$

since $C_J = O(k^{-5})$ by Lemma 5.12. Consequently,

$$\phi_{2k}\left(\frac{n}{l}\right) \ll \frac{1}{\sqrt{k}\,(n/l)^{1/4}(1-n/l)^{1/4}}.$$

Finally,

$$\frac{l^{\epsilon}}{\sqrt{l}}\sum_{n\le l/2}\left|\phi_{2k}\left(\frac{n}{l}\right)\right| \ll \frac{l^{\epsilon}}{\sqrt{l}}\sum_{n\le l/2}\frac{(l/n)^{1/4}}{\sqrt{k}} \ll \frac{l^{1/2+\epsilon}}{\sqrt{k}}. \qquad \square$$

Combining all the results above we obtain the following asymptotic formula.

Theorem 6.4. *For any* $\epsilon > 0$ *and* $l \ll k^2$ *with* $k \equiv 0 \pmod 2$ *we have*

$$M_2(l;0,0) = \frac{2\tau(l)}{\sqrt{l}}(2\log k - \log l - 2\log 2\pi + 2\gamma) + O\left(\frac{l^{1/2+\epsilon}}{\sqrt{k}}\right). \tag{6.4}$$

7. Error terms on average

The Liouville–Green method allows also proving sharp asymptotic formulas on average, as we show in this section. The main goal is to provide an alternative proof for the result (1.1) by Iwaniec and Sarnak. To this end, we estimate the error terms averaged over k with a test function $h \in C_0^\infty(\mathbb{R}^+)$, which is non-negative, compactly supported on $[\theta_1, \theta_2]$ such that $\theta_2 > \theta_1 > 0$ and

$$\|h^{(n)}\|_1 \ll 1 \quad \text{for all } n \geq 0. \tag{7.1}$$

Let

$$H := \int_0^\infty h(y)\,dy, \qquad H_1 := \int_0^\infty h(y)\log y\,dy. \tag{7.2}$$

We denote the averaged moments as follows:

$$A_1(l) := \sum_k h\left(\frac{4k}{K}\right) \sum_{f\in H_{4k}(1)}^h \lambda_f(l) L_f(1/2), \tag{7.3}$$

$$A_2(l) := \sum_k h\left(\frac{4k}{K}\right) \sum_{f\in H_{4k}(1)}^h \lambda_f(l) L_f^2(1/2). \tag{7.4}$$

Lemma 7.1. *The following formula holds:*

$$\begin{aligned} A_2(l) = {}& \frac{2\tau(l)}{\sqrt{l}} \frac{HK}{4}\left(2\log K - \log l - 2\log 8\pi + 2\gamma + 2\frac{H_1}{H}\right) \\ &+ O\left(\frac{1}{\sqrt{l}}\right) + 2\sum_k h\left(\frac{4k}{K}\right)\frac{1}{\sqrt{l}}\sum_{n=1}^\infty \tau(n)\tau(n+l)\Phi_{2k}\left(\frac{l}{n+l}\right) \\ &+ \sum_k h\left(\frac{4k}{K}\right)\frac{1}{\sqrt{l}}\sum_{n=1}^{l-1}\tau(n)\tau(l-n)\phi_{2k}\left(\frac{n}{l}\right). \end{aligned} \tag{7.5}$$

Proof. We average over k the result of Lemma 6.1. To compute the main term, we use [29, Eq. 5.11.2]:

$$\frac{\Gamma'}{\Gamma}(k) \sim \log k - \frac{1}{2k} - \sum_{r=1}^\infty \frac{B_{2r}}{2rk^{2r}},$$

where B_{2r} are the Bernoulli numbers. Note that

$$\sum_k h\left(\frac{4k}{K}\right)\frac{1}{k} \ll 1.$$

By Poisson's summation formula we have

$$\begin{aligned} 2\sum_k h\left(\frac{4k}{K}\right)\log k &= 2\sum_n \int_{-\infty}^\infty h\left(\frac{4x}{K}\right)\log x \exp(-2\pi i n x)\,dx \\ &= 2\frac{K}{4}\sum_n \int_{-\infty}^\infty h(y)(\log y + \log K - \log 4)\exp\left(\frac{-2\pi i n y K}{4}\right)dy. \end{aligned}$$

Note that in the last expression the summand corresponding to $n = 0$ is equal to

$$2\left(\frac{HK}{4}(\log K - \log 4) + \frac{H_1 K}{4}\right).$$

If $n \neq 0$ we integrate by parts $a \geq 2$ times and estimate the expression by its absolute value, obtaining

$$\int_{-\infty}^{\infty} \frac{\partial^a}{\partial y^a}\left(h(y)\log\frac{yK}{4}\right)\frac{1}{(nK)^a}\,dy \ll \frac{\log K}{(nK)^a}.$$

Similarly,

$$\sum_k h\left(\frac{4k}{K}\right) = \frac{HK}{4} + O\left(\frac{1}{K^a}\right). \qquad \square$$

Lemma 7.2. *For $l \ll K^{2-\epsilon}$ and any $A > 0$,*

$$\sum_k h\left(\frac{4k}{K}\right)\frac{1}{\sqrt{l}}\sum_{n=1}^{\infty}\tau(n)\tau(n+l)\Phi_{2k}\left(\frac{l}{n+l}\right) \ll K^{-A}. \tag{7.6}$$

Proof. Averaging the result of Lemma 6.2 we obtain

$$\sum_k h\left(\frac{4k}{K}\right)\frac{1}{\sqrt{l}}\sum_{n=1}^{\infty}\tau(n)\tau(n+l)\Phi_{2k}\left(\frac{l}{n+l}\right)$$
$$\ll l^{\epsilon}\sum_k h\left(\frac{4k}{K}\right)\left(\frac{1}{l^{1/4}k^{1/2}} + \frac{l^{1/2}}{k^{3/2}}\right)\exp\left(-ck/\sqrt{l}\right) \ll K^{-A}. \qquad \square$$

Lemma 7.3. *For any $\epsilon > 0$, any $a \geq 2$ and $l \ll K^{2-\epsilon}$,*

$$\sum_k h\left(\frac{4k}{K}\right)\frac{1}{\sqrt{l}}\sum_{n=1}^{l-1}\tau(n)\tau(l-n)\phi_{2k}\left(\frac{n}{l}\right) \ll \frac{l^{a/2-1/4+\epsilon}}{K^{a+1/2}}K + \frac{l^{\epsilon}}{\sqrt{lK}}K + \frac{l^{1/2+\epsilon}}{K^{7/2}}K.$$

Proof. According to (5.3) we have

$$\phi_{2k}\left(\frac{n}{l}\right) = \phi_{2k}\left(\frac{l-n}{l}\right).$$

If l is odd, then

$$\sum_{n=1}^{l-1}\tau(n)\tau(l-n)\phi_{2k}\left(\frac{n}{l}\right) = 2\sum_{n=1}^{l/2}\tau(n)\tau(l-n)\phi_{2k}\left(\frac{n}{l}\right),$$

and if l is even we have

$$\sum_{n=1}^{l-1}\tau(n)\tau(l-n)\phi_{2k}\left(\frac{n}{l}\right) = 2\sum_{n=1}^{l/2}\tau(n)\tau(l-n)\phi_{2k}\left(\frac{n}{l}\right) - \phi_{2k}(1/2)\tau^2(l/2).$$

The contribution of $\phi_{2k}(1/2)\tau^2(l/2)$ can be estimated using the formula (5.15):

$$\frac{\tau^2(l/2)}{\sqrt{l}}\sum_k h\left(\frac{4k}{K}\right)\phi_{2k}(1/2) = (-1)^k\frac{\tau^2(l/2)}{\sqrt{l}}\sum_k h\left(\frac{4k}{K}\right)\frac{2\sqrt{\pi}\,\Gamma(k)}{\Gamma(k+1/2)}$$
$$\ll l^{-1/2+\epsilon}\sum_k h\left(\frac{4k}{K}\right)\frac{1}{\sqrt{k}} \ll l^{-1/2+\epsilon}\frac{1}{\sqrt{K}}\sum_k h\left(\frac{4k}{K}\right) \ll \frac{l^{\epsilon}}{l^{1/2}K^{1/2}}K.$$

Next we estimate

$$\sum_k h\left(\frac{4k}{K}\right)\frac{1}{\sqrt{l}}\sum_{n=1}^{l/2}\tau(n)\tau(l-n)\phi_{2k}\left(\frac{n}{l}\right) \ll \frac{l^{\epsilon}}{l^{1/2}}\sum_{n=1}^{l/2}\left|\sum_k h\left(\frac{4k}{K}\right)\phi_{2k}\left(\frac{n}{l}\right)\right|.$$

Applying the formula (5.44) we have

$$\phi_{2k}\left(\frac{n}{l}\right) = \frac{C_Y Z_Y(4\arcsin^2(\sqrt{n/l})) + C_J Z_J(4\arcsin^2(\sqrt{n/l}))}{(2\arcsin(\sqrt{n/l}))^{1/2}(2n/l)^{1/4}(1-n/l)^{1/4}}.$$

Let $u := 2k-1/2$, $\xi := 4\arcsin^2(\sqrt{n/l})$. Then using (5.41) and (5.59) we obtain

$$C_J Z_J(\xi) = O\left(\frac{\sqrt{\xi}\,J_0(u\sqrt{\xi})}{k^5}\right).$$

Since $u\sqrt{\xi} = (4k-1)\arcsin(\sqrt{n/l}) \gg K/\sqrt{l} \gg 1$ when $l \ll K^{2-\epsilon}$, we have

$$J_0(u\sqrt{\xi}) \ll \frac{1}{(u\sqrt{\xi})^{1/2}} \ll \frac{1}{\sqrt{k}\,(\arcsin(\sqrt{n/l}))^{1/2}}.$$

Therefore, the contribution of the term with $C_J Z_J$ is bounded by

$$\frac{l^{\epsilon}}{\sqrt{l}}\sum_{n=1}^{l/2}\frac{(l/n)^{1/4}}{(\arcsin(\sqrt{n/l}))^{1/2}}\sum_k h\left(\frac{4k}{K}\right)\frac{\arcsin(\sqrt{n/l})|J_0(u\sqrt{\xi})|}{k^5}$$
$$\ll \frac{l^{\epsilon}}{\sqrt{l}}\sum_{n=1}^{l/2}\left(\frac{l}{n}\right)^{1/4}\sum_k h\left(\frac{4k}{K}\right)\frac{1}{k^{11/2}} \ll \frac{l^{1/2+\epsilon}}{K^{11/2}}K.$$

Now we estimate the contribution of the term with $C_Y Z_Y$, namely

$$\frac{l^{\epsilon}}{\sqrt{l}}\sum_{n=1}^{l/2}\frac{(l/n)^{1/4}}{(\arcsin(\sqrt{n/l}))^{1/2}}\left|\sum_k h\left(\frac{4k}{K}\right)C_Y Z_Y(\xi)\right|.$$

To this end, we use the series representation (5.61) for C_Y with a sufficiently large n so that the error term is negligible. All main terms can be estimated in the same way and the largest contribution comes from the first summand -2π.

The function $Z_Y(\xi)$ is defined by (5.40). Using (5.42) the error term is majorized by

$$\epsilon_{3,1}(u,\xi) \ll \frac{\sqrt{\xi}\,|Y_0(u\sqrt{\xi})|}{u^3},$$

and therefore its contribution is bounded by

$$\frac{l^{1/2+\epsilon}}{K^{7/2}}K.$$

On the interval $(0,\pi^2/4)$ the functions $B_0(\xi)$, $A_1(\xi)$ are bounded, independent of k and non-oscillatory (see (5.38), (5.39)).

Y-Bessel functions have oscillatory behavior. According to [9, Eq. 8.451(2)] we have

$$Y_v(z)=\sqrt{\frac{2}{\pi z}}\sin\Big(z-\frac{\pi v}{2}-\frac{\pi}{4}\Big)\Big(\sum_{s=0}^{s_1-1}\frac{(-1)^s\Gamma(v+2s+1/2)}{(2z)^{2s}(2s)!\Gamma(v-2s+1/2)}+R_1\Big)$$
$$+\sqrt{\frac{2}{\pi z}}\cos\Big(z-\frac{\pi v}{2}-\frac{\pi}{4}\Big)\Big(\sum_{s=0}^{s_1-1}\frac{(-1)^s\Gamma(v+2s+3/2)}{(2z)^{2s+1}(2s+1)!\Gamma(v-2s-1/2)}+R_2\Big),$$

where $R_1=O(z^{-2s_1})$, $R_2=O(z^{-2s_1-1})$ by [9, Eq. 8.451(7,8)]. Since $u\sqrt{\xi}>1$ the only difference between $Y_0(u\sqrt{\xi})$ and $Y_1(u\sqrt{\xi})$ is the $\pi/2$ shift in the oscillating multiples. Thus it is sufficient to consider only $Y_0(u\sqrt{\xi})$.

The contribution of R_1, R_2 is majorized by

$$\frac{l^\epsilon}{\sqrt{l}}\sum_{n=1}^{l/2}\sum_k\frac{h(4k/K)}{(k\sqrt{n/l})^{2s_1+1/2}}\ll\frac{l^\epsilon}{\sqrt{l}}\Big(\frac{l}{K^2}\Big)^{s_1+1/4}K\qquad\text{for } s_1\ge 2.$$

It is sufficient to estimate

$$E:=\frac{l^\epsilon}{\sqrt{l}}\sum_{n=1}^{l/2}\Big(\frac{l}{n}\Big)^{1/4}\Big|\sum_k h\Big(\frac{4k}{K}\Big)\frac{\sin\big((4k-1)\arcsin(\sqrt{n/l})-\pi/4\big)}{\sqrt{4k-1}}\Big|.$$

By the Poisson summation formula [13, Theorem 4.4],

$$\sum_k h\Big(\frac{4k}{K}\Big)\frac{\sin\big((4k-1)\arcsin(\sqrt{n/l})-\pi/4\big)}{\sqrt{4k-1}}=\sum_{m\in\mathbb{Z}}I(m),$$

where

$$I(m):=\int_{-\infty}^{+\infty}h\Big(\frac{4y}{K}\Big)\frac{\sin\big((4y-1)\arcsin(\sqrt{n/l})-\pi/4\big)}{\sqrt{4y-1}}e^{-2\pi iym}\,dy.$$

Let $g(y):=\frac{1}{4}Ky(-2\pi m\pm 4\arcsin(\sqrt{n/l}))$. Then writing the sine in terms of exponentials we have

$$I(m)\ll K\Big|\int_{-\infty}^{\infty}h(y)e^{ig(y)}\frac{dy}{\sqrt{yK-1}}\Big|.$$

Integration by parts $a\ge 2$ times yields

$$I(m)\ll\sqrt{K}\Big|\int_{-\infty}^{+\infty}\frac{\partial^a}{\partial y^a}\Big(\frac{h(y)}{\sqrt{y-1/K}}\Big)\frac{e^{ig(y)}dy}{K^a|-\pi m/2\pm\arcsin(\sqrt{n/l})|^a}\Big|.$$

Note that $0 < \arcsin(\sqrt{n/l}) \le \pi/4$. If $m \ne 0$ we have

$$I(m) \ll \sqrt{K}\,(Km)^{-a} \quad \text{and} \quad I(0) \ll \sqrt{K}\,(K\arcsin(\sqrt{n/l}))^{-a}.$$

Consequently,

$$E \ll l^{-1/2+\epsilon} K^{1/2-a} \sum_{n=1}^{l/2} \left(\frac{l}{n}\right)^{1/4} \left(1 + \frac{1}{(\arcsin(\sqrt{n/l}))^{a}}\right)$$
$$\ll l^{-1/2+\epsilon} K^{1/2-a} \sum_{n=1}^{l/2} \left(\frac{l}{n}\right)^{1/4+a/2} \ll l^{a/2-1/4+\epsilon} K^{-a+1/2}. \qquad \square$$

Finally, we obtain the main result of this section.

Theorem 7.4. *For any $\epsilon > 0$, any $a \ge 2$, and $l \ll K^2$,*

$$A_2(l) = \frac{2\tau(l)}{\sqrt{l}} \frac{HK}{4}(2\log K - \log l - 2\log 8\pi + 2\gamma + 2H_1/H)$$
$$+ O\left(K\left(\frac{l^{a/2-1/4+\epsilon}}{K^{a+1/2}} + \frac{l^{\epsilon}}{\sqrt{lK}} + \frac{l^{1/2+\epsilon}}{K^{7/2}}\right)\right). \tag{7.7}$$

The asymptotic formula for the averaged first moment follows from Theorem 3.1.

Theorem 7.5. *There exist $c_1, c_2 > 0$ such that for $l \ll K$ we have*

$$A_1(l) = \frac{2}{\sqrt{l}} \frac{HK}{4} + O\left(\frac{K}{\sqrt{l}}\left(c_1 \frac{l}{K}\right)^{c_2 K}\right). \tag{7.8}$$

Remark. By more careful calculations, the error term in the above formula can be improved, as shown in [3, Theorem 1.1].

8. Mollification and non-vanishing at the critical point

In this section we apply the technique of mollification in order to establish effective non-vanishing results. We mainly use the methods of [20, 22].

8.1. The choice of mollifier

We choose a mollifier of the type (see [15, 22])

$$M(f) = \sum_{m \le M} x_m \lambda_f(m) m^{-1/2}, \tag{8.1}$$

where

$$x_m = \frac{\mu(m)}{\rho(m)} P\left(\frac{\log(M/m)}{\log M}\right), \tag{8.2}$$

$$\rho(m) = \prod_{p|m}(1 + 1/p), \qquad P(x) = x^2. \tag{8.3}$$

If M is not an integer then [22, Lemma 2.1]

$$\delta_{m<M} P\left(\frac{\log(M/m)}{\log M}\right) = \frac{2}{2\pi i}(\log M)^{-2} \int_{(3)} \frac{M^s}{m^s}\frac{ds}{s^3}. \tag{8.4}$$

Lemma 8.1. *Let $k \equiv 0 \pmod 2$ and $M = k^{\Delta}$. For any $\epsilon > 0$ there is $k_0 = k_0(\epsilon)$ such that for every $k \geq k_0$ the inequality*

$$\sum_{f \in H_{2k}(1)}^{h} M^2(f) \ll \log M \tag{8.5}$$

holds for any $\Delta < 1 - \epsilon$.

Proof. Consider

$$\sum_{f \in H_{2k}(1)}^{h} M^2(f) = \sum_{m_1, m_2 \leq M} \frac{x_{m_1} x_{m_2}}{\sqrt{m_1 m_2}} \sum_{f \in H_{2k}(1)}^{h} \lambda_f(m_1)\lambda_f(m_2).$$

The inner sum can be estimated using the Petersson trace formula and [33, Lemma 2.1]:

$$\sum_{f \in H_{2k}(1)}^{h} \lambda_f(m_1)\lambda_f(m_2) = \delta(m_1, m_2) + O(e^{-k}) \quad \text{for } m_1 m_2 < k^2/10^4.$$

Note that $x_m \ll 1$, and therefore

$$\sum_{f \in H_{2k}(1)}^{h} M^2(f) \ll \sum_{m \leq M} \frac{1}{m} \ll \log M. \qquad \square$$

Averaging over k we prove the following estimate.

Lemma 8.2. *Let $M = K^{\Delta}$. For any $\epsilon > 0$ there is $K_0 = K_0(\epsilon)$ such that for every $K \geq K_0$ the inequality*

$$\sum_k h\left(\frac{4k}{K}\right) \sum_{f \in H_{4k}(1)}^{h} M^2(f) \ll K \log M \tag{8.6}$$

holds for any $\Delta < 1 - \epsilon$.

8.2. The first mollified moment

Lemma 8.3. *Let $k \equiv 0 \pmod 2$ and $M = k^{\Delta}$. For any $\epsilon > 0$ there is $k_0 = k_0(\epsilon)$ such that for every $k \geq k_0$ the asymptotic formula*

$$M_1 := \sum_{f \in H_{2k}(1)}^{h} M(f) L_f(1/2) = \frac{4\zeta(2)}{\log M} + O((\log M)^{-2}) \tag{8.7}$$

holds for any $\Delta < 1 - \epsilon$.

Proof. By Theorem 3.1,

$$\sum_{f\in H_{2k}(1)}^{h} M(f)L_f(1/2) = 2\sum_{m\le M}\frac{x_m}{m} + O\biggl(\sum_{m\le M}\frac{x_m}{m}\Bigl(\frac{2\pi em}{k}\Bigr)^k\biggr).$$

Since

$$\sum_{m\le M}\frac{x_m}{m}\Bigl(\frac{2\pi em}{k}\Bigr)^k \ll (2\pi eM/k)^k = (2\pi e)^k k^{k(\Delta-1)},$$

the error term is negligible for any $\Delta < 1-\epsilon$. Applying (8.2) and (8.4), we have

$$2\sum_{m\le M}\frac{x_m}{m} = \frac{4}{(\log M)^2}\frac{1}{2\pi i}\int_{(3)}\frac{M^s}{s^3}\sum_{m=1}^{\infty}\frac{\mu(m)}{\rho(m)m^{1+s}}\,ds.$$

Consider the sum over m:

$$\sum_{m=1}^{\infty}\frac{\mu(m)}{\rho(m)m^{1+s}} = \frac{\alpha(s)}{\zeta(s+1)},$$

where

$$\alpha(s) = \prod_p \frac{1+1/p-1/p^{s+1}}{(1+1/p)(1-1/p^{s+1})}$$

converges absolutely for $\Re s > -1$, and $\alpha(0) = \zeta(2)$. The resulting integral

$$\frac{4}{(\log M)^2}\frac{1}{2\pi i}\int_{(3)}\frac{M^s\alpha(s)}{s^3\zeta(s+1)}\,ds$$

has a double pole at $s=0$. We cross this pole by moving the contour of integration to

$$\Re s = -\frac{c}{\log(3+|\Im s|)},$$

where $c>0$ is a constant sufficiently small so that there is no zero of $\zeta(s+1)$ to the right of the contour. Then the integral is bounded by

$$\int_0^{\infty} M^{-c/\log(3+t)}(3+t)^{-3+\epsilon}\,dt = \int_{\log 3}^{\infty} e^{x(-2+\epsilon)-cx^{-1}\log M}\,dx.$$

Using the saddle point method, we estimate the last integral as

$$(\log M)^{1/4}e^{-c'\sqrt{\log M}}.$$

Finally, the residue at $s=0$ is equal to

$$\frac{4\zeta(2)}{\log M} + O((\log M)^{-2}). \qquad \square$$

Lemma 8.4. *Let $M = K^{\Delta}$. For any $\epsilon > 0$ there is $K_0 = K_0(\epsilon)$ such that for every $K \ge K_0$ one has*

$$A_1 := \sum_k h\Bigl(\frac{4k}{K}\Bigr)\sum_{f\in H_{4k}(1)}^{h} M(f)L_f(1/2) = \frac{HK}{4}\frac{4\zeta(2)}{\log M} + O(K(\log M)^{-2}) \tag{8.8}$$

for any $\Delta < 1-\epsilon$.

Proof. By Theorem 7.5, for some absolute constants $c_1, c_2 > 0$,

$$A_1 = \frac{HK}{4} 2 \sum_{m \le M} \frac{x_m}{m} + O\left(K\left(\frac{c_1 M}{K}\right)^{c_2 K}\right).$$

The error term is negligible for any $\Delta < 1 - \epsilon$. The main term was evaluated in Lemma 8.3. □

8.3. The second mollified moment

Lemma 8.5. *Let $k \equiv 0 \pmod 2$ and $M = k^{\Delta}$. For any $\epsilon > 0$ there is $k_0 = k_0(\epsilon)$ such that for every $k \ge k_0$ the asymptotic formula*

$$M_2 := \sum_{f \in H_{2k}(1)}^{h} M^2(f) L_f^2(1/2) = \frac{16\zeta^2(2)}{(\log M)^2}(1 + 1/\Delta) + O((\log M)^{-3}) \tag{8.9}$$

holds for any $\Delta < 1/4 - \epsilon$.

Proof. Using the multiplicativity property (2.6) we have

$$\sum_{f \in H_{2k}(1)}^{h} M^2(f) L_f^2(1/2) = \sum_{b \le M} \frac{1}{b} \sum_{m_1, m_2 \le M/b} \frac{x_{m_1 b} x_{m_2 b}}{\sqrt{m_1 m_2}} M_2(m_1 m_2).$$

By Theorem 6.4 the contribution of the error term in $M_2(m_1 m_2)$ is negligible for any $\Delta < 1/4 - \epsilon$: indeed,

$$\sum_{b \le M} \frac{1}{b} \sum_{m_1, m_2 \le M/b} \frac{(m_1 m_2)^{1/2+\epsilon}}{\sqrt{m_1 m_2 k}} \ll \frac{M^{2+\epsilon}}{\sqrt{k}} = k^{2\Delta - 1/2 + \epsilon}.$$

The main term of $M_2(m_1 m_2)$ is

$$2 \frac{\tau(m_1 m_2)}{\sqrt{m_1 m_2}} \left(2 \frac{\Gamma'}{\Gamma}(k) - \log(m_1 m_2) + 2\gamma - 2\log(2\pi)\right).$$

Therefore, the largest contribution comes from

$$2 \frac{\tau(m_1 m_2)}{\sqrt{m_1 m_2}} \log \frac{k^2}{m_1 m_2}.$$

Therefore, we need to compute

$$2 \sum_{b \le M} \frac{1}{b} \sum_{m_1, m_2 \le M/b} \frac{\tau(m_1 m_2) x_{m_1 b} x_{m_2 b}}{m_1 m_2} \log \frac{k^2}{m_1 m_2}.$$

Using $\tau(m_1 m_2) = \sum_{d | (m_1, m_2)} \mu(d) \tau(m_1/d) \tau(m_2/d)$, we have

$$2 \sum_{n \le M} \frac{1}{n} \sum_{d | n} \frac{\mu(d)}{d} \sum_{m_1, m_2 \le M/n} \frac{\tau(m_1) \tau(m_2) x_{m_1 n} x_{m_2 n}}{m_1 m_2} \log \frac{k^2}{d^2 m_1 m_2}.$$

The last expression splits into two parts:

$$P_1 := 4 \sum_{n \le M} \frac{1}{n} \sum_{d|n} \frac{\mu(d)}{d} \sum_{m_1, m_2 \le M/n} \frac{\tau(m_1)\tau(m_2) x_{m_1 n} x_{m_2 n}}{m_1 m_2} \log(k/m_1),$$
$$P_2 := -4 \sum_{n \le M} \frac{1}{n} \sum_{d|n} \frac{\mu(d)\log d}{d} \sum_{m_1, m_2 \le M/n} \frac{\tau(m_1)\tau(m_2) x_{m_1 n} x_{m_2 n}}{m_1 m_2}.$$

The main contribution comes from P_1 due to the additional factor of $\log(k/m_1)$. By Cauchy's integral formula

$$\log(k/m_1) = \frac{1}{2\pi i} \int_{C_\delta} \frac{k^z}{m_1^z z^2}\, dz,$$

where C_δ is the circle of radius δ around 0. Using (8.2) and (8.4), we have

$$P_1 = \frac{16}{(\log M)^4} \frac{1}{(2\pi i)^3} \int_{(3)} \int_{(3)} \int_{C_\delta} \sum_{n=1}^{\infty} \frac{\phi(n)}{n^{2+s_1+s_2}} M^{s_1+s_2}$$
$$\times \sum_{m_1, m_2 = 1}^{\infty} \frac{\tau(m_1)\tau(m_2)\mu(m_1 n)\mu(m_2 n)}{\rho(m_1 n)\rho(m_2 n) m_1^{s_1+z+1} m_2^{s_2+1}} \frac{k^z dz}{z^2} \frac{ds_1}{s_1^3} \frac{ds_2}{s_2^3}.$$

Let

$$\alpha_1(s) := \prod_p \frac{(1 + 1/p - 2p^{-s-1})(1 - 1/p)}{(1 - p^{-s-1})^2},$$
$$\beta_n(s) := \prod_{p|n} \frac{1 + 1/p}{1 + 1/p - 2p^{-s-1}}.$$

Then the sum over m_1 can be computed as follows:

$$\sum_{m_1=1}^{\infty} \frac{\tau(m_1)\mu(m_1 n)}{\rho(m_1 n) m_1^{s_1+z+1}} = \frac{\mu(n)}{\rho(n)} \sum_{(m_1, n) = 1} \frac{\tau(m_1)\mu(m_1)}{\rho(m_1) m_1^{s_1+z+1}}$$
$$= \frac{\mu(n)}{\rho(n)} \prod_{(p,n)=1} \left(1 - \frac{2}{p^{s_1+z+1}(1 + 1/p)}\right) = \frac{\mu(n)}{\rho(n)} \frac{\zeta(2)}{\zeta^2(s_1+z+1)} \beta_n(s_1+z)\alpha_1(s_1+z).$$

Similarly,

$$\sum_{m_2=1}^{\infty} \frac{\tau(m_2)\mu(m_2 n)}{\rho(m_2 n) m_2^{s_2+1}} = \frac{\mu(n)}{\rho(n)} \frac{\zeta(2)}{\zeta^2(s_2+1)} \beta_n(s_2)\alpha_1(s_2).$$

Now the sum over n is equal to

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n^{2+s_1+s_2}} \frac{\mu^2(n)}{\rho^2(n)} \beta_n(s_1+z)\beta_n(s_2)$$
$$= \sum_{n=1}^{\infty} \frac{\phi(n)\mu^2(n)}{n^{2+s_1+s_2}} \prod_{p|n} (1 + 1/p - 2p^{-s_1-z-1})^{-1} (1 + 1/p - 2p^{-s_2-1})^{-1}$$
$$= \prod_p \left(1 + \frac{\left(1 - \frac{1}{p}\right)\left(1 + \frac{1}{p} - 2p^{-s_1-z-1}\right)^{-1}}{p^{s_1+s_2+1}\left(1 + \frac{1}{p} - 2p^{-s_2-1}\right)}\right) = \zeta(s_1+s_2+1)\alpha_2(s_1, s_2, z),$$

where

$$\alpha_2(s_1,s_2,z)=\prod_p\left(1+\frac{\left(1-\frac{1}{p}\right)\left(1+\frac{1}{p}-\frac{2}{p^{s_1+z+1}}\right)^{-1}}{p^{s_1+s_2+1}\left(1+\frac{1}{p}-\frac{2}{p^{s_2+1}}\right)}\right)\left(1-\frac{1}{p^{s_1+s_2+1}}\right).$$

Let us denote

$$\alpha(s_1,s_2,z):=\alpha_1(s_1+z)\alpha_1(s_2)\alpha_2(s_1,s_2,z).$$

Note that $\alpha(s_1,s_2,z)$ converges absolutely for $\Re s_1,\Re s_2,\Re z>-\epsilon$ for some $\epsilon>0$ and $\alpha(0,0,0)=1$. As a result,

$$P_1=\frac{16\zeta^2(2)}{(\log M)^4}\frac{1}{(2\pi i)^3}\int_{(3)}\int_{(3)}\int_{C_\delta}M^{s_1+s_2}\frac{\alpha(s_1,s_2,z)\zeta(s_1+s_2+1)}{\zeta^2(s_1+z+1)\zeta^2(s_2+1)}\frac{k^z dz}{z^2}\frac{ds_1}{s_1^3}\frac{ds_2}{s_2^3}.$$

Let γ_i denote the contour

$$\Re s_i=-\frac{c}{\log(3+|\Im s_i|)},$$

and

$$f(s_1,s_2,z):=M^{s_1+s_2}\alpha(s_1,s_2,z)\frac{\zeta(s_1+s_2+1)}{\zeta^2(s_1+z+1)\zeta^2(s_2+1)}\frac{k^z}{z^2s_1^3s_2^3}.$$

We start by evaluating the integral over z:

$$\begin{aligned}I&:=\frac{1}{(2\pi i)^3}\int_{(3)}\int_{(3)}\int_{C_\delta}f(s_1,s_2,z)\,dz\,ds_1\,ds_2\\&=\frac{1}{(2\pi i)^2}\int_{(3)}\int_{(3)}\operatorname{res}_{z=0}f(s_1,s_2,z)\,ds_2\,ds_1.\end{aligned}$$

Further, we move the contours of integration to $(-\infty,-\delta)\cup c_\delta\cup(\delta,\infty)$, where $\delta>0$ is a small positive number and c_δ is a semicircle in the right half-plane. Note that the function $f(s_1,s_2,z)$ has a pole not only at $s_1=0$ but also at $s_1+s_2=0$. Moving the contour of integration to the line γ_1, we cross a pole at $s_1=0$. Accordingly, by [6, Corollary 2.4.2, p. 55]

$$\begin{aligned}I=&\frac{1}{2\pi i}\int_{(3)}\operatorname{res}_{\substack{s_1=0\\z=0}}f(s_1,s_2,z)\,ds_2+\frac{1}{2\pi i}\int_{\gamma_1}\operatorname{res}_{\substack{s_2=-s_1\\z=0}}f(s_1,s_2,z)\,ds_1\\&+\frac{1}{(2\pi i)^2}\int_{\gamma_1}\int_{(3)}\operatorname{res}_{z=0}f(s_1,s_2,z)\,ds_2\,ds_1.\end{aligned}$$

Then, as a next step, we move the contour of integration over s_2 to the line γ_2, getting

$$\begin{aligned}I=&\frac{1}{2\pi i}\int_{\gamma_1}\operatorname{res}_{\substack{s_2=0\\z=0}}f(s_1,s_2,z)\,ds_1+\frac{1}{2\pi i}\int_{\gamma_2}\operatorname{res}_{\substack{s_1=0\\z=0}}f(s_1,s_2,z)\,ds_2\\&+\frac{1}{(2\pi i)^2}\int_{\gamma_1}\int_{\gamma_2}\operatorname{res}_{z=0}f(s_1,s_2,z)\,ds_2\,ds_1\\&+\frac{1}{2\pi i}\int_{\gamma_1}\operatorname{res}_{\substack{s_2=-s_1\\z=0}}f(s_1,s_2,z)\,ds_1+\operatorname{res}_{s_1=s_2=z=0}f(s_1,s_2,z).\end{aligned}$$

The contribution of the first three integrals above is negligible and can be estimated similarly to the proof of Lemma 8.3. The fourth integral can be bounded by a constant and therefore its contribution to P_1 is $O((\log M)^{-4})$. The main term is given by the residue at $s_1 = s_2 = z = 0$. The function $f(s_1, s_2, z)$ has a simple pole at $s_2 = 0$. Hence

$$\mathrm{res}_{s_1=s_2=z=0} f(s_1, s_2, z) = 2\pi i\, \mathrm{res}_{s_1=z=0} \frac{k^z M^{s_1} \zeta(s_1+1)\alpha(s_1, 0, z)}{z^2 s_1^3 \zeta^2(s_1+z+1)}.$$

Next, we compute the residue at $z = 0$, where the resulting function has a double pole. Finally, evaluating the residue at the triple pole $s_1 = 0$ we find that

$$\begin{aligned} P_1 &= \frac{16\zeta^2(2)}{(\log M)^4}\left(\log k \log M + (\log M)^2\right) + O((\log M)^{-3}) \\ &= \frac{16\zeta^2(2)}{(\log M)^2}(\Delta^{-1} + 1) + O((\log M)^{-3}). \end{aligned}$$

Similarly, using the representation

$$\log d = \frac{1}{2\pi i} \int_{C_\delta} \frac{d^z}{z^2}\, dz,$$

we prove that $P_2 = O((\log M)^{-3})$. □

Lemma 8.6. *Let $M = K^\Delta$. For any $\epsilon > 0$ there is $K_0 = K_0(\epsilon)$ such that for every $K \geq K_0$ the asymptotic formula*

$$\begin{aligned} A_2 &:= \sum_k h\left(\frac{4k}{K}\right) \sum_{f \in H_{4k}(1)}^h M^2(f) L_f^2(1/2) \\ &= \frac{HK}{4} \frac{16\zeta^2(2)}{(\log M)^2}(1 + 1/\Delta) + O(K(\log M)^{-3}) \end{aligned} \tag{8.10}$$

holds for any $\Delta < 1 - \epsilon$.

Proof. Consider

$$A_2 = \sum_{b \leq M} \frac{1}{b} \sum_{m_1, m_2 \leq M/b} \frac{x_{m_1 b} x_{m_2 b}}{\sqrt{m_1 m_2}} A_2(m_1 m_2),$$

where the asymptotics of $A_2(m_1 m_2)$ is given by Theorem 7.4. Accordingly, the contribution of the error term is bounded by

$$K M^\epsilon \left(\frac{M^{a+1/2}}{K^{a+1/2}} + \frac{1}{\sqrt{K}} + \frac{M^2}{K^{7/2}} \right)$$

for any $a \geq 2$. This is negligible if $\Delta < 1 - \epsilon$. The main term can be evaluated similarly to Lemma 8.5. □

8.4. Non-vanishing for the individual weight

Theorem 8.7. *For any $\epsilon > 0$ there exists $k_0 = k_0(\epsilon)$ such that for any $k \geq k_0$ and $k \equiv 0 \pmod 2$ we have*

$$\sum_{\substack{f \in H_{2k}(1) \\ L_f(1/2) \geq (\log k)^{-2}}}^{h} 1 \geq \frac{1}{5} - \epsilon. \tag{8.11}$$

Proof. Asymptotic formulas for the first and second mollified moments are given by (8.7) and (8.10). Accordingly, the largest admissible length of mollifier is $\Delta < 1/4 - \epsilon$. Applying (8.5), we estimate

$$\begin{aligned} \widetilde{M}_1 &:= \sum_{f \in H_{2k}(1)}^{h} M(f) L_f(1/2) \delta_{L_f(1/2) < b(k)(\log k)^{-1/2}} \\ &\leq \Big(\sum_{\substack{f \in H_{2k}(1) \\ L_f(1/2) < b(k)(\log k)^{-1/2}}}^{h} L_f^2(1/2) \Big)^{1/2} \Big(\sum_{f \in H_{2k}(1)}^{h} M^2(f) \Big)^{1/2} \\ &\leq b(k)(\log k)^{-1/2} \Big(\sum_{f \in H_{2k}(1)}^{h} M^2(f) \Big)^{1/2} \leq b(k). \end{aligned}$$

Taking $b(k) = (\log k)^{-3/2}$ we have

$$\sum_{\substack{f \in H_{2k}(1) \\ L_f(1/2) \geq (\log k)^{-2}}}^{h} 1 \geq \frac{(M_1 - \widetilde{M}_1)^2}{M_2} \geq \frac{\Delta}{1 + \Delta}$$

for any $\Delta < 1/4 - \epsilon$. The result follows. □

8.5. Non-vanishing on average

Theorem 8.8. *For any $\epsilon > 0$ there is $K_0 = K_0(\epsilon)$ such that for any $K \geq K_0$ we have*

$$\frac{4}{HK} \sum_{k} h\left(\frac{4k}{K}\right) \sum_{\substack{f \in H_{4k}(1) \\ L_f(1/2) \geq (\log k)^{-2}}}^{h} 1 \geq \frac{1}{2} - \epsilon. \tag{8.12}$$

Proof. Note that

$$\sum_{k} h\left(\frac{4k}{K}\right) \sim \frac{HK}{4} \quad \text{as } K \to \infty.$$

The Cauchy–Schwarz inequality and the estimate (8.6) yield

$$\widetilde{A}_1 := \sum_{k} h\left(\frac{4k}{K}\right) \sum_{\substack{f \in H_{4k}(1) \\ L_f(1/2) < b(k)(\log k)^{-1/2}}}^{h} M(f) L_f(1/2) \ll K b(k).$$

Choosing $b(k) = (\log k)^{-3/2}$ and applying the Cauchy–Schwarz inequality twice, we obtain

$$\sum_k h\left(\frac{4k}{K}\right) \sum_{f\in H_{4k}(1)}^h L_f(1/2)\delta_{L_f(1/2)\geq(\log k)^{-2}}$$
$$\leq \sum_k h\left(\frac{4k}{K}\right)\sqrt{\sum_{f\in H_{4k}(1)}^h L_f^2(1/2)}\sqrt{\sum_{f\in H_{4k}(1)}^h \delta_{L_f(1/2)\geq(\log k)^{-2}}}$$
$$\leq \left(\sum_k h\left(\frac{4k}{K}\right)\sum_{f\in H_{4k}(1)}^h L_f^2(1/2)\right)^{1/2}\left(\sum_k h\left(\frac{4k}{K}\right)\sum_{\substack{f\in H_{4k}(1)\\ L_f(1/2)\geq(\log k)^{-2}}}^h 1\right)^{1/2}.$$

Therefore, by Lemmas 8.4 and 8.6 we have

$$\frac{4}{HK}\sum_k h\left(\frac{4k}{K}\right)\sum_{\substack{f\in H_{4k}(1)\\ L_f(1/2)\geq(\log k)^{-2}}}^h 1 \geq \frac{\left(\frac{4}{HK}A_1 - \frac{4}{HK}\widetilde{A}_1\right)^2}{\frac{4}{HK}A_2} \geq \frac{\Delta}{1+\Delta}$$

for any $\Delta < 1-\epsilon$. □

8.6. Removing the harmonic weight

In order to state the version of Theorem 8.7 for the natural average we apply the techniques developed by Kowalski and Michel [19] prove [20].

Lemma 8.9. *Let α_f be a sequence of complex numbers such that*

$$\sum_{f\in H_{2k}(1)}^h |\alpha_f| \ll (\log k)^A \quad \text{for some } A>0, \tag{8.13}$$

$$\max_{f\in H_{2k}(1)} |\omega_f\alpha_f| \ll k^{-\delta} \quad \text{for some } \delta>0. \tag{8.14}$$

Let $x = k^\epsilon$ and

$$\omega_f(x) := \sum_{n\leq x}\frac{\rho_f(n)}{n} = \sum_{dl^2\leq x}\frac{\lambda_f(d^2)}{dl^2}, \tag{8.15}$$

where $\rho_f(n)$ is defined by (2.8). Then there exists $\kappa = \kappa(\epsilon,\delta) > 0$ such that

$$\sum_{f\in H_{2k}(1)}\alpha_f = \frac{|H_{2k}(1)|}{\zeta(2)}\sum_{f\in H_{2k}(1)}^h \omega_f(x)\alpha_f + O(k^{1-\kappa}). \tag{8.16}$$

Proof. Combining the formula

$$|H_{2k}(1)| = (2k-1)/12 + O(1)$$

with (2.12) and the bound of Hoffstein–Lockhart [10] on $L(\mathrm{sym}^2 f, 1)$, we conclude that

$$\frac{1}{\omega_f} = \frac{L(\mathrm{sym}^2 f,1)}{\zeta(2)}|H_{2k}(1)| + O((\log k)^3). \tag{8.17}$$

Using (8.17) and (8.13) we obtain

$$\sum_{f\in H_{2k}(1)} \alpha_f = \sum_{f\in H_{2k}(1)}^{h} \frac{\alpha_f}{\omega_f} = \frac{|H_{2k}(1)|}{\zeta(2)} \sum_{f\in H_{2k}(1)}^{h} \alpha_f L(\mathrm{sym}^2 f, 1) + O(k^{1-\kappa}). \tag{8.18}$$

Now the key idea is to replace $L(\mathrm{sym}^2 f, 1)$ by a short Dirichlet polynomial $\omega_f(x)$ defined by (8.15). Let us also introduce

$$\omega_f(x, y) := \sum_{x<n\leq y} \frac{\rho_f(n)}{n} = \sum_{x<dl^2\leq y} \frac{\lambda_f(d^2)}{dl^2}.$$

It follows from [25, Lemmas 2.3] that for a sufficiently large constant a (one can take, for example, $a = 10$) and $y = k^a$,

$$L(\mathrm{sym}^2 f, 1) = \omega_f(x) + \omega_f(x, y) + O(k^{-1+\epsilon}). \tag{8.19}$$

Substituting (8.19) to (8.18) we obtain

$$\sum_{f\in H_{2k}(1)} \alpha_f = \frac{|H_{2k}(1)|}{\zeta(2)} \sum_{f\in H_{2k}(1)}^{h} \alpha_f \omega_f(x) + \frac{|H_{2k}(1)|}{\zeta(2)} \sum_{f\in H_{2k}(1)}^{h} \alpha_f \omega_f(x, y) + O(k^{1-\kappa}). \tag{8.20}$$

Repeating the arguments of [20, Proposition 2], applying (8.13) and (8.14), and using instead of [20, Lemma 3] its analogue [25, Lemmas 2.5] in the weight aspect, we obtain

$$\frac{|H_{2k}(1)|}{\zeta(2)} \sum_{f\in H_{2k}(1)}^{h} \alpha_f \omega_f(x, y) \ll k^{1-\kappa}.$$

This completes the proof. □

Note that the main terms in the asymptotic formulas for the twisted moments in the weight aspect have only minor changes compared with the main terms in the level aspect. Thus we can follow closely the approach of [21, Sec. 5].

Theorem 8.10. *For any $\epsilon > 0$ there exists $k_0 = k_0(\epsilon)$ such that for any $k \geq k_0$ and $k \equiv 0 \pmod 2$ we have*

$$\frac{1}{|H_{2k}(1)|} \sum_{\substack{f\in H_{2k}(1)\\ L_f(1/2)\geq(\log k)^{-2}}} 1 \geq \frac{1}{5} - \epsilon. \tag{8.21}$$

Proof. The proof is similar to that of Theorem 8.7 and is based on the asymptotic formulas for the first and second mollified moments. Combining Theorems 3.1 and 6.4, and Lemma 8.9, and choosing the same mollifier as in [21, Sec. 5], we obtain, for $\Delta < 1/4-\epsilon$,

$$\sum_{f\in H_{2k}(1)} M(f)L_f(1/2) = 2\zeta(2)|H_{2k}(1)|(1 + O(k^{-\epsilon_2})),$$

$$\sum_{f\in H_{2k}(1)} M^2(f)L_f^2(1/2) = 2\zeta^2(2)|H_{2k}(1)|\frac{2\log k + 2\log M}{\log M}\left(1 + O\left(\frac{\log\log k}{\log k}\right)\right),$$

where as usual $M = k^{\Delta}$ is the length of the mollifier and $\epsilon, \epsilon_2 > 0$. Note that Lemma 8.9 can be applied for the first and second mollified moments only if the conditions (8.13) and (8.14) are satisfied for $\alpha_f = M(f)L_f(1/2)$ and $\alpha_f = M^2(f)L_f^2(1/2)$. To show that (8.13) holds we use the Cauchy–Schwarz inequality, the estimate (8.5) and Theorem 6.4 in the case of the first moment. In the case of the second moment we proceed as in Lemma 8.5. To show that (8.14) is satisfied we apply the subconvexity bound $L_f(1/2) \ll k^{1/3+\epsilon}$ due to Jutila–Motohashi [17] and Peng [32]. This bound shows that the condition (8.14) is satisfied for any $\Delta < 1/3$. □

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