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Alexander I. Bufetov · Yanqi Qiu · Alexander Shamov

# Kernels of conditional determinantal measures and the Lyons–Peres completeness conjecture

*To Alexei Ivanovich Zykin (13.06.1984–22.04.2017), the organizer of the Yaroslavl summer school where our work had started, in grateful memory, with a deep sense of loss, this paper is respectfully dedicated*

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**Abstract.** The main result of this paper, Theorem 1.4, establishes a conjecture of Lyons and Peres: for a determinantal point process governed by a self-adjoint reproducing kernel, the system of kernels sampled at the points of a random configuration is complete in the range of the kernel. A key step in the proof, Lemma 1.9, states that conditioning on the configuration in a subset preserves the determinantal property, and the main Lemma 1.10 is a new local property for kernels of conditional point processes. In Theorem 1.6 we prove the triviality of the tail $\sigma$-algebra for determinantal point processes governed by self-adjoint kernels.

**Keywords.** Determinantal point processes, Lyons–Peres completeness conjecture, conditional measures, tail triviality, measure-valued martingales, operator-valued martingales

## 1. Introduction

### *1.1. The zero set of a Gaussian analytic function on the disc is a uniqueness set for the Bergman space*

Consider the random series

$$\mathfrak{f}_{\mathbb{D}}(z) = \sum_{n=0}^{\infty} f_n z^n, \tag{1.1}$$

A. I. Bufetov: Aix-Marseille Université, Centrale Marseille, CNRS, Institut de Mathématiques de Marseille, UMR7373, 39 Rue F. Joliot Curie, 13453 Marseille, France; Steklov Mathematical Institute of RAS, Moscow, Russia; Institute for Information Transmission Problems, Moscow, Russia; and The Chebyshev Laboratory, Saint-Petersburg State University, Saint-Petersburg, Russia; e-mail: bufetov@mi.ras.ru, alexander.bufetov@univ-amu.fr

Y. Qiu: Institute of Mathematics, AMSS, CAS and Hua Loo-Keng Key Laboratory of Mathematics, Beijing, China; and CNRS, Institut de Mathématiques de Toulouse, Université Paul Sabatier, Toulouse, France; e-mail: yanqi.qiu@hotmail.com, yanqi.qiu@amss.ac.cn

A. Shamov: Department of Mathematics, Weizmann Institute of Science, Israel;
e-mail: trefoils@gmail.com

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where the coefficients $f_n$ are independent identically distributed complex Gaussian random variables with expectation 0 and variance 1. The series (1.1) has radius of convergence 1 almost surely and defines a holomorphic function on the open unit disc $\mathbb{D}$. Let $Z(\mathfrak{f}_{\mathbb{D}})$ be the zero set of (1.1):

$$Z(\mathfrak{f}_{\mathbb{D}}) = \{z \in \mathbb{D} : \mathfrak{f}_{\mathbb{D}}(z) = 0\}.$$

Denote $A^2(\mathbb{D})$ the Bergman space of holomorphic functions on $\mathbb{D}$ square-integrable with respect to the Lebesgue measure Leb. A subset $X \subset \mathbb{D}$ is called a *uniqueness set* for $A^2(\mathbb{D})$ if a function $h \in A^2(\mathbb{D})$ satisfying $h\restriction_X = 0$ must be the zero function. In this particular case, our main result is

**Theorem 1.1.** *Almost surely, $Z(\mathfrak{f}_{\mathbb{D}})$ is a uniqueness set for $A^2(\mathbb{D})$.*

In other words, almost surely, $Z(\mathfrak{f}_{\mathbb{D}})$ cannot be a zero set of a function in $A^2(\mathbb{D})$. Theorem 1.1 is a direct corollary of our main result, Theorem 1.4, formulated below, since, by the Peres–Virág Theorem [28], the random subset $Z(\mathfrak{f}_{\mathbb{D}}) \subset \mathbb{D}$ is a realization of the determinantal point process on $\mathbb{D}$ governed by the reproducing kernel of $A^2(\mathbb{D}) \subset L^2(\mathbb{D}, \mathrm{Leb})$ given by

$$K_{\mathbb{D}}(z, w) = \frac{1}{\pi(1 - z\bar{w})^2}.$$

**Remark.** After the work on this paper was finished, we became aware of the result of Lyons and Zhai [24], who prove in particular in a different way that $Z(\mathfrak{f}_{\mathbb{D}})$ is almost surely a uniqueness set for $A^2(\mathbb{D})$.

For brevity, the set of zeros of a non-zero function in $A^2(\mathbb{D})$ will be called an $A^2(\mathbb{D})$-*zero set*. Various necessary and sufficient conditions for a subset of the disc to be an $A^2(\mathbb{D})$-zero set exist in the literature. For example, by Hedenmalm–Korenblum–Zhu [16, Theorem 4.7], for any $A^2(\mathbb{D})$-zero set $Z$ the Blaschke-type condition

$$\sum_{z \in Z,\, z \neq 0} \frac{1 - |z|}{\left[\log \frac{1}{1-|z|}\right]^{1+\varepsilon}} < \infty \tag{1.2}$$

holds for any $\varepsilon > 0$. Note, however, that for our random set $Z(\mathfrak{f}_{\mathbb{D}})$, we have

$$\mathbb{E}\left(\sum_{z \in Z(\mathfrak{f}_{\mathbb{D}})} \frac{1 - |z|}{\left[\log \frac{1}{1-|z|}\right]^{1+\varepsilon}}\right) = \int_{\mathbb{D}} \frac{1 - |z|}{\left[\log \frac{1}{1-|z|}\right]^{1+\varepsilon}} \frac{1}{\pi(1 - |z|^2)^2}\, \mathrm{d}A(z) < \infty,$$

where $\mathrm{d}A$ is the Lebesgue measure on the unit disc, hence $Z(\mathfrak{f}_{\mathbb{D}})$ satisfies (1.2) almost surely.

We next observe that $Z(\mathfrak{f}_{\mathbb{D}})$ is neither a *sampling* nor an *interpolating* set for $A^2(\mathbb{D})$. Recall that a discrete subset $Z \subset \mathbb{D}$ is called an $A^2(\mathbb{D})$-*sampling set* if there exist $C_1, C_2 > 0$ such that for any $g \in A^2(\mathbb{D})$ we have

$$C_1 \|g\|^2_{A^2(\mathbb{D})} \leq \sum_{z \in Z} |g(z)|^2 (1 - |z|^2)^2 \leq C_2 \|g\|^2_{A^2(\mathbb{D})}.$$

By definition, an $A^2(\mathbb{D})$-sampling set is also an $A^2(\mathbb{D})$-uniqueness set. A discrete subset $Z = \{z_1, \dots, z_j, \dots\} \subset \mathbb{D}$ is called an $A^2(\mathbb{D})$-*interpolating set* if for any sequence $\{a_j\}$ in $\mathbb{C}$ such that $\{a_j(1 - |z_j|^2)\} \in \ell^2$, there exists $g \in A^2(\mathbb{D})$ such that $g(z_j) = a_j$ for all $j$. Deleting a finite number of points from any uniqueness set for $A^2(\mathbb{D})$ does not change the uniqueness property of the set, so a function in $A^2(\mathbb{D})$ cannot vanish at all points of a uniqueness set except a finite subset, which means that a uniqueness set for $A^2(\mathbb{D})$ is never $A^2(\mathbb{D})$-interpolating.

**Proposition 1.2.** *The subset $Z(\mathfrak{f}_{\mathbb{D}})$ is almost surely neither $A^2(\mathbb{D})$-sampling nor $A^2(\mathbb{D})$-interpolating.*

To see that the set $Z(\mathfrak{f}_{\mathbb{D}})$ is not sampling, we will use Seip's [36, Theorem 7.1], which says that any $A^2(\mathbb{D})$-sampling set is a finite union of sets uniformly separated with respect to the Lobachevskian distance.

**Lemma 1.3.** *Almost surely, $Z(\mathfrak{f}_{\mathbb{D}})$ cannot be expressed as a finite union of uniformly separated sets.*

Lemma 1.3, proved in Section 8.3 below with the use of ergodicity, under the measure $\mathbb{P}_{K_{\mathbb{D}}}$, of the action of one-parameter groups of isometries of the Lobachevsky plane (this ergodicity is due to Hough–Krishnapur–Peres–Virág [19, Proposition 2.3.7]), implies Proposition 1.2.

### 1.2. An outline of the main results

*1.2.1. The Lyons–Peres completeness conjecture.* Let $E$ be a locally compact $\sigma$-compact Polish space and let $\mathrm{Conf}(E)$ be the space of locally finite configurations on $E$. Let $\mu$ be a $\sigma$-finite Radon measure on $E$, let $K$ be the kernel of a locally trace class positive contraction acting on the complex Hilbert space $L^2(E, \mu)$, and let $\mathbb{P}_K$ be the corresponding determinantal measure on $\mathrm{Conf}(E)$ (the precise definitions are recalled in Section 1.3).

Assume moreover that $K$ is a locally trace class orthogonal projection onto a closed subspace $H$ of $L^2(E, \mu)$; in other words, $K$ is the reproducing kernel of a reproducing kernel Hilbert space $H \subset L^2(E, \mu)$. For $x \in E$, introduce a function $K_x \in L^2(E, \mu)$ by the formula

$$K_x(t) := K(t, x), \qquad t \in E. \tag{1.3}$$

Our main result, Theorem 1.4, establishes

**The Lyons–Peres Completeness Conjecture.** *For $\mathbb{P}_K$-almost every $X \in \mathrm{Conf}(E)$, we have*

$$\overline{\mathrm{span}}^{L^2(E,\mu)}\{K_x : x \in X\} = H. \tag{1.4}$$

Lyons [21, Theorem 7.11] proved that the completeness of reproducing kernels holds when $E$ is countable and formulated the general statement as Conjecture 4.6 in [22]. Ghosh [13] established the conjecture under the important additional assumption that the determinantal point process $\mathbb{P}_K$ is number rigid in the sense of Ghosh and Peres. While many determinantal point processes are indeed number rigid (see Ghosh [13] for the sine-process, Ghosh and Peres [14] for the Ginibre ensemble, [4] for processes governed by the Airy, the Bessel and more general integrable kernels, [7] for stationary processes, [9] for generalized Ginibre ensembles), our zero set $Z(\mathfrak{f}_{\mathbb{D}})$ is not: indeed, Holroyd and Soo [18] showed that the point process $Z(\mathfrak{f}_{\mathbb{D}})$ is insertion and deletion tolerant, the opposite of being number rigid. For determinantal point processes associated with generalized Bergman spaces on $\mathbb{D}$, insertion and deletion tolerance is established in [9] and the Radon–Nikodym derivative of the Palm measure with respect to the initial measure is given explicitly as a generalized multiplicative functional.

*1.2.2. Outline of the proof of the Lyons–Peres completeness conjecture.* The key ingredient in our proof of the Lyons–Peres completeness conjecture is the preservation of the determinantal property under conditioning with respect to the configuration in a subset and the explicit description of a suitable correlation kernel of this conditional determinantal measure.

An informal explanation of our proof is as follows. Suppose that the locally trace class operator $K$ is an orthogonal projection onto a closed subspace $H \subset L^2(E, \mu)$. For $\mathbb{P}_K$-almost every configuration $X \in \mathrm{Conf}(E)$, define

$$H(X) := H \ominus \overline{\mathrm{span}}^{L^2(E,\mu)}\{K_x : x \in X\}.$$

For $f \in H$, $x \in E$ write $f(x) = \langle f, K_x\rangle$. Then $H(X)$ is the space of all functions $f \in H$ vanishing on $X$.

Proposition 2.6 below shows that for any precompact subset $S \subset E$, the kernel of the orthogonal projection onto the space

$$\chi_{E\setminus S} H(X \cap S) = \{\chi_{E\setminus S} h : h \in H(X \cap S)\}$$

corresponds to the conditional measure (which is again determinantal) of $\mathbb{P}_K$ with respect to the condition that the restriction of the random configuration onto $S$ coincides with $X \cap S$.

Our key step, Lemma 1.9, is an extension of Proposition 2.6 to conditioning on any Borel subset of $E$, in particular, on a subset with precompact complement. Suppose for contradiction that there exists a subset $\Omega_0 \subset \mathrm{Conf}(E)$ with $\mathbb{P}_K(\Omega_0) > 0$ such that for any $X \in \Omega_0$, the equality (1.4) is violated and thus there exists $f \in H(X) \setminus \{0\}$. Since $f$ is non-zero, we can find a precompact subset $B \subset E \setminus X$ such that $\chi_B f$ is a non-zero element of $L^2(B, \mu)$.

(i) By Lemma 1.9, the conditional measure of our point process $\mathbb{P}_K$, denoted later by $\mathbb{P}_K(\cdot \mid X, E \setminus B)$, with respect to the condition that the restriction of the random configuration to $E \setminus B$ coincides with $X \cap (E \setminus B)$, is determinantal and is induced by a specific kernel $K^{[X,E\setminus B]}$. That is,

$$\mathbb{P}_K(\cdot \mid X, E \setminus B) = \mathbb{P}_{K^{[X,E\setminus B]}}.$$

(ii) By our explicit description of the kernel $K^{[X,\, E\setminus B]}$ in Lemma 1.9, the assumption $f \in H(X)$ implies (see Lemma 6.1) that the function $\chi_B f$ is a fixed point of the operator $K^{[X,E\setminus B]}$, that is,

$$K^{[X,E\setminus B]}(\chi_B f) = \chi_B f.$$

(iii) Since $\chi_B f$ is a non-zero element in $L^2(B,\mu)$, we have

$$\mathbb{P}_K(\#_B = 0 \mid X, E \setminus B) = \mathbb{P}_{K^{[X,E\setminus B]}}(\#_B = 0) = \det(1 - K^{[X,E\setminus B]}) = 0$$

for almost all $X$. Here $\#_B$ denotes the random variable that assigns to each configuration the number of points of this configuration inside $B$.

(iv) Since $X$ has no particles in $B$, the Fubini theorem implies that $\mathbb{P}_K(\#_B = 0 \mid X, E\setminus B) > 0$ for almost all $X$ (see Lemma 6.2). This contradiction settles the Lyons–Peres completeness conjecture.

*1.2.3. Kernels of conditional determinantal point processes.* The preservation of the determinantal property under conditioning with respect to the configuration in a subset will be proved using a specific sequence of *conditional kernels* of determinantal point processes which is a kernel-valued martingale and the proof of the martingale property of the conditional kernels relies on a new *local property* of those kernels. In the following, we informally explain the martingale and local properties of our conditional kernels.

Given a Borel probability measure $\mathbb{P}$ on $\mathrm{Conf}(E)$ and a Borel subset $C \subset E$, the measure $\mathbb{P}(\cdot \mid X, C)$ on the space $\mathrm{Conf}(E \setminus C)$ is defined as the conditional measure of $\mathbb{P}$ with respect to the condition that the restriction of our random configuration to $C$ coincides with $X \cap C$ (see Section 2.2 below for the detailed definition).

Lemma 1.9 establishes that, for any determinantal point process $\mathbb{P}_K$ induced by a *self-adjoint* locally trace class kernel $K$, the conditional measures $\mathbb{P}_K(\cdot \mid X, C)$ are themselves determinantal and governed by explicitly given self-adjoint kernels. For a *precompact* subset $B \subset E$, the determinantal property for $\mathbb{P}_K(\cdot \mid X, B)$ follows from the characterization of Palm measures for determinantal processes due to Shirai–Takahashi [38] and the characterization of induced determinantal processes [3], [6]. For $X \in \mathrm{Conf}(E)$, in Definition 1.8 below we introduce a specific self-adjoint kernel $K^{[X,B]}$ governing the measure $\mathbb{P}_K(\cdot \mid X, B)$.

In order to prove that conditioning preserves the determinantal property, we shall show that, along an increasing or a decreasing sequence of precompact subsets $B_n$, our kernels $K^{[X,B_n]}$ form a martingale after a suitable compression. The one-step martingale property (corresponding to the case of two precompact subsets $B_0 = \emptyset$ and $B_1 = B$) for spanning trees is due to Benjamini, Lyons, Peres and Schramm [1] and for processes on general discrete phase spaces to Lyons [21]. It seems to be essential for the argument of Benjamini, Lyons, Peres and Schramm [1] and Lyons [21] that the phase space be discrete; we do not see how to extend their argument to continuous phase spaces. Moreover, it requires some effort to deduce the full martingale property from the one-step martingale property.

Our proof of the martingale property of the conditional kernels relies on a new local property for the kernels $K^{[X,B]}$ which we now informally explain. If $B \subset C \subset E$, then conditioning on the restriction of the configuration to $B$ commutes with the natural projection map $X \mapsto X \cap C$ from $\mathrm{Conf}(E)$ to $\mathrm{Conf}(C)$. This commutativity manifests itself at the level of the kernels chosen in Definition 1.8 below: we have $\chi_C K^{[X\cap B,\, B]}\chi_C = (\chi_C K \chi_C)^{[X\cap C,\, B]}$. Our local property states that instead of $\chi_C$ one can take a much more general projection $Q$, and the relation still holds. More precisely, let $Q : L^2(E, \mu) \to L^2(E, \mu)$ be an orthogonal projection such that $\mathrm{Ran}(Q) \subset L^2(E \setminus B, \mu)$ and $QKQ$ is locally trace class. In Lemma 1.10 below we shall see that

$$\big((Q + \chi_B)K(Q + \chi_B)\big)^{[X,B]} = (Q + \chi_B)K^{[X,B]}(Q + \chi_B) = QK^{[X,B]}Q. \tag{1.5}$$

Applying (1.5) to a one-dimensional projection operator $Q$, we find that, for an arbitrary $\varphi \in L^2(E \setminus B, \mu)$, the quantity $\langle K^{[X,B]}\varphi, \varphi\rangle$ is a martingale indexed by $B$ (with respect to the partial order of inclusion)—see (4.8) below. Using the *Radon–Nikodym property* for the space of trace class operators, we obtain an operator-valued martingale that converges, along an increasing sequence of precompact subsets of $E$, almost surely in the space of locally trace class operators. As an immediate consequence, we prove that for determinantal point processes governed by self-adjoint kernels, conditioning on the configuration in any Borel subset preserves the determinantal property (see Lemma 1.9).

*1.2.4. Triviality of the tail σ-algebra.* As an application of the local property for the conditional kernels, in Theorem 1.6 we establish the triviality of the tail $\sigma$-algebra for determinantal point processes governed by self-adjoint kernels. Lyons [21] proved tail-triviality in the discrete setting, extending the argument of Benjamini–Lyons–Peres–Schramm [1] for spanning trees, and conjectured that tail triviality holds in full generality [22, Conjecture 3.2]. The argument of [1] and [21] relies on an estimate for the decay of the variance of the conditional kernel; using the local property of Lemma 1.10, we establish a similar variance estimate in full generality (see Lemma 7.3), and obtain the desired triviality of the tail $\sigma$-algebra. The local property of conditional kernels thus allows us to carry out the proof of tail triviality in a unified way for both the continuous and the discrete setting.

The triviality of the tail $\sigma$-algebra for general determinantal point processes with self-adjoint kernels is the main result of the independent work by [27]. The argument of [27] is completely different from ours: Osada–Osada [27] construct a special family of discrete approximations of continuous determinantal point processes and derive the triviality of the tail $\sigma$-algebra in the continuous setting from the theorem of Lyons by approximation. Another approach, due to Lyons [23], for establishing the triviality of the tail $\sigma$-algebra in the continuous setting also deduces it from the discrete result using Goldman's transference principle (Goldman [15, Proposition 12] and Lyons [22, Section 3.6]).

### *1.3. Formulation of the main results*

Let $E$ be a locally compact $\sigma$-compact Polish space, equipped with a metric such that any bounded set is relatively compact, and endowed with a positive $\sigma$-finite Radon measure $\mu$. Let $\mathrm{Conf}(E)$ be the space of locally finite configurations on $E$. A *point process* on $E$ is by definition a Borel probability measure on $\mathrm{Conf}(E)$. Let $K : L^2(E, \mu) \to L^2(E, \mu)$ be a bounded self-adjoint *locally trace class* operator with $\mathrm{spec}(K) \subset [0, 1]$. A theorem obtained by Macchi [25] and Soshnikov [42], as well as by Shirai and Takahashi [37], gives a unique point process on $E$, denoted by $\mathbb{P}_K$, such that for any compactly supported bounded measurable function $g : E \to \mathbb{C}$, we have

$$\mathbb{E}_{\mathbb{P}_K}\Big[\prod_{x\in X}(1 + g(x))\Big] = \det\big(1 + \mathrm{sgn}(g)|g|^{1/2} \cdot K \cdot |g|^{1/2}\big)_{L^2(\mu)}, \quad \mathrm{sgn}(g) = \frac{g}{|g|}.$$

Here $\det(1 + S)$ denotes the Fredholm determinant of the operator $1 + S$ (see, e.g., Simon [40]).

The locally trace class self-adjoint operator $K$ is an integral operator. Following Soshnikov [42], we fix a Borel subset $E_0 \subset E$ with $\mu(E \setminus E_0) = 0$ and *fix* a Borel function $K : E_0 \times E_0 \to \mathbb{C}$, our kernel, in such a way that for any $k \in \mathbb{N}$ and any bounded Borel subset $B \subset E$, we have

$$\mathrm{tr}((\chi_B K \chi_B)^k) = \int_{B^k} K(x_1, x_2)K(x_2, x_3)\cdots K(x_k, x_1)\,\mathrm{d}\mu(x_1)\cdots \mathrm{d}\mu(x_k). \tag{1.6}$$

**Theorem 1.4.** *If $K$ is a locally trace class orthogonal projection onto a closed subspace $H$ of $L^2(E, \mu)$, then for $\mathbb{P}_K$-almost every $X \in \mathrm{Conf}(E)$, the functions $K_x$ defined by (1.3) satisfy*

$$\overline{\mathrm{span}}^{L^2(E,\mu)}\{K_x : x \in X\} = H.$$

If we fix a realization for each $h \in H$ in such a way that the equation $h(x) = \langle h, K_x\rangle$ holds for every $x \in E_0$ and every $h \in H$, then Theorem 1.4 can equivalently be reformulated as follows:

**Corollary 1.5.** *For $\mathbb{P}_K$-almost every $X \in \mathrm{Conf}(E)$, if $h \in H$ satisfies $h\restriction_X = 0$, then $h = 0$.*

**Theorem 1.6.** *Let $B_1 \subset B_2 \subset \cdots \subset E$ be an increasing exhausting sequence of bounded Borel subsets of $E$. The $\sigma$-algebra $\bigcap_{n\in\mathbb{N}} \mathcal{F}(E \setminus B_n)$ is trivial with respect to $\mathbb{P}_K$.*

**Corollary 1.7.** *The point process $\mathbb{P}_K$ has trivial tail $\sigma$-algebra.*

**Remark.** Our assumption on $\sigma$-compactness of $E$ is not essential: in the argument below, one could everywhere replace "relatively compact" (here equivalent to "bounded") by "having finite weight with respect to the measure $K(x, x)\mathrm{d}\mu(x)$". On the other hand, the assumption of *self-adjointness* is used throughout. It would be interesting to obtain similar results on conditional measures for more general determinantal kernels.

*1.3.1. The key lemma.* Let $K : L^2(E, \mu) \to L^2(E, \mu)$ be a bounded self-adjoint *locally trace class* operator with $\mathrm{spec}(K) \subset [0, 1]$. Recall that we fix a Borel subset $E_0 \subset E$ with $\mu(E \setminus E_0) = 0$ and a Borel function $K : E_0 \times E_0 \to \mathbb{C}$, the kernel of the operator $K$, satisfying (1.6).

**Definition 1.8.** For any bounded Borel subset $B \subset E$, we define canonical conditional kernels $K^{[X,B]}$ with respect to the conditioning on the configuration in $B$ as follows:

- For $p \in E_0$, define a kernel $K^p$, for $(x, y) \in E_0 \times E_0$, by the formula

$$K^p(x, y) := \begin{cases} K(x, y) - \dfrac{K(x, p)K(p, y)}{K(p, p)} & \text{if } K(p, p) > 0, \\ 0 & \text{if } K(p, p) = 0. \end{cases} \tag{1.7}$$

- For an $n$-tuple $(p_1, \ldots, p_n) \in E_0^n$, define a kernel $K^{p_1,\ldots,p_n} = (\cdots(K^{p_1})^{p_2}\cdots)^{p_n}$ as follows (cf. Shirai–Takahashi [38, Corollary 6.6]). Given $x, y \in E_0$, write $p_0 = x$, $q_0 = y$, $q_i = p_i$ for $1 \leq i \leq n$, and set

$$K^{p_1,\ldots,p_n}(x, y) := \begin{cases} \dfrac{\det[K(p_i, q_j)]_{0\leq i,j\leq n}}{\det[K(p_i, p_j)]_{1\leq i,j\leq n}} & \text{if } \det[K(p_i, p_j)]_{1\leq i,j\leq n} > 0, \\ 0 & \text{if } \det[K(p_i, p_j)]_{1\leq i,j\leq n} = 0. \end{cases} \tag{1.8}$$

- For a bounded Borel subset $B \subset E$ and $X \in \mathrm{Conf}(E)$ such that $X \cap B = \{p_1, \ldots, p_l\} \subset E_0$, define

$$K^{[X,B]} = \begin{cases} \chi_{E\setminus B} K^{p_1,\ldots,p_l}(1 - \chi_B K^{p_1,\ldots,p_l})^{-1}\chi_{E\setminus B} & \text{if } 1 - \chi_B K^{p_1,\ldots,p_l} \text{ is invertible}, \\ 0 & \text{if } 1 - \chi_B K^{p_1,\ldots,p_l} \text{ is not invertible}. \end{cases} \tag{1.9}$$

If $1-\chi_B K^{p_1,\ldots,p_l}$ is invertible, then so is $1-\chi_B K^{p_1,\ldots,p_l}\chi_B$. It would follow that the contractive operator $\chi_B K^{p_1,\ldots,p_l}\chi_B = (\chi_B K^{p_1,\ldots,p_l})(\chi_B K^{p_1,\ldots,p_l})^*$, and hence $\chi_B K^{p_1,\ldots,p_l}$, is strictly contractive (see the inequalities (3.6) and (3.7) below for the details and see also [5, Section 2.14] for similar discussions). Therefore, the series

$$K^{[X,B]} = \chi_{E\setminus B}\sum_{n=0}^{\infty} K^{p_1,\ldots,p_l}(\chi_B K^{p_1,\ldots,p_l})^n \chi_{E\setminus B} \tag{1.10}$$

converges in the operator norm topology.

In what follows, we will also deal with the kernel of the operator $K^{[X,B]}$ as a two-variable Borel function on $E_0 \times E_0$. This is possible since we may fix a specific Borel realization of the kernels of the operators in the series (1.10) as follows. Using (1.8), we fix the kernel of the operator $\chi_{E\setminus B}K^{p_1,\ldots,p_l}\chi_{E\setminus B}$ as the two-variable function on $E_0 \times E_0$ given by

$$(x, y) \mapsto \chi_{E\setminus B}(x)\chi_{E\setminus B}(y)K^{p_1,\ldots,p_l}(x, y).$$

For any integer $n \geq 1$, by writing $T_n = (\chi_B K^{p_1,\ldots,p_l})^{n-1}$, we have

$$K^{p_1,\ldots,p_l}(\chi_B K^{p_1,\ldots,p_l})^n = K^{p_1,\ldots,p_l}T_n\chi_B K^{p_1,\ldots,p_l},$$

and the kernel for the operator $K^{p_1,\dots,p_l} T_n \chi_B K^{p_1,\dots,p_l}$ is given by

$$\begin{aligned}\int_E \int_E & K^{p_1,\dots,p_l}(x, z_1) T_n(z_1, z_2) \chi_B(z_2) K^{p_1,\dots,p_l}(z_2, y)\, \mathrm{d}\mu(z_2)\, \mathrm{d}\mu(z_1) \\ &= \int_E \overline{K^{p_1,\dots,p_l}(z_1, x)} \big[T_n\big(\chi_B(\cdot) K^{p_1,\dots,p_l}(\cdot, y)\big)\big](z_1)\, \mathrm{d}\mu(z_1) \\ &= \big\langle T_n\big(\chi_B(\cdot) K^{p_1,\dots,p_l}(\cdot, y)\big),\ K^{p_1,\dots,p_l}(\cdot, x)\big\rangle_{L^2(E,\mu)}, \end{aligned} \tag{1.11}$$

where in the above integrals, $T_n(z_1, z_2)$ is the kernel of the operator $T_n$ and in particular, for $n = 1$, we use the convention $T_n(z_1, z_2) d\mu(z_2) = \delta_{z_1 = z_2}$. Therefore, we may fix the kernel of the operator $\chi_{E\setminus B} K^{p_1,\dots,p_l} (\chi_B K^{p_1,\dots,p_l})^n \chi_{E\setminus B}$ as the two-variable function on $E_0 \times E_0$ given by the last term in (1.11). In particular, for $(x, y) \in E_0 \times E_0$, we will use the formula

$$\begin{aligned} K^{[X,B]}(x, y) &= \chi_{E\setminus B}(x) \chi_{E\setminus B}(y) K^{p_1,\dots,p_l}(x, y) \\ &+ \chi_{E\setminus B}(x) \chi_{E\setminus B}(y) \Big\langle \Big(\sum_{n=1}^{\infty} (\chi_B K^{p_1,\dots,p_l})^{n-1}\Big)(\chi_B(\cdot) K^{p_1,\dots,p_l}(\cdot, y)),\ K^{p_1,\dots,p_l}(\cdot, x)\Big\rangle_{L^2(E,\mu)} \end{aligned} \tag{1.12}$$

as our specific Borel realization of the kernel of the operator $K^{[X,B]}$.

**Remark.** We will see in Proposition 2.5 below that under the assumption that $B$ is bounded, the kernel $K^{[X,B]}$ defined above is the correlation kernel (locally trace class kernel), inducing a determinantal point process which is exactly the conditional measure of $\mathbb{P}_K$, the condition being that the configuration on $B$ coincides with $X \cap B$. In particular, for $\mathbb{P}_K$-almost every $X$, we have $X \cap B = \{p_1, \dots, p_l\} \subset E_0$, and $1 - \chi_B K^{p_1,\dots,p_l}$ is invertible. The second case $K^{[X,B]} = 0$ has probability zero. Note that the range of $K^{[X,B]}$ is contained in $L^2(E \setminus B, \mu)$ and we have

$$K^{[X,B]} = \chi_{E\setminus B} K^{[X,B]} \chi_{E\setminus B}. \tag{1.13}$$

For any Borel subset $W \subset E$, *not necessarily bounded*, consider the Borel surjection $\pi_W : \mathrm{Conf}(E) \to \mathrm{Conf}(W)$ given by $X \mapsto X \cap W$. Fibres of this mapping can be identified with $\mathrm{Conf}(E \setminus W)$. For a Borel probability measure $\mathbb{P}$ on $\mathrm{Conf}(E)$, the measure $\mathbb{P}(\cdot \mid X, W)$ on $\mathrm{Conf}(E \setminus W)$ is defined as the conditional measure of $\mathbb{P}$ with respect to the condition that the restriction of our random configuration to $W$ coincides with $\pi_W(X)$. More formally, the measures $\mathbb{P}(\cdot \mid X, W)$ are conditional measures, in the sense of Rokhlin [32], of our initial measure $\mathbb{P}$ on fibres of the measurable partition induced by the surjection $\pi_W$.

Denote by $\mathscr{L}_1(L^2(E, \mu))$ the space of trace class operators on $L^2(E, \mu)$ and by $\mathscr{L}_{1,\mathrm{loc}}(L^2(E, \mu))$ the space of bounded and locally trace class operators on $L^2(E, \mu)$. For more details on trace class operators on a Hilbert space, we refer to Simon [41, Chapter 1]. The space $\mathscr{L}_{1,\mathrm{loc}}(L^2(E, \mu))$ is equipped with the topology induced by the semi-norms $T \mapsto \|\chi_B T \chi_B\|_1$, where $\|\cdot\|_1$ is the trace class norm and $B$ ranges over bounded Borel subsets of $E$.

For any Borel subset $W \subset E$, we denote by $\mathcal{F}(W) := \sigma(\#_A : A \subset W)$ the $\sigma$-algebra on $\mathrm{Conf}(E)$ generated by the mappings $\#_A : \mathrm{Conf}(E) \to \mathbb{R}$ defined by $\#_A(X) := \#(X \cap A)$, where $A$ ranges over all bounded Borel subsets of $W$. We are now ready to formulate our key lemma.

**Lemma 1.9.** *Let $W \subset E$ be a Borel subset and let $B_1 \subset B_2 \subset \cdots \subset W$ be an increasing exhausting sequence of bounded Borel subsets of $W$. For $\mathbb{P}_K$-almost every $X \in \mathrm{Conf}(E)$ there exists a positive self-adjoint contraction $K^{[X,W]} \in \mathscr{L}_{1,\mathrm{loc}}(L^2(E \setminus W, \mu))$ such that*

$$\chi_{E\setminus W} K^{[X,B_n]} \chi_{E\setminus W} \xrightarrow[\text{in } \mathscr{L}_{1,\mathrm{loc}}(L^2(E\setminus W,\mu))]{n\to\infty} K^{[X,W]}$$

*and*

$$\mathbb{P}_K(\cdot \mid X, W) = \mathbb{P}_{K^{[X,W]}}.$$

**Remark.** For a concrete case of the conditional measure of determinantal point processes, the reader is also referred to [10] for conditional measures of generalized Ginibre point processes.

*1.3.2. The local property and the martingale lemma.* At the centre of our argument lies

**Lemma 1.10** (First local property of conditional kernels). *Let $B \subset E$ be a bounded Borel subset and let $Q$ be an orthogonal projection, acting in $L^2(E, \mu)$, such that* $\mathrm{Ran}(Q) \subset L^2(E \setminus B, \mu)$ *and the operator $QKQ$ is locally trace class. For $\mathbb{P}_K$-almost every $X \in \mathrm{Conf}(E)$, we have*

$$\big((Q + \chi_B)K(Q + \chi_B)\big)^{[X,B]} = (Q + \chi_B)K^{[X,B]}(Q + \chi_B) = QK^{[X,B]}Q. \tag{1.14}$$

**Remark.** The formula (1.14) is a strengthening, at the level of kernels, of the general property of point processes that conditioning on the restriction to a subset commutes with the forgetting projection onto a larger subset; see Proposition 2.4 below. The local property can be interpreted in terms of Neretin's formalism [26]: a determinantal measure is viewed as a "determinantal state" on a specially constructed algebra, and in order that conditional states themselves be determinantal the local property must be valid. The local property can thus be seen as the non-commutative analogue of the fact that the operation of conditioning commutes with the operation of restriction of a configuration to a subset.

Let $A, B$ be disjoint bounded Borel subsets of $E$. It is a general property of point processes that conditioning first on $A$ and then on $B$ amounts to a single conditioning on $A \cup B$. A manifestation of this general property at the level of kernels is

**Lemma 1.11** (Second local property of conditional kernels). *Let $A, B$ be disjoint bounded Borel subsets of $E$. For $\mathbb{P}_K$-almost every $X \in \mathrm{Conf}(E)$, we have*

$$(K^{[X,A]})^{[X,B]} = (K^{[X,B]})^{[X,A]} = K^{[X,A\cup B]}.$$

**Remark.** The second local property stated in Lemma 1.11 will be proved using the elementary local property at the measure-theoretic level of taking the conditional measures and the first local property stated in Lemma 1.10 applied to a family of rank-one orthogonal projections $Q$.

Using the local properties, we establish the following key martingale property of the kernels $K^{[X,B]}$.

**Lemma 1.12.** *Let $W \subset E$ be a Borel subset let $B_1 \subset B_2 \subset \cdots \subset W$ be an increasing exhausting sequence of bounded Borel subsets of $W$. The sequence of random variables*

$$(\chi_{E\setminus W} K^{[X,B_n]} \chi_{E\setminus W})_{n\in\mathbb{N}}$$

*is an $(\mathcal{F}(B_n))_{n\in\mathbb{N}}$-adapted operator-valued martingale defined on the probability space* $(\mathrm{Conf}(E), \mathcal{F}(E), \mathbb{P}_K)$.

By definition, we have $K^{[X,B]} = K^{[X\cap B,\, B]}$. Hence the mapping $X \mapsto K^{[X,B]}$ is an $\mathcal{F}(B)$-measurable operator-valued random variable defined on the probability space $(\mathrm{Conf}(E), \mathcal{F}(E), \mathbb{P}_K)$. Lemma 1.12 is equivalent to the claim that, for any $\varphi \in L^2(E \setminus W, \mu)$, the sequence $(\langle \chi_{E\setminus W} K^{[X,B_n]} \chi_{E\setminus W}\varphi, \varphi\rangle)_{n\in\mathbb{N}}$ is an $(\mathcal{F}(B_n))_{n\in\mathbb{N}}$-adapted real-valued martingale defined on the probability space $(\mathrm{Conf}(E), \mathcal{F}(E), \mathbb{P}_K)$. This notion of being a martingale is equivalent to the general notion of Fréchet-space-valued martingales (see Pisier [31]).

## 2. Conditional processes and martingales

### 2.1. *Martingales and the Radon–Nikodym property*

*2.1.1. Vector-valued and measure-valued martingales.* Let $(\Omega, \mathscr{F}, (\mathscr{F}_n)_{n=1}^{\infty}, \mathbf{P})$ be a filtered probability space. Let $\mathfrak{B}$ be a Banach space. A map $F : \Omega \to \mathfrak{B}$ is called *Bochner measurable* if there exists a sequence $F_n$ of measurable (in the usual sense) step functions such that $F_n(\omega) \to F(\omega)$ almost everywhere. For any $1 \le p < \infty$, we denote by $L^p(\Omega, \mathscr{F}, \mathbf{P}; \mathfrak{B})$ the set of all Bochner measurable functions $F : \Omega \to \mathfrak{B}$ such that $\int_\Omega \|F(\omega)\|_{\mathfrak{B}}^p \, \mathbf{P}(d\omega) < \infty$. The space $L^p(\Omega, \mathscr{F}, \mathbf{P}; \mathfrak{B})$ is a Banach space with the norm

$$\|F\|_{L^p(\mathfrak{B})} := \left(\int_\Omega \|F(\omega)\|_{\mathfrak{B}}^p \, \mathbf{P}(d\omega)\right)^{1/p}.$$

The algebraic tensor product $L^p(\Omega, \mathscr{F}, \mathbf{P}) \otimes \mathfrak{B}$ is dense in $L^p(\Omega, \mathscr{F}, \mathbf{P}; \mathfrak{B})$. The operator

$$\mathbb{E}[\cdot \mid \mathscr{F}_n] \otimes \mathrm{Id}_{\mathfrak{B}} : L^p(\Omega, \mathscr{F}, \mathbf{P}) \otimes \mathfrak{B} \to L^p(\Omega, \mathscr{F}, \mathbf{P}) \otimes \mathfrak{B}$$

extends uniquely to a bounded linear operator on $L^p(\Omega, \mathscr{F}, \mathbf{P}; \mathfrak{B})$, for which we keep the name "conditional expectation" and the notation, thus obtaining the operator $\mathbb{E}[\cdot \mid \mathscr{F}_n] : L^p(\Omega, \mathscr{F}, \mathbf{P}; \mathfrak{B}) \to L^p(\Omega, \mathscr{F}, \mathbf{P}; \mathfrak{B})$. A sequence $(R_n)_{n=1}^{\infty}$ in $L^p(\Omega, \mathscr{F}, \mathbf{P}; \mathfrak{B})$ is called an *$(\mathscr{F}_n)_{n=1}^{\infty}$-adapted martingale* if $R_n = \mathbb{E}[R_{n+1} \mid \mathscr{F}_n]$ for any $n \in \mathbb{N}$.

Assume now that $\mathfrak{B}$ is a *separable* space. Then there exists a countable subset $D$ of the unit ball of the dual space $\mathfrak{B}^*$ such that for any $x \in \mathfrak{B}$, we have $\|x\| = \sup_{\xi\in D} |\xi(x)|$. We will need the Pettis measurability theorem for separable Banach spaces.

**Proposition 2.1** ([29, p. 278]). *A function $F : \Omega \to \mathfrak{B}$ is Bochner measurable with respect to $\mathscr{F}$ if and only if for any $\xi \in D$, the scalar function $\omega \mapsto \xi(F(\omega))$ is $\mathscr{F}$-measurable. A sequence $(R_n)_{n=1}^{\infty}$ in $L^p(\Omega, \mathscr{F}, \mathbf{P}; \mathfrak{B})$ is an $(\mathscr{F}_n)_{n=1}^{\infty}$-adapted martingale if and only if for any $\xi \in D$, the sequence $(\xi(R_n))_{n=1}^{\infty}$ is an $(\mathscr{F}_n)_{n=1}^{\infty}$-adapted martingale.*

In this paper, we apply Proposition 2.1 in the particular case when $\mathfrak{B} = \mathscr{L}_1(L^2(E, \mu))$, the space of trace class operators on $L^2(E, \mu)$ and $D$ is the set of contractive finite rank operators on $L^2(E, \mu)$. Martingales in $\mathscr{L}_{1,\mathrm{loc}}(L^2(E, \mu))$ are reduced to the previous case by restricting to $L^2(B, \mu)$ with $B$ a bounded Borel subset of $E$.

Let $(T, \mathscr{A})$ be a topological space equipped with the $\sigma$-algebra of Borel subsets of $T$. We denote by $\mathfrak{P}(T, \mathscr{A})$ the space of Borel probability measures on $(T, \mathscr{A})$. A Borel map $M : \Omega \to \mathfrak{P}(T, \mathscr{A})$ is called a *random probability measure*. Equivalently, we assume that for any $A \in \mathscr{A}$, the map $\omega \mapsto M(\omega, A) := M(\omega)(A)$ is measurable. For more details, see Kallenberg [20, Section 1.2]. A sequence $(M_n)_{n=1}^{\infty}$ of random probability measures is called an $(\mathscr{F}_n)_{n=1}^{\infty}$-*adapted measure-valued martingale* on $(T, \mathscr{A})$ if for any $A \in \mathscr{A}$, the sequence $(M_n(\cdot, A))_{n\in\mathbb{N}}$ is a usual $(\mathscr{F}_n)_{n=1}^{\infty}$-adapted martingale.

*2.1.2. The Radon–Nikodym property.* In proving convergence of conditional kernels, we will use the Radon–Nikodym property for the space of trace class operators. Here we briefly recall the Radon–Nikodym property for Banach spaces; see Dunford–Pettis [12], Phillips [30] and Chapter 2 in Pisier's recent monograph [31] for a more detailed exposition.

Let $\mathfrak{B}$ be a Banach space. Let $(\Omega, \mathscr{F})$ be a measurable space. Any $\sigma$-additive map $m : \mathscr{F} \to \mathfrak{B}$ is called a ($\mathfrak{B}$-valued) *vector measure*. A vector measure $m$ is said to have *finite total variation* if

$$\sup\Big\{\sum_{i=1}^{n} \|m(A_i)\|_{\mathfrak{B}} : \Omega = \bigsqcup_{i=1}^{n} A_i \text{ is a measurable partition of } \Omega\Big\} < \infty.$$

Given a probability measure $\mathbf{P}$ on $(\Omega, \mathscr{F})$, we say that the vector measure $m$ is absolutely continuous with respect to $\mathbf{P}$ if there exists a non-negative function $w \in L^1(\Omega, \mathscr{F}, \mathbf{P})$ such that

$$\|m(A)\|_{\mathfrak{B}} \leq \int_A w \, \mathrm{d}\mathbf{P} \quad \text{for any } A \in \mathscr{F}.$$

**Definition 2.2.** A Banach space $\mathfrak{B}$ is said to have the *Radon–Nikodym property* if for any probability space $(\Omega, \mathscr{F}, \mathbf{P})$ and any $\mathfrak{B}$-valued measure $m$ on $(\Omega, \mathscr{F})$, with $m$ having finite total variation and being absolutely continuous with respect to $\mathbf{P}$, there exists a Bochner integrable function $F_m \in L^1(\Omega, \mathscr{F}, \mathbf{P}; \mathfrak{B})$ such that

$$m(A) = \int_A F_m \, \mathrm{d}\mathbf{P} \quad \text{for any } A \in \mathscr{F}.$$

By Pisier [31, Theorem 2.9], the Radon–Nikodym property for a Banach space $\mathfrak{B}$ is equivalent to either one of the following requirements:

(i) Every $\mathfrak{B}$-valued martingale which is bounded in $L^1(\Omega, \mathscr{F}, \mathbf{P}; \mathfrak{B})$ converges almost surely.

(ii) Every uniformly integrable $\mathfrak{B}$-valued martingale which is bounded in the space $L^1(\Omega, \mathscr{F}, \mathbf{P}; \mathfrak{B})$ converges almost surely and in $L^1(\Omega, \mathscr{F}, \mathbf{P}; \mathfrak{B})$.
(iii) For any $p > 1$, every $\mathfrak{B}$-valued martingale which is bounded in $L^p(\Omega, \mathscr{F}, \mathbf{P}; \mathfrak{B})$ converges almost surely and in $L^p(\Omega, \mathscr{F}, \mathbf{P}; \mathfrak{B})$.

The Banach space $\mathscr{L}_1(L^2(E, \mu))$ of trace class operators on $L^2(E, \mu)$ has the Radon–Nikodym property. More precisely, Pisier [31, Corollary 2.15] proves that if $\mathfrak{B}$ is separable and is a dual space of another Banach space, then $\mathfrak{B}$ has the Radon–Nikodym property. The separable space $\mathscr{L}_1(L^2(E, \mu))$ is the dual space of the space of compact operators on $L^2(E, \mu)$, and therefore we have

**Proposition 2.3.** *The space $\mathscr{L}_1(L^2(E, \mu))$ has the Radon–Nikodym property.*

Note that the first characterization of the Radon–Nikodym property of $\mathscr{L}_1(L^2(E, \mu))$ will be used in the proof of Lemma 1.9. Each of the characterizations (ii) or (iii) of the Radon–Nikodym property can be used to obtain the equality (5.9).

### 2.2. *Conditional measures of point processes*

Let $E$ be a locally compact $\sigma$-compact Polish space, endowed with a positive $\sigma$-finite Radon measure $\mu$. We assume that the metric on $E$ is such that any bounded set is relatively compact (see Hocking and Young [17, Theorem 2-61]).

A *configuration* $X = \{x_i\}$ on $E$ is by definition a *locally finite* countable subset of $E$, possibly with multiplicities. A configuration is called *simple* if all points in it have multiplicity one. Let $\mathrm{Conf}(E)$ denote the set of all configurations on $E$. The mapping $X \mapsto N_X := \sum_i \delta_{x_i}$ embeds $\mathrm{Conf}(E)$ into the space of Radon measures on $E$. Under the *vague topology* (that is, the topology that corresponds to the *vague convergence* of Radon measures: a sequence of Radon measures $\mu_n$ on $E$ converges vaguely to a limit Radon measure $\mu_\infty$ if $\lim_{n\to\infty} \int_E f\, \mathrm{d}\mu_n = \int_E f \mathrm{d}\mu_\infty$ for any *compactly supported* continuous function $f$ on $E$), the space $\mathrm{Conf}(E)$ is again a Polish space (see, e.g., Daley and Vere-Jones [11, Theorem 9.1.IV]). By definition, a *point process* on $E$ is a Borel probability measure $\mathbb{P}$ on $\mathrm{Conf}(E)$. We call $\mathbb{P}$ *simple* if $\mathbb{P}(\{X : X \text{ is simple}\}) = 1$.

For a Borel subset $W \subset E$, let $\mathcal{F}(W)$ be the $\sigma$-algebra on $\mathrm{Conf}(E)$ generated by all mappings $X \mapsto \#_B(X) := \#(X \cap B)$, where $B \subset W$ is a bounded Borel subset; the algebra $\mathcal{F}(E)$ coincides with the Borel $\sigma$-algebra on $\mathrm{Conf}(E)$.

Take a Borel subset $W \subset E$. A Borel probability measure $\mathbb{P}$ on $\mathrm{Conf}(E)$ can be viewed as a measure on $\mathrm{Conf}(W) \times \mathrm{Conf}(E \setminus W)$; we shall sometimes write $\mathbb{P} = \mathbb{P}_{W,\, E\setminus W}$ to stress dependence on $W$.

Denote by $(\pi_W)_*(\mathbb{P})$ the image measure of $\mathbb{P}$ under the surjective mapping $\pi_W : \mathrm{Conf}(E) \to \mathrm{Conf}(W)$ defined by $\pi_W(X) = X \cap W$. By disintegrating the probability measure $\mathbb{P}_{W,W^c}$, for $(\pi_W)_*(\mathbb{P})$-almost every configuration $X_0 \in \mathrm{Conf}(W)$ there exists a probability measure, denoted by $\mathbb{P}(\cdot \mid X_0, W)$, supported on $\{X_0\} \times \mathrm{Conf}(E \setminus W) \subset \mathrm{Conf}(E)$, such that

$$\mathbb{P}_{W,\, E\setminus W} = \int_{\mathrm{Conf}(W)} \mathbb{P}(\cdot \mid X_0, W)\, (\pi_W)_*(\mathbb{P})(\mathrm{d}X_0).$$

The measure $\mathbb{P}(\cdot \mid X_0, W)$ is referred to as the *conditional measure* on $\mathrm{Conf}(E \setminus W)$ or the *conditional point process* on $E \setminus W$ of $\mathbb{P}$, the condition being that the configuration on $W$ coincides with $X_0$. In what follows, we also write

$$\mathbb{P}(\cdot \mid X, W) := \mathbb{P}(\cdot | X \cap W, W) \quad \text{for } \mathbb{P}\text{-almost every configuration } X \in \mathrm{Conf}(E).$$

Moreover, for a random variable $f \in L^1(\mathrm{Conf}(E), \mathbb{P})$, we will write

$$\mathbb{E}_{\mathbb{P}}(f \mid X, W) := \mathbb{E}_{\mathbb{P}}[f \mid \mathcal{F}(W)](X \cap W).$$

**Proposition 2.4.** *Let $W_1, W_2$ be disjoint Borel subsets of $E$. For $\mathbb{P}$-almost every $X \in \mathrm{Conf}(E)$, we have*

$$(\pi_{W_1 \cup W_2})_*[\mathbb{P}](\cdot \mid X, W_1) = (\pi_{W_1 \cup W_2})_*[\mathbb{P}(\cdot \mid X, W_1)]. \tag{2.1}$$

*In other words, for fixed disjoint Borel subsets $W_1, W_2$ of $E$, the following two operations on point processes commute:*

- *taking the conditional measure of a point process, with the condition being that the configuration on $W_1$ coincides with $X \cap W_1$;*
- *taking the push-forward measure of a point process under the map $\pi_{W_1 \cup W_2} : \mathrm{Conf}(E) \to \mathrm{Conf}(W_1 \cup W_2)$.*

*Proof.* First we have

$$\mathbb{P} = \int_{\mathrm{Conf}(E)} \mathbb{P}(\cdot \mid X, W_1)\, \mathbb{P}(\mathrm{d}X),$$
$$(\pi_{W_1 \cup W_2})_*[\mathbb{P}] = \int_{\mathrm{Conf}(E)} (\pi_{W_1 \cup W_2})_*[\mathbb{P}(\cdot \mid X, W_1)]\, \mathbb{P}(\mathrm{d}X).$$

Since $\mathbb{P}(\cdot \mid X, W_1)$ is supported on the subset $\{Y \in \mathrm{Conf}(E) : Y \cap W_1 = X \cap W_1\}$, and $(\pi_{W_1 \cup W_2})_*[\mathbb{P}(\cdot \mid X, W_1)]$ is supported on $\{Z \in \mathrm{Conf}(W_1 \cup W_2) : Z \cap W_1 = X \cap W_1\}$, by the uniqueness of conditional measures we get (2.1). □

The conditional measure $\mathbb{P}(\cdot \mid X, W)$ is by definition supported on $\{X \cap W\} \times \mathrm{Conf}(E \setminus W)$, we may therefore consider $\mathbb{P}(\cdot \mid X, W)$ as a measure on $\mathrm{Conf}(E \setminus W)$. Further identifying the set $\mathrm{Conf}(E \setminus W)$ in a natural way with the subset $\mathrm{Conf}(E, E \setminus W) := \{X \in \mathrm{Conf}(E) : X \cap W = \emptyset\} \subset \mathrm{Conf}(E)$ when necessary, we may also view $\mathbb{P}(\cdot \mid X, W)$ as a measure on $\mathrm{Conf}(E)$ supported on $\mathrm{Conf}(E, E \setminus W)$.

### 2.3. Palm measures

The $n$-th *correlation measure* $\rho_{n,\mathbb{P}}$ of a point process $\mathbb{P}$ on $E$, if it exists, is the unique $\sigma$-finite Borel measure on $E^n$ satisfying

$$\rho_{n,\mathbb{P}}(A_1^{k_1} \times \cdots \times A_j^{k_j}) = \int_{\mathrm{Conf}(E)} \prod_{i=1}^{j} \frac{\#(X \cap A_i)!}{(\#(X \cap A_i) - k_i)!}\, \mathrm{d}\mathbb{P}(X)$$

for all bounded disjoint Borel subsets $A_1, \dots, A_j \subset E$ and positive integers $k_1, \dots, k_j$ with $k_1 + \cdots + k_j = n$. Here if $\#(X \cap A_i) < k_i$, we set $\#(X \cap A_i)!/(\#(X \cap A_i) - k_i)! = 0$.

For example, the $n$-th correlation measure of a determinantal process $\mathbb{P}_K$ is given by

$$\rho_{n,\mathbb{P}_K}(\mathrm{d}x_1 \cdots \mathrm{d}x_n) = \det\left(K(x_i, x_j)\right)_{1 \le i,j \le n} \cdot \mu^{\otimes n}(\mathrm{d}x_1 \cdots \mathrm{d}x_n),$$

where $K(x, y)$ is the integral kernel of the operator $K$ satisfying (1.6).

Assume that $\mathbb{P}$ is a simple point process on $E$ such that $\rho_{n,\mathbb{P}}$ exists for any $n \in \mathbb{N}$. The reduced $n$-th order *Campbell measure* $\mathscr{C}^!_{n,\mathbb{P}}$ of $\mathbb{P}$ is a $\sigma$-finite measure on $E^n \times \mathrm{Conf}(E)$ satisfying

$$\int_{E^n \times \mathrm{Conf}(E)} F(x, X)\, \mathscr{C}^!_{n,\mathbb{P}}(\mathrm{d}x \times \mathrm{d}X) = \int_{\mathrm{Conf}(E)} \Big[ \sum_{x \in X^n}^{\#} F(x, X \setminus \{x_1, \dots, x_n\}) \Big] \mathbb{P}(\mathrm{d}X)$$

for any Borel function $F : E^n \times \mathrm{Conf}(E) \to \mathbb{R}^+$. Here $\sum^{\#}$ is the summation over all *ordered n-tuples* $(x_1, \dots, x_n)$ with distinct coordinates $x_1, \dots, x_n \in X$. Disintegrating $\mathscr{C}^!_{n,\mathbb{P}}(\mathrm{d}x \times \mathrm{d}X)$, we obtain

$$\int_{E^n \times \mathrm{Conf}(E)} F(x, X)\, \mathscr{C}^!_{n,\mathbb{P}}(\mathrm{d}x \times \mathrm{d}X) = \int_{E^n} \rho_{n,\mathbb{P}}(\mathrm{d}x) \int_{\mathrm{Conf}(E)} F(x, X)\, \mathbb{P}^x(\mathrm{d}X), \quad (2.2)$$

where the probability measures $\mathbb{P}^x$ are defined for $\rho_{n,\mathbb{P}}$-almost every $x \in E^n$ and are called *reduced Palm measures* of $\mathbb{P}$. In what follows, by *Palm measures* we always mean reduced Palm measures. Note that the Palm measure $\mathbb{P}^{x_1,\dots,x_n}$ is invariant under permutation of the coordinates in $(x_1, \dots, x_n)$. Therefore, for a configuaration $X$ and a bounded subset $B \subset E$, we will write $\mathbb{P}^{X \cap B}$ for the Palm measure of $\mathbb{P}$ corresponding to the points of $X \cap B$, that is,

$$\mathbb{P}^{X \cap B} := \mathbb{P}^{x_1,\dots,x_n} \quad \text{provided that} \quad X \cap B = \{x_1, \dots, x_n\}. \quad (2.3)$$

### *2.4. Determinantal point processes, conditioning on bounded subsets*

In this section, we state an elementary result which gives, for determinantal point processes, the form of conditional measures with respect to restricting the configuration to a *bounded subset* $B \subset E$.

**Proposition 2.5.** *Assume that* $K : L^2(E, \mu) \to L^2(E, \mu)$ *is a bounded self-adjoint locally trace class operator with* $\mathrm{spec}(K) \subset [0, 1]$. *Let* $B \subset E$ *be a bounded Borel subset. Then for* $\mathbb{P}_K$*-almost every* $X \in \mathrm{Conf}(E)$*, the conditional point process* $\mathbb{P}_K(\cdot \mid X, B)$ *is again a determinantal point process on* $E \setminus B$*, induced by a correlation kernel* $K^{[X,B]}$ *defined in* (1.9).

If we assume moreover that the kernel $K$ is an orthogonal projection, then the kernel $K^{[X,B]}$ defined in (1.9) has the following meaning.

**Proposition 2.6.** *If $K$ is the orthogonal projection onto a closed subspace $H \subset L^2(E, \mu)$, then the kernel $K^{p_1,\dots,p_n}$ corresponds to the orthogonal projection from $L^2(E, \mu)$ onto the subspace*

$$H(p_1, \dots, p_n) := \{h \in H : h(p_1) = \cdots = h(p_n) = 0\}.$$

*Moreover, for a bounded Borel subset $B \subset E$, the operator $K^{[X,B]}$ is the orthogonal projection onto the closure of the subspace*

$$\chi_{E \setminus B} H(X \cap B) = \{\chi_{E \setminus B} h : h \in H(X \cap B)\}.$$

The proofs of Propositions 2.5 and 2.6 will be given in Section 8.1 in the Appendix.

## 3. The local property: proof of Lemmas 1.10, 1.11

### 3.1. *Proof of Lemma 1.10*

Let $B \subset E$ be a bounded Borel subset and let $Q : L^2(E, \mu) \to L^2(E, \mu)$ be an orthogonal projection whose range satisfies $\mathrm{Ran}(Q) \subset L^2(E \setminus B, \mu)$ and such that $QKQ$ is locally trace class. Introduce a positive contractive locally trace class operator $R$ by the formula

$$R = R(K, B, Q) := (Q + \chi_B)K(Q + \chi_B). \tag{3.1}$$

**Remark.** The introduction of the operator $R$ is on the one hand in order to simplify the notation $(Q + \chi_B)K(Q + \chi_B)$, and on the other hand, in the proof of Lemma 1.11, we will use in particular an auxiliary determinantal point process $\mathbb{P}_R$ when $Q$ is the rank-one orthogonal projection, that is, $Q = \varphi \otimes \overline{\varphi}$ with $\varphi \in L^2(E \setminus B, \mu)$. In this case, the corresponding determinantal point process $\mathbb{P}_R$ has the same law as $\mathbb{P}_K$ when restricted to $B$, and the expected number of particles outside $B$ for $\mathbb{P}_R$ will be particularly useful for us in proving equalities of conditional kernels; see the remark after the proof of Lemma 1.11 for more details.

Recall that in Section 1.3, we fixed a Borel subset $E_0 \subset E$ such that $\mu(E \setminus E_0) = 0$ and the kernel $K(x, y)$ is well-defined on $E_0 \times E_0$. Recall also the notation introduced in Definition 1.8.

**Lemma 3.1.** *Let $R$ be the operator introduced in (3.1). For any $p \in B \cap E_0$, $R^p = (Q + \chi_B)K^p(Q + \chi_B)$. More generally, for $(p_1, \dots, p_n) \in (B \cap E_0)^n$, we have*

$$R^{p_1,\dots,p_n} = (Q + \chi_B)K^{p_1,\dots,p_n}(Q + \chi_B).$$

*In particular,*

$$R^{X \cap B} = (Q + \chi_B)K^{X \cap B}(Q + \chi_B) \quad \textit{for } \mathbb{P}_K\textit{-almost every } X \in \mathrm{Conf}(E).$$

*Proof.* First of all, the kernel of the operator $R$ can be chosen such that for any $p \in B \cap E_0$,

$$R(\cdot, p) = (Q + \chi_B)[K(\cdot, p)]. \tag{3.2}$$

That is, the function $x \mapsto R(x, p)$ is given by the image of the function $x \mapsto K(x, p)$ under the operator $Q + \chi_B$. Indeed, since $p \in B \cap E_0$ and $\mathrm{Ran}(Q) \subset L^2(E \setminus B, \mu)$, we have

$$[QKQ](\cdot, p) = 0 \quad \text{and} \quad [\chi_B KQ](\cdot, p) = 0.$$

Therefore, for $p \in B \cap E_0$,

$$R(\cdot, p) = [(Q + \chi_B)K\chi_B](\cdot, p) = [QK\chi_B](\cdot, p) + \chi_B(\cdot)K(\cdot, p). \tag{3.3}$$

Now for any $\varphi \in L^2(B, \mu)$,

$$[QK\chi_B\varphi](x) = Q\left[\int_B K(\cdot, y)\varphi(y)\,\mathrm{d}\mu(y)\right](x) = \int_B Q[K(\cdot, y)](x)\varphi(y)\,\mathrm{d}\mu(y).$$

That means the kernel of $QK\chi_B$ is given by $(x, y) \mapsto Q[K(\cdot, y)](x)\chi_B(y)$. Thus for $p \in B \cap E_0$, we may take $[QK\chi_B](\cdot, p) = Q[K(\cdot, p](\cdot)$. Combining this with (3.3), we obtain the desired equality (3.2).

Now since $R(p, p) = K(p, p)$, we have

$$\begin{aligned} R^p &= R - \frac{R(\cdot, p) \otimes \overline{R(\cdot, p)}}{R(p, p)} \\ &= (Q + \chi_B)K(Q + \chi_B) - \frac{(Q + \chi_B)[K(\cdot, p)] \otimes \overline{(Q + \chi_B)[K(\cdot, p)]}}{K(p, p)} \\ &= (Q + \chi_B)\left[K - \frac{K(\cdot, p) \otimes \overline{K(\cdot, p)}}{K(p, p)}\right](Q + \chi_B) = (Q + \chi_B)K^p(Q + \chi_B). \end{aligned}$$

The formula for $R^{p_1,\ldots,p_n}$ follows immediately by induction on $n$. □

**Lemma 3.2.** *Let $\widetilde{K} : L^2(E, \mu) \to L^2(E, \mu)$ be a bounded self-adjoint locally trace class operator with* $\mathrm{spec}(\widetilde{K}) \subset [0, 1]$. *Let $B$ be a bounded Borel subset of $E$ such that* $\mathbb{P}_{\widetilde{K}}(\#_B = 0) > 0$. *Let $Q : L^2(E, \mu) \to L^2(E, \mu)$ be an orthogonal projection satisfying* $\mathrm{Ran}(Q) \subset L^2(E \setminus B, \mu)$ *and such that $Q\widetilde{K}Q$ is locally trace class. Let $\widetilde{R} = (Q + \chi_B)\widetilde{K}(Q + \chi_B)$ be the operator introduced as in* (3.1). *Then*

$$\chi_{E\setminus B}\widetilde{R}(1 - \chi_B\widetilde{R})^{-1}\chi_{E\setminus B} = Q\big(\chi_{E\setminus B}\widetilde{K}(1 - \chi_B\widetilde{K})^{-1}\chi_{E\setminus B}\big)Q. \tag{3.4}$$

*Proof.* The gap probability $\mathbb{P}_{\widetilde{K}}(\#_B = 0)$ is given by

$$\mathbb{P}_{\widetilde{K}}(\#_B = 0) = \mathbb{P}_{\widetilde{K}}(\{X : X \cap B = \emptyset\}) = \det(1 - \chi_B\widetilde{K}\chi_B) > 0. \tag{3.5}$$

It follows that $1 - \chi_B\widetilde{K}\chi_B$ is invertible and hence 1 is not an eigenvalue of $\chi_B\widetilde{K}\chi_B$. But since $\chi_B K\chi_B$ is a priori a positive contraction and $\chi_B\widetilde{K}\chi_B$ is compact, its norm coincides with its maximal eigenvalue. Hence $\chi_B\widetilde{K}\chi_B$ is strictly contractive. But we also have

$$\|\chi_B\widetilde{K}\chi_B\| = \|(\chi_B\widetilde{K}^{1/2})(\chi_B\widetilde{K}^{1/2})^*\| = \|\chi_B\widetilde{K}^{1/2}\|^2 < 1. \tag{3.6}$$

Hence

$$\|\chi_B\widetilde{K}\| \le \|\chi_B\widetilde{K}^{1/2}\|\,\|\widetilde{K}^{1/2}\| < 1. \tag{3.7}$$

Therefore, both $\chi_B \widetilde{K}$ and $\chi_B \widetilde{R} = \chi_B \widetilde{K}(Q + \chi_B)$ are strictly contractive. In particular, the operators on both the left hand side and the right hand side of (3.4) are well-defined.

Since $Q$ commutes with $\chi_{E\setminus B}$, we have

$$\chi_{E\setminus B}\widetilde{R}\chi_{E\setminus B} = Q\chi_{E\setminus B}\widetilde{K}\chi_{E\setminus B}Q \quad \text{and} \quad \chi_{E\setminus B}\widetilde{R}\chi_B = Q\chi_{E\setminus B}\widetilde{K}\chi_B.$$

By definition of $\widetilde{R}$, we have $\chi_B\widetilde{R}\chi_B = \chi_B\widetilde{K}\chi_B$. Therefore, for $n \geq 1$,

$$\begin{aligned}
&\chi_{E\setminus B}\widetilde{R}(\chi_B\widetilde{R})^n\chi_{E\setminus B} \\
&\qquad = \chi_{E\setminus B}\widetilde{R}(\chi_B\widetilde{R})\cdots(\chi_B\widetilde{R})\chi_{E\setminus B} = \chi_{E\setminus B}\widetilde{R}\chi_B(\chi_B\widetilde{R}\chi_B)^{n-1}\chi_B\widetilde{R}\chi_{E\setminus B} \\
&\qquad = Q\chi_{E\setminus B}\widetilde{K}\chi_B(\chi_B\widetilde{K}\chi_B)^{n-1}\chi_B\widetilde{K}\chi_{E\setminus B}Q = Q\chi_{E\setminus B}\widetilde{K}(\chi_B\widetilde{K})^n\chi_{E\setminus B}Q.
\end{aligned}$$

Now since $\chi_B\widetilde{R}$ and $\chi_B\widetilde{K}$ are both strictly contractive, by using the above equality we can finally write

$$\begin{aligned}
\chi_{E\setminus B}\widetilde{R}(1-\chi_B\widetilde{R})^{-1}\chi_{E\setminus B} &= \sum_{n=0}^{\infty}\chi_{E\setminus B}\widetilde{R}(\chi_B\widetilde{R})^n\chi_{E\setminus B} = \sum_{n=0}^{\infty}Q\chi_{E\setminus B}\widetilde{K}(\chi_B\widetilde{K})^n\chi_{E\setminus B}Q \\
&= Q\chi_{E\setminus B}\widetilde{K}(1-\chi_B\widetilde{K})^{-1}\chi_{E\setminus B}Q.
\end{aligned}$$ □

*Conclusion of the proof of Lemma 1.10.* We want to apply Lemma 3.2 to the operator $\widetilde{K} = K^{X\cap B}$. By our assumption, the orthogonal projection $Q$ with $\mathrm{Ran}(Q) \subset L^2(E \setminus B, \mu)$ has $QKQ$ locally trace class. For the definition (1.7) of $K^p$, we have $K^p \leq K$ in the operator sense. Then by iterating, $\widetilde{K} = K^{X\cap B} \leq K$ and hence $Q\widetilde{K}Q \leq QKQ$ in the operator sense. This implies in particular that $Q\widetilde{K}Q$ is locally trace class. By Lemma 3.1, for $\mathbb{P}_K$-almost every $X \in \mathrm{Conf}(E)$,

$$\widetilde{R} = (Q + \chi_B)\widetilde{K}(Q + \chi_B) = (Q + \chi_B)K^{X\cap B}(Q + \chi_B) = R^{X\cap B}. \tag{3.8}$$

On the other hand, by Propositions 8.1 and 2.5,

$$\mathbb{P}_K(\cdot \mid X, B) = \overline{(\mathbb{P}_K)^{X\cap B}\restriction_{\mathrm{Conf}(E\setminus B)}} = \overline{\mathbb{P}_{K^{X\cap B}}\restriction_{\mathrm{Conf}(E\setminus B)}} \quad \text{for } \mathbb{P}_K\text{-a.e. } X \in \mathrm{Conf}(E).$$

By definition (8.1) of the normalized restriction measure $\overline{\mathbb{P}_{K^{X\cap B}}\restriction_{\mathrm{Conf}(E\setminus B)}}$, we must have

$$\mathbb{P}_{K^{X\cap B}}(\#_B = 0) = \mathbb{P}_{K^{X\cap B}}(\mathrm{Conf}(E \setminus B)) > 0 \quad \text{for } \mathbb{P}_K\text{-a.e. } X \in \mathrm{Conf}(E).$$

Thus the assumptions of Lemma 3.2 are satisfied. By (3.4) and (3.8), for $\mathbb{P}_K$-almost every $X \in \mathrm{Conf}(E)$,

$$\begin{aligned}
&\chi_{E\setminus B}R^{X\cap B}(1-\chi_B R^{X\cap B})^{-1}\chi_{E\setminus B} = \chi_{E\setminus B}\widetilde{R}(1-\chi_B\widetilde{R})^{-1}\chi_{E\setminus B} \\
&\qquad = Q\big(\chi_{E\setminus B}\widetilde{K}(1-\chi_B\widetilde{K})^{-1}\chi_{E\setminus B}\big)Q = Q\big(\chi_{E\setminus B}K^{X\cap B}(1-\chi_B K^{X\cap B})^{-1}\chi_{E\setminus B}\big)Q.
\end{aligned}$$

By definition (1.9) (applied to both $K$ and $R$), the above equality means exactly

$$R^{[X,B]} = QK^{[X,B]}Q \quad \text{for } \mathbb{P}_K\text{-a.e. } X \in \mathrm{Conf}(E). \tag{3.9}$$

By definition (3.1), we thus obtain the desired equality

$$\big((Q+\chi_B)K(Q+\chi_B)\big)^{[X,B]} = QK^{[X,B]}Q \quad \text{for } \mathbb{P}_K\text{-a.e. } X \in \mathrm{Conf}(E).$$

To complete the proof of Lemma 1.10, we only need to observe the elementary equality

$$QK^{[X,B]}Q = (Q+\chi_B)K^{[X,B]}(Q+\chi_B),$$

which follows from (1.13) and thus $\chi_B K^{[X,B]} = K^{[X,B]}\chi_B = 0$. □

### *3.2. Proof of Lemma 1.11*

By assumption, $A, B$ are disjoint bounded subsets of $E$, hence $\chi_{A\cup B} = \chi_A + \chi_B$. Choose an arbitrary unit vector $\varphi \in L^2(E \setminus (A \cup B), \mu)$ and let $Q = \varphi \otimes \overline{\varphi}$ be the orthogonal projection from $L^2(E, \mu)$ onto the one-dimensional subspace spanned by $\varphi$. Define

$$\widehat{R} = R_\varphi := (\chi_A + \chi_B + Q)K(\chi_A + \chi_B + Q).$$

Then by using (3.9), with $R$ and $B$ replaced by $\widehat{R}$ and $A \cup B$ respectively, we obtain

$$\widehat{R}^{[X,\, A\cup B]} = QK^{[X,\, A\cup B]}Q \quad \text{for } \mathbb{P}_K\text{-a.e. } X \in \mathrm{Conf}(E). \tag{3.10}$$

Again by using (3.9) this time with $R$, $Q$ and $B$ replaced by $\widehat{R}$, $Q+\chi_B$ and $A$ respectively, we get

$$\widehat{R}^{[X,\, A]} = (\chi_B + Q)K^{[X,\, A]}(\chi_B + Q) \quad \text{for } \mathbb{P}_K\text{-a.e. } X \in \mathrm{Conf}(E). \tag{3.11}$$

By a further application of (3.9) with $K$ replaced by $K^{[X,A]}$ and then by using (3.11) to replace $R$ in (3.9) by $\widehat{R}^{[X,A]}$, we find that for $\mathbb{P}_K$-almost every $X \in \mathrm{Conf}(E)$ and for $\mathbb{P}_{K^{[X,A]}}$-almost every $Z \in \mathrm{Conf}(E)$,

$$(\widehat{R}^{[X,\, A]})^{[Z,\, B]} = Q(K^{[X,\, A]})^{[Z,\, B]}Q. \tag{3.12}$$

But since $A$ is bounded, Proposition 2.5 implies that $\mathbb{P}_{K^{[X,A]}} = \mathbb{P}_K(\cdot \mid X,\ A)$. This combined with the equalities $\widehat{R}^{[X,\, A]} = \widehat{R}^{[X\cap A,\, A]}$ and $K^{[X,\, A]} = K^{[X\cap A,\, A]}$ implies that the double almost every statement (3.12) is equivalent to

$$(\widehat{R}^{[X,\, A]})^{[X,\, B]} = Q(K^{[X,\, A]})^{[X,\, B]}Q \quad \text{for } \mathbb{P}_K\text{-a.e. } X \in \mathrm{Conf}(E). \tag{3.13}$$

By Proposition 2.5, we also have the following description of conditional measures:

$$\mathbb{P}_{\widehat{R}}(\cdot \mid X,\, A) = \mathbb{P}_{\widehat{R}^{[X,A]}} \text{ and } \mathbb{P}_{\widehat{R}}(\cdot \mid X,\, A\cup B) = \mathbb{P}_{\widehat{R}^{[X,A\cup B]}} \quad \text{for } \mathbb{P}_{\widehat{R}}\text{-a.e. } X \in \mathrm{Conf}(E).$$

The first equality above implies that

$$[\mathbb{P}_{\widehat{R}}(\cdot \mid X,\, A)](\cdot \mid X,\, B) = \mathbb{P}_{\widehat{R}^{[X,\, A]}}(\cdot \mid X,\, B) = \mathbb{P}_{(\widehat{R}^{[X,\, A]})^{[X,\, B]}} \quad \text{for } \mathbb{P}_{\widehat{R}}\text{-a.e. } X \in \mathrm{Conf}(E).$$

Now we may apply the measure-theoretic identity

$$[\mathbb{P}_{\widehat{R}}(\cdot \mid X,\, A)](\cdot \mid X,\, B) = \mathbb{P}_{\widehat{R}}(\cdot \mid X,\, A \cup B) \quad \text{for } \mathbb{P}_{\widehat{R}}\text{-a.e. } X \in \mathrm{Conf}(E)$$

and obtain

$$\mathbb{P}_{\widehat{R}^{[X,A\cup B]}} = \mathbb{P}_{(\widehat{R}^{[X,A]})^{[X,B]}} \quad \text{for } \mathbb{P}_{\widehat{R}}\text{-a.e. } X \in \mathrm{Conf}(E). \tag{3.14}$$

This equality of probability measures implies that for $\mathbb{P}_{\widehat{R}}$-almost every $X \in \mathrm{Conf}(E)$,

$$\begin{aligned}\mathbb{E}_{\mathbb{P}_{\widehat{R}}}[\#(X \cap (E \setminus (A \cup B)) \mid X, A \cup B] &= \mathrm{tr}\big(\chi_{E\setminus(A\cup B)}\widehat{R}^{[X,A\cup B]}\chi_{E\setminus(A\cup B)}\big)\\ &= \mathrm{tr}\big(\chi_{E\setminus(A\cup B)}(\widehat{R}^{[X,A]})^{[X,B]}\chi_{E\setminus(A\cup B)}\big).\end{aligned}$$

Combining this with (3.10), (3.11) and (3.13), we obtain the $\mathbb{P}_{\widehat{R}}$-almost sure equality

$$\mathrm{tr}\big(\chi_{E\setminus(A\cup B)}QK^{[X,A\cup B]}Q\chi_{E\setminus(A\cup B)}\big) = \mathrm{tr}\big(\chi_{E\setminus(A\cup B)}Q(K^{[X,A]})^{[X,B]}Q\chi_{E\setminus(A\cup B)}\big).$$

That is,

$$\langle K^{[X,A\cup B]}\varphi,\varphi\rangle = \langle (K^{[X,A]})^{[X,B]}\varphi,\varphi\rangle \quad \text{for } \mathbb{P}_{\widehat{R}}\text{-a.e. } X \in \mathrm{Conf}(E).$$

Since $\varphi$ is arbitrary and since $L^2(E \setminus (A \cup B), \mu)$ is separable and both $K^{[X,A\cup B]}$ and $(K^{[X,A]})^{[X,B]}$ are supported on $L^2(E \setminus (A \cup B), \mu)$, we obtain

$$K^{[X,A\cup B]} = (K^{[X,A]})^{[X,B]} \quad \text{for } \mathbb{P}_{\widehat{R}}\text{-a.e. } X \in \mathrm{Conf}(E). \tag{3.15}$$

Observe that the equality $\chi_{A\cup B}\widehat{R}\chi_{A\cup B} = \chi_{A\cup B}K\chi_{A\cup B}$ implies the equality $(\pi_{A\cup B})_*(\mathbb{P}_{\widehat{R}}) = (\pi_{A\cup B})_*(\mathbb{P}_K)$. Combining this with (3.15) and the fact that $K^{[X,A\cup B]}$ and $(K^{[X,A]})^{[X,B]}$ are $\mathcal{F}(A \cup B)$-measurable, we get the desired equality

$$K^{[X,A\cup B]} = (K^{[X,A]})^{[X,B]} \quad \text{for } \mathbb{P}_K\text{-a.e. } X \in \mathrm{Conf}(E). \qquad \square$$

**Remark.** Since different correlation kernels may correspond to the same determinantal point process, the coincidence (3.14) of the two determinantal point processes $\mathbb{P}_{\widehat{R}^{[X,A\cup B]}}$ and $\mathbb{P}_{(\widehat{R}^{[X,A]})^{[X,B]}}$ does not imply that $\widehat{R}^{[X,A\cup B]}$ and $(\widehat{R}^{[X,A]})^{[X,B]}$ are the same (if this were true, then the desired equality $K^{[X,A\cup B]} = (K^{[X,A]})^{[X,B]}$ would follow from (3.10) and (3.13) by varying $Q$). Our idea is to derive from (3.14) a useful equality of scalar qunatities and then to complete the proof of Lemma 1.11 by varying $Q = \varphi \otimes \overline{\varphi}$.

## 4. The martingale property: proof of Lemma 1.12

**Proposition 4.1.** *For any bounded Borel subset $B \subset E$, write*

$$\mathbb{E}_{\mathbb{P}_K}(K^{[X,B]}) = \int_{\mathrm{Conf}(E)} K^{[X,B]}\,\mathbb{P}_K(\mathrm{d}X).$$

*Then*

$$\mathbb{E}_{\mathbb{P}_K}(K^{[X,B]}) = \chi_{E\setminus B}K\chi_{E\setminus B}. \tag{4.1}$$

**Remark.** Extending the argument of Benjamini, Lyons, Peres and Schramm [1] for the case of spanning trees, Lyons [21, Lemma 7.17] proved (4.1) when $E$ is discrete and $K$ is an orthogonal projection on $\ell^2(E)$. Our proof, based on the local property, is quite different and works both in the continuous and the discrete settings.

*Proof of Lemma 1.12 assuming Proposition 4.1.* Applying Proposition 4.1 to the kernel $K^{[X,B_n]}$ and the bounded Borel subset $B_{n+1} \setminus B_n \subset E \setminus B_n$, we obtain

$$\mathbb{E}_{\mathbb{P}_{K^{[X,B_n]}}}\big[(K^{[X,\,B_n]})^{[X,B_{n+1}\setminus B_n]}\big] = \chi_{E\setminus B_{n+1}} K^{[X,\,B_n]} \chi_{E\setminus B_{n+1}} \quad \text{for } \mathbb{P}_K\text{-a.e. } X.$$

The equality $\mathbb{P}_{K^{[X,B_n]}} = \mathbb{P}_K(\cdot \mid X, B_n)$ now yields

$$\mathbb{E}_{\mathbb{P}_{K^{[X,B_n]}}}\big[(K^{[X,\,B_n]})^{[X,B_{n+1}\setminus B_n]}\big] = \mathbb{E}_{\mathbb{P}_K}\big[(K^{[X,\,B_n]})^{[X,B_{n+1}\setminus B_n]} \,\big|\, \mathcal{F}(B_n)\big] \quad \text{for } \mathbb{P}_K\text{-a.e. } X.$$

Combining this with Lemma 1.11, we get

$$\mathbb{E}_{\mathbb{P}_K}[K^{[X,B_{n+1}]} \mid \mathcal{F}(B_n)] = \chi_{E\setminus B_{n+1}} K^{[X,B_n]} \chi_{E\setminus B_{n+1}} \quad \text{for } \mathbb{P}_K\text{-a.e. } X.$$

By linearity of the composition on the left and on the right with the operator of multiplication by $\chi_{E\setminus W}$ and the elementary equalities $\chi_{E\setminus W}\cdot\chi_{E\setminus B_{n+1}} = \chi_{E\setminus W}$, we get the desired martingale property:

$$\mathbb{E}[\chi_{E\setminus W} K^{[X,B_{n+1}]} \chi_{E\setminus W} \mid \mathcal{F}(B_n)] = \chi_{E\setminus W} K^{[X,B_n]} \chi_{E\setminus W} \quad \text{for } \mathbb{P}_K\text{-a.e. } X. \qquad \square$$

*Proof of Proposition 4.1.* Let $\varphi \in L^2(E \setminus B, \mu)$ be such that $\|\varphi\|_2 = 1$. We use (3.1) for $Q = \varphi \otimes \overline{\varphi}$, the orthogonal projection onto the one-dimensional space spanned by $\varphi$, and thus set

$$R = (Q + \chi_B)K(Q + \chi_B) = (\varphi \otimes \overline{\varphi} + \chi_B)K(\varphi \otimes \overline{\varphi} + \chi_B). \tag{4.2}$$

We have the clear identity

$$(\pi_B)_*(\mathbb{P}_R) = \mathbb{P}_{\chi_B R \chi_B} = \mathbb{P}_{\chi_B K \chi_B} = (\pi_B)_*(\mathbb{P}_K). \tag{4.3}$$

By Lemma 1.10, for $\mathbb{P}_K$-almost every $X \in \mathrm{Conf}(E)$, we have

$$R^{[X,B]} = QK^{[X,B]}Q = (\varphi \otimes \overline{\varphi})K^{[X,B]}(\varphi \otimes \overline{\varphi}).$$

Since clearly $K^{[X,B]} = K^{[X\cap B,\,B]}$ and $R^{[X,B]} = R^{[X\cap B,\,B]}$, the above equality holds for $\mathbb{P}_R$-almost every $X \in \mathrm{Conf}(E)$. Now recall that $\mathbb{P}_R(\cdot \mid X, B) = \mathbb{P}_{R^{[X,\,B]}}$ for $\mathbb{P}_R$-almost every $X \in \mathrm{Conf}(E)$. Hence

$$\begin{aligned}\mathbb{E}_{\mathbb{P}_R}[\#_{E\setminus B} \mid X, B] &= \mathbb{E}_{\mathbb{P}_{R^{[X,B]}}}[\#_{E\setminus B}] = \mathrm{tr}(\chi_{E\setminus B} R^{[X,B]} \chi_{E\setminus B}) \\ &= \langle K^{[X,B]}\varphi, \varphi\rangle \quad \text{for } \mathbb{P}_R\text{-almost every } X \in \mathrm{Conf}(E).\end{aligned}$$

Note that by the definition (4.2) of $R$ and the assumption that $\varphi$ is supported on $E \setminus B$, we have

$$\mathbb{E}_{\mathbb{P}_R}[\#_{E\setminus B}] = \mathrm{tr}(\chi_{E\setminus B} R \chi_{E\setminus B}) = \mathrm{tr}(QKQ) = \langle K\varphi, \varphi\rangle.$$

On the other hand,

$$\mathbb{E}_{\mathbb{P}_R}[\#_{E\setminus B}] = \mathbb{E}_{\mathbb{P}_R}(\mathbb{E}_{\mathbb{P}_R}[\#_{E\setminus B} \mid X, B]) = \mathbb{E}_{\mathbb{P}_R}(\langle K^{[X,B]}\varphi, \varphi\rangle),$$

whence

$$\mathbb{E}_{\mathbb{P}_R}(\langle K^{[X,B]}\varphi, \varphi\rangle) = \langle K\varphi, \varphi\rangle. \tag{4.4}$$

The relation $K^{[X,B]} = K^{[X\cap B,\, B]}$ implies that $K^{[X,B]}$, when varying $X$, depends only on $X \cap B$, thus

$$\mathbb{E}_{\mathbb{P}_R}(\langle K^{[X,B]}\varphi, \varphi\rangle) = \mathbb{E}_{\mathbb{P}_R}(\langle K^{[X\cap B,\, B]}\varphi, \varphi\rangle) = \mathbb{E}_{(\pi_B)_*\mathbb{P}_R}(\langle K^{[X\cap B,\, B]}\varphi, \varphi\rangle). \tag{4.5}$$

Similarly,

$$\mathbb{E}_{\mathbb{P}_K}(\langle K^{[X,B]}\varphi, \varphi\rangle) = \mathbb{E}_{\mathbb{P}_K}(\langle K^{[X\cap B,\, B]}\varphi, \varphi\rangle) = \mathbb{E}_{(\pi_B)_*\mathbb{P}_K}(\langle K^{[X\cap B,\, B]}\varphi, \varphi\rangle). \tag{4.6}$$

The equalities (4.5) and (4.6) combined with (4.3) imply that

$$\mathbb{E}_{\mathbb{P}_K}(\langle K^{[X,B]}\varphi, \varphi\rangle) = \mathbb{E}_{\mathbb{P}_R}(\langle K^{[X,B]}\varphi, \varphi\rangle). \tag{4.7}$$

Therefore, by combining (4.7) with (4.4), we obtain

$$\mathbb{E}_{\mathbb{P}_K}(\langle K^{[X,B]}\varphi, \varphi\rangle) = \langle K\varphi, \varphi\rangle. \tag{4.8}$$

Since $\varphi$ is an arbitrary norm-one function in $L^2(E \setminus B)$ and since $K^{[X,B]} = \chi_{E\setminus B} K^{[X,B]} \chi_{E\setminus B}$, we obtain (4.1). □

## 5. Proof of Lemma 1.9

**Proposition 5.1.** *Let $W \subset E$ be a Borel subset, and let $B_1 \subset B_2 \subset \cdots \subset W$ be an increasing exhausting sequence of bounded Borel subsets of $W$. The sequence $(\chi_{E\setminus W} K^{[X,B_n]} \chi_{E\setminus W})_{n\in\mathbb{N}}$ converges $\mathbb{P}_K$-almost surely in the space of locally trace class operators.*

*Proof.* Since $K$ is locally of trace class, there exists a function $\psi : E \setminus W \to (0,1]$ such that $\psi^{1/2} K \psi^{1/2}$ is of trace class and for any bounded subset $B \subset E$, we have

$$\inf_{x\in B} \psi(x) > 0. \tag{5.1}$$

Then

$$\mathbb{E}_{\mathbb{P}_K}\Big(\sum_{x\in X} \psi(x)\Big) = \int_E \psi(x) K(x,x)\,\mu(\mathrm{d}x) = \mathrm{tr}(\psi^{1/2} K \psi^{1/2}) = M_\psi < \infty.$$

Denote

$$G(X,n) := \chi_{E\setminus W} K^{[X,B_n]} \chi_{E\setminus W}.$$

Then for any $n \in \mathbb{N}$,

$$\begin{aligned} M_\psi &= \mathbb{E}_{\mathbb{P}_K}\Big(\sum_{x\in X} \psi(x)\Big) = \mathbb{E}_{\mathbb{P}_K}\Big[\mathbb{E}_{\mathbb{P}_K}\Big(\sum_{x\in X} \psi(x) \,\Big|\, \mathcal{F}(B_n)\Big)\Big] \\ &= \mathbb{E}_{\mathbb{P}_K}[\mathrm{tr}(\psi^{1/2} G(X,n) \psi^{1/2})]. \end{aligned} \tag{5.2}$$

By the martingale property of the sequence $(G(X,n))_{n\in\mathbb{N}}$ and the equality (5.2), the sequence $(\psi^{1/2} G(X,n) \psi^{1/2})_{n\in\mathbb{N}}$ forms a *bounded* martingale in $L^1(\mathbb{P}_K, \mathscr{L}_1(L^2(E,\mu)))$. By Proposition 2.3, the Banach space $\mathscr{L}_1(L^2(E,\mu))$ has the Radon–Nikodym property.

Therefore there exists a measurable function $F(X,\infty)$ with values in $\mathscr{L}_1(L^2(E,\mu))$ such that

$$\psi^{1/2}G(X,n)\psi^{1/2} \xrightarrow[\mathbb{P}_K\text{-a.s.}]{\text{in } \mathscr{L}_1(L^2(E,\mu))} F(X,\infty). \tag{5.3}$$

The assumption (5.1) implies that $\psi^{-1/2}F(X,\infty)\psi^{-1/2}\in\mathscr{L}_{1,\mathrm{loc}}(L^2(E,\mu))$. Moreover, (5.1) implies that for any bounded subset $B\subset E$, since $\chi_B\psi^{-1/2}$ is bounded, the convergence (5.3) implies

$$\begin{aligned}\chi_B G(X,n)\chi_B &= \chi_B\psi^{-1/2}[\psi^{1/2}G(X,n)\psi^{1/2}]\chi_B\psi^{-1/2}\\ &\xrightarrow[\mathbb{P}_K\text{-a.s.}]{\text{in } \mathscr{L}_1(L^2(E,\mu))} \chi_B\psi^{-1/2}F(X,\infty)\psi^{-1/2}\chi_B.\end{aligned}$$

Since $B$ is an arbitrary bounded subset of $E$, the above convergence means exactly that

$$\chi_{E\setminus W}K^{[X,B_n]}\chi_{E\setminus W} = G(X,n) \xrightarrow[\mathbb{P}_K\text{-a.s.}]{\text{in } \mathscr{L}_{1,\mathrm{loc}}(L^2(E,\mu))} \psi^{-1/2}F(X,\infty)\psi^{-1/2}. \tag{5.4}$$

□

*Proof of Lemma 1.9.* By (8.14), for $\mathbb{P}_K$-almost every $X\in\mathrm{Conf}(E)$,

$$(\pi_{E\setminus W})_*[\mathbb{P}_K(\cdot\mid X,B_n)] \xrightarrow[\text{weakly}]{n\to\infty} \mathbb{P}_K(\cdot\mid X,W). \tag{5.5}$$

By Proposition 2.5, for $\mathbb{P}_K$-almost every $X\in\mathrm{Conf}(E)$,

$$(\pi_{E\setminus W})_*[\mathbb{P}_K(\cdot\mid X,B_n)] = \mathbb{P}_{\chi_{E\setminus W}K^{[X,B_n]}\chi_{E\setminus W}}. \tag{5.6}$$

Combining (5.4)–(5.6) with the fact that the convergence of correlation kernels in $\mathscr{L}_{1,\mathrm{loc}}(L^2(E,\mu))$ implies the weak convergence of the corresponding determinantal measures, we complete the proof of Lemma 1.9. □

**Remark.** Under the assumption of Proposition 5.1, the limit operator

$$\lim_{n\to\infty}\chi_{E\setminus W}K^{[X,B_n]}\chi_{E\setminus W} \tag{5.7}$$

is a locally trace class positive contractive operator and thus is a valid kernel for a determinantal point process. In Proposition 5.1, we have already shown that the limit operator (5.7) is locally trace class. The contractivity of the limit operator (5.7) follows from the contractivity of $K^{[X,B_n]}$ for all $n$ and the simple observation: locally trace class convergence implies strong operator convergence, and strong operator convergence preserves contractivity.

The limit operator (5.7) almost surely does not depend on the specific choice of the function $\psi$. Indeed, replace $\psi$ by another $\widehat{\psi}: E\setminus W\to(0,1]$ such that $\widehat{\psi}^{1/2}K\widehat{\psi}^{1/2}$ is of trace class and $\inf_{x\in B}\widehat{\psi}(x)>0$ for any bounded subset $B\subset E$. The limit relation (5.3) becomes

$$\widehat{\psi}^{1/2}G(X,n)\widehat{\psi}^{1/2} \xrightarrow[\mathbb{P}_K\text{-a.s.}]{\text{in } \mathscr{L}_1(L^2(E,\mu))} \widehat{F}(X,\infty).$$

This relation combined with (5.3) implies that $\widehat{F}(X,\infty) = \frac{\widehat{\psi}^{1/2}}{\psi^{1/2}} F(X,\infty) \frac{\widehat{\psi}^{1/2}}{\psi^{1/2}}$. Therefore, the final limit in (5.4) becomes

$$\widehat{\psi}^{-1/2}\widehat{F}(X,\infty)\widehat{\psi}^{-1/2} = \psi^{-1/2}F(X,\infty)\psi^{-1/2},$$

proving the independence of the limit operator from the choice of $\psi$.

Moreover, the limit operator (5.7) almost surely does not depend on the choice of the exhausting sequence $\{B_n\}$ of bounded subsets of $W$. Indeed, let $B_1 \subset B_2 \subset \cdots \subset W$ be an exhausting sequence of bounded Borel subsets of $W$. Fix any positive function $\psi : E \setminus W \to (0,1]$ such that $\psi^{1/2}K\psi^{1/2}$ is of trace class and for any bounded $B \subset E$, we have $\inf_{x\in B} \psi(x) > 0$. Then

$$(\psi^{1/2}K^{[X,B_n]}\psi^{1/2})_{n\in\mathbb{N}} \tag{5.8}$$

is an $\mathscr{L}_1(L^2(E \setminus W,\mu))$-valued martingale which is bounded in $L^2(\mathrm{Conf}(E),\mathbb{P};\mathscr{L}_1(L^2(E \setminus W,\mu)))$. In particular, the sequence converges in $L^2(\mathrm{Conf}(E),\mathbb{P};\mathscr{L}_1(L^2(E\setminus W,\mu)))$. Recall that $\mathscr{L}_1(L^2(E\setminus W,\mu))$ has the Radon–Nikodym property. Using the characterization (iii) of the Radon–Nikodym property in Section 2.1.2 we obtain, for $\mathbb{P}_K$-almost every $X \in \mathrm{Conf}(E)$,

$$\psi^{1/2}K^{[X,B_n]}\psi^{1/2} = \mathbb{E}_{\mathbb{P}_K}\Big[\lim_{\ell\to\infty}\psi^{1/2}K^{[X,B_\ell]}\psi^{1/2}\,\Big|\,\mathcal{F}(B_n)\Big]. \tag{5.9}$$

For the $L^2(\mathrm{Conf}(E),\mathbb{P};\mathscr{L}_1(L^2(E \setminus W,\mu)))$-boundedness of the sequence (5.8), we write

$$\|\psi^{1/2}K^{[X,B_n]}\psi^{1/2}\|_{\mathscr{L}_1(L^2(E\setminus W,\mu))} = \mathrm{tr}(\psi^{1/2}K^{[X,B_n]}\psi^{1/2}) = \mathbb{E}_{\mathbb{P}_K}\Big(\sum_{x\in X}\psi(x)\,\Big|\,X,B_n\Big).$$

By Jensen's inequality and the inequalities $0 \le \psi(x)^2 \le \psi(x)$ and $K(x,x)K(y,y) - |K(x,y)|^2 \le K(x,x)K(y,y)$,

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}_K}(\|\psi^{1/2}K^{[X,B_n]}\psi^{1/2}\|^2_{\mathscr{L}_1(L^2(E\setminus W,\mu))}) &= \mathbb{E}_{\mathbb{P}_K}\bigg(\Big[\mathbb{E}_{\mathbb{P}_K}\Big(\sum_{x\in X}\psi(x)\,\Big|\,X,B_n\Big)\Big]^2\bigg)\\
&\le \mathbb{E}_{\mathbb{P}_K}\bigg(\Big(\sum_{x\in X}\psi(x)\Big)^2\bigg) = \mathbb{E}_{\mathbb{P}_K}\Big(\sum_{x,y\in X,\,x\neq y}\psi(x)\psi(y)\Big) + \mathbb{E}_{\mathbb{P}_K}\Big(\sum_{x\in X}\psi(x)^2\Big)\\
&\le \mathbb{E}_{\mathbb{P}_K}\Big(\sum_{x,y\in X,\,x\neq y}\psi(x)\psi(y)\Big) + \mathbb{E}_{\mathbb{P}_K}\Big(\sum_{x\in X}\psi(x)\Big)\\
&\le \Big(\int_E K(x,x)\psi(x)\,\mathrm{d}\mu(x)\Big)^2 + \int_E K(x,x)\psi(x)\,\mathrm{d}\mu(x)\\
&= [\mathrm{tr}(\psi^{1/2}K\psi^{1/2})]^2 + \mathrm{tr}(\psi^{1/2}K\psi^{1/2}),
\end{aligned}$$

which proves the desired boundedness.

Now take another exhausting sequence $\{\widehat{B}_n\}$ of bounded subsets of $W$. By (5.9) applied to the new exhausting sequence $\{\widetilde{B}_n = B_n \cup \widehat{B}_n\}$ of bounded subsets of $W$, we find that for $\mathbb{P}_K$-almost every $X \in \mathrm{Conf}(E)$,

$$\psi^{1/2} K^{[X,\widetilde{B}_n]} \psi^{1/2} = \mathbb{E}_{\mathbb{P}_K}\Big[\lim_{\ell\to\infty} \psi^{1/2} K^{[X,\widetilde{B}_\ell]} \psi^{1/2} \,\Big|\, \mathcal{F}(\widetilde{B}_n)\Big].$$

Then using the martingale structure between $\psi^{1/2} K^{[X,B_n]} \psi^{1/2}$ and $\psi^{1/2} K^{[X,\widetilde{B}_n]} \psi^{1/2}$ for the nested sets $B_n \subset \widetilde{B}_n$ we have, for $\mathbb{P}_K$-almost every $X \in \mathrm{Conf}(E)$,

$$\psi^{1/2} K^{[X,B_n]} \psi^{1/2} = \mathbb{E}_{\mathbb{P}_K}\Big[\psi^{1/2} K^{[X,\widetilde{B}_n]} \psi^{1/2} \,\Big|\, \mathcal{F}(B_n)\Big].$$

Therefore, by combining the previous equalities, for $\mathbb{P}_K$-almost every $X \in \mathrm{Conf}(E)$,

$$\begin{aligned}\lim_{n\to\infty} \psi^{1/2} K^{[X,B_n]} \psi^{1/2} &= \lim_{n\to\infty} \mathbb{E}_{\mathbb{P}_K}[\psi^{1/2} K^{[X,\widetilde{B}_n]} \psi^{1/2} \,|\, \mathcal{F}(B_n)] \\ &= \lim_{n\to\infty} \mathbb{E}_{\mathbb{P}_K}\Big(\mathbb{E}_{\mathbb{P}_K}\Big[\lim_{\ell\to\infty} \psi^{1/2} K^{[X,\widetilde{B}_\ell]} \psi^{1/2} \,\Big|\, \mathcal{F}(\widetilde{B}_n)\Big] \,\Big|\, \mathcal{F}(B_n)\Big) \\ &= \lim_{n\to\infty} \mathbb{E}_{\mathbb{P}_K}\Big(\lim_{\ell\to\infty} \psi^{1/2} K^{[X,\widetilde{B}_\ell]} \psi^{1/2} \,\Big|\, \mathcal{F}(B_n)\Big) = \lim_{\ell\to\infty} \psi^{1/2} K^{[X,\widetilde{B}_\ell]} \psi^{1/2}.\end{aligned}$$

The same argument will show that for $\mathbb{P}_K$-almost every $X \in \mathrm{Conf}(E)$,

$$\lim_{n\to\infty} \psi^{1/2} K^{[X,\widehat{B}_n]} \psi^{1/2} = \lim_{\ell\to\infty} \psi^{1/2} K^{[X,\widetilde{B}_\ell]} \psi^{1/2}$$

and thus

$$\lim_{n\to\infty} \psi^{1/2} K^{[X,B_n]} \psi^{1/2} = \lim_{n\to\infty} \psi^{1/2} K^{[X,\widehat{B}_n]} \psi^{1/2}.$$

## 6. Proof of Theorem 1.4

Recall that we have fixed a realization of our kernel, namely, a Borel function $K(x, y)$ defined on the set $E_0 \times E_0$, where $\mu(E \setminus E_0) = 0$. In this section, we make the additional assumption that $K$ is the orthogonal projection onto a subspace $H \subset L^2(E, \mu)$. Recalling (1.3), we fix a realization for each $h \in H$ in such a way that $h(x) = \langle h, K_x\rangle$ for all $x \in E_0$ and $h \in H$. Given any configuration $X \in \mathrm{Conf}(E)$ and a bounded Borel subset $B \subset E$, we set $H(X) := \{h \in H : h\restriction_X \equiv 0\}$ and $\chi_B H(X) := \{\chi_B h : h \in H(X)\} \subset L^2(E, \mu)$. The subspace $H(X)$ is of course closed, but $\chi_B H(X)$ need not be closed.

Fix an exhausting sequence $E_1 \subset E_2 \subset \cdots \subset E \setminus B$ of bounded Borel subsets of $E \setminus B$, and denote

$$F_n = E \setminus (B \cup E_n).$$

Since $B$ is bounded, we have $\mathscr{L}_{1,\mathrm{loc}}(L^2(B, \mu)) = \mathscr{L}_1(L^2(B, \mu))$. By Lemma 1.9, for $\mathbb{P}_K$-almost every $X \in \mathrm{Conf}(E)$ there exists a positive contraction $K^{[X,E\setminus B]} \in \mathscr{L}_1(L^2(B, \mu))$ such that

$$\chi_B K^{[X,E_n]} \chi_B \xrightarrow[\text{in } \mathscr{L}_1(L^2(B,\mu))]{n\to\infty} K^{[X,E\setminus B]}, \tag{6.1}$$

$$\mathbb{P}_K(\cdot \,|\, X, E \setminus B) = \mathbb{P}_{K^{[X,E\setminus B]}}. \tag{6.2}$$

**Lemma 6.1.** *For $\mathbb{P}_K$-almost every $X \in \mathrm{Conf}(E)$, we have $K^{[X,E\setminus B]}(\chi_B h) = \chi_B h$ for any $h \in H(X \cap (E \setminus B))$.*

*Proof.* For any $n \in \mathbb{N}$, since $E_n \subset E \setminus B$, by definition, we have $H(X \cap (E \setminus B)) \subset H(X \cap E_n)$. Since $E_n$ is bounded and $E \setminus E_n = B \cup F_n$, by Proposition 2.6 the operator $K^{[X,E_n]}$ is the orthogonal projection from $L^2(E,\mu)$ onto the closure of the subspace $\chi_{E\setminus E_n} H(X \cap E_n) = \chi_{B\cup F_n} H(X \cap E_n)$. By the limit relation (6.1) (and the elementary fact that $\mathscr{L}_1$-norm convergence implies operator norm convergence), for $\mathbb{P}_K$-almost every $X \in \mathrm{Conf}(E)$ we have

$$K^{[X,E\setminus B]}(\chi_B h) = \lim_{n\to\infty} (\chi_B K^{[X,E_n]} \chi_B)(\chi_B h) = \lim_{n\to\infty} \chi_B K^{[X,E_n]}(\chi_B h)$$

for any $h \in H(X \cap (E \setminus B))$, where the two limits are in the sense of $L^2$-convergence. Then using the equalities $\chi_B h = \chi_{B\cup F_n} h - \chi_{F_n} h$, $K^{[X,E_n]}(\chi_{B\cup F_n} h) = \chi_{B\cup F_n} h$ and the relation

$$\|\chi_B K^{[X,E_n]}(\chi_{F_n} h)\|_2 \le \|\chi_{F_n} h\|_2 \xrightarrow{n\to\infty} 0,$$

we obtain the desired equality

$$\begin{aligned} K^{[X,E\setminus B]}(\chi_B h) &= \lim_{n\to\infty} \chi_B K^{[X,E_n]}(\chi_{B\cup F_n} h - \chi_{F_n} h) = \chi_B h - \lim_{n\to\infty} \chi_B K^{[X,E_n]}(\chi_{F_n} h) \\ &= \chi_B h, \end{aligned}$$

where the two limits are in the sense of $L^2$-convergence. □

**Lemma 6.2.** *Let $\mathbb{P}$ be a point process on $E$. Then for any bounded Borel subset $B \subset E$,*

$$\mathbb{P}\big(\#_B = \#(X \cap B) \,\big|\, X, E \setminus B\big) > 0 \quad \textit{for } \mathbb{P}\textit{-a.e. } X \in \mathrm{Conf}(E). \tag{6.3}$$

*Proof.* First of all, decomposing $X = Y \cup Z$, $Y \in \mathrm{Conf}(B)$, $Z \in \mathrm{Conf}(E \setminus B)$, we can rewrite the statement as follows:

$$\mathbb{P}\big(\{W \in \mathrm{Conf}(B) : \#(W) = \#(Y)\} \,|\, Z, E \setminus B\big) > 0 \tag{6.4}$$

for $(\pi_{E\setminus B})_*(\mathbb{P})$-almost every $Z \in \mathrm{Conf}(E \setminus B)$ and $\mathbb{P}(\cdot \,|\, Z, E \setminus B)$-almost every $Y \in \mathrm{Conf}(B)$. We make a simple general claim: given an integer-valued measurable function $f$ on a probability space $(\Omega, \mathbf{P})$, for $\mathbf{P}$-almost every $y \in \Omega$ we have $\mathbf{P}\{x : f(x) = f(y)\} > 0$. Indeed, if we define $N = \{n \in \mathbb{Z} : \mathbf{P}\{x : f(x) = n)\} = 0\}$, then the relation $\mathbf{P}\{x : f(x) = f(y)\} > 0$ fails only if $f(y) \in N$, and

$$\mathbf{P}\{y : f(y) \in N\} = \sum_{n\in N} \mathbf{P}\{y : f(y) = n\} = 0.$$

Now for $(\pi_{E\setminus B})_*(\mathbb{P})$-almost every $Z \in \mathrm{Conf}(E \setminus B)$, by taking $\Omega = \mathrm{Conf}(B)$, $\mathbf{P} = \mathbb{P}(\,\cdot\,| Z, E \setminus B)$, $f = \#_B$, we obtain the desired statement (6.4). □

*Proof of Theorem 1.4.* Fix a countable dense subset $T$ of $E$ and let $S_n$ be an enumeration of balls with rational radii centred at $T$:

$$\{S_n : n \in \mathbb{N}\} = \{B(x,q) : x \in T,\ q \in \mathbb{Q}\}. \tag{6.5}$$

Since the family (6.5) is countable, by Lemmas 6.1 and 6.2 there exists a measurable subset $\mathcal{A} \subset \mathrm{Conf}(E)$ such that

- $\mathbb{P}_K(\mathcal{A}) = 1$;
- for all $X \in \mathcal{A}$ and all $n \in \mathbb{N}$, the conditional measures $\mathbb{P}_K(\cdot \mid X, E \setminus S_n)$ and the conditional kernels $K^{[X,E\setminus S_n]}$ are defined and satisfy

$$\mathbb{P}_K(\cdot \mid X, E \setminus S_n) = \mathbb{P}_{K^{[X,E\setminus S_n]}}; \tag{6.6}$$

- for all $X \in \mathcal{A}$ and all $n \in \mathbb{N}$, we have

$$K^{[X,E\setminus S_n]}(\chi_{S_n} h) = \chi_{S_n} h \quad \text{for any } h \in H(X \cap (E \setminus S_n)); \tag{6.7}$$

- for all $X \in \mathcal{A}$ and all $n \in \mathbb{N}$.

$$\mathbb{P}_K\big(\#_{S_n} = \#(X \cap S_n) \,\big|\, X, E \setminus S_n\big) > 0. \tag{6.8}$$

We now show that the above measurable subset $\mathcal{A} \subset \mathrm{Conf}(E)$ has the desired property: $H(X) = \{0\}$ for any $X \in \mathcal{A}$. Take a fixed configuration $X \in \mathcal{A}$ and assume, for contradiction, that there exists $h_0 \in H(X)$, $h_0 \neq 0$. Clearly, since $X$ is a discrete countable subset, there exists $n_0 \in \mathbb{N}$ such that

$$h_0 \restriction_{S_{n_0}} \neq 0 \quad \text{and} \quad X \cap S_{n_0} = \emptyset.$$

Therefore, $\chi_{S_{n_0}} h_0 \neq 0$ and $H(X) = H(X \cap (E \setminus S_{n_0}))$ and hence $h_0 \in H(X \cap (E \setminus S_{n_0}))$. In view of the assumption (6.7) on $\mathcal{A}$, the non-zero function $\chi_{S_{n_0}} h_0$ satisfies $K^{[X,E\setminus S_{n_0}]}(\chi_{S_{n_0}} h_0) = \chi_{S_{n_0}} h_0$, whence 1 is an eigenvalue of the operator $K^{[X,E\setminus S_{n_0}]}$. In particular,

$$\det(1 - K^{[X,E\setminus S_{n_0}]}) = 0.$$

On the other hand, the relations (6.6), (6.8) together with the gap probability formula (3.5) imply that

$$\begin{aligned}\det(1 - K^{[X,E\setminus S_{n_0}]}) &= \mathbb{P}_{K^{[X,E\setminus S_{n_0}]}}(\#_{S_{n_0}} = 0) = \mathbb{P}_K(\#_{S_{n_0}} = 0 \mid X, E \setminus S_{n_0}) \\ &= \mathbb{P}_K\big(\#_{S_{n_0}} = \#(X \cap S_{n_0}) \,\big|\, X, E \setminus S_{n_0}\big) > 0.\end{aligned}$$

We thus obtain a contradiction and Theorem 1.4 is proved completely. □

**Remark.** When $K$ is the orthogonal projection with image $H \subset L^2(E)$, we have seen in Proposition 2.6 that for our exhausting sequence $E_1 \subset E_2 \subset \cdots \subset E \setminus B$ of bounded Borel subsets of $E \setminus B$, the operator $K^{[X,E_n]}$ is the orthogonal projection onto the closure of the subspace $\chi_{E\setminus E_n} H(X \cap E_n)$. The convergence of the sequence of contractions $\chi_B K^{[X,E_n]} \chi_B)$ requires proof. Whether the sequence $K^{[X,E_n]}$ itself converges is not clear to us. The limit operator

$$K^{[X,\,E\setminus B]} = \lim_{n\to\infty} \chi_B K^{[X,\,E_n]} \chi_B$$

is in general not an orthogonal projection (cf. e.g. [8, Corollary 3.13] for the Bergman kernel): the operator $K^{[X,E\setminus B]}$, acting on $L^2(B,\mu)$, is trace class by the boundedness of $B$, but in the example of the Bergman kernel the range of $K^{[X,E\setminus B]}$ has infinite dimension and therefore $K^{[X,E\setminus B]}$ is not an orthogonal projection.

## 7. Triviality of the tail $\sigma$-algebra: proof of Theorem 1.6

**Definition 7.1.** Fix any increasing exhausting sequence $D_1 \subset D_2 \subset \cdots \subset E$ of bounded Borel subsets of $E$. For any Borel subset $W \subset E$, set

$$K^{[X,W]} := \lim_{n\to\infty} \chi_{E\setminus W} K^{[X,\, W\cap D_n]} \chi_{E\setminus W}.$$

The convergence takes place in $\mathscr{L}_{1,\mathrm{loc}}(L^2(E,\mu))$ by Proposition 5.1.The kernel $K^{[X,W]}$ is well-defined for $\mathbb{P}_K$-almost every $X$. For fixed $W$, the limit is almost surely independent of the choice of the sequence $(D_n)_{n=1}^{\infty}$.

**Proposition 7.2.** *Fix a bounded Borel subset $B \subset E$ and let $E \setminus B \supset W_1 \supset W_2 \supset \cdots$ be any decreasing sequence of Borel subsets. Then $(\chi_B K^{[X,W_n]}\chi_B)_{n\in\mathbb{N}}$ is an $(\mathcal{F}(W_n))_{n\in\mathbb{N}}$-adapted reverse martingale defined on the probability space* $(\mathrm{Conf}(E), \mathcal{F}(E), \mathbb{P}_K)$.

*Proof.* It suffices to prove that for any $\phi \in L^2(B,\mu)$, the sequence $(\langle K^{[X,W_n]}\phi, \phi\rangle)_{n\in\mathbb{N}}$ is an $(\mathcal{F}(W_n))_{n\in\mathbb{N}}$-adapted reverse martingale on $(\mathrm{Conf}(E), \mathcal{F}(E), \mathbb{P}_K)$. By definition, for any $n \in \mathbb{N}$,

$$\langle K^{[X,W_n]}\phi, \phi\rangle = \lim_{k\to\infty} \langle K^{[X,W_n\cap D_k]}\phi, \phi\rangle \quad \mathbb{P}_K\text{-almost surely.} \tag{7.1}$$

Since all the operators $K^{[X,W_n]}$ are contractive, by the bounded convergence theorem the convergence (7.1) takes place in $L^1(\mathbb{P}_K)$ as well. Fix $n \in \mathbb{N}$. For any $\varepsilon > 0$, let $k \in \mathbb{N}$ be large enough that

$$\begin{aligned} &\|\langle K^{[X,W_n]}\phi, \phi\rangle - \langle K^{[X,W_n\cap D_k]}\phi, \phi\rangle\|_{L^1(\mathbb{P}_K)} \le \varepsilon,\\ &\|\langle K^{[X,W_{n+1}]}\phi, \phi\rangle - \langle K^{[X,W_{n+1}\cap D_k]}\phi, \phi\rangle\|_{L^1(\mathbb{P}_K)} \le \varepsilon. \end{aligned} \tag{7.2}$$

For fixed $n \in \mathbb{N}$, the sequence

$$\Big(\mathbb{E}_{\mathbb{P}_K}\big[\langle K^{[X,W_n]}\phi, \phi\rangle \,\big|\, \mathcal{F}(W_{n+1}\cap D_k)\big]\Big)_{k=1}^{\infty}$$

is a martingale that converges in $L^1$-norm to $\mathbb{E}_{\mathbb{P}_K}[\langle K^{[X,W_n]}\phi, \phi\rangle \,|\, \mathcal{F}(W_{n+1})]$. We can therefore choose $k$ large enough that

$$\Big\|\mathbb{E}_{\mathbb{P}_K}\big[\langle K^{[X,W_n]}\phi, \phi\rangle \,\big|\, \mathcal{F}(W_{n+1})\big] - \mathbb{E}_{\mathbb{P}_K}\big[\langle K^{[X,W_n]}\phi, \phi\rangle \,\big|\, \mathcal{F}(W_{n+1}\cap D_k)\big]\Big\|_{L^1(\mathbb{P}_K)} \le \varepsilon.$$

Since $W_{n+1}\cap D_k \subset W_n \cap D_k$ and $D_k$ is bounded, Lemma 1.12 implies

$$\mathbb{E}_{\mathbb{P}_K}\big[\langle K^{[X,W_n\cap D_k]}\phi, \phi\rangle \,\big|\, \mathcal{F}(W_{n+1}\cap D_k)\big] = \langle K^{[X,W_{n+1}\cap D_k]}\phi, \phi\rangle,$$

whence

$$\begin{aligned} &\Big\|\mathbb{E}_{\mathbb{P}_K}\big[\langle K^{[X,W_n]}\phi, \phi\rangle \,\big|\, \mathcal{F}(W_{n+1})\big] - \langle K^{[X,W_{n+1}]}\phi, \phi\rangle\Big\|_{L^1(\mathbb{P}_K)}\\ &\le 2\varepsilon + \Big\|\mathbb{E}_{\mathbb{P}_K}\big[\langle K^{[X,W_n]}\phi, \phi\rangle \,\big|\, \mathcal{F}(W_{n+1}\cap D_k)\big] - \langle K^{[X,W_{n+1}\cap D_k]}\phi, \phi\rangle\Big\|_{L^1(\mathbb{P}_K)}\\ &\le 3\varepsilon + \Big\|\mathbb{E}_{\mathbb{P}_K}\big[\langle K^{[X,W_n\cap D_k]}\phi, \phi\rangle \,\big|\, \mathcal{F}(W_{n+1}\cap D_k)\big] - \langle K^{[X,W_{n+1}\cap D_k]}\phi, \phi\rangle\Big\|_{L^1(\mathbb{P}_K)} = 3\varepsilon, \end{aligned} \tag{7.3}$$

and we obtain the desired reverse martingale relation $\mathbb{E}_{\mathbb{P}_K}[\langle K^{[X,W_n]}\phi, \phi\rangle \,|\, \mathcal{F}(W_{n+1})] = \langle K^{[X,W_{n+1}]}\phi, \phi\rangle$. □

**Lemma 7.3.** *For any bounded Borel subset $B \subset E$ and $\phi \in L^2(E \setminus B, \mu)$, we have*

$$\mathrm{Var}_{\mathbb{P}_K}[\langle K^{[X,B]}\phi, \phi\rangle] \leq \|\phi\|_2^2 \cdot \|\chi_B K\phi\|_2^2, \tag{7.4}$$

*where $\|\cdot\|_2$ is the Hilbert norm on $L^2(E, \mu)$.*

We first prove Lemma 7.3 when $K$ is an orthogonal projection. This part of the proof is similar to the argument of Benjamini, Lyons, Peres and Schramm [1, Lemma 8.6] and Lyons [21, Lemma 7.18]. The proof of Lemma 7.3 in full generality proceeds by reduction to the case of projections (the usual argument of extending the phase space must be slightly modified in the continuous setting) and is postponed to the end of the section.

*Proof of Lemma 7.3 when $K$ is an orthogonal projection.* By homogeneity, we may assume that $\|\phi\|_2 \leq 1$. Since $K$ is an orthogonal projection, by [5, Proposition 2.5] so is $K^{[X,B]}$ for $\mathbb{P}_K$-almost every $X \in \mathrm{Conf}(E)$. By Proposition 4.1, we have

$$\begin{aligned}
\mathrm{Var}_{\mathbb{P}_K}[\langle K^{[X,B]}\phi, \phi\rangle] &= \mathbb{E}_{\mathbb{P}_K}\big|\langle (K^{[X,B]} - \chi_{E\setminus B}K\chi_{E\setminus B})\phi, \phi\rangle\big|^2 \\
&\leq \mathbb{E}_{\mathbb{P}_K}\big(\|(K^{[X,B]} - \chi_{E\setminus B}K\chi_{E\setminus B})\phi\|_2^2\big) \\
&= \mathbb{E}_{\mathbb{P}_K}\big(\|K^{[X,B]}\phi\|_2^2 - \langle K^{[X,B]}\phi, \chi_{E\setminus B}K\chi_{E\setminus B}\phi\rangle \\
&\quad - \langle \chi_{E\setminus B}K\chi_{E\setminus B}\phi, K^{[X,B]}\phi\rangle + \|\chi_{E\setminus B}K\chi_{E\setminus B}\phi\|_2^2\big) \\
&= \mathbb{E}_{\mathbb{P}_K}\big(\langle K^{[X,B]}\phi, \phi\rangle - \langle K^{[X,B]}\phi, \chi_{E\setminus B}K\chi_{E\setminus B}\phi\rangle \\
&\quad - \langle \chi_{E\setminus B}K\chi_{E\setminus B}\phi, K^{[X,B]}\phi\rangle + \|\chi_{E\setminus B}K\chi_{E\setminus B}\phi\|_2^2\big) \\
&= \langle \chi_{E\setminus B}K\chi_{E\setminus B}\phi, \phi\rangle - \|\chi_{E\setminus B}K\chi_{E\setminus B}\phi\|_2^2 = \langle K\phi, \phi\rangle - \|\chi_{E\setminus B}K\phi\|_2^2 \\
&= \|K\phi\|_2^2 - \|\chi_{E\setminus B}K\phi\|_2^2 = \|\chi_B K\phi\|_2^2.
\end{aligned} \tag{7.5}$$

□

**Proposition 7.4.** *Fix any $\ell \in \mathbb{N}$. Then $(\chi_{D_\ell}K^{[X,E\setminus D_{n+\ell}]}\chi_{D_\ell})_{n\in\mathbb{N}}$ is an $(\mathcal{F}(E\setminus D_{n+\ell}))_{n\in\mathbb{N}}$-adapted reverse martingale defined on the probability space* $(\mathrm{Conf}(E), \mathcal{F}(E), \mathbb{P}_K)$*, and*

$$\chi_{D_\ell}K^{[X,E\setminus D_{n+\ell}]}\chi_{D_\ell} \xrightarrow[\mathbb{P}_K\text{-a.s. in } \mathscr{L}_1(L^2(E,\mu)) \text{ and in } L^2(\mathbb{P}_K;\, \mathscr{L}_1(L^2(E,\mu)))]{n\to\infty} \chi_{D_\ell}K\chi_{D_\ell}. \tag{7.6}$$

*For any $\ell \in \mathbb{N}$,*

$$\mathbb{E}_{\mathbb{P}_K}\Big[\mathbb{P}_K(\cdot \mid X, E\setminus D_\ell)\,\Big|\, \bigcap_{n=1}^{\infty}\mathcal{F}(E\setminus D_{n+\ell})\Big] = (\pi_{D_\ell})_*(\mathbb{P}_K) \quad \mathbb{P}_K\text{-a.s.}, \tag{7.7}$$

*and, for any $\mathcal{A} \in \mathcal{F}(D_\ell)$,*

$$\lim_{n\to\infty} \mathbb{E}_{\mathbb{P}_K}\big|\mathbb{E}_{\mathbb{P}_K}[\chi_{\mathcal{A}} \mid \mathcal{F}(E\setminus D_{n+\ell})] - \mathbb{P}_K(\mathcal{A})\big| = 0. \tag{7.8}$$

*Proof.* The reverse martingale property of the sequence follows from Proposition 7.2. Set

$$\mathscr{T} := \bigcap_{n=1}^{\infty}\mathcal{F}(E\setminus D_{n+\ell}). \tag{7.9}$$

Since a Banach space valued *reverse* martingale converges (see, e.g., Pisier [31, p. 34]), we obtain

$$\chi_{D_\ell} K^{[X, E\setminus D_{n+\ell}]} \chi_{D_\ell} \xrightarrow[\mathbb{P}_K\text{-almost surely in } \mathscr{L}_1(L^2(E,\mu)) \text{ and in } L^2(\mathbb{P}_K;\mathscr{L}_1(L^2(E,\mu)))]{n\to\infty} \mathbb{E}_{\mathbb{P}_K}[\chi_{D_\ell} K^{[X,E\setminus D_{1+\ell}]}\chi_{D_\ell} \mid \mathscr{T}].$$

Set

$$G_\infty(X) = \mathbb{E}_{\mathbb{P}_K}[\chi_{D_\ell} K^{[X,E\setminus D_{1+\ell}]}\chi_{D_\ell} \mid \mathscr{T}].$$

In particular, for any $\phi \in L^2(D_\ell, \mu)$ with $\|\phi\|_2 \le 1$, we have

$$\langle G_\infty(X)\phi, \phi\rangle = \mathbb{E}_{\mathbb{P}_K}[\langle K^{[X,E\setminus D_{1+\ell}]}\phi, \phi\rangle \mid \mathscr{T}] \quad \mathbb{P}_K\text{-almost surely.}$$

By Definition 7.1 and the inequality $|\langle K^{[X,(E\setminus D_{n+\ell})\cap D_k]}\phi,\phi\rangle| \le 1$, which holds $\mathbb{P}_K$-almost surely, for any $n \in \mathbb{N}$ we have

$$\langle K^{[X,(E\setminus D_{n+\ell})\cap D_k]}\phi,\phi\rangle \xrightarrow{k\to\infty} \langle K^{[X,E\setminus D_{n+\ell}]}\phi,\phi\rangle \quad \mathbb{P}_K\text{-a.s. and in } L^2(\mathbb{P}_K). \tag{7.10}$$

Similarly,

$$\langle K^{[X,E\setminus D_{n+\ell}]}\phi,\phi\rangle \xrightarrow{n\to\infty} \langle G_\infty(X)\phi,\phi\rangle \quad \mathbb{P}_K\text{-a.s. and in } L^2(\mathbb{P}_K). \tag{7.11}$$

In particular, since $(E \setminus D_{1+\ell}) \cap D_k$ are bounded for all $k \in \mathbb{N}$, we can apply Proposition 4.1 to obtain

$$\begin{aligned}\mathbb{E}_{\mathbb{P}_K}\langle G_\infty(X)\phi,\phi\rangle &= \mathbb{E}_{\mathbb{P}_K}[\langle K^{[X,E\setminus D_{1+\ell}]}\phi,\phi\rangle] = \lim_{k\to\infty} \mathbb{E}_{\mathbb{P}_K}[\langle K^{[X,(E\setminus D_{1+\ell})\cap D_k]}\phi,\phi\rangle] \\ &= \langle K\phi,\phi\rangle.\end{aligned}$$

Now by Lemma 7.3 and the assumption $\|\phi\|_2 \le 1$, we have

$$\mathrm{Var}_{\mathbb{P}_K}(\langle K^{[X,(E\setminus D_{n+\ell})\cap D_k]}\phi,\phi\rangle) \le \|\chi_{(E\setminus D_{n+\ell})\cap D_k} K\phi\|_2^2 \le \|\chi_{E\setminus D_{n+\ell}} K\phi\|_2^2.$$

The convergences (7.10), (7.11) yield

$$\mathrm{Var}_{\mathbb{P}_K}(\langle K^{[X,E\setminus D_{n+\ell}]}\phi,\phi\rangle) = \lim_{k\to\infty} \mathrm{Var}_{\mathbb{P}_K}(\langle K^{[X,(E\setminus D_{n+\ell})\cap D_k]}\phi,\phi\rangle) \le \|\chi_{E\setminus D_{n+\ell}} K\phi\|_2^2,$$

$$\mathrm{Var}_{\mathbb{P}_K}(\langle G_\infty(X)\phi,\phi\rangle) = \lim_{n\to\infty} \mathrm{Var}_{\mathbb{P}_K}(\langle K^{[X,E\setminus D_{n+\ell}]}\phi,\phi\rangle) \le \limsup_{n\to\infty} \|\chi_{E\setminus D_{n+\ell}} K\phi\|_2^2 = 0.$$

Consequently, $\langle G_\infty(X)\phi,\phi\rangle = \langle K\phi,\phi\rangle$ $\mathbb{P}_K$-almost surely. Since $\chi_{D_\ell} G_\infty(X)\chi_{D_\ell} = G_\infty(X)$ and since $\phi$ is arbitrarily chosen from the separable unit sphere in $L^2(D_\ell,\mu)$, we obtain the desired equality

$$G_\infty(X) = \chi_{D_\ell} K \chi_{D_\ell} \quad \mathbb{P}_K\text{-a.s.}$$

Finally, Proposition 8.3 implies that

$$(\pi_{D_\ell})_*[\mathbb{P}_K(\cdot \mid X, E\setminus D_{n+\ell})] = \mathbb{E}_{\mathbb{P}_K}\big[\mathbb{P}_K(\cdot \mid X, E\setminus D_\ell) \,\big|\, \mathcal{F}(E\setminus D_{n+\ell})\big] \quad \mathbb{P}_K\text{-a.s.,}$$

and

$$(\pi_{D_\ell})_*[\mathbb{P}_K(\cdot \mid X, E \setminus D_{n+\ell})] \xrightarrow[\text{weakly}]{n\to\infty} \mathbb{E}_{\mathbb{P}_K}\big[\mathbb{P}_K(\cdot \mid X, E \setminus D_\ell) \,\big|\, \mathscr{T}\big] \quad \mathbb{P}_K\text{-a.s.} \tag{7.12}$$

But the convergence (7.6) implies that

$$\begin{aligned}(\pi_{D_\ell})_*[\mathbb{P}_K(\cdot \mid X, E \setminus D_{n+\ell})] &= \mathbb{P}_{\chi_{D_\ell} K^{[X,E\setminus D_{n+\ell}]}\chi_{D_\ell}} \xrightarrow[\text{weakly}]{n\to\infty} \mathbb{P}_{\chi_{D_\ell} K \chi_{D_\ell}} \\ &= (\pi_{D_\ell})_*(\mathbb{P}_K) \quad \mathbb{P}_K\text{-a.s.}\end{aligned} \tag{7.13}$$

Now (7.12) and (7.13) yield (7.7). Martingale convergence for a bounded random variable implies (7.8). □

*Proof of Theorem 1.6.* Take $D_n := B_n$. We prove that the $\sigma$-algebra $\mathscr{T}$ in (7.9) is trivial with respect to $\mathbb{P}_K$. Take an event $\mathcal{A} \in \mathscr{T}$. For $\varepsilon > 0$, find $\ell \in \mathbb{N}$ large enough and $\mathcal{A}_1 \in \mathfrak{F}(D_\ell)$ such that $\mathbb{P}_K(\mathcal{A}_1 \triangle \mathcal{A}) < \varepsilon/3$. By (7.8), we have

$$\lim_{n\to\infty} \mathbb{E}_{\mathbb{P}_K}\big|\mathbb{E}_{\mathbb{P}_K}[\chi_{\mathcal{A}_1} \mid \mathfrak{F}(E \setminus D_{n+\ell})] - \mathbb{P}_K(\mathcal{A}_1)\big| = 0.$$

Now find $n \in \mathbb{N}$ large enough that

$$\mathbb{E}_{\mathbb{P}_K}\big|\mathbb{E}_{\mathbb{P}_K}[\chi_{\mathcal{A}_1} \mid \mathfrak{F}(E \setminus D_{n+\ell})] - \mathbb{P}_K(\mathcal{A}_1)\big| \le \varepsilon/3.$$

It follows that for any $\mathcal{A}_2 \in \mathfrak{F}(E \setminus D_{n+\ell})$, we have

$$\begin{aligned}&|\mathbb{P}_K(\mathcal{A}_1 \cap \mathcal{A}_2) - \mathbb{P}_K(\mathcal{A}_1)\mathbb{P}_K(\mathcal{A}_2)| \\ &\qquad = \big|\mathbb{E}_{\mathbb{P}_K}\big(\chi_{\mathcal{A}_2}\mathbb{E}_{\mathbb{P}_K}[\chi_{\mathcal{A}_1} \mid \mathfrak{F}(E \setminus D_{n+\ell})]\big) - \mathbb{E}_{\mathbb{P}_K}\big(\chi_{\mathcal{A}_2}\mathbb{P}_K(\mathcal{A}_1)\big)\big| \\ &\qquad = \big|\mathbb{E}_{\mathbb{P}_K}\big(\chi_{\mathcal{A}_2}\big[\mathbb{E}_{\mathbb{P}_K}[\chi_{\mathcal{A}_1} \mid \mathfrak{F}(E \setminus D_{n+\ell})] - \mathbb{P}_K(\mathcal{A}_1)\big]\big)\big| \\ &\qquad \le \mathbb{E}_{\mathbb{P}_K}\big(\big|\mathbb{E}_{\mathbb{P}_K}[\chi_{\mathcal{A}_1} \mid \mathfrak{F}(E \setminus D_{n+\ell})] - \mathbb{P}_K(\mathcal{A}_1)\big|\big) \le \varepsilon/3.\end{aligned} \tag{7.14}$$

Finally, we obtain

$$\begin{aligned}&|\mathbb{P}_K(\mathcal{A} \cap \mathcal{A}_2) - \mathbb{P}_K(\mathcal{A})\mathbb{P}_K(\mathcal{A}_2)| \\ &\qquad \le 2\mathbb{P}_K(\mathcal{A}_1 \triangle \mathcal{A}) + |\mathbb{P}_K(\mathcal{A}_1 \cap \mathcal{A}_2) - \mathbb{P}_K(\mathcal{A}_1)\mathbb{P}_K(\mathcal{A}_2)| \le \varepsilon.\end{aligned}$$

Taking $\mathcal{A}_2 = \mathcal{A}$, we obtain $\mathbb{P}_K(\mathcal{A}) = (\mathbb{P}_K(\mathcal{A}))^2$, whence $\mathbb{P}_K(\mathcal{A})$ is either 0 or 1, as desired. □

*Proof of Lemma 7.3 in the general case.* Fix a bounded Borel subset $B \subset E$ and a function $\phi \in L^2(E \setminus B, \mu)$ such that $\|\phi\|_2 = 1$. Recalling (3.1), set

$$R(K, B, \phi) = (\phi \otimes \overline{\phi} + \chi_B)K(\phi \otimes \overline{\phi} + \chi_B). \tag{7.15}$$

By Lemma 1.10,

$$\langle R(K, B, \phi)^{[X,B]}\phi, \phi\rangle = \langle K^{[X,B]}\phi, \phi\rangle \quad \text{for } \mathbb{P}_K\text{-a.e. } X \in \mathrm{Conf}(E).$$

By definition, $K^{[X,B]} = K^{[X\cap B,\, B]}$ and similarly $R(K, B, \phi)^{[X,B]} = R(K, B, \phi)^{[X\cap B,\, B]}$. In particular,

$$\langle R(K, B, \phi)^{[X,B]}\phi, \phi\rangle = \langle K^{[X,B]}\phi, \phi\rangle \quad \text{for } (\pi_B)_*(\mathbb{P}_K) = \mathbb{P}_{\chi_B K \chi_B}\text{-a.e. } X \in \mathrm{Conf}(B), \tag{7.16}$$

and

$$\begin{aligned}\mathrm{Var}_{\mathbb{P}_K}[\langle K^{[X,B]}\phi, \phi\rangle] &= \mathrm{Var}_{\mathbb{P}_{\chi_B K \chi_B}}[\langle K^{[X,B]}\phi, \phi\rangle] \\ &= \mathrm{Var}_{\mathbb{P}_{\chi_B R(K,B,\phi)\chi_B}}\langle R(K, B, \phi)^{[X,B]}\phi, \phi\rangle.\end{aligned} \tag{7.17}$$

**Claim** (see Lyons [22, Section 3.3]). *Let* $\mathbf{m}$ *be the counting measure on* $\mathbb{N}$. *There exists a locally trace class orthogonal projection operator* $\widetilde{K} \in \mathscr{L}_{1,\mathrm{loc}}(L^2(E \sqcup \mathbb{N}, \mu \oplus \mathbf{m}))$ *such that* $K = \chi_E \widetilde{K} \chi_E$.

Indeed, the canonical orthogonal projection dilation of $K$ on $L^2(E, \mu) \oplus L^2(E, \mu)$ is given by the formula

$$\begin{bmatrix} K & \sqrt{K - K^2} \\ \sqrt{K - K^2} & 1 - K \end{bmatrix},$$

but it is not in general locally trace class. Since $L^2(E, \mu)$ is separable and all infinite-dimensional separable Hilbert spaces are isometrically isomorphic, there exists a unitary operator $U : L^2(E, \mu) \to \ell^2(\mathbb{N}) = L^2(\mathbb{N}, \mathbf{m})$, and we set

$$\widetilde{K} := \begin{bmatrix} 1 & 0 \\ 0 & U \end{bmatrix}\begin{bmatrix} K & \sqrt{K - K^2} \\ \sqrt{K - K^2} & 1 - K \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & U^{-1} \end{bmatrix}. \tag{7.18}$$

Now since $\widetilde{K}$ is an orthogonal projection, we can apply (7.5) to finish the proof of Lemma 7.3 as follows. Consider the subset $B \subset E$ as a subset of $E \sqcup \mathbb{N}$, and consider the function $\phi \in L^2(E \setminus B, \mu)$ as an element in $L^2((E \sqcup \mathbb{N}) \setminus B, \mu \oplus \mathbf{m})$ (just extend the definition of $\phi$ so that it vanishes on $\mathbb{N}$). Applying (7.5) to the kernel $\widetilde{K}$ and the determinantal measure $\mathbb{P}_{\widetilde{K}}$, we obtain

$$\mathrm{Var}_{\mathbb{P}_{\widetilde{K}}}[\langle \widetilde{K}^{[X,B]}\phi, \phi\rangle] \leq \|\chi_B \widetilde{K}\phi\|_2^2.$$

For the term on the right hand side, we have

$$\chi_B \widetilde{K}\phi = \chi_B K\phi. \tag{7.19}$$

Similar to the definition (7.15) of $R(K, B, \phi)$, set

$$R(\widetilde{K}, B, \phi) = (\phi \otimes \overline{\phi} + \chi_B)\widetilde{K}(\phi \otimes \overline{\phi} + \chi_B). \tag{7.20}$$

By the definition (7.18) of the kernel $\widetilde{K}$, we have $\chi_E \widetilde{K} \chi_E = K$. Then by using the elementary equality

$$\phi \otimes \overline{\phi} + \chi_B = (\phi \otimes \overline{\phi} + \chi_B)\chi_E = \chi_E(\phi \otimes \overline{\phi} + \chi_B),$$

we obtain, by recalling (7.15) and (7.20),

$$\begin{aligned} R(\widetilde{K}, B, \phi) &= (\phi \otimes \overline{\phi} + \chi_B)\widetilde{K}(\phi \otimes \overline{\phi} + \chi_B) \\ &= (\phi \otimes \overline{\phi} + \chi_B)\chi_E \widetilde{K} \chi_E(\phi \otimes \overline{\phi} + \chi_B) \\ &= (\phi \otimes \overline{\phi} + \chi_B)K(\phi \otimes \overline{\phi} + \chi_B) = R(K, B, \phi). \end{aligned} \tag{7.21}$$

The equality (7.21), combined with (7.16) (applied to both $\widetilde{K}$ and $K$), implies

$$\begin{aligned} \langle \widetilde{K}^{[X,B]}\phi, \phi\rangle &\overset{\mathbb{P}_{\chi_B \widetilde{K} \chi_B}\text{-a.s.}}{=\!=\!=} \langle R(\widetilde{K}, B, \phi)^{[X,B]}\phi, \phi\rangle = \langle R(K, B, \phi)^{[X,B]}\phi, \phi\rangle \\ &\overset{\mathbb{P}_{\chi_B K \chi_B}\text{-a.s.}}{=\!=\!=} \langle K^{[X,B]}\phi, \phi\rangle. \end{aligned}$$

The equality $\chi_B \widetilde{K} \chi_B = \chi_B K \chi_B$ implies $\mathbb{P}_{\chi_B \widetilde{K} \chi_B} = \mathbb{P}_{\chi_B K \chi_B}$. Therefore,

$$\begin{aligned} \mathrm{Var}_{\mathbb{P}_K}[\langle K^{[X,B]}\phi, \phi\rangle] &= \mathrm{Var}_{\mathbb{P}_{\chi_B K \chi_B}}[\langle K^{[X,B]}\phi, \phi\rangle] = \mathrm{Var}_{\mathbb{P}_{\chi_B \widetilde{K} \chi_B}}[\langle \widetilde{K}^{[X,B]}\phi, \phi\rangle] \\ &= \mathrm{Var}_{\mathbb{P}_{\widetilde{K}}}[\langle \widetilde{K}^{[X,B]}\phi, \phi\rangle]. \end{aligned} \tag{7.22}$$

Combining (7.19) and (7.22), we obtain the desired inequality (7.4). □

## 8. Appendix

*8.1. Conditioning on bounded subsets of determinantal point processes: proofs of Propositions 2.5 and 2.6*

Let $W \subset E$ be a Borel subset, not necessarily bounded. Recall that we identify the sets

$$\mathrm{Conf}(W) \simeq \mathrm{Conf}(E, W) := \{X \in \mathrm{Conf}(E) : X \subset W\} \subset \mathrm{Conf}(E).$$

Therefore, given a point process $\mathbb{P}$ on $E$, that is, a Borel probability on $\mathrm{Conf}(E)$, we may set

$$\overline{\mathbb{P}\restriction_{\mathrm{Conf}(W)}} := \begin{cases} \dfrac{\mathbb{P}\restriction_{\mathrm{Conf}(W)}}{\mathbb{P}(\mathrm{Conf}(W))} & \text{if } \mathbb{P}(\mathrm{Conf}(W)) > 0, \\ 0 & \text{if } \mathbb{P}(\mathrm{Conf}(W)) = 0. \end{cases} \tag{8.1}$$

Recall also the notation $\mathbb{P}^{X\cap B}$ introduced in (2.3) for the Palm measure of the point process $\mathbb{P}$ with respect to the points inside $X \cap B$.

We have the following description of conditional measures of general point process with respect to restricting the configuration to a bounded subset $B \subset E$.

**Proposition 8.1.** *Let $B \subset E$ be a bounded Borel subset. If $\mathbb{P}$ is a simple point process on $E$ admitting correlation measures of all orders, then $\mathbb{P}(\cdot \mid X, B) = \overline{\mathbb{P}^{X\cap B}\restriction_{\mathrm{Conf}(E\setminus B)}}$ for $\mathbb{P}$-almost every $X \in \mathrm{Conf}(E)$. In particular, for $\mathbb{P}$-almost every $X \in \mathrm{Conf}(E)$, we have $\mathbb{P}^{X\cap B}(\mathrm{Conf}(E \setminus B)) > 0$.*

*Proof.* Let $\mathrm{Conf}_n(E) = \{X \in \mathrm{Conf}(E) : \#X = n\}$ and define $\mathrm{Conf}_n(B)$ similarly. Via the natural map $E^n \to \mathrm{Conf}_n(E)$ defined by $(x_1, \ldots, x_n) \mapsto \{x_1, \ldots, x_n\}$, we define a measure $\rho^{\#}_{n,\mathbb{P}}$ on $\mathrm{Conf}_n(E)$ as the push-forward of the correlation measure $\rho_{n,\mathbb{P}}$ and define a $\sigma$-finite measure $\mathscr{C}^{\#}_{n,\mathbb{P}}$ on $\mathrm{Conf}_n(E) \times \mathrm{Conf}(E)$ as the push-forward of the $n$-th order Campbell measure $\mathscr{C}^{!}_{n,\mathbb{P}}$. The formula (2.2) implies that

$$\mathscr{C}^{\#}_{n,\mathbb{P}}(\mathrm{d}\mathfrak{p} \times \mathrm{d}X_1) = \rho^{\#}_{n,\mathbb{P}}(\mathrm{d}\mathfrak{p})\mathbb{P}^{\mathfrak{p}}(\mathrm{d}X_1). \tag{8.2}$$

By convention, we set $\rho^{\#}_{0,\mathbb{P}}(\mathrm{d}\mathfrak{p}) := \delta_{\emptyset}$ and $C^{\#}_{0,\mathbb{P}} := \delta_{\emptyset} \otimes \mathbb{P}$, where $\delta_{\emptyset}$ is the Dirac measure at the empty configuration $\emptyset$, i.e., the unique element $\emptyset \in \mathrm{Conf}_0(E)$. Equivalently, for any positive Borel function $H : \mathrm{Conf}_n(E) \times \mathrm{Conf}(E) \to \mathbb{R}^+$,

$$\int_{\mathrm{Conf}_n(E)\times\mathrm{Conf}(E)} H(\mathfrak{p}, X_1)\, \mathscr{C}^{\#}_{n,\mathbb{P}}(\mathrm{d}X_0 \times \mathrm{d}X_1) = \int_{\mathrm{Conf}(E)} \Big[ \sum_{x\in X^n}^{\#} H(\{x_1, \dots, x_n\}, X \setminus \{x_1, \dots, x_n\}) \Big] \mathbb{P}(\mathrm{d}X),$$

where the summation $\sum^{\#}$ is taken over all *ordered $n$-tuples* $(x_1, \dots, x_n)$ with distinct coordinates $x_1, \dots, x_n \in X$. In particular, when $n = 0$, this equality reads: for every $H : \mathrm{Conf}_0(E) \times \mathrm{Conf}(E) \to \mathbb{R}^+$,

$$\int_{\mathrm{Conf}_0(E)\times\mathrm{Conf}(E)} H(\mathfrak{p}, X_1)\, \mathscr{C}^{\#}_{0,\mathbb{P}}(\mathrm{d}\mathfrak{p} \times \mathrm{d}X_1) = \int_{\mathrm{Conf}(E)} H(\emptyset, X)\, \mathbb{P}(\mathrm{d}X).$$

The boundedness of $B \subset E$ implies that $\mathrm{Conf}(B) = \bigsqcup_{n=0}^{\infty} \mathrm{Conf}_n(B)$. Hence

$$\begin{aligned}\mathrm{Conf}(E) &\simeq \mathrm{Conf}(B) \times \mathrm{Conf}(E \setminus B)\\ &= \Big(\bigsqcup_{n=0}^{\infty} \mathrm{Conf}_n(B)\Big) \times \mathrm{Conf}(E \setminus B) = \bigsqcup_{n=0}^{\infty}\big(\mathrm{Conf}_n(B) \times \mathrm{Conf}(E \setminus B)\big).\end{aligned}$$

For any $n = 0, 1, \dots,$ let $H : \mathrm{Conf}_n(E) \times \mathrm{Conf}(E) \to \mathbb{R}^+$ be any non-negative Borel function supported on $\mathrm{Conf}_n(B) \times \mathrm{Conf}(E \setminus B) \subset \mathrm{Conf}_n(E) \times \mathrm{Conf}(E)$. Then for any $X \in \mathrm{Conf}(E)$, we have

$$\sum_{x\in X^n}^{\#} H(\{x_1, \dots, x_n\}, X \setminus \{x_1, \dots, x_n\}) = n! \cdot \chi_{\{\#(X\cap B)=n\}} \cdot H(X \cap B, X \cap (E \setminus B)).$$

When $n = 0$, this equality reads $H(\emptyset, X) = \chi_{\{X\cap B=\emptyset\}} \cdot H(X \cap B, X \cap (E \setminus B))$. By definition of $\mathscr{C}^{\#}_{n,\mathbb{P}}$, we get

$$\begin{aligned}\int_{\mathrm{Conf}_n(E)\times\mathrm{Conf}(E)} & H(\mathfrak{p}, X_1)\, \mathscr{C}^{\#}_{n,\mathbb{P}}(\mathrm{d}\mathfrak{p} \times \mathrm{d}X_1)\\ &= \int_{\mathrm{Conf}(E)} \Big[ \sum_{x\in X^n}^{\#} H(\{x_1, \dots, x_n\}, X \setminus \{x_1, \dots, x_n\}) \Big] \mathbb{P}(\mathrm{d}X)\\ &= n! \cdot \int_{\mathrm{Conf}(E)} \chi_{\{\#(X\cap B)=n\}} \cdot H(X \cap B, X \cap (E \setminus B))\, \mathbb{P}(\mathrm{d}X)\\ &= n! \cdot \int_{\mathrm{Conf}_n(B)\times\mathrm{Conf}(E\setminus B)} H(\mathfrak{p}, X_1)\, \mathbb{P}_{B,E\setminus B}(\mathrm{d}\mathfrak{p} \times \mathrm{d}X_1).\end{aligned}$$

The above equality, combined with (8.2), yields

$$\begin{aligned}\mathbb{P}_{B,E\setminus B}\!\restriction_{\mathrm{Conf}_n(B)\times\mathrm{Conf}(E\setminus B)}(\mathrm{d}\mathfrak{p}\times \mathrm{d}X_1) &= \frac{1}{n!}\mathscr{C}^{\#}_{n,\mathbb{P}}\!\restriction_{\mathrm{Conf}_n(B)\times\mathrm{Conf}(E\setminus B)}(\mathrm{d}\mathfrak{p}\times \mathrm{d}X_1)\\ &= \frac{1}{n!}\rho^{\#}_{n,\mathbb{P}}\!\restriction_{\mathrm{Conf}(B)}(\mathrm{d}\mathfrak{p})\mathbb{P}^{\mathfrak{p}}\!\restriction_{\mathrm{Conf}(E\setminus B)}(\mathrm{d}X_1)\\ &= \frac{\mathbb{P}^{\mathfrak{p}}(\mathrm{Conf}(E\setminus B))}{n!}\rho^{\#}_{n,\mathbb{P}}\!\restriction_{\mathrm{Conf}(B)}(\mathrm{d}\mathfrak{p})\overline{\mathbb{P}^{\mathfrak{p}}\!\restriction_{\mathrm{Conf}(E\setminus B)}}(\mathrm{d}X_1).\end{aligned}$$

Consequently,

$$\mathbb{P}_{B,E\setminus B}(\mathrm{d}\mathfrak{p}\times \mathrm{d}X_1) = \left(\sum_{n=0}^{\infty}\frac{\mathbb{P}^{\mathfrak{p}}(\mathrm{Conf}(E\setminus B))}{n!}\rho^{\#}_{n,\mathbb{P}}\!\restriction_{\mathrm{Conf}(B)}(\mathrm{d}\mathfrak{p})\right)\overline{\mathbb{P}^{\mathfrak{p}}\!\restriction_{\mathrm{Conf}(E\setminus B)}}(\mathrm{d}X_1).$$

This implies both the formula for $\pi_B(\mathbb{P})(\mathrm{d}\mathfrak{p})$ and the formula for $\mathbb{P}(\mathrm{d}X_1\,|\,\mathfrak{p},B) = \mathbb{P}_{B,E\setminus B}(\mathrm{d}X_1\,|\,\mathfrak{p},B)$:

$$\pi_B(\mathbb{P})(\mathrm{d}\mathfrak{p}) = \sum_{n=0}^{\infty}\frac{\mathbb{P}^{\mathfrak{p}}(\mathrm{Conf}(E\setminus B))}{n!}\rho^{\#}_{n,\mathbb{P}}\!\restriction_{\mathrm{Conf}(B)}(\mathrm{d}\mathfrak{p}), \tag{8.3}$$

$$\mathbb{P}(\mathrm{d}X_1\,|\,\mathfrak{p},B) = \overline{\mathbb{P}^{\mathfrak{p}}\!\restriction_{\mathrm{Conf}(E\setminus B)}}(\mathrm{d}X_1) \quad \text{for } \pi_B(\mathbb{P})\text{-a.e. } \mathfrak{p}\in\mathrm{Conf}(B). \tag{8.4}$$

Hence we get the desired relation $\mathbb{P}(\cdot\,|\,X,B) = \overline{\mathbb{P}^{X\cap B}\!\restriction_{\mathrm{Conf}(E\setminus B)}}$ for $\mathbb{P}$-almost every $X\in \mathrm{Conf}(E)$. □

**Remark.** Kallenberg [20, Section 12.3] defined the *compound Campbell* measure of $\mathbb{P}$ on $\mathrm{Conf_{fin}}(E)\times\mathrm{Conf}(E)$ by

$$\mathscr{C}^{\#}_{\mathbb{P}}(\mathrm{d}\mathfrak{p}\times \mathrm{d}X_1) := \sum_{n=0}^{\infty}\frac{1}{n!}\mathscr{C}^{\#}_{n,\mathbb{P}}(\mathrm{d}\mathfrak{p}\times \mathrm{d}X_1),$$

where $\mathrm{Conf_{fin}}(E) = \bigsqcup_{n=0}^{\infty}\mathrm{Conf}_n(E)$.

To prove Proposition 2.5, we will also need the description of the normalized restriction of a determinantal measure on $\mathrm{Conf}(E)$ to $\mathrm{Conf}(E\setminus B)\subset\mathrm{Conf}(E)$, which is given by the following

**Lemma 8.2** (see [3], [6, Propositions 2.1 and 2.2] and [5, Propositions 2.3 and 2.5]). *Let $\widetilde{K}: L^2(E,\mu)\to L^2(E,\mu)$ be a bounded self-adjoint locally trace class operator with* $\mathrm{spec}(\widetilde{K})\subset[0,1]$. *Assume that $B\subset E$ is a bounded subset such that the operator $1-\chi_B\widetilde{K}$ is invertible. Then:*

(i) *The measure*

$$\overline{\mathbb{P}_{\widetilde{K}}\!\restriction_{\mathrm{Conf}(E\setminus B)}} \tag{8.5}$$

*is a determinantal point process induced by the kernel*

$$\chi_{E\setminus B}\widetilde{K}(1-\chi_B\widetilde{K})^{-1}\chi_{E\setminus B}. \tag{8.6}$$

*In notation,*

$$\overline{\mathbb{P}_{\widetilde{K}}\restriction_{\mathrm{Conf}(E\setminus B)}} = \mathbb{P}_{\chi_{E\setminus B}\widetilde{K}(1-\chi_B\widetilde{K})^{-1}\chi_{E\setminus B}}. \tag{8.7}$$

(ii) *If moreover* $\widetilde{K}$ *is an orthogonal projection, then the operator* (8.6) *is the orthogonal projection from* $L^2(E,\mu)$ *onto the closure of*

$$\chi_{E\setminus B}\mathrm{Ran}(\widetilde{K}) = \{\chi_{E\setminus B} f : f \in \mathrm{Ran}(\widetilde{K})\}.$$

*Proof.* Recall that for any bounded linear operators $T, S$ on a Hilbert space, the invertibility of $1 - TS$ and the invertibility of $1 - ST$ are equivalent. Therefore, the assumption that $1 - \chi_B\widetilde{K} = 1 - \chi_B \cdot \chi_B\widetilde{K}$ is invertible implies that $1 - \chi_B\widetilde{K}\chi_B$ is invertible and hence $\det(1 - \chi_B\widetilde{K}\chi_B) \neq 0$. Now from the gap probability formula for determinantal point processes, we have

$$\mathbb{P}_{\widetilde{K}}(\mathrm{Conf}(E\setminus B)) = \det(1 - \chi_B\widetilde{K}\chi_B) > 0.$$

Since

$$\prod_{x\in X}\chi_{E\setminus B}(x) = \chi_{\mathrm{Conf}(E\setminus B)}(X),$$

we have

$$\overline{\mathbb{P}_{\widetilde{K}}\restriction_{\mathrm{Conf}(E\setminus B)}} = \frac{\prod_{x\in X}\chi_{E\setminus B}(x)\cdot\mathbb{P}_{\widetilde{K}}}{\int_{\mathrm{Conf}(E)}\prod_{x\in Y}\chi_{E\setminus B}(x)\cdot\mathbb{P}_{\widetilde{K}}(dY)}.$$

Now item (i) reads

$$\frac{\prod_{x\in X}\chi_{E\setminus B}(x)\cdot\mathbb{P}_{\widetilde{K}}}{\int_{\mathrm{Conf}(E)}\prod_{x\in Y}\chi_{E\setminus B}(x)\cdot\mathbb{P}_{\widetilde{K}}(dY)} = \mathbb{P}_{\chi_{E\setminus B}\widetilde{K}(1-\chi_B\widetilde{K})^{-1}\chi_{E\setminus B}}, \tag{8.8}$$

which follows from [3], [6, Proposition 2.1] or [5, Proposition 2.3] (by taking the function $g$ in [6, Proposition 2.1] or in [5, Proposition 2.3] to be the characteristic function $\chi_{E\setminus B}$). Item (ii) follows from [6, Proposition 2.2] (see also [5, Proposition 2.5]). □

**Remark.** In the discrete setting, see also Borodin and Rains [2] and Lyons [21] for the statements in Lemma 8.2.

**Remark.** For the reader's convenience, we include the proof of (8.8) under the additional assumption

$$\chi_B\widetilde{K} \in \mathscr{L}_1(L^2(E,\mu)).$$

Take any bounded measurable function $f$ on $E$ such that $(f-1)\widetilde{K} \in \mathscr{L}_1(L^2(E,\mu))$. Then

$$(f\chi_{E\setminus B} - 1)\widetilde{K} = \chi_{E\setminus B}(f-1)\widetilde{K} - \chi_B\widetilde{K} \in \mathscr{L}_1(L^2(E,\mu)).$$

By direct computation,

$$1 + (f\chi_{E\setminus B} - 1)\widetilde{K} = \left[1 + (f-1)\chi_{E\setminus B}\widetilde{K}(1-\chi_B\widetilde{K})^{-1}\right]\cdot(1-\chi_B\widetilde{K}).$$

Therefore, using elementary properties of Fredholm determinants, we get

$$\begin{aligned}\det(1+(f\chi_{E\setminus B}-1)\widetilde{K}) &= \det\big(1+(f-1)\chi_{E\setminus B}\widetilde{K}(1-\chi_B\widetilde{K})^{-1}\big)\det(1-\chi_B\widetilde{K})\\ &= \det\big(1+(f-1)\chi_{E\setminus B}\widetilde{K}(1-\chi_B\widetilde{K})^{-1}\chi_{E\setminus B}\big)\det(1-\chi_B\widetilde{K}),\end{aligned}$$

which in turn implies

$$\begin{aligned}&\frac{\int_{\mathrm{Conf}(E)}\prod_{x\in X}f(x)\cdot\prod_{x\in X}\chi_{E\setminus B}(x)\,\mathbb{P}_{\widetilde{K}}(\mathrm{d}X)}{\int_{\mathrm{Conf}(E)}\prod_{x\in X}\chi_{E\setminus B}(x)\,\mathbb{P}_{\widetilde{K}}(\mathrm{d}X)}\\ &\qquad = \int_{\mathrm{Conf}(E)}\Big(\prod_{x\in X}f(x)\Big)\mathbb{P}_{\chi_{E\setminus B}\widetilde{K}(1-\chi_B\widetilde{K})^{-1}\chi_{E\setminus B}}(\mathrm{d}X),\end{aligned}$$

and the equality (8.8) follows immediately.

*Proof of Proposition 2.5.* Since the determinantal point process $\mathbb{P}_K$ is a simple point process, by Proposition 8.1 for $\mathbb{P}_K$-almost all $X\in\mathrm{Conf}(E)$ we have $\mathbb{P}_K^{X\cap B}(\mathrm{Conf}(E\setminus B)) > 0$ and

$$\mathbb{P}_K(\cdot\,|\,X,B) = \overline{\mathbb{P}_K^{X\cap B}\restriction_{\mathrm{Conf}(E\setminus B)}}. \tag{8.9}$$

Recall that the measure $\mathbb{P}_K^{X\cap B}$, introduced in (2.3), is the Palm measure of $\mathbb{P}_K$ with respect to the points inside $X\cap B$. By Shirai and Takahashi [38, Theorem 1.7], for $\mathbb{P}_K$-almost every $X\in\mathrm{Conf}(E)$, the Palm measure $\mathbb{P}_K^{X\cap B}$ is a determinantal point process on $E$, induced by the correlation kernel

$$K^{X\cap B} = K^{p_1,\dots,p_n} \quad \text{if } X\cap B=\{p_1,\dots,p_n\}, \tag{8.10}$$

where $K^{p_1,\dots,p_n}$ is defined by (1.8). In notation,

$$\mathbb{P}_K^{X\cap B} = \mathbb{P}_{K^{X\cap B}}. \tag{8.11}$$

The above identity combined with $\mathbb{P}_K^{X\cap B}(\mathrm{Conf}(E\setminus B)) > 0$ for $\mathbb{P}_K$-almost every $X\in\mathrm{Conf}(E)$ implies that

$$\det(1-\chi_B K^{X\cap B}\chi_B) = \mathbb{P}_{K^{X\cap B}}(\mathrm{Conf}(E\setminus B)) = \mathbb{P}_K^{X\cap B}(\mathrm{Conf}(E\setminus B)) > 0$$

for $\mathbb{P}_K$-almost every $X\in\mathrm{Conf}(E)$. This in turn implies that $1-\chi_B K^{X\cap B}\chi_B$ and hence $1-\chi_B K^{X\cap B}$ is invertible. So Lemma 8.2 in the Appendix is applicable to the kernel $K^{X\cap B}$ and the subset $B\subset E$. Combining (8.7) with $\widetilde{K}$ replaced by $K^{X\cap B}$ and the equalities (8.9), (8.11), we obtain

$$\mathbb{P}_K(\cdot\,|\,X,B) = \overline{\mathbb{P}_K^{X\cap B}\restriction_{\mathrm{Conf}(E\setminus B)}} = \overline{\mathbb{P}_{K^{X\cap B}}\restriction_{\mathrm{Conf}(E\setminus B)}} = \mathbb{P}_{\chi_{E\setminus B}K^{X\cap B}(1-\chi_B K^{X\cap B})^{-1}\chi_{E\setminus B}}.$$

Now using the definition (8.10) of $K^{X\cap B}$ and the definition (1.9) of $K^{[X,B]}$, we obtain

$$\chi_{E\setminus B}K^{X\cap B}(1-\chi_B K^{X\cap B})^{-1}\chi_{E\setminus B} = K^{[X,B]}.$$

This completes the proof of Proposition 2.5. □

*Proof of Proposition 2.6.* The first assertion of Proposition 2.6 can be proved by induction on $n$, by noting that $K^{p_1,\dots,p_n} = ((K^{p_1})^{\cdots})^{p_n}$. In particular, when $n = 1$, the equality $K^{p_1}(x, y) = K(x, y) - \frac{K(x,p_1)K(p_1,y)}{K(p_1,p_1)}$ implies that $K^{p_1} = K - \Pi_{K_{p_1}}$ where $\Pi_{K_{p_1}}$ is the rank-one orthogonal projection onto the linear space spanned by the function $K_{p_1}(\cdot) = K(\cdot, p_1)$. Therefore, $K^{p_1}$ is the orthogonal projection onto $H(p_1)$.

The second assertion of Proposition 2.6 is an immediate consequence of Lemma 8.2(ii). □

### 8.2. *Martingales corresponding to conditional processes*

Let $\mathbb{P}$ be a point process on $E$ and let $W \subset E$ be a Borel subset of $E$. Let $W_1 \subset W_2 \subset \cdots \subset W$ be an increasing sequence of Borel subsets of $W$ such that $W = \bigcup_{n=1}^{\infty} W_n$.

**Proposition 8.3.** *The sequence* $((\pi_{E\setminus W})_*[\mathbb{P}(\cdot \mid X, W_n)])_{n\in\mathbb{N}}$ *is an* $(\mathcal{F}(W_n))_{n\in\mathbb{N}}$*-adapted martingale defined on the probability space* $(\mathrm{Conf}(E), \mathcal{F}(E), \mathbb{P})$*. Moreover,*

$$(\pi_{E\setminus W})_*[\mathbb{P}(\cdot \mid X, W_n)] = \mathbb{E}_{\mathbb{P}}\big[\mathbb{P}(\cdot \mid X, W) \,\big|\, \mathcal{F}(W_n)\big] \quad \textit{for } \mathbb{P}\textit{-a.e. } X \in \mathrm{Conf}(E). \tag{8.12}$$

*In particular, by the martingale convergence theorem, for all Borel subsets* $\mathcal{A} \subset \mathrm{Conf}(E \setminus W)$ *and any* $1 \le p < \infty$*,*

$$\big((\pi_{E\setminus W})_*[\mathbb{P}(\cdot \mid X, W_n)]\big)(\mathcal{A}) \xrightarrow[\mathbb{P}\text{-a.s. and in } L^p(\mathrm{Conf}(E),\mathbb{P})]{n\to\infty} \mathbb{P}(\mathcal{A} \mid X, W). \tag{8.13}$$

*Moreover, for* $\mathbb{P}$*-almost every* $X \in \mathrm{Conf}(E)$*,*

$$(\pi_{E\setminus W})_*[\mathbb{P}(\cdot \mid X, W_n)] \xrightarrow[\text{weakly}]{n\to\infty} \mathbb{P}(\cdot \mid X, W). \tag{8.14}$$

**Remark.** In general, (8.13) cannot be strengthened to the claim that for $\mathbb{P}$-almost every $X \in \mathrm{Conf}(E)$, we have $((\pi_{E\setminus W})_*[\mathbb{P}(\cdot \mid X, W_n)])(\mathcal{A}) \xrightarrow{n\to\infty} \mathbb{P}(\mathcal{A} \mid X, W)$, for **all** Borel subsets $\mathcal{A} \subset \mathrm{Conf}(E \setminus W)$.

We prepare a simple lemma. Let $\Omega_i$, $i = 1, 2, \dots,$ and $\Omega^*$ be standard Borel spaces. Fix $n \in \mathbb{N}$ and denote

$$x := (x_i)_{i=1}^{\infty} \quad \text{and} \quad t =: (x_i)_{i \ge n+1},$$

while $z$ will stand for the coordinate on $\Omega^*$. Let $Q(\mathrm{d}x \times \mathrm{d}z)$ be a Borel probability measure on $(\prod_{i=1}^{\infty} \Omega_i) \times \Omega^*$. For any $n \in \mathbb{N}$, let $q_n(x_1, \dots, x_n; \mathrm{d}z)$ be the marginal on $\Omega^*$ of the conditional measure $Q(\mathrm{d}t \times \mathrm{d}z \mid x_1, \dots, x_n)$.

**Lemma 8.4.** *We have*

$$q_n(x_1, \dots, x_n; \mathrm{d}z) = \mathbb{E}[Q(\mathrm{d}z \mid x_1, \dots, x_n, t) \mid x_1, \dots, x_n].$$

*Proof.* Denote by $Q_n$ the marginal measure of $Q$ on $\Omega_1 \times \cdots \times \Omega_n$. Let $Q_\infty$ be the marginal measure of $Q$ on $\prod_{i=1}^{\infty} \Omega_i$. By definition of conditional measures, we have

$$Q(\mathrm{d}x \times \mathrm{d}z) = Q_\infty(\mathrm{d}x) Q(\mathrm{d}z \,|\, x_1, \dots, x_n, t),$$
$$Q(\mathrm{d}x \times \mathrm{d}z) = Q_n(\mathrm{d}x_1 \cdots \mathrm{d}x_n) Q(\mathrm{d}t \times \mathrm{d}z \,|\, x_1, \dots, x_n).$$

And also

$$\mathbb{E}[Q(\mathrm{d}z \,|\, x_1, \dots, x_n, t) \,|\, x_1, \dots, x_n] = \int_{t \in \prod_{i=n+1}^{\infty} \Omega_i} Q(\mathrm{d}z \,|\, x_1, \dots, x_n, t) Q_\infty(\mathrm{d}t \,|\, x_1, \dots, x_n).$$

Since $Q_\infty(\mathrm{d}x) = Q_n(\mathrm{d}x_1 \cdots \mathrm{d}x_n) Q_\infty(\mathrm{d}t \,|\, x_1, \dots, x_n)$, we get

$$Q(\mathrm{d}x \times \mathrm{d}z) = Q_n(\mathrm{d}x_1 \cdots \mathrm{d}x_n) Q_\infty(\mathrm{d}t \,|\, x_1, \dots, x_n) Q(\mathrm{d}z \,|\, x_1, \dots, x_n, t).$$

Consequently,

$$Q(\mathrm{d}t \times \mathrm{d}z \,|\, x_1, \dots, x_n) = Q_\infty(\mathrm{d}t \,|\, x_1, \dots, x_n) Q(\mathrm{d}z \,|\, x_1, \dots, x_n, t).$$

By definition, we have

$$\begin{aligned} q_n(x_1, \dots, x_n; \mathrm{d}z) &= \int_{t \in \prod_{i=n+1}^{\infty} \Omega_i} Q(\mathrm{d}t \times \mathrm{d}z \,|\, x_1, \dots, x_n) \\ &= \int_{t \in \prod_{i=n+1}^{\infty} \Omega_i} Q_\infty(\mathrm{d}t \,|\, x_1, \dots, x_n) Q(\mathrm{d}z \,|\, x_1, \dots, x_n, t) \\ &= \mathbb{E}[Q(\mathrm{d}z \,|\, x_1, \dots, x_n, t) \,|\, x_1, \dots, x_n]. \qquad \square \end{aligned}$$

*Proof of Proposition 8.3.* Apply Lemma 8.4 to $\Omega_i = \mathrm{Conf}(W_i \setminus W_{i-1})$. □

### 8.3. *Mixing for Möbius transformations acting on* $(\mathrm{Conf}(\mathbb{D}), \mathbb{P}_{K_\mathbb{D}})$ *and proof of Lemma 1.3*

For any $n \in \mathbb{N}$ and any $\varepsilon > 0$, we have

$$\mathbb{P}\big(\#(Z(\mathfrak{f}_\mathbb{D}) \cap \{z \in \mathbb{D} : |z| \leq \varepsilon\}) \geq n\big) > 0.$$

To conclude the proof of Lemma 1.3, it suffices to establish the ergodicity of the distribution of $Z(\mathfrak{f}_\mathbb{D})$ under the group $\mathrm{Aut}(\mathbb{D})$ of Möbius transformations, in other words, the group of isometries of the Lobachevsky plane. We prove mixing for hyperbolic and parabolic one-dimensional subgroups of $\mathrm{Aut}(\mathbb{D})$.

**Lemma 8.5.** *If* $\gamma \in \mathrm{Aut}(\mathbb{D})$ *is either hyperbolic or parabolic, then the dynamical system* $(\mathrm{Conf}(\mathbb{D}), \mathbb{P}_{K_\mathbb{D}}, \gamma)$ *is strongly mixing.*

*Proof.* Fix an increasing sequence $r_k$ in $(0, 1)$ such that $\lim_k r_k = 1$. Let $\mathcal{A}, \mathcal{B}$ be any fixed measurable subsets in $\mathrm{Conf}(\mathbb{D})$. For any $\varepsilon > 0$, there exist $\mathcal{A}_\varepsilon, \mathcal{B}_\varepsilon \subset \mathrm{Conf}(\mathbb{D})$ and a compact subset $C_\varepsilon \subset \mathbb{D}$ such that $\mathcal{A}_\varepsilon, \mathcal{B}_\varepsilon$ are both $\mathcal{F}(C_\varepsilon)$-measurable and

$$\mathbb{P}_{K_\mathbb{D}}(\mathcal{A} \triangle \mathcal{A}_\varepsilon) \leq \varepsilon, \quad \mathbb{P}_{K_\mathbb{D}}(\mathcal{B} \triangle \mathcal{B}_\varepsilon) \leq \varepsilon. \tag{8.15}$$

Since $\mathbb{P}_{K_\mathbb{D}}$ is $\gamma$-invariant, we have

$$\sup_{n\in\mathbb{N}} |\mathbb{P}_{K_\mathbb{D}}(\mathcal{A} \cap \gamma^{-n}(\mathcal{B})) - \mathbb{P}_{K_\mathbb{D}}(\mathcal{A}_\varepsilon \cap \gamma^{-n}(\mathcal{B}_\varepsilon))| \leq 2\varepsilon. \tag{8.16}$$

For any $r_k$ denote $\mathbb{D}_{r_k} := \{z \in \mathbb{D} : |z| < r_k\}$. By the assumption on $\gamma$, for any $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that

$$\gamma^{-n}(C_\varepsilon) \cap \mathbb{D}_{r_k} = \emptyset \quad \text{for all } n \geq n_k.$$

It follows that for any $n \geq n_k$,

$$\mathbb{P}_{K_\mathbb{D}}(\mathcal{A}_\varepsilon \cap \gamma^{-n}(\mathcal{B}_\varepsilon)) = \mathbb{E}_{\mathbb{P}_{K_\mathbb{D}}}\big(\chi_{\gamma^{-n}(\mathcal{B}_\varepsilon)} \mathbb{E}_{\mathbb{P}_{K_\mathbb{D}}}[\chi_{\mathcal{A}_\varepsilon} | \mathcal{F}(\mathbb{D} \setminus \mathbb{D}_{r_k})]\big).$$

Therefore, for any $k$,

$$\begin{aligned}
&\limsup_{n\to\infty} |\mathbb{P}_{K_\mathbb{D}}(\mathcal{A}_\varepsilon \cap \gamma^{-n}(\mathcal{B}_\varepsilon)) - \mathbb{P}_{K_\mathbb{D}}(\mathcal{A}_\varepsilon)\mathbb{P}_{K_\mathbb{D}}(\mathcal{B}_\varepsilon)| \\
&\quad = \limsup_{n\to\infty} \big|\mathbb{E}_{\mathbb{P}_{K_\mathbb{D}}}\big(\chi_{\gamma^{-n}(\mathcal{B}_\varepsilon)} \mathbb{E}_{\mathbb{P}_{K_\mathbb{D}}}[\chi_{\mathcal{A}_\varepsilon} \mid \mathcal{F}(\mathbb{D} \setminus \mathbb{D}_{r_k})]\big) - \mathbb{E}_{\mathbb{P}_{K_\mathbb{D}}}(\chi_{\gamma^{-n}(\mathcal{B}_\varepsilon)} \mathbb{E}_{\mathbb{P}_{K_\mathbb{D}}}[\chi_{\mathcal{A}_\varepsilon}])\big| \\
&\quad \leq \limsup_{n\to\infty} \mathbb{E}_{\mathbb{P}_{K_\mathbb{D}}} |\chi_{\gamma^{-n}(\mathcal{B}_\varepsilon)} \mathbb{E}_{\mathbb{P}_{K_\mathbb{D}}}[\chi_{\mathcal{A}_\varepsilon} \mid \mathcal{F}(\mathbb{D} \setminus \mathbb{D}_{r_k})] - \chi_{\gamma^{-n}(\mathcal{B}_\varepsilon)} \mathbb{E}_{\mathbb{P}_{K_\mathbb{D}}}[\chi_{\mathcal{A}_\varepsilon}]| \\
&\quad \leq \mathbb{E}_{\mathbb{P}_{K_\mathbb{D}}} |\mathbb{E}_{\mathbb{P}_{K_\mathbb{D}}}[\chi_{\mathcal{A}_\varepsilon} \mid \mathcal{F}(\mathbb{D} \setminus \mathbb{D}_{r_k})] - \mathbb{E}_{\mathbb{P}_{K_\mathbb{D}}}[\chi_{\mathcal{A}_\varepsilon}]|.
\end{aligned}$$

Theorem 1.6 now implies

$$\lim_{k\to\infty} \mathbb{E}_{\mathbb{P}_{K_\mathbb{D}}} |\mathbb{E}_{\mathbb{P}_{K_\mathbb{D}}}[\chi_{\mathcal{A}_\varepsilon} \mid \mathcal{F}(\mathbb{D} \setminus \mathbb{D}_{r_k})] - \mathbb{E}_{\mathbb{P}_{K_\mathbb{D}}}[\chi_{\mathcal{A}_\varepsilon}]| = 0$$

and hence

$$\limsup_{n\to\infty} |\mathbb{P}_{K_\mathbb{D}}(\mathcal{A}_\varepsilon \cap \gamma^{-n}(\mathcal{B}_\varepsilon)) - \mathbb{P}_{K_\mathbb{D}}(\mathcal{A}_\varepsilon)\mathbb{P}_{K_\mathbb{D}}(\mathcal{B}_\varepsilon)| = 0. \tag{8.17}$$

Combining (8.15)–(8.17), we obtain

$$\lim_{n\to\infty} \mathbb{P}_{K_\mathbb{D}}(\mathcal{A} \cap \gamma^{-n}(\mathcal{B})) = \mathbb{P}_{K_\mathbb{D}}(\mathcal{A})\mathbb{P}_{K_\mathbb{D}}(\mathcal{B})$$

and thus complete the proof of the strong mixing property of the dynamical system $(\mathrm{Conf}(\mathbb{D}), \mathbb{P}_{K_\mathbb{D}}, \gamma)$. □

*Proof of Lemma 1.3.* We need to show that almost surely,

$$\sup_{\gamma\in\mathrm{Aut}(\mathbb{D})} \#(Z(\mathfrak{f}_\mathbb{D}) \cap \gamma^{-1}(\mathbb{D}_\varepsilon)) = \infty \quad \text{for all } \varepsilon \in (0, 1) \cap \mathbb{Q}.$$

Since $(0, 1) \cap \mathbb{Q}$ is countable, we only need to show that for any fixed $\varepsilon \in (0, 1) \cap \mathbb{Q}$,

$$\sup_{\gamma\in\mathrm{Aut}(\mathbb{D})} \#(Z(\mathfrak{f}_\mathbb{D}) \cap \gamma^{-1}(\mathbb{D}_\varepsilon)) = \infty \quad \text{almost surely.} \tag{8.18}$$

Now fix any $\varepsilon \in (0,1) \cap \mathbb{Q}$. The distribution of $Z(\mathfrak{f}_{\mathbb{D}}) \cap \mathbb{D}_\varepsilon$ is given by the determinantal measure induced by the kernel $\chi_{\mathbb{D}_\varepsilon} K_{\mathbb{D}} \chi_{\mathbb{D}_\varepsilon}$. Since $\mathrm{rank}(\chi_{\mathbb{D}_\varepsilon} K_{\mathbb{D}} \chi_{\mathbb{D}_\varepsilon}) = \infty$, for any $\ell \in \mathbb{N}$ we have

$$\mathbb{P}(\#(Z(\mathfrak{f}_{\mathbb{D}}) \cap \mathbb{D}_\varepsilon) \geq \ell) > 0.$$

If $\gamma_0 \in \mathrm{Aut}(\mathbb{D})$ is hyperbolic or parabolic, then Lemma 8.5 implies that the dynamical system $(\mathrm{Conf}(\mathbb{D}), \mathbb{P}_{K_{\mathbb{D}}}, \gamma_0)$ is ergodic, whence for any $\ell \in \mathbb{N}$, the relation

$$\#(Z(\mathfrak{f}_{\mathbb{D}}) \cap \gamma_0^{-n}(\mathbb{D}_\varepsilon)) \geq \ell$$

holds for infinitely many $n$'s on a set of full measure. Since $\ell$ is arbitrary, the desired equality (8.18) follows. □

We conclude this section with a conjecture on the asymptotic density of zeros of Gaussian analytic functions. Let $F$ be a finite subset of the unit circle $\mathbb{T}$ and $\mathfrak{s}_F$ be the corresponding Stolz star domain, which, by definition, is the union, over all $z \in F$, of the Euclidean convex hulls of the unions $\{z\} \cup \{w \in \mathbb{D} : |w| \leq 1/\sqrt{2}\}$. Let $\{I_k\}_k$ be the complementary arcs of the subset $F$ in $\mathbb{T}$, and set

$$\widehat{k}(F) := 1 - \sum_k \frac{|I_k|}{2\pi} \log \frac{|I_k|}{2\pi}.$$

For a countable subset $X \subset \mathbb{D}$ without accumulation points in the interior of the disc, following [16, Chapter 4, Definition 4.9] write

$$D^+(X) := \frac{1}{2} \limsup_{\widehat{k}(F)\to\infty} \frac{\sum_n \{1 - |x|^2 : x \in \mathfrak{s}_F \cap X\}}{\widehat{k}(F)},$$
$$D^-(X) := \frac{1}{2} \liminf_{\widehat{k}(F)\to\infty} \frac{\sum_n \{1 - |x|^2 : x \in \mathfrak{s}_F \cap X\}}{\widehat{k}(F)}.$$

For $p > 1$, let $A^p(\mathbb{D})$ be the $L^p$-version of Bergman space. Theorem 4.31 and Corollary 4.38 in [16] state that

$$X \text{ is an } A^{2+\varepsilon}(\mathbb{D})\text{-zero set for some } \varepsilon > 0 \quad \text{if and only if} \quad D^+(X) < 1/2,$$
$$X \text{ is an } A^{2-\varepsilon}(\mathbb{D})\text{-zero set for all } \varepsilon > 0 \quad \text{if and only if} \quad D^+(X) \leq 1/2.$$

**Conjecture.** $D^+(Z(\mathfrak{f}_{\mathbb{D}})) = 1/2$ *almost surely.*

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