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# Local eigenvalue statistics of one-dimensional random nonselfadjoint pseudodifferential operators

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**Abstract.** We consider a class of one-dimensional nonselfadjoint semiclassical pseudodifferential operators, subject to small random perturbations, and study the statistical properties of their (discrete) spectra, in the semiclassical limit $h \to 0$. We compare two types of random perturbation: a random potential vs. a random matrix. Hager and Sjöstrand showed that, with high probability, the local spectral density of the perturbed operator follows a semiclassical form of Weyl's law, depending on the value distribution of the principal symbol of our pseudodifferential operator.

Beyond the spectral density, we investigate the full local statistics of the perturbed spectrum, and show that it satisfies a form of universality: the statistics only depends on the local spectral density, and of the type of random perturbation, but it is independent of the precise law of the perturbation. This local statistics can be described in terms of the Gaussian Analytic Function, a classical ensemble of random entire functions.

**Keywords.** Spectral theory of nonselfadjoint operators, random operators, pseudodifferential operators, random analytic functions

## Contents



## 1. Introduction

The spectral analysis of linear operators acting on a Hilbert space is much developed in the case of selfadjoint operators: one can then use powerful tools, like the spectral the-

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orem, or variational methods. This fact has been very useful in mathematical physics, for example in quantum mechanics, where the natural operators (quantum observables, Hamiltonian) are selfadjoint. However, nonselfadjoint operators also appear in mathematical physics, and deserve to be investigated. For instance, in quantum mechanics, the study of scattering systems naturally leads to the concept of quantum resonances, which appear as the (complex valued) poles of the analytic continuation of the scattering matrix (or of the resolvent of the Hamiltonian) into the so-called nonphysical sheet of the complex energy plane. These resonances may also be obtained as *bona fide* eigenvalues of a nonselfadjoint operator, obtained from the initial selfadjoint Hamiltonian through a complex dilation procedure [1, 2]. Still in quantum mechanics, when considering the evolution of a "small system" in contact with an "environment", one can be led to express the effective dynamics of the small system through a nonselfadjoint *Lindblad operator* [34]. In statistical mechanics, the evolution of the system may be described by a linear operator, which is often nonselfadjoint: the Fokker–Planck, or the linearized Boltzmann equation typically contain convective as well as dissipative terms, leading to nonselfadjoint operators. In hydrodynamics, the operators appearing when linearizing the Navier–Stokes equation in the vicinity of some specific solution are generally not selfadjoint [19].

When studying evolution problems generated by linear operators, one is naturally led to analyze the spectrum of that operator. Yet, in the nonselfadjoint case, the connection between the long time evolution and a spectrum of complex eigenvalues is not so obvious as in the selfadjoint case, since eigenstates do not form an orthonormal family. This difficulty of relating spectrum and dynamics is linked with a characteristics of nonselfadjoint operators, namely the possible strong *instability* of their spectrum with respect to small perturbations, a phenomenon nowadays commonly called *pseudospectral effect*. Traditionally this spectral instability was considered as a drawback, since it can be at the source of immense numerical errors [14]. However, as we will see below, analyzing this instability can also exhibit interesting phenomena. Numerical analysis studies, e.g. by L. N. Trefethen [47], somewhat changed the perspective of this instability problem: they showed that considering the *pseudospectrum* of the (nonselfadjoint) operator—that is the region where the norm of the resolvent operator exceeds some (large) threshold—is often more relevant than considering its spectrum, and can reveal important dynamical information. As an example, when studying a certain class of nonlinear diffusion equations, Sandsteede–Scheel [39], Raphael–Zworski [38] and Galkowski [18] showed that the pseudospectrum of the (nonselfadjoint) linearization of the equation can explain the finite time blow-up of the solutions to the full nonlinear equation, while the mere study of its spectrum would suggest a stable evolution.

In physical situations, an "ideal" evolution operator can be perturbed by many different sources, most of them uncontrolled by the experimentalist. Hence, it seems relevant to set up a model of *random* perturbations, and to investigate how the spectrum of our initial operator reacts to the addition of such perturbations. The spectrum of the perturbed operator thereby becomes random; in case this spectrum is discrete, it forms a *random point process* on the complex plane, which can be investigated by probabilistic methods. This is what we will do in this article, for a particular class of nonselfadjoint operators. Namely, we will focus on *semiclassical pseudodifferential operators with complex valued* *symbols*, and with some ellipticity assumption ensuring that the spectrum is discrete (at least in some region of the complex plane). Here "semiclassical" means that our operators depend on a parameter $h \in ]0, 1]$ (often referred to as "Planck's parameter"), and that we will be interested in the asymptotic (*semiclassical*) regime $h \searrow 0$. This small parameter will provide us with a natural threshold to define the pseudospectrum, and thereby to measure the spectral instability. The spectrum of these operators is in general very sensitive to perturbations: as we will see, in many examples the spectrum of the initial operator is localized along 1-dimensional curves in the complex plane, while the spectrum of the perturbed operator fills up an open domain of $\mathbb{C}$ (called the *classical spectrum*), defined by the symbol of our unperturbed operator. This filling up of the classical spectrum through perturbation has been studied in a series of works by Hager [24, 23], Sjöstrand [42, 41, 25] and Bordeaux-Montrieux [5] (see also [8] for a similar phenomenon in the framework of Toeplitz operators on the 2-dimensional torus). These authors show that the spectrum of the randomly perturbed operator satisfies, with high probability, a complex valued version of Weyl's law: the density of eigenvalues near a given "complex energy" $z_0$ inside the classical spectrum is approximately given by $(2\pi h)^{-1} D(z_0)$, where $D(z_0) > 0$ is the *classical density* at the energy $z_0$, associated with the symbol of our initial operator.

This Weyl's law counts the eigenvalues in any given region of $\mathbb{C}$, independent of $h$, it therefore describes the spectrum at the *macroscopic scale*. Since the spectral density is of order $h^{-1}$, it is reasonable to think that the typical distance between nearest eigenvalues should be of order $h^{1/2}$, which we will call the *microscopic scale*. Our aim in the present article is to investigate the distribution of eigenvalues at this microscopic scale, from a statistical point of view; in other words, we aim at studying the *local spectral statistics*, for our family of randomly perturbed operators, in particular the type of statistical correlations between nearby eigenvalues. A first result on these correlations has been obtained by the second named author [48], who computed the 2-point correlation between the eigenvalues of our randomly perturbed operator in the case of Gaussian perturbations.

In this article we will give a full description of these local statistics, expressed in terms of a certain *Gaussian analytic function*. In particular, we will prove a partial form of *universality* with respect to the law of the random perturbation.

Before stating our results more precisely, and to provide some motivation, let us recall some background on the topic of spectral statistics, from a mathematical physics perspective. In the 1950s Wigner [50] had the idea, when studying the spectra of complicated Hamiltonian operators in nuclear physics, to replace these (very structured) operators by large (nonstructured) random matrices. Those random matrices could not reproduce the large scale density fluctuations of the nuclear spectra, which depend on specific features of the system, but they could (empirically) reproduce the local statistical properties of the spectra, at the scale of the mean spacing between eigenvalues. Wigner and Dyson [13] understood that these *local* statistical properties only depend on certain global symmetries of the Hamiltonian, like time reversal invariance, but not on the fine details of the Hamiltonian: these statistical properties were thus said to be *universal*. In the 1980s, this universality conjecture was extended to simpler Hamiltonians, namely Laplacians on Euclidean domains with specific shapes: Bohigas–Giannoni–Schmidt [4] observed that if the billiard flow in the domain is "chaotic", then the local spectral statistics of the corresponding Laplacian correspond to Dyson's Gaussian Orthogonal ensemble of random matrices. In parallel, a large variety of non-Gaussian random Hermitian matrix ensembles were developed and studied, notably the Wigner random matrices (all entries are i.i.d., up to Hermitian symmetry), for which the local spectral statistics was recently shown to be identical with that of the Gaussian ensembles [15], another manifestation of universality.

What about nonselfadjoint operators? Various random ensembles of nonhermitian matrices have also been introduced in the theoretical physics literature. The main objective has been to understand the distribution of quantum resonances for various types of scattering or dissipative systems; see for instance [17, 52, 32, 21] (a short recent review can be found in [16]). For most of these models, the focus has been to derive the mean spectral density, without investigating the correlations between the eigenvalues. The "historical" nonhermitian random matrix model, for which the full eigenvalue statistics has been derived in closed form, is the complex Ginibre ensemble [20], where all entries are i.i.d. complex Gaussian; the nearby eigenvalues then exhibit a statistical repulsion between themselves, similar to the case of Dyson's Gaussian Unitary Ensemble of Hermitian matrices. For certain non-Gaussian ensembles, recent results [7, 46] have been obtained on the eigenvalue distribution at the microscopic scale, including some partial universality results.

Let us also mention a model studied recently by Capitaine and Bordenave [6] (see also [10]), namely the case of a large Jordan block perturbed by a Ginibre random matrix: the authors prove that most eigenvalues of the perturbed matrix lie close to the unit circle, but they also show that the "outliers" (the relatively few eigenvalues away from the unit circle) are statistically distributed like the zeros of a "hyperbolic" Gaussian analytic function (GAF). A similar result was proved by Sjöstrand and the second named author [44] in the case of a nonselfadjoint bi-diagonal matrix, perturbed by a small Ginibre matrix. In these two models, GAFs appear because the perturbation is chosen to be Gaussian.

Our results will also involve Gaussian analytic functions, but of "Euclidean" type. In our case, these GAFs will describe the bulk of the spectrum, as opposed to a few outliers; also, in our case Gaussian functions appear in the limit, even though the perturbation operator or potential is not necessarily Gaussian distributed.

### *1.1. Presentation of the results for a simple model case*

Before stating our results in full generality, we will illustrate them by first focussing on a simple case. Call $h \in ]0, 1]$ *Planck's parameter*, and consider the semiclassical complex harmonic oscillator

$$P_h := -h^2\partial_x^2 + ix^2 \quad \text{acting on } L^2(\mathbb{R}). \tag{1.1}$$

The (semiclassical) principal symbol of $P_h$ is given by the function

$$p(x,\xi) = \xi^2 + ix^2 \quad \text{on the phase space } \mathbb{R}^2 \ni \rho = (x,\xi). \tag{1.2}$$

We call the set

$$\Sigma := \overline{p(\mathbb{R}^2)} \subset \mathbb{C} \quad \text{the \textit{classical spectrum} of } P_h. \tag{1.3}$$

Here $\Sigma$ is the upper right quadrant of $\mathbb{C}$. The spectrum of $P_h$ is purely discrete, and is contained in $\Sigma$ (it is explicitly given by $\{z_n = e^{i\pi/4}h(2n+1);\ n \in \mathbb{N}\}$). Take an open subset $\Omega \Subset \mathring{\Sigma}$. Then, for any $z = X + iY \in \Omega$, an important data for our construction will be the structure of the "energy shell"[1] $p^{-1}(z) \subset \mathbb{R}^2$. Since $p : \mathbb{R}^2 \to \mathbb{C}$ is a local diffeomorphism for $z \in \Omega$, this energy shell consists of a discrete set of points; in the case of the harmonic oscillator, $p^{-1}(z)$ consists of four distinct points:

$$\begin{aligned} \rho_+^1 &= (Y^{1/2}, -X^{1/2}), \qquad \rho_+^2 = (-Y^{1/2}, X^{1/2}),\\ \rho_-^1 &= (Y^{1/2}, X^{1/2}), \qquad \rho_+^2 = (-Y^{1/2}, -X^{1/2}). \end{aligned} \tag{1.4}$$

We have labelled those points according to the sign of the Poisson bracket $\{\mathrm{Re}\, p, \mathrm{Im}\, p\}(\rho) = 4x\xi$: at the points $\rho_+^j$ the bracket is negative, while at the points $\rho_-^j$ it is positive. From this bracket condition, one can construct [9, 11], for each $z \in \Omega$ and $j = 1, 2$, a semiclassical family of functions $(e_+^j(z,h) \in L^2(\mathbb{R}))_{h\in]0,1]}$, $\|e_+^j(z,h)\| = 1$, satisfying[2]

$$\|(P_h - z)e_+^j(z,h)\| = \mathcal{O}(h^\infty), \tag{1.5}$$

and such that $e_+^j(z,h)$ is microlocalized at the point $\rho_+^j(z)$.[3] (Here and in the entire text, all norms without index are either norms in $L^2$ or in $\mathcal{B}(L^2)$, the space of bounded linear operators $L^2 \to L^2$). We call each family $(e_+^j(z,h))$ an $h^\infty$-*quasimode* of $P - z$, or for short a quasimode of $P - z$. Similarly, there exist quasimodes $e_-^j(z,h)$ for the adjoint operator $(P_h - z)^*$, microlocalized at the points $\rho_-^j(z)$. From the quasimode equation (1.5) it is easy to exhibit an operator $Q$ of norm 1 and a parameter $\delta = \delta(h) = \mathcal{O}(h^\infty)$ such that the perturbed operator $P_h + \delta Q$ has an eigenvalue at $z$ (for instance, if we call the error $r_+^j = (P_h - z)e_+^j$, we may take the rank 1 operator $\delta Q = -r_+^j \otimes (e_+^j)^*$). The possibility to create an eigenvalue at $z$ upon a very small ($\mathcal{O}(h^\infty)$) perturbation indicates that $z$ is in a region of strong spectral instability for $P_h$ when $h \ll 1$. Since $z$ was chosen arbitrarily in the interior of $\Sigma$, this whole region is therefore a zone of spectral instability; for this reason, we call $\mathring{\Sigma}$ the ($h^\infty$-)*pseudospectrum of* $P_h$.

Let us now explain how we construct *random* perturbations, following [25]. Let $\{e_k\}_{k\in\mathbb{N}}$ denote an orthonormal basis of $L^2(\mathbb{R})$ consisting of the eigenfunctions of the *nonsemiclassical* harmonic oscillator $H = -\partial_x^2 + x^2$, and let $\{q_{jk}\}_{j,k\in\mathbb{N}}$, $\{v_j\}_{j\in\mathbb{N}}$ be independent and identically distributed (i.i.d.) complex Gaussian random variables with expectation 0 and variance 1 (that is, with distribution $\mathcal{N}_{\mathbb{C}}(0,1)$). Let $N(h) = C_1/h^2$, with $C_1 > 0$ large enough. Using these data, we define two types of random operators $Q$:

1. **A random, Ginibre-type matrix**

$$Q = M_\omega = \frac{1}{N(h)} \sum_{0\le j,k<N(h)} q_{j,k}\, e_j \otimes e_k^*$$

[1] We will refer to the values $p(x,\xi)$ as "energies", even though they are complex.

[2] The notation $= \mathcal{O}(h^\infty)$ means that for any $N$, there exists $C_N > 0$ such that for all $h \in ]0,1]$, the left hand side is bounded above by $C_N h^N$.

[3] This microlocalization means that the function $x \mapsto e_+^j(x; z, h)$ is concentrated near $x_+^j(z)$ when $h \to 0$, while its semiclassical Fourier transform $(\mathcal{F}_h e_+^j(z,h))(\xi)$ is concentrated near $\xi_+^j(z)$.

2. **A random (complex valued) potential**

$$Q = V_\omega = \frac{1}{N(h)} \sum_{0 \le j < N(h)} v_j \, e_j$$

(more precisely, $Q$ is the operator of multiplication by the potential $V_\omega$).

The coupling parameter $\delta = \delta(h)$ will be assumed to be in the range

$$h^M \le \delta \le h^\kappa, \tag{1.6}$$

where $\kappa > 3$, and $M > \kappa$ is an arbitrarily large but fixed constant. Although the random operator $Q$ and $\delta$ depend on $h$, we will omit this dependence in our notations. We are interested in the spectrum of the perturbed operator

$$P_h^\delta = P_h + \delta Q,$$

where the random operator $Q$ is either $M_\omega$ or $V_\omega$. Note that since the operator $Q$ is bounded on $L^2$, the spectrum of $P_h^\delta$ remains purely discrete. More quantitatively, with probability exponentially close to 1 as $h \to 0$, we have the bounds $\|M_\omega\|_{\mathrm{HS}} \le Ch^{-1}$ and $\|V_\omega\|_\infty \le Ch^{-1}$ [23, 25].

Our objective will be to study the spectrum of $P_h^\delta$ in a microscopic neighbourhood of some given point $z_0 \in \Omega$. As explained in the previous section, the probabilistic Weyl's law [23, 25] shows that the typical density of eigenvalues near $z_0$ is of order $h^{-1}$, so we expect nearby eigenvalues to be at distances $\sim h^{1/2}$ from one another. In order to test the statistical correlations between nearby eigenvalues, we zoom to the scale $h^{1/2}$ at the point $z_0$, by defining the *rescaled spectral point process*

$$\mathcal{Z}^Q_{h,z_0} := \sum_{z \in \mathrm{Spec}(P_h + \delta Q)} \delta_{(z - z_0) h^{-1/2}}.$$

Our main result is that, in the semiclassical limit, this rescaled point process converges in distribution to the point process formed by the zeros of a certain *random analytic function*. The building block of this random function is the (Euclidean) Gaussian analytic function (GAF), which we now review.

### *1.2. The Gaussian analytic function*

Let $(\alpha_n)_{n \in \mathbb{N}}$ be i.i.d. normal complex Gaussian random variables, i.e. $\alpha_n \sim \mathcal{N}_{\mathbb{C}}(0, 1)$. For a given $\sigma > 0$, we consider the random entire series

$$g_\sigma(w) := \sum_{n=0}^{\infty} \alpha_n \frac{\sigma^{n/2} w^n}{\sqrt{n!}}, \qquad w \in \mathbb{C}. \tag{1.7}$$

With probability 1, this series converges absolutely on the full plane, and defines a Gaussian analytic function (GAF) on $\mathbb{C}$: $g_\sigma$ is a random entire function such that for any $n \in \mathbb{N}$

and any $w_1, \dots, w_n \in \mathbb{C}$ the random vector $(g_\sigma(w_1), \dots, g_\sigma(w_n))$ is a centred complex Gaussian,

$$(g_\sigma(w_1), \dots, g_\sigma(w_n)) \sim \mathcal{N}_{\mathbb{C}}(0, \Gamma), \tag{1.8}$$

where the *covariance matrix* $\Gamma \in \mathrm{GL}_n(\mathbb{C})$ has entries

$$\Gamma_{i,j} = \mathbb{E}\big[g_\sigma(w_i)\overline{g_\sigma(w_j)}\big] =: K_\sigma(w_i, \overline{w}_j) = \exp(\sigma w_i \overline{w}_j). \tag{1.9}$$

The function $\mathbb{C}^2 \ni (u, v) \mapsto K_\sigma(u, \overline{v})$ is called the *covariance kernel* of the GAF $g_\sigma$; it completely determines its distribution. As a result, $K_\sigma$ also completely determines the distribution of

$$\mathcal{Z}_{g_\sigma} := \sum_{w \in g_\sigma^{-1}(0)} \delta_w,$$

the random point process defined by the zeros of the GAF $g_\sigma$ (see for instance [30]). In Section 6, we will review basic notions and results concerning zero point processes of random analytic functions, making the above statements more precise.

The GAF zero process $\mathcal{Z}_{g_\sigma}$ has interesting geometric properties. Its covariance kernel shows that for any $w_0 \in \mathbb{C}$, the translated function $g_\sigma(w + w_0)$ is equal in distribution to the function $e^{\sigma(w\bar{w}_0 + |w_0|^2)} g_\sigma(w)$, which has the same zeros as $g_\sigma(w)$; hence the zero process $\mathcal{Z}_{g_\sigma}$ is translation invariant on $\mathbb{C}$. The average density (1-point function) of $\mathcal{Z}_{g_\sigma}$ is thus constant over the plane, it is equal to $\sigma/\pi$ (see Section 2.5.3). The linear dependence in $\sigma$ is coherent with the scaling covariance $g_\sigma(w) \stackrel{d}{=} g_1(\sqrt{\sigma} w)$: dilating the zero process $\mathcal{Z}_{g_1}$ by $1/\sqrt{\sigma}$ multiplies the average density by $\sigma$.

Let us give a short historical background of the GAF. It has appeared in the context of holomorphic representations of quantum mechanics, when investigating the properties of *random states*. In the framework of Toeplitz quantization on a compact Kähler manifold $M$, one defines a positive holomorphic line bundle $L$ over $M$, and for any integer $N \geq 1$ a "quantum" Hilbert space $\mathcal{H}_N$ is formed by the holomorphic sections of the bundle $L^{\otimes N}$; the limit $N \to \infty$ is interpreted as a semiclassical limit. In the case of the 1-dimensional projective space $M = \mathbb{C}P^1$, which is the phase space of the spin, Hannay [26] defined a natural ensemble of random holomorphic sections in $\mathcal{H}_N$, and studied the point process formed by their zeros (topological constraints force any section to have exactly $N$ zeros). He explained how to compute the $k$-point correlation function of this process, and explicitly computed the limit (after microscopic rescaling) of the 2-point correlation function, which coincides with the 2-point function of the GAF. A few years later, Bleher–Schiffman–Zelditch [3] proved that, for a general Toeplitz quantization $(M, L)$, the zeros of random holomorphic sections converge, when $N \to \infty$, to a *universal* process depending only on the dimension of $M$. In dimension 1, this process is given by the zero process of the GAF.

We are now ready to state our theorem concerning the spectrum of $P_h^\delta$.

**Theorem 1.1** (Complex harmonic oscillator). *Fix $z_0 = X_0 + iY_0 \in \mathring{\Sigma}$, and define the classical density for the symbol $p(x, \xi) = \xi^2 + ix^2$ at the points $\rho_\pm^j \in p^{-1}(z_0)$:*

$$\sigma(z_0) := \frac{1}{|\{\mathrm{Re}\, p, \mathrm{Im}\, p\}(\rho_\pm^j(z_0))|} = \frac{1}{4\sqrt{X_0 Y_0}}, \qquad j = 1, 2.$$

*For $h \in ]0,1]$, let the random perturbation $Q$ be either $M_\omega$ or $V_\omega$, and take $\delta$ in the interval (1.6). Then, for any domain $O \Subset \mathbb{C}$, the rescaled spectral point process at $z_0$ converges in distribution:*

$$\mathcal{Z}^{Q}_{h,z_0} \xrightarrow{d} \mathcal{Z}_{G_{z_0,Q}} \qquad \text{on } O \qquad \text{as } h \to 0.$$

*Here $\mathcal{Z}_{G_{z_0,Q}}$ is the zero point process for the random entire function $G_{z_0,Q}$ described below:*

(1) *if $Q = V_\omega$ then*

$$G_{z_0,V}(w) = g^1_{z_0}(w) g^2_{z_0}(w), \qquad w \in \mathbb{C},$$

*where $g^1_{z_0}, g^2_{z_0}$ are two independent copies of the GAF $g_{\sigma(z_0)}$;*

(2) *if $Q = M_\omega$ then*

$$G_{z_0,M}(w) = \det\left(g^{i,j}_{z_0}(w)\right)_{1\le i,j\le 2}, \qquad w \in \mathbb{C},$$

*where $g^{i,j}_{z_0}$, $1 \le i, j \le 2$, are four independent copies of the GAF $g_{\sigma(z_0)}$.*

The convergence in distribution of point processes is described more explicitly in Theorem 2.5. As we will explain in Section 2.5.2, this convergence implies that all $k$-point measures converge as well to the limiting ones.

## 2. Main results—general framework

The above theorem can be generalized to a large class of 1-dimensional nonselfadjoint $h$-pseudodifferential operators, and with random perturbations which are not necessarily Gaussian. We first present the class of unperturbed operators we will be dealing with.

### 2.1. Semiclassical framework

We begin by recalling the definition of the *pseudospectrum* of an operator, an important notion which quantifies its spectral instability.

Let $P : L^2 \to L^2$ be a densely defined closed linear operator, with resolvent set $\rho(P)$ and spectrum $\mathrm{Spec}(P) = \mathbb{C} \setminus \rho(P)$. For any $\varepsilon > 0$, we define the *$\varepsilon$-pseudospectrum* of $P$ by

$$\mathrm{Spec}_\varepsilon(P) := \mathrm{Spec}(P) \cup \{z \in \rho(P);\ \|(P-z)^{-1}\| > \varepsilon^{-1}\}. \tag{2.1}$$

When $\varepsilon$ is small, the set (2.1) describes a region of spectral instability of the operator $P$, since any point in the $\varepsilon$-pseudospectrum of $P$ lies in the spectrum of a certain $\varepsilon$-perturbation of $P$ [14]. Indeed, $\mathrm{Spec}_\varepsilon(P)$ can also be defined by

$$\mathrm{Spec}_\varepsilon(P) = \bigcup_{\substack{Q\in\mathcal{B}(L^2)\\ \|Q\|<1}} \mathrm{Spec}(P + \varepsilon Q). \tag{2.2}$$

A third, equivalent definition of the $\varepsilon$-pseudospectrum of $P$ is via the existence of approximate solutions to the eigenvalue problem $P - z$:

$$z \in \mathrm{Spec}_\varepsilon(P) \iff \exists u_z \in D(P) : \|(P - z)u_z\| < \varepsilon \|u_z\|, \tag{2.3}$$

where $D(P)$ denotes the domain of $P$. Such a state $u_z$ is called an $\varepsilon$-quasimode, or simply a *quasimode* of $P - z$.

Next, let us fix the type of unperturbed operators we will consider in this paper. We will use the notation $\rho = (x, \xi) \in \mathbb{R}^2$ for phase space points. We start by considering an *order function* $m \in \mathcal{C}^\infty(\mathbb{R}^2, [1, \infty[)$, namely a function satisfying the following growth conditions:

$$\exists C_0 \geq 1,\ \exists N_0 > 0 : \quad m(\rho) \leq C_0 \langle \rho - \mu \rangle^{N_0} m(\mu), \quad \forall \rho, \mu \in \mathbb{R}^2, \tag{2.4}$$

with the usual "Japanese brackets" notation $\langle \rho - \mu \rangle := \sqrt{1 + |\rho - \mu|^2}$. To this order function is associated a semiclassical symbol class [12, 51]:

$$S(\mathbb{R}^2, m) = \{q \in \mathcal{C}^\infty(\mathbb{R}^2_\rho \times ]0, 1]_h);\ \forall \alpha \in \mathbb{N}^2,\ \exists C_\alpha : \\ |\partial_\rho^\alpha q(\rho; h)| \leq C_\alpha m(\rho),\ \forall \rho \in \mathbb{R}^2,\ \forall h \in ]0, 1]\}. \tag{2.5}$$

We assume that the symbol $p \in S(\mathbb{R}^2, m)$ is "classical", namely it satisfies an asymptotic expansion in the limit $h \to 0$:

$$p(\rho; h) \sim p_0(\rho) + h p_1(\rho) + \cdots \quad \text{in } S(\mathbb{R}^2, m), \tag{2.6}$$

where each $p_j \in S(\mathbb{R}^2, m)$ is independent of $h$. In this case we call $p_0$ the (semiclassical) *principal symbol* of $p$. We then define two subsets of $\mathbb{C}$ associated with $p_0$:

$$\Sigma := \overline{p_0(\mathbb{R}^2)}, \qquad \Sigma_\infty := \{z \in \Sigma;\ \exists (\rho_j)_{j \geq 1} : |\rho_j| \to \infty,\ p_0(\rho_j) \to z\}. \tag{2.7}$$

$\Sigma$ is the classical spectrum of the operator $P_h$ defined below, while $\Sigma_\infty$ can be called the classical spectrum at infinity. Furthermore, we suppose that the principal symbol $p_0$ is *elliptic* at some "energy" $z_{\mathrm{out}} \in \mathbb{C} \setminus \Sigma$:

$$\exists C_0 > 0 : \quad |p_0(\rho) - z_{\mathrm{out}}| \geq m(\rho)/C_0, \quad \forall \rho \in \mathbb{R}^2. \tag{2.8}$$

For $h \in ]0, 1]$ we let $P_h$ denote the $h$-Weyl quantization of the symbol $p$,

$$P_h u(x) = p^w(x, hD_x; h)u(x) = \frac{1}{2\pi h} \iint e^{\frac{i}{h}(x-y)\cdot\xi} p\left(\frac{x+y}{2}, \xi; h\right) u(y)\, dy\, d\xi, \tag{2.9}$$

which makes sense for $u$ in $\mathcal{S}(\mathbb{R}^d)$, the Schwartz space. The closure of $P_h$ as an unbounded operator on $L^2$ has the dense domain $H(m) := (P_h - z_{\mathrm{out}})^{-1}(L^2(\mathbb{R})) \subset L^2(\mathbb{R})$; we will still denote this closed operator by $P_h$. Moreover, we will denote by $\|u\|_m := \|(P_h - z_{\mathrm{out}})u\|$ the associated norm on $H(m)$.[4]

[4] Although this norm depends on the choice of the symbol $p - z_{\mathrm{out}}$, it is equivalent to the norm defined from any elliptic operator in $q \in S(m)$, so that the space $H(m)$ only depends on the order function $m$.

Let $\widetilde{\Omega} \subset \mathbb{C}$ be open simply connected, not entirely contained in $\Sigma$, and such that $\overline{\widetilde{\Omega}} \cap \Sigma_\infty = \emptyset$. Then the spectrum of $P_h$ inside $\widetilde{\Omega}$ has the following properties in the semiclassical limit [23, 25]:

- for $h > 0$ small enough, $\mathrm{Spec}(P_h) \cap \widetilde{\Omega}$ is discrete,
- for all $\varepsilon > 0$, there is $h(\varepsilon) > 0$ such that

$$\mathrm{Spec}(P_h) \cap \widetilde{\Omega} \subset \Sigma + D(0, \varepsilon), \qquad \forall 0 < h < h(\varepsilon). \tag{2.10}$$

Here, $D(0, \varepsilon) \subset \mathbb{C}$ denotes the open disc of radius $\varepsilon > 0$ centred at 0.

In this work we will study the spectrum of small random perturbations of $P_h$, in the semiclassical limit $h \to 0$, in the interior of $\Sigma \cap \widetilde{\Omega}$.

### 2.2. *Pseudospectrum and the energy shell*

Let $\widetilde{\Omega}$ be as above and let

$$\Omega \Subset \widetilde{\Omega} \cap \mathring{\Sigma} \text{ be open, simply connected.} \tag{2.11}$$

Recall that $p_0$ is the principal symbol of $p$ (see (2.6)). We assume that

$$\text{for every } \rho \in p_0^{-1}(\overline{\Omega}), \text{ the 1-forms } dp_0, d\overline{p_0} \text{ are linearly independent.} \tag{2.12}$$

Since the dimension $d = 1$, this condition is equivalent to

$$\text{for every } \rho \in p_0^{-1}(\overline{\Omega}), \ \{\mathrm{Re}\, p_0, \mathrm{Im}\, p_0\} \neq 0, \tag{2.13}$$

where $\{\cdot, \cdot\}$ denotes the Poisson bracket:

$$\{p, q\}(\rho) := \partial_\xi p(\rho)\, \partial_x q(\rho) - \partial_\xi q(\rho)\, \partial_x p(\rho), \qquad \rho = (x, \xi) \in \mathbb{R}^2.$$

It was observed by Dencker, Sjöstrand and Zworski [11] and Sjöstrand [43] that since $\Omega$ is relatively compact and simply connected, (2.12), or equivalently (2.13), implies that there exists $J \in \mathbb{N}^*$ depending only on $\Omega$ such that for any $z \in \Omega$, the “energy shell” $p_0^{-1}(z)$ consists of exactly $2J$ points:

$$\begin{aligned} p_0^{-1}(z) = \{\rho_\pm^j(z);\ j = 1, \ldots, J\} \quad \text{with} \quad & \pm\{\mathrm{Re}\, p, \mathrm{Im}\, p\}(\rho_\pm^j(z)) < 0, \\ & \rho_\pm^i(z) \neq \rho_\pm^j(z) \text{ if } i \neq j, \end{aligned} \tag{HYP}$$

and the points $\rho_\pm^j(z) = (x_\pm^j(z), \xi_\pm^j(z))$ depend smoothly on $z$.

We shall make the further (generic) assumption

$$\forall z \in \Omega, \qquad x_\pm^i(z) \neq x_\pm^j(z) \ \text{ if } \ i \neq j, \tag{HYP-x}$$

which will play a role when studying the perturbation by a random potential.

Davies [9] and Dencker, Sjöstrand and Zworski [11] showed that (HYP) implies, for each $z \in \Omega$ and each $j = 1, \ldots, J$, the existence of an $h^\infty$-quasimode for $P_h - z$

(resp. $(P_h - z)^*$), microlocalized at $\rho_+^j(z)$ (resp. $\rho_-^j(z)$). We will denote those modes by $e_\pm^j = e_\pm^j(z;h) \in L^2(\mathbb{R})$ and normalize them as $\|e_\pm^j\| = 1$; they satisfy

$$\|(P_h - z)e_+^j\| = \mathcal{O}(h^\infty) \quad \text{and} \quad \mathrm{WF}_h(e_+^j) = \{\rho_+^j(z)\}, \tag{2.14}$$

respectively

$$\|(P_h - z)^* e_-^j\| = \mathcal{O}(h^\infty) \quad \text{and} \quad \mathrm{WF}_h(e_-^j) = \{\rho_-^j(z)\}. \tag{2.15}$$

Recall that for $u = (u(h))_{h\in]0,1]}$ a bounded family in $L^2$, its semiclassical wavefront set $\mathrm{WF}_h(u)$ denotes the phase space region where $u$ is $h$-microlocalized:

$$\mathrm{WF}_h(u) := \complement\{(x,\xi) \in \mathbb{R}^2;\ \exists a \in C_c^\infty(\mathbb{R}^2) : a(x,\xi) = 1,\ \|a^w(x, hD_x)u(h)\|_{L^2} = \mathcal{O}(h^\infty)\}$$

where $a^w$ denotes the Weyl quantization of $a$.

In view of the characterization (2.3) of the pseudospectrum, we see that the assumption (2.12) implies that $\Omega$ is contained in the $h^\infty$-pseudospectrum of $P_h$, a spectrally highly unstable region.

### 2.3. *Adding a random perturbation*

We will now consider random perturbations of the operator $P_h$ which are given by either a random matrix or a random potential, generalizing a little the constructions made in Section 1.1. As in that section, we let $(e_k)_{k\in\mathbb{N}}$ be the orthonormal eigenbasis of the (nonsemiclassical) harmonic oscillator $H = -\partial_x^2 + x^2$.

**Remark 2.1.** This choice of orthonormal basis is convenient for us, but it is far from unique. It will become clear later that what we need is a family of states (not necessarily orthonormal) such that the first $N(h)$ states microlocally cover a sufficiently large part of phase space, namely a neighbourhood of $p_0^{-1}(\Omega)$. We also need to avoid states which would have a large overlap with some of the quasimodes $e_\pm^j$ (cf. (2.14), (2.15)). We refer the reader in particular to the proofs of Propositions 7.3 and 8.4 below.

Let $\alpha$ be a complex valued random variable defined on some probability space $(\mathcal{M}, \mathcal{F}, \mathbb{P})$, with the properties

$$\mathbb{E}[\alpha] = 0, \qquad \mathbb{E}[\alpha^2] = 0, \qquad \mathbb{E}[|\alpha|^2] = 1, \qquad \mathbb{E}[|\alpha|^{4+\epsilon_0}] < \infty, \tag{2.16}$$

where $\varepsilon_0 > 0$ is an arbitrarily small but fixed constant. Here, $\mathbb{E}[\cdot]$ denotes the expectation with respect to the probability measure $\mathbb{P}$. The Markov inequality implies the following tail estimate: there exists a constant $\kappa_\alpha > 0$ such that

$$\mathbb{P}[|\alpha| \geq \gamma] \leq \kappa_\alpha\, \gamma^{-(4+\varepsilon_0)}, \qquad \forall \gamma > 0. \tag{2.17}$$

**Remark 2.2.** For instance, the complex centred Gaussian random variable of (1.7) satisfies the above assumptions.

**Random matrix.** Let $N(h) = C_1/h^2$ with $C_1 > 0$ large enough (we will be more precise about this condition later). Let $q_{j,k}$, $0 \le j, k < N(h)$, be independent copies of the random variable $\alpha$ satisfying the conditions (2.16). We consider the random matrix

$$M_\omega = \frac{1}{N(h)} \sum_{0 \le j,k < N(h)} q_{j,k}\, e_j \otimes e_k^*, \tag{RM}$$

where $e_j \otimes e_k^* u = (u|e_k)e_j$ for $u \in L^2(\mathbb{R})$. For some coupling parameter $0 < \delta \ll 1$, we define the randomly perturbed operator

$$P_M^\delta = P_h + \delta M_\omega. \tag{2.18}$$

**Random potential.** Take $N(h) = C_1/h^2$ with $C_1 > 0$ as above. Let $v_j$, $0 \le j < N(h)$, be independent copies of the random variable $\alpha$. Still using the same orthonormal family $(e_k)_{k \in \mathbb{N}}$, we define the random function

$$V_\omega = \frac{1}{N(h)} \sum_{0 \le j < N(h)} v_j\, e_j. \tag{RP}$$

For $0 < \delta \ll 1$, consider the perturbed operator

$$P_V^\delta = P_h + \delta V_\omega. \tag{2.19}$$

We call this perturbation a "random potential", even though $V_\omega$ is complex valued. When we consider this type of perturbation, we will make an additional symmetry assumption:

$$p(x, \xi; h) = p(x, -\xi; h). \tag{SYM}$$

This hypothesis implies that we can group the points forming $p_0^{-1}(z)$ (see (HYP)) in pairs such that $\rho_\pm^j = (x^j, \pm\xi^j)$. As a result, the centres of microlocalization of the quasimodes $e_+^j$ and $e_-^j$ are located on the same fibre $T_{x^j}^*\mathbb{R} = \{(x_j, \xi);\ \xi \in \mathbb{R}\}$.

**Remark 2.3.** We could relax the assumption (SYM) into requiring this symmetry only at the level of the principal symbol, i.e. $p_0(x, \xi) = p_0(x, -\xi)$. However, for simplicity of presentation we prefer to make the above stronger hypothesis.

**Restricting to bounded perturbations.** For both types of perturbations, it will be easier for us to restrict the random variables to large discs $D(0, C/h)$, i.e. assume that

$$|v_i|, |q_{i,j}| \le C/h, \qquad 0 \le i, j < N(h), \qquad \text{for some } C > 0 \text{ sufficiently large.} \tag{2.20}$$

This restriction induces the boundedness of the perturbations $M_\omega$, $V_\omega$. Indeed, on this restricted probability space we have the bound

$$\|M_\omega\|_{\mathrm{HS}} \le C h^{-1}, \tag{2.21}$$

where $\|M_\omega\|_{\mathrm{HS}}$ denotes the Hilbert–Schmidt norm of $M_\omega$. In the case of the random potential,

$$\|V_\omega\|_\infty \le C h^{-1}. \tag{2.22}$$

We note that even for unrestricted random variables, these bounds on the perturbations hold with high probability. Indeed, using (2.17) to estimate the probability that (2.20) holds, we deduce that (2.21) holds with probability $\geq 1 - C_2 h^{\varepsilon_0}$, and that (2.22) occurs with probability $\geq 1 - C_2 h^{2+\varepsilon_0}$ for some $C_2 > 0$.

Finally, we will take the coupling parameter $\delta = \delta(h)$ in the same interval as in (1.6).

We will see in Section 5 that the spectra of $P_M^\delta$ and $P_V^\delta$ in $\Omega$ are purely discrete. The principal aim of this paper is to show that the statistical properties of these spectra, in a microscopic neighbourhood of any $z_0 \in \Omega$, are universal, in a sense that we will specify later on.

Since $p_0 - z$ is elliptic for every $z \in \mathbb{C} \setminus \Sigma$, the resolvent norm satisfies $\|(P_h - z)^{-1}\| = \mathcal{O}(1)$, uniformly for $z$ in compact subsets of $\mathbb{C} \setminus \Sigma$, as $h \to 0$. In view of (2.21), (2.22) and (1.6), we are considering random perturbations of size $\|\delta M_\omega\|_{\mathrm{HS}} \ll h^2$ and $\|\delta V_\omega\|_\infty \ll h^2$. Therefore, in view of the characterization (2.2) of the pseudospectrum, the spectra of $P_M^\delta$ and $P_V^\delta$ are contained in $\Sigma + D(0, \epsilon)$ for any given $\epsilon > 0$ and $h > 0$ small enough. Moreover, since $\Omega \Subset \mathring{\Sigma}$, we will not feel the effects of the boundary of $\Sigma$; we will simply say that $\Omega$ lies in the *bulk* of the spectrum of the perturbed operator.

### 2.4. *Probabilistic Weyl's law and local statistics*

In a series of works by Hager [24, 23, 25] and Sjöstrand [42, 41], the authors considered randomly perturbed operators $P^\delta$ as given in (2.18) and (2.19). Under more restrictive assumptions on the random variables than (2.16), they have shown the following result.

**Theorem 2.4** (Probabilistic Weyl's law). *Let $\Omega$ be as in (2.11), (2.12). Let $\Gamma \Subset \Omega$ be open with $\mathcal{C}^2$ boundary. Let $P_h^\delta$ be either of the randomly perturbed operators $P_M^\delta$ or $P_V^\delta$ with $\delta$ as in (1.6) with $\kappa > 0$ sufficiently large. Then, in the limit $h \to 0$,*

$$\#(\mathrm{Spec}(P_h^\delta) \cap \Gamma) = \frac{1}{2\pi h}\left(\iint_{p_0^{-1}(\Gamma)} dx\, d\xi + o(1)\right) \quad \textit{with probability} \geq 1 - Ch^\eta, \tag{2.23}$$

*for some fixed $\eta > 0$.*

The authors also give an explicit control over both the error term in Weyl's law, and the error term in the probability estimate.

This probabilistic Weyl's law shows that, with probability close 1, the number of eigenvalues of the perturbed operator $P_h^\delta$ in any *fixed* subset of $\Omega$ is of order $\asymp h^{-1}$. Hence, the spectrum of $P_h^\delta$ will spread across $\Omega$, with an average spacing between nearby eigenvalues of order $h^{1/2}$.

Figure 1 illustrates this behaviour of $P_h = -h^2\partial_x^2 + \mathrm{e}^{3ix}$ acting on the torus $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. We draw random perturbations $M_\omega$, $V_\omega$ and plot some region of the spectra of $P_M^\delta$ and $P_V^\delta$, in the interior of $\Sigma$. Both spectra are grossly uniform over the plotted region, yet in the case of $P_V^\delta$ (right plot) the distribution of the eigenvalues seems a bit "less uniform" than in the case of $P_M^\delta$ (left plot), in particular it allows the presence of small clusters of very near eigenvalues.

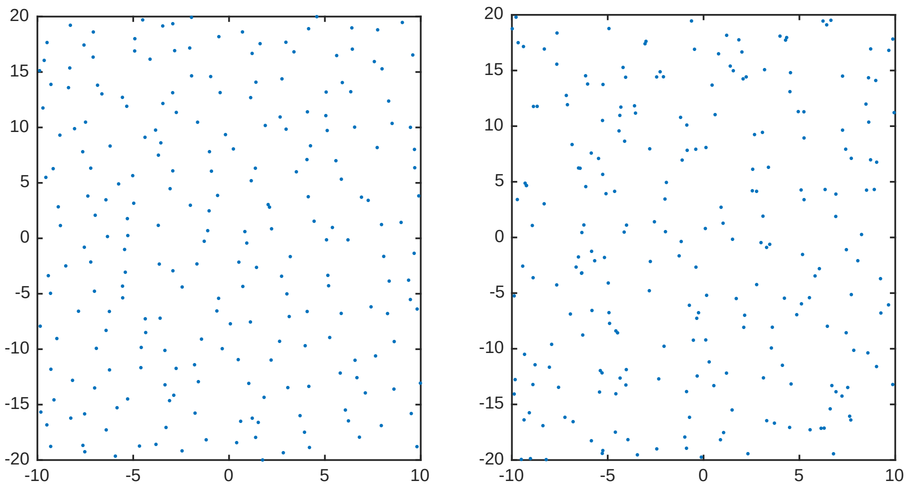

**Fig. 1.** Numerically computed spectra of the operators $-h^2\partial_x^2 + e^{3ix} + \delta Q$ acting on $L^2(\mathbb{T})$, with $h = 10^{-3}$ and $\delta = 10^{-12}$. The perturbation $Q$ is either a Gaussian random matrix $M_\omega$ (left), or a Gaussian random potential $V_\omega$ (right). The region plotted is the same, it is part of the bulk (the units on the axes are arbitrary). In the region the number of quasimodes is $2J$, $J = 6$.

To quantify this difference of uniformity between the spectra of $P_M^\delta$ and $P_V^\delta$, we study the local statistics of the eigenvalues, that is, the statistics of the eigenvalues on the scale of their mean level spacing. For this purpose, we fix a point $z_0 \in \Omega$. In both cases $Q = M_\omega$ and $Q = V_\omega$, we view the rescaled spectrum of the randomly perturbed operator $P_Q^\delta$ as a random point process

$$\mathcal{Z}_{h,z_0}^Q := \sum_{z \in \mathrm{Spec}(P_Q^\delta)} \delta_{(z-z_0)h^{-1/2}}, \qquad Q = M_\omega \quad \text{or} \quad Q = V_\omega, \tag{2.24}$$

where the eigenvalues are counted according to their algebraic multiplicities.

Notice that the rescaled eigenvalues $(z_j - z_0)h^{-1/2}$ have a mean spacing of order $\asymp 1$. The principal aim of this paper is to show that, under the assumption (2.16) on the random coefficients, in the limit $h \to 0$ the correlation functions of the processes $\mathcal{Z}_{h,z_0}^M$ and $\mathcal{Z}_{h,z_0}^V$ are *universal*, in the sense that they

- *depend* only on the structure of the energy shell $p_0^{-1}(z)$ and on the type of random perturbation used, either $M_\omega$ or $V_\omega$;
- *are independent* of the law of the random variable $\alpha$ used to define the random perturbations, as long as $\alpha$ satisfies (2.16).

Finally, let us stress that our results concern solely the eigenvalues in the bulk of the spectrum of $P_h^\delta$, that is, in the interior of the $h^\infty$-pseudospectrum of $P_h$. Near the boundary of that pseudospectrum, we expect the statistical properties of the eigenvalues to change drastically. It has been shown by the second author [49] in the case of a model operator that the probabilistic Weyl's law breaks down in the vicinity of $\partial\Sigma$, in fact, the density of eigenvalues explodes near that boundary.

### 2.5. *Perturbation by a random potential*

We begin with the case of a perturbation (2.19) by a random potential $V_\omega$. In Weyl's law of Theorem 2.4, the main term on the right hand side can be easily expressed in terms of the *classical spectral density*, pull-back of the symplectic measure on $T^*\mathbb{R}$ through the symbol $p_0$:

$$\int_{p_0^{-1}(\Gamma)} dx\,d\xi = \int_\Gamma (p_0)_*(dx\,d\xi)$$

(the Lebesgue measure $dx\,d\xi$ on $\mathbb{R}^2$ is also the measure induced by the symplectic form on $T^*\mathbb{R} \cong \mathbb{R}^2$).

From the structure (HYP) of the energy shell $p_0^{-1}(z)$, the classical spectral density at the energy $z$ can be expressed as follows:

$$(p_0)_*(dx\,d\xi) = \sum_{j=1}^{J} (\sigma_+^j(z) + \sigma_-^j(z)) L(dz), \qquad \sigma_\pm^j(z) = \frac{1}{\mp\{\mathrm{Re}\, p_0, \mathrm{Im}\, p_0\}(\rho_\pm^j(z))}. \tag{2.25}$$

Here $L$ denotes the Lebesgue measure on $\mathbb{C}$. In other words, each point $\rho_\pm^j$ of the energy shell provides a density component $\sigma_\pm^j(z) > 0$, which depends smoothly on $z \in \Omega$.

If we additionally assume the symmetry (SYM) and group the points so that $\rho_\pm^j = (x^j, \pm\xi^j)$, we find that $\sigma_+^j(z) = \sigma_-^j(z)$ for all $j = 1, \dots, J$.

2.5.1. *Universal limiting point process.* Let us now state our main theorem for the perturbed operators $P_V^\delta$. It provides the asymptotic behaviour of the rescaled spectral point processes $\mathcal{Z}_{h,z_0}^V$ in the semiclassical limit.

**Theorem 2.5.** *Let $p$ be as in (2.6) satisfying (2.12) and (SYM). Let $\Omega \Subset \mathring{\Sigma}$ be as in (2.11), and choose $z_0 \in \Omega$. Then, for any bounded open set $O \Subset \mathbb{C}$, the rescaled spectral point processes at $z_0$ converge in distribution:*

$$\mathcal{Z}_{h,z_0}^V \xrightarrow{d} \mathcal{Z}_{G_{z_0}} \quad \text{in } O \quad \text{as } h \to 0.$$

*This convergence means that for any test function $\phi \in \mathcal{C}_c(O, \mathbb{R})$,*

$$\langle \mathcal{Z}_{h,z_0}^V, \phi\rangle = \sum_{z \in \mathrm{Spec}(P_h^\delta)} \phi((z - z_0)h^{-1/2}) \xrightarrow{d} \langle \mathcal{Z}_{G_{z_0}}, \phi\rangle = \sum_{z \in G_{z_0}^{-1}(0)} \phi(z) \quad \text{as } h \to 0.$$

*Here $\mathcal{Z}_{G_{z_0}}$ is the zero point process for the random analytic function*

$$G_{z_0}(z) = \prod_{j=1}^{J} g_{z_0}^j(z), \qquad z \in \mathbb{C},$$

*where the $g_{z_0}^j$ are $J$ independent GAFs $g_{z_0}^j \sim g_{\sigma_+^j(z_0)}$ (see Section 1.2), with $\sigma_+^i(z_0)$ the local spectral densities given in (2.25).*

For the reader's convenience, in Section 6 we present a short review of the probabilistic notions used in this paper, such as convergence in distribution. The definition and basic properties of the GAFs have been presented in Section 1.2.

This theorem tells us that at any given point $z_0 \in \Omega$ in the bulk of the pseudospectrum, the rescaled spectral point process converges, as $h \to 0$, to the point process given by the zeros of the product of $J$ independent GAFs. This limiting point process is the superposition of $J$ independent processes, each generated by a GAF $g^j_{z_0}$. The latter only depends on the part of the classical spectral density coming from the pair of points $\rho^j_\pm = (x^j, \pm\xi^j)$. In particular, this limiting process is independent of the precise probability distribution of the coefficients $(v_j)$, as long as it satisfies (2.16), or of the orthonormal family $(e_j)$ used to generate the random potential $V_\omega$; this process only depends on the cardinality $2J$ of the energy shell $p_0^{-1}(z)$ and of the local spectral densities $\{\sigma^j_+(z_0);\ j = 1, \dots, J\}$.

It is known that the zero process of a single GAF exhibits a local repulsion between the nearby points (see Section 2.5.3). On the other hand, as a superposition of $J$ *independent* point processes, the limiting process $\mathcal{Z}_{G_{z_0}}$ authorizes the presence of *clusters* of at most $J$ points very close to one another, confirming our observations on the right plot of Fig. 1 (for the operator and plotted region considered, we have $J = 6$). In the next section we will analyze this clustering by computing the correlation functions between the points of the process.

*2.5.2. Scaling limit of the k-point measures.* An explicit way to quantify the statistical correlations between $k$ nearby eigenvalues of $P_V^\delta$ consists in defining the *k-point measures* of the point process $\mathcal{Z}^V_{h,z_0}$. These are positive measures $\mu_h^{k,V,z_0}$ on $O^k$, where $O$ is the open domain as in Theorem 2.5. These measures are defined through their action on an arbitrary test function $\phi \in \mathcal{C}_c(O^k, \mathbb{R}_+)$ as follows:

$$\mathbb{E}[(\mathcal{Z}^V_{h,z_0})^{\otimes k}(\phi)] = \mathbb{E}\Big[\sum_{z_1,\dots,z_k \in \mathrm{Spec}(P^\delta)} \phi((z_1 - z_0)h^{-1/2}, \dots, (z_k - z_0)h^{-1/2})\Big]$$
$$=: \int_{O^k} \phi(w)\, \mu^{k,V}_{h,z_0}(dw). \tag{2.26}$$

In practice, one often studies these measures away from the generalized diagonal $\Delta = \{z \in \mathbb{C}^k; \exists\, i \neq j : z_i = z_j\}$, in order to avoid trivial self-correlations. Hence the test functions we will use below will be chosen in $\mathcal{C}_c(O^k \setminus \Delta, \mathbb{R}_+)$.

When these $k$-point measures are absolutely continuous with respect to the Lebesgue measure on $\mathbb{C}^k$, we call their densities the *k-point functions*.

**Theorem 2.6.** *Let $\mu^{k,V}_{h,z_0}$ be the k-point measure of $\mathcal{Z}^V_{h,z_0}$, defined in (2.26), and let $\mu^{k,V}_{z_0}$ be the k-point measure of the point process $\mathcal{Z}_{G_{z_0}}$, given in Theorem 2.5. Then, for any domain $O \Subset \mathbb{C}$ and for all $\phi \in \mathcal{C}_c(O^k \setminus \Delta, \mathbb{R}_+)$,*

$$\int_{O^k\setminus\Delta} \phi(w)\, \mu^{k,V}_{h,z_0}(dw) \to \int_{O^k\setminus\Delta} \phi(w)\, \mu^{k,V}_{z_0}(dw), \qquad h \to 0.$$

*Moreover, $\mu_{z_0}^{k,V}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{C}^k$. Its density $d_{z_0}^{k,V}$ is given by*

$$d_{z_0}^{k,V}(w_1,\ldots,w_k) = \sum_{\substack{\alpha\in\mathbb{N}^J \\ \sum_j \alpha_j = k}} \sum_{\tau\in\mathfrak{S}_k} \frac{1}{\alpha!} \prod_{j=1}^{J} d_{g_j}^{\alpha_j}(w_{\tau(\alpha_1+\cdots+\alpha_{j-1}+1)},\ldots,w_{\tau(\alpha_1+\cdots+\alpha_j)}), \quad (2.27)$$

*where $\mathfrak{S}_k$ is the symmetric group on $k$ elements, and for all $1 \le j \le J$ and all $r \in \mathbb{N}^*$,*

$$d_{g^j}^{r}(w) = \frac{\mathrm{perm}[C_j^r(w) - B_j^r(w)(A_j^r)^{-1}(w)(B_j^r)^*(w)]}{\det \pi A_j^r(w)}, \quad \textit{while} \quad d_{g^j}^{0}(w) \equiv 1. \quad (2.28)$$

*Here,* perm *denotes the permanent of a matrix; $A_j^r, B_j^r, C_j^r$ are complex $r\times r$-matrices given by*

$$(A_j^r)_{n,m} = K^j(w_n,\bar{w}_m),\ \ (B_j^r)_{n,m} = (\partial_w K^j)(w_n,\bar{w}_m),\ \ (C_j^r)_{n,m} = (\partial^2_{w\bar{w}} K^j)(w_n,\bar{w}_m),$$

*where $K^j(w,\bar{w}) = \exp(\sigma_+^j(z_0)w\bar{w})$ is the covariance function of the GAFs $g_{z_0}^j$ appearing in Theorem 2.5.*

The function $d_{g^j}^r(z)$ in (2.28) is the $r$-point function for the zero process of the Gaussian analytic function $g^j$. The limiting $k$-point functions are thus obtained by concatenating the $r$-point functions ($1 \le r \le k$) of the $J$ GAFs $g^j$ associated with the points $\rho_\pm^j$ of the energy shell. The zeros associated with different points $\rho_\pm^j$ are uncorrelated with one another.

A result by Nazarov and Sodin [36, Theorem 1.1] implies the following estimate for the $r$-point densities of a single GAF.

**Proposition 2.7** ([36]). *Let $O \Subset \mathbb{C}$ be a bounded domain. Let $(g^j = g_{z_0}^j)_{1\le j\le J}$ be the GAFs appearing in Theorem 2.5, and let $d_{g^j}^r(w)$, $1 \le r \le k$, be the corresponding $r$-point functions as in (2.28). Then there exists a constant $C = C(r, g^j, O) > 1$ such that, for any configuration of pairwise distinct points $w_1,\ldots,w_k \in O$,*

$$C^{-1}\prod_{i<j}|w_i - w_j|^2 \le d_{g^j}^r(w_1,\ldots,w_k) \le C\prod_{i<j}|w_i - w_j|^2.$$

This estimate shows that the zeros of a GAF enjoy a *statistical* (*quadratic*) *repulsion* at short distance, namely they are very unlikely to approach one another much more than the mean distance.

In formula (2.27) we see that if $k > J$, each summand has at least one factor $d_{g^j}^{\alpha_j}$ with $\alpha_j \ge 2$. Hence, Theorem 2.6 and Proposition 2.7 lead to the following

**Corollary 2.8.** *Let $O \Subset \mathbb{C}$ be a bounded domain, let $k > J$, and let $d_{z_0}^{k,V}(w)$ be as in (2.27). Then there exists a positive constant $C = C(r, O)$ such that, for any configuration of pairwise distinct points $w_1,\ldots,w_k \in O$,*

$$d_{z_0}^{k,V}(w_1,\ldots,w_k) \le C\sum_{i<j}|w_i - w_j|^2.$$

We have seen in Theorem 2.6 that the limiting point process of the rescaled eigenvalues is given by the superposition of $J$ independent processes given by the zeros of independent Gaussian analytic functions. Due to this independence, $k$ points, each originating from a different GAF process, may approach each other without any statistical repulsion: this allows the formation of clusters of at most $J$ points. As a result, for $k \leq J$ the limiting $k$-point functions do not decay to zero as the distances between the $k$ points get smaller: this allows the presence of clusters of at most $J$ points. This behaviour is made more explicit in the next section in the case $k = 2$.

On the other hand, if $k > J$ then at least two points must originate from the same GAF process, and therefore statistically repel each other when approaching each other. This is exactly what Corollary 2.8 tells us: the probability to find more than $J$ points close together decays at least quadratically with the distance. As a result, finding small clusters containing more than $J$ eigenvalues is very unlikely.

*2.5.3. 2-point correlation function.* The 2-point *correlation function* of a point process is defined by the 2-point function, renormalized by the local 1-point functions (or local average densities):

$$K_{z_0}^{2,Q}(w_1, w_2) = \frac{d_{z_0}^{2,Q}(w_1, w_2)}{d_{z_0}^{1,Q}(w_1) d_{z_0}^{1,Q}(w_2)}, \qquad w_1 \neq w_2 \in O, \qquad Q = V_\omega,\ M_\omega.$$

By Theorem 2.6, the limiting local 1-point function $d_{z_0}^{1,V}(w)$ is a constant function, given by

$$d_{z_0}^{1,V}(w) = \sum_{j=1}^{J} \frac{\sigma_+^j(z_0)}{\pi}, \qquad \forall w \in O.$$

This average density of eigenvalues (at the *microscopic* scale near $z_0$) exactly corresponds to the *macroscopic* density predicted by the probabilistic Weyl's law in Theorem 2.4 (see also (2.25)).

The limiting 1-point and 2-point functions of the zero process generated by a single GAF $g_\sigma$ (see Section 1.2) are given by

$$d_{g_\sigma}^1(w_1) = \frac{\sigma}{\pi}, \qquad \text{respectively} \qquad d_{g_\sigma}^2(w_1, w_2) = \left(\frac{\sigma}{\pi}\right)^2 \kappa\left(\frac{\sigma |w_1 - w_2|^2}{2}\right),$$

with the scaling function

$$\kappa(t) := \frac{(\sinh^2 t + t^2)\cosh t - 2t \sinh t}{\sinh^3 t}, \qquad \forall t \geq 0. \tag{2.29}$$

The function $\kappa(\sigma |w_1 - w_2|^2/2)$ describes the 2-point correlation function of the zeros of the GAF $g_\sigma$. A remarkable property of this function is its isotropy: it only depends on the distance between the points $w_1, w_2$. In Figure 2 we plot the function $t \mapsto \kappa(t^2)$; it behaves like $\kappa(t^2) = t^2(1 + \mathcal{O}(t^4))$ when $t \to 0$, which reflects the quadratic repulsion between the nearby zeros of $g_\sigma$. On the other hand, when $t \gg 1$ it converges exponentially fast to unity, showing a fast decorrelation between the zeros at large distances.

To our knowledge, the function $\kappa$ was first computed by Hannay [26], as the scaling limit 2-point correlation function for the zeros of certain ensembles of random polynomials. In the work by Bleher, Shiffman and Zelditch [3], $\kappa$ describes the scaling limit 2-point correlation function for the zeros of random holomorphic sections of large powers of a positive Hermitian line bundle over a compact complex Kähler surface.

In the present work, $\kappa$ appears as a building block for the limit 2-point correlation function of the eigenvalues of $P_V^\delta$:

$$K_{z_0}^{2,V}(w_1, w_2) = 1 + \sum_{j=1}^{J} \frac{(\sigma_+^j(z_0))^2}{(\sum_{j=1}^{J} \sigma_+^j(z_0))^2} \left[ \kappa\left( \frac{\sigma_+^j(z_0)|w_1 - w_2|^2}{2} \right) - 1 \right]. \tag{2.30}$$

Let us study this 2-point correlation function more closely:

**Long range decorrelation.** For $|w_1 - w_2| \gg 1$, as $h \to 0$, the 2-point correlation function converges exponentially fast to unity:

$$K_{z_0}^{2,V}(w_1, w_2) = 1 + \mathcal{O}\big(\mathrm{e}^{-\min_j \sigma_+^j(z_0)|w_1 - w_2|^2}\big).$$

This shows that two points at distances $|w_1 - w_2| \gg 1$ are statistically uncorrelated.

**A weak form of repulsion.** When $|w_1 - w_2| \ll 1$, in the limit $h \to 0$, there is a weak form of repulsion between two nearby eigenvalues,

$$K_{z_0}^{2,V}(w_1, w_2) = 1 - \sum_{j=1}^{J} \frac{\sigma_+^j(z_0)^2}{(\sum_{l=1}^{J} \sigma_+^l(z_0))^2} \left[ 1 - \frac{\sigma_+^j(z_0)|w_1 - w_2|^2}{2} + \mathcal{O}(|w_1 - w_2|^4) \right]. \tag{2.31}$$

This formula shows that the probability of finding two rescaled eigenvalues $w_1, w_2$ at distance $\ll 1$ is smaller than the one of finding them at large distances: pairs of rescaled

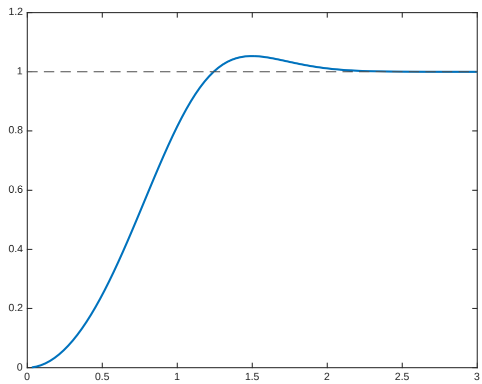


**Fig. 2.** Plot of the function $t \mapsto \kappa(t^2)$ (see (2.29)).

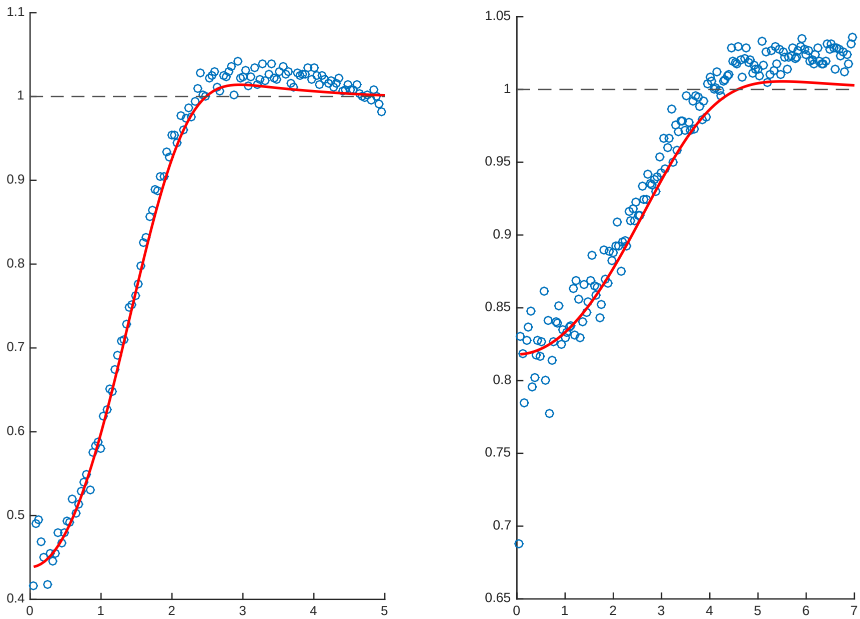


**Fig. 3.** Blue points: rescaled 2-point correlation functions near the energy $z_0 = 1.6$, obtained by numerically computing the spectra of the operators $P_{h,1}$ (left) and $P_{h,3}$ (right) perturbed by a Gaussian random potential $\delta V_\omega$. Red curves: scaling limit 2-point correlation functions $K^{2,V}_{z_0}$ for both operators, as given in (2.30); the horizontal coordinate is the rescaled square distance $|w_1 - w_2|^2$.

eigenvalues show a weak repulsion at short distance. However, the correlation function does not converge to zero when $|w_1 - w_2| \to 0$, but to the positive value $1 - \sum_{j=1}^{J} \sigma_+^j(z_0)^2 / (\sum_{l=1}^{J} \sigma_+^l(z_0))^2$. This weak repulsion can be explained by the fact that the random function $G_{z_0}$ is the product of $J$ independent GAFs: two zeros $w_1, w_2$ will not repel each other if they originate from different GAFs, while they will repel quadratically if they come from the same GAF. The net result is this weak form of repulsion. The larger the number of quasimodes $J$, the weaker this repulsion becomes, since two zeros $w_1, w_2$ chosen at random will have a smaller chance to come from the same GAF.

In Figure 3 we compare the limiting 2-point correlation functions $K^{2,V}_{z_0}$ with the one obtained from numerical spectra of two operators on the torus $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$:

$$P^{\delta}_{h,q} = -h^2 \partial_x^2 + e^{iqx} + \delta V_\omega, \qquad q = 1, 3, \qquad x \in \mathbb{T}. \tag{2.32}$$

We took the parameters $h = 10^{-3}$, $\delta = 10^{-12}$, and the Gaussian random potential $V_\omega$ as in Section 1.1. We use operators defined on $\mathbb{T}$ because they are numerically easier to diagonalize than operators on $\mathbb{R}$. For each operator $P_{h,q}$, we drew 1000 samples of the random potential $V_\omega$, and computed the corresponding spectra of $P^{\delta}_{V}$, then extracted from these spectra the correlation function.

The analysis of the principal symbols $p_{q,0}$ shows that the classical spectrum is, in both cases, given by $\Sigma = \mathbb{R}_+ + U(1)$. At the energy $z_0 = 1.6$ (clearly located in the "bulk"), the operator $P_{h,q} - z_0$ admits $J = 2q$ quasimodes. Figure 3 compares the numerically obtained 2-point correlation functions (shown as blue dots) of the operators $P^\delta_{h,1}$ (left) and $P^\delta_{h,3}$ (right), with the theoretical scaling limit described in (2.31). For the two operators, the theoretical curve fits the numerical points quite well, including at short distances $|w_1 - w_2| \ll 1$.

### 2.6. *Perturbation by a random matrix*

We now describe the situation where the operator $P_h$ is perturbed by a small random matrix $\delta M_\omega$, as described in (RM) and (2.18). In this case we do not need to assume the symmetry property (SYM) for the symbol $p_0$.

*2.6.1. Universal limiting point process.* Here as well, we can prove a convergence of the rescaled spectral point process $\mathcal{Z}^M_{h,z_0}$ (see (2.24)) towards a limiting zero process when $h \to 0$.

**Theorem 2.9.** *Let $p$ be as in (2.6) satisfying (2.12). Let $\Omega \Subset \mathring{\Sigma}$ be as in (2.11). Choose $z_0 \in \Omega$. Then, for any bounded open set $O \Subset \mathbb{C}$, the rescaled spectral point process $\mathcal{Z}^M_{h,z_0}$ converges in distribution towards the zero point process associated with a random analytic function $\tilde{G}_{z_0}$ described below:*

$$\mathcal{Z}^M_{h,z_0} \xrightarrow{d} \mathcal{Z}_{\tilde{G}_{z_0}} \quad \text{on } O \quad \text{as } h \to 0. \tag{2.33}$$

*The random function $\tilde{G}_{z_0}$ is defined as*

$$\tilde{G}_{z_0}(w) := \det\,(g^{i,j}_{z_0}(w))_{1\le i,j\le J}, \qquad w \in \mathbb{C},$$

*where $g^{i,j}_{z_0}$, for $1 \le i, j \le J$, are $J^2$ independent GAFs $g^{i,j}_{z_0} \sim g_{\sigma^{i,j}_{z_0}}$, for the parameters*

$$\sigma^{i,j}_{z_0} = \tfrac{1}{2}(\sigma^i_+(z_0) + \sigma^j_-(z_0)). \tag{2.34}$$

*The local classical densities $\sigma^i_\pm(z_0)$ associated with the points $\rho^i_\pm(z_0)$ were defined in (2.25).*

Theorem 2.9 tells us that at any given point $z_0 \in \widetilde{\Omega} \cap \mathring{\Sigma}$ in the bulk of the pseudospectrum, the local rescaled point process of the eigenvalues of $P^\delta_M$ is given, in the limit $h \to 0$, by the zero process associated with the determinant of a $J \times J$ matrix, whose entries are independent GAFs. The GAF situated at the entry $i, j$ of the matrix only depends on the local classical densities of the points $\rho^i_+(z_0)$ and $\rho^j_-(z_0)$.

The limiting point process $\mathcal{Z}_{\tilde{G}_{z_0}}$ features some partial form of universality: it is independent of the precise law of the entries of the perturbation $M_\omega$ (2.16), but only depends on the cardinality $2J$ of the energy shell $p_0^{-1}(z)$, and on the local classical densities

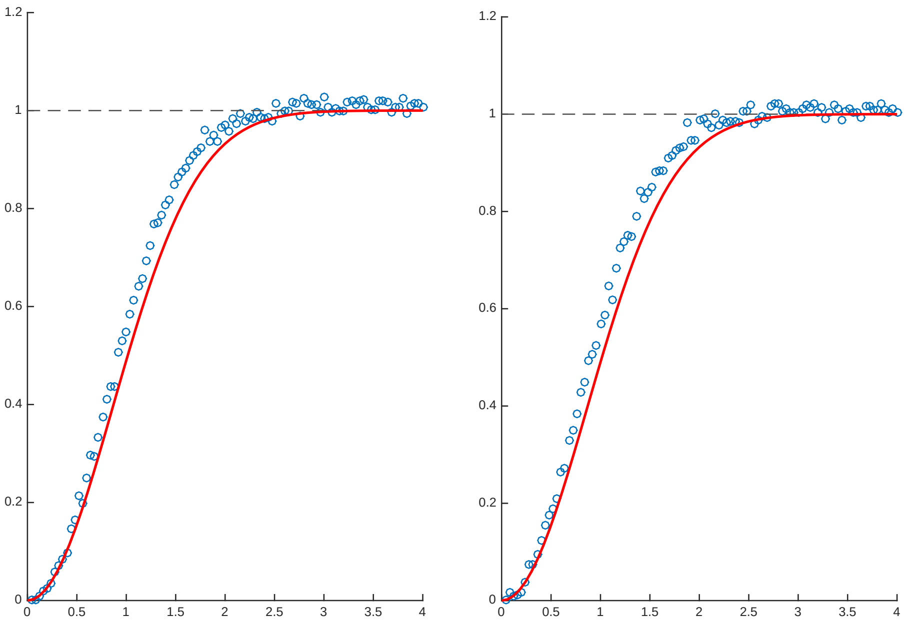


**Fig. 4.** Blue points: values of the 2-point correlation functions, obtained by numerically computing the spectra of the operators $P_{h,1}$ (left) and $P_{h,3}$ (right) perturbed by a Gaussian random matrix $\delta M_\omega$. The parameters $z_0, h, \delta$ are as in Figure 3. Red curves: the 2-point correlation function for the Ginibre ensemble, $K_{z_0}^{2,\mathrm{Gin}}$, as given in (2.37). The horizontal coordinate is the rescaled square distance $|w_1 - w_2|^2$.

$\{\sigma_\pm^j(z_0);\ j = 1, \ldots, J\}$ (notice that in absence of the symmetry (SYM), the densities $\sigma_+^j(z_0)$ and $\sigma_-^j(z_0)$ are a priori unrelated).

The limiting process $\mathcal{Z}_{\tilde{G}_{z_0}}$ is different from the universal limit $\mathcal{Z}_{G_{z_0}}$ studied in the previous section. In particular the function $\tilde{G}_{z_0}$ is not given by a simple product of GAFs, but by a more complicated expression, namely a determinant. As we will see below, we expect the zeros of $\tilde{G}_{z_0}$ to exhibit a quadratic repulsion between nearby points, as opposed to the zeros of the function $G_{z_0}$ in Theorem 2.5.

*2.6.2. Scaling limit k-point measures.* A direct consequence of the convergence of the zero processes $\mathcal{Z}_{h,z_0}^M$ is the convergence of their $k$-point measures to those of the limiting point process.

**Corollary 2.10.** *Let $\mu_{h,z_0}^{k,M}$ be the k-point measure of $\mathcal{Z}_{h,z_0}^M$, defined as in (2.26), and let $\mu_{z_0}^{k,M}$ be the k-point measure of the point process $\mathcal{Z}_{\tilde{G}_{z_0}}$ described in Theorem 2.9. Then, for any open connected domain $O \Subset \mathbb{C}$ and for all $\phi \in \mathcal{C}_c(O^k \setminus \Delta, \mathbb{R}_+)$,*

$$\int_{O^k\setminus\Delta} \phi(w)\, \mu_{h,z_0}^{k,M}(dw) \to \int_{O^k\setminus\Delta} \phi(w)\, \mu_{z_0}^{k,M}(dw), \qquad h \to 0.$$

One can calculate the densities of the limiting 1-point measures $\mu_{z_0}^{1,M}$:

$$d_{z_0}^{1,M}(w) = \sum_{i=1}^{J} \frac{\sigma_+^i(z_0) + \sigma_-^i(z_0)}{2\pi}, \qquad \forall w \in O. \tag{2.35}$$

Not surprisingly, this microscopic density is exactly the rescaling of the macroscopic spectral density at $z_0$ predicted by the probabilistic Weyl's law in Theorem 2.4 (see also (2.25)). In the case of an operator $P_h$ satisfying the symmetry assumption (SYM), this microscopic density is equal to the one obtained for the operator perturbed by a random potential: for such symmetric symbols, the microscopic densities $d_{z_0}^{1,V}$ and $d_{z_0}^{1,M}$ coincide, and therefore they cannot distinguish between the type of perturbation imposed on $P_h$.

On the other hand, we believe that for $k > 1$, the $k$-point densities $d_{z_0}^{k,Q}$ (equivalently, the $k$-point correlation functions $K_{z_0}^{k,Q}$) can distinguish between the two types of perturbation (still assuming (SYM) for the symbol). We have not been able to compute in closed form the densities of the limiting $k$-point measures $d_{z_0}^{k,M}$ associated with the random function $\tilde{G}_{z_0}$; however, the numerical experiments presented in Figure 4, as well as Proposition 2.7, lead us to the following

**Conjecture 2.11.** *The $k$-point densities $d_{z_0}^{k,M}$ of the zero point process of the random function $\tilde{G}_{z_0}$ described in Theorem 2.9 exhibit a quadratic repulsion at short distance. Namely, for any open set $O \Subset \mathbb{C}$, there exists a constant $C > 1$ depending only on $O$ and $k$ such that, for all pairwise distinct points $w_1, \dots, w_k \in O$,*

$$C^{-1} \prod_{i<j} |w_i - w_j|^2 \le d_{z_0}^{k,M}(w_1, \dots, w_k) \le C \prod_{i<j} |w_i - w_j|^2.$$

In Figure 4 we compare numerical values of the 2-point correlation function with the 2-point correlation function of a well-known spectral point process on $\mathbb{C}$, namely the spectrum of large Ginibre random matrices. This ensemble corresponds to random matrices $M_\omega$ alone, when the entries are i.i.d. Gaussian $\sim \mathcal{N}_{\mathbb{C}}(0,1)$, in the limit $h \to 0$, or equivalently the limit of large matrices. It has been known since the work of Ginibre [20] that the eigenvalues of these matrices repel each other quadratically at short distance. When the eigenvalues are rescaled so that the mean local density is $d^1(w) = \sigma/\pi$, the 2-point correlation function takes the simple form

$$K_\sigma^{2,\mathrm{Gin}}(w_1, w_2) = 1 - \exp(-\sigma |w_1 - w_2|^2). \tag{2.36}$$

Hence, in view of our local density (2.35), we draw in Fig. 4 the 2-point function

$$K_{z_0}^{2,\mathrm{Gin}}(w_1, w_2) = 1 - \exp\left[-\frac{1}{2} \sum_{i=1}^{J} (\sigma_+^i(z_0) + \sigma_-^i(z_0)) |w_1 - w_2|^2\right]. \tag{2.37}$$

This function is markedly different from the scaling function $\kappa(t^2)$ corresponding to the zero GAF process (2.29). It seems rather close to our experimental data of Fig. 4, even though we observe a deviation for values $|w_1 - w_2| \sim 1$. Is this deviation due to the finite value of $h$ used in our numerical experiment? Or does the deviation persist when $h \to 0$, that is, in the limiting correlation function $K^{2,M}_{z_0}$? We conjecture that the latter correlation function $K^{2,M}_{z_0}$ differs from the (appropriately rescaled) Ginibre function $K^{2,\mathrm{Gin}}_{z_0}$, but that it becomes closer and closer to it when the number $J$ of quasimodes increases (a property which purely concerns the classical symbol $p_0$). Indeed, when $J \gg 1$ the function $\tilde{G}_{z_0}$ is the determinant of a large matrix of independent GAFs. In any case, computing the $k$-point densities for the process $\mathcal{Z}_{G_{z_0}}$ seems to us to be an interesting open problem.

*Translation invariance*

One easy property of the limiting point processes obtained in Theorems 2.6 and 2.9 is that they are homogeneous and isotropic. This property is naturally inherited from the translation invariance of the zero process of individual GAFs, as mentioned in Section 1.2.

**Proposition 2.12.** *The limiting point processes $\mathcal{Z}_{G_{z_0}}$ and $\mathcal{Z}_{\tilde{G}_{z_0}}$ obtained in Theorems 2.6 and 2.9 are invariant in distribution under the action of the group of translations and rotations on $\mathbb{C}$. More precisely, for arbitrary $\alpha, \beta \in \mathbb{C}$ with $|\alpha| = 1$ let us define the plane isometry $\tau(w) = \alpha w + \beta$, $w \in \mathbb{C}$. Then the zero processes of $G_{z_0}$ and $\tilde{G}_{z_0}$ satisfy*

$$\mathcal{Z}_{G_{z_0}} \stackrel{d}{=} \mathcal{Z}_{G_{z_0}\circ\tau}, \qquad \mathcal{Z}_{\tilde{G}_{z_0}} \stackrel{d}{=} \mathcal{Z}_{\tilde{G}_{z_0}\circ\tau}.$$

## 2.7. *Sketch and key ideas of the proof*

The proof of the main results has two distinct parts. The first part uses linear algebra and semiclassical methods to reduce the eigenvalue problem of the infinite-dimensional operator $P_h$ to a nonlinear spectral problem expressed in terms of a finite-dimensional matrix (called the *effective Hamiltonian*), which depends nonlinearly on the spectral parameter. This reduction will be applied to the randomly perturbed operators $P^\delta_Q$ as well. The reduction is based on the construction of quasimodes of the unperturbed operator, which we perform in Section 3. In Section 5 we use these quasimodes to construct a *well-posed Grushin problem* for the operators $P_h$ or $P^\delta_Q$, which leads to the effective Hamiltonian. The spectrum of the random operator $P^\delta_Q$ is now obtained as the zero locus of the determinant of the effective Hamiltonian; in the case of the randomly perturbed operator $P^\delta_Q$, this determinant is a certain type of random analytic function.

In the second part of the argument, we analyze the statistical properties of this random analytic function. First, we rescale the spectral parameter near a given point $z_0$ to the scale of the average spacing between eigenvalues. Then we show that, for each type of perturbation, the determinant of the effective Hamiltonian (after some “change of gauge”) converges in distribution to the universal random analytic function stated in Theorem 2.5, resp. Theorem 2.9. In Section 6 we provide an overview of the notions and results from the theory of random analytic functions used in this paper. Sections 7 and 8 then complete the proofs of our main theorems.

Let us now give some more details on the successive steps.

*2.7.1. Part I: Reduction to an effective Hamiltonian.* By (2.12) for all $z \in \Omega$ and for each point $\rho_\pm^j(z)$, $j = 1, \dots, J$, in the energy shell $p^{-1}(z)$, we can construct an $h^\infty$-quasimode $e_\pm^j$ for the problem $P_h - z$ resp. $(P_h - z)^*$ as in (2.14)–(2.15). These quasimodes are microlocalized at $\rho_\pm^j$, i.e. $\mathrm{WF}_h(e_\pm^j) = \{\rho_\pm^j\}$. The quasimodes $e_+^j$ (resp. $e_-^j$) essentially span the space of singular values of $P_h - z$ (resp. $(P_h - z)^*$) smaller than $h/C$. This property will be used later to extend the operator $P_h - z$ to a well-posed Grushin problem.

*Almost holomorphic quasimodes.* The quasimodes are constructed in Section 3, their relevant properties are gathered in Proposition 3.5. For this construction, near each point $\rho_+ = (x_+, \xi_+) \in p^{-1}(z)$ we use the Malgrange preparation theorem (see Section 3.1) to factorize our operator $P - z$, microlocally near $\rho_+$, into a simple form $\widetilde{P} = hD_x + g^+(x, z)$. The WKB-method (see Section 3.3) then allows one to construct a state

$$e_+^{\mathrm{hol}}(x, z; h) = a_+(x, z; h)\mathrm{e}^{\frac{i}{h}\phi_+(x,z)} \tag{2.38}$$

satisfying the quasimode equation

$$\|(P_h - z)e_+^{\mathrm{hol}}\| = \mathcal{O}(h^\infty)\|e_+^{\mathrm{hol}}\|. \tag{2.39}$$

For a fixed $z$ this WKB construction is standard [12], but we also need to control how the quasimodes depend on the parameter $z$. In [24] the author treated the case where the symbol $p$ in (2.6) is analytic. Here we are only assuming the symbol to be smooth, and we take particular care in Proposition 3.5 to construct quasimodes depending on $z$ in an *almost holomorphic* way, at least near the reference point $z_0$ where we study the spectrum. The almost holomorphy near $z_0$ takes the form

$$\mathrm{e}^{-\frac{1}{h}\Phi_+(z;h)}\|\partial_{\bar{z}}\, e_+^{\mathrm{hol}}(z; h)\| = \mathcal{O}(h^{-1}|z - z_0|^\infty + h^\infty). \tag{2.40}$$

Because we will eventually focus on $z$ in an $\mathcal{O}(h^{1/2})$ neighbourhood of $z_0$, the right hand side will effectively be $\mathcal{O}(h^\infty)$.

This holomorphy implies that the states $e_+^{\mathrm{hol}}(z; h)$ are not $L^2$ normalized for all $z$. Indeed, we show that for $z$ in a neighbourhood of $z_0$, their norm takes the form

$$\|e_+^{\mathrm{hol}}(z; h)\| = \mathrm{e}^{\frac{1}{h}\Phi_+(z;h)} \tag{2.41}$$

with a phase function

$$\Phi_+(z; h) = \Phi_{+,0}(z) + \mathcal{O}(h \log h), \qquad \Phi_{+,0}(z) := -\operatorname{Im}\phi_+(x_+(z), z).$$

The normalized quasimode for $P_h - z$, microlocalized at $\rho_+$, as in (2.14), can then be defined as $e_+(z; h) = \mathrm{e}^{-\frac{1}{h}\Phi_+(z;h)}\, e_+^{\mathrm{hol}}(z; h)$.

Similarly, for the adjoint problem $P_h - z^*$, we construct WKB states $e_-^{\mathrm{hol}}(z; h)$ which are almost anti-holomorphic with respect to $z$ and their normalized version $e_-(z; h) = \mathrm{e}^{-\frac{1}{h}\Phi_-(z;h)}\, e_-^{\mathrm{hol}}(z; h)$.

*Interaction between the quasimodes.* In Section 4 we analyse the overlaps ("interactions") between nearby quasimodes. These interactions will be relevant when computing the covariance of the components of the effective Hamiltonian. Since two points $\rho_+^j(z)$, $\rho_+^k(z)$ ($j \neq k$) remain at finite distance, the corresponding quasimodes are essentially orthogonal to each other. On the other hand, we will need to control the interactions between the quasimodes microlocalized at $\rho_+^j(z)$ and $\rho_+^j(w)$ when the energies $z, w$ are close to one another (in practice, $|z-w| = \mathcal{O}(h^{1/2})$). In Propositions 4.1 and 4.3 we exploit the almost (anti-)holomorphy of $e_\pm^{\mathrm{hol},j}$ to show that

$$(e_+^{j,\mathrm{hol}}(z)|e_+^{j,\mathrm{hol}}(w)) = \mathrm{e}^{\frac{2}{h}\Psi_+^j(z,w;h)} + \text{small}, \qquad \Psi_+^j(z,w;h) = \Psi_{+,0}^j(z,w) + \mathcal{O}(h\log h). \tag{2.42}$$

Here $\Psi_{+,0}^j(z,w)$ is a polarization of the phase function $\Phi_{+,0}(z)$, almost $z$-holomorphic and almost $w$-anti-holomorphic near $\{(z_0,z_0)\}$. In particular it satisfies the second-order Taylor expansion

$$\Psi_{+,0}(z_0+\zeta_1, z_0+\zeta_2) = \sum_{|\alpha|\leq 2} (\partial_z^{\alpha_1}\partial_{\bar z}^{\alpha_2}\Phi_{+,0})(z_0)\frac{\zeta_1^{\alpha_1}\bar\zeta_2^{\alpha_2}}{\alpha!} + \mathcal{O}(|\zeta|^3). \tag{2.43}$$

**Remark 2.13.** In the case of perturbation by a random potential $\delta V_\omega$, we will also need to compute the interactions between the *squared* functions $(e_-^j(x,z))^2$, namely estimate scalar products of the form $((e_-^j(z))^2|(e_+^j(w)))^2$ (see Section 4.2).

*Grushin problem for the perturbed operator $P_h^\delta$.* The next step in the proof is to use the quasimodes $e_\pm(z;h)$ to construct a well-posed Grushin problem for the operator $P_h - z$. As reviewed in [45], in order to analyze the small singular values of a $z$-dependent operator $P(z) = P - z : \mathcal{H}_1 \to \mathcal{H}_2$ (in particular the spectrum of $P$), the general idea of setting up a Grushin problem is to extend this operator to an operator of the form

$$\mathcal{P}(z) := \begin{pmatrix} P(z) & R_-(z) \\ R_+(z) & 0 \end{pmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_- \to \mathcal{H}_2 \oplus \mathcal{H}_+,$$

where $\mathcal{H}_\pm$ (resp. $R_\pm$) are well-chosen auxiliary spaces (resp. operators). The Grushin problem is said to be *well-posed* if the extended operator $\mathcal{P}(z)$ is bijective for the range of $z$ under study, with good control on its inverse. Roughly speaking, the role of $R_+(z)$ is to map the quasi-kernel of $P(z)$ to the auxiliary space, while $R_-(z)$ maps the latter to the quasi-cokernel of $P(z)$; both actions finally make $\mathcal{P}(z)$ invertible.

In the case where $\dim \mathcal{H}_- = \dim \mathcal{H}_+ < \infty$, one decomposes the inverse operator blockwise as

$$\begin{pmatrix} P(z) & R_- \\ R_+ & 0 \end{pmatrix}^{-1} = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{pmatrix} =: \mathcal{E}(z).$$

The key observation, going back to Schur's complement formula, is the following: the initial operator $P(z)$ is invertible if and only if the finite rank operator $E_{-+}(z) : \mathcal{H}_+ \to \mathcal{H}_-$ is invertible, in which case both inverses are related by

$$P(z)^{-1} = E(z) - E_+(z)E_{-+}^{-1}(z)E_-(z).$$

The finite rank operator $E_{-+}(z)$ is often called an *effective Hamiltonian* for the original problem $P(z)$. It depends in a nonlinear way on the spectral parameter $z$, but it has the advantage of being finite-dimensional. In a sense, $E_{-+}(z)$ encapsulates, in a minimal way, the spectral properties of $P$. If the spectrum of $P$ is discrete in the $z$-range under study, its eigenvalues can be obtained as the zeros of $\det E_{-+}(z)$ (with multiplicities).

In our case we construct our Grushin problem using the normalized quasimodes: we take

$$R_+(z) : H(m) \to \mathbb{C}^J, \quad (R_+(z)u)_j = (u|e_+^j(z)), \qquad j = 1, \dots, J,$$
$$R_-(z) : \mathbb{C}^J \to L^2, \qquad R_-(z)u_- = \sum_{j=1}^{J} u_-(j)e_-^j(z).$$

The roles of $R_\pm(z)$ are quite transparent: $R_+(z)$ indeed maps the quasi-kernel of $P - z$ (the quasimodes $e_+^j(z)$) to the auxiliary space, while $R_-(z)$ constructs the quasi-cokernel of $(P - z)^*$. We thus obtain a well-posed Grushin problem $\mathcal{P}(z)$ (see Proposition 5.1). Note that this construction was already performed in [24].

After restricting our random variables to discs of radius $Ch^{-1}$, the perturbations $\delta Q$ of $P_h$ are small in norm: $\|\delta Q\| \ll 1$. As a result, the Grushin problem is still well-posed if we replace $P_h$ by the perturbed operator $P_Q^\delta$ (see Proposition 5.3). The eigenvalues of $P_Q^\delta - z$ are then given by the zeros of $\det(E_{-+}^\delta(z))$, where $E_{-+}^\delta(z)$ is the perturbed effective Hamiltonian. In Section 5.2 we compute this effective Hamiltonian:

$$\delta^{-J} \det(E_{-+}^\delta(z)) = (-1)^J \det\big[(Qe_+^j(z)|e_-^i(z))_{i,j\le J} + \mathcal{O}(\delta h^{-5/2})\big].$$

A crucial feature of this expression is that the effective Hamiltonian is dominated by the random perturbation, in spite of the fact that the latter is of size $\propto \delta$, which is a small parameter. However, the unperturbed effective Hamiltonian $E_{-+}(z)$ is actually of size $\mathcal{O}(h^\infty)$, allowing a perturbation of size $h^N$ to be comparatively large.

The quasimodes used in the definition of the effective Hamiltonian were normalized, hence $E_{-+}^\delta(z)$ is not holomorphic. In Section 5.2 we show that by multiplying $\det(E_{-+}^\delta(z))$ by an appropriate nonvanishing function of $z$, we obtain a holomorphic function, of the following form:

$$G^\delta(z; h) = (1 + R_1) \det\big[h^{-1/2}(Qe_+^{j,\mathrm{hol}}(z)|e_-^{i,\mathrm{hol}}(z))_{i,j\le J} + R_2\big], \tag{2.44}$$

with $R_1, R_2$ some small ($z$-dependent) error terms. Not surprisingly, the normalized quasimodes have been replaced by their almost (anti-)holomorphic counterparts in the expression. The entries of the matrix on the right hand side are dominated by the scalar products $(Qe_+^{j,\mathrm{hol}}(z)|e_-^{i,\mathrm{hol}}(z))$, which represent the *coupling* between the quasimodes through the perturbation operator $Q$. Remember that these quasimodes are microlocalized at different phase space points $\rho_+^j(z)$, $\rho_-^i(z)$; hence, the coupling will be nonnegligible only if the perturbation operator $Q$ is able to "transport mass" from one point to the other.

**Remark 2.14.** Here a major difference occurs between the two types of perturbations. The random operator $M_\omega$ (eq. (RM)) will typically be able to couple any pair of quasimodes $(e_+^{j,\mathrm{hol}}, e_-^{i,\mathrm{hol}})$, leading to a full $J \times J$ matrix. On the other hand, multiplication by a random potential $V_\omega$ will not be able to couple quasimodes localized at different positions $x_+^j \neq x_-^i$. This is the reason why, in this case, we need to assume the symmetry property (SYM), which ensures that each quasimode $e_+^j$ admits a dual quasimode $e_-^j$ with $x_+^j = x_-^j$. The further assumption (HYP-x) ensures that no other quasimode will be localized at $x_+^j$: this property makes the matrix $(V_\omega e_+^{j,\mathrm{hol}}(z)|e_-^{i,\mathrm{hol}}(z))_{i,j\le J}$ approximately diagonal. This diagonal structure will lead to a limiting random determinant given by the product of $J$ independent GAFs, each corresponding to one of the diagonal entries. On the other hand, for a random matrix perturbation, the full matrix $(M_\omega e_+^{j,\mathrm{hol}}(z)|e_-^{i,\mathrm{hol}}(z))_{i,j\le J}$ will lead to a full matrix of GAFs.

*2.7.2. Part II: Convergence to Gaussian analytic functions.* In the second part we study the point process consisting of the zeros of the random analytic function $G^\delta(z;h)$ of (2.44). Performing the rescaling $z = z_w := z_0 + h^{1/2}w$, with $w$ in some bounded open set $O \Subset \mathbb{C}$, we are led to study the zeros of the rescaled random function

$$F_h^\delta(w) := G^\delta(z_0 + h^{1/2}w; h) = (1 + R_1)\det[(f_{i,j}^{\delta,h}(w))_{i,j\le J} + R_2]. \tag{2.45}$$

The terms $R_1$ and $R_2$ are small, they converge to 0 in probability sufficiently quickly (see Corollary 7.4 and Lemma 7.5), hence the expression is dominated by $\det(f_{i,j}^{\delta,h}(w))_{i,j\le J}$, where $f_{i,j}^{\delta,h}(w) = h^{-1/2}(Qe_+^{j,\mathrm{hol}}(z_w)|e_-^{i,\mathrm{hol}}(z_w))$.

In Section 6 we collect some general notions and results concerning random holomorphic functions and the associated zero processes. The key observation [40, Proposition 2.3] (see also Proposition 6.11) is that if a sequence of random holomorphic functions $f_n$ converges in distribution to a random holomorphic function $f$ (which is almost surely $\not\equiv 0$), then the zero point processes of $f_n$ converge in distribution to the zero point process of $f$.

Therefore, we need to show that the function $F_h^\delta$ (actually, after multiplication by appropriate "gauge" factors) converges in distribution to the random analytic function $G_{z_0}$ in the case of Theorem 2.5, resp. to $\widetilde{G}_{z_0}$ in the case of Theorem 2.9. The first case is treated in Section 8, the second in Section 7. In this sketch we mostly describe the second case $Q = M_\omega$, and highlight the differences with the perturbation by a random potential.

*Covariances.* To show the convergence of $F_h^\delta$, we will need to show that the entries $f_{i,j}^{\delta,h}(w)$ converge in distribution to $J^2$ independent GAFs.

The assumptions (2.16) on the coefficients of the random matrix $M_\omega$ imply that at each point $w$ the random variable $f_{i,j}^{\delta,h}(w)$ is centred. The second step is to compute the covariances

$$\mathbb{E}\big[f_{i,j}^{\delta,h}(v)\overline{f_{l,k}^{\delta,h}(w)}\big] = h^{-1}\mathbb{E}\big[(M_\omega e_+^{j,\mathrm{hol}}(z_v)|e_-^{i,\mathrm{hol}}(z_v))(e_-^{k,\mathrm{hol}}(z_w)|M_\omega e_+^{l,\mathrm{hol}}(z_w))\big].$$

Expanding the random operator $M_\omega$ in the orthonormal family $(e_m)_{m<N(h)}$ leads to

$$\mathbb{E}\big[f^{\delta,h}_{i,j}(v)\overline{f^{\delta,h}_{l,k}(w)}\big]=h^{-1}(e^{j,\mathrm{hol}}_{+}(z_v)|\Pi_{N(h)}e^{k,\mathrm{hol}}_{+}(z_w))(e^{l,\mathrm{hol}}_{-}(z_w)|\Pi_{N(h)}e^{i,\mathrm{hol}}_{-}(z_v)),\tag{2.46}$$

where $\Pi_{N(h)}$ is the orthogonal projector on the space spanned by $(e_m)_{m<N(h)}$. From our assumption on this orthonormal basis, this projector is equivalent to the identity microlocally near $\Omega$; since all our quasimodes are microlocalized inside $\Omega$, the projectors $\Pi_{N(h)}$ may be removed from the scalar products, up to negligible errors. The covariance is hence expressed in terms of the interations between neighbouring quasimodes.

From our analysis of these interactions, in Proposition 7.1 we deduce the following expressions for the covariances:

$$\begin{aligned}\mathbb{E}[f^{\delta,h}_{i,j}(v)\overline{f^{\delta,h}_{l,k}(w)}]&\approx h^{-1}(e^{j,\mathrm{hol}}_{+}(z_v)|e^{k,\mathrm{hol}}_{+}(z_w))\,(e^{l,\mathrm{hol}}_{-}(z_w)|e^{i,\mathrm{hol}}_{-}(z_v))\\&\approx\delta_{i,l}\delta_{j,k}\mathrm{e}^{2(\partial^2_{z\bar z}\Phi^j_{+,0}(z_0)+\partial^2_{z\bar z}\Phi^i_{-,0}(z_0))v\overline{w}}\mathrm{e}^{F_{i,j}(v)+\overline{F_{i,j}(w)}}\\&=\delta_{i,l}\delta_{j,k}\mathrm{e}^{\frac12(\sigma^j_{+}(z_0)+\sigma^i_{-}(z_0))v\overline{w}}\mathrm{e}^{F_{i,j}(v)+\overline{F_{i,j}(w)}}\\&=:K^{i,j}(v,\overline{w})\mathrm{e}^{F_{i,j}(v)+\overline{F_{i,j}(w)}}.\end{aligned}\tag{2.47}$$

The Kronecker factors already hint at the fact that the random functions $f^{\delta,h}_{i,j}$ and $f^{\delta,h}_{l,k}$ are statistically independent if $(i,j)\neq(l,k)$. To obtain the second line, we have expanded the phase function $\Psi_{\pm,0}$ describing the interaction to second order (see the Taylor expansion (2.43)), and have separated the mixed term $v\bar w$ from the separated terms $|v|^2$, $|w|^2$ which we grouped in the functions $F_{i,j}(\bullet)$.

To obtain the third line of (2.47) we used the relation between the phase function $\Phi^j_{\pm,0}$ describing the $L^2$ norm of the quasimode, and the local classical density $\sigma^j_{\pm}$ (see (2.25)). We indeed show in Section 3.5 that

$$\partial^2_{z\bar z}\Phi^j_{\pm,0}(z_0)=\tfrac14\sigma^j_{\pm}(z_0).$$

In the mixed term we recognize the covariance of the GAF $g_{\sigma^{ij}}$ with parameter $\sigma^{ij}=\frac12(\sigma^j_{+}+\sigma^i_{-})$ (see (1.9)). Hence, the whole expression corresponds to the covariance of the modified GAF

$$f^{\mathrm{GAF}}_{i,j}:=\mathrm{e}^{F_{i,j}}\,g_{\sigma^{ij}}.$$

**Remark 2.15.** In the case of a random potential (see Section 8), we perform a similar computation in Proposition 8.2. In that case the covariance will involve scalar products of the type $((e^{j,\mathrm{hol}}_{-}(z_v))^2|(e^{k,\mathrm{hol}}_{-}(z_w))^2)$, which will be nonnegligible only if $j=k$; we also recover the covariance of a GAF multiplied by a gauge factor.

*Convergence to a Gaussian function.* Computing the covariances is not sufficient to prove the convergence in distribution of the random functions $f^{\delta,h}_{i,j}$ towards the modified GAFs $f^{\mathrm{GAF}}_{i,j}$. The entries of $M_\omega$ are in general not Gaussian, so neither are the functions $f^{\delta,h}_{i,j}$. How does one prove the convergence of a sequence of random functions?

Prokhorov's Theorem 6.7 shows that to prove the convergence of random functions, it is enough to prove the convergence in the sense of finite-dimensional distributions (see Definition 6.5): namely, for any $n \in \mathbb{N}^*$ and any set of points $(w_1, \dots, w_n) \in O^n$, we need to show that the random complex vector

$$(f_{i,j}^{\delta,h}(w_1), \dots, f_{i,j}^{\delta,h}(w_n)) \text{ converges in distribution to } (f_{i,j}^{\mathrm{GAF}}(w_1), \dots, f_{i,j}^{\mathrm{GAF}}(w_n)).$$

This type of convergence will be denoted by $f_{i,j}^{\delta,h} \xrightarrow{\mathrm{fd}} f_{i,j}^{\mathrm{GAF}}$. We actually need to show that the various functions $f_{i,j}^{\delta,h}$ are asymptotically independent from one another, and converge towards independent GAFs $f_{i,j}^{\mathrm{GAF}}$.

The random operator $M_\omega$ is in general not Gaussian, so to obtain convergences to Gaussian vectors, we need to apply a suitable version of the central limit theorem (Theorem 6.12). The application of the CLT relies on the fact that each quasimode $e_\pm^j$ has nonnegligible overlaps with many of the basis states $e_m$ used to construct $M_\omega$. Thanks to this property, the higher moments of the $f_{i,j}^{\delta,h}$ will involve sums over many i.i.d. random variables $\alpha$, and hence to a Gaussian law (see Proposition 7.3).

Taking into account the small error terms and applying Prokhorov's theorem, this leads to the convergence in distribution of the full determinant

$$F_h^\delta(\bullet) \xrightarrow{d} \det((f_{i,j}^{\mathrm{GAF}}(\bullet))_{1\le i,j\le J}) \quad \text{when } h \to 0.$$

Finally, at the end of Section 7.5 we use the fact that the "gauge" functions split into $F_{i,j}(v) = \phi_-^i(v) + \phi_+^j(v)$. This splitting allows one to extract the gauge factors $e^{F_{i,j}}$ from the random matrix as follows:

$$(f_{i,j}^{\mathrm{GAF}}(v))_{i,j} = \mathrm{diag}(\mathrm{e}^{\phi_-^i(v)})\,(g_{\sigma^{ij}}(v))_{i,j}\ \mathrm{diag}(\mathrm{e}^{\phi_+^j(v)}).$$

The determinant of the diagonal matrices never vanishes, so the zero process is that of the determinant of the matrix of GAFs $g_{\sigma^{ij}}$, as in Theorem 2.9.

**Remark 2.16.** In the case of a random potential, the major difference lies in the fact that the off-diagonal entries in (2.45) are negligible; this leads to a product of $J$ independent functions, each converging to a GAF (compare Corollaries 7.4 and 8.5).

*$k$-correlation functions.* The convergence of the point processes implies the convergence of the $k$-point correlation measures. It remains to compute the latter, as given in Theorem 2.6, Corollary 2.10 and Proposition 2.12; the computations are performed in Sections 7.6 and 8.4. In the case of a perturbation by a random operator, we can only compute the 1-point function, whereas in the case of a random potential we obtain explicit formulas for the limiting $k$-point correlation functions.

## 3. Quasimodes

Our main objective in this section is the construction of the $h^\infty$-quasimodes $e_\pm^j(z_0)$ for the operator $P_h - z_0$ (resp. $(P_h - z_0)^*$), which will be used to set up the Grushin problem in the next section.

For this purpose we will first factorize our symbol $p$ into a "nice" form, using semiclassical analysis and the Malgrange preparation theorem in Section 3.1. This factorized form will allow for very explicit expressions of our quasimodes, which will naturally exhibit an almost holomorphic (resp. anti-holomorphic) dependence on the spectral parameter $z$. In Section 3.2 we recall the notion of almost holomorphy and we provide some results needed for the construction of the quasimodes in Section 3.3. Section 3.4 contains additional remarks on the construction of quasimodes in case we assume the symmetry property (SYM). Finally, Section 3.5 provides a link between the quasimode phase functions and the symplectic volume on $T^*\mathbb{R}$, which will be used later.

### *3.1. Malgrange preparation theorem*

We start by Moyal-factorizing the symbol $p$ of the operator $P_h$ in a neighbourhood of each of the points $\rho_\pm^j = (x_\pm^j, \xi_\pm^j) \in p_0^{-1}(z_0)$ (see (HYP)). The method presented is an adaptation of [23].

**Proposition 3.1.** *Let $p(h)$ in $S(\mathbb{R}^2, m)$ as in (2.6) satisfy (HYP), let $\Omega \Subset \mathbb{C}$ be as in (2.11), and let $z_0 \in \Omega$. For $j = 1, \dots, J$, let $U_\pm^j$ be open neighbourhoods of $\rho_\pm^j(z_0)$. Then there exists an open neighbourhood $W(z_0) \subset \Omega$ of $z_0$, open sets $V_\pm^j \subset U_\pm$ containing $\rho_\pm^j(\overline{W(z_0)})$, and symbols in $S(V_\pm^j, 1)$:*

$$q^{\pm,j}(x, \xi, z; h) \sim \sum_{k\geq 0} h^k q_k^{\pm,j}(x, \xi, z), \qquad g^{\pm,j}(x, z; h) \sim \sum_{k\geq 0} h^k g_k^{\pm,j}(x, z), \tag{3.1}$$

*depending smoothly on $z \in W(z_0)$, such that for all $z \in W(z_0)$,*

$$\begin{aligned} p(x, \xi; h) - z &\sim q^{+,j}(x, \xi, z; h) \,\#\, (\xi + g^{+,j}(x, z; h)) \qquad \text{in } S(V_+^j, m), \\ p(x, \xi; h) - z &\sim (\xi + g^{-,j}(x, z; h)) \,\#\, q^{-,j}(x, \xi, z; h) \qquad \text{in } S(V_-^j, m). \end{aligned} \tag{3.2}$$

*Furthermore, the principal symbols satisfy $q_0^{\pm,j}(x_\pm^j(z), \xi_\pm^j(z), z) \neq 0$ and $g_0^{\pm,j}(x_\pm^j(z), z) = -\xi_\pm^j(z)$.*

We recall that # indicates the Moyal product, which translates the operator composition to the symbolic level [12, Chapter 7]: for any symbols $a_j \in S(\mathbb{R}^2, \widetilde{m}_j)$, for $j = 1, 2$,

$$a_1^w \circ a_2^w = (a_1 \,\#\, a_2)^w.$$

The Moyal product $\# : S(\mathbb{R}^2, \widetilde{m}_1) \times S(\mathbb{R}^2, \widetilde{m}_2) \to S(\mathbb{R}^2, \widetilde{m}_1\widetilde{m}_2)$ is a bilinear and continuous map.

*Proof of Proposition 3.1.* We will focus on a single point $\rho_+^j(z)$, and will omit the $\pm$ and $j$ sub/superscripts in the proof. The case of the points $\rho_-^j$ can be treated identically.

The condition (HYP) implies that for any $z \in \Omega$, we have $p_0(\rho(z)) - z = 0$ and $\partial_\xi p_0(\rho(z)) \neq 0$. Fix $z_0 \in \Omega$. By the Malgrange preparation theorem [29, Theorem

7.5.5], there exist open neighbourhoods $V \subset U$ of $\rho(z_0)$ and $W(z_0) \subset \mathbb{C}$ of $z_0$, as well as smooth functions $q_0 \in \mathcal{C}^\infty(V \times W(z_0))$ and $g_0 \in \mathcal{C}^\infty(\pi_x(V) \times W(z_0))$, such that

$$p_0(x,\xi) - z = q_0(x,\xi,z)(\xi + g_0(x,z)) \quad \text{for all } (x,\xi,z) \in V \times W(z_0), \tag{3.3}$$

and $q_0(x(z_0), \xi(z_0), z_0) \neq 0$, while $g_0(x(z_0), z_0) = -\xi(z_0)$. We can suppose that $q_0 \neq 0$ in $V \times W(z_0)$ by possibly shrinking $V$ and $W(z_0)$. Up to shrinking $W(z_0)$ we may also assume that $\rho(W(z_0)) \subset V$, so that $g_0(x(z), z) = -\xi(z)$ for all $z \in W(z_0)$.

Next, we make the formal Ansatz

$$q(x,\xi,z;h) = \sum_{k\geq 0} h^k q_k(x,\xi,z), \qquad g(x,z;h) = \sum_{k\geq 0} h^k g_k(x,z),$$

and group together the terms of the Moyal product

$$q(x,\xi,z;h) \# (\xi + g(x,z;h)) = \mathrm{e}^{\frac{ih}{2}(D_\xi D_y - D_x D_\eta)} q(x,\xi,z;h)(\eta + g(y,z;h))|_{\substack{y=x\\ \eta=\xi}}$$

with the same power of $h$. The symbols $q_k$, $g_k$ can then be computed by induction. For $N \geq 1$, assume we already know $q_k, g_k$ for $0 \leq k < N$. Equating the coefficient of $h^N$ of the above asymptotic expansion to the symbol $p_N$, we obtain

$$G_N(x,\xi,z) = \frac{q_N(x,\xi,z)}{q_0(x,\xi,z)}(\xi + g_0(x,z)) + g_N(x,z), \tag{3.4}$$

where

$$G_N(x,\xi,z) = \frac{1}{q_0(x,\xi,z)}\Bigg(p_N(x,\xi) - \sum_{l=1}^{N-1} q_{N-l}(x,\xi,z) g_l(x,z) \\ + \sum_{k=0}^{N-1}\left(\frac{i}{2}(D_\xi D_y - D_x D_\eta)\right)^{N-k}\left(q_k(x,\xi,z)\eta + \sum_{r=0}^{k} q_{k-r}(x,\xi,z) g_r(y,z)\right)\Bigg)\Bigg|_{\substack{y=x\\ \eta=\xi}}. \tag{3.5}$$

Notice that $G_N$ only depends on $q_k$, $g_k$, for $k < N$, and on $p_N$. We can then determine the functions $q_N$ and $g_N$: since

$$\xi(z) + g_0(x(z), z) = 0, \qquad \partial_\xi(\xi + g_0(x,z)) = 1,$$

the Malgrange preparation theorem implies the existence of smooth functions $q_N/q_0$ and $g_N$ in $V \times W(z_0)$ satisfying (3.4). This way we obtain all symbols $q_k \in \mathcal{C}^\infty(V \times W(z_0))$, $g_k \in \mathcal{C}^\infty(\pi_x(V) \times W(z_0))$, $k \geq 1$, which allows us to construct full symbols $q(x,\xi,h)$ and $g(x,h)$ by Borel summation, which satisfy (3.1) and (3.2). □

### *3.2. Almost holomorphic extensions*

We did not assume the symbol $p(x,\xi,h)$ to be real analytic in the variables $x,\xi$, so the functions $q(\rho, z)$ and $g(x,z)$ constructed in Proposition 3.1 are, a priori, not holomorphic in $z$. Yet, we show below that they are *almost holomorphic* near the classical energy shell.

We begin by recalling the notion of almost holomorphic extension of a smooth function. It has been introduced by Hörmander [28] and Nirenberg [37] in different contexts.

**Definition 3.2.** Let $X \subset \mathbb{C}^n$ be an open set and let $\Gamma \subset X$ be closed. If $f \in \mathcal{C}^\infty(X)$, we say that $f$ is *almost holomorphic on* $\Gamma$ if $\partial_{\bar{z}} f$ vanishes to infinite order there, i.e. for any $N \in \mathbb{N}$ there exists a constant $C_N > 0$ such that for all $z$ in a small neighbourhood of $\Gamma$ in $X$,

$$|\partial_{\bar{z}} f(z)| \leq C_N \operatorname{dist}(z, \Gamma)^N.$$

In this case we write $\partial_{\bar{z}} f(z) = \mathcal{O}(\operatorname{dist}(z, \Gamma)^\infty)$. In the case $\Gamma = X \cap \mathbb{R}^n$, we simply say that $f$ is almost holomorphic.

If $f, g \in \mathcal{C}^\infty(X)$ are almost holomorphic at $\Gamma$ and if $f - g$ vanishes to infinite order there, then we say that $f$ and $g$ are equivalent at $\Gamma$. If $\Gamma = X \cap \mathbb{R}^n$, then we simply say that $f$ and $g$ are equivalent and we write $f \sim g$.

Any $f \in \mathcal{C}^\infty(X \cap \mathbb{R}^n)$ admits an almost holomorphic extension, uniquely determined up to equivalence (see e.g. [28, 35]). Before we continue, we recall parts of a technical lemma from [35, Lemma 1.5].

**Lemma 3.3.** *Let $\Omega \subset \mathbb{R}^d$ be an open set and suppose that $u \in \mathcal{C}^\infty(\Omega)$. Let $v(x)$ be a Lipschitz continuous function on $\Omega$. Suppose that for any $\Omega' \Subset \Omega$ and all $N \in \mathbb{N}$, we have*

$$|u(x)| \leq C_{N,\Omega'} |v(x)|^N, \qquad x \in \Omega'.$$

*Then for any $\Omega' \Subset \Omega$, any multi-index $\alpha \in \mathbb{N}^d$ and any $N \in \mathbb{N}$, there is a constant $C_{N,\Omega',\alpha}$ such that*

$$|v(x)|^{|\alpha|} |D_x^\alpha u(x)| \leq C_{N,\Omega',\alpha} |v(x)|^N, \qquad x \in \Omega'.$$

We will use this lemma in the construction of almost holomorphic extensions of the function $g_0(x, z)$ from Proposition 3.1.

**Lemma 3.4.** *Under the hypotheses and notations of Proposition 3.1, let $\widetilde{g}_0^{\pm,j}$ be an almost $x$-holomorphic extension of $g_0^{\pm,j}$ for $j = 1, \dots, J$. Then there exists an open bounded neighbourhood $W(z_0)$ of $z_0$, open bounded sets $X_\pm^j \subset \mathbb{R}$ and small complex neighbourhoods $\widetilde{X}_\pm^j$ of $X_\pm^j$ such that $x_\pm^j(\overline{W(z_0)}) \subset X_\pm^j$, and such for any $N \in \mathbb{N}$ and any $\alpha, \beta, \gamma \in \mathbb{N}$, there exists a constant $C_N^{\alpha,\beta,\gamma} > 0$ such that*

$$|\partial_x^\alpha \partial_z^\beta \partial_{\bar{z}}^{\gamma+1} \widetilde{g}_0^{\pm,j}(x, z)| \leq C_N^{\alpha,\beta,\gamma} |x - x_\pm^j(z)|^N, \qquad z \in W(z_0),\ x \in \widetilde{X}_\pm^j. \tag{3.6}$$

*Moreover, for any higher order symbol $g_k^{\pm,j}$, $k \in \mathbb{N}$, for any $\alpha \in \mathbb{N}$ and any $N \in \mathbb{N}$, there exists a constant $C_{N,\alpha,k,j}^\pm > 0$ such that*

$$|\partial_x^\alpha \partial_{\bar{z}}\, g_k^{\pm,j}(x, z)| \leq C_{N,\alpha,k,j}^\pm |x - x_\pm^j(z)|^N, \qquad x \in X_\pm^j,\ z \in W(z_0). \tag{3.7}$$

In particular (3.6) implies

$$|\partial_{\bar z} g_0^{\pm,j}(x,z)| \le C_N |x - x_\pm^j(z)|^N, \qquad x \in X_\pm^j,\ z \in W(z_0). \tag{3.8}$$

*Proof of Lemma 3.4.* Again, we focus only on the "+" case and omit the superscripts $j$ and $+$. Let $q_0(x,\xi,z) \in \mathcal{C}^\infty(V \times W(z_0))$ and $g_0(x,z) \in \mathcal{C}^\infty(\pi_x(V) \times W(z_0))$ be as in Proposition 3.1, with $V$ an open bounded set containing $\rho_+(\overline{W(z_0)})$. As in the proof of that proposition, we may suppose that $q_0(x,\xi,z) \neq 0$ in $V \times W(z_0)$.

For a small bounded complex neighbourhood $\widetilde V$ of $V$, we construct an almost $(x,\xi)$-holomorphic extension $\widetilde q_0(x,\xi,z)$ of $q_0(x,\xi,z)$ defined on $\widetilde V \times W(z_0)$ such that

$$\widetilde q_0(x,\xi,z) \neq 0, \qquad (x,\xi,z) \in \widetilde V \times W(z_0). \tag{3.9}$$

Although $z$ appears as a parameter in the construction, we can ensure that $\widetilde q_0$ is smooth in $z$, and that any function $\partial_z^\alpha \partial_{\bar z}^\beta \widetilde q_0$ is an almost $(x,\xi)$-holomorphic extension of $\partial_z^\alpha \partial_{\bar z}^\beta q_0$.

Similarly, we construct, for $x \in \pi_x(\widetilde V)$, an almost $x$-holomorphic extension $\widetilde g_0(x,z)$ of $g_0(x,z)$. Using these two functions to extend the equation (3.3), we obtain the function

$$\widetilde p_0(x,\xi,z) := \widetilde q_0(x,\xi,z)(\xi + \widetilde g_0(x,z)), \qquad (x,\xi,z) \in \widetilde V \times W(z_0). \tag{3.10}$$

$\widetilde p_0(x,\xi,z)$ is naturally an almost $(x,\xi)$-holomorphic extension of $p_0(x,\xi) - z$. It is also the case of its $\partial_{\bar z}$ derivative, so that

$$\partial_{\bar z} \widetilde p_0(x,\xi,z) = \mathcal{O}(|\mathrm{Im}\,(x,\xi)|^\infty). \tag{3.11}$$

By (HYP), we have $\partial_\xi p_0(\rho_+(z)) \neq 0$ for all $z \in W(z_0)$. By possibly shrinking $V$, $W(z_0)$ and $\widetilde V$ we can arrange that $\rho_+(\overline{W(z_0)}) \subset V$ and $\partial_\xi \widetilde p_0(x,\xi,z) \neq 0$ for all $(x,\xi,z) \in \widetilde V \times W(z_0)$.

Recall from Proposition 3.1 that $g_0(x_+(z),z) = -\xi_+(z)$ for all $z \in W(z_0)$. Hence, by possibly shrinking $V$ and $W(z_0)$ and by restricting $\widetilde g_0(\cdot,z)$ to an open bounded convex complex neighbourhood $\widetilde X$ of $X := \pi_x(V) \subset \mathbb{R}$ with $\widetilde X \Subset \pi_x(\widetilde V)$, we can arrange that $x_+(\overline{W(z_0)}) \subset X$ and $(x, -\widetilde g_0(x,z)) \in \widetilde V$ for all $x \in \widetilde X$.

Taking $\xi = -\widetilde g_0(x,z)$ in (3.10) and then taking the $\partial_{\bar z}$ derivative of that equation, taking into account (3.11), we get, for all $x \in \widetilde X$ and all $z \in W(z_0)$,

$$\partial_\xi \widetilde p_0(x, -\widetilde g_0(x,z)) \partial_{\bar z} \widetilde g_0(x,z) + \partial_{\bar\xi} \widetilde p_0(x, -\widetilde g_0(x,z)) \overline{\partial_z \widetilde g_0(x,z)} = \mathcal{O}(|\mathrm{Im}\,(x,\xi)|^\infty).$$

Since $\widetilde p_0$ is almost $(x,\xi)$-holomorphic, while $\partial_\xi \widetilde p_0$ is bounded away from zero, we find, for any $N \in \mathbb{N}$,

$$|\partial_{\bar z} \widetilde g_0(x,z)| \le C_N(|\mathrm{Im}\, \widetilde g_0(x,z)|^N + |\mathrm{Im}\, x|^N), \qquad x \in \widetilde X,\ z \in W(z_0).$$

Since $\widetilde g_0(x_+(z),z) = -\xi_+(z) \in \mathbb{R}$, one has $\mathrm{Im}\,\widetilde g_0(x_+(z),z) = 0$; and since $\widetilde g_0$ is a bounded smooth function on $\overline{\widetilde X} \times \overline{W(z_0)}$, it follows by Taylor expansion that

$$|\partial_{\bar z} \widetilde g_0(x,z)| \le C'_N |x - x_+(z)|^N, \qquad x \in \widetilde X,\ z \in W(z_0). \tag{3.12}$$

This proves (3.6) in the case $\alpha = \beta = \gamma = 0$. Now, observe that since $x - x_+(z)$ is a smooth function of $(x, z)$, it follows by Lemma 3.3 that after slightly shrinking $\widetilde{X}$ and $W(z_0)$, for any $\alpha, \beta, \gamma \in \mathbb{N}$,

$$|\partial_x^\alpha \partial_z^\beta \partial_{\bar{z}}^{\gamma+1} \widetilde{g}_0(x, z)| \leq C_N^{\alpha,\beta,\gamma} |x - x_+(z)|^N. \tag{3.13}$$

In particular, restricting $x$ to the value $x_+(z) \in X_+$, we find

$$\forall \alpha, \beta, \gamma \in \mathbb{N}, \quad (\partial_x^\alpha \partial_z^\beta \partial_{\bar{z}}^{\gamma+1} g_0)(x_+(z), z) = 0, \quad z \in W(z_0). \tag{3.14}$$

Next, using (3.3) and $g_0(x_+(z), z) = -\xi_+(z)$ (see Proposition 3.1), we deduce by a direct computation of $\{\bar{p}_0, p_0\}(\rho_+(z))$ that

$$\mathrm{Im}(\partial_x g_0)(x_+(z), z) = \frac{\{\bar{p}_0, p_0\}(\rho_+(z))}{2i |q_0(\rho_+(z), z)|^2} < 0, \tag{3.15}$$

where the last inequality is a consequence of (HYP).

Finally, by differentiating (3.10) with respect to $x$ and $\bar{z}$ and by evaluating it at the point $(\rho_+(z), z)$ we find $(\partial_{\bar{z}} \widetilde{q}_0)(\rho_+(z), z) = 0$. By repeated differentiation of (3.10) and using the Leibniz rule, this generalizes to

$$\forall \eta \in \mathbb{N}^2, \quad \partial_\rho^\eta \partial_{\bar{z}} \widetilde{q}_0(\rho, z)\restriction_{\rho=\rho_+(z)} = 0. \tag{3.16}$$

Let us now consider higher order symbols. We recall that for any $N > 0$, the symbols $q_N \in \mathcal{C}^\infty(V \times W(z_0))$ and $g_N \in \mathcal{C}^\infty(\pi_x(V) \times W(z_0))$ were constructed by solving the equation (3.4), with $G_N$ defined in (3.5) in terms of lower order symbols. We will also construct almost holomorphic extensions of these symbols iteratively.

Assume that for all $0 \leq k < N$, we have extended $q_k$ (resp. $g_k$) to an almost $\rho$-holomorphic function $\widetilde{q}_k$ (resp. almost $x$-holomorphic function $\widetilde{g}_k$). Injecting these extensions on the right hand side (3.5) (the derivatives $D_x$, $D_\xi$ etc. being now understood as holomorphic derivatives) defines $\widetilde{G}_N$, an almost holomorphic extension of $G_N$. Then we extend $q_N$ to $\widetilde{q}_N$. Injecting in (3.4) $\widetilde{G}_N$, $\widetilde{q}_N$ and the previously defined extensions $\widetilde{q}_0$, $\widetilde{g}_0$ ends up with the definition of $\widetilde{g}_N$, which almost holomorphically extends $g_N$.

Let us show by induction that the symbols satisfy

$$\forall \alpha \in \mathbb{N}, \quad \partial_x^\alpha \partial_{\bar{z}} \widetilde{g}_N(x, z)\restriction_{x=x_+(z)} = 0, \tag{3.17}$$

$$\forall \eta \in \mathbb{N}^2, \quad \partial_\rho^\eta \partial_{\bar{z}} \widetilde{q}_N(\rho, z)\restriction_{\rho=\rho_+(z)} = 0. \tag{3.18}$$

We already know from (3.13) and (3.16) that this is the case at the level $N = 0$. Differentiating the equation (3.5) defining $\widetilde{G}_N$ with respect to $\bar{z}$ and $\rho$, one finds that

$$\forall \eta \in \mathbb{N}^2, \quad \partial_\rho^\eta \partial_{\bar{z}} \widetilde{G}_N(\rho, z)\restriction_{\rho=\rho_+(z)} = 0. \tag{3.19}$$

As we did before, taking $\xi = -\widetilde{g}_0(x, z)$ in (3.4), differentiating with respect to $\bar{z}$ and $x$, and evaluating the expression at $x = x_+(z)$, results in the following identities:

$$\forall \alpha \in \mathbb{N}, \quad \partial_x^\alpha \partial_{\bar{z}} \widetilde{G}_N(\rho, z)\restriction_{\rho=\rho_+(z)} = \partial_x^\alpha \partial_{\bar{z}} \widetilde{g}_N(x, z)\restriction_{x=x_+(z)}.$$

Then (3.19) shows that the above expression vanishes, proving (3.17).

Let us now treat the symbol $\widetilde{q}_N$. Differentiating the right hand side of (3.4) with respect to $\bar{z}$ and $\rho$, and evaluating at $\rho = \rho_+(z)$, we find, using (3.19) and (3.17),

$$\partial_\rho^\eta \partial_{\bar{z}} \left( \frac{\widetilde{q}_N(x,\xi,z)}{\widetilde{q}_0(x,\xi,z)} (\xi + \widetilde{g}_0(x,z)) \right) \restriction_{\rho=\rho_+(z)} = 0.$$

Our knowledge of $\widetilde{g}_0$ and $\widetilde{q}_0$ at the point $\rho_+(z)$ implies the desired equation (3.18).

Using a Taylor expansion near $x = x_+(z)$ and taking into account the $x$-almost holomorpy of $\partial_{\bar{z}} g_N(x,z)$, the equation (3.17) leads to the property

$$\forall \alpha \in \mathbb{N}, \quad \partial_x^\alpha \partial_{\bar{z}} \widetilde{g}_N(x,z) = \mathcal{O}(|x - x_+(z)|^\infty), \quad x \in \widetilde{X},\ z \in W(z_0).$$

When restricting to real arguments $x \in X$, this gives the required equation (3.7). □

### *3.3. Construction of the quasimodes*

From now on we will always assume that the symbol $p_0$ satisfies the hypothesis (HYP) (each energy shell $p^{-1}(z)$ consists of $2J$ different points). Using the factorization of Proposition 3.1, we will construct quasimodes for the operators $P_h - z$ and $(P_h - z)^*$ following the WKB method.

**Proposition 3.5** (Almost holomorphic quasimodes). *Let $p(\cdot;h) \in S(\mathbb{R}^2, m)$ be as in (2.6) and satisfy (HYP). Let $\Omega \Subset \mathbb{C}$ be as in (2.11), and take $z_0 \in \Omega$. Let $W(z_0)$ and $X_\pm^j$, $j = 1,\dots,J$, be as in Proposition 3.1. Let $\chi_\pm^j \in \mathcal{C}_0^\infty(X_\pm^j, [0,1])$ be such that $\chi_\pm^j \equiv 1$ in a small neighbourhood of $x_\pm^j(\overline{W(z_0)})$. Then there exist functions*

$$\begin{aligned} e_\pm^{j,\mathrm{hol}}(x,z;h) &= a_\pm^j(x,z;h)\, \chi_\pm^j(x) \mathrm{e}^{\frac{i}{h}\varphi_\pm(x,z)}, \quad x \in \mathbb{R},\ z \in W(z_0), \\ \varphi_+^j(x,z) &= -\int_{x_+^j(z_0)}^x g_0^{j,+}(y,z)dy, \quad \varphi_-^j(x,z) = -\int_{x_-^j(z_0)}^x \overline{g_0^{j,-}(y,z)}\,dy. \end{aligned} \tag{3.20}$$

*Here $g_0^{\pm,j}$ are as in Proposition 3.1, depending smoothly on $x \in X_\pm^j$ and $z \in W(z_0)$. The symbol $a_\pm^j(x,z;h) \sim (a_0^{\pm,j}(x,z) + h a_1^{\pm,j}(x,z) + \cdots)$ is also smooth in $x, z$, with all derivatives uniformly bounded as $h \to 0$.*

*Moreover, the states $e_\pm^{j,\mathrm{hol}}$ have the following properties:*

(1) *Their $L^2$ norms satisfy*

$$\begin{aligned} \|e_\pm^{j,\mathrm{hol}}(z;h)\| &= \mathrm{e}^{\frac{1}{h}\Phi_\pm^j(z;h)}, \\ \Phi_\pm^j(z;h) &= -\operatorname{Im}\varphi_\pm^j(x_\pm^j(z),z) + h\log(h^{1/4}A_\pm^j(z;h)), \end{aligned} \tag{3.21}$$

*with $\operatorname{Im}\varphi_\pm^j(x_\pm^j(z),z) \leq 0$, with equality iff $z = z_0$, and $A_\pm^j(z;h) \sim A_0^{j,\pm}(z) + hA_1^{j,\pm}(z) + \cdots$ depending smoothly on $z$ such that all derivatives with respect to $z, \bar{z}$ are bounded when $h \to 0$.*

(2) *The states* $e_{\pm}^{j,\mathrm{hol}}$ *are almost* $z$*-holomorphic (resp. almost* $z$*-anti-holomorphic) at* $z_0$*, in the sense that*

$$\begin{aligned} \mathrm{e}^{-\frac{1}{h}\Phi_{+}^{j}(z;h)}\|\partial_{\bar z} e_{+}^{j,\mathrm{hol}}(z;h)\| &= \mathcal{O}(h^{-1}|z-z_0|^{\infty}+h^{\infty}), \quad z\in W(z_0),\\ \mathrm{e}^{-\frac{1}{h}\Phi_{-}^{j}(z;h)}\|\partial_{z} e_{-}^{j,\mathrm{hol}}(z;h)\| &= \mathcal{O}(h^{-1}|z-z_0|^{\infty}+h^{\infty}), \quad z\in W(z_0). \end{aligned} \tag{3.22}$$

(3) *The corresponding normalized states,*

$$e_{\pm}^{j}(x,z;h) := e_{\pm}^{j,\mathrm{hol}}(x,z;h)\mathrm{e}^{-\frac{1}{h}\Phi_{\pm}^{j}(z;h)}, \tag{3.23}$$

*are* $h^{\infty}$*-quasimodes of* $P_h - z$*, resp.* $(P_h - z)^*$*:*

$$\|(P_h - z)e_{+}^{j}(z;h)\| = \mathcal{O}(h^{\infty}), \qquad \|(P_h - z)^* e_{-}^{j}(z;h)\| = \mathcal{O}(h^{\infty}). \tag{3.24}$$

(4) *For any cutoffs* $\psi_{\pm}^{j} \in \mathcal{C}_0^{\infty}(\mathbb{R}^2,[0,1])$ *such that* $\psi_{\pm}^{j} \equiv 1$ *near* $\rho_{\pm}^{j}(W(z_0))$*, and any order function* $m'$*,*

$$\|(1-(\psi_{\pm}^{j})^{w})e_{\pm}^{j}(z;h)\|_{H(m')} = \mathcal{O}(h^{\infty}), \qquad j=1,\dots,J, \qquad z\in W(z_0). \tag{3.25}$$

*In all equations above, the* $\mathcal{O}(h^{\infty})$ *remainders are uniform in* $z \in W(z_0)$*.*

For future use, we intentionally introduced two versions of quasimodes: the normalized ones $e_{+}^{j}(z;h)$, and the almost holomorphic ones $e_{\pm}^{j,\mathrm{hol}}(z;h)$.

*Proof of Proposition 3.5.* We will give the proof only in the "+" case, since the "−" case is similar. We will suppress the superscript $j$ until further notice. We begin with the following result:

**Lemma 3.6.** *Let* $p(\cdot;h) \in S^2(\mathbb{R}^2, m)$ *be as in (2.6) and satisfy (HYP). Let* $\Omega \Subset \mathbb{C}$ *be as in (2.11), and let* $z_0 \in \Omega$*. Let* $W(z_0)$ *and* $X_+$ *with* $x_+(\overline{W(z_0)}) \subset X_+$ *be as in Proposition 3.1. Let* $g^+(x,z;h)$ *be the symbol constructed in Proposition 3.1. Then the equation*

$$(hD_x + g^+(x))f_+(x,z;h) = 0, \qquad (x,z)\in X_+\times W(z_0), \tag{3.26}$$

*admits a solution* $f_+^{\mathrm{hol}}(x,z;h)$ *of the form*

$$\begin{aligned} f_+^{\mathrm{hol}}(x,z;h) &= a_+(x,z;h)\mathrm{e}^{\frac{i}{h}\varphi_+(x,z)}, \qquad (x,z)\in X_+\times W(z_0),\\ \text{with}\quad \varphi_+(x) &= -\int_{x_+(z_0)}^{x} g_0^+(y,z)\,dy. \end{aligned} \tag{3.27}$$

*The symbol* $a_+(x,z;h) \sim a_0^+(x,z) + ha_1^+(x,z) + \cdots$ *depends smoothly on* $x$ *and* $z$*, with all derivatives bounded as* $h \to 0$*. Moreover, for all* $z \in W(z_0)$ *and any* $\alpha \in \mathbb{N}$*,*

$$(\partial_x^{\alpha}\partial_{\bar z}a^+)(x_+(z),z;h) = \mathcal{O}(|z-z_0|^{\infty}+h^{\infty}). \tag{3.28}$$

*Proof.* For any $h \in ]0,1]$ and $z \in W(z_0)$, the first order equation (3.26) can be easily solved by the Ansatz

$$f_+^{\mathrm{hol}}(x,z;h) := \exp\Big(-\frac{i}{h}\int_{x_0}^x g^+(y,z;h)\,dy\Big),$$

where we choose to take the reference point $x_0 = x_+(z_0)$ independent of $z$. Taking into account the expansion (3.1) of the symbol $g^+$, its primitive may be expanded as

$$-\int_{x_0}^x g^+(y,z;h)\,dy \sim \varphi_0^+(x,z) + h\varphi_1^+(x,z) + \cdots, \qquad \varphi_k^+(x,z) := -\int_{x_0}^x g_k^+(y,z)\,dy.$$

Separating the first term $\varphi_+ = \varphi_0^+$ from the subsequent ones, we may write

$$f_+^{\mathrm{hol}}(x,z;h) = \mathrm{e}^{\frac{i}{h}\varphi_+(x,z)}\, a^+(x,z;h)$$

with $a^+(h) \in C^\infty(X_+ \times W(z_0))$ admitting an expansion

$$a^+(h) \sim a_0^+ + ha_1^+ + h^2 a_2^+ + \cdots.$$

Each term $a_j^+ \in \mathcal{C}^\infty(X_+ \times W(z_0))$ depends on the functions $\{\varphi_k^+;\ 1 \le k \le j+1\}$.

Alternatively, the expansion $\sum h^j a_j^+$ can be constructed order by order through a WKB construction (see e.g. [12]): one iteratively solves the *transport equations*

$$-i\partial_x a_n^+(x,z) + \sum_{k=0}^{n} g_{n+1-k}^+(x,z)a_k^+(x,z) = 0, \qquad n \ge 0,$$

by the expressions

$$\begin{aligned} a_0^+(x,z) &= \mathrm{e}^{-i\int_{x_0}^x g_1^+(y,z;h)\,dy}, \\ a_n^+(x,z) &= -i\, a_0^+(x,z)\int_{x_0}^x \sum_{k=0}^{n-1} \frac{g_{n+1-k}^+ a_k^+}{a_0^+}(y,z)\,dy. \end{aligned} \tag{3.29}$$

Let us make some remarks about the phase function $\varphi_+$. It is the unique solution to the *eikonal equation* $\partial_x\varphi_+(x,z) + g_0^+(x,z) = 0$, satisfying the boundary condition $\varphi_+(x_0,z) = 0$. By Proposition 3.1, $\partial_x\varphi_+(x_+(z);z) = \xi_+(z) \in \mathbb{R}$, therefore $x_+(z)$ is a critical point of $\operatorname{Im}\varphi_+$. Furthermore, by (3.15),

$$\operatorname{Im}\partial_x^2\varphi_+(x_+(z),z) = -\operatorname{Im}(\partial_x g_0^+)(x_+(z),z) > 0, \tag{3.30}$$

showing that $x_+(z)$ is a nondegenerate critical point of $\operatorname{Im}\varphi_+(\cdot,z)$. By possibly shrinking $W(z_0)$ and $X_+$, we can arrange that $x_+(\overline{W(z_0)}) \subset X_+$ and that (3.30) holds for all $x \in X_+$, so that $x_+(z)$ is the unique critical point of $\operatorname{Im}\varphi_+(\cdot,z)$ in $X_+$.

By iteratively differentiating the equations (3.29) and using the estimates (3.7) on $g_k$, we find that for any $n \in \mathbb{N}$ and any $\alpha \in \mathbb{N}$,

$$(\partial_x^\alpha \partial_{\bar z} a_n^+)(x,z) = \mathcal{O}(|x - x_+(z)|^\infty + |x_+(z_0) - x_+(z)|^\infty), \qquad x \in X_+,\ z \in W(z_0).$$

By Taylor expanding $x_+(z)$ around $z = z_0$, at the critical points the above estimate becomes

$$(\partial_x^\alpha \partial_{\bar z} a_n^+)(x_+(z), z) = \mathcal{O}(|z - z_0|^\infty).$$

Summing over all the symbols, we finally control the full symbol, as in (3.28). □

**Remark 3.7.** In the "−" case we construct a solution for $(hD_x + \overline{g^-(x,z)}) f_- = 0$. Hence, the phase function reads $\varphi_-(x;z) = -\int_{x_-(z_0)}^x \overline{g_0^-}(y,z)\,dy$. Moreover, the transport equations depend on $\overline{g_0^-}(y,z)$, which is almost anti-holomorphic in $z$ at the point $(x_+(z), z)$. Hence, for any $\alpha \in \mathbb{N}$ we obtain $(\partial_x^\alpha \partial_z a^-)(x_-(z), z; h) = \mathcal{O}(|z-z_0|^\infty + h^\infty)$.

Let us now proceed with the proof of Proposition 3.5. Let $\chi_+ \in \mathcal{C}_0^\infty(X_+, [0,1])$ be such that $\chi_+ \equiv 1$ in a small neighbourhood of $x_+(W(z_0))$. We define the function

$$e_+^{\text{hol}}(x,z) := \chi_+(x)\, f_+^{\text{hol}}(x,z;h), \qquad (x,z) \in X_+ \times W(z_0),$$

smoothly extended by $e_+^{\text{hol}}(x,z) = 0$ for $x \in \mathbb{R} \setminus X_+$. Recall from (3.30) and the discussion afterwards that $x_+(z)$ is the unique critical point of $\operatorname{Im}\varphi(\cdot, z)$ in $X_+$, and is a nondegenerate minimum point. In particular $\operatorname{Im}\varphi_+(x,z) - \operatorname{Im}\varphi_+(x_+(z),z) \geq 0$ for all $x \in X_+$, with a strict inequality for $x \neq x_+(z)$. Hence, applying the method of stationary phase, we find that

$$\|e_+^{\text{hol}}\| = h^{1/4} A_+(z;h) \mathrm{e}^{\frac{1}{h}\Phi_{+,0}(z)} =: \mathrm{e}^{\frac{1}{h}\Phi_+(z;h)} \tag{3.31}$$

with the phase function

$$\Phi_{+,0}(z) = -\operatorname{Im}\varphi_+(x_+(z), z)$$

and a symbol $A_+(z;h) \sim A_0^+(z) + hA_1^+(z) + \cdots$ depending smoothly on $z$, with all derivatives in $z, \bar z$ uniformly bounded when $h \to 0$. The principal symbol

$$A_0^+(z) = \left( \frac{\pi |a_0^+(x_+(z), z)|^2}{\operatorname{Im}\partial_x^2 \varphi_+(x_+(z), z)} \right)^{1/4} > 0. \tag{3.32}$$

It follows from (3.30) and the property $\varphi_+(x_+(z_0), z) = 0$ that $\Phi_{+,0}(z) \geq 0$ for any $z \in W(z_0)$, with equality precisely when $z = z_0$. Hence, for points such that $|z - z_0| \geq 1/C$, the norms of the states $e_+^{\text{hol}}(z;h)$ are exponentially large.

Let us now prove that the state $e_+^{\text{hol}}(z;h)$ is almost $z$-holomorphic at $z_0$. Using the equations (3.7), (3.20) and a Taylor expansion of $x(z)$ at $z_0$, we easily obtain

$$\forall \alpha \in \mathbb{N}, \quad (\partial_x^\alpha \partial_{\bar z}\varphi)(x(z), z) = \mathcal{O}(|x(z) - x(z_0)|^\infty) = \mathcal{O}(|z - z_0|^\infty). \tag{3.33}$$

By (3.30) and the ensuing discussion, $x_+(z)$ is the unique minimum point of $x \mapsto \operatorname{Im}\varphi(x,z)$ on the support of $\chi_+$ and it is nondegenerate by (3.30). Hence, by combining (3.31), (3.33) and (3.28), we get

$$\begin{aligned} \mathrm{e}^{-\frac{1}{h}\Phi_+} \|\partial_{\bar z} e_+^{\text{hol}}\| &\leq \mathrm{e}^{-\frac{1}{h}\Phi_+} \|(\partial_{\bar z} a_+)\chi_+ \mathrm{e}^{\frac{i}{h}\varphi_+}\| + \mathrm{e}^{-\frac{1}{h}\Phi_+} \|a_+\chi_+(ih^{-1}\partial_{\bar z}\varphi_+)\mathrm{e}^{\frac{i}{h}\varphi_+}\| \\ &= \mathcal{O}(h^{-1}|z - z_0|^\infty + h^\infty), \end{aligned} \tag{3.34}$$

which is the estimate (3.22) of Proposition 3.5.

Let us now check point (3) of the proposition. Using Lemma 3.6, we have

$$(hD_x + g^+)\chi_+ f_+^{\mathrm{hol}} = [hD_x, \chi_+] f_+^{\mathrm{hol}}. \tag{3.35}$$

Using $x_+(z) \notin \operatorname{supp} \chi'_+$ and the fact that $-\operatorname{Im} \varphi_+(x, z)$ reaches its maximum only at $x = x_+(z)$, we find the norm estimate

$$\|[hD_x, \chi_+] f_+^{\mathrm{hol}}\| = \mathcal{O}(\mathrm{e}^{-1/Ch})\mathrm{e}^{\frac{1}{h}\Phi_{+,0}(z)},$$

which shows that

$$\|(hD_x + g^+(z))e_+^{\mathrm{hol}}(z)\| = \mathcal{O}(\mathrm{e}^{-C/h})\|e_+^{\mathrm{hol}}(z)\|. \tag{3.36}$$

Next, using (3.31), we define the $L^2$ normalized state as in (3.23): $e_+(x, z; h) := e_+^{\mathrm{hol}}(x, z; h)\mathrm{e}^{-\frac{1}{h}\Phi_+(z;h)}$. This state is in $\mathcal{C}_c^\infty(X_+)$, and depends smoothly on $z \in W(z_0)$. It microlocalizes at the point $\rho_+(z)$, as shown in the next

**Lemma 3.8.** *Under the assumptions of Lemma 3.6, take any order function $\widetilde{m}$, an arbitrary point $z \in W(z_0)$, and choose a cutoff function $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^2, [0, 1])$ such that $\psi = 1$ near $\rho_+(z)$. Then*

$$\|(1 - \psi)^w e_+(z; h)\|_{H(\widetilde{m})} = \mathcal{O}(h^\infty). \tag{3.37}$$

*Proof.* We smoothly extend $g_+ \in \mathcal{C}^\infty(X_+)$ to[5] $\widetilde{g}_+ \in \mathcal{C}^\infty(\mathbb{R})$ such that $\widetilde{g}_+(x, z; h) = -\frac{i}{C}(x - x_+(z))$ for $|x| \geq C$, $C > 0$ large enough, and such that $\xi + \widetilde{g}_+ \in S(\mathbb{R}^2, \langle\rho\rangle)$ is elliptic outside $\rho_+(z)$. This is possible due to the monotonicity property (3.30). Thus, $hD_x + \widetilde{g}_+ = (\xi + \widetilde{g}_+)^w$.

Since $\operatorname{supp} \chi_+ \subset X_+$, we have $\widetilde{g}_+ = g_+$ on the support of $e_+(z; h)$. Hence, by (3.35),

$$\begin{aligned} \eta &:= (hD_x + \widetilde{g}^+)^*(hD_x + \widetilde{g}^+)e_+, \\ \eta &= \mathrm{e}^{-\frac{1}{h}\Phi_+(z;h)}\big([hD_x, [hD_x, \chi_+]] f_+^{\mathrm{hol}} - 2i \operatorname{Im} \widetilde{g}^+ [hD_x, \chi_+] f_+^{\mathrm{hol}}\big). \end{aligned} \tag{3.38}$$

We see that for any $\alpha, \beta \in \mathbb{N}$,

$$x^\alpha \partial_x^\beta \eta(x, z; h) = d_{\alpha,\beta}(x, z; h)h^{-n}\mathrm{e}^{-\frac{1}{h}\Phi_+(z;h)}\mathrm{e}^{\frac{i}{h}\varphi_+(x,z)}$$

for some symbol $d_{\alpha,\beta}(x, z; h) \sim d^0_{\alpha,\beta}(x, z) + hd^1_{\alpha,\beta}(x, z) + \cdots$ depending smoothly on $x$ and $z$, with all derivatives bounded as $h \to 0$. Moreover, its support with respect to $x$ is contained in $\operatorname{supp} \partial\chi_+$. Since $x_+(z)$ is not in this support, and since $-\operatorname{Im} \varphi_+(x, z)$ reaches its maximum only at $x = x_+(z)$ (cf. (3.30)), we deduce that for any order function $\widetilde{m}$, $\|\eta\|_{H(\widetilde{m})} = \mathcal{O}(\mathrm{e}^{-C/h})$.

Taking into account the cutoff $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^2, [0, 1])$ of the lemma, we choose a "thinner" cutoff $\widetilde{\chi} \in \mathcal{C}_0^\infty(\mathbb{R}^2, [0, 1])$ such that $\widetilde{\chi} = 1$ near $\rho_+(z)$, while $\psi = 1$ near $\operatorname{supp} \widetilde{\chi}$. We use it to define the operator

$$q^w = (hD_x + \widetilde{g}^+)^*(hD_x + \widetilde{g}^+) + \widetilde{\chi}^w. \tag{3.39}$$

[5] This notation should not be confused with the notation $\widetilde{g}_0^+$ used for the almost holomorphic extension of $g_0^+$ in Lemma 3.4.

The symbol of this operator,

$$q(x,\xi,z;h) = |\xi + \widetilde{g}_0^+(x,z)|^2 + \widetilde{\chi}(x,\xi) + \mathcal{O}(h)_{S(\mathbb{R}^2,\langle\rho\rangle^2)},$$

is elliptic in $S(\mathbb{R}^2, \langle\rho\rangle^2)$. Hence, for $h > 0$ small enough, $q^w$ admits a bounded inverse $b^w$ with $b \in S(\mathbb{R}^2, \langle\rho\rangle^{-2})$. Notice that by (3.38) and (3.39),

$$e_+ = b^w\eta + (b \# \widetilde{\chi})^w e_+.$$

We may now apply the operator $(1-\psi)^w$ to this decomposition. Since $(1-\psi)^w$ and $b^w$ are bounded on $H(\widetilde{m})$, it follows that $\|((1-\psi)\# b)^w\eta\|_{H(\widetilde{m})} = \mathcal{O}(\mathrm{e}^{-C/h})$.

Moreover, since our cutoff $\psi$ is 1 near $\operatorname{supp}\widetilde{\chi}$, pseudodifferential calculus shows that for any order function $\widetilde{m}$,

$$(1-\psi)\# b \# \widetilde{\chi} = \mathcal{O}(h^\infty)_{S(\mathbb{R}^2,\widetilde{m}^{-1})},$$

and as a consequence $\|(1-\psi)^w b^w \widetilde{\chi}^w e_+\|_{H(\widetilde{m})} = \mathcal{O}(h^\infty)$.

Adding the two contributions, we get the estimate (3.37). □

Combining Proposition 3.1 with Lemmata 3.6 and 3.8, we may use any cutoff $\psi \in C_0^\infty(\mathbb{R}^2)$ with $\psi = 1$ near $\rho_+(z)$ to show that $e_+(z)$ is a quasimode of $P_h - z$:

$$\begin{aligned}(P_h - z)e_+ &= (P_h - z)\psi^w e_+ + (P_h - z)(1-\psi)^w e_+\\ &= (\psi \# q^+ \# (\xi + g^+))^w e_+ + [P_h, \psi^w]e_+ + \mathcal{O}(h^\infty)\\ &= \mathcal{O}(h^\infty).\end{aligned} \tag{3.40}$$

Here the first term in the second line is $\mathcal{O}(h^\infty)$ by (3.36). The second term is $\mathcal{O}(h^\infty)$ by Lemma 3.8, since the wavefront set of the commutator is disjoint from a fixed neighbourhood of $\rho_+(z)$. This proves the quasimode property (3.24) of the state $e_+^j(z;h)$.

This concludes the proof of Proposition 3.5 in the "+" case. The "−" case can be proved following the same steps to construct a quasimode for $(P_h - z)^*$, using the microlocal factorization of this operator into $(q^{-,w})^*(hD_x + \overline{g^-})$ by Proposition 3.1. □

### 3.4. *Quasimodes for symmetric symbols*

Let us now assume that the symbol $p$ satisfies the symmetry property (SYM):

$$p(x,\xi;h) = p(x,-\xi;h).$$

Then the formal adjoint $(P_h - z)^*$ satisfies

$$(P_h - z)^* = \Gamma(P_h - z)\Gamma, \quad \text{where we recall that} \quad (\Gamma u)(x) := \overline{u(x)}. \tag{3.41}$$

Moreover, (SYM) implies that if $\rho = (x,\xi) \in p_0^{-1}(z)$ with $\{\operatorname{Re} p, \operatorname{Im} p\}(\rho) < 0$, as in (HYP), then $(x,-\xi) \in p_0^{-1}(z)$ as well, with $\{\operatorname{Re} p, \operatorname{Im} p\}(x,-\xi) > 0$. The hypotheses (HYP), (HYP-x) can thus be written as

$$\begin{aligned}&p_0^{-1}(z) = \{\rho_\pm^j(z) = (x_\pm^j(z), \xi_\pm^j(z)) := (x^j(z), \pm\xi^j(z));\ j = 1,\dots,J\},\ \text{with}\\ &\pm\{\operatorname{Re} p, \operatorname{Im} p\}(\rho_\pm^j(z)) < 0,\ \text{and } x^i \neq x^j \text{ if } i \neq j.\end{aligned} \tag{3.42}$$

For $j = 1, \dots, J$, let us consider the "+" quasimode $e_+^{j,\mathrm{hol}}$ constructed in Proposition 3.5. We may define the corresponding "−" quasimode as

$$e_-^{j,\mathrm{hol}}(z;h) := \Gamma e_+^{j,\mathrm{hol}}(z;h). \tag{3.43}$$

It is then clear from Proposition 3.5 that for all $z \in W(z_0)$,

$$\|e_-^{j,\mathrm{hol}}(z;h)\| = \|e_+^{j,\mathrm{hol}}(z;h)\| = \mathrm{e}^{\frac{1}{h}\Phi_+^j(z;h)}. \tag{3.44}$$

Moreover, by (3.40) and (3.41), it is obvious that the normalized state

$$e_-^j(z;h) = e_-^{j,\mathrm{hol}}(z;h)\mathrm{e}^{-\frac{1}{h}\Phi_+^j(z;h)} = \Gamma e_+^j(z;h) \tag{3.45}$$

is a quasimode of $(P_h - z)^*$:

$$\|(P_h - z)^* e_-^j(z;h)\| = \mathcal{O}(h^\infty).$$

Using the fact that complex conjugation microlocally acts as $(x, \xi) \mapsto (x, -\xi)$, we also find from Lemma 3.8 that for all $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ with $\psi = 1$ near $\rho_-^j(z)$,

$$\|(1 - \psi^w)e_-^j(z;h)\|_{H(m')} = \mathcal{O}(h^\infty). \tag{3.46}$$

Later we will have to deal with the squared states

$$(e_+^j(x,z;h))^2 = \chi_+^j(x)^2\, a_+^j(x,z;h)^2 \mathrm{e}^{\frac{2i}{h}\varphi_+(x,z)}\mathrm{e}^{-\frac{2}{h}\Phi_+(z;h)}.$$

A simple computation using (3.31) shows that the $L^2$ norm of this state is of order $Ch^{-1/4}$. Using the same method as in Lemma 3.8, one shows that this state is microlocalized at $(x_+^j(z), 2\xi_+^j(z))$.

Similarly, the squared state $(e_-^j(z;h))^2$ is microlocalized at the point $(x_-^j(z), 2\xi_-^j(z))$. Namely, for any $\psi_2 \in \mathcal{C}_0^\infty(\mathbb{R}^2, [0,1])$ with $\chi_2 = 1$ near the point $(x_-^j(z), 2\xi_-^j(z))$,

$$\|(1 - \psi_2^w)(e_-^j(z;h))^2\|_{H(m')} = \mathcal{O}(h^\infty). \tag{3.47}$$

### *3.5. Relation to symplectic volume*

In this section we will study the phase functions $\Phi_\pm^j(z;h)$ governing the $L^2$ norms of the holomorphic quasimodes (see Proposition 3.5), and show how they are connected with the symplectic volume in $T^*\mathbb{R}$. This link will be crucial when computing the universal limiting GAFs described in Theorems 2.5 and 2.9, as well as for the $k$-point correlation functions in Theorem 2.6 (see also (2.30)), and in Corollary 2.10 (see also (2.35)). Indeed, we will see in Sections 7.3 and 8.1 that the covariances of these GAFs involve the "interactions" between neighbouring quasimodes; these interactions will be studied in Section 4. We will see (for instance in Proposition 4.1) that they can be described by polarizing the phase function $\Phi_\pm^j(z;h)$. After rescaling the spectral parameter to the microscale, the dominant part of this polarization will be a term of the form $\partial^2_{z\bar z}\Phi_\pm^j(z_0;h)z\bar w$ (see (7.25) and (8.19)), which in turn is determined (3.54) by the Lebesgue density of $p_0(|d\xi \wedge dx|)$ of (2.25).

We will strongly use the almost holomorphy of the symbol $g^{\pm,j}$ expressed in Lemma 3.4. From the expression (3.21) for the phase function, let us write

$$\Phi^j_\pm(z;h) = \Phi^j_{\pm,0}(z) + h\log(h^{1/4}A^j_\pm(z;h)), \qquad \Phi^j_{\pm,0}(z) := -\operatorname{Im}\varphi^j_\pm(x^j_\pm(z), z). \tag{3.48}$$

From now on, we will focus on the function $\Phi^j_+(z;h)$, and omit the super/subscript $^j_+$ on all quantities.

By (3.33) we have

$$\begin{aligned}\partial_z \Phi_0(z) &= \partial_z \frac{\overline{\varphi(x,z)} - \varphi(x,z)}{2i}\Big|_{x=x(z)} - (\partial_z x(z)) \operatorname{Im}\partial_x\varphi(x,z)\restriction_{x=x(z)} \\ &= \frac{i}{2}\partial_z\varphi(x,z)\restriction_{x=x(z)} + \mathcal{O}(|z-z_0|^\infty).\end{aligned} \tag{3.49}$$

To obtain the last line we have used (3.33), and the fact that $x(z)$ is a critical point of $x \mapsto \operatorname{Im}\partial_x\varphi(x,z)$. Indeed, by Proposition 3.1 and (3.20), we have

$$\partial_x\varphi(x,z) = -g_0(x,z), \quad \text{in particular} \quad \partial_x\varphi(x(z),z) = \xi(z).$$

Differentiating this last expression with respect to $z$, we get

$$(\partial^2_{xx}\varphi)(x(z),z)\,\partial_z x(z) + (\partial^2_{xz}\varphi)(x(z),z) = \partial_z\xi(z). \tag{3.50}$$

Differentiating with respect to $\bar z$ and using (3.33), we obtain

$$(\partial^2_{xx}\varphi)(x(z),z)\,\partial_{\bar z} x(z) + \mathcal{O}(|z-z_0|^\infty) = \partial_{\bar z}\xi(z).$$

Eq. (3.33) is a form of almost holomorphy of $\varphi$ at the point $(x(z_0), z_0)$. Using (3.13) allows us to further differentiate it with respect to $z$:

$$(\partial^2_{z\bar z}\varphi)(x,z)\restriction_{x=x(z)} = -\int_{x(z_0)}^{x(z)} (\partial^2_{z\bar z}g_0)(y,z)\,dy = \mathcal{O}(|z-z_0|^\infty),$$

where we have used $|x(z) - x(z_0)| = \mathcal{O}(|z-z_0|)$. Let us now fix $x = x(z)$, and differentiate $\varphi(x(z),z)$ with respect to $z, \bar z$. Using the above estimate, the almost holomorphy of $\varphi$ and expression (3.50), we compute

$$\partial^2_{z\bar z}(\varphi(x(z),z)) = \partial_z\xi(z)\partial_{\bar z}x(z) + \xi(z)\partial^2_{z\bar z}x(z) + \mathcal{O}(|z-z_0|^\infty).$$

Notice that $\xi(z)\partial^2_{z\bar z}x(z) \in \mathbb{R}$, since $\xi(z)$ and $x(z)$ are real-valued smooth functions and $\partial^2_{z\bar z}$ is a real differential operator. Thus, taking the imaginary part of the above equation produces

$$\frac{2}{i}\partial^2_{z\bar z}\Phi_0(z) = \partial_z\xi(z)\partial_{\bar z}x(z) - \partial_{\bar z}\xi(z)\partial_z x(z) + \mathcal{O}(|z-z_0|^\infty). \tag{3.51}$$

We now restore the $+$, $j$ notations, and write this expression using 2-forms:

$$-d\xi_+^j \wedge dx_+^j(z) = \big(\partial_z \xi_+^j(z)\partial_{\bar z} x_+^j(z) - \partial_{\bar z}\xi_+^j(z)\partial_z x_+^j(z)\big)\, d\bar z \wedge dz$$
$$= \big(4\partial^2_{z\bar z}\Phi^j_{+,0}(z) + \mathcal{O}(|z-z_0|^\infty)\big)\,\frac{d\bar z \wedge dz}{2i}. \qquad (3.52)$$

This expressions provides the connection between the volume form in phase space, $d\xi \wedge dx$, and the volume form in energy space, $\frac{d\bar z \wedge dz}{2i}$. One can perform the symmetric computations for the functions $\Phi^j_{-,0}(z)$, and obtain

$$d\xi_-^j \wedge dx_-^j(z) = \big(\partial_{\bar z} \xi_-^j(z)\partial_{z} x_-^j(z) - \partial_{z}\xi_-^j(z)\partial_{\bar z} x_-^j(z)\big)\, d\bar z \wedge dz$$
$$= \big(4\partial^2_{z\bar z}\Phi^j_{-,0}(z) + \mathcal{O}(|z-z_0|^\infty)\big)\,\frac{d\bar z \wedge dz}{2i}. \qquad (3.53)$$

Let us now express the factor $4\partial^2_{z\bar z}\Phi^j_{+,0}(z)$ in terms of the symbol $p_0$. Differentiating the identity $p_0(\rho^j_\pm(z)) = z$ with respect to $z$ or $\bar z$, we obtain the linear system

$$\begin{cases} \partial_\xi p_0(\rho^j_\pm)\partial_z \xi^j_\pm + \partial_x p_0(\rho^j_\pm)\partial_z x^j_\pm = 1, \\ \partial_\xi \overline{p}_0(\rho^j_\pm)\partial_z \xi^j_\pm + \partial_x \overline{p}_0(\rho^j_\pm)\partial_z x^j_\pm = 0, \end{cases}$$

which can be solved by

$$\partial_z \xi^j_\pm(z) = \frac{-\partial_x \overline{p}_0}{\{\overline{p}_0, p_0\}}(\rho^j_\pm(z)), \qquad \partial_z x^j_\pm(z) = \frac{\partial_\xi \overline{p}_0}{\{\overline{p}_0, p_0\}}(\rho^j_\pm(z)).$$

Using the fact that $x^j_\pm$, $\xi^j_\pm$ are real, we deduce by (3.51), and by a similar computation for $\Phi^j_{-,0}$, that

$$4\partial^2_{z\bar z}\Phi^j_{\pm,0}(z) = \left(\frac{\mp 1}{\frac{1}{2i}\{\overline{p}_0, p_0\}(\rho^j_\pm(z))} + \mathcal{O}(|z-z_0|^\infty)\right).$$

It follows from (HYP) that the first term on the RHS is positive, hence the functions $\Phi^j_\pm$ are strictly subharmonic near $z_0$.

Identifying the Lebesgue measure $L(dz)$ on the energy plane with the volume form $\frac{d\bar z \wedge dz}{2i}$, and denoting by $|d\xi \wedge dx|$ the symplectic volume on $T^*\mathbb{R}$, the expression (3.52) (resp. (3.53)) relates an infinitesimal volume in energy space to an infinitesimal volume near the phase space point $\rho^j_+$ (resp. $\rho^j_-$). Adding the contributions of all the $2J$ points in $p_0^{-1}(z)$, we obtain the following expression for the push-forward of the symplectic volume $|d\xi \wedge dx|$ through the principal symbol $p_0$:

$$(p_0)_*(|d\xi \wedge dx|) = \sum_{j=1}^{J} (\sigma^j_+(z) + \sigma^j_-(z)) L(dz),$$
$$\sigma^j_\pm(z) := 4\partial^2_{z\bar z}\Phi^j_{\pm,0}(z) + \mathcal{O}(|z-z_0|^\infty). \qquad (3.54)$$

Observe that in case we assume the symmetry (SYM), we get $\sigma^j_+(z) = \sigma^j_-(z)$.

**Remark 3.9.** The error term $\mathcal{O}(|z-z_0|^\infty)$ will be very small in the following, because we will investigate values of $z$ in some $h^{1/2}$-neighbourhood of $z_0$.

## 4. Interaction between the quasimodes

In this section we study the "interactions" (scalar products) between nearby quasimodes. It interactions will be fundamental in the derivation of the limiting point processes; they will determine the covariances of the limiting Gaussian analytic functions (see Sections 7.3, 7.5 as well as 8.1, 8.3).

In our notations, the $L^2$ scalar product $(u|v)$ is linear in $u$ and antilinear in $v$. The assumption (HYP) implies that we may shrink the neighbourhood $W(z_0)$ (see Proposition 3.5) to ensure that

$$\rho^j_\pm(\overline{W(z_0)}) \cap \rho^k_\pm(\overline{W(z_0)}) = \emptyset \quad \text{for any } j \neq k. \tag{4.1}$$

Thus, from (4.1) and the microlocalization property (3.25), we have

$$(e^j_+(z)|e^k_+(z)) = \delta_{jk} + \mathcal{O}(h^\infty), \quad (e^j_-(z)|e^k_-(z)) = \delta_{jk} + \mathcal{O}(h^\infty), \tag{4.2}$$

uniformly for all $z \in W(z_0)$ (here $\delta_{jk}$ denotes the Kronecker symbol).

More generally, for any $z, w \in W(z_0)$ one obtains

$$(e^j_\pm(z)|e^k_\pm(w)) = \mathcal{O}(h^\infty) \quad \text{for any } j \neq k. \tag{4.3}$$

This quasi-orthogonality reflects the fact that for $j \neq k$, the points $\rho^j_\pm(z)$ and $\rho^k_\pm(w)$ remain at positive distance from each other, uniformly when $z, w \in W(z_0)$, as embodied by (4.1).

The next subsection will be devoted to the computation of the "diagonal" scalar products $(e^j_\pm(z)|e^j_\pm(w))$.

Notice that if our symbol satisfies the further hypothesis (HYP-x), we may even assume (up to shrinking $W(z_0)$) that the cutoff functions $\chi^j_\pm$ used to construct our quasimodes have disjoint supports:

$$\operatorname{supp} \chi^j_\pm \cap \operatorname{supp} \chi^k_\pm = \emptyset \quad \text{for any } j \neq k, \tag{4.4}$$

in which case the remainders $\mathcal{O}(h^\infty)$ in the above estimates vanish.

### *4.1. Overlaps between nearby quasimodes*

The scalar product between quasimodes localized on nearby points is given in the following propositions. The first one deals with the overlaps between the "+" quasimodes.

**Proposition 4.1.** *Let $(e^{j,\mathrm{hol}}_+(z))_{z\in W(z_0)}$ be the quasimodes constructed in Proposition 3.5. Recall the notation $\Phi^j_{+,0}(z;h)$ from (3.48). Then, for $|z-w| \le c$ with $c > 0$ sufficiently small,*

$$(e^{j,\mathrm{hol}}_+(z)|e^{j,\mathrm{hol}}_+(w)) = \mathrm{e}^{\frac{2}{h}\Psi^j_+(z,w;h)} + \mathcal{O}(h^\infty)\mathrm{e}^{\frac{1}{h}\Phi^j_{+,0}(z)+\frac{1}{h}\Phi^j_{+,0}(w)} \tag{4.5}$$

*with*

$$\Psi^j_+(z,w;h) = \Psi^j_{+,0}(z,w) + \frac{h}{2}\log(h^{1/2}b^j_+(z,w;h)). \tag{4.6}$$

*Here $\Psi^j_{+,0}(z,w)$ is almost $z$-holomorphic and almost $w$-anti-holomorphic at $\{(z_0,z_0)\}$, and $b^j_+(z,w;h) \sim b^{+,j}_0(z,w) + hb^{+,j}_1(z,w) + \cdots$ is smooth in $z$ and $w$, with any derivative uniformly bounded as $h \to 0$. Moreover,*

- $\Psi^j_+(w,z;h) = \overline{\Psi^j_+(z,w;h)}$,
- $\Psi^j_{+,0}(z,z) = \Phi^j_{+,0}(z)$,
- $b^j_+(z,z;h) = A^j_+(z,h)^2$ *with* $A^j_+(z,h)$ *given in Proposition 3.5,*
- *for* $|\zeta_i| \le c$, $i = 1,2$, *with* $c > 0$ *sufficiently small, and for any* $N \in \mathbb{N}$,

$$\Psi_{+,0}(z_0+\zeta_1, z_0+\zeta_2) = \sum_{|\alpha|\le N} (\partial_z^{\alpha_1}\partial_{\bar z}^{\alpha_2}\Phi_{+,0})(z_0)\frac{\zeta_1^{\alpha_1}\bar\zeta_2^{\alpha_2}}{\alpha!} + \mathcal{O}(|\zeta|^{N+1}),$$

*where* $\alpha \in \mathbb{N}^2$ *and* $|\alpha| = \alpha_1 + \alpha_2$.

The second proposition, symmetric to the previous one, deals with the overlaps between nearby "−" quasimodes.

**Proposition 4.2.** *Let $(e^{j,\text{hol}}_-(z)_{z\in W(z_0)}$ be the quasimodes constructed in Proposition 3.5. Then, for $|z-w| \le c$ with $c > 0$ sufficiently small,*

$$(e^{j,\text{hol}}_-(z)|e^{j,\text{hol}}_-(w)) = \mathrm{e}^{\frac{2}{h}\Psi^j_-(z,w;h)} + \mathcal{O}(h^\infty)\mathrm{e}^{\frac{1}{h}\Phi^j_{-,0}(z)+\frac{1}{h}\Phi^j_{-,0}(w)} \tag{4.7}$$

*with*

$$\Psi^j_-(z,w;h) = \Psi^j_{-,0}(z,w) + \frac{h}{2}\log(h^{1/2}b^j_-(z,w;h)). \tag{4.8}$$

*Here $\Psi^j_{-,0}(z,w)$ is almost $z$-anti-holomorphic and almost $w$-holomorphic at $\{(z_0,z_0)\}$, and $\Phi^j_{-,0}(z;h)$ is the function defined in (3.48). The symbol $b^j_-(z,w;h) \sim b^{-,j}_0(z,w) + hb^{-,j}_1(z,w) + \cdots$ is smooth in $z$ and $w$, with any derivative uniformly bounded as $h \to 0$. Moreover,*

- $\Psi^j_-(w,z;h) = \overline{\Psi^j_-(z,w;h)}$,
- $\Psi^j_{-,0}(z,z) = \Phi^j_{-,0}(z)$,
- $b^j_-(z,z;h) = A^j_-(z,h)^2$ *with* $A^j_-(z,h)$ *as in Proposition 3.5,*
- *for* $|\zeta_i| \le c$, $i = 1,2$, *with* $c > 0$ *small enough, and for any* $N \in \mathbb{N}$,

$$\Psi_{-,0}(z_0+\zeta_1, z_0+\zeta_2) = \sum_{|\alpha|\le N} (\partial_{\bar z}^{\alpha_1}\partial_z^{\alpha_2}\Phi_{-,0})(z_0)\frac{\bar\zeta_1^{\alpha_1}\zeta_2^{\alpha_2}}{\alpha!} + \mathcal{O}(|\zeta|^{N+1}).$$

*Proof of Proposition 4.1.* Until further notice we drop the superscript $j$. By Proposition 3.5 and (3.48) we find that for $z, w \in W(z_0)$,

$$\begin{aligned}(e^{\text{hol}}_+(z)|e^{\text{hol}}_+(w)) &= \mathrm{e}^{\frac{1}{h}\Phi_{+,0}(z)+\frac{1}{h}\Phi_{+,0}(w)}I(z,w),\\ I(z,w) &= \int \chi_+(x)\, a_+(x,z;h)\,\overline{a_+(x,w;h)}\mathrm{e}^{\frac{i}{h}\Theta(x,z,w)}\,dx,\end{aligned} \tag{4.9}$$

with the phase function

$$\begin{aligned}\Theta(x,z,w) &= \varphi_+(x,z) - \overline{\varphi_+(x,w)} + i\Phi_{+,0}(z) + i\Phi_{+,0}(w)\\ &=: 2\psi(x,z,w) + i\Phi_{+,0}(z) + i\Phi_{+,0}(w).\end{aligned} \tag{4.10}$$

By (3.48), (3.30) and the ensuing discussion, we have $\operatorname{Im}\varphi(x,z) \geq \operatorname{Im}\varphi(x_+(z),z)$, with equality only if $x = x_+(z)$. As a result, for all $x \in \operatorname{supp}\chi_+$ and $z, w \in W(z_0)$ we have

$$\operatorname{Im}\Theta(x,z,w) \geq 0, \tag{4.11}$$

with equality on the submanifold $\{(x_+(z), z, z)\}$ composed of the critical points with $\partial_x\psi(x_+(z), z, z) = 0$. Actually (3.30) implies that

$$\operatorname{Im}(\partial^2_{xx}\psi)(x,z,z) > 0 \quad \text{for all } x \in \operatorname{supp}\chi_+, \tag{4.12}$$

To estimate the integral $I(z,w)$ we will apply the method of stationary phase for complex valued phase functions [35, Theorem 2.3].

Let us fix a value $z \in W(z_0)$, and view $w \in W(z_0)$ as a parameter. In the case $w = z$ the stationary point $\partial_x\Theta$ occurs at $x = x_+(z)$, with a real valued phase value. In order to estimate $I(z,w)$ for $w \neq z$, we extend $\psi(\cdot,z,w)$ almost holomorphically in $x$, to a function $\widetilde{\psi}(x,z,w)$ defined in some complex neighbourhood $x \in \widetilde{X}$ of $\operatorname{supp}\chi_+$. From (4.10) and (3.20), this can be done by using the almost $x$-holomorphic extension $\widetilde{g}_0^+$ of $g_0^+$ constructed in Lemma 3.4, and then defining $\widetilde{\varphi}_+(x,z) = -\int_{x_0}^x \tilde{g}_0(y,z)\,dy$ by taking as contour of integration the straight line connecting $x_0$ to $x \in \widetilde{X}$, and then take

$$\widetilde{\psi}(x,z,w) = \widetilde{\varphi}_+(x,z) - \overline{\widetilde{\varphi}_+(\bar{x},w)}. \tag{4.13}$$

Notice that the exact relation $\partial_x\varphi_+(x,z) = -g_0^+(x,z)$ is then replaced by an approximate one: for any $\alpha, \beta \in \mathbb{N}$,

$$\partial_z^\alpha\partial_{\bar{z}}^\beta\partial_x\widetilde{\varphi}_+(x,z) = -\partial_z^\alpha\partial_{\bar{z}}^\beta\widetilde{g}_0^+(x,z) + \mathcal{O}_{\alpha,\beta}(|\operatorname{Im}x|^\infty), \qquad z \in W(z_0),\ x \in \widetilde{X}, \tag{4.14}$$

with the implied constants uniform in $z \in W(z_0)$.

As long as $|z-w|$ is sufficiently small (hence, up to shrinking $W(z_0)$), the nondegeneracy condition (4.12) extends to this complex neighbourhood:

$$\partial^2_{xx}\widetilde{\psi}(x,z,w) \neq 0, \qquad x \in \widetilde{X},\ z,w \in W(z_0),\ ,$$

where $\partial^2_{xx}$ denotes the holomorphic derivative. This ensures that for any $z, w \in W(z_0)$, $\widetilde{\psi}(\cdot,z,w)$ admits a unique, nondegenerate critical point $x = x^c(z,w) \in \widetilde{X}$, satisfying

$$\partial_x\widetilde{\psi}(x^c(z,w), z, w) = 0, \tag{4.15}$$

where $\partial_x$ denotes the holomorphic derivative. Notice that when $w = z$, the critical point $x^c(z,z) = x(z)$ is real valued.

We also need almost holomorphic extensions with respect to $x$ of the symbols $a_+(\cdot,z;h)$ to $\tilde{a}_+(\cdot,z;h)$, and replace $a_+\overline{a_+}$ in the integral $I(z,w)$ by the almost holomorphic expression $\tilde{a}_+(x,z;h)\overline{\tilde{a}_+(\bar{x},w;h)}$.

We can now apply the stationary phase theorem [35, Theorem 2.3] to the integral $I(z,w)$, and obtain the expansion

$$I(z,w) = h^{\frac{1}{2}}\, b_+(z,w;h)\, \mathrm{e}^{\frac{1}{h}(2\Psi_{+,0}(z,w)-\Phi_{+,0}(z)-\Phi_{+,0}(w))} + \mathcal{O}(h^\infty), \qquad z,w\in W(z_0). \tag{4.16}$$

Here $\Psi_{+,0}(z,w)$ is given by

$$\Psi_{+,0}(z,w) = i\widetilde{\psi}(x^c(z,w),z,w), \tag{4.17}$$

in particular on the diagonal $\Psi_{+,0}(z,z)=\Phi_{+,0}(z)$.

The symbol $b_+(z,w;h)\sim b_{+,0}(z,w)+hb_{+,1}(z,w)+\cdots$ can be expressed in terms of derivatives of $\widetilde{a_+}(x,z;h)\overline{\widetilde{a_+}(\bar{x},w;h)}$ with respect to $x$ at the point $x=x^c(z,w)$; it depends smoothly on $z$ and $w$, uniformly as $h\to 0$. The normalization $I(z,z)=1$ set by (4.9), (3.21) shows that we may take $b_+(z,z;h)=A_+(z;h)^2$, with $A_+(z;h)$ as in Proposition 3.5.

As explained in the proof of [35, Theorem 2.3], the arbitrariness in the almost holomorphic extensions $\widetilde{\psi}$, $\tilde{a}_+$ is absorbed in the $\mathcal{O}(h^\infty)$ remainder. Let us check the symmetry relations satisfied by the functions $\Psi_{+,0}$ and $\Psi_+ := \Psi_{+,0}+\frac{h}{2}\log(h^{1/2}b_+)$. The symmetry between $z$ and $\bar{w}$ in (4.13) and (4.17) shows that $x^c(z,w)=\overline{x^c(w,z)}$, hence

$$\Psi_{+,0}(w,z) = \overline{\Psi_{+,0}(z,w)}. \tag{4.18}$$

The expression of $b_+$ shows that this symmetry also concerns the "full" function $\Psi_+(z,w;h)=\overline{\Psi_+(w,z;h)}$.

Let us now show that the function $\Psi_{+,0}$ of (4.17) is the "polarization" of the real valued function $\Phi_{+,0}$. We first show that the critical point $x^c(z,w)$ is almost $z$-holomorphic and almost $w$-anti-holomorphic at the diagonal $\{(z,z);\ z\in W(z_0)\}$. Differentiating (4.15) with respect to $z$ and $\bar{z}$, one finds by (4.13) that

$$\begin{aligned} \partial_z x^c(z,w) &= \frac{-(\partial^2_{xz}\widetilde{\varphi}_+)(x^c(z,w),z)+\mathcal{O}(|\mathrm{Im}\, x^c(z,w)|^\infty)}{2\partial^2_{xx}\widetilde{\psi}(x^c(z,w),z,w)},\\ \partial_{\bar{z}} x^c(z,w) &= \frac{-(\partial^2_{x\bar{z}}\widetilde{\varphi}_+)(x^c(z,w),z)+\mathcal{O}(|\mathrm{Im}\, x^c(z,w)|^\infty)}{2\partial^2_{xx}\widetilde{\psi}(x^c(z,w),z,w)}, \end{aligned} \tag{4.19}$$

where we have used that $\widetilde{\psi}$ is almost $x$-holomorphic. Now, (4.14) and Lemma 3.4 imply that

$$\begin{aligned} \partial_{x\bar{z}}\widetilde{\varphi}_+(x^c(z,w),z) &= \mathcal{O}(|x^c(z,w)-x_+(z)|^\infty), \qquad \text{hence}\\ \partial_{\bar{z}}x^c(z,w) &= \mathcal{O}(|x^c(z,w)-x_+(z)|^\infty) = \mathcal{O}(|z-w|^\infty), \end{aligned}$$

where for the last equality we have used the smoothness of $x^c$ in $z,w$.

Using (4.17), (4.13) and (3.33), we then infer that the function $\Psi_{+,0}(z,w)$ is almost $z$-holomorphic at the point $z=w=z_0$:

$$\begin{aligned} -i\partial_{\bar{z}}\Psi_{+,0}(z,w) &= (\partial_x\widetilde{\psi})(x^c(z,w),z,w)\partial_{\bar{z}}x^c(z,w)\\ &\quad + (\partial_{\bar{x}}\widetilde{\psi})(x^c(z,w),z,w)\,\overline{\partial_z x^c(z,w)} + \tfrac{1}{2}\partial_{\bar{z}}\widetilde{\varphi}_+(x^c(z,w),z)\\ &= \mathcal{O}(|z-w|^\infty+|z-z_0|^\infty), \end{aligned}$$

From the symmetry (4.18), this function is also almost $w$-anti-holomorphic at that point. Taking into account the identity $\Psi_{+,0}(z,z) = \Phi_{+,0}(z)$, we deduce that the Taylor expansion of $\Psi_{+,0}$ at $(z_0, z_0)$ takes the following form: for any $N \in \mathbb{N}$,

$$\Psi_{+,0}(z_0+\zeta_1, z_0+\zeta_2) = \sum_{|\alpha|\le N} (\partial_z^{\alpha_1}\partial_{\bar z}^{\alpha_2}\Phi_{+,0})(z_0)\frac{\zeta_1^{\alpha_1}\overline{\zeta}_2^{\alpha_2}}{\alpha!} + \mathcal{O}(|\zeta|^{N+1}), \qquad \zeta_j \in W(z_0)-z_0,\ j=1,2.$$

It is in this sense that $\Psi_{+,0}$ "polarizes" $\Phi_{+,0}$. In particular, keeping the expansion up to second order, we find the following identity:

$$2\operatorname{Re}\Psi_{+,0}(z,w) - \Phi_{+,0}(z) - \Phi_{+,0}(w) = -\partial^2_{z\bar z}\Phi_{+,0}(z_0)|z-w|^2 + \mathcal{O}((|z-z_0|, |w-z_0|)^3). \qquad \square$$

The proof of Proposition 4.2 is identical to the one of Proposition 4.1, so we omit it.

### 4.2. *Symmetric symbols*

Assuming (SYM), the additional symmetry of the symbol $p(x,\xi;h)$, we will also need to compute the interaction between the squared quasimodes $(e_-^j(z))^2$, when considering perturbations by a random potential. The construction of these quasimodes was discussed in Section 3.4. The properties of these squared quasimodes are very similar to those of the original $e_-(z)$, the main difference being that most quantities will now bear a subscript $s$ (for "squared").

Notice that by (3.42), (3.47) and the assumption (4.4), we have, for all $z, w \in W(z_0)$,

$$((e_-^j(z))^2 | (e_-^k(w))^2) = 0 \quad \text{for } j \neq k. \tag{4.20}$$

As in the proof of Proposition 3.5, one obtains by the method of stationary phase

$$\begin{aligned} &\|(e_-^{j,\mathrm{hol}}(z))^2\| = \mathrm{e}^{\frac{1}{h}\Phi_s^j(z;h)},\\ &\Phi_s^j(z;h) := \Phi_{s,0}^j(z) + h\log(h^{1/4}A_s^j(z;h)), \qquad \Phi_{s,0}^j(z) := 2\Phi_{+,0}^j(z), \end{aligned} \tag{4.21}$$

where $\Phi_{+,0}^j(z)$ is as in (3.48) and, in view of (3.42), $x_+^j = x_-^j = x^j$. We recall that $\Phi_{+,0}^j(z) \ge 0$, with equality if and only if $z = z_0$, and $A_s^j(z;h) \sim A_0^{j,s}(z) + hA_1^{j,s}(z) + \cdots$ depends smoothly on $z$, with all derivatives with respect to $z$ and $\bar z$ bounded when $h \to 0$. Moreover,

$$A_0^{j,s}(z) = \left(\frac{\pi |a_0^+(x_+(z);z)|^4}{2\operatorname{Im}\partial^2_{xx}\varphi_+(x_+(z),z)}\right)^{1/4} > 0, \tag{4.22}$$

where $a_0^+$ is as in (3.29). One may compare the expression for this norm with the one in (3.31), (3.32) for $\|e_-^{j,\mathrm{hol}}(z)\|$.

The following proposition describes the overlaps between nearby squared quasimodes. The expressions are very similar to the ones of the previous section.

**Proposition 4.3.** *Suppose that (SYM) holds and that the neighbourhood $W(z_0) \Subset \Omega$, as in Proposition 3.5, satisfies (4.1), (4.4). Let $\Phi_s^j(z;h)$, $z \in W(z_0)$, be as in (4.21). Then, for $|z-w| \leq c$ with $c > 0$ sufficiently small,*

$$((e_-^{j,\mathrm{hol}}(z))^2 | (e_-^{j,\mathrm{hol}}(w))^2) = \mathrm{e}^{\frac{2}{h}\Psi_s^j(z,w;h)} + \mathcal{O}(h^\infty)\mathrm{e}^{\frac{1}{h}\Phi_{s,0}^j(z)+\frac{1}{h}\Phi_{s,0}^j(w)} \tag{4.23}$$

*with*

$$\Psi_s^j(z,w;h) = \Psi_{s,0}^j(z,w) + \frac{h}{2}\log(h^{1/2} b_s^j(z,w;h)), \tag{4.24}$$

*where $\Psi_{s,0}^j(z,w)$ is almost $z$-anti-holomorphic and almost $w$-holomorphic at $\{(z_0,z_0)\}$ and $b_s^j(z,w;h) \sim b_0^{s,j}(z,w) + h b_1^{s,j}(z,w) + \cdots$ is smooth in $z$ and $w$ with all derivatives bounded as $h \to 0$. Moreover,*

- $\overline{\Psi_s^j(z,w;h)} = \Psi_s^j(w,z;h)$,
- $\Psi_{s,0}^j(z,z) = \Phi_{s,0}^j(z) = 2\Phi_{+,0}^j(z)$,
- $b_s^j(z,z;h) \sim A_s^j(z,h)^2$ *with* $A_s^j(z,h)$ *as in (4.21),*
- *for* $|\zeta_i| \leq c$, $i = 1,2$, *with* $c > 0$ *small enough, and for any* $N \in \mathbb{N}$,

$$\Psi_{s,0}^j(z_0+\zeta_1, z_0+\zeta_2) = 2\sum_{|\alpha|\leq N} (\partial_{\bar z}^{\alpha_1}\partial_z^{\alpha_2}\Phi_{+,0})(z_0)\frac{\overline{\zeta}_1^{\alpha_1}\zeta_2^{\alpha_2}}{\alpha!} + \mathcal{O}(|\zeta|^{N+1}).$$

*Proof.* The proof exactly mimics the one of Proposition 4.1. □

### 4.3. Finite rank truncation of the quasimodes

In this section we show that the quasimodes $\{e_\pm^j(z);\, z \in \Omega\}$ essentially live in a finite-dimensional subspace of $L^2(\mathbb{R})$, which we build using the orthonormal eigenbasis $(e_m)_{m\in\mathbb{N}}$ of the harmonic oscillator $H = -\partial_x^2 + x^2$, corresponding to the eigenvalues $\lambda_m = 2m+1$. Roughly speaking, we show that if $C_1 > 0$ is chosen large enough, the truncated basis $(e_m)_{m\leq C_1/h^2}$ microlocally covers a part of phase space which contains the region $p^{-1}(\Omega)$, inside which all our quasimodes are microlocalized.

For any $N \in \mathbb{N}$, let $\Pi_N$ be the orthogonal projector on the subspace of $L^2$ spanned by the states $(e_m)_{0\leq m\leq N}$. In the following lemma we show that if $N$ is chosen large enough, this projection essentially does not modify our quasimodes.

**Lemma 4.4.** *Let $\{e_\pm^j(z);\, z \in W(z_0)\}$ be the normalized quasimodes constructed in Proposition 3.5. Then, if $C_1 > 0$ is chosen sufficiently large, taking $N(h) = C_1/h^2$ we have*

$$\forall z \in W(z_0),\ \forall j = 1,\dots,J, \quad \|(1-\Pi_{N(h)})e_\pm^j(z)\| = \mathcal{O}(h^\infty). \tag{4.25}$$

*In the case of a symmetric symbol (SYM), the same estimate applies to the squared quasi-modes:*

$$\forall z \in W(z_0),\ \forall j = 1,\dots,J, \quad \|(1-\Pi_{N(h)})(e_\pm^j(w))^2\| = \mathcal{O}(h^\infty). \tag{4.26}$$

*In both cases the $\mathcal{O}(h^\infty)$ remainder is uniform in $z \in W(z_0)$ and $h \in\, ]0,1]$.*

*Proof.* We will focus on proving the first estimate, the estimate for squared quasimodes being very similar.

Let $U \Subset \mathbb{R}^2$ be a bounded open set such that

$$\bigcup_{j=1}^{J} \rho_{\pm}^{j}(\overline{W(z_0)}) \subset U. \tag{4.27}$$

Our proof will amount to showing that if $C_1$ is chosen large enough, the projector $\Pi_{N(h)}$ is microlocally equal to the identity in a neighbourhood of $U$.

Let $\psi \in \mathcal{C}_c^{\infty}(\mathbb{R}^2, [0,1])$ be such that $\psi \equiv 1$ in a neighbourhood of $\overline{U}$ and $\operatorname{supp} \psi \subset B(0,R) \subset \mathbb{R}^2$ for some sufficiently large $R > 0$ . Then, by the microlocalization property (3.25), we have

$$\|(1-\psi^w)e_{\pm}^{j}(z)\| = \mathcal{O}(h^{\infty}), \qquad \text{uniformly in } z \in W(z_0). \tag{4.28}$$

For our aims, it will suffice to prove that there exists a constant $C_1 > 0$ such that

$$\forall m > N(h) = C_1/h^2, \qquad \|\psi^w(x,hD_x)e_m\| = \mathcal{O}(\lambda_m^{-\infty}), \tag{4.29}$$

where the implied constants are uniform in $m > N(h)$ and $h \in ]0,1]$. Indeed, the above bounds imply that

$$\|(1-\Pi_{N(h)})\psi^w e_{\pm}^{i}(z)\|^2 = \sum_{m>N(h)} |(\psi^w e_{\pm}^{i}(z)|e_m)|^2 = \mathcal{O}(h^{\infty}). \tag{4.30}$$

Together with (4.28), this shows the desired estimate (4.25).

The right hand side $\mathcal{O}(\lambda_m^{-\infty})$ in (4.29) is sharper than the standard remainder $\mathcal{O}(h^{\infty})$ produced by the $h$-pseudodifferential calculus (remember that $\lambda_m \geq C_1/h^2$). This hints at the fact that to obtain a remainder $\mathcal{O}(\lambda_m^{-\infty})$, we should use a semiclassical calculus whose small parameter is

$$\tilde{h} = \tilde{h}_m := \lambda_m^{-1},$$

rather than $h$. We will do it by rescaling the coordinates appropriately (see e.g. [51, Theorem 6.5] for a similar rescaling procedure). Namely, we let $U_{\lambda_m}$ be the unitary transformation on $L^2(\mathbb{R})$ given by $U_{\lambda_m}u(x) = \lambda_m^{1/4}u(\lambda_m^{1/2}x)$. The nonsemiclassical harmonic oscillator can be written $H = q^w(x, D_x)$, where $q(x,\xi) = \xi^2 + x^2$. A direct computation shows that

$$U_{\lambda_m}(H-\lambda_m)U_{\lambda_m}^{-1} = \lambda_m(q^w(x,\lambda_m^{-1}D_x)-1) = \tilde{h}^{-1}(q^w(x,\tilde{h}D_x)-1). \tag{4.31}$$

Let us insert the dilation $U_{\lambda_m}$ in the expression we want to estimate:

$$\|\psi^w(x,hD_x)e_m\| = \|U_{\lambda_m}\psi^w(x,hD_x)U_{\lambda_m}^{-1}\,\tilde{e}_m\|, \qquad \tilde{e}_m = U_{\lambda_m}e_m. \tag{4.32}$$

The rescaled state $\tilde{e}$ satisfies the eigenvalue equation $(q^w(x,\tilde{h}D_x)-1)\tilde{e}_m = 0$. From standard elliptic estimates (see e.g. [51, Theorem 6.4]), the state $\tilde{e}_m$ is $\tilde{h}$-microlocalized at the energy shell $q^{-1}(1)$, that is, the unit circle in $\mathbb{R}^2$. As a consequence, for any $\chi \in S(1)$ supported inside $B(0, 2/3) \subset \mathbb{R}^2$, we will have

$$\chi^w(x, \tilde{h}D_x)\tilde{e}_m = \mathcal{O}(\tilde{h}^\infty). \tag{4.33}$$

Let us come back to (4.32), and interpret $U_{\lambda_m}\psi^w(x, hD_x)U_{\lambda_m}^{-1}$ as a pseudodifferential operator in the $\tilde{h}$-calculus:

$$U_{\lambda_m}\psi^w(x, hD_x)U_{\lambda_m}^{-1} = \tilde{\psi}^w(x, \tilde{h}D_x) \quad \text{for the symbol } \tilde{\psi}(x, \xi) = \psi(\tilde{h}^{-1/2}x, h\tilde{h}^{-1/2}\xi).$$

After this rescaling, the required estimate (4.29) has been transformed to

$$\tilde{\psi}^w(x, \tilde{h}D_x)\tilde{e}_m = \mathcal{O}(\tilde{h}^\infty). \tag{4.34}$$

The symbol $\psi$ is supported in the square $\{|x|, |\xi| \leq R\}$, so the rescaled symbol $\tilde{\psi}$ will be supported in the rectangle

$$\{|x| \leq R\tilde{h}^{1/2},\ |\xi| \leq R\tilde{h}^{1/2}/h \leq R(2C_1)^{-1/2}\},$$

where in the second inequality we have used our assumption that $\lambda_m \geq 2C_1/h^2$. We choose $C_1$ large enough, so that this rectangle is contained inside $B(0, 1/2)$. However, we cannot directly apply the estimate (4.33) to our symbol $\tilde{\psi}$, because the latter is not in the class $S(1)$, its derivatives growing, when $\tilde{h} \to 0$, as

$$|\partial_x^\alpha \partial_\xi^\beta \tilde{\psi}| \leq C\tilde{h}^{-(|\alpha|+|\beta|)/2}.$$

These estimates indicate that $\tilde{\psi}$ belongs to the exotic class $S_{1/2}(1)$ in the $\tilde{h}$-calculus (see e.g. [51, Chapter 4]). Even though the Moyal product between two symbols in the class $S_{1/2}(1)$ does not usually admit an asymptotic expansion, the Moyal product of a symbol $\tilde{\psi} \in S_{1/2}(1)$ and a symbol $\chi \in S(1)$ does admit an expansion in powers of $\tilde{h}$. As a result, if we take a symbol $\tilde{\chi} \in S(1)$ supported inside $B(0, 2/3)$ with $\tilde{\chi} \equiv 1$ inside $B(0, 1/2)$, the Moyal product expansion for $\tilde{\psi} \# \tilde{\chi}$ implies that

$$\tilde{\psi}(x, \tilde{h}D_x)\, \tilde{\chi}(x, \tilde{h}D_x) = \tilde{\psi}(x, \tilde{h}D_x) + \mathcal{O}(\tilde{h}^\infty). \tag{4.35}$$

Since supp $\tilde{\chi} \subset B(0, 2/3)$, the operator $\tilde{\chi}^w(x, \tilde{h}D_x)$ satisfies (4.33). Combined with (4.35), this finally gives the estimate (4.34), which completes the proof of the lemma. □

The previous lemma implies that each quasimode can be essentially reconstructed by taking linear combinations of $N(h)$ basis states $e_m$. This number is not optimal, but the following lemma shows that quasimodes cannot be reconstructed from only a few basis states. This fact will be relevant in Section 7.5.

**Lemma 4.5.** *Let $e_\pm^{j,\mathrm{hol}}(z)$ be the quasimodes constructed in Proposition 3.5. Take as before the orthonormal eigenbasis $(e_m)_{m\in\mathbb{N}}$ of the harmonic oscillator $H$, associated with the eigenvalues $\{\lambda_m = 2m + 1\}$. Then*

$$\forall m \geq 0, \quad (h^{-1/4}e_\pm^{j,\mathrm{hol}}(z)|e_m) = \mathcal{O}(h^{1/4})(2m+1)^{-1/6}\mathrm{e}^{\frac{1}{h}\Phi_{\pm,0}^j(z)}, \tag{4.36}$$

*where* $\Phi^j_{\pm,0}(z)$ *is given in* (3.48). *Similarly, the squared quasimodes satisfy*

$$\forall m \geq 0, \quad (h^{-1/4}(e_-^{j,\mathrm{hol}}(z))^2|e_m) = \mathcal{O}(h^{1/4})\lambda_m^{-1/6}\mathrm{e}^{\frac{2}{h}\Phi^j_{+,0}(z)}.$$

*In both estimates the implied constant is uniform in* $m \in \mathbb{N}$, $h \in \,]0,1]$ *and* $z \in W(z_0)$.

*Proof.* We only handle the case of nonsquared quasimodes, the other case being similar. By the Hölder inequality,

$$|(h^{-1/4}e_\pm^{j,\mathrm{hol}}(z)|e_m)| \leq \|h^{-1/4}e_\pm^{j,\mathrm{hol}}(z)\|_{L^1}\|e_m\|_{L^\infty}.$$

Using (3.30), the method of stationary phase yields

$$\|h^{-1/4}e_\pm^{j,\mathrm{hol}}(z)\|_{L^1} = \mathcal{O}(h^{1/4})\mathrm{e}^{\frac{1}{h}\Phi^j_{+,0}(z)},$$

where $\Phi^j_{+,0}(z)$ is as in (3.48). By [33, Corollary 3.2], there exists a constant $C > 0$ such that for all $m \in \mathbb{N}$,

$$\|e_m\|_\infty \leq C\lambda_m^{-1/6}\|e_m\|_{L^2}. \qquad \square$$

**Remark 4.6.** The choice of the orthonormal basis $(e_m)_{m\geq 0}$ used to define the truncation operator is rather arbitrary, as explained in [23, 25]. What is needed is that for $N = N(h)$ large enough, the projection $\Pi_{N(h)}$ on the subspace spanned by a collection of $N(h)$ of those states is microlocally equivalent to the identity in some given bounded region $U \subset \mathbb{R}^2$. Any orthonormal basis of $L^2(\mathbb{R})$ will have this property, but the number $N(h)$ of necessary states will depend on the choice of basis.

In [23] the author used the eigenbasis of the semiclassical harmonic oscillator $q^w(x, hD_x)$, in which case it was sufficient to include only $\mathcal{O}(h^{-1})$ eigenstates. Our choice to use the nonsemiclassical harmonic oscillator $H$ requires including a larger number $N(h) \geq C_1 h^{-2}$ of states. This choice was guided by the extra requirement that each quasimode $e^j_\pm(z)$ should essentially decompose into *many* basis states $e_m$ (as shown by Lemma 4.5), a fact which will be important in Section 7.5 when applying the Central Limit Theorem.

## 5. Setting up the Grushin problem

We begin by giving a short refresher on Grushin problems. They have become an important tool in spectral theory and are employed in a vast number of works, especially when dealing with spectral studies of nonselfadjoint operators. As reviewed in [45], the central idea is to analyze the operator $P(z) = P - z : \mathcal{H}_1 \to \mathcal{H}_2$ by extending it into a larger operator of the form

$$\begin{pmatrix} P(z) & R_-(z) \\ R_+(z) & 0 \end{pmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_- \to \mathcal{H}_2 \oplus \mathcal{H}_+,$$

where $\mathcal{H}_\pm$ (resp. $R_\pm(z)$) are well-chosen auxiliary spaces (resp. operators). The Grushin problem is said to be *well-posed* if this matrix of operators is invertible for the range of $z$

under study, with good control on its inverse. In the cases where $\dim \mathcal{H}_- = \dim \mathcal{H}_+ < \infty$, one writes the inverse operator blockwise as

$$\begin{pmatrix} P(z) & R_-(z) \\ R_+(z) & 0 \end{pmatrix}^{-1} = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{pmatrix}.$$

The key observation, going back to Schur's complement formula, is the following: the operator $P(z) : \mathcal{H}_1 \to \mathcal{H}_2$ is invertible if and only if the finite rank operator $E_{-+}(z) : \mathcal{H}_+ \to \mathcal{H}_-$ is invertible, in which case both inverses are related by

$$P(z)^{-1} = E(z) - E_+(z)E_{-+}(z)^{-1}E_-(z).$$

For this reason, the finite rank operator $E_{-+}(z)$ is often called an *effective Hamiltonian* for the original problem $P(z)$. As opposed to $P(z)$, it depends in a nonlinear way on the variable $z$, but has the advantage of being finite-dimensional. $E_{-+}(z)$ encapsulates, in a compact way, the spectral properties of $P$.

### *5.1. Grushin problem for our unperturbed nonselfadjoint operator*

Hager [23] showed how to construct an efficient Grushin problem for our nonselfadjoint operator $P_h$, using the quasimodes constructed in Proposition 3.5.

**Proposition 5.1** (Unperturbed Grushin). *Let $p(\cdot; h) \in S(\mathbb{R}^2, m)$ be as in (2.6) and satisfy (2.8), (HYP). Let a small energy range $W(z_0) \Subset \mathbb{C}$ satisfy (4.1), and consider the corresponding normalized quasimodes $\{e_\pm^j(z);\ z \in W(z_0)\}$, $j = 1, \dots, J$, constructed in Proposition 3.5. For $z \in W(z_0)$ let*

$$\mathcal{P}(z) = \begin{pmatrix} P_h - z & R_-(z) \\ R_+(z) & 0 \end{pmatrix} : H(m) \times \mathbb{C}^J \to L^2 \times \mathbb{C}^J$$

*with*

$$(R_+(z)u)_j := (u|e_+^j(z)), \quad u \in H(m), \qquad R_-(z)u_- := \sum_{j=1}^{J} u_-^j e_-^j(z), \quad u_- \in \mathbb{C}^J. \tag{5.1}$$

*Then $\mathcal{P}(z)$ is bijective for all $z \in W(z_0)$, with bounded inverse denoted by*

$$\mathcal{E}(z) = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{pmatrix} : L^2 \times \mathbb{C}^J \to H(m) \times \mathbb{C}^J.$$

*The blocks $E_\pm(z)$ have the following forms:*

$$\begin{aligned} (E_-(z)v)_j &= (v|e_-^j(z)) + \mathcal{O}(h^\infty)v, && v \in L^2, \\ E_+(z)v_+ &= \sum_{j=1}^{J} v_+^j e_+^j(z) + \mathcal{O}(h^\infty)v_+, && v_+ \in \mathbb{C}^J. \end{aligned} \tag{5.2}$$

*The blocks of $\mathcal{E}(z)$ admit the following bounds in the semiclassical limit:*

$$\begin{aligned}\|E(z)\|_{L^2\to H(m)} &= \mathcal{O}(h^{-1/2}), \quad & \|E_+(z)\|_{\mathbb{C}^J\to L^2} &= \mathcal{O}(1),\\ \|E_-(z)\|_{L^2\to\mathbb{C}^J} &= \mathcal{O}(1), & \|E_{-+}(z)\|_{\mathbb{C}^J\to\mathbb{C}^J} &= \mathcal{O}(h^\infty),\end{aligned} \tag{5.3}$$

*uniformly for $z \in W(z_0)$.*

*Proof.* Follow [23, proof of Proposition 4.1] line by line, with obvious changes. □

**Remark 5.2.** As explained in Section 2.7.1, the role of $R_+(z)$ is to "absorb" the elements in the quasi-kernel of $P(z)$, namely the linear combinations of the quasimodes $\{e_+^j(z);\ 1 \le j \le J\}$; similarly, the role of $R_-(z)$ is to "fill" the quasi-cokernel of $P(z)$, using the auxiliary space $\mathbb{C}^J$. As a result, the extended operator $\mathcal{P}(z)$ has neither quasi-kernel nor quasi-cokernel, and is thus invertible, with the norm of its inverse being under control.

### 5.2. *Grushin problem for the perturbed operator*

We wish to study the eigenvalues of

$$P^\delta = P_h + \delta Q, \tag{5.4}$$

where $\delta > 0$ satisfies (1.6) and $Q$ is given by a random matrix $M_\omega$, as in (2.18), or by a random potential $V_\omega$, as in (2.19).

Recall from (2.20) that we want to restrict the random variables used to construct $M_\omega$ and $V_\omega$, to the disc of radius $C/h$. If we denote by

$$\mathrm{PD}_N(0, R) = D(0, R)^N \subset \mathbb{C}^N$$

the $N$-dimensional polydisc of radius $R$ centred at 0, then the restricted probability space $\mathcal{M}_h \subset \mathcal{M}$ in the case of the random matrix $M_\omega$, resp. of the random polynomial $V_\omega$, is defined by the following events:

$$q = (q_{j,k})_{j,k<N(h)} \in \mathrm{PD}_{N(h)^2}(0, C/h), \tag{5.5}$$

$$\text{resp.} \quad v = (v_j)_{j<N(h)} \in \mathrm{PD}_{N(h)}(0, C/h). \tag{5.6}$$

By the tail estimate (2.17) and the fact that $N(h) = C_1/h^2$, this restriction holds with high probability as $h \to 0$, for both the matrix and potential cases. More precisely, there exists a constant $C_2 > 0$ such that

$$\begin{aligned}\mathbb{P}[q \in \mathrm{PD}_{N(h)^2}(0, C/h)] &\ge 1 - \kappa N(h)^2 h^{4+\varepsilon_0} = 1 - C_2 h^{\varepsilon_0},\\ \mathbb{P}[v \in \mathrm{PD}_{N(h)}(0, C/h)] &\ge 1 - \kappa N(h) h^{4+\varepsilon_0} = 1 - C_2 h^{2+\varepsilon_0}.\end{aligned} \tag{5.7}$$

In their restricted spaces, the random matrix, resp. random potential, satisfy the bounds

$$\|M_\omega\|_{\mathrm{HS}} = N(h)^{-1}\Big(\sum_{i,j<N(h)} |q_{i,j}|^2\Big)^{1/2} \le Ch^{-1}, \tag{5.8}$$

$$\text{resp.} \quad \|V_\omega\|_\infty \le N(h)^{-1} \sum_{n<N(h)} |v_n|\,\|e_n\|_\infty \le Ch^{-1}. \tag{5.9}$$

This proves the estimates (2.21) and (2.22) on the size of the perturbations in the restricted space.

Until further notice we will work in this restricted probability space $\mathcal{M}_h$, so that (5.8), resp. (5.9) holds. Using Proposition 5.1, we obtain a well-posed Grushin problem for the perturbed operator $P^\delta$.

**Proposition 5.3** (Perturbed Grushin). *Let $P_h$ and $W(z_0)$ be as in Proposition 5.1, and $P^\delta$ be the perturbed operator as in (5.4) with $\delta > 0$ satisfying (1.6). For any $z \in W(z_0)$ let*

$$\mathcal{P}^\delta(z) = \begin{pmatrix} P^\delta - z & R_-(z) \\ R_+(z) & 0 \end{pmatrix} : H(m) \times \mathbb{C}^J \to L^2 \times \mathbb{C}^J$$

*with the operators $R_+(z)$, $R_-(z)$ defined in (5.1). Then, for any realization of the perturbation $Q_\omega$ in the restricted probability space (5.7), the operator $P^\delta(z)$ is bijective with bounded inverse*

$$\mathcal{E}^\delta(z) = \begin{pmatrix} E^\delta(z) & E_+^\delta(z) \\ E_-^\delta(z) & E_{-+}^\delta(z) \end{pmatrix} : L^2 \times \mathbb{C}^J \to H(m) \times \mathbb{C}^J.$$

*Moreover, the perturbed blocks $E_\pm^\delta$, $E_{-+}^\delta$ are related with the unperturbed ones of Proposition 5.1 as follows:*

$$E_-^\delta = E_- + \mathcal{O}(\delta h^{-3/2}), \qquad E_+^\delta = E_+ + \mathcal{O}(\delta h^{-3/2}), \tag{5.10}$$

*and*

$$E_{-+}^\delta = E_{-+} - \delta E_- Q E_+ + \mathcal{O}(\delta^2 h^{-5/2}), \tag{5.11}$$

*uniformly with respect to the perturbation in the restricted probability space.*

*Proof.* The result follows from an application of the Neumann series. Let $\mathcal{E}$ be as in Proposition 5.1. Then

$$\mathcal{P}^\delta \mathcal{E} = \mathcal{P}\mathcal{E} + \begin{pmatrix} \delta Q & 0 \\ 0 & 0 \end{pmatrix} \mathcal{E} = 1 + \begin{pmatrix} \delta Q E & \delta Q E_+ \\ 0 & 0 \end{pmatrix} =: 1 + K.$$

Putting together the bounds (1.6), (5.8) (resp. (5.9)) and (5.3), we get

$$\|K\| \leq \delta \|Q\| (\|E\| + \|E_+\|) = \mathcal{O}(\delta h^{-3/2}) \ll 1, \qquad h \to 0.$$

Thus, $1 + K$ can be inverted by the Neumann series, which provides the inverse $\mathcal{E}^\delta = \mathcal{E}(1 + K)^{-1}$ for $\mathcal{P}^\delta$:

$$\begin{aligned} \mathcal{E}^\delta &:= \begin{pmatrix} E^\delta(z) & E_+^\delta(z) \\ E_-^\delta(z) & E_{-+}^\delta(z) \end{pmatrix} \\ &= \begin{pmatrix} \sum_{n=0}^\infty (-1)^n E(\delta Q E)^n & \sum_{n=0}^\infty (-1)^n (\delta E Q)^n E_+ \\ \sum_{n=0}^\infty (-1)^n E_-(\delta Q E)^n & E_{-+} + \delta \sum_{n=1}^\infty (-1)^n E_-(\delta Q E)^{n-1} Q E_+ \end{pmatrix}. \end{aligned} \tag{5.12}$$

The estimates (5.10), (5.11) easily follow, which concludes the proof. □

Using Propositions 3.5, 5.1 and (5.11), we get

$$(E_{-+}^\delta(z))_{i,j} = -\delta(Q e_+^j | e_-^i) + \mathcal{O}(\delta h^\infty) + \mathcal{O}(h^\infty) + \mathcal{O}(\delta^2 h^{-5/2}). \tag{5.13}$$

**Remark 5.4.** A salient feature of the expansion (5.13) is that, with high probability, the matrix $E^{\delta}_{-+}$ is dominated by the first term on the right hand side, namely is the random perturbation term. This fact could appear strange, since the perturbation had a small norm $\propto \delta$. However, the unperturbed matrix $E_{-+}(z)$ is even smaller ($\mathcal{O}(h^{\infty})$), a consequence of the strong pseudospectral effect. This dominance of the random perturbation in our effective Hamiltonian is crucial in the analysis of the perturbed spectrum.

Using the assumption (1.6) on the size of $\delta$ and taking the determinant, we get

$$\det[\delta^{-1}E^{\delta}_{-+}(z)] = (-1)^J \det\big[(Qe^j_+(z)|e^i_-(z))_{i,j\leq J} + \mathcal{O}(\delta h^{-5/2})\big]. \tag{5.14}$$

As explained in Section 2.7.1, the spectrum of $P^{\delta}$ corresponds to the zero locus of the function $z \mapsto \det[E^{\delta}_{-+}(z)]$. Since we used the normalized quasimodes, this function is not holomorphic in $z \in W(z_0)$. Because we want to make a connection with $z$-analytic functions, we will transform this determinant into a $z$-holomorphic function (without modifying its zero locus), using the fact that it satisfies a $\overline{\partial}$-equation. To see this, we take the $\partial_{\bar{z}}$ derivative of the equation $\mathcal{P}^{\delta}\mathcal{E}^{\delta} = 1_{L^2\times\mathbb{C}^J}$, and get

$$\partial_{\bar{z}}\mathcal{E}^{\delta} = -\mathcal{E}^{\delta}\partial_{\bar{z}}\mathcal{P}^{\delta}\mathcal{E}^{\delta}.$$

The lower right block of this matrix reads

$$\partial_{\bar{z}}E^{\delta}_{-+} = -E^{\delta}_{-+}(\partial_{\bar{z}}R_+)E^{\delta}_{+} - E^{\delta}_{-}(\partial_{\bar{z}}R_-)E^{\delta}_{-+}.$$

This, together with the identity $\partial_{\bar{z}}\log(\det E^{\delta}_{-+}) = \operatorname{tr}[(E^{\delta}_{-+})^{-1}\partial_{\bar{z}}E^{\delta}_{-+}]$ then yields

$$\begin{aligned}\partial_{\bar{z}}\det(E^{\delta}_{-+}) &= -\operatorname{tr}[(\partial_{\bar{z}}R_+)E^{\delta}_{+} + E^{\delta}_{-}(\partial_{\bar{z}}R_-)]\det(E^{\delta}_{-+})\\ &=: -k^{\delta}\det(E^{\delta}_{-+}).\end{aligned} \tag{5.15}$$

Let us study the factor $k^{\delta}(z)$. Using the expressions (5.1), (5.2), (5.10), we find

$$\begin{aligned}((\partial_{\bar{z}}R_+)E^{\delta}_{+})_{jj} &= (e^j_+(z)|\partial_z e^j_+(z)) + \mathcal{O}(\delta h^{-5/2}),\\ (E^{\delta}_{-}\partial_{\bar{z}}R_-))_{jj} &= (\partial_{\bar{z}}e^j_-(z)|e^j_-(z)) + \mathcal{O}(\delta h^{-5/2}).\end{aligned}$$

Here, we have used the fact (easily following from the expressions for the quasimodes in Proposition 3.5) that $\|\partial_z e^j_+(z)\| = \mathcal{O}(h^{-1})$, $\|\partial_{\bar{z}}e^j_-(z)\| = \mathcal{O}(h^{-1})$; for instance,

$$\partial_z e^j_+ = \mathrm{e}^{-\frac{1}{h}\Phi^j_+}(\partial_z e^{j,\mathrm{hol}}_+ - h^{-1}(\partial_z\Phi^j_+)e^{j,\mathrm{hol}}_+) = \mathcal{O}(h^{-1})_{L^2}. \tag{5.16}$$

Putting these estimates together, we obtain

$$k^{\delta} = \sum_{j=1}^{J}[(e^j_+|\partial_z e^j_+) + (\partial_{\bar{z}}e^j_-|e^j_-)] + \mathcal{O}(\delta h^{-5/2}), \tag{5.17}$$

uniformly in $z \in W(z_0)$.

We now want to compute the function $k^\delta(z)$, using the properties of the almost holomorphic quasimodes $e_\pm^{j,\mathrm{hol}}(z)$ we constructed in Proposition 3.5. Taking the $\partial_{\bar z}$ derivative of $\|e_+^{j,\mathrm{hol}}(z)\|^2 = \mathrm{e}^{2\Phi_+^j(z)/h}$ (see (3.21)), we get

$$\begin{aligned}2h^{-1}\partial_{\bar z}\Phi_+^j &= \mathrm{e}^{-\frac{2}{h}\Phi_+^j}\big((\partial_{\bar z}e_+^{j,\mathrm{hol}}|e_+^{j,\mathrm{hol}}) + (e_+^{j,\mathrm{hol}}|\partial_z e_+^{j,\mathrm{hol}})\big)\\ &= \mathrm{e}^{-\frac{2}{h}\Phi_+^j}(e_+^{j,\mathrm{hol}}|\partial_z e_+^{j,\mathrm{hol}}) + \mathcal{O}(h^{-1}|z-z_0|^\infty + h^\infty),\end{aligned} \tag{5.18}$$

where in the last line we have used the almost holomorphy property (3.22). Taking the $z$-derivative of $e_+^j(z) = \mathrm{e}^{-\Phi_+^j(z)/h}e_+^{j,\mathrm{hol}}(z)$ (see (3.23)), we find

$$\partial_z e_+^j = -h^{-1}\partial_z\Phi_+^j\, e_+^j + \mathrm{e}^{-\frac{1}{h}\Phi_+^j}\partial_z e_+^{j,\mathrm{hol}},$$

and hence

$$(e_+^j|\partial_z e_+^j) = -h^{-1}\partial_{\bar z}\Phi_+^j + \mathrm{e}^{-\frac{2}{h}\Phi_+^j}(e_+^{j,\mathrm{hol}}|\partial_z e_+^{j,\mathrm{hol}}).$$

Using this identity together with (5.18), we obtain

$$(e_+^j|\partial_z e_+^j) = h^{-1}\partial_{\bar z}\Phi_+^j + \mathcal{O}(h^{-1}|z-z_0|^\infty + h^\infty). \tag{5.19}$$

A similar computation shows that

$$\begin{aligned}(\partial_{\bar z}e_-^j|e_-^j) &= -h^{-1}\partial_{\bar z}\Phi_-^j + \mathrm{e}^{-\frac{2}{h}\Phi_-^j}(\partial_{\bar z}e_-^{j,\mathrm{hol}}|e_-^{j,\mathrm{hol}})\\ &= h^{-1}\partial_{\bar z}\Phi_-^j + \mathcal{O}(h^{-1}|z-z_0|^\infty + h^\infty),\end{aligned} \tag{5.20}$$

which finally results in the following expression for the function $k^\delta(z)$:

$$k^\delta(z) = h^{-1}\sum_{j=1}^J(\partial_{\bar z}\Phi_+^j(z) + \partial_{\bar z}\Phi_-^j(z)) + \mathcal{O}(h^{-1}|z-z_0|^\infty + \delta h^{-5/2}).$$

The function $k^\delta(z)$ depends smoothly on $z \in W(z_0)$, and the equation $\partial_{\bar z}l^\delta = hk^\delta$ can be solved exactly in $W(z_0)$ (see e.g. [27, Theorems 1.2.2, 1.4.4] or [22, p. 6]) with a solution of the form

$$l^\delta(z) = -2Jh\log(h^{1/4}) + \sum_{j=1}^J\big(\Phi_+^j(z;h) + \Phi_-^j(z;h)\big) + \mathcal{O}(|z-z_0|^\infty + \delta h^{-3/2}). \tag{5.21}$$

Here we have added the constant term $-2Jh\log(h^{1/4})$ in order to balance the behaviour of $\Phi_\pm^j(z_0;h) = h\log(h^{1/4}) + \mathcal{O}(1)$. From (5.15) we obtain the following holomorphic function in $z \in W(z_0)$:

$$\begin{aligned}G^\delta(z;h) &:= (-\delta)^{-J}\mathrm{e}^{\frac{1}{h}l^\delta(z)}\det E_{-+}^\delta(z)\\ &= \mathrm{e}^{\frac{1}{h}l^\delta(z)}\det[(Qe_+^j(z)|e_-^i(z))_{i,j\le J} + \mathcal{O}(\delta h^{-5/2})].\end{aligned} \tag{5.22}$$

It will be convenient to introduce the diagonal matrices containing the norms of the almost holomorphic quasimodes:

$$\Lambda_\pm = \Lambda_\pm(z) := \operatorname{diag}(h^{-1/4}\mathrm{e}^{\frac{1}{h}\Phi_\pm^j(z)})_{j=1,\dots,J}. \tag{5.23}$$

We infer from (5.21) that

$$\mathrm{e}^{\frac{1}{h}l^{\delta}(z)} = \det(\Lambda_-\Lambda_+)\big(1 + \mathcal{O}(h^{-1}|z - z_0|^{\infty} + \delta h^{-5/2})\big).$$

Injecting this expression in (5.22) and using (3.23), we finally express $G^{\delta}$ in terms of the almost holomorphic quasimodes:

$$\begin{aligned} G^{\delta}(z;h) &= (1 + R_1)\det\big[\Lambda_-(Qe_+^{j}(z)|e_-^{i}(z))_{i,j\le J}\Lambda_+ + \Lambda_-\mathcal{O}(\delta h^{-5/2})\Lambda_+\big] \\ &= (1 + R_1)\det\big[(Qh^{-1/4}e_+^{j,\mathrm{hol}}(z)|h^{-1/4}e_-^{i,\mathrm{hol}}(z))_{i,j\le J} + R_2\big], \end{aligned} \tag{5.24}$$

where the two remainder terms satisfy, uniformly for $z \in W(z_0)$,

$$\begin{aligned} R_1 &:= R_1(z;h) = \mathcal{O}(h^{-1}|z - z_0|^{\infty} + \delta h^{-5/2}), \\ R_2 &:= R_2(z;h) = \Lambda_-\mathcal{O}(\delta h^{-5/2})\Lambda_+. \end{aligned} \tag{5.25}$$

It seems natural that the almost holomorphic, resp. almost anti-holomorphic quasimodes appear in the expression of the holomorphic function $G^{\delta}$. Indeed, if $e_j^+(z)$ (resp. $e_j^-(z)$)) were exactly holomorphic (resp. exactly anti-holomorphic), then the dominant entries $(Qe_+^{j,\mathrm{hol}}(z)|e_-^{i,\mathrm{hol}}(z))$ in the determinant would be holomorphic as well.

The important output of this section is that the eigenvalues of $P^{\delta}$ in $W(z_0)$ are given (with multiplicities) by the zeros of the holomorphic function $G^{\delta}(z;h)$; for this reason, we will call this holomorphic function an *effective spectral determinant* for the operator $P^{\delta}$. Our future task will thus be to analyze the zeros of this function. We will do it in Sections 7 and 8, after recalling general properties of random analytic functions.

## 6. Random analytic functions

In this section we provide background material and references concerning the theory of random analytic functions, which are needed for the proofs in Sections 7 and 8. We begin by recalling some standard notions and facts about random analytic functions and stochastic processes, as discussed for instance in [31, 30].

Let $O \subset \mathbb{C}$ be an open, simply connected domain, and let $\mathcal{H}(O)$ denote the space of holomorphic functions on $O$. Given an exhaustion on $O$ by compact subsets $K_j \Subset O$, we endow $\mathcal{H}(O)$ with the metric

$$d(f,g) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{\|f - g\|_{K_j}}{1 + \|f - g\|_{K_j}}, \tag{6.1}$$

where $\|f\|_{K_j} := \max_{z\in K_j} |f(z)|$. This metric induces the topology of uniform convergence on compact sets. This makes $\mathcal{H}(O)$ a complete separable metric space, in other words a *Polish space*, and we may equip it with its Borel $\sigma$-algebra $\mathcal{B}(\mathcal{H}(O))$. This makes $(\mathcal{H}(O), \mathcal{B}(\mathcal{H}(O)))$ a measurable space.

**Definition 6.1.** Let $(\mathcal{M}, \mathcal{A}, \nu)$ be a probability space. Then any measurable map

$$f : (\mathcal{M}, \mathcal{A}) \to (\mathcal{H}(O), \mathcal{B}(\mathcal{H}(O)))$$

is called a $\mathbb{C}$-*valued stochastic process on* $O$ *with paths in* $\mathcal{H}(O)$, or simply a *random analytic function on* $O$.

The Borel $\sigma$-algebra $\mathcal{B}(\mathcal{H}(O))$ is equal to the $\sigma$-algebra generated by the evaluation maps

$$\pi_z : \mathcal{H}(O) \to \mathbb{C}, \quad \pi_z g := g(z), \quad z \in O,$$

that is, it is the smallest $\sigma$-algebra in $\mathcal{H}(O)$ such that $\pi_z$ is measurable for every $z \in O$. As a consequence, a function $f : (\mathcal{M}, \mathcal{A}) \to (\mathcal{H}(O), \mathcal{B}(\mathcal{H}(O)))$ is measurable if and only if $\pi_z f : (\mathcal{M}, \mathcal{A}) \to (\mathbb{C}, \mathcal{B}(\mathbb{C}))$ is measurable for all $z \in O$.

Hence, any measurable function $f : \mathcal{M} \to \mathcal{H}(O)$ can be regarded as well as a measurable function from $O \times \mathcal{M}$ to $\mathbb{C}$:

$$f(z, \omega) = \pi_z f(\omega), \quad (z, \omega) \in O \times \mathcal{M}.$$

Due to the above measurability property, $f$ can be considered as a collection of random complex variables $(f(z))_{z\in O}$. The *finite-dimensional distributions* of $f$ describe the joint laws of finite vectors $(f(z_1), \dots, f(z_k))$. More precisely, for $(z_1, \dots, z_k) \in O^k$, the joint distribution of the random vector $(f(z_1), \dots, f(z_k)) \in \mathbb{C}^k$ is the probability measure on $\mathbb{C}^k$ defined by

$$\mu_{z_1,\dots,z_k} = (f(z_1), \dots, f(z_k))_*(\nu), \quad z_1, \dots, z_k \in O, \ k \in \mathbb{N}^*.$$

The next result states that these finite-dimensional distributions fully determine the law of a random analytic function, i.e. the direct image measure $f_*\nu$ on $\mathcal{H}(O)$ (see e.g. [31, Proposition 2.2]). Below the notation $X \overset{d}{=} Y$ between two random variables denotes equality in distribution: the laws of $X$ and $Y$ are equal to one another.

**Theorem 6.2.** *Let $f$ and $g$ be two random analytic functions. Then*

$$f \overset{d}{=} g \iff (f(z_1), \dots, f(z_k)) \overset{d}{=} (g(z_1), \dots, g(z_k)), \ \forall (z_1, \dots, z_k) \in O^k, \ \forall k \in \mathbb{N}^*.$$

Next, let us recall that a $\mathbb{C}^k$-valued random variable $X$ is said to have a *centred symmetric complex Gaussian distribution* with covariance matrix $\Sigma_k \in \mathrm{GL}_k(\mathbb{C})$, for short $X \sim \mathcal{N}_{\mathbb{C}}(0, \Sigma_k)$, if its distribution is given by

$$X_*\nu = (\det \pi \Sigma_k)^{-1} \mathrm{e}^{-X^* \Sigma_k^{-1} X} L(dX),$$

where $L(dX)$ is the Lebesgue measure on $\mathbb{C}^k$. The covariance matrix $\Sigma_k$ must be Hermitian and positive definite. This Gaussian distribution is characterized by its means and variances:

$$\mathbb{E}[X_i] = 0, \quad \mathbb{E}[X_i X_j] = 0, \quad \mathbb{E}[X_i \overline{X_j}] = (\Sigma_k)_{ij}, \quad 1 \le i, j \le k.$$

**Definition 6.3** (Gaussian analytic function, GAF). Let $O \subset \mathbb{C}$ be an open simply connected complex domain. A random analytic function $f$ on $O$ is called a *Gaussian analytic function* on $O$ if its finite-dimensional distributions are centred symmetric complex Gaussian, i.e. for all $k \in \mathbb{N}^*$ and all $z_1, \dots, z_k \in O$, we have $(f(z_1), \dots, f(z_k)) \sim \mathcal{N}_{\mathbb{C}}(0, \Sigma_k)$ for some covariance matrix $\Sigma_k$.

The matrix $\Sigma_k$ depends on $(z_1, \ldots, z_k)$. Each of its entries $(\Sigma_k)_{ij}$ is given by the covariance kernel

$$K(z_i, \bar{z}_j) := \mathbb{E}\big[f(z_i)\overline{f(z_j)}\big],$$

which is a $z_i$-holomorphic and $z_j$-anti-holomorphic function on $O \times O$. Hence, the complete distribution of the GAF is fully characterized by the covariance kernel.

For more details on the theory of Gaussian analytic functions we refer the reader to [30].

### 6.1. *Sequences of random analytic functions*

We now review some notions and results concerning convergence of sequences of random analytic functions.

**Definition 6.4.** Let $X_n$, $n \in \mathbb{N}$, and $X$ be random variables defined on probability spaces $(\mathcal{M}_n, \mathcal{F}_n, \nu_n)$, $n \in \mathbb{N}$, respectively $(\mathcal{M}, \mathcal{F}, \nu)$ and taking values in a Polish space $(S, d)$. We say that $(X_n)_n$ *converges in distribution* to $X$ if the induced probability measures on $S$ converge in the weak-$*$ topology:

$$(X_n)_*\nu_n \xrightarrow{w^*} X_*\nu, \qquad n \to \infty,$$

equivalently if for any $\phi \in \mathcal{C}_b(S, \mathbb{R})$,

$$\int_{\mathcal{M}_n} \phi(X_n)\, d\nu_n \to \int_{\mathcal{M}} \phi(X)\, d\nu, \qquad n \to \infty.$$

When this is the case we simply write $X_n \xrightarrow{d} X$.

**Definition 6.5.** Let $O \subset \mathbb{C}$ be an open, simply connected domain. Let $f_n$, $n \in \mathbb{N}$, and $f$ be random analytic functions on $O$ (not necessarily defined on the same probability space). We say that $f_n$ *converges in the sense of finite-dimensional distributions* to $f$ if for all $k \geq 1$ and all $(z_1, \ldots, z_k) \in O^k$,

$$(f_n(z_1), \ldots, f_n(z_k)) \xrightarrow{d} (f(z_1), \ldots, f(z_k)), \qquad n \to \infty.$$

When this is the case we write $f_n \xrightarrow{\text{fd}} f$.

As discussed in Theorem 6.2, the distribution of each random analytic function is uniquely determined by its finite-dimensional distributions. However, the convergence of a sequence of random analytic functions in the sense of finite-dimensional distributions does *not* in general imply the convergence in distribution of the sequence of random functions. To achieve this implication, one needs to add a *tightness condition*, which provides the relative compactness of the sequence.

**Definition 6.6.** Let $(\mu_n)_n$ be a sequence of probability measures on some Polish space $(S, \mathcal{B}(S))$. The sequence $(\mu_n)_n$ is said to be *tight* if

$$\sup_{K \Subset S} \liminf_{n\to\infty} \mu_n(K) = 1,$$

where the supremum is taken over all compact sets $K \Subset S$.

Similarly, a sequence of random variables $(X_n)_n$ taking values in $S$ is called *tight* if

$$\sup_{K \Subset S} \liminf_{n\to\infty} \mathbb{P}[X_n \in K] = 1,$$

where $\mathbb{P}[X_n \in K] = (X_n)_* \nu_n(K) := \mu_n(K)$ is the standard notation for the probability measure induced by $X_n$ on $S$.

Tightness ensures that the probability measures $\mu_n$ do not "escape to infinity" when $n \to \infty$: for any $\epsilon > 0$, there exists a compact set $K_\epsilon \Subset S$ and $n_\epsilon \in \mathbb{N}$ such that $\mu_n(K_\epsilon) \geq 1 - \epsilon$ for any $n \geq n_\epsilon$.

An important result due to Prokhorov (see e.g. [31, Theorem 14.3]) is the following.

**Theorem 6.7** (Prokhorov). *For any sequence $(X_n)_n$ of random variables taking values in a Polish space, tightness is equivalent to relative compactness in distribution, i.e. the sequence $(\mu_n)_n$ of probability measures on $S$ induced by $(X_n)_n$ is relatively compact in the weak-$*$ topology.*

**Remark 6.8.** As a consequence of this theorem, for a tight sequence of probability measures on a Polish space $S$, convergence with respect to the weak-$*$ topology of $\mathcal{C}_b(S)'$ (where test functions are continuous and bounded) is equivalent to convergence with respect to the weak-$*$ topology of $\mathcal{C}_c(S)'$ (test functions are continuous with compact supports). The latter topology is sometimes referred to as the *vague topology*.

Shirai [40, Proposition 2.5] provides a useful criterion for the tightness of sequences of random analytic functions:

**Proposition 6.9** ([40]). *Let $f_n$, $n \in \mathbb{N}$, and $f$ be random analytic functions on an open simply connected set $O \subset \mathbb{C}$. Suppose that for any compact set $K \Subset O$, the sequence $(\|f_n\|_{L^\infty(K)})_n$ of random real variables is tight, i.e.*

$$\lim_{r\to\infty} \limsup_{n\to\infty} \mathbb{P}[\|f_n\|_{L^\infty(K)} > r] = 0.$$

*Then $(f_n)_n$ is tight in the space of random analytic functions on $O$, and*

$$f_n \xrightarrow{\text{fd}} f \text{ as } n \to \infty \quad \text{implies} \quad f_n \xrightarrow{d} f \text{ as } n \to \infty.$$

*This property naturally extends to a family of functions $(f_h)_h$ depending on a continuous parameter $h \in ]0, h_0]$: it holds for any sequence $(f_{h_n})_n$ such that $h_n \to 0$ as $n \to \infty$.*

### 6.2. *Point processes given by the zeros of a random analytic function*

Let us first recall the definition of a random point process. Let $O \subset \mathbb{C}$ be an open connected domain, and let $M(O)$ denote the space of complex valued, locally finite Borel measures on $O$, which we endow with the vague topology of $\mathcal{C}_c(O)'$. This topology is metrizable, and it makes $M(O)$ a Polish space, which we can equip with its Borel $\sigma$-algebra. An $M(O)$-valued random variable is called a *random measure* on $O$.

Inside $M(O)$ we distinguish the space $PM(O)$ of integer valued measures in $M(O)$, which forms a closed subspace of $M(O)$. Any element $\mu \in PM(O)$ is a point measure: it can be expressed as

$$\mu = \sum_{z_i} \delta_{z_i}$$

where $\delta_{z_i}$ denotes the Dirac measure on the point $z_i \in O$. The points $\{z_i\}_i$ form a finite or countable set, which has no accumulation point (yet each $z_i$ can be repeated finitely many times). From this characterization, a $PM(O)$-valued random variable is called a *point process* on $O$.

Let $f$ be a nontrivial analytic function on $O$, and let

$$\mathcal{Z}_f := \sum_{\lambda \in f^{-1}(0)} \delta_\lambda \tag{6.2}$$

be the point measure defined by its set of zeros (counted with multiplicities). If $f$ is a random analytic function (with $f \not\equiv 0$ a.s.), then $\mathcal{Z}_f$ defines a point process on $O$, which we will call a *zero point process*. This is indeed a point process: $f$ is measurable and for every test function $\varphi \in \mathcal{C}_c(O, \mathbb{R})$, the functional $\pi_\varphi : \mathcal{H}(O)\backslash\{0\} \to \mathbb{R}$, $\pi_\varphi(f) = \langle \mathcal{Z}_f, \varphi\rangle$, is continuous on $(\mathcal{H}(O) \setminus \{0\}, d)$, for $d$ the metric defined in (6.1). This was observed for instance by Shirai [40].

**Remark 6.10.** An easy extension is that for any $M \in \mathbb{N}$ and any tensor product function $\varphi^{\otimes M} \in \bigotimes_{j=1}^{M} \mathcal{C}_c(O, \mathbb{R})$, the linear mapping

$$\pi_{\varphi^{\otimes M}} : (\mathcal{H}(O) \setminus \{0\})^M \to \mathbb{R}, \qquad (f_1, \dots, f_M) \mapsto \langle \mathcal{Z}_{f_1}, \varphi_1\rangle \cdots \langle \mathcal{Z}_{f_M}, \varphi_M\rangle,$$

is continuous with respect to the metric $\widetilde{d}(f, g) = \sum_{i=1}^{M} d(f^i, g^i)$ on $\mathcal{H}(O)^M$.

The following result is essentially due to Shirai [40, Proposition 2.3].

**Proposition 6.11.** *Let $O \subset \mathbb{C}$ be an open, simply connected domain. Let $f_n$, $n \in \mathbb{N}$, and $f$ be random analytic functions on $O$, not necessarily defined on the same probability space. Suppose that $f_n, f \not\equiv 0$ almost surely, and $f_n$ converges in distribution to $f$. Then the zero point processes $\mathcal{Z}_{f_n}$ converge in distribution to $\mathcal{Z}_f$. Moreover, for any $\varphi^{\otimes M} \in \bigotimes_{j=1}^{M} \mathcal{C}_c(O, \mathbb{R})$, we have the following convergence of real random variables:*

$$\langle \mathcal{Z}_{f_n}, \varphi_1\rangle \cdots \langle \mathcal{Z}_{f_n}, \varphi_M\rangle \xrightarrow{d} \langle \mathcal{Z}_f, \varphi_1\rangle \cdots \langle \mathcal{Z}_f, \varphi_M\rangle.$$

*Proof.* The convergence in distribution of the point processes $\mathcal{Z}_{f_n}$ to $\mathcal{Z}_f$ is equivalent to

$$\langle \mathcal{Z}_{f_n}, \varphi\rangle \xrightarrow{d} \langle \mathcal{Z}_f, \varphi\rangle$$

for all $\varphi \in \mathcal{C}_c(O, \mathbb{R})$. As discussed in Remark 6.10, the mapping $\pi_{\varphi^{\otimes M}}$ is continuous. Hence, the continuous mapping theorem, stating that convergence in distribution of random variables is preserved under continuous mappings between metric spaces (see e.g. [31, Theorem 3.27]), implies the claimed results. □

### 6.3. A Central Limit Theorem for complex valued random variables

The random matrix $M_\omega$, respectively the random potential $V_\omega$ forming the perturbation are generally not Gaussian (see our general assumptions (2.16) on the law of the variables generating $M_\omega$ and $V_\omega$). In order to obtain a limiting Gaussian random analytic function, we will need to apply some Central Limit Theorem. Below we present the version of the CLT for complex valued random variables which we will use.

**Theorem 6.12.** *Let $\sigma > 0$ and let $\xi \sim \mathcal{N}_{\mathbb{C}}(0, \sigma)$ be a complex Gaussian random variable with mean 0 and variance $\sigma$. Let $\{\xi_{nj}\}_{n\in\mathbb{N},1\leq j\leq N(n)}$ be a triangular array, with $N(n) \to \infty$ as $n \to \infty$, of row-wise independent complex-valued random variables, satisfying*

(i) $\sum_{j=1}^{N(n)} |\mathbb{E}[\xi_{nj}]| \to 0$ *as* $n \to \infty$,

(ii) $\sum_{j=1}^{N(n)} \mathbb{E}[\xi_{nj}^2] \to 0$ *as* $n \to \infty$,

(iii) $\sum_{j=1}^{N(n)} \mathbb{E}[|\xi_{nj}|^2] \to \sigma$ *as* $n \to \infty$,

(iv) *for any* $\varepsilon > 0$, $\sum_{j=1}^{N(n)} \mathbb{E}[|\xi_{nj}|^2 \mathbf{1}_{\{|\xi_{nj}|>\varepsilon\}}] \to 0$ *as* $n \to \infty$.

*Then* $\sum_{j=1}^{N(n)} \xi_{nj} \xrightarrow{d} \xi$ *as* $n \to \infty$.

**Remark 6.13.** Condition (iv) is known as the Lindeberg condition for a CLT. A simpler version of the above theorem was presented in [40, Proposition 4.2], assuming that all random variables $\xi_{nj}$ have expectation 0 and are such that the real random variables $\mathrm{Re}\,\xi_{nj}$ and $\mathrm{Im}\,\xi_{nj}$ are independent and have the same variance. Notice that these extra assumptions imply conditions (i) and (ii).

*Proof of Theorem 6.12.* The proof is standard. It can be obtained by a direct modification of the proof of the well-known central limit theorem under the Lindeberg condition (see [31, Theorem 4.12]) for a linear combination of the real and imaginary part of $\xi_{ni}$. □

## 7. Local statistics of the eigenvalues of $P_h$ perturbed by a random matrix $M_\omega$

In this section we prove Theorem 2.9 and Corollary 2.10, which concern the local spectral statistics for the randomly perturbed operator

$$P^\delta = P_h + \delta M_\omega, \tag{7.1}$$

where $M_\omega$ is the random matrix defined in (RM). To do this we will begin by rescaling the determinant of the holomorphic effective Hamiltonian (5.24), which describes the eigenvalues of $P^\delta$, to the scale of the average spacing between nearby eigenvalues, obtaining the rescaled random analytic function $F_h^\delta(w)$ in (7.12). Next, we will use the results of Section 4 to study the covariances of the matrix elements of the effective Hamiltonian. This will then be used to show that the random function $F_h^\delta(w)$ is tight (see Section 7.4). Next, we show in Section 7.5 that (after a change of gauge) the function $F_h^\delta(w)$ converges in finite-dimensional distributions to the limiting function $\widetilde{G}_{z_0}(w)$ defined in Theorem 2.9. Together with the tightness and Prokhorov's Theorem 6.7, this proves the convergence in distribution of Theorem 2.9. In Section 7.6, we finally prove Corollary 2.10.

### *7.1. On restricted random variables*

We recall that we restrict ourselves to the restricted probability space $\mathcal{M}_h$ defined by (5.5). This restricting leads to defining restricted random variables $q^h = (q^h_{j,k})_{j,k\leq N(h)}$, equipped with the conditional probability

$$\mathbb{P}[q^h \in A] = \mathbb{P}[q \in A \mid q \in \mathrm{PD}_{N(h)^2}(0, C/h)], \qquad \forall A \subset \mathbb{C}^{N(h)^2}. \tag{7.2}$$

Because the variables $q_{j,k}$, $j, k < N(n)$, are i.i.d. with the law of the variable $\alpha$ (2.16), and the restriction to $\mathcal{M}_h$ is a product of identical restrictions, the restricted variables $q^h_{j,k}$ are i.i.d. as well, with common law described by the restricted variable $\alpha^h$:

$$\mathbb{P}[\alpha^h \in A] = \mathbb{P}[\alpha \in A \mid |\alpha| \leq C/h] = \frac{\mathbb{P}[\{\alpha \in A\} \cap \{|\alpha| \leq C/h\}]}{\mathbb{P}[|\alpha| \leq C/h]}, \qquad \forall A \subset \mathbb{C}. \tag{7.3}$$

Let us study the law of the restricted variables $\alpha^h$. By (2.17), the denominator in (7.3) takes values

$$\mathbb{P}[|\alpha| \leq C/h] = 1 + \mathcal{O}(h^{4+\varepsilon_0}). \tag{7.4}$$

The first two conditions in (2.16) imply that

$$\mathbb{E}[\alpha^n \mathbb{1}_{\{|\alpha|\leq C/h\}}] = -\mathbb{E}[\alpha^n \mathbb{1}_{\{|\alpha|>C/h\}}], \qquad n = 1, 2. \tag{7.5}$$

Furthermore, by Fubini's Theorem, we have the following useful identities for $n = 1, 2$:

$$\begin{aligned}\mathbb{E}[|\alpha|^n \, \mathbb{1}_{\{|\alpha|>C/h\}}] &= \int_0^\infty \mathbb{P}[\{|\alpha| > t^{1/n}\} \cap \{|\alpha| > C/h\}]\, dt \\ &= \mathcal{O}(h^{4+\varepsilon_0-n}),\end{aligned} \tag{7.6}$$

where we have used the tail estimate (2.17) to obtain the last line.

Using (7.3)–(7.6), we obtain the following estimate of the average and variance of the restricted random variable:

$$\begin{aligned}\left|\mathbb{E}[(\alpha^h)^n]\right| &= \frac{1}{\mathbb{P}[|\alpha| \leq C/h]} |\mathbb{E}[\alpha^n \mathbb{1}_{\{|\alpha|>C/h\}}]| \\ &\leq \frac{1}{\mathbb{P}[|\alpha| \leq C/h]} \mathbb{E}[|\alpha^n| \mathbb{1}_{\{|\alpha|>C/h\}}] = \mathcal{O}(h^{4+\varepsilon_0-n}).\end{aligned}$$

Thus, the first two identities in (2.16) are replaced by

$$\mathbb{E}[\alpha^h] = \mathcal{O}(h^{3+\varepsilon_0}), \qquad \mathbb{E}[(\alpha^h)^2] = \mathcal{O}(h^{2+\varepsilon_0}). \tag{7.7}$$

Finally, using (2.16), (2.17) as well as (7.4)–(7.6), we obtain the variance

$$\mathbb{E}[|\alpha^h|^2] = \frac{1}{\mathbb{P}[|\alpha| \leq C/h]} \big(\mathbb{E}[|\alpha|^2] - \mathbb{E}[|\alpha|^2 \mathbb{1}_{\{|\alpha|>C/h\}}]\big) = 1 + \mathcal{O}(h^{2+\varepsilon_0}). \tag{7.8}$$

Finally, with (2.16) it is easy to check that the $(4 + \epsilon_0)$-moment of $\alpha^h$ remains uniformly bounded when $h \to 0$:

$$\mathbb{E}[|\alpha^h|^{4+\epsilon_0}] \leq C, \qquad \forall h \in ]0, 1]. \tag{7.9}$$

### 7.2. Rescaling our spectral determinant

Next, let $\Omega \Subset \mathring{\Sigma}$ be as in (2.11) and pick a $z_0 \in \Omega$. By the Grushin problem constructed in Proposition 5.3, the eigenvalues of the perturbed operator $P^\delta$, in the small neighborhood $W(z_0)$ of $z_0$ constructed in Proposition 5.1, are precisely the zeros of the random holomorphic function obtained in (5.24):

$$G^\delta(z;h) = (1 + R_1(z;h)) \det\big[(M_\omega h^{-1/4} e_+^{j,\mathrm{hol}}(z) | h^{-1/4} e_-^{i,\mathrm{hol}}(z))_{i,j\leq J} + R_2(z;h)\big], \tag{7.10}$$

with the remainder terms estimated in (5.25).

Since the spectrum of $P^\delta$ in $W(z_0)$ is discrete, $G^\delta(\cdot;h) \not\equiv 0$ in $W(z_0)$. To compute the local spectral statistics near the point $z_0$, we rescale the spectral parameter $z$ by the average distance between nearest neighbours, which is of order $\asymp h^{1/2}$, i.e. we write

$$z = z_w = z_0 + h^{1/2} w, \qquad w \in \mathbb{C}. \tag{7.11}$$

For an eigenvalue $z$ of $P^\delta$ in $W(z_0)$, we will call the corresponding $w$ a *rescaled eigenvalue*, which forms the rescaled spectrum. We will focus on the eigenvalues in a *microscopic* neighbourhood of $z_0$. Namely, we will consider an arbitrary open, connected and bounded set $O \Subset \mathbb{C}$, and only consider the points $w \in O$. For $h > 0$ small enough, the rescaled eigenvalues are precisely the zeros of the $w$-holomorphic function

$$\begin{aligned} F_h^\delta(w) &:= G^\delta(z_w;h) \\ &= (1 + \widetilde{R}_1(w;h)) \det[(f_{i,j}^{\delta,h}(w))_{i,j\leq J} + \widetilde{R}_2(w;h)], \qquad w \in O, \end{aligned} \tag{7.12}$$

with

$$\begin{aligned} &f_{i,j}^{\delta,h}(w) := (M_\omega h^{-1/4} e_+^{j,\mathrm{hol}}(z_w) | h^{-1/4} e_-^{i,\mathrm{hol}}(z_w)), \qquad 0 \leq i,j \leq J, \\ &\widetilde{R}_1(w;h) := R_1(z_w;h), \qquad \widetilde{R}_2(w;h) := R_2(z_w;h). \end{aligned} \tag{7.13}$$

We may call $F_h^\delta$ a *rescaled spectral determinant*. We have $F_h^\delta \not\equiv 0$ in $O$, and hence, by the discussion in Section 6.1, the random measure

$$\mathcal{Z}_h := \sum_{w \in (F_h^\delta)^{-1}(0) \cap O} \delta_w \tag{7.14}$$

is a well-defined point process on $O$ (the zeros are repeated according to their multiplicities). This is the rescaled spectral point process $\mathcal{Z}_{h,z_0}^M$ of (2.24), restricted to the set $O$. Our next goal is to analyze the statistical properties of this point process.

### 7.3. Covariance

In this section we study the covariance kernel of the random functions $f_{i,j}^{\delta,h}(w)$ defined over $w \in O \Subset \mathbb{C}$. This kernel will be a crucial ingredient in the analysis of $\mathcal{Z}_h$.

**Proposition 7.1.** *Let $\sigma_\pm^j(z_0)$ be the classical densities at the point $z_0$, as in (3.54); let $f_{i,j}^{\delta,h}(\bullet)$, for $0 \leq i, j \leq J$, be the random functions as in (7.13). The covariance kernel of those functions admits the following expression, uniformly for $v, w \in O$:*

$$\mathbb{E}\big[f_{i,j}^{\delta,h}(v)\overline{f_{l,k}^{\delta,h}(w)}\big]\mathrm{e}^{-F_{i,j}(v;h)-\overline{F_{l,k}(w;h)}} = \delta_{i,l}\delta_{j,k}K^{i,j}(v,\overline{w})(1+\mathcal{O}(h^{1/2}))+\mathcal{O}(h^2). \tag{7.15}$$

*Here $\delta_{i,j}$ is the Kronecker symbol. The most important part of the above formula is the kernel*

$$K^{i,j}(v,\overline{w}) = \exp\big(\tfrac{1}{2}(\sigma_+^j(z_0)+\sigma_-^i(z_0))\,v\overline{w}\big). \tag{7.16}$$

*The other ingredients are the "gauge functions"*

$$F_{i,j}(v;h) = \phi_+^j(v;h)+\phi_-^i(v;h), \tag{7.17}$$

*where*

$$\phi_\pm^j(v;h) = \tfrac{1}{2}[\log A_\pm^j(z_0;h)+(\partial_{zz}^2\Phi_{\pm,0}^j)(z_0)v^2], \tag{7.18}$$

*with $A_\pm^j(z_0;h) \sim A_0^{j,\pm}(z_0)+hA_1^{j,\pm}(z_0)+\cdots$ as in Proposition 3.5 and $\Phi_{\pm,0}^j(z)$ is given in (3.48).*

We already noticed that $K^{i,j}(v,\overline{w})$ is the covariance kernel for the GAF in Theorem 2.9 (see (2.34)). We recall from (3.32) that $A_0^{j,\pm}(z_0) > 0$. The above proposition implies that

$$\mathbb{E}\big[f_{i,j}^{\delta,h}(v)\overline{f_{l,k}^{\delta,h}(w)}\big]\mathrm{e}^{-F_{i,j}(v;h)-\overline{F_{l,k}(w;h)}} \xrightarrow{h\to 0} \delta_{i,l}\delta_{j,k}\,K^{i,j}(v,\overline{w}), \tag{7.19}$$

uniformly for $v, w \in O$. The "gauge factors" $\mathrm{e}^{F_{i,j}(\bullet;h)}$ never vanish and are deterministic, so the point process could as well be defined as the zero set of the random holomorphic function $\mathrm{e}^{F_{i,j}(\bullet;h)}f_{i,j}^{\delta,h}(\bullet)$.

*Proof of Proposition 7.1.* For $v, w \in O$, we have, by (7.13) and (RM),

$$hK_h^{i,j,l,k}(v,\overline{w}) := h\mathbb{E}\big[f_{i,j}^{\delta,h}(v)\overline{f_{l,k}^{\delta,h}(w)}\big] = \sum_{n,m,n',m'}\mathbb{E}\big[q_{n,m}^h\,\overline{q_{n',m'}^h}\big]\zeta_{n,m}^{i,j}(v)\,\overline{\zeta_{n',m'}^{l,k}(w)}, \tag{7.20}$$

where all indices are summed in the range $[0, N(h))$, and we use the notation

$$\zeta_{n,m}^{i,j}(w) = (e_+^{j,\mathrm{hol}}(z_w)|e_m)(e_n|e_-^{i,\mathrm{hol}}(z_w)), \tag{7.21}$$

and $z_w = z_0+h^{1/2}w$ as in (7.11). By (7.7), (7.8) and the independence of the coefficients $q_{n,m}^h$, the expression (7.20) is equal to

$$(1+\mathcal{O}(h^{2+\varepsilon_0}))\Big(\sum_m (e_+^{j,\mathrm{hol}}(z_v)|e_m)(e_m|e_+^{k,\mathrm{hol}}(z_w))\Big)\Big(\sum_n (e_-^{l,\mathrm{hol}}(z_w)|e_n)(e_n|e_-^{i,\mathrm{hol}}(z_v))\Big)$$
$$+\mathcal{O}(h^{6+2\varepsilon_0})\sum_{n,m,n',m'}|(e_+^{j,\mathrm{hol}}(z_v)|e_m)|\,|(e_n|e_-^{i,\mathrm{hol}}(z_v))|\,|(e_+^{k,\mathrm{hol}}(z_w)|e_{m'})|\,|(e_{n'}|e_-^{l,\mathrm{hol}}(z_w))|.$$

From Lemma 4.4,

$$\sum_{n<N(h)} |(e_+^{j,\mathrm{hol}}(z_v)|e_n)| \le N(h)^{1/2}\|e_+^{j,\mathrm{hol}}(z_v)\|(1+\mathcal{O}(h^\infty)).$$

Since $N(h)=\mathcal{O}(h^{-2})$, and by another application of Lemma 4.4, we get

$$\begin{aligned} hK_h^{i,j,l,k}(v,\overline{w}) &= (e_+^{j,\mathrm{hol}}(z_v)|e_+^{k,\mathrm{hol}}(z_w))(e_-^{l,\mathrm{hol}}(z_w)|e_-^{i,\mathrm{hol}}(z_v)) \\ &\quad + \mathcal{O}(h^{2+\varepsilon_0})\|e_+^{j,\mathrm{hol}}(z_v)\|\,\|e_-^{i,\mathrm{hol}}(z_v)\|\,\|e_+^{k,\mathrm{hol}}(z_w)\|\,\|e_-^{l,\mathrm{hol}}(z_w)\|. \end{aligned} \tag{7.22}$$

Using Proposition 3.5, Proposition 4.1, (7.22) and (4.3), we get the expression

$$\begin{aligned} hK_h^{i,j,l,k}(v,\overline{w}) &= \delta_{i,l}\delta_{j,k}\mathrm{e}^{\frac{2}{h}(\Psi_+^j(z_v,z_w;h)+\Psi_-^i(z_w,z_v;h))} \\ &\quad + \mathcal{O}(h^{2+\varepsilon_0})\mathrm{e}^{\frac{1}{h}(\Phi_+^j(z_v;h)+\Phi_+^k(z_w;h)+\Phi_-^i(z_v;h)+\Phi_-^l(z_w;h))}. \end{aligned} \tag{7.23}$$

Now, recall from (3.20) and (3.48) that the phase functions $\Phi_{\pm,0}^j$ were "centred" at the point $z_0$, so that

$$\Phi_{\pm,0}^j(z_0)=0, \qquad \partial_z\Phi_{\pm,0}^j(z_0)=0, \qquad (\partial_{\overline{z}}\Phi_{\pm,0}^j)(z_0)=0.$$

Taking into account that $\partial^\alpha \log A_\pm^j(\bullet;h) = \mathcal{O}(1)$ in $W(z_0)$, the Taylor expansion of $\Phi_\pm^j(\bullet;h)$ around $z_0$ gives

$$\frac{1}{h}\Phi_\pm^j(z_v;h) = \log(h^{1/4}) + (\partial_{z\overline{z}}^2\Phi_{\pm,0}^j)(z_0)v\overline{v} + \phi_\pm^j(v;h) + \overline{\phi_\pm^j(v;h)} + \mathcal{O}(h^{1/2}), \tag{7.24}$$

where we use the notation (7.18).

Similarly, by Proposition 4.1, we have

$$\begin{aligned} \frac{1}{h}\Psi_+^j(z_v,z_w;h) &= \log(h^{1/4}) + (\partial_{z\overline{z}}^2\Phi_{+,0}^j)(z_0)v\overline{w} + \phi_\pm^j(v;h) + \overline{\phi_+^j(w;h)} + \mathcal{O}(h^{1/2}) \\ &= \log(h^{1/4}) + \tfrac{1}{4}\sigma_+^j(z_0)v\overline{w} + \phi_+^j(v;h) + \overline{\phi_+^j(w;h)} + \mathcal{O}(h^{1/2}), \end{aligned} \tag{7.25}$$

where in the second line we have used (3.54). We also have

$$\frac{1}{h}\Psi_-^j(z_w,z_v;h) = \log(h^{1/4}) + \tfrac{1}{4}\sigma_-^j(z_0)\overline{w}v + \overline{\phi_-^j(w;h)} + \phi_-^j(v;h) + \mathcal{O}(h^{1/2}). \tag{7.26}$$

In all estimates the error terms are uniform in $z,w \in O$. Thus, combining (7.23) with (7.24)–(7.25) and using the notation (7.17), we obtain

$$K_h^{i,j,l,k}(v,\overline{w}) = \delta_{i,l}\delta_{j,k}\mathrm{e}^{\frac{1}{2}(\sigma_+^j(z_0)+\sigma_-^i(z_0))v\overline{w}}\mathrm{e}^{F_{i,j}(v;h)+\overline{F_{l,k}(w;h)}+\mathcal{O}(h^{1/2})} + \mathcal{O}(h^{2+\varepsilon_0}).$$

Notice that the factor $h$ on the left hand side of (7.23) is facing four factors $h^{1/4}$ on the right hand side, so we removed them all. This estimate gives the equation (7.15) of the proposition. ◻

*7.4. Tightness of the rescaled spectral determinant*

We will now show that the family $(F_h^\delta(\bullet))_{h\to 0}$ of random analytic functions on $O$, defined in (7.12), is tight, namely the function $F_h^\delta$ has a small probability to be large on $O$, uniformly as $h \to 0$.

**Proposition 7.2.** *There exists $h_0 > 0$ such that the family of random analytic functions $(F_h^\delta(\bullet))_{0<h\le h_0}$ defined in (7.12) is tight in the sense of Proposition 6.9.*

*Proof.* Recall the estimates (5.25) for the remainders $R_1, R_2$; for $z \in O$, we get $|z-z_0| = \mathcal{O}(h^{1/2})$, so all terms $|z-z_0|^\infty = \mathcal{O}(h^\infty)$. Moreover, the expansion (7.24) for $\frac{1}{h}\Phi_\pm^j(z_v; h)$ implies that the diagonal matrices $\Lambda_\pm(z_v)$ of (5.23) are of order $\mathcal{O}(1)$, so that

$$\widetilde{R}_1(v; h) = \mathcal{O}(\delta h^{-5/2}) = \mathcal{O}(h^{1/2}), \qquad \widetilde{R}_2(v; h) = \mathcal{O}(\delta h^{-5/2}) = \mathcal{O}(h^{1/2}), \tag{7.27}$$

uniformly in $v \in O$ and in the restricted probability space $\mathcal{M}_h$. Here we have used the assumption (1.6) on the perturbation parameter $\delta$.

Let $K \Subset O$ be some compact subset. Pick $\varepsilon > 0$ small enough such that the $\varepsilon$-neighbourhood satisfies $K_\varepsilon = K + \overline{D(0,\varepsilon)} \Subset O$. By Proposition 7.1, for $h_0 > 0$ small enough,

$$\sup_{0<h\le h_0} \mathbb{E}[\| f_{i,j}^{h,\delta}\|^2_{L^2(K_\varepsilon)}] \le C(K_\varepsilon) < \infty. \tag{7.28}$$

Since $F_h^\delta$ is holomorphic, Hardy's convexity theorem [40, Lemma 2.6] implies that for any $p > 0$,

$$\|F_h^\delta\|^p_{L^\infty(K)} \le (\pi\varepsilon^2)^{-1} \int_{K_\varepsilon} |F_h^\delta(w)|^p \, L(dw). \tag{7.29}$$

To estimate the size of $F_h^\delta$, we will use the following inequality, valid for any $J \times J$ matrix $M$:

$$|\det(M)| \le \|M\|^J_{\mathrm{HS}}. \tag{7.30}$$

Markov's inequality shows that for any $r > 0$,

$$\begin{aligned}\sup_{0<h<h_0} \mathbb{P}[\|F_h^\delta\|_{L^\infty(K)} > r] &= \sup_{0<h<h_0} \mathbb{P}[\|F_h^\delta\|^{2/J}_{L^\infty(K)} > r^{2/J}] \\ &\le \sup_{0<h<h_0} r^{-2/J}\mathbb{E}[\|F_h^\delta\|^{2/J}_{L^\infty(K)}].\end{aligned}$$

From (7.29), the definition (7.12) of $F_h^\delta$ and the algebraic inequality (7.30), the expectation $\mathbb{E}[\|F_h^\delta\|^{2/J}_{L^\infty(K)}]$ is bounded above by

$$(\pi\varepsilon^2)^{-1} \sup_{0<h<h_0} \mathbb{E}\Big[ \|1+\widetilde{R}_1\|^{2/J}_{L^\infty(K_\varepsilon)} \int_{K_\varepsilon} 2\big(\|(f_{i,j}^{h,\delta}(w))_{i,j\le J}\|^2_{\mathrm{HS}} + \|\widetilde{R}_2(w;h)\|^2_{\mathrm{HS}}\big)\, L(dw)\Big].$$

Finally, using the estimates (7.27) on the $\widetilde{R}_i$ and the uniform bounds (7.28), we get

$$\sup_{0<h<h_0} \mathbb{P}[\|F_h^\delta\|_{L^\infty(K)} > r] \le C(K,\varepsilon) r^{-2/J}. \tag{7.31}$$

This proves the tightness of the family $(F_h^\delta)_{0<h\le h_0}$. □

*7.5. Weak convergence to a Gaussian analytic function*

Next we will show that the random analytic function $F_h^\delta$ converges in distribution to a *Gaussian* analytic function when $h \to 0$. By Proposition 6.9 and Section 7.4, it is sufficient to prove the convergence of the finite-dimensional distributions of $F_h^\delta$. We recall that, in general, the coefficients $q_{i,j}$ are not Gaussian distributed.

We begin with the following result:

**Proposition 7.3.** *Let $f_{i,j}^{h,\delta}$ be as in (7.13), and let $K^{i,j}(z,\overline{w})$ be as in (7.16). Then*

$$(f_{i,j}^{h,\delta};\, 1 \le i,j \le J) \xrightarrow{\text{fd}} (f_{i,j}^{\mathrm{GAF}};\, 1 \le i,j \le J), \qquad h \to 0. \tag{7.32}$$

*Here $f_{i,j}^{\mathrm{GAF}}$ are independent Gaussian analytic functions with covariance kernels*

$$K^{i,j}(v,\overline{w})\mathrm{e}^{F_{i,j}(v;0)+\overline{F_{i,j}(w;0)}}, \qquad z,w \in O. \tag{7.33}$$

*$K^{i,j}$ was defined in (7.16), and the function $F_{i,j}(w;0)$ is the limit as $h \searrow 0$ of the function defined in (7.17)–(7.18).*

Before proving this proposition, we deduce an immediate consequence. We recall the expression (7.12) for the rescaled spectral determinant:

$$F_h^\delta(w) = (1+\widetilde{R}_1(w;h))\det[(f_{i,j}^{\delta,h}(w))_{i,j\le J} + \widetilde{R}_2(w;h)],$$

where both terms $\widetilde{R}_i$ are $\mathcal{O}(h^{1/2})$, uniformly on the restricted probability space.

**Corollary 7.4.** *Under the notations of Proposition 7.3, we have*

$$F_h^\delta \xrightarrow{d} \det((f_{i,j}^{\mathrm{GAF}})_{i,j}), \qquad h \to 0. \tag{7.34}$$

*Proof.* We start by proving that the perturbation of (7.32) by $\widetilde{R}_2$ is irrelevant for the limit:

$$((f_{i,j}^{h,\delta} + (\widetilde{R}_2)_{i,j})_{1\le i,j\le J} \xrightarrow{\text{fd}} (f_{i,j}^{\mathrm{GAF}})_{1\le i,j\le J}, \qquad h \to 0. \tag{7.35}$$

Equivalently, for any $L \in \mathbb{N}^*$ and any $(w_1,\dots,w_L) \in O^L$, we want to show that

$$(f_{i,j}^{h,\delta}(w_l) + (\widetilde{R}_2)_{i,j}(w_l))_{\substack{1\le i,j\le J\\ 1\le l\le L}} \xrightarrow{d} (f_{i,j}^{\mathrm{GAF}}(w_l))_{\substack{1\le i,j\le J\\ 1\le l\le L}}, \qquad h \to 0. \tag{7.36}$$

Proposition 7.3 proves the convergence of the left hand side. The uniform bounds (7.27) imply that the $J^2L$-vector $(\widetilde{R}_2(w_l))_{1\le l\le L}$ converges to zero everywhere as $h \to 0$, therefore it converges in probability. The application of Lemma 7.5 below, in the case $N = J^2L$, thus proves the convergence (7.36).

In a second step, from Definition 6.4, it is easy to check that the convergence in distribution is preserved under composition with a continuous function [31, Theorem 4.27]. As a result, we infer from (7.36) the convergence

$$D_h^\delta(w) := \det((f_{i,j}^{h,\delta})_{i,j} + \widetilde{R}_2) \xrightarrow{\text{fd}} \det((f_{i,j}^{\mathrm{GAF}})_{i,j}), \qquad h \to 0. \tag{7.37}$$

Third, we split

$$F_h^\delta = (1+\widetilde{R}_1)D_h^\delta = D_h^\delta + \widetilde{R}_1 D_h^\delta.$$

For any $(w_1,\dots,w_L) \in O^L$, we use the tightness of $(D_h^\delta(w_l)))_{1\le L}$ and the uniform decay of the $\widetilde{R}_1(w_l)$ to show that the $\mathbb{C}^L$-random vector $(\widetilde{R}_1(w_l)\, D_h^\delta(w_l))_{1\le l\le L}$ converges to zero in probability as $h \to 0$. One more application of Lemma 7.5 then proves the convergence of $(F_h^\delta(w_l))_{1\le L}$, and thus of $F_h^\delta(w)$ to the right hand side of (7.34) in the sense of finite-dimensional distributions.

Finally, since $F_h^\delta$ is a tight sequence of random analytic functions (see Section 7.4), Proposition 6.9 implies its convergence in distribution stated in the corollary. □

**Lemma 7.5.** *Let $(X_n)_{n\in\mathbb{N}}$, $X$ and $(R_N)_{n\in\mathbb{N}}$ be random vectors in $\mathbb{C}^N$, such that $X_n \stackrel{d}{\to} X$, while $R_n$ converges in probability to zero as $n\to\infty$:*

$$\forall \varepsilon > 0 \qquad \lim_{n\to\infty} \mathbb{P}[|R_n| > \varepsilon] = 0. \tag{7.38}$$

*Then $X_n + R_n \stackrel{d}{\to} X$.*

*Proof.* By a standard result [31, Lemma 4.8], the convergence $X_n \stackrel{d}{\to} X$ implies that $(X_n)_{n\in\mathbb{N}}$ is a tight sequence of random vectors. From the assumption (7.38), it also easily follows that $(X_n + R_n)_{n\in\mathbb{N}}$ is tight.

As a result, from Definition 6.4 and Remark 6.8, to test the convergence of the latter sequence it suffices to consider compactly supported test functions. Let us thus choose $\phi \in \mathcal{C}_c(\mathbb{C}^N,\mathbb{R})$ and fix some arbitrary $\varepsilon > 0$. Since the function $\phi$ is uniformly continuous, there exists a $\delta > 0$ such that for any $|X-Y| < \delta$, if we split the expectation as

$$\begin{aligned}|\mathbb{E}[\phi(X_n+R_n)-\phi(X_n)]| \le |&\mathbb{E}[\phi(X_n+R_n)-\phi(X_n) \mid |R_n|<\delta]|\mathbb{P}[|R_n|<\delta]\\ &+ |\mathbb{E}[\phi(X_n+R_n)-\phi(X_n) \mid |R_n|\ge\delta]|\mathbb{P}[|R_n|\ge\delta],\end{aligned}$$

the first term is bounded above by $\varepsilon/2$. In turn, the assumption (7.38) allows us to choose $n_\delta > 0$ such that for all $n \ge n_\delta$, $\mathbb{P}[|R_n| \ge \delta] < \varepsilon/(4\|\phi\|_{L^\infty})$; the second term is then bounded above by $\varepsilon/2$. □

*Proof of Proposition 7.3.* The proof is an adaptation of the proof of [40, Theorem 4.4]. Our goal is to prove the convergence (7.36), in absence of the terms $\widetilde{R}_2$. By the Cramér–Wold Theorem [31, Corollary 5.5], it suffices to show that for any $\lambda = (\lambda_l^{i,j};\, 1\le i,j\le J,\ 1\le l\le L) \in \mathbb{C}^{J^2L}$, the complex valued random variable

$$S(\lambda) := \sum_{i,j,l} \lambda_l^{i,j} f_{i,j}^{h,\delta}(w_l)$$

converges in distribution, as $h \to 0$, to the complex Gaussian random variable

$$S^{\mathrm{GAF}}(\lambda) := \sum_{i,j,l} \lambda_l^{i,j} f_{i,j}^{\mathrm{GAF}}(w_l). \tag{7.39}$$

Let us write $S(\lambda)$ in terms of our restricted complex random variables $q_{n,m}^h$:

$$S(\lambda) = \sum_{n,m<N(h)} q_{n,m}^h G_{n,m}, \qquad G_{n,m} = \sum_{i,j,l} \lambda_l^{i,j} h^{-1/2} \zeta_{n,m}^{i,j}(w_l), \tag{7.40}$$

where we have used the notation of (7.21). Since the coefficients $q^h_{n,m}$ are not Gaussian, to prove that $S(\lambda) \xrightarrow{d} S^{\mathrm{GAF}}(\lambda)$ we will invoke the Central Limit Theorem 6.12. We thus need to check that the family $(S(\lambda))_{0<h\le h_0}$ of random variables satisfies the four conditions stated in the theorem.

Let us first estimate the average of $S(\lambda)$: from the left of (7.7), we get

$$\begin{aligned}|\mathbb{E}[S(\lambda)]| &\le \mathcal{O}(h^{3+\varepsilon_0}) \sum_{n,m<N(h)} |G_{n,m}| \\ &\le \mathcal{O}_\lambda(h^{3+\varepsilon_0}) \sum_{i,j,l} \sum_{m<N(h)} |(h^{-1/4} e_+^{j,\mathrm{hol}}(z_{w_l})|e_m)| \sum_{n<N(h)} |(h^{-1/4} e_-^{i,\mathrm{hol}}(z_{w_l})|e_n)| \\ &\le \mathcal{O}_\lambda(h^{3+\varepsilon_0}) N(h) \sum_{i,j,l} \|h^{-1/4} e_+^{j,\mathrm{hol}}(z_{w_l})\| \, \|(h^{-1/4} e_-^{i,\mathrm{hol}}(z_{w_l})\| \\ &\le \mathcal{O}_\lambda(h^{1+\varepsilon_0}). \end{aligned} \tag{7.41}$$

Here $\mathcal{O}_\lambda$ just indicates that the implied constant depends on the vector $\lambda$. To obtain the third line we have used twice the Cauchy–Schwarz inequality, and for the last one the fact that $\|h^{-1/4} e_\pm^{j,\mathrm{hol}}(z_w)\| = \mathcal{O}(1)$ uniformly when $w \in O$. This proves point (i) in Theorem 6.12.

Let us now check condition (ii) of the theorem. Using the right relation of (7.7), we obtain the bound

$$\begin{aligned}\Big| \sum_{n,m<N(h)} \mathbb{E}[(q^h_{n,m})^2] G^2_{n,m} \Big| &\le \sum_{n,m<N(h)} |\mathbb{E}[(q^h_{n,m})^2] G^2_{n,m}| \le \mathcal{O}(h^{2+\varepsilon_0}) \sum_{m,n} |G_{n,m}|^2 \\ &\le \mathcal{O}_\lambda(h^{2+\varepsilon_0}) \sum_{i,j,l} \sum_{m<N(h)} |(h^{-1/4} e_+^{j,\mathrm{hol}}(z_{w_l})|e_m)|^2 \sum_{n<N(h)} |(h^{-1/4} e_-^{i,\mathrm{hol}}(z_{w_l})|e_n)|^2 \\ &\le \mathcal{O}_\lambda(h^{2+\varepsilon_0}). \end{aligned} \tag{7.42}$$

The computation of the variance of $S(\lambda)$ needs more care. By (7.8) and (7.40),

$$\begin{aligned}\mathbb{E}\Big[ \sum_{n,m<N(h)} |q^h_{n,m} G_{n,m}|^2 \Big] &= \mathbb{E}[|\alpha^h|^2] \sum_{n,m<N(h)} |G_{n,m}|^2 \\ &= (1 + \mathcal{O}(h^{2+\varepsilon_0})) \sum_{i,j,l,r,s,t} \lambda_l^{i,j} \overline{\lambda_t^{r,s}} \sum_{n,m<N(h)} h^{-1} \zeta^{i,j}_{n,m}(w_l) \overline{\zeta^{r,s}_{n,m}(w_t)}. \end{aligned}$$

By (7.21) and Lemma 4.4, this is equal to

$$\begin{aligned}(1 + \mathcal{O}(h^{2+\varepsilon_0})) \sum_{i,j,l,r,s,t} \lambda_l^{i,j} \overline{\lambda_t^{r,s}} \, h^{-1} (e_+^{j,\mathrm{hol}}(z_{w_l})|e_+^{s,\mathrm{hol}}(z_{w_t})) \\ \times (e_-^{r,\mathrm{hol}}(z_{w_t})|e_-^{i,\mathrm{hol}}(z_{w_l})) + \mathcal{O}_\lambda(h^\infty). \end{aligned} \tag{7.43}$$

To control the remainder we have used again $\|e_\pm^{k,\mathrm{hol}}(z_w)\| = \mathcal{O}(h^{1/4})$ uniformly for $w \in O$. From the quasi-orthogonality (4.3) we see that the only nonnegligible terms in (7.43) should satisfy $i = r$ and $j = s$. Using the notations of Proposition 7.1, we then obtain, as $h \to 0$,

$$\mathbb{E}\Big[\sum_{n,m<N(h)} |q^h_{n,m} G_{n,m}|^2\Big] \to \sum_{i,j,l,t} \lambda^{i,j}_l \overline{\lambda^{i,j}_t} K^{i,j}(w_l, w_t) \mathrm{e}^{F_{i,j}(w_l;0)+\overline{F_{i,j}(w_t;0)}} =: \sigma(\lambda, w). \tag{7.44}$$

In order to check condition (iv) of Theorem 6.12, we use the following connection with $(4+\varepsilon_0)$-moments: for any $\varepsilon > 0$,

$$\sum_{n,m<N(h)} \mathbb{E}[|q^h_{n,m} G_{n,m}|^2 \mathbb{1}_{\{|q^h_{n,m} G_{n,m}|>\varepsilon\}}] \le \varepsilon^{-(2+\varepsilon_0)} \sum_{n,m<N(h)} \mathbb{E}[|q^h_{n,m} G_{n,m}|^{4+\varepsilon_0}].$$

From (7.9) and the Hölder inequality, the above right hand side is bounded above by

$$\begin{aligned} &C \sum_{n,m<N(h)} \Big|\sum_{i,j,l} \lambda^{i,j}_l \, h^{-1/2} \zeta^{i,j}_{n,m}(w_l)\Big|^{4+\varepsilon_0} \\ &\le C_\lambda \sum_{i,j,l} \Big(\sum_{m<N(h)} |(h^{-1/4} e^{j,\mathrm{hol}}_+(z_{w_l})|e_m)|^{4+\varepsilon_0}\Big)\Big(\sum_{n<N(h)} |(h^{-1/4} e^{i,\mathrm{hol}}_-(z_{w_l})|e_n)|^{4+\varepsilon_0}\Big). \end{aligned}$$

Splitting $|\bullet|^{4+\varepsilon_0} = |\bullet|^2 |\bullet|^{2+\varepsilon_0}$, using the bound (4.36) of Lemma 4.5 on the individual overlaps and the fact that $(e_m)$ forms an orthonormal basis, one finds that the above quantity is bounded above by

$$\begin{aligned} &= \mathcal{O}_\lambda(h^{1+\varepsilon_0/2}) \sum_{i,j,l} \|h^{-1/4} e^{j,\mathrm{hol}}_+(z_{w_l}))\|^2 \|h^{-1/4} e^{i,\mathrm{hol}}_-(z_{w_l})\|^2 \mathrm{e}^{\frac{2}{h}\Phi^i_{-,0}(z_{w_l})} \mathrm{e}^{\frac{2}{h}\Phi^j_{+,0}(z_{w_l})} \\ &= \mathcal{O}_\lambda(h^{1+\varepsilon_0/2}). \end{aligned}$$

In the last line we have used the fact that both the norms and the exponentials are $\mathcal{O}(1)$ uniformly for $w \in O$. This checks condition (iv) of Theorem 6.12.

**Remark 7.6.** It is especially to control this $(4+\varepsilon_0)$-moment that we need all the overlaps $(h^{-1/4} e^{j,\mathrm{hol}}_\pm(z_w)|e_m)$ to be small, and for this very reason we chose our auxiliary basis $(e_m)_{m\in\mathbb{N}}$ to have different microlocalization properties than our quasimodes.

Havind checked that the four conditions of Theorem 6.12 are satisfied by the sum $S(\lambda)$ of (7.40), we may apply this CLT to show that $S(\lambda)$ converges in distribution to the complex Gaussian random variable $\mathcal{N}_\mathbb{C}(0, \sigma(\lambda, w)^2)$, with variance given in (7.44). On the other hand, since the $(f^{\mathrm{GAF}}_{i,j})_{i,j\le J}$ are independent Gaussian analytic functions with covariance kernel (7.33), the sum $S^{\mathrm{GAF}}(\lambda)$ of (7.39) is a complex centred Gaussian variable, with variance

$$\begin{aligned} \mathbb{E}\Big[\Big|\sum_{i,j,l} \lambda^{i,j}_l f^{\mathrm{GAF}}_{i,j}(w_l)\Big|^2\Big] &= \sum_{i,j,l,r,s,t} \lambda^{i,j}_l \overline{\lambda^{r,s}_t} \, \mathbb{E}\big[f^{\mathrm{GAF}}_{i,j}(w_l) \overline{f^{\mathrm{GAF}}_{r,s}(w_t)}\big] \\ &= \sum_{i,j,l,t} \lambda^{i,j}_l \overline{\lambda^{i,j}_t} K^{i,j}(w_l, w_t) \mathrm{e}^{F_{i,j}(w_l;0)+\overline{F_{i,j}(w_t;0)}} \\ &= \sigma(\lambda, w). \end{aligned} \tag{7.45}$$

Since a complex centred Gaussian random variable is uniquely determined by its variance, we conclude that $S(\lambda) \xrightarrow{d} S^{\mathrm{GAF}}(\lambda)$. This completess the proof of the proposition. □

Let us come back to the result of Corollary 7.4 and Proposition 7.3. The covariance kernels of the GAFs $f_{i,j}^{\mathrm{GAF}}$ were defined (see (7.33), (7.17), (7.18)) by

$$K^{i,j}(v,\overline{w})\mathrm{e}^{F_{i,j}(v;0)}\mathrm{e}^{\overline{F_{i,j}(w;0)}}, \qquad F_{i,j}(v;0) = \phi_-^i(v;0) + \phi_+^j(v;0),$$
$$\phi_\pm^i(v;0) = \tfrac{1}{2}[\log A_0^{j,\pm}(z_0) + (\partial_{zz}^2 \Phi_{\pm,0}^j)(z_0)\, v^2].$$

Since $\mathrm{e}^{F_{i,j}(\bullet;0)}$ is a nonvanishing deterministic holomorphic function on $O$, we define the random analytic function

$$g_{z_0}^{i,j}(\bullet) := f_{i,j}^{\mathrm{GAF}}(\bullet)\mathrm{e}^{-F_{i,j}(\bullet;0)}, \qquad i,j = 1,\dots,J.$$

Then $\{g_{z_0}^{i,j};\, i,j = 1,\dots,J\}$ are independent GAFs on $O$, with covariance kernels $K^{i,j}(v,\overline{w})$, given in Theorem 2.9. If we define the diagonal matrices

$$\Lambda_\pm(v) = \mathrm{diag}\,((\mathrm{e}^{-\phi_\pm^i(v;0)})_{1\le i\le J}),$$

we get equality of random analytic functions:

$$T(v) := \det((f_{i,j}^{\mathrm{GAF}}(v))_{i,j}) = \det(\Lambda_-(v)\Lambda_+(v))\det((g_{z_0}^{i,j}(v))_{i,j})$$
$$:= \det(\Lambda_-(v)\Lambda_+(v))\tilde{G}_{z_0}(v).$$

Since $\det(\Lambda_-\Lambda_+)(v)$ never vanishes, the zero point processes associated with the random functions $T$ and $\tilde{G}_{z_0}$ coincide. Hence, from Corollary 7.4 we infer that

$$\mathcal{Z}_{F_h^\delta} \xrightarrow{d} \mathcal{Z}_T = \mathcal{Z}_{\tilde{G}_{z_0}} \text{ when } h \to 0. \tag{7.46}$$

Together with the discussion at the beginning Section 7 and the fact that $\mathcal{Z}_{F_h^\delta} = \mathcal{Z}_{h,z_0}^M$ represents the set of rescaled eigenvalues of $P^\delta$, this concludes the proof of Theorem 2.9. □

### 7.6. *k-point measures*

In this subsection we show that the $k$-point measures $\mu_k$ of the point process $\mathcal{Z}_{h,z_0}^M$, defined in (2.26), converge to the $k$-point measures $\mu_k$ of the point process $\mathcal{Z}_{\tilde{G}_{z_0}}$ defined in Theorem 2.9. We begin with a technical

**Lemma 7.7.** *Let $F_h^\delta(w)$ be as in (7.12), and let $\tilde{G}_{z_0}(w)$ be as in Theorem 2.9, with $w \in O$. Then, for any $K \Subset O$, the distribution of the numbers $n_F^h(K)$ (resp. $n_{\tilde{G}_{z_0}}(K)$) of zeros of $F_h^\delta$ (resp. $\tilde{G}_{z_0}$) in $K$ has exponential tail: there exist constants $C_1, C_2 > 0$ such that for any $\lambda > 0$,*

$$\forall h \le h_0, \quad \mathbb{P}[n_F^h(K) > \lambda] \le C_1\mathrm{e}^{-C_2\lambda}, \quad \mathbb{P}[n_{G_{z_0}}(K) > \lambda] \le C_1\mathrm{e}^{-C_2\lambda}.$$

*Proof.* From [30, Theorem 3.2.1], to prove the first inequality it suffices to show that, for some $c, b > 0$, the random analytic function $F_h^\delta$ satisfies

$$\mathbb{E}[|F_h^\delta(w)|^{\pm c}] \leq b \quad \text{uniformly for } w \in O \text{ and } h \leq h_0, \tag{7.47}$$

and the second inequality requires a similar estimate for the random function $\tilde{G}_{z_0}(w)$.

Fix $w \in O$. Recall the bounds (7.27) on $\widetilde{R}_1(w)$, $\widetilde{R}_2(w)$. We start with the bound:

$$\begin{aligned}
\mathbb{E}[|F_h^\delta(w)|^{2/J}] &\leq C_1 \mathbb{E}[|\det((f_{i,j}^{\delta,h}(w))_{i,j} + \widetilde{R}_2(w))|^{2/J}] \\
&\leq C_1 \mathbb{E}[\|(f_{i,j}^{\delta,h}(w))_{i,j} + \widetilde{R}_2(w)\|_{\mathrm{HS}}^2] \\
&\leq 2C_1 \mathbb{E}[\|(f_{i,j}^{\delta,h}(w))_{i,j}\|_{\mathrm{HS}}^2 + \|\widetilde{R}_2(w)\|_{\mathrm{HS}}^2] \\
&\leq 2C_1 \sum_{i,j=1}^{J} [K^{i,j}(w,\bar{w}) \mathrm{e}^{2\,\mathrm{Re}\, F_{i,j}(w;h)}(1 + \mathcal{O}(h^{1/2})) + \mathcal{O}(h)] \\
&\leq \mathcal{O}(1) \quad \text{uniformly in } w \in O \text{ and } h \leq h_0,
\end{aligned} \tag{7.48}$$

where in the second line we have used the inequality (7.30), and in the two last inequalities we have used Proposition 7.1.

We also need to check that the inverse function $F_h^\delta(w)^{-1}$ is not too large on average:

$$\begin{aligned}
\mathbb{E}[|F_h^\delta(w)|^{-1/J}] &= \int_0^\infty \mathbb{P}[|F_h^\delta(w)|^{-1/J} \geq t]\, dt \\
&= \int_0^\infty \mathbb{P}[|F_h^\delta(w)|^{1/J} \leq \tau]\tau^{-2}\, d\tau.
\end{aligned} \tag{7.49}$$

From the convergence in distribution (7.34) and the Portmanteau Theorem [31, Theorem 3.25],

$$\limsup_{h\to 0} \mathbb{P}[|F_h^\delta(w)|^{1/J} \leq \tau] \leq \mathbb{P}[|\det(f_{i,j}^{\mathrm{GAF}}(w))_{i,j}|^{1/J} \leq \tau]. \tag{7.50}$$

Our next goal is to compute the probability on the right hand side. We thus fix some $w \in O$ and study the random determinant at $w$. From (7.45), the variables $(f_{i,j}^{\mathrm{GAF}}(w))_{1\leq i,j\leq J}$ are independent centred complex Gaussian variables, with variances

$$\begin{aligned}
K^{i,j}(w,\bar{w})\mathrm{e}^{2\,\mathrm{Re}\, F_{i,j}(w;0)} &= \left(\mathrm{e}^{\frac{1}{2}\sigma_-^i(z_0)|w|^2 + 2\,\mathrm{Re}\,\phi_-^i(w;0)}\right)\left(\mathrm{e}^{\frac{1}{2}\sigma_+^j(z_0)|w|^2 + 2\,\mathrm{Re}\,\phi_+^j(w;0)}\right) \\
&=: \beta_-^i(w)\beta_+^j(w) \neq 0.
\end{aligned}$$

Using this factorization of the variances, we set $\beta_\pm(w) = \mathrm{diag}((\beta_\pm^i(w))_{1\leq i\leq J})$, which allows us to factorize the limiting random determinant as follows:

$$\det(f_{i,j}^{\mathrm{GAF}}(w)) = \det\beta_+(w)\det\beta_-(w)\ \det(v_1(w),\ldots,v_J(w)), \tag{7.51}$$

$$v_j(w) = (v_{1j}(w),\ldots,v_{Jj}(w))^T, \tag{7.52}$$

where the entries $(v_{ij}(w))_{1\leq i,j\leq J}$ are i.i.d. complex Gaussian random variables with distribution $\mathcal{N}_{\mathbb{C}}(0,1)$ (in other words, the matrix $(v_{ij}(w))$ is a Ginibre random matrix of size $J\times J$). Until further notice we suppress $w$ in the notation. Each column $v_j$ is a Gaussian vector in $\mathbb{C}^J$, with the identity $I$ as covariance matrix, i.e. with distribution $\mathcal{N}_{\mathbb{C}}(0,I)$. As a result, the real variable $r=|v_{ij}|^2$ has the exponential distribution $f(r)dr$ with

$$f(r)=\mathrm{e}^{-r}H(r),\qquad H(r)=\mathbb{1}_{[0,\infty[}(r),\qquad \text{with Fourier transform } \hat{f}(\rho)=\frac{1}{1+i\rho}.$$

Since the components $(v_{ij})_{i\leq J}$ are independent, the squared norm $|v_j|^2$ is distributed according to the $J$-th convolution power $f*\cdots*f(r)dr=f^{*J}dr$. A direct computation shows that

$$f^{*J}(r)=\frac{r^{J-1}\mathrm{e}^{-r}}{(J-1)!}H(r),$$

which is the $\chi^2_{2J}$ distribution in the variable $2r$. We now compute the law of $|\det(v_1,\ldots,v_J)|$. For this, we perform $J-1$ linear operations on the matrix $(v_1,\ldots,v_J)$, setting $\widetilde{v}_1=v_1$, and iteratively for $j=2,\ldots,J$, taking for $\widetilde{v}_j$ the orthogonal projection of $v_j$ onto the orthogonal complement of the space $V_{j-1}:=\mathrm{span}_{\mathbb{C}}\{v_1,\ldots,v_{j-1}\}$. Elementary linear algebra then allows us to write

$$|\det(v_1,\ldots,v_J)|=|\det(\widetilde{v}_1,\ldots,\widetilde{v}_J)|=\prod_{j=1}^{J}|\widetilde{v}_j|.$$

Once $v_1,\ldots,v_{j-1}$, hence $V_{j-1}$, are chosen, the vector $\widetilde{v}_j$ is distributed as a complex Gaussian random vector in $V_j^{\perp}\equiv\mathbb{C}^{J-j+1}$, with distribution $\mathcal{N}_{\mathbb{C}}(0,I)$ (this follows from the fact that the initial distribution $\mathcal{N}_{\mathbb{C}}(0,I)$ is invariant under unitary transformations on $\mathbb{C}^J$). As a result $\{|\widetilde{v}_j|^2;\ 1\leq j\leq J\}$ are independent random variables with $\chi^2_{2(J-j+1)}$ distributions.

Setting $\eta=\eta(w):=(\det\beta_+(w)\beta_-(w))^{1/J}$ and using the fact that the $|\widetilde{v}_j|^2$ are independent, by (7.51) and a straightforward computation we get

$$\begin{aligned}\mathbb{P}[|\det(f^{\mathrm{GAF}}_{i,j}(w))|^{1/J}\leq\tau]&=\mathbb{P}[|\det(v_1,\ldots,v_J)|^{1/J}\leq\tau/\eta]\\&=\mathbb{P}\Big[\Big(\prod_{j=1}^{J}|\widetilde{v}_j|^2\Big)^{1/J}\leq(\tau/\eta)^2\Big]=1-\mathbb{P}\Big[\Big(\prod_{j=1}^{J}|\widetilde{v}_j|^2\Big)^{1/J}>(\tau/\eta)^2\Big]\\&\leq 1-\prod_{j=1}^{J}\mathbb{P}[|\widetilde{v}_j|^2>(\tau/\eta)^2]=1-\mathrm{e}^{-\frac{\tau^2}{\eta^2}}\prod_{j=1}^{J-1}\sum_{k=0}^{J-j}\frac{(\tau/\eta)^{2(J-j-k)}}{(J-j-k)!}.\end{aligned}$$

Each sum on the right hand side is larger than or equal to 1, hence so is their product, and we get

$$\mathbb{P}[|\det(f^{\mathrm{GAF}}_{i,j}(w))|^{1/J}\leq\tau]\leq 1-\mathrm{e}^{-\frac{\tau^2}{\eta^2}J}\leq\frac{\tau^2}{\eta(w)^2}J.$$

Combining this with (7.49), (7.50) and splitting the integral into $[0,1]\cup[1,\infty[$, we obtain

$$\mathbb{E}[|F^{\delta}_h(w)|^{-1/J}]=\mathcal{O}(1)\qquad\text{uniformly in } w\in O \text{ and } h\leq h_0. \tag{7.53}$$

Together with (7.48), this proves the bounds (7.47) with $c = 1/J$, hence the first inequality of the lemma. An easy adaptation of the above computations shows that the function $\tilde{G}_{z_0}(w)$ satisfies similar bounds, and hence the second inequality. □

Using Lemma 7.7, we see that for all $\varphi \in \mathcal{C}_c(O, \mathbb{R}_+)$ and any $p > 0$ and $h < h_0$,

$$\mathbb{E}[|\langle \mathcal{Z}_{F_h^\delta}, \varphi\rangle|^p] \le \|\varphi\|_\infty^p \mathbb{E}[(n_F^h(\operatorname{supp}\varphi))^p] \le \|\varphi\|_\infty^p \int_0^\infty \mathbb{P}[(n_F^h(\operatorname{supp}\varphi))^p \ge t]\, dt$$
$$\le C_1 \|\varphi\|_\infty^p \int_0^\infty \mathrm{e}^{-C_2 t^{1/p}}\, dt \le C(\varphi, p) < \infty.$$

Hence, for all $\varphi_l \in \mathcal{C}_c(O, \mathbb{R}_+)$, $l = 1, \dots, k$, the positive random variable $\langle \mathcal{Z}_{F_h^\delta}, \varphi_1\rangle \cdots \langle \mathcal{Z}_{F_h^\delta}, \varphi_k\rangle$ is integrable, uniformly in $h \le h_0$. By Proposition 6.11 and (7.46) it then follows that

$$\mathbb{E}[\langle \mathcal{Z}_{F_h^\delta}, \varphi_1\rangle \cdots \langle \mathcal{Z}_{F_h^\delta}, \varphi_k\rangle] \to \mathbb{E}[\langle \mathcal{Z}_{\tilde{G}_{z_0}}, \varphi_1\rangle \cdots \langle \mathcal{Z}_{\tilde{G}_{z_0}}, \varphi_k\rangle] \quad \text{as } h \to 0.$$

Since linear combinations of functions in $\bigotimes_{j=1}^k \mathcal{C}_c(O, \mathbb{R}_+)$ form a dense set in $\mathcal{C}_c(O^k, \mathbb{R}_+)$, we have obtained the following

**Theorem 7.8.** *Take any open, relatively compact connected domain $O \Subset \mathbb{C}$. Let $\mu_{h,z_0}^{k,M}$ (resp. $\mu_{z_0}^{k,M}$) be the k-point density measure (see (2.26)) of the point process $\mathcal{Z}_{h,z_0}^M$ (resp. $\mathcal{Z}_{\tilde{G}_{z_0}}$) defined on $O$ in Theorem 2.9. Then, for any $\varphi \in \mathcal{C}_c(O^k, \mathbb{R}_+)$,*

$$\int_{O^k} \varphi\, d\mu_{h,z_0}^{k,M} \to \int_{O^k} \varphi\, d\mu_{z_0}^{k,M}, \qquad h \to 0.$$

If we choose our test functions supported away from the generalized diagonal, this theorem gives Corollary 2.10.

To end this section on matrix perturbations, let us compute the formula (2.35) for the 1-point density. The Lelong formula states that, in the sense of distributions,

$$\mathcal{Z}_{G_{z_0}} = \frac{1}{\pi} \partial_{\bar{w}} \partial_w \log |\det \tilde{G}_{z_0}|^2.$$

Hence, the 1-point measure is given by

$$\mu_1(dw) = \frac{1}{\pi} \partial_{\bar{w}} \partial_w \mathbb{E}[\log |\det \tilde{G}_{z_0}(w)|^2] L(dw). \tag{7.54}$$

Recall from Theorem 2.6 that for any fixed $w \in O$, the matrix elements $g_{z_0}^{i,j}(w)$ of $\tilde{G}_{z_0}(w)$ are complex Gaussian variables with variance $\mathrm{e}^{\frac{1}{2}(\sigma_+^j(z_0) + \sigma_-^i(z_0))|w|^2}$. If we consider the diagonal matrices

$$\widetilde{\Lambda}_\pm(w) = \operatorname{diag}(\mathrm{e}^{\frac{1}{4}\sigma_\pm^i(z_0)|w|^2};\ 1 \le i \le J),$$

then the elements of the matrix $\widetilde{G}(w) := \widetilde{\Lambda}_-(w)^{-1} \tilde{G}_{z_0}(w) \widetilde{\Lambda}_+(w)^{-1}$ are i.i.d. $\sim \mathcal{N}_{\mathbb{C}}(0, 1)$. As a consequence, $\mathbb{E}[\log |\det \widetilde{G}(w)|^2]$ is independent of $w$. When applying the Lelong formula (7.54), the differentiations will only act on $\log|\det\widetilde{\Lambda}_-(w)|^2| + \log|\det\widetilde{\Lambda}_+(w)|^2$, and yield the 1-point density

$$d_{z_0}^{1,M}(z) = \frac{1}{2\pi}\sum_{i=1}^{J}(\sigma_+^j(z_0) + \sigma_-^i(z_0)). \qquad \square$$

## 8. Local statistics of the eigenvalues of $P_h$ perturbed by a random potential $V_\omega$

In this section we are interested in the local statistics of the eigenvalues of the operator perturbed by a random potential,

$$P^\delta = P_h + \delta V_\omega, \tag{8.1}$$

where the symbol of the initial operator $P_h$ satisfies the symmetry assumption (SYM), and the random potential $V_\omega$ is defined in (RP).

We will prove Theorem 2.5 on the limiting process. The steps of the proof are the same as for the proof of Theorem 2.9 in Section 7. However, some details are different, in particular the random potential acts locally in space, and can therefore only connect quasi-modes localized near the same point $x_i$. As a result, the effective Hamiltonian $E_{-+}^\delta(z)$ obtained through the Grushin problem is essentially a *diagonal matrix*, as opposed to the case of a random matrix perturbation, leading to the factorized limiting zero process of Theorem 2.5. Finally, we provide the proofs of Theorem 2.6 in Subsection 8.4, and of Proposition 2.12 in Subsection 8.5.

As explained in (2.20), we restrict the random variables used in the construction of $V_\omega$ to large polydiscs:

$$v = (v_k)_{k<N(h)} \in \mathrm{PD}_{N(h)}(0, C/h), \tag{8.2}$$

which implies the estimate (5.9). As in Section 7, consider the restricted probability space $\mathcal{M}_h$, and draw the i.i.d. restricted random variables $(v_k^h)_{k<N(h)}$, distributed according to the law of $\alpha^h$, (7.3). As in Section 7, we pick a $z_0 \in \Omega$. The Grushin problem constructed in Proposition 5.3, leading to (5.24), shows that the eigenvalues of the perturbed operator $P^\delta$ in $W(z_0)$, a relatively compact neighbourhood of $z_0$, are given by the zeros of the holomorphic function

$$G^\delta(z;h) = (1+R_1(z;h))\det\big[(V_\omega h^{-1/4}e_+^{j,\mathrm{hol}}(z)|h^{-1/4}e_-^{i,\mathrm{hol}}(z))_{i,j\le J}+R_2(z;h)\big], \tag{8.3}$$

with the error terms $R_1$, $R_2$ satisfying the bounds (5.25):

$$\begin{aligned} R_1(z;h) &= \mathcal{O}(|z-z_0|^\infty + \delta h^{-5/2}),\\ (R_2(z;h))_{i,j} &= \mathrm{e}^{\frac{1}{h}(\Phi^i_{+,0}(z)+\Phi^j_{-,0}(z)+\mathcal{O}(|z-z_0|^\infty))}\mathcal{O}(\delta h^{-3}), \end{aligned} \tag{8.4}$$

uniformly in $z \in W(z_0)$ and in the restricted probability space. Since the spectrum of $P^\delta$ in $W(z_0)$ is discrete, $G^\delta(\bullet;h) \not\equiv 0$ in $W(z_0)$.

Recall from (3.43) that the assumption (SYM) implies that the $\pm$ quasimodes are complex conjugate to one another:

$$e_-^{i,\mathrm{hol}}(z;h) = \overline{e_+^{i,\mathrm{hol}}(z;h)}, \tag{8.5}$$

The further assumptions (HYP-x), (4.4) directly imply that

$$\forall i \neq j,\ \forall z, z' \in W(z_0), \qquad (V_\omega e_+^{i,\mathrm{hol}}(z) | e_-^{j,\mathrm{hol}}(z')) = 0, \tag{8.6}$$

showing that the dominant matrix on the right hand side of (8.3) is diagonal.

As in the previous section, we rescale the spectral parameter around $z_0$ by the factor $h^{1/2}$, setting $z = z_w = z_0 + h^{1/2} w$, and we restrict $w$ to $O \Subset \mathbb{C}$, a bounded open connected set. For $h < h_0$, the rescaled eigenvalues in $O$ are precisely the zeros of the holomorphic function

$$\begin{aligned} F_h^\delta(w) &:= h^{J/4} G^\delta(z_w; h) \\ &= (1 + \widetilde{R}_1(w;h)) \det[\operatorname{diag}(f_j^{\delta,h}(z_w);\ j = 1, \dots, J) + \widetilde{R}_2(w;h)], \end{aligned} \tag{8.7}$$

with the notations

$$\begin{aligned} f_j^{\delta,h}(w) &:= (V_\omega | h^{-1/4} (e_-^{j,\mathrm{hol}}(z_w))^2), \qquad 1 \leq j \leq J, \\ \widetilde{R}_1(w;h) &:= R_1(z_w;h), \\ \widetilde{R}_2(w;h) &:= h^{1/4} R_2(z_w;h). \end{aligned} \tag{8.8}$$

The need for the normalization by the factor of $h^{J/4}$ will become apparent later on.

**Remark 8.1.** A direct consequence of the locality of the random potential is that the dominant term in (8.7) is given by a diagonal matrix. In the case of perturbation by a random matrix, we had found a full matrix (7.12) instead.

From now on we fix the bounded open domain $O \Subset \mathbb{C}$, and take $h_0 > 0$ small enough such that (8.7) is well-defined for all $h \leq h_0$. From the above discussion, $F_h^\delta \not\equiv 0$ is a random holomorphic function in $O$, so according to Section 6.1 the random measure

$$\mathcal{Z}_h := \sum_{w \in (F_h^\delta)^{-1}(0)} \delta_w \tag{8.9}$$

is a well-defined point process on $O$. Our aim is to study the statistical properties of this process in the limit $h \to 0$.

## *8.1. Covariance*

In this section we study the covariance of the random functions $f_j^{\delta,h}$.

**Proposition 8.2.** *Let $(f_i^{\delta,h})_{1\leq i\leq J}$ be the rescaled random functions on $O$, defined in (8.8). Then, for any $u, w \in O$, the covariance kernel satisfies*

$$\mathbb{E}\big[f_j^{\delta,h}(u)\overline{f_k^{\delta,h}(w)}\big]\mathrm{e}^{-2\phi_s^j(u;h)-2\overline{\phi_s^k(w;h)}} = \delta_{j,k}K^j(u,\overline{w})(1+\mathcal{O}(h^{1/2}))+\mathcal{O}(h^2),$$

*where*

$$K^j(u,\overline{w}) = \mathrm{e}^{\sigma_+^j(z_0)u\bar{w}}, \tag{8.10}$$

*with $\sigma_+(z_0)$ the classical density defined in (3.54), and*

$$\phi_s^j(u;h) = \tfrac{1}{2}[\log A_s^j(z_0;h) + 2(\partial_{zz}^2\Phi_{+,0}^j)(z_0)u^2], \tag{8.11}$$

*with $A_s^j(z_0;h) \sim A_0^{j,s}(z_0) + hA_1^{j,s}(z_0) + \cdots$ as defined in Section 4.2. The error terms are uniform in $u, w \in O$.*

Recall from (4.22) that $A_0^{j,\pm}(z_0) > 0$, so this proposition implies that

$$\mathbb{E}\big[f_j^{\delta,h}(u)\overline{f_k^{\delta,h}(w)}\big]\mathrm{e}^{-2\phi_s^j(u;h)-2\overline{\phi_s^k(w;h)}} \to \delta_{j,k}K^j(u,\overline{w}) \tag{8.12}$$

uniformly in $v, w \in O$ as $h \to 0$.

*Proof of Proposition 8.2.* The proof parallels that of Proposition 7.1. We define the following function on $O \times O$:

$$\begin{aligned} h^{1/2}K_h^{j,k}(u,\overline{w}) &:= h^{1/2}\mathbb{E}\big[f_j^{\delta,h}(u)\overline{f_k^{\delta,h}(w)}\big] \\ &= \sum_{n,m=0}^{N(h)-1}\mathbb{E}\big[v_n^h\overline{v_m^h}\big]\zeta_n^j(u)\,\overline{\zeta_m^k(w)}, \end{aligned} \tag{8.13}$$

where

$$\zeta_n^j(u) = (e_n|(e_-^{j,\mathrm{hol}}(z_u))^2). \tag{8.14}$$

Using the law of the restricted coefficients $v_j^h$, we estimate (8.13) by

$$\begin{aligned} &(1+\mathcal{O}(h^{2+\varepsilon_0}))\sum_m ((e_-^{k,\mathrm{hol}}(z_w)^2|e_m)(e_m|(e_-^{j,\mathrm{hol}}(z_u))^2) \\ &\qquad + \mathcal{O}(h^{6+\varepsilon_0})\sum_{n,m}\big|((e_-^{j,\mathrm{hol}}(z_u))^2|e_n)\big|\,\big|((e_-^{k,\mathrm{hol}}(z_w))^2|e_m)\big|. \end{aligned} \tag{8.15}$$

**Remark 8.3.** Here, the correlations are only determined by the squared quasimodes $(e_-^j)^2$. This is different from the case of perturbation by a random matrix, where the covariance involved all possible combinations of interactions between the quasimodes $e_-^j$ and between the $e_+^j$ (see (7.22)).

By (4.26) in Lemma 4.4,

$$\sum_{n<N(h)}\big|((e_-^{j,\mathrm{hol}}(z_u))^2|e_n)\big| \leq N(h)^{1/2}\|(e_-^{j,\mathrm{hol}}(z_u))^2\|(1+\mathcal{O}(h^\infty)).$$

Here, we have also used $h^{-1/4}\|(e_-^{j,\mathrm{hol}}(z_u))^2\| \asymp 1$, which follows directly from (4.21) and (8.18) below. Using this to estimate the term of order $h^6$ and applying Lemma 4.4 to the first term in (8.15), one gets

$$h^{1/2}K_h^{j,k}(u,\overline{w}) = ((e_-^{k,\mathrm{hol}}(z_w))^2|(e_-^{j,\mathrm{hol}}(z_u))^2) \\ + \mathcal{O}(h^2)\|(e_-^{k,\mathrm{hol}}(z_w))^2\|\,\|(e_-^{j,\mathrm{hol}}(z_u))^2\|. \tag{8.16}$$

Applying Proposition 4.3 and (4.20) to this equation, we obtain

$$h^{1/2}K_h^{j,k}(u,\overline{w}) = \delta_{j,k}\exp\Big(\frac{2}{h}\Psi_s^j(\overline{z_w},z_u;h)\Big) \\ + \mathcal{O}(h^2)\exp\Big(\frac{1}{h}(\Phi_s^k(z_w;h)+\Phi_s^j(z_u;h))\Big). \tag{8.17}$$

Next, recall the expansion (4.21) for the phase functions and write, similar to (3.48),

$$\Phi_s^j(z) = 2\Phi_{+,0}^j(z) + h\log A_s^j(z;h), \qquad \Phi_{+,0}^j(z_0) = -\operatorname{Im}\varphi_+^j(x_+^j(z),z).$$

By (3.20) and (3.49), at the point $z_0$ the phase function satisfies

$$\Phi_{+,0}^j(z_0) = (\partial_z\Phi_{+,0}^j)(z_0) = (\partial_{\bar z}\Phi_{+,0}^j)(z_0) = 0.$$

Moreover, by the discussion after (4.21) we see that $\partial^\alpha h\log(h^{1/4}A_s^j(z;h)) = \mathcal{O}(h)$ for all $\alpha\in\mathbb{N}^2$, uniformly in $z\in W(z_0)$. Thus, by Taylor expanding around $z_0$ we have, for $u\in O$ and $h<h_0$,

$$\frac{1}{h}\Phi_s^j(z_u;h) = \log h^{1/4} + 2(\partial_{z\bar z}^2\Phi_{+,0}^j)(z_0)u\overline{u} + \phi_s^j(u;h) + \phi_s^j(u;h) + \mathcal{O}(h^{1/2}), \tag{8.18}$$

uniformly in $u\in O$. Here $\phi_s^j(u;h)$ is as in the statement of the proposition. Similarly, by Proposition 4.3 and (3.54), for $u,w\in O$ we have

$$\frac{1}{h}\Psi_s^j(z_u,\overline{z_w};h) = \log h^{1/4} + 2(\partial_{z\bar z}^2\Phi_{+,0}^j)(z_0)u\overline{w} + \phi_s^j(u;h) + \overline{\phi_s^j(w;h)} + \mathcal{O}(h^{1/2}) \\ = \log h^{1/4} + \tfrac{1}{2}\sigma_+^j(z_0)u\overline{w} + \phi_s^j(u;h) + \overline{\phi_s^j(w;h)} + \mathcal{O}(h^{1/2}), \tag{8.19}$$

where the error term is uniform in $u,w\in O$. Thus, combining (8.17) with (8.18)–(8.19), and using the fact that $\phi_s^j(u;h)$ is uniformly bounded for $u\in O$, we obtain the estimate of the proposition. □

### 8.2. *Tightness*

We will follow the same arguments as in Section 7.4 to show the tightness of the sequence $(f_j^{h,\delta}(w))_{h<h_0}$ of random analytic functions defined in (8.8), in the limit $h\to 0$.

Recall that the "error terms" satisfy (5.25), uniformly in $z \in W(z_0)$ and in the restricted probability space. Similar to (7.24), it follows from Proposition 3.5 and Taylor expansion that $\mathrm{e}^{\frac{1}{h}(\Phi^j_{+,0}(z_w)+\Phi^j_{-,0}(z_w))} = \mathcal{O}(1)$, uniformly in $w \in O$. By (5.25), (8.8) we deduce the bounds

$$\widetilde{R}_1(w;h) = \mathcal{O}(\delta h^{-3/2}), \qquad \widetilde{R}_2(w;h) = \mathcal{O}(\delta h^{-11/4}), \tag{8.20}$$

uniformly in $w \in O$ and in the restricted probability space.

Let $K \Subset O$, and choose $\varepsilon > 0$ small enough that $K_\varepsilon = K + \overline{D(0,\varepsilon)} \Subset O$. By Proposition 8.2, for $h_0 > 0$ small enough we have

$$\sup_{0<h<h_0} \mathbb{E}[\| f_j^{h,\delta}\|^2_{L^2(K_\varepsilon)}] < C(K_\varepsilon) < \infty. \tag{8.21}$$

Since $F_h^\delta$ is holomorphic, we show as for (7.29) that

$$\| F_h^\delta\|^p_{L^\infty(K)} \le C_{K_\varepsilon} \int_{K_\varepsilon} |F_h^\delta(w)|^p \, L(dw). \tag{8.22}$$

Using Markov's inequality, (8.20) and (8.7) in combination with (8.22) for $p = 2/J$, one finds that for $h_0 >$ small enough,

$$\begin{aligned} \sup_{0<h<h_0} \mathbb{P}[\| F_h^\delta\|^2_{L^\infty(K)} > r] &\le r^{-1/J} C_{K_\varepsilon} \sup_{0<h<h_0} \mathbb{E}\left[\int_{K_\varepsilon} |F_h^\delta(w)|^{2/J} \, L(dw)\right] \\ &\le r^{-1/J}\Big(C_1 C_{K_\varepsilon} \sup_{0<h<h_0} \sum_{j=1}^J \mathbb{E}[|f_j^{h,\delta}(w)|^2] \, L(dw) + C_2\Big) \le \mathcal{O}(r^{-1/J}), \end{aligned} \tag{8.23}$$

where all the constants are independent of $r > 0$. In the above we have used the algebraic bounds

$$\begin{aligned} &|\det[\mathrm{diag}(f_j^{\delta,h}(w))_{j=1,\dots,J} + \widetilde{R}_2(w;h)]|^{2/J} \\ &\qquad \le 2\|\mathrm{diag}(f_j^{\delta,h}(w))_{j=1,\dots,J}\|^2_{\mathrm{HS}} + 2\|\widetilde{R}_2(w;h)\|^2_{\mathrm{HS}} \\ &\qquad \le 2\sum_{j=1}^J |f_j^{h,\delta}(w)|^2 + 2\|\widetilde{R}_2(w;h)\|^2_{\mathrm{HS}}. \end{aligned}$$

Hence, in view of Proposition 6.9, we conclude that $F_h^\delta$ is a tight sequence of random analytic functions.

### *8.3. Weak convergence to a Gaussian analytic function*

In this section we will show that the random analytic functions $F_h^\delta$ (see (8.7)) converge in distribution, when $h \to 0$, to a product of independent Gaussian analytic functions. We start by the following result, analogue of Proposition 7.3:

**Proposition 8.4.** *The random functions $f_j^{h,\delta}$ defined in (8.8) satisfy*

$$(f_j^{h,\delta})_{1\leq j\leq J} \xrightarrow{\text{fd}} (f_j^{\text{GAF}})_{1\leq j\leq J}, \qquad h\to 0. \tag{8.24}$$

*Here $(f_j^{\text{GAF}})_{1\leq j\leq J}$ are independent Gaussian analytic functions on $O$ with covariance kernels*

$$K^j(u,\overline{w})\mathrm{e}^{2\phi_s^j(u;0)+2\overline{\phi_s^j(w;0)}}, \qquad u,w\in O, \tag{8.25}$$

*where $K^j$ is the kernel in (8.10), while $\phi_s^j(u;0)$ is the quadratic polynomial from (8.11).*

We have the following immediate consequence, proved similarly to Corollary 7.4:

**Corollary 8.5.** *Under the assumptions of Proposition 8.4, we have*

$$F_h^\delta \xrightarrow{d} \prod_{j=1}^J f_j^{\text{GAF}}, \qquad h\to 0. \tag{8.26}$$

*Proof of Proposition 8.4.* The proof is similar to that of Proposition 7.3. For $L\in\mathbb{N}^*$ let $w_1,\dots,w_L\in O$ and $\lambda=(\lambda_l^j; 1\leq j\leq J, 1\leq l\leq L)\in\mathbb{C}^{J\cdot L}$. Consider the complex valued random variable

$$S(\lambda) := \sum_{j,l}\lambda_l^j f_j^{h,\delta}(w_l) = \sum_{n<N(h)} v_n^h G_n, \qquad G_n = \sum_{j,l}\lambda_l^j h^{-1/4}\zeta_n^j(z_{w_l}),$$

where $\zeta_n^j$ is as in (8.14). We are going to show that for any such $\lambda$, $S(\lambda)$ converges in distribution to the complex valued random variable

$$S^{\text{GAF}}(\lambda) := \sum_{j,l}\lambda_l^j f_j^{\text{GAF}}(w_l).$$

By the Cramér–Wold Theorem this implies the convergence (8.24).

To prove the limit $S(\lambda)\xrightarrow{d} S^{\text{GAF}}(\lambda)$, we will use the Central Limit Theorem 6.12. We thus need to check that the family $(S(\lambda))_{0<h\leq h_0}$ of random variables satisfies the four conditions of the theorem, remembering that the variables $v_j^h$ are distributed like $\alpha^h$ in (7.3).

We begin by estimating the average of $S(\lambda)$. From (7.7), we get

$$\begin{aligned}\sum_{n<N(h)}|\mathbb{E}[v_n^h G_n]| &\leq \mathcal{O}(h^{3+\varepsilon_0})\sum_{n<N(h)}|G_n|\\ &\leq \mathcal{O}(h^{3+\varepsilon_0})\sum_{j,l}\sum_{n<N(h)}\left|(h^{-1/4}(e_-^{j,\text{hol}}(z_{w_l})^2|e_n)\right|\\ &\leq \mathcal{O}(h^{3+\varepsilon_0})N(h)\sum_{j,l}\|h^{-1/4}(e_-^{j,\text{hol}}(z_{w_l}))^2\|\\ &\leq \mathcal{O}(h^{1+\varepsilon_0}).\end{aligned} \tag{8.27}$$

In the third inequality we have used two Cauchy–Schwarz inequalities, and in the last one the fact that $\|h^{-1/4}(e_-^{j,\mathrm{hol}}(z_w))^2\| = \mathcal{O}(1)$ uniformly when $w \in O$. This proves point (i) in Theorem 6.12.

Let us now check condition (ii). Using again (7.7) and (4.26), we get the bound

$$\Big|\sum_{n<N(h)} \mathbb{E}[(v_n^h)^2 G_n^2]\Big| \leq \mathcal{O}(h^{2+\varepsilon_0}) \sum_n |G_n|^2$$
$$\leq \mathcal{O}(h^{2+\varepsilon_0}) \sum_{j,l} \sum_{n<N(h)} \big|(h^{-1/4}(e_-^{j,\mathrm{hol}}(z_{w_l}))^2|e_n)\big|^2 \leq \mathcal{O}(h^{2+\varepsilon_0}). \tag{8.28}$$

Next, we compute the variance of $S(\lambda)$. By (7.7) and (7.8),

$$\mathbb{E}\Big[\sum_{n<N(h)} |v_n^h G_n|^2\Big] = (1+\mathcal{O}(h^{2+\varepsilon_0})) \sum_{j,l,s,t} \lambda_l^j \overline{\lambda_t^s} \sum_{n<N(h)} h^{-1} \zeta_n^j(z_{w_l}) \overline{\zeta_n^s(z_{w_t})},$$

which, by (8.14) and Lemma 4.4 is equal to

$$(1+\mathcal{O}(h^{2+\varepsilon_0})) \sum_{j,l,s,t} \lambda_l^j \overline{\lambda_t^s} h^{-1/2}((e_-^{j,\mathrm{hol}}(z_{w_t}))^2|(e_-^{s,\mathrm{hol}}(z_{w_l}))^2) + \mathcal{O}(h^\infty). \tag{8.29}$$

Here, we have also used $\|h^{-1/4}\, e_\pm^{k,\mathrm{hol}}(z_{w_l})\| = \mathcal{O}(1)$. By (4.20) we see that the terms in (8.29) with $j \neq s$ vanish. Similar to the proof of Proposition 8.2, we then obtain the limit

$$\mathbb{E}\Big[\sum_{n<N(h)} |v_n^h G_n|^2\Big] \xrightarrow{h\to 0} \sum_{j,l,t} \lambda_l^j \overline{\lambda_t^j} K^j(w_l, \overline{w}_t) \mathrm{e}^{2\phi_s^j(w_l;0)+\overline{2\phi_s^j(w_t;0)}} := \sigma(\lambda, w), \tag{8.30}$$

which settles condition (iii).

To check condition (iv), we use as in (7.5) a bound by $(4+\epsilon_0)$-moments: for any $\varepsilon > 0$,

$$\sum_{n<N(h)} \mathbb{E}[|v_n^h G_n|^2 \mathbb{1}_{\{|v_n^h G_n|>\varepsilon\}}] < \varepsilon^{-(2+\varepsilon_0)} \sum_{n<N(h)} \mathbb{E}[|v_n^h G_n|^{4+\varepsilon_0}] \to 0. \tag{8.31}$$

By (7.9) and the Hölder inequality, we find

$$\sum_{n<N(h)} \mathbb{E}[|v_n^h G_n|^{4+\varepsilon_0}] \leq C_\lambda \sum_{j,l} \sum_{n<N(h)} |h^{-1/4} \zeta_n^j(z_{w_l})|^{4+\varepsilon_0}$$
$$= C_\lambda \sum_{j,l} \sum_{m<N(h)} \big|(h^{-1/4}(e_-^{j,\mathrm{hol}}(z_{w_l}))^2|e_m)\big|^{4+\varepsilon_0}.$$

Using the fact that $(e_m)$ is an orthonormal basis of $L^2(\mathbb{R})$, as well as the uniform estimates $\mathcal{O}(h^{1/4})$ for the overlaps (see Lemma 4.5), one obtains

$$\sum_{n<N(h)} \mathbb{E}[|v_n^h G_n|^{4+\varepsilon_0}] = \mathcal{O}(h^{(2+\varepsilon_0)/4}) \sum_{j,l} \|(h^{-1/4}(e_-^{j,\mathrm{hol}}(z_{w_l}))^2\|^2 \mathrm{e}^{\frac{4}{h}\Phi_{+,0}^j(z_{w_l})}$$
$$= \mathcal{O}(h^{(2+\varepsilon_0)/4}).$$

In the last line we have used again uniform bounds for the norms and the exponentials $e^{\frac{4}{h}\Phi^j_{+,0}}$ throughout $O$, as in (8.18), to see that the exponentials are of order $\mathcal{O}(1)$.

After checking the four conditions, we may apply the Central Limit Theorem 6.12 to show that $S(\lambda)$ converges in distribution to the random Gaussian variable $\sim \mathcal{N}_{\mathbb{C}}(0, \sigma(\lambda, w))$, with variance given in (8.30).

On the other hand, since $(f_j^{\mathrm{GAF}})_{j\le J}$ are independent Gaussian analytic functions with covariance kernel (8.25), it follows that $\sum_{j,l} \lambda_l^j f_j^{\mathrm{GAF}}(w_l)$ is a complex centred Gaussian variable with variance

$$\begin{aligned}\mathbb{E}\Big[\Big|\sum_{j,l} \lambda_l^j f_j^{\mathrm{GAF}}(w_l)\Big|^2\Big] &= \sum_{j,l,s,t} \lambda_l^j \overline{\lambda_t^s}\, \mathbb{E}\big[f_j^{\mathrm{GAF}}(w_l)\overline{f_s^{\mathrm{GAF}}(w_t)}\big] \\ &= \sum_{j,l,t} \lambda_l^j \overline{\lambda_t^j} K^j(w_l, \overline{w}_t) e^{2\phi_s^j(w_l;0)+\overline{2\phi_s^j(w_t;0)}} = \sigma(\lambda, w), \end{aligned} \tag{8.32}$$

where in the last line we recover the expression (8.30). We conclude that $S(\lambda) \xrightarrow{d} S^{\mathrm{GAF}}(\lambda)$ for any $\lambda \in \mathbb{C}^{JL}$, which implies the convergence (8.24). □

The GAFs $f_j^{\mathrm{GAF}}$ appearing in Proposition 8.4 and Corollary 8.5 have covariance kernels $K^j(u, \bar{w})e^{2\phi_s^j(u;0)}e^{\overline{2\phi_s^j(w;0)}}$. Since $e^{2\phi_s^j(u;0)}$ is a nonvanishing deterministic holomorphic function on $O$, we may divide by this "gauge factor" to obtain

$$g_{z_0}^j(w) := f_j^{\mathrm{GAF}}(w)e^{-2\phi_s^j(w;0)},$$

a Gaussian analytic function on $O$ with covariance kernel $K^j(u, \bar{w})$, as in the statement of Theorem 2.5. Moreover, the $J$ random functions $(g_{z_0}^j(w))_{1\le j\le J}$ are independent of each other.

It is clear that the random holomorphic functions $\prod_{j=1}^J f_j^{\mathrm{GAF}}(w)$ and $G_{z_0}(w) = \prod_{j=1}^J g_{z_0}^j(w)$ admit the same zero process $\mathcal{Z}_{G_{z_0}}$ on $O$. Hence, by (8.26) and Proposition 6.11,

$$\mathcal{Z}_{F_h^\delta} \xrightarrow{d} \mathcal{Z}_{G_{z_0}}, \qquad h \to 0.$$

Taking into account the discussion at the beginning of Section 7 and the expression (7.16) for the covariance kernel, this completes the proof of Theorem 2.5.

### *8.4. Correlation functions*

Let $\mu_k^h = \mu_{h,z_0}^{k,V}$ be the $k$-point density measure of the process $\mathcal{Z}_h^V$, and $\mu_k = \mu_{z_0}^{k,V}$ be the $k$-point density measure of the point process $\mathcal{Z}_{G_{z_0}}$, defined in Theorem 2.6. Following the same arguments as in Section 7.6, we obtain the first part of Theorem 2.6: for any test function $\varphi \in \mathcal{C}_c(O^k \setminus \Delta, \mathbb{R}_+)$,

$$\int \varphi\, d\mu_k^h \to \int \varphi\, d\mu_k \quad \text{as } h \to 0.$$

Recall from Theorem 2.5 that $G_{z_0}(z) = \prod_{j=1}^{J} g^j_{z_0}(z)$ where the $g_j = g^j_{z_0}$ are independent Gaussian analytic functions. To complete the proof of Theorem 2.6, we will show, using the simple product structure of $G_{z_0}(z)$, that the $k$-point measure $\mu_k$ has a continuous Lebesgue density $d^k_G$, called the $k$-point density, which can be explicitly computed, using the expressions of the $k'$-point densities of the GAFs $g_j$. This is not surprising, remembering that the process $\mathcal{Z}_{G_{z_0}}$ is the superposition of the $J$ independent processes $\mathcal{Z}_{g_j}$. This contrasts with the case of the point process associated with random matrix perturbations of $P_h$ (see Theorem 2.9), for which the computation of the limiting $k$-densities remains an open problem.

Denoting $\Delta_w = \partial_{\bar{w}}\partial_w$ and $w = (w_1, \dots, w_k)$, the generalization of (7.54) reads

$$\mu_k(dw) = (2\pi)^{-k}\Big(\prod_{i=1}^{k} \Delta_{w_i}\Big)\mathbb{E}[\log|G_{z_0}(w_1)|\cdots\log|G_{z_0}(w_k)|]\, L(dw)$$
$$=: d^k_G(w)\, L(dw), \tag{8.33}$$

in the sense of distributions. This follows from the Poincaré Lelong formula and Fubini's theorem (see e.g. [30]). Applying the product structure of $G_{z_0}$, we obtain

$$d^k_G(w) = \sum_{\beta_1,\dots,\beta_k=1}^{J} \Big(\prod_{i=1}^{k} \Delta_{w_i}\Big)\mathbb{E}[\log|g_{\beta_1}(w_1)|\cdots\log|g_{\beta_k}(w_k)|]. \tag{8.34}$$

Since the $g_j$ are mutually independent, for each term $\beta = (\beta_1, \dots, \beta_k)$ the expectation in (8.34) factorizes, each factor grouping together the identical GAF $g_{\beta_i} = g_j$. Upon applying the relevant derivatives $\Delta_{w_i}$, each such factor yields a certain density associated with the GAF $g_j$.

According to [30, Cor. 3.4.2], for any Gaussian analytic function $f$ on an open set $U \subset \mathbb{C}$ with covariance kernel $K_f(u, \overline{w})$, if $\det(K_f(w_i, \overline{w_j})_{1\le i,j\le k})$ does not vanish anywhere on $U^k \setminus \Delta$, then the $k$-point measure of $\mathcal{Z}_f$ has a continuous Lebesgue density $d^k_f(w)$, given by

$$d^k_f(w) = (2\pi)^{-k}\Big(\prod_{i=1}^{k} \Delta_{w_i}\Big)\mathbb{E}[\log|f(w_1)|\cdots\log|f(w_k)|]$$
$$= \frac{\operatorname{perm}(C(w) - B(w)A^{-1}(w)B^*(w))}{\pi^k \det A(w)}, \tag{8.35}$$

where perm$(\cdot)$ denotes the permanent of a matrix and $A$, $B$, $C$ are complex $k \times k$ matrices with entries

$$A_{i,j}(w) = K(w_i, \overline{w_j}), \quad B_{i,j}(w) = (\partial_w K)(w_i, \overline{w_j}), \quad C_{i,j}(w) = (\partial_{w\overline{w}} K)(w_i, \overline{w_j}).$$

Let us investigate the $k$-densities $d^k_{g_j}$ associated with the individual processes $\mathcal{Z}_{g_j}$. Theorem 2.5 shows that for any $j = 1, \dots, J$ and any $k \ge 1$, the covariance kernel of the GAF $g_j(w)$ leads to the determinant

$$\det A(w) = \det((K^j(w_n, \overline{w}_m))_{1\le n,m\le k}) = \det((\mathrm{e}^{\sigma^j_+(z_0) w_n \overline{w}_m})_{1\le n,m\le k}).$$

This determinant vanishes if and only if $w_i = w_j$ for some $i \neq j$, that is, if $w$ is in the diagonal $\Delta$. Therefore, the $k$-point density functions $d^k_{g_j}$ exist, and can be expressed in the form (8.35).

Any $k$-density is symmetric with respect to permutations of its entries: denoting by $\mathfrak{S}_k$ the permutation group of $k$ elements,

$$d^k_f(w) = \frac{1}{k!} \sum_{\tau \in \mathfrak{S}_k} d^k_f(w_{\tau(1)}, \dots, w_{\tau(k)}). \tag{8.36}$$

Let us come back to the decomposition (8.34). First, write

$$d^k_G(w) = \frac{1}{k!} \sum_{\tau \in \mathfrak{S}_k} d^k_G(w_{\tau(1)}, \dots, w_{\tau(k)}). \tag{8.37}$$

Then, simple combinatorics allows us decompose the index set in (8.34) as follows:

$$\begin{aligned} \{1 \le \beta_1, \dots, \beta_k \le J\} &= \bigsqcup_{\alpha \in \mathbb{N}^J,\, |\alpha| = k} A_\alpha, \\ A_\alpha &= \{\beta \in \mathbb{N}^k;\ \text{for each } j = 1, \dots, J, \text{ there are } \alpha_j \text{ indices } \beta_i = j\}. \end{aligned} \tag{8.38}$$

Here $|\alpha| = \alpha_1 + \dots + \alpha_J$ for $\alpha \in \mathbb{N}^J$. Applying the decomposition (8.38) and the permutation symmetry to (8.34), we write

$$d^k_G(w) = \sum_{\alpha \in \mathbb{N}^J,\, |\alpha| = k} \sum_{\beta \in A_\alpha} \frac{1}{k!} \sum_{\tau \in \mathfrak{S}_k} \Big(\prod_{i=1}^k \Delta_{w_i}\Big) \mathbb{E}[\log |g_{\beta_1}(w_{\tau(1)})| \cdots \log |g_{\beta_k}(w_{\tau(k)})|].$$

Since we sum over all permutations $\tau \in \mathfrak{S}_k$, and the sum over each $\beta \in A_\alpha$ corresponds to all possible orderings for a fixed configuration $\alpha$, it follows together with the first line of (8.35) that

$$d^k_G(w) = \sum_{\alpha \in \mathbb{N}^J,\, |\alpha| = k} \frac{1}{k!} \binom{k}{\alpha} \sum_{\tau \in \mathfrak{S}_k} \prod_{j=1}^J d^{\alpha_j}_{g_j}(w_{\tau(\alpha_1 + \dots + \alpha_{j-1} + 1)}, \dots, w_{\tau(\alpha_1 + \dots + \alpha_j)}),$$

where $\binom{k}{\alpha} = \frac{k!}{\alpha!} := \frac{k!}{\alpha_1! \cdots \alpha_J!}$ is the multinomial coefficient. This is exactly the expression (2.27), and this completes the proof of Theorem 2.6. $\square$

To end this presentation, we show the explicit formula (2.30) for the 2-point correlation function of the limiting point process $\mathcal{Z}_{G_{z_0}}$. Notice first that

$$d^1_G(w) = \sum_{j=1}^J d^1_{g_j}(w).$$

From (8.35) one calculates directly that

$$d^1_{g_j}(w) = \frac{1}{\pi} \partial_w \partial_{\bar{w}} \log K^j(w, \overline{w}) = \frac{\sigma^j_+(z_0)}{\pi}.$$

Hence,

$$d^1_G(w) = \frac{1}{\pi}\sum_{j=1}^{J}\sigma_+^j(z_0) \quad \text{for all } w \in \mathbb{C}.$$

Next, as a particular case of (2.27), we find

$$d^2(z_1,z_2) = \sum_{j=1}^{J} d^2_{g_j}(z_1,z_2) + \sum_{\substack{i,j=1\\ i\neq j}}^{J} d^1_{g_i}(z_1)d^1_{g_j}(z_2).$$

A cumbersome but straightforward calculation, using (8.35), shows that for any $w_1 \neq w_2$,

$$d^2_{g_j}(w_1,w_2) = \left(\frac{\sigma_+^j(z_0)}{\pi}\right)^2 \kappa\left(\frac{\sigma_+^j(z_0)}{2}|w_1-w_2|^2\right),$$

where

$$\kappa(t) = \frac{(\sinh^2 t + t)\cosh t - 2t\sinh t}{\sinh^3 t}, \qquad t \geq 0.$$

A Taylor expansion shows that $\kappa(t) = t(1+\mathcal{O}(t^2))$ as $t \to 0^+$, and $\kappa(t) = 1+\mathcal{O}(t^2 \mathrm{e}^{-2t})$ as $t \to \infty$. This expression has been obtained before in [26, 3]. We then obtain the following formula for the 2-point correlation function of $\mathcal{Z}_{G_{z_0}}$, as stated in (2.30):

$$\begin{aligned} K^2(w_1,w_2) &:= \frac{d^2(w_1,w_2)}{d^1(w_1)d^1(w_1)} \\ &= 1 + \sum_{j=1}^{J}\frac{(\sigma_+^j(z_0))^2}{(\sum_{j=1}^{J}\sigma_+^j(z_0))^2}\left[\kappa\left(\frac{\sigma_+^j(z_0)}{2}|w_1-w_2|^2\right) - 1\right]. \end{aligned}$$

For $|w_1 - w_2| \gg 1$, we have the asymptotics

$$K^2(w_1,w_2) = 1 + \mathcal{O}\big((\sigma_+^j(z_0)|w_1-w_2|^2)^2 \mathrm{e}^{-\min_j \sigma_+^j(z_0)|w_1-w_2|^2}\big),$$

while for $|w_1 - w_2| \ll 1$,

$$K^2(w_1,w_2) = 1 - \sum_{j=1}^{J}\frac{(\sigma_+^j(z_0))^2}{(\sum_{j=1}^{J}\sigma_+^j(z_0))^2}\left[1 - \frac{\sigma_+^j(z_0)}{2}|z_1-z_2|^2(1+\mathcal{O}(|z_1-z_2|^4))\right].$$

In particular, this 2-point correlation does not vanish when $|w_1 - w_2| \to 0$.

### *8.5. Invariance by isometries*

In this last subsection, we prove Proposition 2.12, which expresses the invariance of our limiting processes with respect to the direct isometries of $\mathbb{C}$, for both types of random perturbations.

We first show this invariance in the case of the potential perturbation, namely the case of Theorem 2.5. A direct isometry takes the form $\tau(w) = \alpha w + \beta$ with $\alpha, \beta \in \mathbb{C}$, $|\alpha| = 1$. By the continuous mapping theorem [31, Theorem 3.27] and the discussion after (6.2), it is sufficient to show that there exists a deterministic holomorphic function $\Phi(w)$ such that

$$G_{z_0} \circ \tau \overset{d}{=} G_{z_0} \mathrm{e}^{\Phi} \quad \text{as random holomorphic functions on } \mathbb{C}. \tag{8.39}$$

Recall that $G_{z_0}(w) = \prod_{j=1}^{J} g_j(w)$, where $(g_j)_{1\le j\le J}$ are $J$ independent GAFs on $\mathbb{C}$, with covariance kernels

$$K^j(v, \overline{w}) = \mathrm{e}^{\sigma_+^j(z_0) v\overline{w}}. \tag{8.40}$$

Then the translated functions $(g_j \circ \tau)_{1\le j\le J}$ are $J$ independent Gaussian analytic functions on $\mathbb{C}$, with covariance kernels

$$K^j(\tau(v), \overline{\tau(w)}) = \mathrm{e}^{\sigma_+^j(z_0) v\overline{w}}\, \mathrm{e}^{\phi_+^i(v) + \overline{\phi_+^i(w)}}, \qquad \phi_+^j(w) = \sigma_+^i(z_0)\big(\alpha\overline{\beta} w + \tfrac{1}{2}|\beta|^2\big). \tag{8.41}$$

Hence, the GAFs $g_j \circ \tau$ and $g_j \mathrm{e}^{\phi_+^j}$ are equal in distribution, since they have the same covariance kernel. By the continuous mapping theorem, the random analytic functions $G_{z_0} \circ \tau$ and $G_{z_0}\mathrm{e}^{\Phi}$ are equal in distribution if we take $\Phi(w) = \sum_{j=1}^{J} \phi_+^j(w)$.

The case of the process $\mathcal{Z}_{\tilde{G}_{z_0}}$ of Theorem 2.9 is treated similarly. Each entry $g_{i,j} = g_{z_0}^{i,j}$ of the matrix defining $\tilde{G}_{z_0}$ is a GAF of kernel $K_{\sigma_{z_0}^{ij}}(v, \overline{w})$ with $\sigma_{z_0}^{ij} = \frac{1}{2}(\sigma_+^i(z_0) + \sigma_-^j(z_0))$. Therefore, upon composing with $\tau$, we obtain a shifted kernel

$$K_{\sigma_{z_0}^{ij}}(\tau(v), \overline{\tau(w)}) = K_{\sigma_{z_0}^{ij}}(v, \overline{w}) e^{\phi^{ij}(v) + \overline{\phi^{ij}(w)}}, \qquad \phi^{ij}(v) = \tfrac{1}{2}\phi_+^i(v) + \tfrac{1}{2}\phi_-^j(v),$$

where we have used the notation of (8.41) for $\phi_+^i$, and a corresponding one for $\phi_-^j$. Therefore, the GAF $g_{i,j} \circ \tau$ has the same distribution as the GAF $g_{i,j} e^{\phi^{ij}} = g_{i,j} e^{\frac{1}{2}\phi_+^i + \frac{1}{2}\phi_-^j}$. Thanks to this product structure, the determinant $\tilde{G}_{z_0} \circ \tau$ can be factorized as

$$\det((g_{ij} e^{\frac{1}{2}\phi_+^i(v) + \frac{1}{2}\phi_-^j})_{i,j}) = \det(\Lambda_+^\tau) \det((g_{ij})_{i,j}) \det(\Lambda_-^\tau), \qquad \Lambda_\pm^\tau = \mathrm{diag}(e^{\frac{1}{2}\phi_\pm^i}).$$

Since the matrices $\Lambda_\pm^\tau$ are nonsingular, the zero process of the left hand side is identical to that of $\det((g_{ij})_{i,j}) = \tilde{G}_{z_0}$. □

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