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Exceptional directions for the Teichmüller geodesic flow and Hausdorff dimension

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Abstract. We prove that for every flat surface ω, the Hausdorff dimension of the set of directions in which Teichmüller geodesics starting from ω exhibit a definite amount of deviation from the correct limit in Birkhoff's and Oseledets' theorems is strictly less than 1. This theorem extends a result by Chaika and Eskin who proved that such sets have measure 0. We also prove that the Hausdorff dimension of the directions in which Teichmüller geodesics diverge on average in a stratum is bounded above by $1/2$, strengthening a classical result due to Masur. Moreover, we show that the Hausdorff codimension of the set of non-weakly-mixing IETs with permutation $(d, d-1, \dots, 1)$, where $d \geq 5$ is an odd number, is at least $1/2$, thus strengthening a result by Avila and Leguil. Combined with a recent result of Chaika and Masur, this shows that the Hausdorff dimension of this set is exactly $1/2$.

Keywords. Flat surfaces, Hausdorff dimension, ergodic theorems, divergent trajectories

Contents

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1. Introduction

The problem of determining the size of the set of points with non-dense orbits under a partially hyperbolic transformation has a long history. These include orbits which escape to infinity, remain confined inside a proper compact set or simply miss a given open set. In the most studied setting, the transformation preserves a natural ergodic measure and hence these non-dense orbits have measure zero. Thus, it is natural to ask whether different types of non-dense orbits are more abundant than others with respect to other notions of size among which *Hausdorff dimension* is the most common.

Many instances of this problem have been studied for algebraic partially hyperbolic flows on homogeneous spaces. For such flows, Margulis conjectured in his 1990 ICM address that orbits with closure a compact subset of a (non-compact) homogeneous space that misses a countable set of points have full Hausdorff dimension [Mar91, Conjectures A, B]. A full resolution of these conjectures was provided in subsequent papers of Kleinbock and Margulis [KM96] and Kleinbock and Weiss [KW13]. This phenomenon of abundance of non-dense orbits also takes place in the setting of hyperbolic dynamical systems. For example, Urbański [Urb91] showed that non-dense orbits of Anosov flows on compact manifolds have full Hausdorff dimension. Then Dolgopyat [Dol97] studied the Hausdorff dimension of orbits of Anosov flows and diffeomorphisms which do not accumulate on certain low entropy subsets. It was shown that these trajectories have full Hausdorff dimension in many cases.

On the other hand, non-dense orbits of divergence type tend to be less abundant. In the homogeneous setting, it was shown in [KKLM17] that the divergent-on-average trajectories for certain flows on $\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$ do not have full Hausdorff dimension. In fact, an explicit upper bound on the Hausdorff dimension is given, generalizing earlier papers by Cheung [Che11] and Cheung and Chevallier [CC16]. In the setting of strata of quadratic differentials, Masur showed that the Hausdorff dimension of the set of directions in which Teichmüller geodesics diverge in moduli space is bounded above by $1/2$. This set includes directions in which the associated translation flow on the surface is not uniquely ergodic.

In this article, we quantify the abundance of non-dense orbits in the setting of Teichmüller dynamics. Theorem 1.8 is the analogue of the result of [KKLM17] on the dimension of the directions in which orbits of the Teichmüller geodesic flow are divergent on average. It provides a strengthening of Masur's result mentioned above. As for non-dense orbits, we study the more general problem concerning the set of directions at a fixed basepoint in which trajectories exhibit a definite amount of deviation from the correct limit in Birkhoff's and Oseledets' theorems. Theorems 1.1 and 1.5 show that the Hausdorff dimension of these sets of directions is bounded away from 1 uniformly as the basepoint varies in the complement of certain proper submanifolds of the stratum. In particular, this implies that the intersection of the set of orbits which miss a given open set with

any Teichmüller disk in the complement of these finitely many proper submanifolds has positive Hausdorff codimension (see Corollary 1.3).

These results generalize prior work of Chaika and Eskin [CE15] in which the aforementioned exceptional sets were shown to have measure 0. The work of Chaika and Eskin was used in [DHL14] to study the diffusion rate of billiard orbits in periodic wind-tree models. It was shown that for any choice of side lengths of the periodic rectangular obstacles, diffusion of orbits has a constant polynomial rate in almost every direction. Theorems 1.1 and 1.5 imply that the directions exhibiting a definite deviation from the expected diffusion rate do not have full Hausdorff dimension. Prior to the work of Chaika and Eskin, Athreya and Forni [AF08] established a polynomial bound on the deviation of Birkhoff averages of sufficiently regular functions along orbits of translation flows on flat surfaces in almost every direction. This full measure set of directions was chosen so that the average of a certain continuous function along the Teichmüller flow orbits is close to its expected value. Theorem 1.1 can be used to show that the directions which deviate by a definite amount from this bound are of dimension < 1.

It is well known that Teichmüller dynamics is closely tied to interval exchange transformations. In particular, Theorem 1.8 allows us to derive a lower bound on the Hausdorff codimension of the set of non-weakly-mixing IETs with permutation $(d, d-1, \ldots, 1)$, where d is an odd number. In combination with the result of [CM20] establishing the upper bound, this allows us to compute the precise Hausdorff codimension.

Formulation of results

Let $g \geq 1$ and let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be an integral partition of $2g-2$. An *abelian differential* is a pair (M, ω), where M is a Riemann surface of genus g and ω is a holomorphic 1-form on M whose zeros have multiplicities $\alpha_1, \ldots, \alpha_n$. Throughout this paper, $\mathcal{H}_1(\alpha)$ will denote a stratum of abelian differentials with area 1 with respect to the induced area form on M. We refer to points of $\mathcal{H}_1(\alpha)$ as *translation surfaces*. For the sake of brevity, we will often refer to ω itself as an element of $\mathcal{H}_1(\alpha)$.

We recall that there are well-defined local coordinates on a stratum, called *period coordinates* (e.g., see [FM14, Section 2.3] for details), such that all changes of coordinates are given by affine maps. In period coordinates, $\mathrm{SL}_2(\mathbb{R})$ acts naturally on each copy of $\mathbb{C}$. Moreover, the closure of any $\mathrm{SL}_2(\mathbb{R})$ orbit is an affine invariant manifold [EMM15], i.e., a closed subset of $\mathcal{H}_1(\alpha)$ that is invariant under the $\mathrm{SL}_2(\mathbb{R})$ action and looks like an affine subspace in period coordinates. Therefore, it is the support of an ergodic $\mathrm{SL}_2(\mathbb{R})$-invariant probability measure.

The action of the following one-parameter subgroups of $\mathrm{SL}_2(\mathbb{R})$ will be referred to throughout the article:

$$g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \quad r_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad h_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \quad \check{h}_s = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}.$$

We recall that the actions of g_t, r_θ, h_s and $\check{h}_s$ correspond to the Teichmüller geodesic flow, the rotation of the flat surface by the angle θ, and the expanding and contracting horocycle flows, respectively.

Birkhoff's ergodic theorem

Chaika and Eskin [CE15] proved that for any translation surface $(M, \omega) \in \mathcal{H}_1(\alpha)$, and any continuous compactly supported function f on $\mathcal{H}_1(\alpha)$, for almost all $\theta \in [0, 2\pi]$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(g_t r_\theta w)\, dt = \int_{\mathcal{M}} f\, d\nu_{\mathcal{M}}, \tag{1.1}$$

where $\mathcal{M} = \overline{\mathrm{SL}_2(\mathbb{R})\omega}$ is the smallest affine invariant manifold containing ω, and $\nu_{\mathcal{M}}$ is the affine measure whose support is $\mathcal{M}$.

In this paper, we show that the Hausdorff dimension of the set of directions exhibiting a definite amount of deviation from the correct limit in (1.1) is strictly less than 1.

Theorem 1.1. *Suppose $\mathcal{M} \subseteq \mathcal{H}_1(\alpha)$ is an affine invariant submanifold and $\nu_{\mathcal{M}}$ is the affine measure whose support is $\mathcal{M}$. Then, for any bounded continuous function f on $\mathcal{M}$ and any $\varepsilon > 0$, there exist affine invariant submanifolds $\mathcal{N}_1, \ldots, \mathcal{N}_k$, properly contained in $\mathcal{M}$, and $0 < \delta < 1$ such that for all $\omega \in \mathcal{M} \setminus \bigcup_{i=1}^k \mathcal{N}_i$, the Hausdorff dimension of the set*

$$\left\{\theta \in [0, 2\pi] : \limsup_{T\to\infty} \left| \frac{1}{T} \int_0^T f(g_t r_\theta \omega)\, dt - \int_{\mathcal{M}} f\, d\nu_{\mathcal{M}} \right| \geq \varepsilon \right\}$$

is at most δ.

Remark 1.2. We note that the upper bound in Theorem 1.1 is uniform as the basepoint ω varies in the complement of finitely many proper affine invariant submanifolds in $\mathcal{M}$. This, in particular, includes points ω whose $\mathrm{SL}_2(\mathbb{R})$ orbit is not dense in $\mathcal{M}$.

In Theorem 6.7, we obtain a version of Theorem 1.1 for discrete Birkhoff averages which is needed for later applications. It is worth noting that the exceptional sets considered in Theorem 1.1 are non-empty in most examples and can, in fact, have positive Hausdorff dimension. By using the results in [KW04], one can find a compact set K such that the Hausdorff dimension of the trajectories which are completely contained in K is at least $1 - \delta'$ for some $0 < \delta' < 1$. By taking f to be supported in the complement of K and to have $\nu_{\mathcal{M}}(f) \neq 0$, these bounded trajectories will belong to the exceptional set for all ε sufficiently small. A similar argument shows that directions in which geodesics diverge on average (Definition 1.7) belong to the exceptional sets of compactly supported function with non-zero average.

Using the uniform dimension estimate in Theorem 1.1, we obtain the following corollary.

Corollary 1.3. *Suppose $\mathcal{M} \subseteq \mathcal{H}_1(\alpha)$ is an affine invariant submanifold and $\nu_{\mathcal{M}}$ is the affine measure whose support is $\mathcal{M}$. Then, for any bounded continuous function f on $\mathcal{M}$ and any $\varepsilon > 0$, there exist affine invariant submanifolds $\mathcal{N}_1, \ldots, \mathcal{N}_k$, properly contained in $\mathcal{M}$, and $0 < \delta < 1$ such that for all $\omega \in \mathcal{M} \setminus \bigcup_{i=1}^k \mathcal{N}_i$, the Hausdorff dimension of the set*

$$\left\{x \in \mathrm{SL}_2(\mathbb{R})\omega : \limsup_{T\to\infty} \left| \frac{1}{T} \int_0^T f(g_t x)\, dt - \int_{\mathcal{M}} f\, d\nu_{\mathcal{M}} \right| \geq \varepsilon \right\}$$

is at most $2 + \delta$.

In particular, by a standard approximation argument, we see that for any non-empty open subset U of a connected component $\mathcal{C}$ of the stratum $\mathcal{H}_1(\alpha)$, the Hausdorff dimension of the set

$$\{x \in \mathrm{SL}_2(\mathbb{R})\omega : g_t x \notin U \text{ for all } t > 0\}$$

is strictly less than the dimension of $\mathrm{SL}_2(\mathbb{R})$, which is 3, this being true uniformly over all Teichmüller curves $\mathrm{SL}_2(\mathbb{R})\omega$ in the complement of finitely many lower-dimensional invariant submanifolds of $\mathcal{C}$.

The scheme suggested in this paper is quite flexible and can be applied to get similar results about the Hausdorff dimension in various settings. In particular, we believe that using our approach for the proof of Theorem 1.1, one should be able to answer the following question affirmatively.

Question 1.4. *Suppose $\mathcal{M} \subseteq \mathcal{H}_1(\alpha)$ is an affine invariant submanifold and $\nu_{\mathcal{M}}$ is the affine measure whose support is $\mathcal{M}$. Let f be a bounded continuous function on $\mathcal{M}$ and $\varepsilon > 0$. Is the Hausdorff dimension of the set*

$$\left\{x \in \mathcal{M} : \limsup_{T\to\infty} \left|\frac{1}{T}\int_0^T f(g_t x)\,dt - \int_{\mathcal{M}} f\,d\nu_{\mathcal{M}}\right| \geq \varepsilon\right\}$$

strictly less than the dimension of $\mathcal{M}$?

Notice that the affirmative answer will imply that for any non-empty open subset U of a connected component $\mathcal{C}$ of the stratum $\mathcal{H}_1(\alpha)$, the Hausdorff dimension of $\{x \in \mathcal{C} : g_t x \notin U \text{ for all } t > 0\}$ is strictly less than the dimension of $\mathcal{C}$.

Oseledets' theorem for the Kontsevich–Zorich cocycle

The next object of our study is the Lyapunov exponents of the Kontsevich–Zorich cocycle. Consider the Hodge bundle whose fiber over every point $(X, \omega) \in \mathcal{H}_1(\alpha)$ is the cohomology group $H^1(X, \mathbb{R})$. Let $\mathrm{Mod}(X)$ be the *mapping class group*, i.e., the group of isotopy classes of orientation preserving homeomorphisms of X. Fix a fundamental domain in the Teichmüller space for the action of $\mathrm{Mod}(X)$. Consider the cocycle $\tilde{A} : \mathrm{SL}_2(\mathbb{R}) \times \mathcal{H}_1(\alpha) \to \mathrm{Mod}(X)$, where for x in the fundamental domain, $\tilde{A}(g, x)$ is the element of $\mathrm{Mod}(X)$ that is needed to return the point gx to the fundamental domain. Then the *Kontsevich–Zorich cocycle* $A(g, x)$ is defined by

$$A(g, x) = \rho(\tilde{A}(g, x)),$$

where $\rho : \mathrm{Mod}(X) \to \mathrm{Sp}(2g, \mathbb{Z})$ is given by the induced action of $\mathrm{Mod}(X)$ on cohomology. We recall the notion of a strongly irreducible $\mathrm{SL}_2(\mathbb{R})$ cocycle.

Definition (Strongly irreducible cocycle). Let (X, ν) be a probability space admitting an action of a locally compact group G which leaves ν invariant. Let $\pi : V \to X$ be a vector bundle over X on which G acts fiberwise linearly. We say that V admits a *ν-measurable almost invariant splitting* if there exists $n > 1$ such that for ν-almost every x, the fiber $\pi^{-1}(x)$ splits into non-trivial subspaces $V_1(x), \ldots, V_n(x)$ satisfying $V_i(x) \cap V_j(x) = \{0\}$

for all $i \neq j$ and $gV_i(x) = V_i(gx)$ for all i, for ν-almost every $x \in X$ and for almost every $g \in G$ with respect to the (left) Haar measure on G. Finally, the map $x \mapsto V_i(x)$ is required to be ν-measurable for all i.

The G action on V is said to be *strongly irreducible* with respect to ν if the action does not admit any ν-measurable almost invariant splitting.

In this setting, we prove the following statement about deviations in the Lyapunov exponents of the Kontsevich–Zorich cocycle.

Theorem 1.5. *Suppose $\mathcal{M} \subseteq \mathcal{H}_1(\alpha)$ is an affine invariant submanifold and $\nu_{\mathcal{M}}$ is the affine measure whose support is $\mathcal{M}$. Let V be a continuous (on $\mathcal{M}$) $\mathrm{SL}_2(\mathbb{R})$-invariant subbundle of (some exterior power of) the Hodge bundle. Assume that A_V is strongly irreducible with respect to $\nu_{\mathcal{M}}$, where A_V is the restriction of the Kontsevich–Zorich cocycle to V. Then, for any $\varepsilon > 0$, there exist affine invariant submanifolds $\mathcal{N}_1, \ldots, \mathcal{N}_k$, properly contained in $\mathcal{M}$, and $0 < \delta < 1$ such that for all $\omega \in \mathcal{M} \setminus \bigcup_{i=1}^{k} \mathcal{N}_i$, the Hausdorff dimension of the set*

$$\left\{\theta \in [0, 2\pi] : \limsup_{t\to\infty} \frac{\log \|A_V(g_t, r_\theta\omega)\|}{t} \geq \lambda_V + \varepsilon\right\}$$

is at most δ, where λ_V denotes the top Lyapunov exponent for A_V with respect to $\nu_{\mathcal{M}}$.

This complements a result in [CE15] where it is shown that under the same hypotheses, for *every* ω and for Lebesgue almost every $\theta \in [0, 2\pi]$, the following limit exists:

$$\lim_{t\to\infty} \frac{\log \|A_V(g_t, r_\theta\omega)\|}{t} = \lambda_V.$$

It is shown in [EM13, Theorem A.6] that the Kontsevich–Zorich cocycle is in fact semisimple, which means that, after passing to a finite cover, the Hodge bundle splits into $\nu_{\mathcal{M}}$-measurable $\mathrm{SL}_2(\mathbb{R})$-invariant, strongly irreducible subbundles. Moreover, it is shown in [Fil16] that such subbundles can be taken to be continuous (and in fact real analytic) in period coordinates. Additionally, it is well-known that the top Lyapunov exponent of the k^{th} exterior power of the cocycle is the sum of the top k exponents of the cocycle itself. In this manner, we can deduce the deviation statement for *all* Lyapunov exponents by examining the top Lyapunov exponents of exterior powers of the cocycle. The following corollary is the precise statement. For more details, see [CE15, proof of Theorem 1.4].

Corollary 1.6. *Suppose $(M, \omega) \in \mathcal{H}_1(\alpha)$ and $\nu_{\mathcal{M}}$ is the affine measure whose support is $\mathcal{M} = \overline{\mathrm{SL}_2(\mathbb{R})\omega}$. Let A be the Kontsevich–Zorich cocycle over $\mathcal{M}$. Denote by λ_i the Lyapunov exponents of A (with multiplicities) with respect to $\nu_{\mathcal{M}}$. For any $\theta \in [0, 2\pi]$, suppose $\psi_1(t, \theta) \leq \cdots \leq \psi_{2g}(t, \theta)$ are the eigenvalues of the matrix $A^*(g_t, r_\theta\omega)A(g_t, r_\theta\omega)$. Then the Hausdorff dimension of the set*

$$\left\{\theta \in [0, 2\pi] : \limsup_{t\to\infty} \frac{\log \|\psi_i(t, \theta)\|}{t} \geq 2\lambda_i + \varepsilon\right\}$$

is strictly less than 1.

Divergent trajectories

The study of exceptional trajectories in Birkhoff's and Oseledets' theorems naturally leads to studying trajectories which frequently miss large sets with good properties. This problem is closely connected to studying *divergent* geodesics, i.e., geodesics which leave every compact subset of $\mathcal{H}_1(\alpha)$. Masur [Mas92] showed that, for every translation surface ω, the set of directions θ for which $g_t r_\theta \omega$ is divergent has Hausdorff dimension at most $1/2$. Cheung [Che03] showed that this upper bound is optimal by constructing explicit examples for which the bound is realized.

In this paper, we study divergent-on-average geodesics, i.e., geodesics that spend asymptotically zero percent of the time in any compact set.

Definition 1.7. A direction $\theta \in [0, 2\pi]$ corresponds to a *divergent-on-average geodesic* $g_t r_\theta \omega$ if for every compact set $K \subset \mathcal{H}_1(\alpha)$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \chi_K(g_t r_\theta \omega)\, dt = 0.$$

Note that the set of divergent-on-average geodesics contains the set of divergent geodesics. Therefore, Theorem 1.8 below strengthens [Mas92].

Theorem 1.8. *For every translation surface $\omega \in \mathcal{H}_1(\alpha)$, the Hausdorff dimension of the directions θ such that the orbit $(g_t r_\theta \omega)_{t\geq 0}$ is divergent on average in the stratum $\mathcal{H}_1(\alpha)$ is at most $1/2$.*

We note that the set of divergent [on average] orbits of g_t in the stratum properly includes those orbits whose projection diverges [on average] in the moduli space. See also Theorem 3.2 where we consider the set of directions with a prescribed divergence behavior in open strata with finitely many invariant submanifolds removed, which may be of independent interest.

Combining Theorem 1.8 with the results in [BN04], we derive the following bound on the dimension of non-weakly-mixing interval exchange transformations (IETs) whose permutation is of type W. We refer the reader to Section 9 for detailed definitions.

Corollary 1.9. *Suppose π is a type W permutation. Then the Hausdorff codimension of the set of non-weakly-mixing IETs (with respect to the Lebesgue measure) with permutation π is at least $1/2$.*

For $d \in \mathbb{N}$, we say a permutation π on $\{1, \ldots, d\}$ is a *rotation* if $\pi(i+1) = \pi(i)+1 \bmod d$ for $1 \leq i \leq d$. Avila and Forni [AvF07] showed that for any irreducible permutation which is not a rotation, Lebesgue almost every IET is weakly mixing. In [AL18], this result was extended to show that for all such permutations, the non-weakly-mixing IETs have positive Hausdorff codimension. Thus, Corollary 1.9 is an improvement of [AL18] in the case of type W permutations. Moreover, it is shown in [CM20] that if π is the permutation $(d, d-1, \ldots, 1)$ for $d \geq 5$, then the Hausdorff codimension of the set of non-weakly-mixing IETs with permutation π is at most $1/2$ (the case $d = 4$ was considered in [AC15]). When d is odd, the permutation $(d, d-1, \ldots, 1)$ is of type W. Thus, we identify the exact Hausdorff dimension in this case.

Outline of proofs and paper organization

Our general approach is to deduce the desired results (Theorems 1.1, 1.5 and 1.8) from the analogous results for horocycle arcs (Theorems 2.1, 2.2 and 2.3, respectively). The reason is that horocycles are more convenient to work with as the geodesic flow normalizes the horocycle flow in $\mathrm{SL}_2(\mathbb{R})$. This is carried out along with the proof of Corollary 1.3 in Section 2.

The strategy for proving Theorem 2.1 on deviations of Birkhoff averages consists of three main steps. First, we show that the convergence in (1.1) holds uniformly as the basepoint ω varies over compact sets in the complement of finitely many proper affine submanifolds; Theorem 5.1 is the precise statement. This result strengthens a result in [CE15] and may be of independent interest.

Another ingredient involves showing that the Hausdorff dimension of the set of directions whose geodesics frequently miss large compact sets, chosen with the help of a height function, is bounded away from 1. This statement is made precise in Theorem 3.2 whose proof is the main content of Section 3. Using similar techniques, Theorem 2.3 is proved in Section 4.

Theorem 2.1 is proved in Section 6. The idea is to treat a long orbital average as a sum of orbital averages over shorter orbit segments. With the help of Theorem 5.1, we show that most orbit segments which start from a suitably chosen large compact set with good properties will have an orbital average close to the correct limit. Using Theorem 3.2, we control the dimension of those orbit segments which miss our good compact set.

A key step is to show that the sum of such averages over orbit segments behaves like a sum of weakly dependent random variables, which is achieved by Lemma 6.5. This allows us to show that the measure of the badly behaved long orbit averages decays exponentially.

The proof of Theorem 2.2 treating deviations in Oseledets' theorem spans Sections 7 and 8. It follows the same strategy as the one outlined above. It is shown in [CE15] that Oseledets' theorem holds uniformly in the basepoint over large *open* sets for random walk trajectories. Using Egorov's and Luzin's theorems, we translate these results into results about the Teichmüller geodesic flow. This relies on the classical fact that a random walk trajectory is tracked by a geodesic, up to a sublinear error.

Finally, we show that trajectories which frequently miss such a large set with good properties exhibit deviation in the discrete Birkhoff averages of its indicator function. The dimension of those trajectories is in turn controlled by Theorem 6.7.

In Section 9 we prove Corollary 1.9. In Proposition 9.5 we relate the criterion for weak mixing of IETs to a type W permutation in [BN04] and recurrence of Teichmüller geodesics in a stratum. The combination of this relation and our Theorem 1.8 finishes the proof.

2. Preliminaries

2.1. Reduction to horocycles

We explain how to deduce Theorems 1.1, 1.5 and 1.8 from the analogous results for horocycle arcs, Theorems 2.1, 2.2 and 2.3, respectively.

For any $\theta \in [-\pi/4, \pi/4]$, the following equality holds:

$$r_\theta = \check{h}_{-\tan\theta} g_{\log\cos\theta} h_{\tan\theta}.$$

Recall that g_t contracts $\check{h}_{-\tan\theta}$, i.e., $g_t \check{h}_{-\tan\theta} g_{-t} = \check{h}_{-e^{-2t}\tan\theta}$, and $g_t g_{\log\cos\theta} = g_{t+\log\cos\theta}$. Therefore, in each theorem formulated in the introduction, θ belongs to the exceptional set if and only if $\tan\theta$ belongs to the exceptional set in the corresponding theorem formulated below.

Finally, the bounds for the Hausdorff dimensions of the corresponding sets are preserved as the map $\theta \mapsto \tan\theta$ is bi-Lipschitz on $[-\pi/4, \pi/4]$.

Theorem 2.1 (Analogue of Theorem 1.1). *Suppose $\mathcal{M} \subseteq \mathcal{H}_1(\alpha)$ is an affine invariant submanifold and $\nu_{\mathcal{M}}$ is the affine measure whose support is $\mathcal{M}$. Then, for any bounded continuous function f on $\mathcal{M}$ and any $\varepsilon > 0$, there exist affine invariant submanifolds $\mathcal{N}_1, \ldots, \mathcal{N}_k$, properly contained in $\mathcal{M}$, and $\delta \in (0, 1)$ such that for all $\omega \in \mathcal{M} \setminus \bigcup_{i=1}^k \mathcal{N}_i$, the Hausdorff dimension of the set*

$$\left\{ s \in [-1, 1] : \limsup_{T\to\infty} \frac{1}{T} \int_0^T f(g_t h_s \omega)\, dt \geq \int_{\mathcal{M}} f\, d\nu_{\mathcal{M}} + \varepsilon \right\}$$

is at most δ.

We remark that minor modifications of the proof of Theorem 2.1 also yield an upper bound on the Hausdorff dimension of the set of directions for which the lim inf is less than the correct limit by a definite amount. Moreover, the exceptional set in Theorem 1.1 can be written as

$$\begin{aligned}\left\{ \theta \in [0, 2\pi] : \limsup_{T\to\infty} \left| \frac{1}{T} \int_0^T f(g_t r_\theta \omega)\, dt - \nu_{\mathcal{M}}(f) \right| \geq \varepsilon \right\} \\ = \left\{ \theta \in [0, 2\pi] : \limsup_{T\to\infty} \frac{1}{T} \int_0^T f(g_t r_\theta \omega)\, dt \geq \nu_{\mathcal{M}}(f) + \varepsilon \right\} \\ \cup \left\{ \theta \in [0, 2\pi] : \liminf_{T\to\infty} \frac{1}{T} \int_0^T f(g_t r_\theta \omega)\, dt \leq \nu_{\mathcal{M}}(f) - \varepsilon \right\},\end{aligned}$$

where $\nu_{\mathcal{M}}(f) = \int_{\mathcal{M}} f\, d\mathcal{M}$. Thus, Theorem 1.1 follows from the reduction to horocycles, Theorem 2.1, and its variant for the lim inf.

Theorem 2.2 (Analogue of Theorem 1.5). *Suppose $(M, \omega) \in \mathcal{H}_1(\alpha)$ and $\nu_{\mathcal{M}}$ is the affine measure whose support is $\mathcal{M} = \overline{\mathrm{SL}_2(\mathbb{R})\omega}$. Let V be a continuous (on $\mathcal{M}$) $\mathrm{SL}_2(\mathbb{R})$-invariant subbundle of (some exterior power of) the Hodge bundle. Assume that A_V is strongly irreducible with respect to $\nu_{\mathcal{M}}$, where A_V is the restriction of the Kontsevich–Zorich cocycle to V. Then, for any $\varepsilon > 0$, there exist affine invariant submanifolds $\mathcal{N}_1, \ldots, \mathcal{N}_k$, properly contained in $\mathcal{M}$, and $\delta \in (0, 1)$ such that for all $\omega \in \mathcal{M} \setminus \bigcup_{i=1}^k \mathcal{N}_i$, the Hausdorff dimension of the set*

$$\left\{ s \in [-1, 1] : \limsup_{t\to\infty} \frac{\log \|A_V(g_t, h_s\omega)\|}{t} \geq \lambda_V + \varepsilon \right\}$$

is at most δ, where λ_V denotes the top Lyapunov exponent for A_V with respect to $\nu_{\mathcal{M}}$.

Theorem 2.3 (Analogue of Theorem 1.8). *Suppose* $(M, \omega) \in \mathcal{H}_1(\alpha)$. *Then the Hausdorff dimension of the set*

$$\{s \in [-1, 1] : \textit{the geodesic } g_t h_s \omega \textit{ is divergent on average}\}$$

is less than or equal to $1/2$.

2.2. *Properties of Hausdorff dimension*

The exceptional sets we study in this paper are of the form $A = \limsup_{n\to\infty} A_n$, that is,

$$A = \bigcap_{n\geq 1} \bigcup_{l\geq n} A_l$$

for a sequence of subsets A_n of the real line.

In this section, we reduce the problem of finding an upper bound on the Hausdorff dimension of such sets to finding efficient covers of the A_n (see Lemma 2.5).

First, we recall the definition of the Hausdorff dimension. Let A be a subset of a metric space X. For any $\rho, \beta > 0$, we define

$$H_\rho^\beta(A) = \inf\Big\{\sum_{I\in\mathcal{U}} \operatorname{diam}(I)^\beta : \mathcal{U} \text{ is a cover of } A \text{ by balls of diameter } < \rho\Big\}.$$

Then the *β-dimensional Hausdorff measure* of A is defined to be

$$H^\beta(A) = \lim_{\rho\to 0} H_\rho^\beta(A).$$

Definition 2.4. The *Hausdorff dimension* of a subset A of a metric space X is equal to

$$\dim_H(A) = \inf\{\beta \geq 0 : H^\beta(A) = 0\} = \sup\{\beta \geq 0 : H^\beta(A) = \infty\}.$$

The following lemma provides an upper bound on the Hausdorff dimension of a set for which we have efficient covers.

Lemma 2.5. *Let* $\{A_n\}_{n\geq 1}$ *be a collection of subsets of* $\mathbb{R}$. *Suppose there exist constants* $C, C', t > 0$ *and* $\lambda \in (0, 1)$ *such that for each* n, A_n *can be covered with* $Ce^{2(1-\lambda)tn}$ *intervals of radius* $C'e^{-2tn}$. *Then the Hausdorff dimension of the set* $A = \limsup_{n\to\infty} A_n$ *is at most* $1 - \lambda$.

Proof. Let $\beta \in (1-\lambda, 1)$ and let H^β denote the β-dimensional Hausdorff (outer) measure on $\mathbb{R}$. We show that $H^\beta(A) = 0$, and that implies the lemma. For any $\rho \in (0, 1)$, let $n_0 = n_0(\rho)$ be a natural number such that $e^{-2tn} < C\rho$ for all $n \geq n_0$. Notice that n_0 tends to infinity as ρ goes to 0. Denote by $\mathcal{U}_n$ a cover of A_n by $Ce^{2(1-\lambda)tn}$ intervals of radius $C'e^{-2tn}$. Then $\mathcal{U} = \bigcup_{n\geq n_0} \mathcal{U}_n$ is a cover of A for which

$$\begin{aligned}\sum_{I\in\mathcal{U}} \operatorname{diam}(I)^\beta &= \sum_{n\geq n_0} \sum_{I\in\mathcal{U}_n} \operatorname{diam}(I)^\beta = (C')^\beta \sum_{n\geq n_0} \#\mathcal{U}_n e^{-2\beta tn} \\ &\leq (C')^\beta C \sum_{n\geq n_0} e^{2(1-\lambda-\beta)tn},\end{aligned}$$

where $\#\mathcal{U}_n$ is the number of intervals in the cover $\mathcal{U}_n$.

Thus, since $1-\lambda-\beta<0$, we obtain

$$H^{\beta}_{\rho}(A) \le (C')^{\beta} C \sum_{n\ge n_0} e^{2(1-\lambda-\beta)tn} = (C')^{\beta} C \frac{e^{2(1-\lambda-\beta)tn_0}}{1-e^{2(1-\lambda-\beta)t}} \xrightarrow{\rho\to 0} 0.$$

This implies that $H^{\beta}(A)=0$ for all $\beta\in(1-\lambda,1)$. □

Let us also recall some basic useful facts about Hausdorff dimension. The first concerns the dimension of product sets.

Proposition 2.6 ([Mat95, Corollary 8.11]). *If $A, B \subset \mathbb{R}^d$ are Borel sets, then $\dim_H(A\times B)\ge \dim_H(A)+\dim_H(B)$. If in addition the upper packing dimension of B is equal to its Hausdorff dimension, then*

$$\dim_H(A\times B)=\dim_H(A)+\dim_H(B).$$

We remark that the lower bound on the dimension of the product is a classical fact while the upper bound can be obtained directly when B is an open ball, which is the case we will be interested in.

2.3. *Proof of Corollary 1.3*

(We assume Theorem 2.1.) Using a simple approximation argument, we may assume that f is Lipschitz. By a similar argument to the one following Theorem 2.1, it suffices to prove that the set

$$\overline{B}(f,\varepsilon) := \left\{ x\in\mathcal{M} : \limsup_{T\to\infty} \frac{1}{T}\int_0^T f(g_t x)\,dt \ge \int_{\mathcal{M}} f\,dv_{\mathcal{M}} + \varepsilon \right\}$$

has positive Hausdorff codimension in $\mathcal{M}$. Let $\delta>0$ and $\mathcal{N}_1,\dots,\mathcal{N}_k$ be the affine invariant submanifolds properly contained in $\mathcal{M}$ which are provided by Theorem 2.1, depending on f and ε, and suppose $\omega\in\mathcal{M}\setminus\bigcup_{i=1}^k \mathcal{N}_i$.

Since the action of $\mathrm{SL}_2(\mathbb{R})$ is locally free,[1] we can find a small neighborhood of identity $\mathcal{O}_\omega\subset \mathrm{SL}_2(\mathbb{R})$ such that the map $g\mapsto g\omega$ is injective on $\mathcal{O}_\omega$. By making $\mathcal{O}_\omega$ smaller if necessary, we may assume that $\mathcal{O}_\omega$ is the diffeomorphic image of an open bounded neighborhood of 0 in the Lie algebra of $\mathrm{SL}_2(\mathbb{R})$ under the exponential map. In particular, there are bounded neighborhoods $\mathcal{O}^s_\omega$, $\mathcal{O}^c_\omega$ and $\mathcal{O}^u_\omega$ of 0 in $\mathbb{R}$ such that the map

$$(z,r,s)\mapsto \check{h}_z g_r h_s \tag{2.1}$$

is a diffeomorphism from $\mathcal{O}^s_\omega\times\mathcal{O}^c_\omega\times\mathcal{O}^u_\omega$ onto $\mathcal{O}_\omega$. Define

$$\overline{B}(f,\varepsilon)^u_\omega := \left\{ s\in\mathcal{O}^u_\omega : \limsup_{T\to\infty} \frac{1}{T}\int_0^T f(g_t h_s\omega)\,dt \ge \int_{\mathcal{M}} f\,dv_{\mathcal{M}} + \varepsilon \right\}.$$

[1] For every compact set $K\subset\mathcal{M}$, there exists a bounded neighborhood of identity $B\subset \mathrm{SL}_2(\mathbb{R})$ such that the map $(g,x)\mapsto gx$ is injective from $B\times K$ into $\mathcal{M}$.

By Theorem 2.1, the Hausdorff dimension of $\overline{B}(f,\varepsilon)^u_\omega$ is at most $\delta < 1$. Now, suppose that $x = g\omega \in \overline{B}(f,\varepsilon) \cap \mathcal{O}_\omega\omega$ and write $g = \check{h}_z g_r h_s$. Since g_t contracts $\check{h}_z$ and commutes with g_r, using the fact that f is Lipschitz we see that $s \in \overline{B}(f,\varepsilon)^u_\omega$. Conversely, for all $s \in \overline{B}(f,\varepsilon)^u_\omega$ and all $(z,r) \in \mathcal{O}^s_\omega \times \mathcal{O}^c_\omega$, we have $\check{h}_z g_r h_s \omega \in \overline{B}(f,\varepsilon) \cap \mathcal{O}_\omega\omega$.

In particular, we have the identification

$$\overline{B}(f,\varepsilon) \cap \mathcal{O}_\omega\omega \cong \mathcal{O}^s_\omega \times \mathcal{O}^c_\omega \times \overline{B}(f,\varepsilon)^u_\omega$$

under the smooth coordinate map in (2.1). Thus, by Proposition 2.6, since the upper packing dimension of an open interval in $\mathbb{R}$ is equal to its topological and Hausdorff dimension, we get

$$\dim_H(\overline{B}(f,\varepsilon) \cap \mathcal{O}_\omega\omega) = \dim_H(\mathcal{O}^s_\omega \times \mathcal{O}^c_\omega \times \overline{B}(f,\varepsilon)^u_\omega) \le 2 + \delta.$$

The above argument shows that the dimension of the intersection of $\overline{B}(f,\varepsilon)$ with any open subset of an $\mathrm{SL}_2(\mathbb{R})$ orbit in the complement of $\bigcup_{i=1}^k \mathcal{N}_i$ is at most $2 + \delta \neq 3$.

3. The contraction hypothesis and analysis of recurrence

In this section, we study the problem of the Hausdorff dimension of sets of trajectories with prescribed divergence behavior. We prove an abstract result for $\mathrm{SL}_2(\mathbb{R})$ actions on metric spaces which satisfy the *contraction hypothesis* (Definition 3.1) in the terminology of Benoist and Quint [BQ12, Section 2].

Let X be a manifold equipped with a smooth $\mathrm{SL}_2(\mathbb{R})$ action. For $t, \delta > 0$, $N \in \mathbb{N}$, $Q \subset X$ a (compact) set and $x \in X$, define

$$Z_x(Q,N,t,\delta) = \left\{ s \in [-1,1] : \frac{1}{N} \sum_{l=1}^{N} \chi_Q(g_{lt} h_s x) \le 1 - \delta \right\}, \tag{3.1}$$

where χ_Q denotes the indicator function of Q.

Definition 3.1 (The contraction hypothesis). Let Y be a proper $\mathrm{SL}_2(\mathbb{R})$-invariant submanifold of X ($Y = \emptyset$ is allowed). The action of $\mathrm{SL}_2(\mathbb{R})$ on X is said to satisfy the *contraction hypothesis* with respect to Y if there exists a proper, $\mathrm{SO}(2)$-invariant function $\alpha : X \to [1,\infty]$ satisfying the following properties:

(1) $\alpha(x) = \infty$ if and only if $x \in Y$.

(2) There is a constant $\sigma > 0$ such that for all $x \in X \setminus Y$ and all $t > 0$,

$$e^{-\sigma t}\alpha(x) \le \alpha(g_t x) \le e^{\sigma t}\alpha(x). \tag{3.2}$$

(3) There exists a constant $b = b(Y) > 0$ such that for all $a \in (0,1)$ there exists $t_0 = t_0(a) > 1$ such that for all $t > t_0$ and all $x \in X \setminus Y$,

$$\int_0^{2\pi} \alpha(g_t r_\theta x)\, d\theta \le a\alpha(x) + b. \tag{3.3}$$

(4) For all $M \ge 1$, the sets $\overline{\{x \in X : \alpha(x) \le M\}}$, denoted by $X_{\le M}$, form a compact exhaustion of $X \setminus Y$.

The function α is called a *height function*.

We remark that the study of height functions as in Definition 3.1 originated in [EMM98] in the context of homogeneous spaces.

Throughout this section, X is a manifold equipped with a smooth $\mathrm{SL}_2(\mathbb{R})$ action and satisfies the contraction hypothesis with respect to Y, which is a proper $\mathrm{SL}(2,\mathbb{R})$-invariant submanifold of X.

Our goal in this section is to prove the following theorem.

Theorem 3.2. *Given $\delta > 0$, there exist $M_0 = M_0(\delta) > 0$ and $t_0 > 0$ such that for all $M \geq M_0$ and all $t \geq t_0$ such that $e^t \in \mathbb{N}$, there exists $\lambda = \lambda(\delta, t) \in (0,1)$ such that for all $x \in X \setminus Y$, the Hausdorff dimension of the set $\limsup_{N\to\infty} Z_x(X_{\leq M}, N, t, \delta)$ is at most $1-\lambda$, where λ tends to 0 as $t \to \infty$.*

The proof of Theorem 3.2 can be found in Section 3.3. It should be noted that the difference between this theorem and [KKLM17, Theorem 1.5] is the flexibility in the step size t. As a result, the upper bound on the Hausdorff dimension of the set considered depends on t. This flexibility will be important for the proof of Theorem 2.1.

Remark 3.3. The proof of Theorem 3.2 gives explicit information on the value of λ as a function of δ and t. In particular, we show that there exists an absolute constant $C \geq 1$ so that given $\delta > 0$, if we let t_0 be such that (3.3) holds for $a = \frac{1}{2}e^{-2C/\delta}$ and all $t > t_0$, we can take $\lambda = C/(2t)$. We note that in many cases the relationship between a and t_0 in (3) of Definition 3.1 is explicit. This is the case for the function we use in Section 4 to prove Theorem 1.8. In this special case, one can take $t_0 = \frac{1}{1-\eta}\log(a/C_\eta)$ for all $\eta \in (0,1)$, where the constant $C_\eta \geq 1$ depends only on η. See also Remark 3.10 for an explicit choice of M_0 depending on δ. Finally, we note that the restriction $e^t \in \mathbb{N}$ is to avoid minor technicalities and can be easily removed.

3.1. Estimates for integrals over horocycle orbits

In this section we obtain an integral estimate similar to (3.3) for integrals over an entire horocycle orbit. We begin by stating a property of the function which we use throughout our proofs.

Lemma 3.4. *Suppose α is a height function as in Definition 3.1. Then the map $g \mapsto \log\alpha(gx)$ is locally Lipschitz, uniformly over all $x \in X$. More precisely, for every bounded neighborhood of identity $\mathcal{O} \subset \mathrm{SL}_2(\mathbb{R})$, there exists a constant $C_\mathcal{O} \geq 1$ such that for all $g \in \mathcal{O}$ and all $x \in X$,*

$$C_\mathcal{O}^{-1}\alpha(x) \leq \alpha(gx) \leq C_\mathcal{O}\alpha(x).$$

Proof. Let $K = \mathrm{SO}(2)$. The statement follows from the KAK decomposition of $\mathrm{SL}_2(\mathbb{R})$, K-invariance of α and property (2) in Definition 3.1. □

Lemma 3.5. *Let $\alpha : X \to [1,\infty]$ be a height function. Then there is a constant $\bar{b} > 0$ such that for all $\bar{a} \in (0,1)$, there exists $\bar{t_0} = \bar{t_0}(\bar{a}) > 0$ such that for all $t > \bar{t_0}$ and all $x \in X \setminus Y$,*

$$\int_{-1}^{1} \alpha(g_t h_s x)\, ds \leq \bar{a}\alpha(x) + \bar{b}. \tag{3.4}$$

Proof. Let $b = b(Y)$ and for any $a \in (0, 1)$ take $t_0 = t_0(a)$ as in Definition 3.1. Then, by (3.3), for any $t > t_0$ and all $x \in X$,

$$\int_{-\pi/4}^{\pi/4} \alpha(g_t r_\theta x)\, d\theta \le a\alpha(x) + b.$$

For any $\theta \in [-\pi/4, \pi/4]$,

$$g_t r_\theta = \underbrace{\begin{pmatrix} 1 & 0 \\ -e^{-t}\tan\theta & 1 \end{pmatrix}}_{\check{h}_{-e^{-t}\tan\theta}} \underbrace{\begin{pmatrix} \cos\theta & 0 \\ 0 & \sec\theta \end{pmatrix}}_{g_{\log\cos\theta}} g_t h_{\tan\theta}.$$

By Lemma 3.4, there exists a positive constant $c_0 \ge 1$, independent of t, such that for all $\theta \in [-\pi/4, \pi/4]$ and all $x \in X$,

$$\alpha(g_t h_{\tan\theta} x) \le c_0 \alpha(g_t r_\theta x).$$

Thus, we get

$$\int_{-\pi/4}^{\pi/4} \alpha(g_t h_{\tan\theta} x)\, d\theta \le c_0(a\alpha(x) + b).$$

Using the change of variable $s = \tan\theta$ and noting that the Jacobian of this change of variable is bounded from below on $[-\pi/4, \pi/4]$ by $1/2$, we obtain

$$\int_{-1}^{1} \alpha(g_t h_s x)\, ds \le 2c_0(a\alpha(x) + b).$$

That implies the lemma with $\bar{b} = 2c_0 b$, $a = \frac{\bar{a}}{2c_0} \in (0, 1)$ and $\bar{t}_0 = t_0(a)$. □

3.2. Coverings and long excursions

In this section, we aim to find efficient coverings for the set of directions for which geodesics take long excursions outside of certain fixed compact sets. The idea of the proofs of Lemma 3.6 and Proposition 3.7 comes from [Kha20b, Section 4].

Lemma 3.6. *For all $x \in X$, $n \in \mathbb{N}$, $t > 0$ and any closed interval $J \subset [-1, 1]$ of length at least $2e^{-2nt}$, we have*

$$\int_J \alpha(g_{(n+1)t} h_s x)\, ds \le \int_J \int_{-1}^{1} \alpha(g_t h_r g_{nt} h_s x)\, dr\, ds. \tag{3.5}$$

Proof. Let $J + a$ denote the shift of the interval J by $|a|$ to the right if $a \ge 0$ and to the left if $a < 0$.

The length of J is at least $2e^{-2nt}$, therefore, for any $r \in [-1, 1]$, we have

$$J \subseteq (J + re^{-2nt}) \cup (J - re^{-2nt}). \tag{3.6}$$

Using positivity of α, (3.6) and a change of variable, we obtain

$$\begin{aligned}\int_J \alpha(g_{(n+1)t}h_s x)\,ds &= \int_0^1 \int_J \alpha(g_{(n+1)t}h_s x)\,ds\,dr \\ &\le \int_0^1 \int_{J-re^{-2nt}} \alpha(g_{(n+1)t}h_s x)\,ds\,dr + \int_0^1 \int_{J+re^{-2nt}} \alpha(g_{(n+1)t}h_s x)\,ds\,dr \\ &= \int_{-1}^1 \int_{J+re^{-2nt}} \alpha(g_{(n+1)t}h_s x)\,ds\,dr = \int_{-1}^1 \int_J \alpha(g_{(n+1)t}h_{s+re^{-2nt}} x)\,ds\,dr \\ &= \int_{-1}^1 \int_J \alpha(g_t h_r g_{nt} h_s x)\,ds\,dr.\end{aligned}$$

Then, using Fubini's theorem, we obtain the lemma. □

Throughout this section, we fix x in $X \setminus Y$ and use $Z_x(M, N, t)$ to denote the set $Z_x(X_{\le M}, N, t, 1)$ defined in (3.1). Moreover, given $a \in (0, 1)$, let $b > 0$ and let $t_0 = t_0(a) > 1$ be as in Lemma 3.5. The following is the main result of this section.

Proposition 3.7 (cf. [Kha20b, Proposition 4.6]). *There exists a constant $C_1 \ge 1$ (independent of x and a) such that for all $M > C_1 b/a$, all $t \ge t_0$ such that $e^t \in \mathbb{N}$, and all $N \in \mathbb{N}$,*

$$\int_{Z_x(M,N-1,t)} \alpha(g_{Nt}h_s x)\,ds \le (2a)^N \alpha(x) + (2a)^{N-1} b. \tag{3.7}$$

Proof. The proof is the same as that of [Kha20b, Proposition 4.6]. The main difference is that we relax the assumption on the height of x and on the dependence of M on t.

Using property (2) in Definition 3.1, let $C_1 \ge 1$ be such that for all $s \in [-2, 2]$,

$$C_1^{-1} \le \frac{\alpha(h_s x)}{\alpha(x)} \le C_1. \tag{3.8}$$

Consider $M > C_1 b/a$ and $t > t_0$ such that $e^t \in \mathbb{N}$, where b and t_0 are as in Lemma 3.5. Let $y \in X \setminus Y$ be such that $\alpha(y) > b/a$. By Lemma 3.5, we have

$$\int_{-1}^1 \alpha(g_t h_s y)\,ds \le a\alpha(y) + b \le 2a\alpha(y). \tag{3.9}$$

Let $B(M, 0) = [-1, 1]$ and $B(M, k) = Z_x(M, k, t)$ for any $k \in \mathbb{N}$. We note that

$$B(M, 0) \supseteq B(M, 1) \supseteq B(M, 2) \supseteq \cdots. \tag{3.10}$$

For any $k \in \mathbb{N}$, we denote by $\mathcal{P}_k$ the partition of the interval $[-1, 1]$ into e^{2kt} intervals of equal length.

Fix a natural number $n \in [1, N]$. Suppose $J \in \mathcal{P}_{n-1}$ is such that $J \cap B(M, n-1) \ne \emptyset$ and let $s_0 \in J \cap B(M, n-1)$. Then $\alpha(g_{(n-1)t}h_{s_0} x) > M$. Moreover, for every $s \in J$, we have $|s - s_0| \le 2e^{-2(n-1)t}$. Therefore, by (3.8) and the fact that $M > C_1 b/a$,

$$\alpha(g_{(n-1)t}h_s x) = \alpha(h_{e^{2(n-1)t}(s-s_0)} g_{(n-1)t} h_{s_0} x) > b/a.$$

Then, by Lemma 3.6 and (3.9),

$$\int_J \alpha(g_{nt}h_s x)\,ds \le \int_J \int_{-1}^{1} \alpha(g_t h_r g_{(n-1)t} h_s x)\,dr\,ds \le 2a \int_J \alpha(g_{(n-1)t}h_s x)\,ds. \tag{3.11}$$

Note that

$$B(M, n-1) \subseteq \bigcup_{\substack{J\in\mathcal{P}_{n-1}\\ J\cap B(M,n-1)\neq\emptyset}} J.$$

In particular, by (3.11),

$$\begin{aligned}\int_{B(M,n-1)} \alpha(g_{nt}h_s x)\,ds &\le \sum_{\substack{J\in\mathcal{P}_{n-1}\\ J\cap B(M,n-1)\neq\emptyset}} \int_J \alpha(g_{nt}h_s x)\,ds \\ &\le 2a \sum_{\substack{J\in\mathcal{P}_{n-1}\\ J\cap B(M,n-1)\neq\emptyset}} \int_J \alpha(g_{(n-1)t}h_s x)\,ds. \end{aligned}\tag{3.12}$$

Since $e^t \in \mathbb{N}$, for each $1 \le j \le k$ the partition $\mathcal{P}_k$ is a refinement of $\mathcal{P}_j$. Therefore, for any $1 \le m \le N$,

$$\bigcup_{\substack{J\in\mathcal{P}_{N-(m-1)}\\ J\cap B(M,N-1)\neq\emptyset}} J \subseteq \bigcup_{\substack{J\in\mathcal{P}_{N-m}\\ J\cap B(M,N-1)\neq\emptyset}} J. \tag{3.13}$$

Then, by (3.12) for $n = N$, (3.13) for $m = 2$, and the fact that α is non-negative,

$$\int_{B(M,N-1)} \alpha(g_{Nt}h_s x)\,ds \le 2a \sum_{\substack{J\in\mathcal{P}_{N-2}\\ J\cap B(M,N-1)\neq\emptyset}} \int_J \alpha(g_{(N-1)t}h_s x)\,ds. \tag{3.14}$$

As a result, for any $2 \le k \le N$ we have

$$\int_{B(M,N-1)} \alpha(g_{Nt}h_s x)\,ds \le (2a)^{k-1} \sum_{\substack{J\in\mathcal{P}_{N-k}\\ J\cap B(M,N-1)\neq\emptyset}} \int_J \alpha(g_{(N-(k-1))t}h_s x)\,ds. \tag{3.15}$$

The above inequality (3.15) is obtained by induction. The base case is $k = 2$ (see (3.14)). For the induction step, first observe that if $J \cap B(M, N-1) \neq \emptyset$, then $J \cap B(M, n) \neq \emptyset$ for any $n \le N-1$ by (3.10). The induction step thus follows by applying (3.11) with $n = N-(k-1)$ and (3.13) with $m = k+1$.

In particular, since $\mathcal{P}_0 = \{[-1, 1]\}$, applying (3.15) with $k = N$ shows that

$$\int_{B(M,N-1)} \alpha(g_{Nt}h_s x)\,ds \le (2a)^{N-1} \int_{-1}^{1} \alpha(g_t h_s x)\,ds.$$

Finally, by Lemma 3.5 and since α is non-negative, we have the desired inequality

$$\int_{B(M,N-1)} \alpha(g_{Nt}h_s x)\,ds \le (2a)^{N-1}\,[a\alpha(x) + b] \le (2a)^N \alpha(x) + (2a)^{N-1} b. \qquad \square$$

As a corollary, we obtain the following covering result.

Corollary 3.8. *There exists $b > 0$ such that for all $a \in (0,1)$, there exists $t_0 > 1$ such that for all $x \in X$, all $M > C_1^2 b/a$, all $N \in \mathbb{N}$, and all $t > t_0$ such that $e^t \in \mathbb{N}$, the set $Z_x(M,N,t)$ can be covered by $2C_1(2a)^N e^{2tN} \max\{1, \alpha(x)/M\}$ intervals of radius e^{-2tN}, where $C_1 > 1$ is the absolute constant in Proposition 3.7.*

Proof. Let b, C_1 be as in Proposition 3.7 and let $a > 0$. Cover the interval $[-1,1]$ with intervals of radius e^{-2tN}. By Proposition 3.7, for all $M > C_1^2 b/a$, we have

$$\int_{Z_x(C_1^{-1}M,N-1,t)} \alpha(g_{Nt}h_s x)\,ds \le (2a)^N \alpha(x) + (2a)^{N-1} b$$

whenever $t > t_0$ and $e^t \in \mathbb{N}$. By Chebyshev's inequality in L^1, this implies that

$$\nu(Z_x(C_1^{-1}M,N,t)) \le C_1\left[\frac{(2a)^N\alpha(x) + (2a)^{N-1}b}{M}\right] \le 2C_1(2a)^N \max\left\{1, \frac{\alpha(x)}{M}\right\}.$$

Therefore, the number of intervals of radius e^{-2tN} contained in $Z_x(C_1^{-1}M,N,t)$ is at most

$$2C_1(2a)^N e^{2tN} \max\{1, \alpha(x)/M\}.$$

We finish the proof by showing that for any interval I not contained in $Z_x(C_1^{-1}M,N,t)$, we have $I \cap Z_x(M,N,t) = \emptyset$. So let I be such an interval. This means that for some $s_0 \in I$ and some $1 \le l \le N$ we have $\alpha(g_{lt}h_{s_0}x) \le C_1^{-1}M$. This implies that for all $s_1 \in I$,

$$\alpha(g_{lt}h_{s_1}x) = \alpha((g_{\ell t}h_{(s_1-s_0)}g_{-\ell t})g_{\ell t}h_{s_0}x) \le C_1 C_1^{-1} M = M,$$

by the choice of C_1 in (3.8). This completes the proof. □

Proposition 3.9 (cf. [KKLM17, Theorem 1.5]). *Suppose $x \in X \setminus Y$. Then, for any $\delta, a \in (0,1)$ there exist $M_0 > 1$ and $t_0 > 1$, depending only on a, such that for all $M > M_0$, all $t > t_0$ with $e^t \in \mathbb{N}$ and all $N \in \mathbb{N}$, the set $Z_x(X_{\le M}, N, t, \delta)$ can be covered with $2^N C_1^N (2a)^{\delta N} e^{2tN} C(x)$ intervals of radius e^{-2tN}, where C_1 is as in Corollary 3.8 and $C(x) = \max\{1, \alpha(x)/M\}$.*

Proof. We describe the modifications to the proof of [KKLM17, Theorem 1.5] needed to prove the proposition. In the notation of Corollary 3.8, we take $M_0 = C_1^2 b/a$ and let $M > M_0$. The rest of the proof follows the same induction scheme used in [KKLM17, Theorem 1.5] with the base case being Corollary 3.8. The only modification is to skip the steps involving enlarging M depending on the largeness of the step size t and instead work directly with the bound on covers provided by the preceding corollary.

In particular, in the second case of the inductive step in [KKLM17, Theorem 1.5], M is assumed large enough depending on t to apply the covering result of [KKLM17, Corollary 5.2] which only applies to $x \in X$ with $\alpha(x)$ sufficiently large. Since Corollary 3.8 above works for all x, this restriction on M is not needed. □

3.3. Proof of Theorem 3.2

Let C_1 be the constant in Proposition 3.9. Fix $a \in (0, 1/2)$ such that $\delta > -\log(2C_1)/\log(2a)$.

Let $M_0 = M_0(a)$ and $t_0 = t_0(a)$ be as in Proposition 3.9. Let $M > M_0$ and $t > t_0$. Define $Q = X_{\leq M}$, $\gamma = -\log(2a)/(2t)$ and $\beta = \log(2C_1)/(2t)$, i.e., $2C_1 = e^{2\beta t}$. Then, by Proposition 3.9, for all $N \in \mathbb{N}$, we can cover the set $Z_x(Q, N, t, \delta)$ with $C' e^{2tN(1+\beta-\delta\gamma)}$ intervals of radius e^{-2tN}, for some constant C' depending only on x. Note that C' is finite by our assumption that $x \in X \setminus Y$.

Hence, by Lemma 2.5, the Hausdorff dimension of the set $\limsup_{N\to\infty} Z_x(Q, N, t, \delta)$ is at most $1 + \beta - \delta\gamma > 0$. By the choice of a, this upper bound is strictly less than 1. Finally, by definition of β and γ, our upper bound is uniform over all $x \in X \setminus Y$.

Remark 3.10. The proofs of Proposition 3.9 and Theorem 3.2 show that one can choose $M_0 = c'' b e^{c'/\delta}$ in the conclusion of Theorem 3.2, for some positive constants c' and c'', where b is as in Definition 3.1.

4. Hausdorff dimension of the divergent-on-average directions

In this section we prove Theorem 2.3 which implies Theorem 1.8 (see Section 2). Consider the set

$$Z = \{s \in [-1, 1] : g_t h_s \omega \text{ diverges on average}\}.$$

Notice that for all compact sets $Q \subset \mathcal{H}_1(\alpha)$, all $0 \leq \delta \leq 1$ and all $t > 0$,

$$Z \subseteq \liminf_{N\to\infty} Z_\omega(Q, N, t, \delta) := \bigcup_{N_0=1}^{\infty} \bigcap_{N=N_0}^{\infty} Z_\omega(Q, N, t, \delta), \tag{4.1}$$

where $Z_\omega(Q, N, t, \delta)$ is defined in (3.1).

Building on earlier work of [EM01], it is shown in [Ath06] that the $\mathrm{SL}_2(\mathbb{R})$ action on $X = \mathcal{H}_1(\alpha)$ satisfies the contraction hypothesis with respect to $Y = \emptyset$. The precise statement is the following:

Lemma 4.1 ([Ath06, Lemma 2.10]). *For every $0 < \eta < 1$, there exists a function $\alpha_\eta : X \to \mathbb{R}^+$ satisfying items* (1), (2) *and* (4) *of Definition 3.1. Moreover, there are constants $c = c(\eta)$ and $t_0 = t_0(\eta) > 0$ such that for all $t > t_0$, there exists $b = b(t, \eta) > 0$ such that for all $x \in X$,*

$$\frac{1}{2\pi} \int_0^{2\pi} \alpha_\eta(g_t r_\theta x)\, d\theta \leq c e^{-(1-\eta)t} \alpha_\eta(x) + b. \tag{4.2}$$

By using the integral estimate in (4.2) in place of the one in (3.3), we can prove the following analogue of Lemma 3.5.

Corollary 4.2. *For every $0 < \eta < 1$, let $\alpha_\eta : X \to \mathbb{R}^+$ be as in Lemma 4.1. Let σ be as in Definition 3.1. Then there are constants $c' = c'(\eta, \sigma)$ and $t_0 = t_0(\eta) > 0$ such that for all $t > t_0$, there exists $b' = b'(t, \eta) > 0$ such that for all $x \in X$,*

$$\int_{-1}^{1} \alpha_\eta(g_t h_s x)\, ds \le c' e^{-(1-\eta)t} \alpha_\eta(x) + b'. \tag{4.3}$$

In particular, if $\alpha_\eta(x) > b' e^{(1-\eta)t}/c'$, then

$$\int_{-1}^{1} \alpha_\eta(g_t h_s x)\, ds \le 2c' e^{-(1-\eta)t} \alpha_\eta(x). \tag{4.4}$$

As a result, we deduce the upper bound on $\dim_H(Z)$ from a covering result for the sets $Z_\omega(Q, N, t, \delta)$, where Q will be a sublevel set of a height function α_η for $\eta \in (0, 1)$.

Proof of Theorem 2.3. Fixing any choice of the parameter $\eta \in (0, 1)$, by Proposition 3.9 applied with a replaced with $2c'e^{-(1-\eta)t}$ for sufficiently large t, the set $\bigcap_{N \ge N_0} Z_\omega(Q, N, t, \delta)$ can be covered with at most $2^N C_1^N (2c'e^{-(1-\eta)t})^{\delta N} e^{2tN} C(x)$ intervals of radius e^{-2tN}, for each $N \ge N_0$ and for all $N_0 \in \mathbb{N}$. Since the Hausdorff dimension is majorized by the upper box dimension, it follows that

$$\begin{aligned} \dim_H\Big(\bigcap_{N \ge N_0} Z_\omega(Q, N, t, \delta)\Big) &\le \lim_{N\to\infty} \frac{\log(2^N C_1^N (2c'e^{-(1-\eta)t})^{\delta N} e^{2tN} C(x))}{-\log e^{-2Nt}} \\ &\le \frac{\log(4C_1 c')}{2t} + 1 - \frac{\delta(1-\eta)}{2}. \end{aligned}$$

By letting $t \to \infty$ and in view of 4.1, we get $\dim_H(Z) \le 1 - \delta(1-\eta)/2$. By sending δ to 1 and η to 0, we get the desired conclusion. □

5. Uniformity in Birkhoff's theorem

The purpose of this section is to prove a uniform version of the result of Chaika and Eskin [CE15, Theorem 1.1] on the pointwise equidistribution of Teichmüller geodesics with respect to the Lebesgue measure on a horocycle arc. This step is crucial for our Hausdorff dimension estimates in large deviation problems.

Throughout this section, suppose $\mathcal{M} \subset \mathcal{H}_1(\alpha)$ is a fixed $\mathrm{SL}_2(\mathbb{R})$-invariant affine submanifold. For an affine invariant submanifold $\mathcal{N} \subset \mathcal{H}_1(\alpha)$, we denote by $\nu_\mathcal{N}$ the unique $\mathrm{SL}_2(\mathbb{R})$-invariant Lebesgue probability measure with support $\mathcal{N}$. For any bounded continuous function ϕ on $\mathcal{H}_1(\alpha)$, let $\nu_\mathcal{N}(\phi) = \int_{\mathcal{H}_1(\alpha)} \phi\, d\nu_\mathcal{N}$. For any $T > 0$, $s \in [-1, 1]$ and $x \in \mathcal{H}_1(\alpha)$, we denote by $A_s^T(x)$ the measure defined by

$$A_s^T(x)(\varphi) := \frac{1}{T} \int_0^T \varphi(g_t h_s x)\, dt \tag{5.1}$$

for any bounded continuous function φ on $\mathcal{M}$. Similarly, for $N \in \mathbb{N}$ and $l > 0$, we define the measure $S_s^N(x)$ in the following way:

$$S_s^N(x)(\varphi) := \frac{1}{N} \sum_{n=1}^{N} \varphi(g_{ln} h_s x). \tag{5.2}$$

Notice that $S_s^N(x)$ depends on the step size l, though we do not emphasize this in the notation.

For any $\tilde{g} \in \mathrm{SL}_2(\mathbb{R})$, we define

$$\tilde{g} A_s^T(x)(\varphi) := \frac{1}{T} \int_0^T \varphi(\tilde{g} g_t h_s x)\, dt \quad \text{and} \quad \tilde{g} S_s^N(x)(\varphi) := \frac{1}{N} \sum_{n=1}^{N} \varphi(\tilde{g} g_{ln} h_s x). \tag{5.3}$$

Let $C_c^\infty(\mathcal{H}_1(\alpha))$ be the space of smooth compactly supported functions on $\mathcal{H}_1(\alpha)$. For any $\varphi \in C_c^\infty(\mathcal{H}_1(\alpha))$, we define the Sobolev norm $\mathcal{S}(\varphi)$ of φ by

$$\mathcal{S}(\varphi) := \|\varphi\|_{\mathrm{Lip}} + \|\varphi\|_\infty, \tag{5.4}$$

where $\|\phi\|_{\mathrm{Lip}}$ and $\|\varphi\|_\infty$ denote the Lipschitz constant and the maximum of φ on $\mathcal{H}_1(\alpha)$, respectively. Then, for all $g \in \mathrm{SL}_2(\mathbb{R})$ and all $x \in \mathcal{H}_1(\alpha)$, one has

$$|\varphi(gx) - \varphi(x)| \le \mathcal{S}(\varphi) d(g, \mathrm{id}),$$

where $d(\cdot, \cdot)$ denotes some metric on $\mathrm{SL}_2(\mathbb{R})$.

The following theorem is the main result of this section.

Theorem 5.1. *Suppose f is a bounded continuous function on $\mathcal{H}_1(\alpha)$. Then for any $\varepsilon > 0$ there exist finitely many proper affine $\mathrm{SL}_2(\mathbb{R})$-invariant submanifolds of $\mathcal{M}$, denoted by $\mathcal{N}_1, \ldots, \mathcal{N}_l$ such that for any compact set $F \subset \mathcal{M} \setminus \bigcup_{i=1}^{l} \mathcal{N}_i$ and any $\kappa > 0$, there exists $T_0 = T_0(F, \kappa, \varepsilon, f) > 0$ such that for any $x \in F$ and any $T > T_0$,*

$$\left|\{s \in [-1, 1] : |A_s^T(x)(f) - \nu_{\mathcal{M}}(f)| \ge \varepsilon\}\right| < \kappa. \tag{5.5}$$

The proof of Theorem 5.1 (see Section 5.3) is based on a combination of the techniques used to prove [CE15, Theorem 1.1] and [EMM15, Theorem 2.11], paying additional care to the unipotent invariance of limiting distributions. Following the same idea, we also prove the following discrete version of Theorem 5.1 (see Section 5.4 for the proof).

Theorem 5.2. *Suppose f is a bounded continuous function on $\mathcal{H}_1(\alpha)$. Then for any $\varepsilon > 0$ there exist finitely many proper affine $\mathrm{SL}_2(\mathbb{R})$-invariant submanifolds of $\mathcal{M}$, denoted by $\mathcal{N}_1, \ldots, \mathcal{N}_k$, such that for any compact set $F \subset \mathcal{M} \setminus \bigcup_{i=1}^{k} \mathcal{N}_i$ and any $\kappa, l > 0$, there exists $N_0 = N_0(F, \kappa, l, \varepsilon, f) > 0$ such that for any $x \in F$ and any $N > N_0$,*

$$\left|\{s \in [-1, 1] : |S_s^N(x)(f) - \nu_{\mathcal{M}}(f)| \ge \varepsilon\}\right| < \kappa. \tag{5.6}$$

Theorems 5.1 and 5.2 are in the spirit of the results of [EMM15] and [DM93].

5.1. Some finiteness and recurrence results

In this section we formulate some facts that we use throughout Section 5.

The following lemma will provide us with the finite exceptional collection of invariant submanifolds in Theorem 5.1.

Lemma 5.3 ([EMM15, Lemma 3.4]). *Given $\varepsilon > 0$ and $\varphi \in C_c(\mathcal{H}_1(\alpha))$, there exists a finite collection $\mathcal{C}$ of proper affine invariant submanifolds of $\mathcal{M}$ with the following property: if $\mathcal{N} \subset \mathcal{M}$ is an affine invariant submanifold such that $|\nu_{\mathcal{N}}(\varphi) - \nu_{\mathcal{M}}(\varphi)| \geq \varepsilon$, then $\mathcal{N}$ is contained in some $\mathcal{N}' \in \mathcal{C}$.*

The following proposition shows that most geodesic trajectories avoid any given finite collection of proper submanifolds of $\mathcal{M}$.

Proposition 5.4 ([EMM15, Proposition 3.8]). *Given $\varepsilon > 0$ and any (possibly empty) proper affine invariant submanifold $\mathcal{N}$, there exists an open neighborhood $\Omega_{\mathcal{N},\varepsilon}$ of $\mathcal{N}$ with the following property: the complement of $\Omega_{\mathcal{N},\varepsilon}$ is compact and for any compact set $F \subset \mathcal{H}_1(\alpha) \setminus \mathcal{N}$, there exists $T_0 = T_0(F) > 0$ such that for any $T > T_0$ and any $x \in F$,*

$$\int_{-1}^{1} A_s^T(x)(\chi_{\Omega_{\mathcal{N},\varepsilon}})\, ds < \varepsilon, \tag{5.7}$$

where $\chi_{\Omega_{\mathcal{N},\varepsilon}}$ denotes the indicator function of the set $\Omega_{\mathcal{N},\varepsilon}$.

The following discrete version of Proposition 5.4 also holds.

Proposition 5.5. *Given $\varepsilon > 0$ and any (possibly empty) proper affine invariant submanifold $\mathcal{N}$, there exists an open neighborhood $\Omega_{\mathcal{N},\varepsilon}$ of $\mathcal{N}$ with the following property: the complement of $\Omega_{\mathcal{N},\varepsilon}$ is compact and for any compact set $F \subset \mathcal{H}_1(\alpha) \setminus \mathcal{N}$ and any $l > 0$, there exists $N_0 > 0$ such that for any $N > N_0$ and any $x \in F$,*

$$\int_{-1}^{1} S_s^N(x)(\chi_{\Omega_{\mathcal{N},\varepsilon}})\, ds < \varepsilon. \tag{5.8}$$

The proof of Proposition 5.5 is similar to that of Proposition 5.4, i.e., it is a consequence of the contraction hypothesis (see Definition 3.1) shown in [EMM15, Proposition 2.13]. See also [EM04, Lemma 3.1].

Proof. By [EMM15, Proposition 2.13], there exists a height function $f_{\mathcal{N}}$ with $X = \mathcal{H}_1(\alpha)$ and $Y = \mathcal{N}$ (see Definition 3.1). Let $m_F = \sup\{f_{\mathcal{N}}(x) : x \in F\}$. Notice that $m_F \geq 1$ because, by definition, $f_{\mathcal{N}}(x) \geq 1$ for any $x \in \mathcal{H}_1(\alpha)$. Then, by Lemma 3.5, there exists $t_1 > 0$ such that for all $t > t_1$ and all $x \in F$,

$$\int_{-1}^{1} f_{\mathcal{N}}(g_t h_s x)\, ds \leq \frac{1}{m_F} f_{\mathcal{N}}(x) + \bar{b} \leq \bar{b} + 1. \tag{5.9}$$

Moreover, by property (2) in Definition 3.1, there exists $M = M(t_1) > 0$ such that for all $0 \leq t \leq t_1$ and all x,

$$f_{\mathcal{N}}(g_t h_s x) \leq M f_{\mathcal{N}}(x). \tag{5.10}$$

Let $L > 0$ be such that $\frac{\bar{b}+2}{L} < \varepsilon$. Define

$$\Omega_{\mathcal{N},\varepsilon} = \mathrm{Int}(\{x \in \mathcal{H}_1(\alpha) : f_{\mathcal{N}}(x) > L\}),$$

where Int denotes the interior of a set. Then, by property (4) in Definition 3.1, $\Omega_{\mathcal{N},\varepsilon}$ is an open neighborhood of $\mathcal{N}$ with compact complement. Let $N_0 \in \mathbb{N}$ be sufficiently large that

$$\frac{Mm_F t_1}{lN_0} < 1.$$

Then, using the estimates in (5.9) and (5.10), we get

$$\begin{aligned}\int_{-1}^{1} S_s^N(x)(f_{\mathcal{N}})\,ds &= \frac{1}{N}\sum_{1\le n\le t_1/l}\int_{-1}^{1} f_{\mathcal{N}}(g_{ln}h_s x)\,ds + \frac{1}{N}\sum_{t_1/l<n\le N}\int_{-1}^{1} f_{\mathcal{N}}(g_{ln}h_s x)\,ds \\ &\le \frac{Mm_F t_1}{lN} + \bar{b} + 1 \le \bar{b} + 2. \end{aligned} \tag{5.11}$$

Notice that for any $n \in \mathbb{N}$ and $s \in [-1, 1]$,

$$f_{\mathcal{N}}(g_{ln}h_s x) \ge L\chi_{\Omega_{\mathcal{N},\varepsilon}}(g_{ln}h_s x). \tag{5.12}$$

Therefore, by (5.11) and (5.12), we obtain

$$\int_{-1}^{1} S_s^N(x)(\chi_{\Omega_{\mathcal{N},\varepsilon}})\,ds \le \frac{\bar{b}+2}{L} < \varepsilon. \qquad \square$$

5.2. *Effective unipotent invariance*

In this section, we show a quantative version of [CE15, Proposition 3.1] (Proposition 5.7) regarding almost sure unipotent invariance of limit points of measures of the form (5.1). Also, we state an analogue for discrete averages (Proposition 5.8), whose proof is identical to the flow case. See [Kha20a] for a generalization of this phenomenon to semisimple Lie group actions.

Suppose $x \in \mathcal{H}_1(\alpha)$, $\phi \in C_c^\infty(\mathcal{H}_1(\alpha))$ and $\beta \in \mathbb{R}$. For $t > 0$ and $s \in [-1, 1]$, we define

$$f_t(s) = \varphi(g_t h_s x) - \varphi(h_\beta g_t h_s x). \tag{5.13}$$

The following lemma formulated for horocycle arcs is an analogue of Lemma 3.3 in [CE15] which is proved for circle arcs.

Lemma 5.6 (Analogue of [CE15, Lemma 3.3]). *There exists a constant $C_1 > 0$ such that for all $x \in \mathcal{H}_1(\alpha)$, $\phi \in C_c^\infty(\mathcal{H}_1(\alpha))$, $\beta \in \mathbb{R}$ and $t_1, t_2 > 0$,*

$$\left|\int_{-1}^{1} f_{t_1}(s) f_{t_2}(s)\,ds\right| \le C_1 \mathcal{S}(\varphi)^2 e^{-2|t_1-t_2|},$$

where $\mathcal{S}(\phi)$ and $f_t(s)$ are defined in (5.4) and (5.13), respectively.

We note that the proof of Lemma 5.6 is identical to the proof of [CE15, Lemma 3.3] and simpler if one takes into account that the group of elements h_s is normalized by g_t.

Proposition 5.7 (Quantitative version of [CE15, Proposition 3.1]). *Suppose $\beta \in \mathbb{R}$. Then there exists a constant $C > 0$ such that for all $T > 0$, all $x \in \mathcal{H}_1(\alpha)$ and all $\varphi \in C_c^\infty(\mathcal{H}_1(\alpha))$, the Lebesgue measure of the set*

$$\{s \in [-1, 1] : |A_s^T(x)(\varphi) - (h_\beta A_s^T(x))(\varphi)| > \mathcal{S}(\varphi)/T^{1/8}\}$$

is at most $C/T^{1/4}$.

The version of Proposition 5.7 for discrete averages is the following.

Proposition 5.8. *Suppose $\beta \in \mathbb{R}$. Then there exists a constant $C > 0$ such that for all $N > 0$, all $x \in \mathcal{H}_1(\alpha)$ and all $\varphi \in C_c^\infty(\mathcal{H}_1(\alpha))$, the Lebesgue measure of the set*

$$\{s \in [-1, 1] : |S_s^N(x)(\varphi) - (h_\beta S_s^N(x))(\varphi)| > \mathcal{S}(\varphi)/N^{1/8}\}$$

is at most $C/N^{1/4}$.

Proof of Proposition 5.7. By Fubini's theorem and Lemma 5.6, one has

$$\begin{aligned}\int_{-1}^{1} |A_s^T(\varphi) - (h_\beta A_s^T)(\varphi)|^2 \, ds &\le \frac{1}{T^2} \int_{[0,T]^2} \left| \int_{-1}^{1} f_{t_1}(s) f_{t_2}(s) \, ds \right| dt_1 \, dt_2 \\ &= \frac{1}{T^2} \int_{|t_1 - t_2| < T^{1/2}} \left| \int_{-1}^{1} f_{t_1}(s) f_{t_2}(s) \, ds \right| dt_1 \, dt_2 + C_1 \mathcal{S}(\varphi)^2 e^{-2T^{1/2}} \\ &\le \frac{16 \|\varphi\|_\infty^2}{T^{1/2}} + C_1 \mathcal{S}(\varphi)^2 e^{-2T^{1/2}} \le 2C_2 \mathcal{S}(\varphi)^2 T^{-1/2},\end{aligned}$$

where we use the facts that $|f_t(s)| \le 2\|\varphi\|_\infty$, the measure of the region $|t_1 - t_2| < T^{1/2}$ is at most $2T^{3/2}$, and $C_2 > 16C_1$ is a constant such that for all $T > 0$,

$$e^{-2T^{1/2}} \le C_2 T^{-1/2}.$$

Using Chebyshev's inequality, we obtain the proposition. □

5.3. *Proof of Theorem 5.1*

Fix positive constants ε and κ. Let $\mathcal{C}$ be the finite collection of affine invariant submanifolds $\mathcal{N}_1, \ldots, \mathcal{N}_k$ of $\mathcal{M}$ given by Lemma 5.3 applied to the given function f and $\varepsilon/2$. Consider a compact subset $F \subset \mathcal{M} \setminus \bigcup_i \mathcal{N}_i$.

Let $\varepsilon' > 0$ be a sufficiently small number such that $\sqrt{\varepsilon'} < \min\left\{\frac{\kappa}{3}, \frac{\varepsilon}{9\|f\|_\infty}\right\}$. By Proposition 5.4, since $\mathcal{C}$ is a finite collection, there exists an open neighborhood $\Omega_{\mathcal{C},\varepsilon'}$ of $\bigcup_{i=1}^k \mathcal{N}_i$ and $T_0 > 0$ depending on ε' and F such that for all $T > T_0$ and all $x \in F$, we have

$$\int_{-1}^{1} A_s^T(x)(\chi_{\Omega_{\mathcal{C},\varepsilon'}}) \, ds < \varepsilon'$$

and hence, by Chebyshev's inequality in $L^1(ds)$, the measure of the set

$$D(x, T, \varepsilon') = \{s \in [-1, 1] : A_s^T(x)(\chi_{\Omega_{\mathcal{C},\varepsilon'}}) \geq \sqrt{\varepsilon'}\} \tag{5.14}$$

is at most $\sqrt{\varepsilon'}$.

Let $\Phi = \{\varphi_n : n \in \mathbb{N}\} \subset C_c^\infty(\mathcal{H}^1(\alpha))$ be a countable dense collection of functions. For each N, let $\Phi_N = \{\varphi_n : 1 \leq n \leq N, n \in \mathbb{N}\} \subset \Phi$. Consider $\beta_1, \beta_2 \in \mathbb{R}$ with $\beta_1/\beta_2 \notin \mathbb{Q}$. By Proposition 5.7, there exists a constant $C > 0$ such that for all $T > 0$ and all x, the measure of the set

$$B(x, T, \Phi_N) := \bigcup_{n=1}^{N} B(x, T, \varphi_n) \tag{5.15}$$

is at most $CNT^{-1/4}$, where $\mathcal{S}(\cdot)$ is a Sobolev norm (see (5.4)) and

$$B(x, T, \varphi_n) := \left\{ s \in [-1, 1] : |A_s^T(x)(\varphi_n) - (h_{\beta_i} A_s^T(x))(\varphi_n)| > \frac{\mathcal{S}(\varphi_n)}{T^{1/8}} \text{ for } i = 1, 2 \right\}.$$

We prove the theorem by contradiction. Suppose that the conclusion does not hold for our choice of F and κ. Then there exist sequences $x_n \in F$ and $T_n \to \infty$ such that for each $n \in \mathbb{N}$, the set

$$Z_{x_n}(f, T_n) := \{s \in [-1, 1] : |A_s^{T_n}(x)(f) - \nu_{\mathcal{M}}(f)| \geq \varepsilon\} \tag{5.16}$$

has measure at least κ.

By our estimates on the measures of the sets in (5.14)–(5.16), and the choice of ε' such that $\sqrt{\varepsilon'} < \kappa/3$, the following holds. For all n sufficiently large that $CT_n^{-1/8} < \kappa/3$, we have

$$Z_{x_n}(f, T_n) \cap D(x_n, T_n, \varepsilon')^c \cap B(x_n, T_n, \Phi_{T_n^{1/8}})^c \neq \emptyset \tag{5.17}$$

where for a set $A \subset [-1, 1]$, we use A^c to denote its complement in $[-1, 1]$. Therefore, for all n sufficiently large we can choose a point s_n that belongs to the intersection in (5.17). Since the space of Borel measures on $\mathcal{H}_1(\alpha)$ of mass at most 1 is compact in the weak-$*$ topology, after passing to a subsequence if necessary, we may assume that there is a Borel measure ν such that

$$A_{s_n}^{T_n}(x_n) \xrightarrow{\text{weak-}*} \nu.$$

Note that a priori ν may be the 0 measure. We show that this is not the case.

We claim that ν is $\mathrm{SL}_2(\mathbb{R})$-invariant. By Eskin and Mirzakhani's measure classification theorem [EM13], it is sufficient to show that ν is invariant by P, the subgroup of upper triangular matrices. Clearly, ν is invariant by g_t for all t. Moreover, by the dominated convergence theorem, it suffices to show that ν is invariant by h_{β_1} and h_{β_2} as they generate a dense subgroup of $U = \{h_s : s \in \mathbb{R}\}$.

Since smooth functions are dense in the set of compactly supported continuous functions, it suffices to show that for $i = 1, 2$, $h_{\beta_i}\nu(\varphi_k) = \nu(\varphi_k)$, where $h_{\beta_i}\nu(\varphi_k) := \int_{\mathcal{H}_1(\alpha)} \varphi_k(h_{\beta_i}\omega)\, d\nu(\omega)$ for all φ_k in Φ, our countable dense collection of smooth compactly supported functions.

Fix some $\varphi_k \in \Phi$. Note that for all n sufficiently large, $\varphi_k \in \Phi_{T_n^{1/8}}$, and therefore $s_n \notin B(x_n, T_n, \varphi_k)$. As a result,

$$|A^{T_n}_{s_n}(x_n)(\varphi_k) - h_{\beta_i} A^{T_n}_{s_n}(x_n)(\varphi_k)| \le \mathcal{S}(\varphi_k)/T_n \xrightarrow{n\to\infty} 0.$$

Therefore, ν is $\mathrm{SL}_2(\mathbb{R})$-invariant.

Moreover, since $s_n \in Z_{x_n}(f, T_n)$ for all n, we obtain

$$|\nu(f) - \nu_{\mathcal{M}}(f)| \ge \varepsilon.$$

We show that this is not possible. By [EMM15, Proposition 2.16], there are countably many affine invariant submanifolds in $\mathcal{H}_1(\alpha)$. Thus, since ν is $\mathrm{SL}_2(\mathbb{R})$-invariant, it has a countable ergodic decomposition of the form

$$\nu = \sum_{\mathcal{N}\subseteq\mathcal{M}} a_{\mathcal{N}}\nu_{\mathcal{N}},$$

where the sum is taken over all such proper (possibly empty) affine invariant submanifolds and $a_{\mathcal{N}} \in [0, 1]$ for all $\mathcal{N}$. Note that since $s_n \notin D(x_n, T_n, \varepsilon')$, we have

$$\sum_{\substack{\mathcal{N}:\,\exists\mathcal{N}'\in\mathcal{C},\\ \mathcal{N}\subseteq\mathcal{N}'}} a_{\mathcal{N}} \le \nu(\Omega_{\mathcal{C},\varepsilon'}) \le \sqrt{\varepsilon'}.$$

Since the complement of $\Omega_{\mathcal{C},\varepsilon'}$ is compact and $A^{T_n}_{s_n}(x_n)(1 - \chi_{\Omega_{\mathcal{C},\varepsilon'}}) \ge 1 - \sqrt{\varepsilon'}$, the total mass of ν is at least $1 - \sqrt{\varepsilon'}$.

Furthermore, $|\nu_{\mathcal{N}}(f) - \nu_{\mathcal{M}}(f)| < \varepsilon/2$ for all $\mathcal{N}$ not contained in any member of $\mathcal{C}$, by definition of the collection $\mathcal{C}$.

Let $|\nu| := \sum_{\mathcal{N}} a_{\mathcal{N}}$ be the total mass of ν. Then

$$\begin{aligned}
\varepsilon \le |\nu(f) - \nu_{\mathcal{M}}(f)| &= \Big|(1 - |\nu|)\nu_{\mathcal{M}}(f) + \sum_{\mathcal{N}\subseteq\mathcal{M}} a_{\mathcal{N}}(\nu_N(f) - \nu_{\mathcal{M}}(f))\Big| \\
&\le \sqrt{\varepsilon'}\,\|f\|_\infty + \sum_{\substack{\mathcal{N}:\,\exists\mathcal{N}'\in\mathcal{C},\\ \mathcal{N}\subseteq\mathcal{N}'}} a_{\mathcal{N}}|\nu_{\mathcal{N}}(f) - \nu_{\mathcal{M}}(f)| + \sum_{\substack{\mathcal{N}:\,\nexists\mathcal{N}'\in\mathcal{C},\\ \mathcal{N}\subseteq\mathcal{N}'}} a_{\mathcal{N}}|\nu_{\mathcal{N}}(f) - \nu_{\mathcal{M}}(f)| \\
&\le \sqrt{\varepsilon'}\,\|f\|_\infty + 2\|f\|_\infty \nu(\Omega_{\mathcal{C},\varepsilon'}) + |\nu|\varepsilon/2 \le 3\|f\|_\infty\sqrt{\varepsilon'} + \varepsilon/2 < \varepsilon.
\end{aligned}$$

We get the desired contradiction by our choice of ε'.

5.4. Proof of Theorem 5.2

The proof is similar to the proof of Theorem 5.1 (the flow case), and relies on using Propositions 5.5 and 5.8 instead of Propositions 5.4 and 5.7, respectively. The proof also goes by contradiction. Assuming that the conclusion of the theorem does not hold, we construct an $\mathrm{SL}_2(\mathbb{R})$-invariant measure ν. The analysis of its ergodic decomposition implies a contradiction as in Section 5.3.

The following lemma allows us to show that the constructed measure ν is $\mathrm{SL}_2(\mathbb{R})$-invariant.

Lemma 5.9. *Let $l > 0$ and let P_l be the group generated by all elements of the form $g_{ln}h_s$ for $n \in \mathbb{Z}$ and $s \in \mathbb{R}$. Let ν be a P_l ergodic invariant probability measure on $\mathcal{M}$. Then ν is $\mathrm{SL}_2(\mathbb{R})$-invariant.*

Proof. Denote by $\bar{\nu}$ the measure defined by

$$\bar{\nu} := \frac{1}{l}\int_0^l (g_t)_*\nu \, dt, \tag{5.18}$$

where $(g_t)_*\nu$ is the pushforward of ν. Then $\bar{\nu}$ is invariant by the group P of upper triangular matrices. Notice that for any $t \in (0, l)$, $(g_t)_*\nu$ is invariant by the group $U = \{h_s : s \in \mathbb{R}\}$ due to the fact that U is normalized by g_t and ν is invariant by U. That implies that $\bar{\nu}$ is invariant by U as it is a convex combination of U-invariant measures. Similarly, we can show that $\bar{\nu}$ is invariant under the $\mathbb{Z}$ action of g_l. To show the invariance under the group $A = \{g_t : t \in \mathbb{R}\}$, we write $t = ml + r$ for some $m \in \mathbb{Z}$ and $r \in [0, 1)$ and use the invariance by $\{g_{nl} : n \in \mathbb{Z}\}$.

As a result, by [EM13, Theorem 1.4], $\bar{\nu}$ is $\mathrm{SL}_2(\mathbb{R})$-invariant. Thus, $\bar{\nu}$ has the following ergodic decomposition with respect to the $\mathrm{SL}_2(\mathbb{R})$ action:

$$\bar{\nu} = \sum_{\mathcal{N}\subseteq\mathcal{M}} a_{\mathcal{N}}\nu_{\mathcal{N}}, \tag{5.19}$$

where each $\nu_{\mathcal{N}}$ is ergodic under the $\mathrm{SL}_2(\mathbb{R})$ action. But, by Mautner's phenomenon, each $\nu_{\mathcal{N}}$ is ergodic under the action of h_s for all $s \neq 0$.

On the other hand, $(g_t)_*\nu$ is h_s-invariant for all t and s. Hence, (5.18) and (5.19) give two decompositions of $\bar{\nu}$ for the action of h_s, one of which is a countable decomposition into ergodic measures.

Thus, by uniqueness of the ergodic decomposition, there exists a set $A \subseteq [0, l]$ of positive Lebesgue measure $|A|$ and an affine invariant manifold $\mathcal{N}$ such that $a_{\mathcal{N}} = |A|/l$ and

$$\frac{1}{l}\int_A (g_t)_*\nu \, dt = a_{\mathcal{N}}\nu_{\mathcal{N}}.$$

But, by ergodicity of $\nu_{\mathcal{N}}$ under the action of h_s, $(g_t)_*\nu = \nu_{\mathcal{N}}$ for almost every $t \in A$. Since $\nu_{\mathcal{N}}$ is $\mathrm{SL}_2(\mathbb{R})$-invariant, so is ν. □

6. Dimension of directions with large deviations in Birkhoff's theorem

The goal of this section is to prove Theorem 2.1. We also outline the modifications needed to prove Theorem 6.7 in Section 6.6.

In what follows, $\mathcal{M} \subseteq \mathcal{H}_1(\alpha)$ is a fixed affine invariant manifold. By a simple approximation argument, it is enough to prove Theorem 2.1 when f is a Lipschitz function. We let $\mathcal{S}(f)$ denote the Sobolev norm (see (5.4)), and $\nu_{\mathcal{M}}(f) = \int_M f \, d\nu_{\mathcal{M}}$.

Throughout this section we use the following notation. For any positive $\varepsilon, N \in \mathbb{R}$, $M \in \mathbb{N}$ and a subset $Q \subseteq \mathcal{M}$, we define

$$B_\omega(f,N,\varepsilon,M) := \left\{ s \in [-1,1] : \frac{1}{MN}\int_0^{MN} f(g_t h_s \omega)\,dt > \nu_{\mathcal{M}}(f) + \varepsilon \right\},$$
$$Z_\omega(Q,M,N,\varepsilon) := \left\{ s \in [-1,1] : \frac{\#\{0 \le i \le M-1 : g_{iN} h_s \omega \notin Q\}}{M} > \varepsilon \right\}, \tag{6.1}$$
$$B_\omega(f,N,\varepsilon) := \limsup_{M\to\infty} B_\omega(f,N,\varepsilon,M), \qquad Z_\omega(Q,N,\varepsilon) := \limsup_{M\to\infty} Z_\omega(Q,N,\varepsilon,M).$$

Using the boundedness of f and the fact that given $T, N > 0$, we can write $T = MN + K$ for some $M \in \mathbb{N} \cup \{0\}$ and $0 \le K < N$, it is straightforward to check that

$$\left\{ s \in [-1,1] : \limsup_{T\to\infty} \frac{1}{T}\int_0^T f(g_t h_s \omega)\,dt \ge \int_{\mathcal{M}} f\,d\nu_{\mathcal{M}} + \varepsilon \right\} \subseteq B_\omega(f,N,\varepsilon/2).$$

The set on the left-hand side is equal to the exceptional set in Theorem 2.1. Thus, to prove Theorem 2.1, it suffices to show that the dimension of $B_\omega(f,N,\varepsilon)$ is strictly less than 1 for any ε and an appropriate choice of N depending on ε. We note further that the sets $Z_\omega(Q,M,N,\varepsilon)$ are the same as the ones defined in (3.1).

Next, for any $s \in [-1,1]$, $i \in \mathbb{N}$ and positive β, $N \in \mathbb{R}$, we define

$$f_i(s) := \frac{1}{N}\int_{iN}^{(i+1)N} f(g_t h_s \omega)\,dt, \tag{6.2}$$
$$F_i(\beta) := \{ s : f_i(s) > \nu_{\mathcal{M}}(f) + \beta \}. \tag{6.3}$$

Here, we drop the dependence on the basepoint ω from the notation for simplicity.

Strategy

The strategy for proving Theorem 2.1 consists of two steps. The first step is to use Theorem 5.1 to control the measure of the sets $F_i(\beta)$. This is carried out in Lemma 6.2.

The next step is to show that the sets $F_i(\varepsilon/2)$ behave like level sets of independent random variables (Proposition 6.5). This will allow us to bound the measure of finite intersections of these sets. The proof of this independence property also yields a mechanism for controlling the number of intervals needed to cover such a finite intersection using its measure (Lemma 6.6).

In order to apply Theorem 5.2, we need to ensure that our trajectories land in a prechosen compact set. Hence, we are forced to run the above argument but restricted to the "recurrent directions". This restriction to recurrent directions is handled in Lemma 6.1. Applying Theorem 3.2, we control the Hausdorff dimension of the non-recurrent directions.

6.1. *Sets and partitions*

For $N > 0$ and $i \in \mathbb{N}$, let $\mathcal{P}_i$ denote the partition of $[-1,1]$ into intervals of radius e^{-2iN}. Note that when $e^N \in \mathbb{N}$, $(\mathcal{P}_i)_{i\in\mathbb{N}}$ is a refining sequence of partitions, i.e., for all $i < j$, every interval of $\mathcal{P}_i$ is a union of elements of $\mathcal{P}_j$.

For a set $Q \subset \mathcal{H}_1(\alpha)$, define the subpartitions

$$\mathcal{R}_i(Q) = \{J \in \mathcal{P}_i : \exists s \in J, g_{iN} h_s \omega \in Q\}.$$

Let $\mathcal{D}_i(Q) = \mathcal{P}_i \setminus \mathcal{R}_i(Q)$. Here $\mathcal{R}$ indicates recurrence and $\mathcal{D}$ indicates divergence. We note that the definition of $\mathcal{R}_i$ depends on the basepoint ω but we suppress this dependence in our notation.

Lemma 6.1. *Suppose $Q \subset \mathcal{H}_1(\alpha)$. Then, for any $\omega \in \mathcal{H}_1(\alpha)$, $N, \varepsilon > 0$, $0 < \delta \le \frac{\varepsilon}{4\mathcal{S}(f)}$, and $M \in \mathbb{N}$, we have*

$$B_\omega(f, N, \varepsilon, M) \subseteq Z_\omega(Q, M, N, \delta) \cup \bigcup_{\substack{A \subseteq \{0,\dots,M-1\} \\ |A| = \lceil \delta M \rceil}} \bigcap_{i \in A} F_i^{\mathcal{R}}(\varepsilon/2),$$

where $\mathcal{R}_i := \mathcal{R}_i(Q)$ for all $i \in \mathbb{N}$ and

$$F_i^{\mathcal{R}}(\varepsilon/2) = F_i(\varepsilon/2) \cap \bigcup_{J \in \mathcal{R}_i} J.$$

Proof. We claim that

$$B_\omega(f, N, \varepsilon, M) \subseteq \bigcup_{\substack{A \subseteq \{0,\dots,M-1\} \\ |A| > 2\delta M}} \bigcap_{i \in A} F_i(\varepsilon/2). \tag{6.4}$$

This follows from

$$\begin{aligned}
&\frac{1}{MN} \int_0^{MN} f(g_t h_s \omega)\, dt \\
&\qquad = \frac{1}{M} \sum_{i=0}^{M-1} f_i(s) = \frac{1}{M} \sum_{\substack{0 \le i \le M-1 \\ f_i(s) \le \nu_{\mathcal{M}}(f) + \varepsilon/2}} f_i(s) + \frac{1}{M} \sum_{\substack{0 \le i \le M-1 \\ f_i(s) > \nu_{\mathcal{M}}(f) + \varepsilon/2}} f_i(s) \\
&\qquad \le \nu_{\mathcal{M}}(f) + \varepsilon/2 + \frac{\|f\|_\infty}{M} \#\{i : f_i(s) > \nu_{\mathcal{M}}(f) + \varepsilon/2\} \\
&\qquad \le \nu_{\mathcal{M}}(f) + \varepsilon/2 + \frac{\mathcal{S}(f)}{M} \#\{i : s \in F_i(\varepsilon/2)\}.
\end{aligned}$$

Thus, if $s \in B_\omega(f, N, \varepsilon, M)$, then we must have

$$\#\{i : s \in F_i(\varepsilon/2)\} > \frac{\varepsilon}{2\mathcal{S}(f)} M \ge 2\delta M$$

by our choice of δ.

By (6.4), to prove the lemma it suffices to show that

$$Z_\omega(Q, M, N, \delta)^c \cap \bigcup_{\substack{A \subseteq \{0,\dots,M-1\} \\ |A| > 2\delta M}} \bigcap_{i \in A} F_i(\varepsilon/2) \subseteq \bigcup_{\substack{A \subseteq \{0,\dots,M-1\} \\ |A| = \lceil \delta M \rceil}} \bigcap_{i \in A} F_i^{\mathcal{R}}(\varepsilon/2),$$

where for a set $E \subseteq [-1, 1]$, E^c denotes its complement.

The set $Z_\omega(Q, M, N, \delta)$ was defined to be the set of directions s such that $g_{iN}h_s\omega \notin Q$ for at least δM natural numbers $i < M$. Hence,

$$Z_\omega(Q, M, N, \delta)^c \subseteq \bigcup_{\substack{B\subseteq\{0,\dots,M-1\}\\ |B|>(1-\delta)M}} \bigcap_{j\in B} \bigcup_{J\in\mathcal{R}_j} J.$$

Indeed, the right hand side describes the set of directions s which belong to $\bigcup_{J\in\mathcal{R}_j} J$ for at least $(1-\delta)M$ natural numbers $j < M$. By definition of $\mathcal{R}_j$, this certainly contains the set of directions s for which $g_{jN}h_s\omega \in Q$ for at least $(1-\delta)M$ natural numbers $j < M$, which is the set on the left hand side.

Notice that

$$\begin{aligned}\Big[\bigcup_{\substack{A\subseteq\{0,\dots,M-1\}\\ |A|>2\delta M}} \bigcap_{i\in A} F_i(\varepsilon/2)\Big] \cap \Big[\bigcup_{\substack{B\subseteq\{0,\dots,M-1\}\\ |B|>(1-\delta)M}} \bigcap_{j\in B} \bigcup_{J\in\mathcal{R}_j} J\Big] &\subseteq \bigcup_{\substack{A,B\subseteq\{0,\dots,M-1\}\\ |A|>2\delta M\\ |B|>(1-\delta)M}} \Big[\bigcap_{i\in A} F_i(\varepsilon/2) \cap \bigcap_{j\in B} \bigcup_{J\in\mathcal{R}_j} J\Big]\\ &\subseteq \bigcup_{\substack{A,B\subseteq\{0,\dots,M-1\}\\ |A|>2\delta M\\ |B|>(1-\delta)M}} \bigcap_{i\in A\cap B} F_i^{\mathcal{R}}(\varepsilon/2) \subseteq \bigcup_{\substack{A\subseteq\{0,\dots,M-1\}\\ |A|>\delta M}} \bigcap_{i\in A} F_i^{\mathcal{R}}(\varepsilon/2),\end{aligned}$$

where for the last inclusion we use the fact that for $A, B \subseteq \{0, \dots, M-1\}$ with $|A| > 2\delta M$ and $|B| > (1-\delta)M$, we have $|A \cap B| > \delta M$. Moreover,

$$\bigcup_{\substack{A\subseteq\{0,\dots,M-1\}\\ |A|>\delta M}} \bigcap_{i\in A} F_i^{\mathcal{R}}(\varepsilon/2) \subseteq \bigcup_{\substack{A\subseteq\{0,\dots,M-1\}\\ |A|=\lceil\delta M\rceil}} \bigcap_{i\in A} F_i^{\mathcal{R}}(\varepsilon/2).$$

This completes the proof. □

6.2. *Measure bounds for F_i*

The next lemma allows us to control the measure of the proportion of a set F_i in an element of the partition $\mathcal{R}_i(Q)$ for a suitably chosen large compact set with good properties. This will be a direct application of Theorem 5.1.

Let $\mathcal{N}_1, \dots, \mathcal{N}_k$ be proper affine invariant submanifolds as in Theorem 5.1 applied to ε and f. By [EMM15, Proposition 2.13], for any $i = 1, \dots, k$ there exist height functions $f_{\mathcal{N}_i}$ such that for all $\ell > 0$, the sets

$$C_\ell = \overline{\Big\{x \in \mathcal{H}_1(\alpha) : \sum_{i=1}^{k} f_{\mathcal{N}_i}(x) \le \ell\Big\}}$$

are compact. The following is the main result of this section; it is the form of Theorem 5.1 we will use. Recall the definition of the sets $F_i(\beta)$ in (6.3).

Lemma 6.2. *For all $\ell > 0$ and all $a > 0$, there exists $T_0 > 0$ such that for all $N > T_0$, $\beta > \varepsilon$, $i \in \mathbb{N}$, $\omega \in \mathcal{H}_1(\alpha)$ and $J \in \mathcal{R}_i(C_\ell)$, we have*

$$\frac{\nu(J \cap F_i(\beta))}{\nu(J)} \leq a,$$

where ν is the Lebesgue probability measure on $[-1, 1]$.

Proof. Denote by B_1 the ball of radius 1 around the identity in $\mathrm{SL}_2(\mathbb{R})$. Fix $\ell > 0$ and $a > 0$. Let $\ell' > \ell$ be such that

$$B_1 C_\ell \subseteq C_{\ell'}.$$

By Theorem 5.1 applied to f, ε, a and the compact set $F = C_{\ell'} \subset \mathcal{M} \setminus \bigcup_{i=1}^{k} \mathcal{N}_i$, where $\mathcal{N}_i$ are given by that theorem, there exists T_0 such that for all $N > T_0$ and $x \in F$,

$$\left| \left\{ s \in [-1, 1] : \left| \frac{1}{N} \int_0^N f(g_t h_s x)\, dt - \nu_{\mathcal{M}}(f) \right| \geq \varepsilon \right\} \right| < a. \tag{6.5}$$

For any $i \in \mathbb{N}$, we define $\mathcal{R}_i := \mathcal{R}_i(C_\ell)$. Fix $J \in \mathcal{R}_i$. Let $s_0 \in J$ be such that $g_{iN} h_{s_0} \omega \in C_\ell$. By our choice of ℓ', for any $s \in J$,

$$g_{iN} h_s \omega = h_{e^{2iN}(s-s_0)} g_{iN} h_{s_0} \omega \in B_1 C_\ell \subseteq C_{\ell'}.$$

In particular, the above holds for the center c_0 of the interval J. Let $s \in J - c_0$ be such that $s + c_0 \in F_i(\beta)$. Then

$$\nu_M(f) + \beta < \frac{1}{N} \int_0^N f(g_{t+iN} h_{s+c_0} \omega)\, dt = \frac{1}{N} \int_0^N f(g_t h_{e^{2iN}s} g_{iN} h_{c_0} \omega)\, dt.$$

Thus,

$$e^{2iN}\big((J \cap F_i(\beta)) - c_0\big) \subseteq \left\{ s \in [-1, 1] : \left| \frac{1}{N} \int_0^N f(g_t h_s g_{iN} h_{c_0} \omega)\, dt - \nu_{\mathcal{M}}(f) \right| \geq \beta \right\}.$$

Since $g_{iN} h_{c_0} \omega \in C_{\ell'}$, the lemma follows from (6.5). □

The following corollary is an immediate consequence of Lemma 6.2 and the fact that the elements of $\mathcal{R}_i$ are pairwise disjoint.

Corollary 6.3. *For all $\ell > 0$ and all $a > 0$, there exists $T_0 > 0$ such that for all $N > T_0$, $\beta > \varepsilon$ and $i \in \mathbb{N}$,*

$$\nu\Big(F_i(\beta) \cap \bigcup_{J \in \mathcal{R}_i(C_\ell)} J\Big) \leq a.$$

6.3. Independence of the sets F_i

The goal of this section is to prove that the sets $F_i(\beta)$ behave as if they were independent. More precisely, we will prove that the measure of the intersection of such sets is bounded above by the product of their measures, up to a controlled error. Recall the definition of partitions $\mathcal{P}_i$ in Section 6.1.

We start with the following simple but key observation.

Lemma 6.4. *Suppose $i < j$ are natural numbers, and $\beta > 0$. Let $J \in \mathcal{P}_j$ be such that $J \cap F_i(\beta) \neq \emptyset$. Then $J \subseteq F_i\big(\beta - \frac{\mathcal{S}(f)}{N} e^{2(i+1-j)N}\big)$.*

Proof. Let $s \in J \cap F_i(\beta)$. Then, $|s - \eta| \leq e^{-2jN}$ for any $\eta \in J$. Since f is Lipschitz, we deduce that for all $t \in [iN, (i+1)N]$,

$$|f(g_t h_\eta \omega) - f(g_t h_s \omega)| \leq \|f\|_{\mathrm{Lip}}\, d(h_{e^{2t}(s-\eta)}, \mathrm{id}) \leq \mathcal{S}(f) e^{2(t-j)N},$$

where $d(g, h)$ is the metric on $\mathrm{SL}_2(\mathbb{R})$ defined by the maximum absolute value of the entries of the matrix $gh^{-1} - \mathrm{id}$. Averaging the above inequality in t, we get

$$|f_i(\eta) - f_i(s)| \leq \frac{\mathcal{S}(f) e^{-2jN}}{N} \int_{iN}^{(i+1)N} e^{2t}\, dt \leq \frac{\mathcal{S}(f)}{N} e^{2(i+1-j)N},$$

which implies the lemma. □

The following lemma is the main result of this section; the notation is as in Lemma 6.2.

Lemma 6.5 (Independence lemma). *Let $\varepsilon > 0$. Then, for all $\ell > 0$ and all $a > 0$, there exists $T_0 > 0$ such that for all $\omega \in \mathcal{H}_1(\alpha)$, $N > T_0$ with $e^N \in \mathbb{N}$, $\beta > \varepsilon + \mathcal{S}(f)/N$ and finite sets $A \subset \mathbb{N}$, we have*

$$\nu\Big(\bigcap_{i \in A}\Big(F_i(\beta) \cap \bigcup_{J \in \mathcal{R}_i(C_\ell)} J\Big)\Big) \leq a^{|A|},$$

where $|A|$ is the number of elements in A.

Proof. Fix some ℓ and a. Let $T_0 > 0$ be as in Lemma 6.2. Suppose $N > T_0$ and $e^N \in \mathbb{N}$, $A \subset \mathbb{N}$ with $p = |A|$, and $\omega \in \mathcal{H}_1(\alpha)$. Assume that $A = \{1, \ldots, p\}$; we later explain that this assumption causes no loss in generality.

For any $\beta > \varepsilon + \mathcal{S}(f)/N$ and $i \in \mathbb{N}$, we recall that

$$F_i^{\mathcal{R}}(\beta) := F_i(\beta) \cap \bigcup_{J \in \mathcal{R}_i} J.$$

Here, we use $\mathcal{R}_i$ to denote $\mathcal{R}_i(C_\ell)$.

We proceed by induction on p. Since the elements of $\mathcal{R}_p$ are pairwise disjoint, we have

$$\begin{aligned}\nu\Big(\bigcap_{i \in A} F_i^{\mathcal{R}}(\beta)\Big) &= \nu\Big(\bigcup_{J \in \mathcal{R}_p}\Big(J \cap F_p(\beta) \cap \bigcap_{i=1}^{p-1} F_i^{\mathcal{R}}(\beta)\Big)\Big) \\ &= \sum_{J \in \mathcal{R}_p} \nu\Big(J \cap F_p(\beta) \cap \bigcap_{i=1}^{p-1} F_i^{\mathcal{R}}(\beta)\Big).\end{aligned}$$

Moreover,

$$\bigcap_{i=1}^{p-1} F_i^{\mathcal{R}}(\beta) \subseteq \bigcap_{i=1}^{p-1} F_i^{\mathcal{R}}\left(\beta - \frac{\mathcal{S}(f)}{N} e^{2(i+1-p)N}\right). \tag{6.6}$$

Let $J \in \mathcal{R}_p$ be such that $J \cap \bigcap_{i=1}^{p-1} F_i^{\mathcal{R}}(\beta) \neq \emptyset$. Then $J \cap \bigcap_{i=1}^{p-1} F_i(\beta) \neq \emptyset$ and for any $i = 1, \ldots, p-1$ there exists $J' \in \mathcal{R}_i$ such that $J \cap J' \neq \emptyset$. Hence, by Lemma 6.4,

$$J \subseteq \bigcap_{i=1}^{p-1} F_i\left(\beta - \frac{\mathcal{S}(f)}{N} e^{2(i+1-p)N}\right).$$

Since $e^N \in \mathbb{N}$, the partition $\mathcal{P}_j$ is a refinement of $\mathcal{P}_i$ for $i \leq j$. Hence,

$$J \subseteq \bigcap_{i=1}^{p-1} \bigcup_{J' \in \mathcal{R}_i} J'.$$

In particular, we obtain the following base step in our inductive procedure:

$$J \subseteq \bigcap_{i=1}^{p-1} F_i^{\mathcal{R}}\left(\beta - \frac{\mathcal{S}(f)}{N} e^{2(i+1-p)N}\right) \tag{6.7}$$

for all $J \in \mathcal{P}_p$ satisfying $J \cap \bigcap_{i=1}^{p-1} F_i^{\mathcal{R}}(\beta) \neq \emptyset$. Therefore,

$$\begin{aligned}
\nu\left(\bigcap_{i\in A} F_i^{\mathcal{R}}(\beta)\right) &\leq \sum_{\substack{J\in\mathcal{R}_p \\ J\cap\bigcap_{i=1}^{p-1} F_i^{\mathcal{R}}(\beta)\neq\emptyset}} \nu(J \cap F_p(\beta)) \\
&\leq a \sum_{\substack{J\in\mathcal{R}_p \\ J\cap\bigcap_{i=1}^{p-1} F_i^{\mathcal{R}}(\beta)\neq\emptyset}} \nu(J) && \text{by Lemma 6.2} \\
&\leq a\nu\left(\bigcap_{i=1}^{p-1} F_i^{\mathcal{R}}\left(\beta - \frac{\mathcal{S}(f)}{N} e^{2(i+1-p)N}\right)\right) && \text{by (6.7).}
\end{aligned} \tag{6.8}$$

Our choice of β guarantees that for all $1 \leq k \leq p$,

$$\beta - \frac{\mathcal{S}(f)}{N} \sum_{j=0}^{k-1} e^{(i+1-(p-j))N} > \varepsilon.$$

Note here that our assumption that $A = \{1, \ldots, p\}$ maximizes the sum in the above inequality. Moreover, our choice of β guarantees that the above inequality holds where the sum is taken over any set A of natural numbers of cardinality p. In other words, the bound in (6.8) only gets better if we replace A with any other subset of $\mathbb{N}$ of cardinality p.

Hence, by induction on our base measure estimate in (6.8), via repeated application of Lemma 6.4,

$$\begin{aligned}\nu\Big(\bigcap_{i\in A} F_i^{\mathcal{R}}(\beta)\Big) &\le a\nu\Big(\bigcap_{i=1}^{p-1} F_i^{\mathcal{R}}\Big(\beta - \frac{\mathcal{S}(f)}{N}e^{2(i+1-p)N}\Big)\Big)\\ &\le a^2\nu\Big(\bigcap_{i=1}^{p-2} F_i^{\mathcal{R}}\Big(\beta - \frac{\mathcal{S}(f)}{N}e^{2(i+1-p)N} - \frac{\mathcal{S}(f)}{N}e^{2(i+1-(p-1))N}\Big)\Big)\\ &\le \cdots\\ &\le a^k\nu\Big(\bigcap_{i=1}^{p-k} F_i^{\mathcal{R}}\Big(\beta - \frac{\mathcal{S}(f)}{N}\sum_{j=0}^{k-1} e^{2(i+1-(p-j))N}\Big)\Big) \le a^p\end{aligned}$$

as desired. □

6.4. A covering lemma

As a consequence of Lemma 6.5, we obtain the following bound on the number of intervals needed to cover intersections of the recurrent parts of the sets F_i. More precisely, we obtain the following.

Lemma 6.6. *Let $\varepsilon > 0$. Then, for all $\ell > 0$ and $a > 0$, there exists $T_0 > 0$ such that for all $\omega \in \mathcal{H}_1(\alpha)$, $N > T_0$ with $e^N \in \mathbb{N}$, $\beta > \varepsilon + 2\mathcal{S}(f)/N$, $M \in \mathbb{N}$, and finite sets $A \subseteq \{0, \dots, M-1\}$, we have*

$$\#\Big\{J \in \mathcal{P}_M : J \cap \bigcap_{i\in A}\Big(F_i(\beta) \cap \bigcup_{J'\in\mathcal{R}_i(C_\ell)} J'\Big) \neq \emptyset\Big\} \le e^{2MN}a^{|A|}.$$

Proof. Fix ℓ and a. Let $T_0 > 0$ be as in Lemma 6.5, $N > T_0$, $M \in \mathbb{N}$ and $\beta > \varepsilon + 2\mathcal{S}(f)/N$. Suppose $A \subset \{0, \dots, M-1\}$. For each $i \in \mathbb{N}$, let

$$F_i^{\mathcal{R}}(\beta) := F_i(\beta) \cap \bigcup_{J\in\mathcal{R}_i} J.$$

As in the proof of Lemma 6.5, a combination of Lemma 6.4 and the fact that the partitions $\mathcal{P}_i$ form a refining sequence of partitions (by our assumption that $e^N \in \mathbb{N}$) shows that for all $J \in \mathcal{P}_M$,

$$J \cap \bigcap_{i\in A} F_i^{\mathcal{R}}(\beta) \neq \emptyset \implies J \subseteq \bigcap_{i\in A} F_i^{\mathcal{R}}\Big(\beta - \frac{\mathcal{S}(f)}{N}e^{2(i+1-M)N}\Big).$$

In particular, for any $J \in \mathcal{P}_M$ satisfying $J \cap \bigcap_{i\in A} F_i^{\mathcal{R}}(\beta) \neq \emptyset$, one has

$$J \subseteq \bigcap_{i\in A} F_i^{\mathcal{R}}\Big(\beta - \frac{\mathcal{S}(f)}{N}\Big).$$

Therefore, by our condition on β and Lemma 6.5, we get

$$\sum_{\substack{J\in\mathcal{P}_M \\ J\cap\bigcap_{i\in A} F_i^{\mathcal{R}}(\varepsilon/2)\neq\emptyset}} \nu(J) \le \nu\Big(\bigcap_{i\in A} F_i^{\mathcal{R}}(\beta - \mathcal{S}(f)/N)\Big) \le a^{|A|}. \tag{6.9}$$

Recall that $\mathcal{P}_M$ is a partition of $[-1, 1]$ into intervals of radius e^{-2MN}. In particular, for $J \in \mathcal{P}_M$, $\nu(J) = e^{-2MN}$. Combined with (6.9), this implies the lemma. □

6.5. *Proof of Theorem 2.1*

Let us fix the following parameters so that we can apply Lemmas 6.5 and 6.6. Fix $\varepsilon > 0$. Let $\delta, a > 0$ be sufficiently small that

$$\delta \le \frac{\varepsilon}{4\mathcal{S}(f)} \quad \text{and} \quad a^\delta < \frac{1}{2}. \tag{6.10}$$

Let $\mathcal{N}_1, \dots, \mathcal{N}_k$ be proper affine invariant submanifolds as in Theorem 5.1 applied to $\varepsilon/50$ and f. By [EMM15, Proposition 2.13], for any $i = 1, \dots, k$ there exists a height function $f_{\mathcal{N}_i}$. For $\ell > 0$, let

$$C_\ell = \overline{\Big\{x \in \mathcal{H}_1(\alpha) : \sum_{i=1}^k f_{\mathcal{N}_i}(x) \le \ell\Big\}}.$$

The function $\alpha = \sum_{i=1}^k f_{\mathcal{N}_i}$ satisfies all the properties in Definition 3.1 (see [EMM15, Proposition 2.13]). Suppose $\omega \in \mathcal{M} \setminus \bigcup_{i=1}^k \mathcal{N}_i$. Thus, $\alpha(\omega) < \infty$. In particular, Theorem 3.2 applies and guarantees the existence of some $\ell = \ell(\delta)$ and $t_0 > 0$ such that for all $t > t_0$ with $e^t \in \mathbb{N}$, one has

$$\dim_H(Z_\omega(C_\ell, t, \delta)) < 1, \tag{6.11}$$

where $Z_\omega(C_\ell, t, \delta)$ was defined in (6.1), and the bound is uniform over all $\omega \in \mathcal{M} \setminus \bigcup_{i=1}^k \mathcal{N}_i$.

Let $\ell > 0$ be such that (6.11) holds. Let $T_0 > 0$ be as in Lemma 6.6 applied to f and $\varepsilon/50$. Let $N > \max\{T_0, t_0\}$. In the course of the proof, we will enlarge N if necessary, depending only on ε, a and f. We will also always assume that

$$e^N \in \mathbb{N}.$$

Fix some $\omega \in \mathcal{M} \setminus \bigcup_{i=1}^k \mathcal{N}_i$. Recall the definition of the sets F_i (see (6.3)), partitions $\mathcal{P}_i$ and $\mathcal{R}_i := \mathcal{R}_i(C_\ell)$ (see Section 6.1). Note that the $\mathcal{P}_i$ form a refining sequence of partitions. For each $i \in \mathbb{N}$ and $\beta > 0$, define

$$F_i^{\mathcal{R}}(\beta) = F_i(\beta) \cap \bigcup_{J\in\mathcal{R}_i} J.$$

By Lemma 6.1, we get

$$B_\omega(f, N, \varepsilon) \subseteq Z_\omega(C_\ell, N, \delta) \cup \limsup_{M\to\infty} \bigcup_{\substack{A\subseteq\{0,\dots,M-1\}\\ |A|=\lceil \delta M\rceil}} \bigcap_{i\in A} F_i^{\mathcal{R}}(\varepsilon/2). \tag{6.12}$$

Thus, by (6.11), it suffices to bound the Hausdorff dimension of the second set on the right hand side. Let $M \in \mathbb{N}$ and define

$$\mathcal{F}_M^{\mathcal{R}} = \bigcup_{\substack{A\subseteq\{0,\dots,M-1\}\\ |A|=\lceil \delta M\rceil}} \bigcap_{i\in A} F_i^{\mathcal{R}}(\varepsilon/2).$$

The number of sets of the form A in the above union is at most $\binom{M}{\lceil \delta M\rceil}$. Moreover, we may assume N is large enough that

$$\varepsilon/2 > \varepsilon/50 + 2\mathcal{S}(f)/N.$$

Hence, we may apply Lemma 6.6 with $\varepsilon/50$ in place of ε to find that when N is large enough,

$$\begin{aligned}\#\{J \in \mathcal{P}_M : J \cap \mathcal{F}_M^{\mathcal{R}} \neq \emptyset\} &\leq \sum_{\substack{A\subseteq\{0,\dots,M-1\}\\ |A|=\lceil \delta M\rceil}} \#\Big\{J \in \mathcal{P}_M : J \cap \bigcap_{i\in A} F_i^{\mathcal{R}}(\varepsilon/2) \neq \emptyset\Big\} \\ &\leq \binom{M}{\lceil \delta M\rceil} e^{2MN} a^{\delta M} \leq 2^M e^{2MN} a^{\delta M}. \end{aligned} \tag{6.13}$$

Let $\beta = \log(2)/(2N)$ and $\gamma = -\frac{1}{2N}\log(a^\delta)$. Then (6.13) can be rewritten as

$$\#\{J \in \mathcal{P}_M : J \cap \mathcal{F}_M^{\mathcal{R}} \neq \emptyset\} \leq e^{2(1+\beta-\gamma)MN}.$$

By Lemma 2.5, the Hausdorff dimension of $\limsup_M \mathcal{F}_M^{\mathcal{R}}$ is at most $1+\beta-\gamma$. This bound is strictly less than 1 if and only if $a^\delta < 1/2$, which holds by our choice of a in (6.10). Finally, note that our upper bound depends only on f and ε and is uniform in the choice of ω in $\mathcal{M} \setminus \bigcup_{i=1}^k \mathcal{N}_i$. This completes the proof.

6.6. *Deviations of discrete Birkhoff averages*

The same methods as those used to prove Theorem 2.1 also imply the following analogous statement for discrete Birkhoff averages.

Theorem 6.7. *Suppose $\mathcal{M} \subseteq \mathcal{H}_1(\alpha)$ is an affine invariant submanifold and $\nu_{\mathcal{M}}$ is the affine measure whose support is $\mathcal{M}$. Then, for any bounded continuous function f on $\mathcal{M}$ and any $\varepsilon > 0$, there exist affine invariant submanifolds $\mathcal{N}_1, \dots, \mathcal{N}_k$, properly contained in $\mathcal{M}$, and $\delta \in (0, 1)$ such that for all $\omega \in \mathcal{M} \setminus \bigcup_{i=1}^k \mathcal{N}_i$ and all $l > 0$, the Hausdorff dimension of the set*

$$\left\{s \in [-1, 1] : \limsup_{N\to\infty} \left|\frac{1}{N}\sum_{n=1}^{N} f(g_{ln}h_s\omega) - \int_{\mathcal{M}} f \, d\nu_{\mathcal{M}}\right| \geq \varepsilon\right\}$$

is at most δ.

We note that by modifying the definition of the functions f_i in (6.2) to be

$$f_i(s) = \frac{1}{N} \sum_{k=iN}^{(i+1)N} f(g_{lk} h_s \omega)$$

the rest of the proof of Theorem 6.7 follows verbatim as in the case of flows and therefore we omit it.

7. Random walks and Oseledets' theorem

In this section, we recall some results on the growth of the Kontsevich–Zorich cocycle along random walk trajectories on $\mathcal{H}_1(\alpha)$ which were proved in [CE15]. Using the fact that a typical random walk trajectory is tracked by a geodesic up to a sublinear error, we translate such results to results concerning the Teichmüller geodesic flow.

Suppose $(M, \omega) \in \mathcal{H}_1(\alpha)$ and $\nu_{\mathcal{M}}$ is the affine measure whose support is $\mathcal{M} = \overline{\mathrm{SL}_2(\mathbb{R})\omega}$. Let V be a continuous $\mathrm{SL}_2(\mathbb{R})$-invariant subbundle over $\mathcal{H}_1(\alpha)$ of (an exterior power of) the Hodge bundle. Denote by $A_V : \mathrm{SL}_2(\mathbb{R}) \times \mathcal{M} \to \mathrm{GL}(V)$ the restriction of the Kontsevich–Zorich cocycle to V. Let $\|A_V(\cdot, \cdot)\|$ be the Hodge norm on V (see [FM14, Section 3.4]).

Denote by λ_V the top Lyapunov exponent of this cocycle under the Teichmüller geodesic flow with respect to $\nu_{\mathcal{M}}$. In particular, by Oseledets' multiplicative ergodic theorem, for $\nu_{\mathcal{M}}$-almost every $x \in \mathcal{M}$,

$$\lim_{t \to \infty} \frac{\log \|A_V(g_t, x)\|}{t} = \lambda_V.$$

The cocycle A_V satisfies the following (Lipschitz) property with respect to the Hodge norm: there exists a constant $K \in \mathbb{N}$ such that for all $x \in \mathcal{M}$ and all $g \in \mathrm{SL}_2(\mathbb{R})$,

$$\|A_V(g, x)\| \leq \|g\|^K, \tag{7.1}$$

where for $g \in \mathrm{SL}_2(\mathbb{R})$, we use $\|g\|$ to denote the norm of g in its standard action on $\mathbb{R}^2$. This follows from [For02, Lemma 2.1′] (see also [FM14, Corollary 30]). We note that the power K appears since we are considering the action of an exterior power of the cocycle. Moreover, Forni's variational formula for the derivative of the cocycle along geodesics implies (7.1) for general elements of $\mathrm{SL}_2(\mathbb{R})$ by the KAK decomposition, the cocycle property and the fact that $\|A(r_\theta, \cdot)\| = 1$ for all θ.

Since $A_V(\mathrm{id}, x) = \mathrm{id}$ for all x, we see that $A_V(g, x)^{-1} = A_V(g^{-1}, gx)$ for all $g \in \mathrm{SL}_2(\mathbb{R})$ and $x \in \mathcal{M}$. Hence, by (7.1), we get

$$\|A_V(g, x)^{-1}\| \leq \|g^{-1}\|^K. \tag{7.2}$$

We shall need the following facts about matrix norms which follow from the KAK decomposition and the bi-invariance of $\|\cdot\|$ under K.

Lemma 7.1. *There exists a constant $C_1 > 0$ such that for all $g \in \mathrm{SL}_2(\mathbb{R})$,*

(1) $\log \|g\| \leq C_1 d(g, \mathrm{id})$,
(2) $\|g^{-1}\| = \|g\|$,

where d denotes the right invariant metric on $\mathrm{SL}_2(\mathbb{R})$ and id is the identity element.

7.1. Random walks

In the remainder of this section and in the next section, we fix a compactly supported probability measure μ on $\mathrm{SL}_2(\mathbb{R})$ which is $\mathrm{SO}(2)$-bi-invariant and absolutely continuous with respect to the Haar measure. Let $\mathrm{SL}_2(\mathbb{R})^{\mathbb{N}}$ be the space of infinite sequences of elements in $\mathrm{SL}_2(\mathbb{R})$ equipped with the probability measure $\mu^{\mathbb{N}}$. For each n define the random variable $\omega_n : \mathrm{SL}_2(\mathbb{R})^{\mathbb{N}} \to \mathrm{SL}_2(\mathbb{R})$ as

$$(g_1, g_2, \ldots, g_n, \ldots) \mapsto \omega_n = g_n g_{n-1} \cdots g_2 g_1.$$

For any fixed basepoint $x \in \mathcal{H}_1(\alpha)$, the orbit $\{\omega_n x\}_{n\in\mathbb{N}}$ in $\mathcal{H}_1(\alpha)$ is called a *random walk* on $\mathcal{H}_1(\alpha)$.

A measure ν on $\mathcal{H}_1(\alpha)$ is called *μ-stationary* if $\mu * \nu = \nu$ where

$$\mu * \nu = \int_{\mathrm{SL}_2(\mathbb{R})} (g_*\nu)\, d\mu(g).$$

The measure $\nu_{\mathcal{M}}$ is an ergodic μ-stationary measure, i.e., it cannot be written as a non-trivial convex combination of other μ-stationary measures. By a variant of Oseledets' theorem, due to [GM89] in the setting of random walks, there exists $\lambda_V^\mu \in \mathbb{R}$ such that for $\nu_{\mathcal{M}}$-almost every x and for $\mu^{\mathbb{N}}$-almost every $(g_1, g_2, \ldots) \in \mathrm{SL}_2(\mathbb{R})^{\mathbb{N}}$,

$$\lim_{n\to\infty} \frac{\log \|A_V(g_n g_{n-1} \cdots g_1, x)\|}{n} = \lambda_V^\mu.$$

The following sets were introduced in [CE15] as a way to quantify uniformity in the above limit.

The sets $E_{\mathrm{good}}(\varepsilon, L)$

Let $\varepsilon > 0$ and $L \in \mathbb{N}$. Denote by $E_{\mathrm{good}}(\varepsilon, L)$ the set of points $y \in \mathcal{M}$ such that for all $v \in V$, there exists a set $H(v) \subseteq \mathrm{SL}_2(\mathbb{R})^L$ such that

(1) $\mu^L(H(v) > 1 - \varepsilon$,
(2) for all $(g_1, \ldots, g_L) \in H(v)$,

$$\lambda_V^\mu - \varepsilon < \frac{\log \|A_V(g_L \cdots g_1, y)v\|}{L} \le \frac{\log \|A_V(g_L \cdots g_1, y)\|}{L} < \lambda_V^\mu + \varepsilon.$$

The following lemma is an important part of our proof as it is a key step in the proof of the Oseledets part of [CE15].

Lemma 7.2 ([CE15, Lemma 2.11]). *For any fixed $\varepsilon > 0$, the sets $E_{\mathrm{good}}(\varepsilon, L)$ are open and*

$$\lim_{L\to\infty} \nu_M(E_{\mathrm{good}}(\varepsilon, L)) = 1.$$

7.2. From random walks to flows

Since we will be concerned with metric properties of the exceptional set, it will be important for us to translate random walk results into the language of Teichmüller geodesics. It is a classical fact that a generic random walk trajectory, sampled according to a probability measure on $\mathrm{SL}_2(\mathbb{R})$, tracks a Teichmüller geodesic (up to a sublinear error). This is made precise in the following:

Lemma 7.3 ([CE15, Lemma 4.1]). *There exists $\lambda > 0$, depending only on μ, such that there exists a measurable map $\Theta : \mathrm{SL}_2(\mathbb{R})^{\mathbb{N}} \to [-\pi/2, \pi/2]$, defined $\mu^{\mathbb{N}}$-almost everywhere, such that for $\mu^{\mathbb{N}}$-a.e. $\overline{g} = (g_1, g_2, \dots) \in \mathrm{SL}_2(\mathbb{R})^{\mathbb{N}}$,*

$$\lim_{n\to\infty} \frac{\log \|g_{\lambda n} r_{\Theta(\overline{g})} (g_n \cdots g_1)^{-1}\|}{n} = 0. \tag{7.3}$$

Furthermore, $\Theta_* \mu^{\mathbb{N}}$ coincides with the normalized Lebesgue measure. In particular, for any interval $[a, b] \subseteq [-\pi/2, \pi/2]$,

$$\mu^{\mathbb{N}} (\overline{g} : \Theta(\overline{g}) \in [a, b]) = \frac{b-a}{\pi}. \tag{7.4}$$

Remark 7.4. The relationship between the Lyapunov exponent λ_V^μ of the random walk and the Lyapunov exponent λ_V of the Kontsevich–Zorich cocycle under the Teichmüller flow is provided by the parameter λ in Lemma 7.3:

$$\lambda_V = \lambda_V^\mu / \lambda.$$

The following lemma uses Lemma 7.3 to show that geodesic trajectories which start within the sets $E_{\mathrm{good}}(\varepsilon, L)$ also exhibit good properties with respect to the cocycle.

For simplicity, throughout this section we write $A := A_V$.

Lemma 7.5. *There exists a constant $C > 0$, depending only on the constants of the cocycle, such that the following holds: for every $\varepsilon > 0$, there exists $L_0 > 0$ such that for all $L \in \mathbb{N}$ with $L \geq L_0$, all $y \in E_{\mathrm{good}}(\varepsilon, L)$ and all $v \in V$, there exists $\tilde{H}(v) \subseteq [-\pi/2, \pi/2]$ such that for all $\theta \in \tilde{H}(v)$,*

$$\lambda_V - C\varepsilon < \frac{\log \|A(g_{\lambda L}, r_\theta y)\|}{\lambda L} < \lambda_V + C\varepsilon$$

and $\nu(\tilde{H}(v)) > 1 - 3\varepsilon$, where ν is the normalized Lebesgue measure on $[-\pi/2, \pi/2]$.

Proof. Let λ be as in Lemma 7.3. Using Egorov's theorem, we can find a set $\mathcal{U} \subseteq \mathrm{SL}_2(\mathbb{R})^{\mathbb{N}}$ with $\mu^{\mathbb{N}}(\mathcal{U}) > 1 - \varepsilon$ such that the convergence in (7.3) is uniform over $\mathcal{U}$. In particular, we can choose $L \in \mathbb{N}$ so large that for all $\overline{g} \in \mathcal{U}$,

$$\frac{\log \|g_{\lambda L} r_{\Theta(\overline{g})} (g_L \cdots g_1)^{-1}\|}{L} < \varepsilon. \tag{7.5}$$

Fix $y \in E_{\mathrm{good}}(\varepsilon, L)$ and $v \in V$. Let $H(v) \subseteq \mathrm{SL}_2(\mathbb{R})^L$ be as in the definition of $E_{\mathrm{good}}(\varepsilon, L)$. We will regard $H(v)$ as a cylinder subset of $\mathrm{SL}_2(\mathbb{R})^{\mathbb{N}}$ in the natural way.

The set $\tilde{H}(v)$ will be essentially the image of $H(v) \cap \mathcal{U}$ under Θ, except that Θ is only a measurable map. To go around this, we use Luzin's theorem to find a compact set $\mathcal{K} \subset \mathrm{SL}_2(\mathbb{R})^{\mathbb{N}}$ such that $\mu^{\mathbb{N}}(\mathcal{K}) > 1 - \varepsilon$ and the restriction of Θ to $\mathcal{K}$ is continuous. Let

$$\tilde{H}(v) = \Theta(H(v) \cap \mathcal{U} \cap \mathcal{K}).$$

Since Θ is continuous on $\mathcal{K}$ and $\tilde{H}(v)$ is a Borel subset of $\mathcal{K}$, we see that $\tilde{H}(v)$ is Lebesgue measurable. Moreover, by Lemma 7.3,

$$\nu(\tilde{H}(v)) = \mu^{\mathbb{N}}(\Theta^{-1}(\tilde{H}(v))) \geq \mu^{\mathbb{N}}(H(v) \cap \mathcal{U} \cap \mathcal{K}) > 1 - 3\varepsilon.$$

To see that $\tilde{H}(v)$ satisfies the conclusion of the lemma, let $\overline{g} \in H(v) \cap \mathcal{U} \cap \mathcal{K}$. For all L sufficiently large that (7.5) holds for all $\overline{g} \in \mathcal{U}$, define $\varepsilon_L \in \mathrm{SL}_2(\mathbb{R})$ by

$$g_{\lambda L} r_{\Theta(\overline{g})} = \varepsilon_L g_L \cdots g_1$$

with $\varepsilon_L \in \mathrm{SL}_2(\mathbb{R})$. Then, using the cocycle property, we get

$$A(g_{\lambda L}, r_{\Theta(\overline{g})}\omega) = A(\varepsilon_L, g_L \cdots g_1 \omega) A(g_L \cdots g_1, \omega).$$

Hence, since $\overline{g} \in H(v)$, by definition of the set $E_{\mathrm{good}}(\varepsilon, L)$ and by (7.1), we get

$$\begin{aligned} \frac{\log \|A(g_{\lambda L}, r_{\Theta(\overline{g})}\omega)\|}{L} &\leq \frac{\log \|A(\varepsilon_L, g_L \cdots g_1 \omega)\|}{L} + \frac{\log \|A(g_L \cdots g_1, \omega)\|}{L} \\ &\leq \frac{K \log \|\varepsilon_L\|}{L} + \lambda_V^{\mu} + \varepsilon \leq \lambda_V^{\mu} + \varepsilon(1 + K). \end{aligned}$$

Similarly, using (7.1) and Lemma 7.1(2), we get

$$\begin{aligned} \frac{\log \|A(g_{\lambda L}, r_{\Theta(\overline{g})}\omega)\|}{L} &\geq \frac{\log \|A(g_L \cdots g_1, \omega)\|}{L} - \frac{\log \|A(\varepsilon_L, g_L \cdots g_1 \omega)^{-1}\|}{L} \\ &\geq \lambda_V^{\mu} - \varepsilon - \frac{K \log \|\varepsilon_L^{-1}\|}{L} = \lambda_V^{\mu} - \varepsilon - \frac{K \log \|\varepsilon_L\|}{L} \\ &\geq \lambda_V^{\mu} - \varepsilon(1 + K). \end{aligned}$$

Dividing both estimates by λ and noting that by Remark 7.4, $\lambda_V = \lambda_V^{\mu}/\lambda$, we get the desired conclusion. □

As a corollary, we obtain the following statement for horocycles.

Corollary 7.6. *There exists a constant $C_2 > 0$, depending only on the constants of the cocycle, such that the following holds: for every $\varepsilon > 0$, there exists $L_0 > 0$ such that for all $L \in \mathbb{N}$ with $L \geq L_0$, all $y \in E_{\mathrm{good}}(\varepsilon, L)$ and all $v \in V$, there exists $G(v) \subseteq [-2, 2]$ such that for all $s \in G(v)$,*

$$\lambda_V - C_2\varepsilon < \frac{\log \|A(g_{\lambda L}, h_s y)\|}{\lambda L} < \lambda_V + C_2\varepsilon, \tag{7.6}$$

and $|G(v)| \geq 4(1 - 30\varepsilon)$, where $|\cdot|$ is the Lebesgue measure on $[-2, 2]$.

Proof. Fix $\varepsilon > 0$. Suppose $L \in \mathbb{N}$ is so large that the conclusion of Lemma 7.5 holds, $y \in E_{\text{good}}(\varepsilon, L)$ and $v \in V$. Let $\tilde{H}(v) \subseteq [-\pi/2, \pi/2]$ and $C > 0$ be as in the conclusion of Lemma 7.5. Consider

$$G(v) = \tan(\tilde{H}(v)) \cap [-2, 2].$$

We verify that this set is as desired. Let

$$\rho = \tan^{-1}(2).$$

For every $\theta \in \tilde{H}(v) \cap [-\rho, \rho]$ we write $r_\theta = \check{h}_{-\tan\theta} g_{\log\cos\theta} h_{\tan\theta}$. Then, by the cocycle property,

$$\begin{aligned} &A(g_{\lambda L}, r_\theta y) \\ &\quad = A(\check{h}_{-e^{-2\lambda L}\tan\theta} g_{\log\cos\theta}, g_{\lambda L} h_{\tan\theta} y) A(g_{\lambda L}, h_{\tan\theta} y) A(\check{h}_{-\tan\theta} g_{\log\cos\theta}, h_{\tan\theta} y)^{-1} \end{aligned}$$

Therefore, using the Lipschitz property (7.1) and (7.2) and the fact that $\theta \in [-\rho, \rho]$, we get

$$\frac{\log \|A(g_{\lambda L}, r_\theta y)\|}{\lambda L} = \frac{\log \|A(g_{\lambda L}, h_{\tan\theta} y)\|}{\lambda L} + O\left(\frac{1}{L}\right).$$

Thus, for L large enough, and for all $s \in [-2, 2]$ of the form $s = \tan\theta$ with $\theta \in \tilde{H}(v) \cap [-\rho, \rho]$, we can find a constant $C' > 0$, independent of L, such that (7.6) holds.

Let $|\cdot|$ be the Lebesgue measure. Since $|\tilde{H}(v)| \geq \pi(1 - 3\varepsilon)$, we get

$$\frac{|\tilde{H}(v) \cap [-\rho, \rho]|}{2\rho} \geq 1 - \frac{3\varepsilon\pi}{2\rho}.$$

Hence, since the Jacobian of the map $\theta \mapsto \tan\theta$ is bounded by $\sec^2(\rho) = 5$ on $[-\rho, \rho]$, we conclude that

$$|G(v)|/4 \geq 1 - 15\varepsilon\pi/4 \geq 1 - 30\varepsilon. \qquad \square$$

8. Large deviations in Oseledets' theorem

The purpose of this section is to prove Theorem 2.2 concerning the Hausdorff dimension of those directions whose geodesics exhibit deviation of the top Laypunov exponent for the Kontsevich–Zorich cocycle. The structure of the proof is very similar to that of Theorem 2.1. The idea is to relate the directions exhibiting deviation in Oseledets' theorem along a Teichmüler geodesic to the directions exhibiting deviation in Birkhoff's theorem for the indicator function of a large open set with good properties with respect to the cocycle. The proof is written so as to mirror the proof of Theorem 2.1 on deviations in Birkhoff's theorem.

Throughout this section we retain the notation from the previous section and also use the following. For any positive $\varepsilon, L \in \mathbb{R}$ and $M \in \mathbb{N}$, we define

$$\begin{aligned} B(A, L, \varepsilon, M) &:= \left\{ s \in [-1, 1] : \frac{\log \|A(g_{LM}, h_s\omega)\|}{LM} > \lambda_V + \varepsilon \right\}, \\ B(A, L, \varepsilon) &:= \limsup_{M \to \infty} B(A, L, \varepsilon, M). \end{aligned}$$

Using the cocycle property, it is easy to check that for any $L > 0$,

$$B(A, L, \varepsilon/2) \supseteq \left\{ s \in [-1, 1] : \limsup_{t\to\infty} \frac{\log \|A(g_t, h_s\omega)\|}{t} \geq \lambda_V + \varepsilon \right\}.$$

Thus, to prove Theorem 2.2, we prove that the sets $B(A, L, \varepsilon)$ have dimension strictly less than 1 for all $\varepsilon > 0$ and for an appropriate choice of L depending on ε.

For every $s \in [-1, 1]$, $\beta, L > 0$ and $i \in \mathbb{N}$,

$$a_i(s) = \frac{\log \|A(g_L, g_{Li}h_s\omega)\|}{L}, \qquad A_i(\beta) = \{\theta : a_i(s) > \lambda_V + \beta\}.$$

The functions a_i and the sets A_i play the role of the functions f_i (see (6.2)) and the sets F_i (see (6.3)), respectively, in the proof of large deviations in Birkhoff's theorem.

8.1. Sets and partitions

For $L > 0$ and $i \in \mathbb{N}$, let $\mathcal{P}_i$ denote the partition of $[-1, 1]$ into intervals of radius e^{-2iL}. By enlarging L if necessary, we may assume that $e^L \in \mathbb{N}$ and $\mathcal{P}_{i+1}$ is a refinement of $\mathcal{P}_i$ for all i. For $\varepsilon > 0$, define the subpartitions

$$\mathcal{E}_i(\varepsilon, L) = \{J \in \mathcal{P}_i : \exists s \in J,\ g_{iL}h_s\omega \in E_{\text{good}}(\varepsilon, L)\}.$$

Here $\mathcal{E}$ signifies recurrence to the set E_{good}.

The following lemma is an analogue of Lemma 6.1.

Lemma 8.1. *Let $\varepsilon_1, \varepsilon_2, L > 0$, $M \in \mathbb{N}$, and $0 < \delta \leq \varepsilon/(4K)$, where $K \in \mathbb{N}$ is the exponent in (7.1). Then*

$$B(A, L, \varepsilon_1, M) \subseteq Z_\omega(E_{\text{good}}(\varepsilon_2, L), M, L, \delta) \cup \bigcup_{\substack{B\subseteq\{1,\dots,M\}\\ |B|=\lceil\delta M\rceil}} \bigcap_{i\in B} \Big(A_i(\varepsilon_1/2) \cap \bigcup_{J\in\mathcal{E}_i(\varepsilon_2,L)} J \Big),$$

where $Z_\omega(E_{\text{good}}(\varepsilon_2, L), M, L, \delta)$ is defined in (3.1).

Proof. First, we notice that for any $\varepsilon > 0$,

$$B(A, L, \varepsilon, M) \subseteq \bigcup_{\substack{B\subseteq\{1,\dots,M\}\\ |B|>2\delta M}} \bigcap_{i\in B} A_i(\varepsilon/2).$$

Using the cocycle property and submultiplicativity of matrix norms, we have the following inequalities

$$\frac{\log \|A(g_{LM}, h_s\omega)\|}{LM} \leq \frac{1}{M} \sum_{i=1}^{M} \frac{\log \|A(g_L, g_{Li}h_s\omega)\|}{L}.$$

From this point on, using (7.1) to bound $\|A(g_L, \cdot)\|$, the proof is identical to that of Lemma 6.1.

□

8.2. Measure bounds for A_i

The goal of this section is to obtain a uniform bound on the measure of sets of the form $A_i \cap J$ for any $J \in \mathcal{E}_i$ and any i. This step is analogous to Lemma 6.2.

The following is the main result of this section. The key input in the proof is Lemma 7.5.

Lemma 8.2. *Let $C_2 > 0$ be as in Corollary 7.6. Then, for every $\varepsilon > 0$, there exists $L_1 > 0$ such that for all $L \geq L_1$, all $\gamma \geq 2C_2\varepsilon$, all $i \in \mathbb{N}$ and all $J \in \mathcal{E}_i(\varepsilon, L)$,*

$$\frac{\nu(J \cap A_i(\gamma))}{\nu(J)} \leq 120\varepsilon,$$

where ν is the Lebesgue probability measure on $[-1, 1]$.

Proof. Let $L_0 > 0$ and $\lambda > 0$ be as in Corollary 7.6 and Lemma 7.3, respectively. Define $L_1 := L_0/\lambda$. Suppose $\gamma \in \mathbb{R}$ and $L \in \mathbb{N}$ are such that $\gamma \geq 2C_2\varepsilon$ and $L \geq L_1$.

Let $i \in \mathbb{N}$, $J \in \mathcal{E}_i := \mathcal{E}_i(\varepsilon, L)$, and $s_0 \in J$ be such that $y_0 := g_{iL}h_{s_0}\omega \in E_{\text{good}}(\varepsilon, L)$. Let $v \in V$ and $G(v) \subseteq [-2, 2]$ be as in Corollary 7.6. Choose $\eta \in J - s_0$ such that $s_0 + \eta \in A_i(\gamma) \cap J$. Then

$$\lambda_V + \gamma \leq a_i(s_0 + \eta) = \frac{\log \|A(g_L, h_{e^{2iL}\eta}y_0)\|}{L}.$$

Hence, by definition of $G(v)$,

$$e^{2iL}\eta \notin G(v). \tag{8.1}$$

Note that $e^{2iL}(J - s_0)$ is a subinterval of $[-2, 2]$ of length 2. In particular,

$$e^{2iL}\left((A_i(\gamma) \cap J) - s_0\right) \subseteq [-2, 2] \setminus G(v).$$

Thus, since the Lebesgue measure of $G(v)$ is at least $4(1 - 30\varepsilon)$, we get the measure estimate

$$|e^{2iL}\left((A_i(\gamma) \cap J) - s_0\right)| = \frac{\nu(J \cap A_i(\gamma))}{\nu(J)} \leq 120\varepsilon.$$

This concludes the proof in the case $L \in \mathbb{N}$. For $L \geq L_1$ with $L \notin \mathbb{N}$, write $L = \lfloor L \rfloor + \{L\}$ where $\lfloor L \rfloor$ is the largest natural number less than L and $\{L\} = L - \lfloor L \rfloor$. Then, using the cocycle property, submultiplicativity of the norm and the Lipschitz property of the cocycle (7.1), we get

$$\frac{\log \|A(g_L, \cdot)\|}{L} \leq \frac{\log \|A(g_{\lfloor L \rfloor}, \cdot)\|}{L} + O\left(\frac{1}{L}\right).$$

Thus, the conclusion follows in this case from the case when $L \in \mathbb{N}$ by choosing L_1 sufficiently large depending on ε. □

8.3. Independence of the sets A_i

As a consequence of the Lipschitz property of the cocycle (7.1), we are able to prove an analogue of Lemma 6.4.

Lemma 8.3. *There exists a constant $C_3 > 0$, depending only on the constants of the cocycle A, such that the following holds. Suppose $i < j$, where i and j are natural numbers, $L > 0$, and $\gamma > 0$. Let $J \in \mathcal{P}_j$ be such that $J \cap A_i(\gamma) \neq \emptyset$. Then $J \subseteq A_i\big(\gamma - C_3 \frac{e^{2(i+1-j)L}}{L}\big)$.*

Proof. Let $s_0 \in J \cap A_i(\gamma)$. Then, by definition of the partition $\mathcal{P}_j$ in Section 8.1, $|s_0 - \eta| \leq e^{-2jL}$ for any $\eta \in J$.

Using the cocycle property, we have

$$\begin{aligned} A(g_L, g_{iL}h_\eta\omega) &= A(g_L, h_{e^{2iL}(\eta-s_0)}g_{iL}h_{s_0}\omega) \\ &= A(g_L h_{e^{2iL}(\eta-s_0)}, g_{iL}h_{s_0}\omega)A(h_{e^{2iL}(\eta-s_0)}, g_{iL}h_{s_0}\omega)^{-1} \\ &= A(h_{e^{2(i+1)L}(\eta-s_0)}, g_{(i+1)L}h_{s_0}\omega)A(g_L, g_{iL}h_{s_0}\omega)A(h_{e^{2iL}(\eta-s_0)}, g_{iL}h_{s_0}\omega)^{-1}. \end{aligned}$$

Therefore,

$$\begin{aligned} &A(g_L, g_{iL}h_{s_0}\omega) \\ &\quad = A(h_{e^{2(i+1)L}(\eta-s_0)}, g_{(i+1)L}h_{s_0}\omega)^{-1}A(g_L, g_{iL}h_\eta\omega)A(h_{e^{2iL}(\eta-s_0)}, g_{iL}h_{s_0}\omega). \end{aligned} \tag{8.2}$$

Hence, by (7.1), (7.2) and Lemma 7.1, there exists a constant C_1 such that

$$\begin{aligned} a_i(s_0) - a_i(\eta) &\leq \frac{K \log \|h^{-1}_{e^{2(i+1)L}(\eta-s_0)}\|}{L} + \frac{K \log \|h_{e^{2iL}(\eta-s_0)}\|}{L} \\ &\leq \frac{KC_1 d(h_{-e^{2(i+1)L}(\eta-s_0)}, \mathrm{id})}{L} + \frac{KC_1 d(h_{e^{2iL}(\eta-s_0)}, \mathrm{id})}{L} \\ &\leq 2KC_1 \frac{e^{2(i+1-j)L}}{L}, \end{aligned}$$

which concludes the proof. □

As a consequence, we obtain exponential decay in the measure of intersections of the sets A_i, similarly to Lemma 6.5.

Lemma 8.4 (Independence lemma for A_i). *Let $C_3 > 0$ be as in Lemma 8.3 and $C_2 > 0$ be as in Corollary 7.6. Then, for all $\varepsilon > 0$, there exists $L_1 > 0$ such that for all $L \geq L_1$ with $e^L \in \mathbb{N}$, all finite sets $B \subset \mathbb{N}$ and all $\gamma > 2C_2\varepsilon + C_3/L$,*

$$\nu\Big(\bigcap_{i\in B}\Big(A_i(\gamma) \cap \bigcup_{J\in\mathcal{E}_i(\varepsilon,L)} J\Big)\Big) \leq (120\varepsilon)^{|B|}.$$

Proof. The proof is identical to the proof of Lemma 6.5 which is a formal consequence of two results: Lemma 6.2 that gives an upper bound on the measure of F_i, and Lemma 6.4. The analogues of those two results are Lemma 8.2 and Lemma 8.3, respectively. □

8.4. A covering lemma

The following lemma shows existence of efficient covers for intersections of the sets A_i, similarly to Lemma 6.6.

Lemma 8.5. *Let $C_3 > 0$ be as in Lemma 8.3 and $C_2 > 0$ be as in Corollary 7.6. Then, for all $\varepsilon > 0$, there exists $L_1 > 0$ such that for all $L \geq L_1$ with $e^L \in \mathbb{N}$, $M \in \mathbb{N}$, sets $B \subseteq \{1, \dots, M\}$ and $\gamma > 2C_2\varepsilon + 2C_3/L$,*

$$\#\Big\{ J \in \mathcal{P}_{M+1} : J \cap \bigcap_{i \in B} \Big(A_i(\gamma) \cap \bigcup_{J' \in \mathcal{E}_i(\varepsilon, L)} J' \Big) \neq \emptyset \Big\} \leq e^{2L(M+1)} (120\varepsilon)^{|B|},$$

where $|B|$ is the number of elements in B.

Proof. The proof is a direct consequence of Proposition 8.4 and proceeds as the proof of Lemma 6.6. □

8.5. Proof of Theorem 2.2

Fix $\varepsilon > 0$. Suppose $\varepsilon' > 0$ is a sufficiently small number (depending only on ε). Define $\delta := \varepsilon/(4K)$, where K is the exponent in (7.1). By Lemma 7.2, choose $L > 0$ large enough, depending on ε', so that $e^L \in \mathbb{N}$ and

$$\nu_{\mathcal{M}}(E_{\text{good}}(\varepsilon', L)) > 1 - \delta/2. \tag{8.3}$$

Let χ_E denote the indicator function of the open set $E_{\text{good}}(\varepsilon', L)$. Then, using a variant of Urysohn's lemma, we can find a Lipschitz compactly supported continuous function $f : \mathcal{M} \to [0, 1]$ satisfying $f \leq \chi_E$ and

$$\nu_{\mathcal{M}}(f) > 1 - 3\delta/4. \tag{8.4}$$

Moreover, for all $M \in \mathbb{N}$ and all $\omega \in \mathcal{M}$,

$$Z_\omega(E_{\text{good}}(\varepsilon', L), M, L, \delta) \subseteq B_\omega(1 - f, L, \delta - \nu_{\mathcal{M}}(1 - f), M)$$

where these sets are defined in (3.1) and (6.1) (for discrete Birkhoff averages).

Note that $\delta - \nu_{\mathcal{M}}(1 - f) > 0$ by (8.4). Thus, by Theorem 6.7, there exist $0 < \eta < 1$ and finitely many proper affine invariant manifolds $\mathcal{N}_1, \dots, \mathcal{N}_k \subset \mathcal{M}$, depending on f and ε, such that

$$\dim_H \Big(\limsup_M Z_\omega(E_{\text{good}}(\varepsilon', L), M, L, \delta) \Big) \leq \eta < 1 \tag{8.5}$$

uniformly for all $\omega \in \mathcal{M} \setminus \bigcup_{i=1}^k \mathcal{N}_i$. These are the affine manifolds appearing in the conclusion of Theorem 2.2. Now, fix one such ω.

Recall the definition of the sets A_i and partitions $\mathcal{P}_i$ in Section 8.1. By our assumption that $e^L \in \mathbb{N}$, the $\mathcal{P}_i$ form a refining sequence of partitions. For $i \in \mathbb{N}$ and $\gamma > 0$, define

$$\mathcal{A}_i^{\mathcal{E}}(\gamma) := A_i(\gamma) \cap \bigcup_{J \in \mathcal{E}_i(\varepsilon', L)} J.$$

Then, by Lemma 8.1, since $\delta = \varepsilon/(4K)$, we get

$$\limsup_M B(A, L, \varepsilon, M) \subseteq \limsup_M Z_\omega(E_{\text{good}}(\varepsilon', L), M, L, \delta) \cup \limsup_M \bigcup_{\substack{B \subseteq \{1,\dots,M\} \\ |B| = \lceil \delta M \rceil}} \bigcap_{i \in B} \mathcal{A}_i^{\mathcal{E}}(\varepsilon/2).$$

Thus, it remains to control the Hausdorff dimension of the second set on the right side. We apply Lemma 8.5 to ε' in place of ε and $\gamma = \varepsilon/2$. By choosing ε' sufficiently small and L sufficiently large, we can ensure that

$$\varepsilon/2 > 2C_2\varepsilon' + 2C_3/L$$

where C_2 and C_3 are constants depending only on the cocycle as in the statement of Lemma 8.5.

As a result, choosing L sufficiently large with $e^L \in \mathbb{N}$, we can apply Lemma 8.5 and proceed as in the proof of Theorem 2.1 to get

$$\dim_H \Big(\limsup_M \bigcup_{\substack{B \subseteq \{1,\dots,M\} \\ |B| = \lceil \delta M \rceil}} \bigcap_{i \in B} \mathcal{A}_i^{\mathcal{E}}(\varepsilon/2) \Big) \leq 1 + \frac{\log 2 + \delta \log(120\varepsilon')}{2L}.$$

By choosing $\varepsilon' < 2^{-1/\delta}/120$ (thus depending only on ε), we find that this upper bound is strictly less than 1. Moreover, the parameters δ, ε', L appearing in the upper bound above are independent of ω. This completes the proof.

9. Weakly mixing IETs

This section is dedicated to the proof of Corollary 1.9. We first recall some definitions and the results of [BN04] which connect weak mixing properties of IETs with the recurrence of Teichmüller geodesics in an appropriate stratum.

Throughout this section, we fix a natural number $d \geq 2$. Given a permutation π on d letters and $\lambda = (\lambda_1, \dots, \lambda_d) \in \mathbb{R}_+^d$, we define $|\lambda| = \lambda_1 + \cdots + \lambda_d$ and an interval exchange transformation (IET) with permutation π to be the piecewise linear map $T_{\lambda,\pi} : [0, |\lambda|) \to [0, |\lambda|)$ defined as follows: first we partition the interval $[0, |\lambda|]$ into d ordered half-open subintervals I_i so that the length of I_i is equal to λ_i and $T_{\lambda,\pi}$ maps I_i linearly onto $I_{\pi(i)}$ for each $1 \leq i \leq d$. More formally, for $1 \leq i \leq d$ and $x \in I_i$,

$$T_{\lambda,\pi}(x) = x + \sum_{\pi(j) < \pi(i)} \lambda_j - \sum_{j < i} \lambda_j.$$

An IET $T_{\lambda,\pi}$ has finitely many points of possible discontinuity:

$$\beta_0 = 0, \quad \beta_i(\lambda) := \sum_{j \le i} \lambda_j, \quad 1 \le i \le d-1. \tag{9.1}$$

We use β_i to denote $\beta_i(\lambda)$ when the dependence on λ is clear from context. Define (see [Vee78] and [MW14, Section 2.2]) an alternating bilinear form on $\mathbb{R}^d \times \mathbb{R}^d$ by its value on the standard basis elements $\mathbf{e}_i$:

$$Q(\mathbf{e}_i, \mathbf{e}_j) = \begin{cases} 1, & i > j,\ \pi(i) < \pi(j), \\ -1, & i < j,\ \pi(i) > \pi(j), \\ 0, & \text{otherwise}, \end{cases} \tag{9.2}$$

for $1 \le i, j \le d$. Then, for each $\lambda \in \mathbb{R}^d_+$, $1 \le i \le d$ and $x \in [\beta_{i-1}, \beta_i)$, we have

$$T_{\lambda,\pi}(x) - x = Q(\lambda, \mathbf{e}_i).$$

The cone $\mathbb{R}^d_+$ can be viewed as the space of IETs with a given permutation π with a natural Euclidean metric and Lebesgue measure. IETs preserve the Lebesgue measure and we shall refer to ergodic properties (ergodicity, weak mixing, etc.) of IETs with respect to it.

9.1. A criterion for weak mixing

A permutation π on d letters $\{1, \dots, d\}$ is *irreducible* if for every $1 \le j < d$,

$$\pi(\{1, \dots, j\}) \ne \{1, \dots, j\}.$$

Definition 9.1. Suppose π is an irreducible permutation on d letters. Define inductively a finite sequence $\{a_p\}_{p=0,1,\dots,l}$ of natural numbers as follows.

Set $a_0 = 1$. If $a_{p-1} \in \{\pi^{-1}(1), d+1\}$, then set $l = p-1$ and stop. Otherwise, define $a_p = \pi^{-1}(\pi(a_{p-1}) - 1) + 1$. The permutation π is of *type* **W** if $a_l = \pi^{-1}(1)$.

Following [BN04], we say that $T_{\lambda,\pi}$ satisfies *IDOC* (the *infinite distinct orbit condition*) if each discontinuity point β_i has an infinite orbit under $T_{\lambda,\pi}$ and for $i \ne j$, the orbits of β_i and β_j are disjoint.

Using the orbits of the points β_i under the IET $T_{\lambda,\pi}$, we define a sequence of partitions of $[0, |\lambda|)$ as follows: for each $n \ge 1$, P_n denotes the partition into subintervals whose endpoints are the successive elements of the sets

$$D_n = \bigcup_{0 \le k \le n-1} T^{-k}_{\lambda,\pi}(\{\beta_0, \dots, \beta_{d-1}\}).$$

For each n, we define $\epsilon_n(T_{\lambda,\pi})$ to be the length of the shortest interval in the partition P_n. The following criterion of weak mixing was proved in [BN04].

Theorem 9.2 ([BN04, Theorem 5.3]). *Suppose π is a type W permutation and $T_{\lambda,\pi}$ is an ergodic IET satisfying IDOC for some $\lambda \in \mathbb{R}^d_+$. If $\limsup_{n\to\infty} n\epsilon_n\left(T_{\lambda,\pi}\right) > 0$, then $T_{\lambda,\pi}$ is weakly mixing.*

Motivated by this criterion, we will say that an IET $T_{\lambda,\pi}$ *has short intervals* if

$$\lim_{n\to\infty} n\epsilon_n(T_{\lambda,\pi}) = 0. \tag{9.3}$$

9.2. A compactness criterion for strata

Suppose $\mathcal{H}$ is a stratum of abelian differentials. We recall here a description of standard compact subsets of $\mathcal{H}$. Given $\omega \in \mathcal{H}$, denote by $\mathcal{L}_\omega$ the set of all of its saddle connections, i.e., the set of all flat geodesic segments joining a pair of singularities of ω. Then we can naturally regard $\mathcal{L}_\omega$ as a subset of vectors in $\mathbb{C}$. Note that $\mathcal{L}_\omega$ is a discrete set. Moreover, using the standard action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{C}$, for any $g \in \mathrm{SL}_2(\mathbb{R})$ the set $\mathcal{L}_{g\omega}$ can be identified with $g \cdot \mathcal{L}_\omega$.

For $v \in \mathbb{C}$, let $\|v\| = \max\{|\mathrm{Re}(v)|, |\mathrm{Im}(v)|\}$. Now, define a function $\ell : \mathcal{H} \to \mathbb{R}_+$ by

$$\ell(\omega) := \min\{\|v\| : v \in \mathcal{L}_\omega\}. \tag{9.4}$$

For any $\varepsilon > 0$, define

$$\mathcal{K}_\varepsilon := \{\omega \in \mathcal{H} : \ell(\omega) \geq \varepsilon\}. \tag{9.5}$$

It is known that the sets $\mathcal{K}_\varepsilon$ with $\varepsilon > 0$ are compact subsets of $\mathcal{H}$ and that any bounded subset of $\mathcal{H}$ is contained in $\mathcal{K}_\varepsilon$ for some ε.

9.3. Lifts of IETs to translation surfaces

In this subsection, we associate a stratum $\mathcal{H}_\pi$ to an irreducible permutation π in such a way that each IET of the form $T_{\lambda,\pi}$ arises as a first-return map of the vertical foliation of appropriate abelian differentials in $\mathcal{H}_\pi$.

Fix an irreducible permutation π on d letters and let $\lambda \in \mathbb{R}_+^d$ be given. Following [Mas82, Section 3], we associate an abelian differential ω to $T_{\lambda,\pi}$ in the following manner. Define $\lambda^\pi \in \mathbb{R}_+^d$ by $\lambda_i^\pi = \lambda_{\pi(i)}$. For each $1 \leq i < d$, denote by $V_i^\pm \in \mathbb{C}$ the following points:

$$V_i^+ = \beta_i(\lambda) + \sqrt{-1}\sum_{l=1}^{i} \pi(l) - l, \qquad V_i^- = \beta_i(\lambda^\pi) - \sqrt{-1}\sum_{l=1}^{i} \pi^{-1}(l) - l,$$

where $\beta_i(\lambda)$ and $\beta_i(\lambda^\pi)$ are defined in (9.1). Let $V_0^\pm = 0$ and $V_d^\pm = \beta_d(\lambda) = \beta_d(\lambda^\pi)$. Consider the polygon $P(\lambda, \pi)$ in the plane whose sides are the segments $S_i^\pm = V_{i-1}^\pm V_i^\pm$. One checks that S_i^+ and $S_{\pi(i)}^-$ are parallel and of equal length. Thus, gluing these parallel sides by translations defines an abelian differential $\omega(\lambda, \pi)$. Denote by $\mathcal{H}_\pi$ the stratum containing $\omega(\lambda, \pi)$.

Proposition 9.3 ([Mas82, Proposition 3.1]). *The stratum $\mathcal{H}_\pi$ obtained as above depends only on π and is independent of λ.*

Denote by S the surface obtained from $P(\lambda, \pi)$ by gluing its parallel sides as above. Consider the straight line flow on S obtained from the vertical flow on $P(\lambda, \pi)$. As shown in [Mas82], the first return map of this flow to the segment $[0, \beta_d)$ defines an IET which coincides with the IET $T_{\lambda,\pi}$.

It is also possible to produce an IET from an abelian differential. Fix some $\omega \in \mathcal{H}_\pi$ and let $\Sigma(\omega) \subset S$ denote its zeros. Then ω defines a flow on S by moving points at linear

speed in the vertical direction $\mathrm{Im}(\omega)$. Following [MW14, Section 2.3], we say $\gamma \subset S$ is a *judicious transversal* for ω if γ is the image of a smooth embedding of a closed interval into S so that its endpoints belong to Σ and the interior of γ is disjoint from Σ and meets every vertical flow orbit of ω transversally.

Let γ be a judicious transversal for $\omega \in \mathcal{H}_\pi$. Integrating ω over γ endows the transversal with a parametrization and we may identify γ with a subinterval J of $\mathbb{R}$ with left endpoint at 0. Then the first return map to γ of the vertical flow defines an IET on J which we denote $T(\omega, \gamma)$.

Definition 9.4. Given $\lambda \in \mathbb{R}^d_+$, an abelian differential $\omega \in \mathcal{H}_\pi$ is said to be a *lift* of $T_{\lambda,\pi}$ if there exists a judicious transversal γ on S such that the first return map $T(\omega, \gamma)$ coincides with $T_{\lambda,\pi}$.

It should be noted that $\omega(\lambda, \pi)$, constructed at the beginning of this subsection, is a lift of $T_{\lambda,\pi}$.

9.4. Short intervals and recurrence

The following proposition allows us to relate the criterion in Theorem 9.2 to the recurrence of Teichmüller geodesics in strata. A similar result was obtained in [MW14, Proposition 7.2] using a slightly different proof. We include a proof here for completeness.

Proposition 9.5. *Let $\lambda \in \mathbb{R}^d_+$ and suppose that $T_{\lambda,\pi}$ has short intervals. Let $\tilde{\omega} \in \mathcal{H}_\pi$ be a lift of $T_{\lambda,\pi}$. Then the forward geodesic $g_t\tilde{\omega}$ diverges in $\mathcal{H}_\pi$.*

Proof. Fix some $\varepsilon > 0$ and let $n_0 \geq 1$ be such that $n\epsilon_n(T_{\lambda,\pi}) < \varepsilon$ for all $n \geq n_0$. We construct a sequence of saddle connections v_n in $\tilde{\omega}$ so that the length of $g_{\log(n/\sqrt{\varepsilon})}v_n$ is $\ll \sqrt{\varepsilon}$ for all $n \geq n_0$. Since ε is arbitrary, $g_t\tilde{\omega}$ diverges in $\mathcal{H}_\pi$.

For this we use an argument similar to the one found in [Bos85, Section 10]. Denote by I a judicious transversal for $\tilde{\omega}$. Let $P_1, \ldots, P_k$ be a collection of polygons in the plane representing $\tilde{\omega}$ and let $\tilde{I}$ be a lift of I under the covering map $\bigcup P_i \to S$ which glues parallel sides by translations. We recall that S is the surface of genus $g \geq 1$ that supports the abelian differentials in the proposition.

For each $n \geq n_0$, denote by $I_n \subset I$ a subinterval such that

$$|I_n| = \epsilon_n(T_{\lambda,\pi}) < \epsilon/n.$$

We use $\tilde{I}_n$ to denote a lift of I_n inside $\tilde{I}$. Denote by $\mathcal{C}$ the open cylinder consisting of the union of the vertical flow orbits of the points in the interior of I_n up to the n^{th} time these orbits hit the transversal I. By definition of the endpoints of the interval I_n, the cylinder $\mathcal{C}$ contains no zeros of $\tilde{\omega}$.

Let $\tilde{\mathcal{C}}$ denote a lift of $\mathcal{C}$ to the complex plane which we unfold to a parallelogram in the following manner. Let x be an arbitrary point in the interior of $\tilde{I}_n$ and denote $x_t := x + it$ for $t > 0$. Define $t(x, n)$ to be the time $t > 0$ corresponding to the n^{th} return of x to I under the vertical flow.

Next, we define a finite sequence of times $q_i \in (0, t(x,n))$ and polygons L_i with $1 \leq i \leq n$ by induction as follows. Let $L_1 \in \{P_1, \ldots, P_k\}$ denote the polygon containing x. Define

$$q_1 = \inf\{0 < t < t(x,n) : x_t \text{ meets a side of } L_1\}.$$

As the endpoints of I_n are discontinuities of the first-return IET, the set on the right hand side is necessarily non-empty. Let l_1 denote the side of L_1 such that $x_{q_1} \in l_1$. Let r_1 denote the unique side of a polygon $R_1 \in \{P_1, \ldots, P_k\}$ which is identified to l_1 by a translation T_1 (which defines the gluing of parallel sides).

Once $(q_j, L_j, l_j, r_j, T_j, R_j)$ have been defined for all $1 \leq j \leq i-1 < n$, we define

$$q_i = \inf\{q_{i-1} < t < t(x,n) : x_t \text{ meets a side of } T_{i-1} \cdot R_{i-1}\}.$$

Let $L_i = T_{i-1} \cdot R_{i-1}$, and let l_i denote the side of L_i such that $x_{q_i} \in l_i$.

Note that l_i is the image of a side l_i' of a polygon in $\{P_1, \ldots, P_k\}$ by a translation A, i.e., A brings l_i back to a side l_i' of one of the original polygons $\{P_1, \ldots, P_k\}$. Denote by r_i the unique side of a polygon $R_i \in \{P_1, \ldots, P_k\}$ which is identified to l_i' by a translation B. Define the i^{th} translation T_i by $T_i = A \circ B$.

Now, consider the parallelogram

$$\mathcal{P}_n = \{x_t \in \mathbb{C} : x \in \text{Int}(\tilde{I}_n),\ 0 \leq t \leq t(x,n)\}$$

where $\text{Int}(\tilde{I}_n)$ denotes the interior of $\tilde{I}_n$.

By definition of the endpoints of I_n, each of the two vertical sides of $\mathcal{P}_n$ necessarily meets a vertex of one of the polygons $L_1, \ldots, L_n$. On the other hand, the interior of $\mathcal{P}_n$ is free from the vertices of the polygons. In particular, if we let v_n denote a straight line segment joining two of the vertices on the two vertical sides of $\mathcal{P}_n$, we see that v_n represents a saddle connection for x which is entirely contained in $\mathcal{P}_n$.

If we regard v_n as a vector in $\mathbb{C}$, we see that $|\text{Im}(v_n)|$ is at most the height of the parallelogram $\mathcal{P}_n$. Thus, in particular,

$$|\text{Im}(v_n)| \asymp n, \tag{9.6}$$

where the implied constant depends only on the lengths of the sides of the polygons $P_1, \ldots, P_k$. Moreover,

$$|\text{Re}(v_n)| \ll |I_n| \leq \varepsilon/n, \tag{9.7}$$

where the implied constant depends on the angle between the segment $\tilde{I}$ and the horizontal axis, which in turn depends only on the neighborhood $\mathcal{U}_\omega$. Therefore, we see that the length of the saddle connection $g_{\log(n/\sqrt{\varepsilon})} v_n$ is $\ll \sqrt{\varepsilon}$ as desired. □

9.5. Horocycles and lines in the space of IETs

It was shown by Minsky and Weiss [MW14] that short horocycle arcs arise as lifts (in the sense of Definition 9.4) of certain short line segments in $\mathbb{R}^d_+$. This result was used in the work of Athreya and Chaika [AC15] to relate the dimension of divergent directions for the Teichmüller flow to the dimension of non-uniquely-ergodic IETs. We use a similar idea to obtain the following proposition.

Proposition 9.6. *Suppose π is an irreducible permutation on d letters. Then the set of $\lambda \in \mathbb{R}^d_+$ corresponding to uniquely ergodic IETs $T_{\lambda,\pi}$ which are IDOC and satisfy (9.3) has Hausdorff codimension at least $1/2$.*

The proof of Proposition 9.6 will be given in Section 9.6 after some technical preparation. We begin by recalling a result in [MW14] characterizing line segments which lift to short horocycle segments.

Proposition 9.7 ([MW14, Theorem 5.3]). *Suppose $\lambda \in \mathbb{R}^d_+$ is such that $T_{\lambda,\pi}$ is uniquely ergodic and satisfies IDOC. Suppose $\mathbf{b} \in \mathbb{R}^d$ satisfies $Q(\lambda, \mathbf{b}) > 0$. Then there exists an $\varepsilon > 0$, an open neighborhood $\mathcal{O}$ of $(\lambda, \mathbf{b})$ in $\mathbb{R}^d_+ \times \mathbb{R}^d$ and an affine homeomorphism $\mathbf{q} : \mathcal{O} \to \mathcal{H}_\pi$ such that $h_s \mathbf{q}(\lambda, \mathbf{b}) = \mathbf{q}(\lambda + s\mathbf{b}, \mathbf{b})$ for $|s| < \varepsilon$. Moreover, $\mathbf{q}(\lambda, \mathbf{b})$ is a lift of $T_{\lambda,\pi}$ for all $(\lambda, \mathbf{b}) \in \mathbb{R}^d_+ \times \mathbb{R}^d$.*

We remark that Theorem 5.3 in [MW14] is not stated in the form we use here, but our statement follows easily from the original one, the definition of *positive pairs* [MW14, Definition 5.1] and [MW14, Proposition 5.2].

The next lemma shows that the positivity condition $Q(\cdot, \cdot) > 0$ in Proposition 9.7 is not restrictive.

Lemma 9.8. *For Lebesgue almost every $(\lambda, \mathbf{b}) \in \mathbb{R}^d_+ \times \mathbb{R}^d$, $Q(\lambda, \mathbf{b}) \neq 0$.*

Proof. Suppose $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{R}^d_+$. We claim that $Q(\lambda, \mathbf{e}_1) > 0$. Note that $Q(\mathbf{e}_1, \mathbf{e}_1) = 0$ and $Q(\mathbf{e}_j, \mathbf{e}_1) \geq 0$ for all $j > 1$. As $\lambda \in \mathbb{R}^d_+$, this implies that $Q(\lambda, \mathbf{e}_1) \geq 0$.

Now, since π is irreducible, we have $\pi(1) > 1$. Hence, we can find some $j_0 > 1$ such that $\pi(j_0) = 1 < \pi(1)$. It follows that $Q(\lambda, \mathbf{e}_1) \geq \lambda_{j_0} Q(\mathbf{e}_{j_0}, \mathbf{e}_1) = \lambda_{j_0} > 0$.

This shows that the linear form $Q(\lambda, \cdot)$ is not identically zero. That is, the kernel of $Q(\lambda, \cdot)$ has dimension $d - 1$ and thus has measure 0. Hence, the lemma follows by Fubini's theorem. □

9.6. Hausdorff dimension, slicing, and proof of Proposition 9.6

Denote by $\mathrm{Gr}(d, m)$ the Grassmannian of m-dimensional subspaces in $\mathbb{R}^d$ and let $\gamma_{d,m}$ denote a Lebesgue class measure on $\mathrm{Gr}(d, m)$. The space of lines (1-dimensional affine subspaces) in $\mathbb{R}^d$ can be naturally identified with $\mathrm{Gr}(d, d-1) \times \mathbb{R}^{d-1}$ and thus carries a Lebesgue class measure. The following fact about slicing Borel sets of small Hausdorff codimension with lines will be useful for us.

Proposition 9.9 ([Mat95, Theorem 10.8 and Corollary 8.9(3)]). *Suppose $A \subset \mathbb{R}^d$ is a Borel set with $\dim_H(A) > t > d-1$. Then there exists a set $B \subseteq \mathrm{Gr}(d, d-1) \times \mathbb{R}^{d-1}$ of lines in $\mathbb{R}^d$ of positive Lebesgue measure such that for each line $\ell \in B$,*

$$\dim_H(\ell \cap A) \geq t - d + 1.$$

We also recall Frostman's lemma.

Lemma 9.10 ([Mat95, Theorem 8.8]). *Suppose A is a Borel subset of $\mathbb{R}^d$. Let $s > 0$. Then the following are equivalent:*

(1) *$H^s(A) > 0$, where H^s denotes the s-dimensional Hausdorff measure.*
(2) *There exists a Borel measure μ on $\mathbb{R}^d$ with support in A such that $\mu(B(x,r)) \leq r^s$ for all $x \in \mathbb{R}^d$ and $r > 0$, where $B(x,r)$ is the closed ball with center x and radius r.*

The idea of the proof of Proposition 9.6 is the following. First, we use Proposition 9.9 to relate the dimension of the set of interest to the dimension of its intersection with line segments. Then Proposition 9.7 allows us to relate the dimension of sets on line segments to the dimension of subsets of horocycle arcs. Finally, using Proposition 9.5, we show that the sets of interest on the horocycle arcs correspond to points with divergent g_t orbits. As a result, Theorem 2.3 concludes the argument.

The suggested outline of the proof is a modified version of an argument given in [AC15, Section 6]. The main difference is the use of Lemma 9.8 to bypass the use of Rauzy induction [AC15, Lemma 6.5], which we believe makes the approach more direct.

Proof of Proposition 9.6. Denote by A the set of $\lambda \in \mathbb{R}^d_+$ such that $T_{\lambda,\pi}$ is uniquely ergodic, IDOC, and has short intervals, and note that A is Borel measurable. Suppose that for some $0 < c < 1$,

$$\mathrm{codim}_H(A) \leq c.$$

Then, by Proposition 9.9, there exists a positive measure set $\mathcal{L}$ of lines in $\mathbb{R}^d$ such that for each line $\ell \in \mathcal{L}$, the set $\ell \cap A$ has Hausdorff dimension at least $1-c$. By Lemma 9.8, we may assume that for each line $\ell \in \mathcal{L}$ there exists some point $\lambda \in \ell$ such that $Q(\lambda, \mathbf{b}) \neq 0$, where $\mathbf{b}$ is a vector in $\mathbb{R}^d$ parallel to ℓ. Let $\ell \in \mathcal{L}$ be a line that passes through a point $\lambda \in \mathbb{R}^d_+$ and is parallel to $\mathbf{b} \in \mathbb{R}^d$, i.e., $\ell = \{\lambda + s\mathbf{b} : s \in \mathbb{R}\}$, and $Q(\lambda, \mathbf{b}) \neq 0$.

By Lemma 9.10((1)⇒(2)), there exists a measure μ supported on $\ell \cap A$ such that for all $x \in \ell$ and all $r > 0$,

$$\mu(B(x,r)) \leq r^{1-c}.$$

The linearity of Q implies that $Q(\lambda + s\mathbf{b}, \mathbf{b}) \neq 0$ for all $s \neq -Q(\lambda, \mathbf{b})/Q(\mathbf{b}, \mathbf{b})$ and for all $s \in \mathbb{R}$ if $Q(\mathbf{b}, \mathbf{b}) = 0$. Hence, since μ is not a Dirac mass, we can find $x \in \mathrm{supp}\,\mu \subset \ell \cap A$ such that $Q(x, \mathbf{b}) \neq 0$. In particular, $T_{x,\pi}$ is uniquely ergodic, IDOC, and has short intervals. Notice that a priori $\lambda \in \ell$ may not belong to $\mathrm{supp}\,\mu$. By replacing $\mathbf{b}$ with $-\mathbf{b}$ if necessary, we may assume $Q(x, \mathbf{b}) > 0$.

Hence, by Proposition 9.7, we can find $\varepsilon_0 > 0$ and a map $\mathbf{q}$ such that $\mathbf{q}(x + s\mathbf{b}, \mathbf{b}) = h_s(\mathbf{q}(x, \mathbf{b}))$ for $|s| < \varepsilon_0$. But, by Proposition 9.5, since $\mathbf{q}(x + s\mathbf{b}, \mathbf{b})$ is a lift of $T_{x+s\mathbf{b},\pi}$,

the forward g_t orbit of each element of the set $\{\mathbf{q}(x + s\mathbf{b}, \mathbf{b}) : |s| < \varepsilon_0,\ x + s\mathbf{b} \in A\}$ is divergent (on average) in the stratum $\mathcal{H}_\pi$.

By restricting μ to the segment $x+s\mathbf{b}$ with $|s| < \varepsilon_0$ and using Lemma 9.10((2)⇒(1)), we see that the Hausdorff dimension of $A\cap\{x+s\mathbf{b} : |s| < \varepsilon_0\}$ is at least $1-c$. Theorem 2.3 thus implies that $1 - c \le 1/2$. Here, we use the fact that the map $\mathbf{q}$, restricted to the set $A \cap \{x + s\mathbf{b} : |s| < \varepsilon_0\}$, is an affine homeomorphism onto its image, and in particular preserves Hausdorff dimension. □

9.7. Proof of Corollary 1.9

Let π be a type W permutation on $d \ge 4$ letters. By Theorem 9.2,

$$\begin{aligned}\{\lambda \in \mathbb{R}^d_+ : T_{\lambda,\pi} \text{ not weakly mixing}\}&\\ \subseteq \{\lambda : T_{\lambda,\pi} \text{ is not IDOC}\} \cup \{\lambda : T_{\lambda,\pi} \text{ is NUE}\}&\\ \cup \{\lambda : T_{\lambda,\pi} \text{ is IDOC and UE, and has short intervals}\},& \quad (9.8)\end{aligned}$$

where [N]UE denotes [non-]uniquely ergodic, and having short intervals means $T_{\lambda,\pi}$ satisfies (9.3). The last two sets in the above union have codimension at least $1/2$ by [AC15, Theorem 1.6] and Proposition 9.6, respectively.

It is shown in [Kea75] that if the components of λ are linearly independent over $\mathbb{Q}$, then $T_{\lambda,\pi}$ is IDOC. In particular, the set of IETs which are not IDOC is contained in the intersection of the simplex $\mathbb{R}^d_+$ with countably many codimension 1 subspaces of $\mathbb{R}^d$ which are defined over $\mathbb{Q}$. This implies that the set of non-IDOC IETs has Hausdorff codimension at least 1.

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